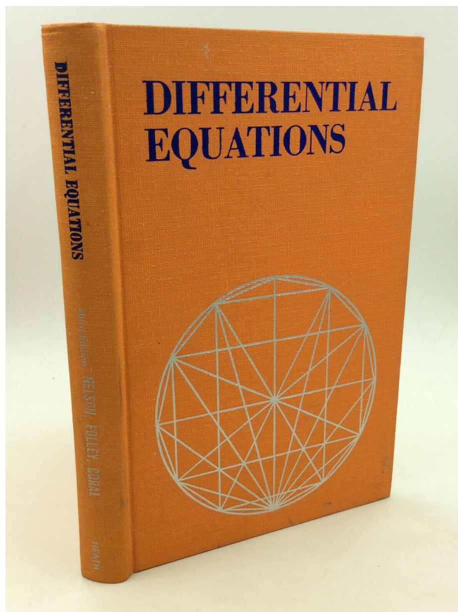
A Solution Manual For

Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964



Nasser M. Abbasi

May 15, 2024

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1.1 problem 1

Internal problem ID [1870]
Internal file name [`OUTPUT/1871_Sunday_June_05_2022_02_36_19_AM_5652881/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{yx + \left(x^2+1\right) y' = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{xy}{x^2+1} \end{aligned}$$

Where $f(x) = -\frac{x}{x^2+1}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= -\frac{x}{x^2+1}\, dx \\
\int \frac{1}{y}\, dy &= \int -\frac{x}{x^2+1}\, dx \\
\ln(y) &= -\frac{\ln(x^2+1)}{2} + c_1 \\
y &= \mathrm{e}^{-\frac{\ln(x^2+1)}{2}+c_1} \\
&= \frac{c_1}{\sqrt{x^2+1}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{\sqrt{x^2+1}} \tag{1}
$$

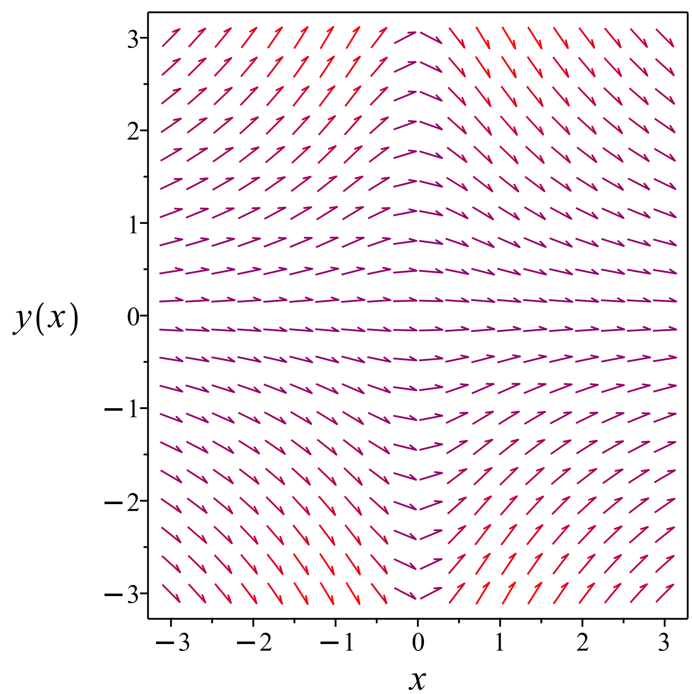


Figure 1: Slope field plot

Verification of solutions

$$
y = \frac{c_1}{\sqrt{x^2+1}}
$$

Verified OK.

1.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{x}{x^2+1} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{xy}{x^2+1} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{x}{x^2+1}dx} \\ &= \sqrt{x^2+1} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\sqrt{x^2+1}\, y\right) &= 0 \end{aligned}$$

Integrating gives

$$\sqrt{x^2+1}\, y = c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{x^2+1}$ results in

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2+1}} \tag{1}$$

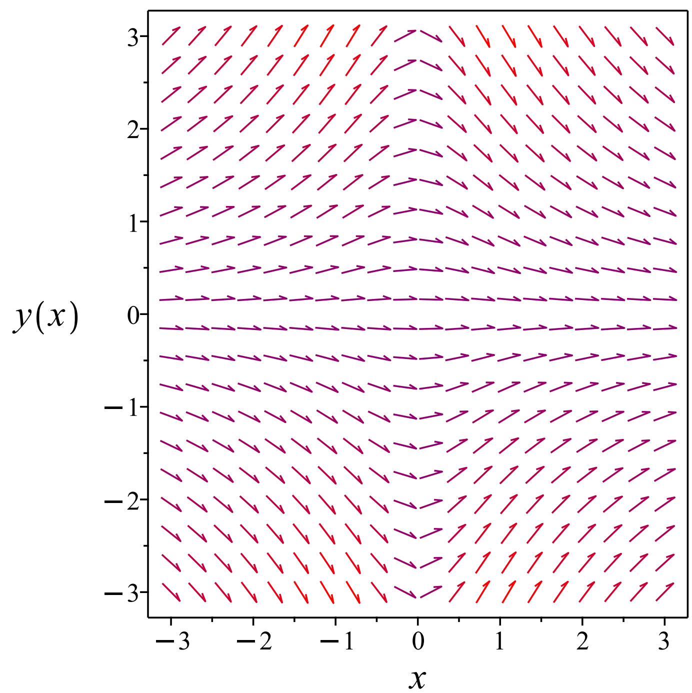


Figure 2: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2 + 1}}$$

Verified OK.

1.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x^2 + \left(x^2 + 1\right)\left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(1 + 2x^2)}{x\,(x^2 + 1)} \end{aligned}$$

Where $f(x) = -\frac{1+2x^2}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1+2x^2}{x\,(x^2+1)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1+2x^2}{x\,(x^2+1)}\,dx \\
\ln(u) &= -\ln(x) - \frac{\ln(x^2+1)}{2} + c_2 \\
u &= \mathrm{e}^{-\ln(x)-\frac{\ln(x^2+1)}{2}+c_2} \\
&= c_2\mathrm{e}^{-\ln(x)-\frac{\ln(x^2+1)}{2}}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2}{x\sqrt{x^2+1}}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{c_2}{\sqrt{x^2+1}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2}{\sqrt{x^2+1}} \tag{1}$$

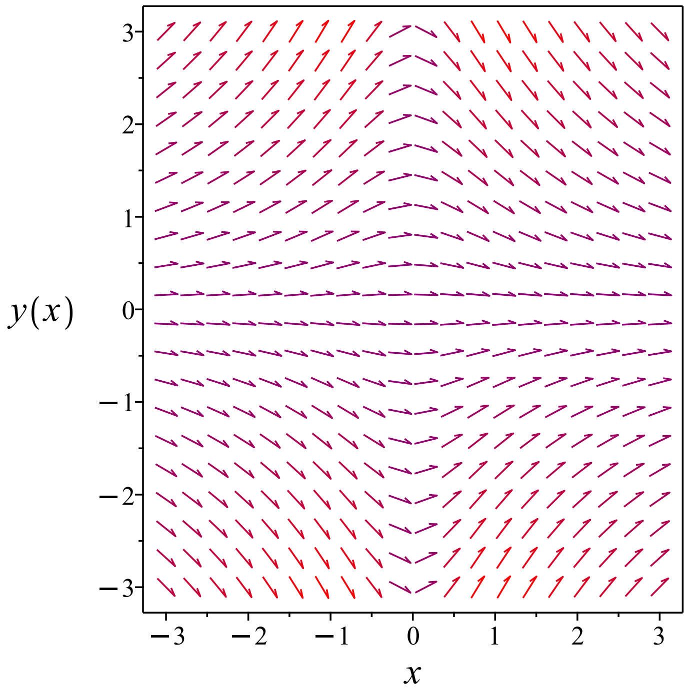


Figure 3: Slope field plot

Verification of solutions

$$y = \frac{c_2}{\sqrt{x^2+1}}$$

Verified OK.

1.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{xy}{x^2+1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sqrt{x^2 + 1}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x^2+1}}} dy\end{aligned}$$

Which results in

$$S = \sqrt{x^2+1}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{xy}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{yx}{\sqrt{x^2+1}} \\ S_y &= \sqrt{x^2+1}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x^2+1}\, y = c_1$$

Which simplifies to

$$\sqrt{x^2+1}\, y = c_1$$

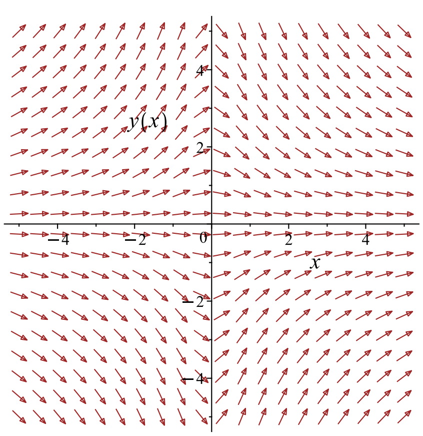
Which gives

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{xy}{x^2+1}$



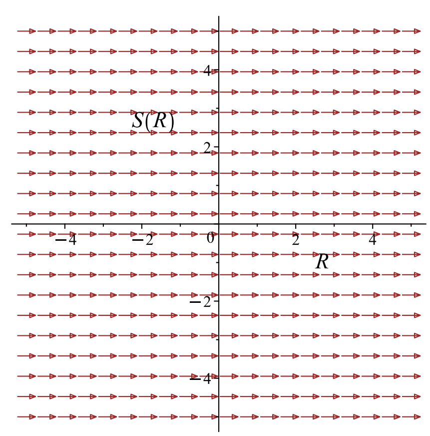
</td><td>

$R = x$

$S = \sqrt{x^2+1}\, y$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2+1}} \tag{1}$$

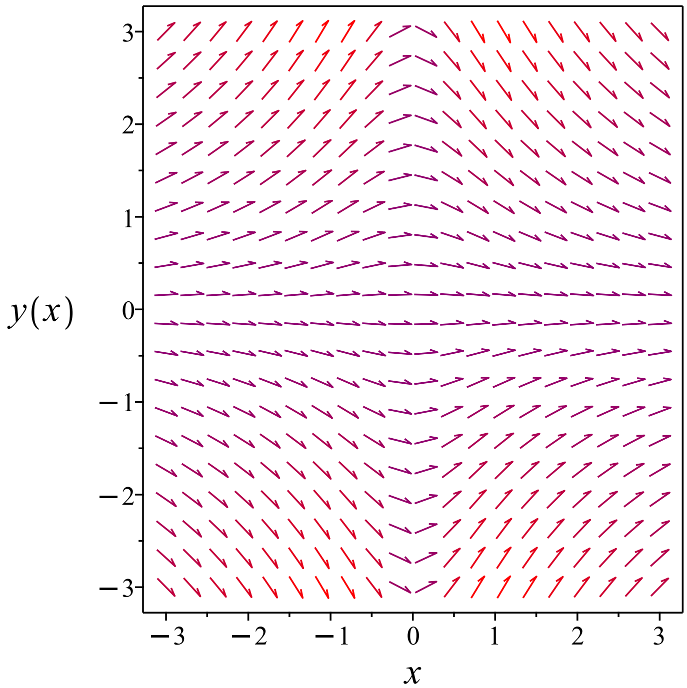


Figure 4: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2 + 1}}$$

Verified OK.

1.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y = \left(\frac{x}{x^2+1}\right)\mathrm{d}x$$
$$\left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(-\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x}{x^2+1}$$
$$N(x,y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{x}{x^2+1} \,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\ln(x^2+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y\\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} - \ln(y)$$

The solution becomes

$$y = e^{-\frac{\ln(x^2+1)}{2} - c_1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = e^{-\frac{\ln(x^2+1)}{2} - c_1} \tag{1}$$

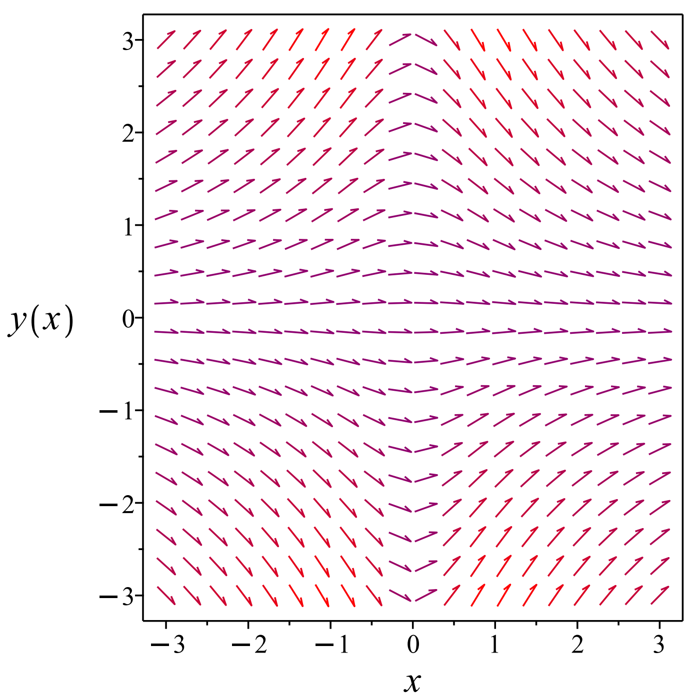


Figure 5: Slope field plot

Verification of solutions

$$y = e^{-\frac{\ln\left(x^2+1\right)}{2}-c_1}$$

Verified OK.

1.1.6 Maple step by step solution

Let's solve

$yx + (x^2+1)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{x}{x^2+1}dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{\ln(x^2+1)}{2} + c_1$

- Solve for y

  $y = e^{-\frac{\ln\left(x^2+1\right)}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve(x*y(x)+(x^2+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 22

```
DSolve[x*y[x]+(x^2+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{\sqrt{x^2+1}}$$
$$y(x) \to 0$$

1.2 problem 2

Internal problem ID [1871]
Internal file name [`OUTPUT/1872_Sunday_June_05_2022_02_36_21_AM_43027433/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy^2 + (y - x^2 y)\, y' = -x}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(y^2+1)}{y\,(x^2-1)} \end{aligned}$$

Where $f(x) = \frac{x}{x^2-1}$ and $g(y) = \frac{y^2+1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{y}}\, dy = \frac{x}{x^2-1}\, dx$$

$$\int \frac{1}{\frac{y^2+1}{y}}\,dy = \int \frac{x}{x^2-1}\,dx$$

$$\frac{\ln\left(y^2+1\right)}{2} = \frac{\ln\left(x-1\right)}{2} + \frac{\ln\left(x+1\right)}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = c_2 \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}$$

Which simplifies to

$$\sqrt{1+y^2} = c_2\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_1}$$

The solution is

$$\sqrt{1+y^2} = c_2\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_1}$$

Summary

The solution(s) found are the following

$$\sqrt{1+y^2} = c_2\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_1} \tag{1}$$

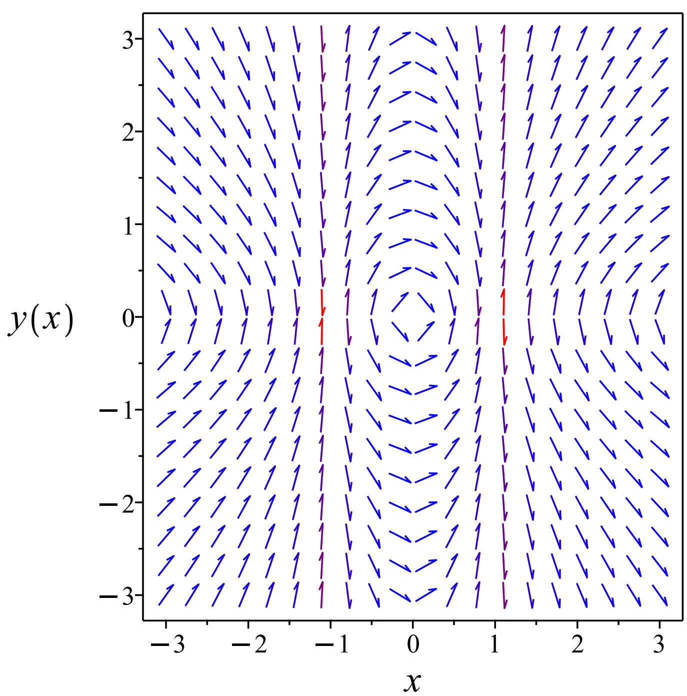


Figure 6: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} = c_2\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_1}$$

Verified OK.

1.2.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^3u(x)^2 + \left(u(x)\,x - x^3u(x)\right)\left(u'(x)\,x + u(x)\right) = -x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2+1}{ux\,(x^2-1)} \end{aligned}$$

Where $f(x) = \frac{1}{x(x^2-1)}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2+1}{u}}\,du &= \frac{1}{x\,(x^2-1)}\,dx \\ \int \frac{1}{\frac{u^2+1}{u}}\,du &= \int \frac{1}{x\,(x^2-1)}\,dx \\ \frac{\ln(u^2+1)}{2} &= -\ln(x) + \frac{\ln(x+1)}{2} + \frac{\ln(x-1)}{2} + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = \mathrm{e}^{-\ln(x)+\frac{\ln(x+1)}{2}+\frac{\ln(x-1)}{2}+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = c_3\mathrm{e}^{-\ln(x)+\frac{\ln(x+1)}{2}+\frac{\ln(x-1)}{2}}$$

Which simplifies to

$$\sqrt{u(x)^2+1} = \frac{c_3\sqrt{x+1}\,\sqrt{x-1}\,\mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u(x)^2+1} = \frac{c_3\sqrt{x+1}\,\sqrt{x-1}\,\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+1} = \frac{c_3\sqrt{x+1}\,\sqrt{x-1}\,\mathrm{e}^{c_2}}{x}$$
$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3\sqrt{x+1}\,\sqrt{x-1}\,\mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3\sqrt{x+1}\,\sqrt{x-1}\,\mathrm{e}^{c_2}}{x} \tag{1}$$

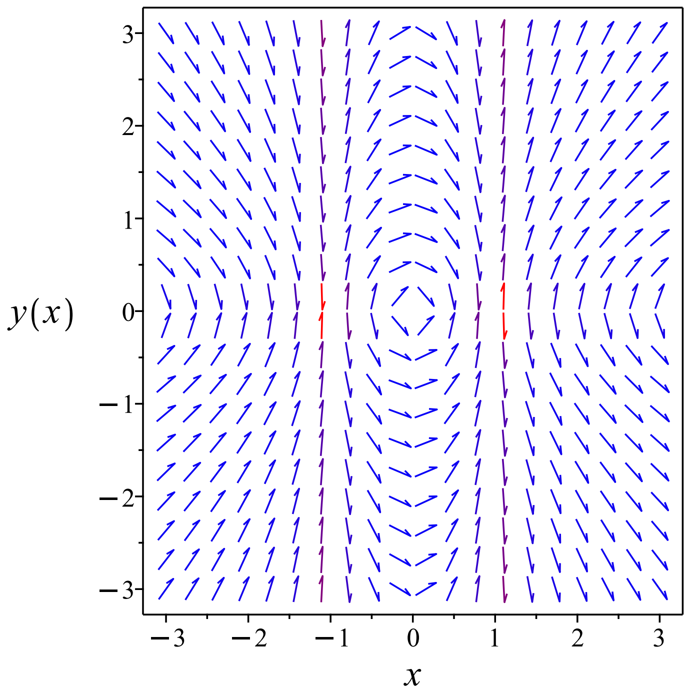


Figure 7: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3\sqrt{x+1}\,\sqrt{x-1}\,\mathrm{e}^{c_2}}{x}$$

Verified OK.

1.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x(y^2+1)}{y\,(x^2-1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \frac{x^2-1}{x} \\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^2-1}{x}} dx\end{aligned}$$

Which results in

$$S = \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x(y^2+1)}{y(x^2-1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x}{x^2 - 1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2 + 1)}{2} + c_1 \tag{4}$$

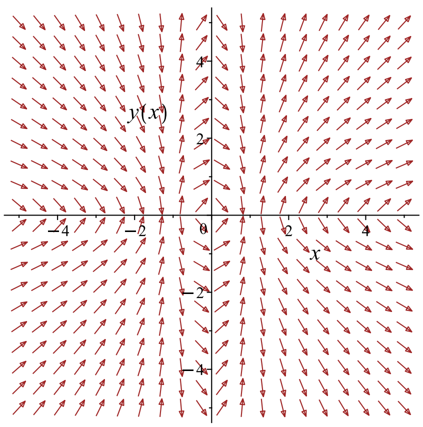
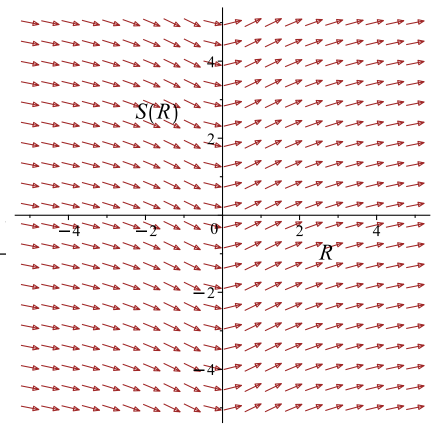
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x - 1)}{2} + \frac{\ln(x + 1)}{2} = \frac{\ln(1 + y^2)}{2} + c_1$$

Which simplifies to

$$\frac{\ln(x - 1)}{2} + \frac{\ln(x + 1)}{2} = \frac{\ln(1 + y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(y^2+1)}{y(x^2-1)}$ 	$R = y$ $S = \frac{\ln (x-1)}{2} + \frac{\ln (x+}{2}$	$\frac{dS}{dR} = \frac{R}{R^2+1}$ 

Summary

The solution(s) found are the following

$$\frac{\ln (x-1)}{2} + \frac{\ln (x+1)}{2} = \frac{\ln (1+y^2)}{2} + c_1 \tag{1}$$

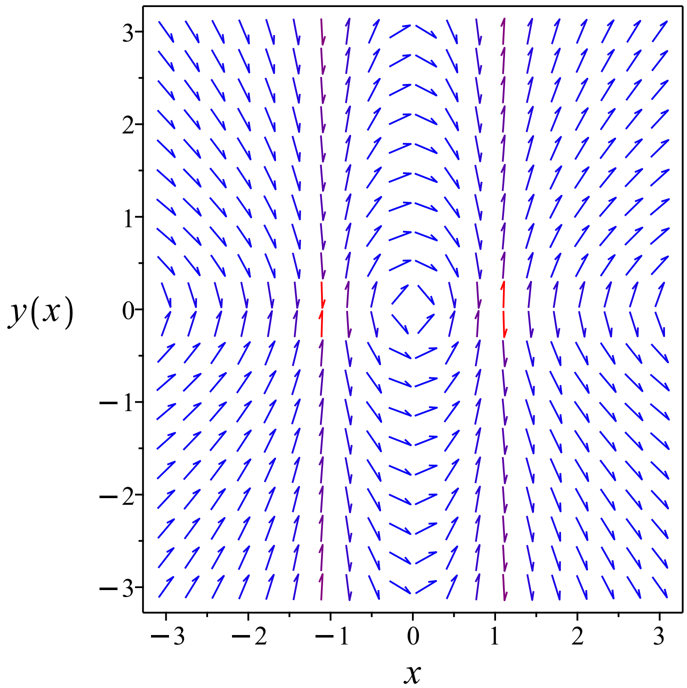


Figure 8: Slope field plot

Verification of solutions

$$\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}=\frac{\ln(1+y^2)}{2}+c_1$$

Verified OK.

1.2.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x(y^2+1)}{y(x^2-1)} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{x}{x^2-1}y+\frac{x}{x^2-1}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y+f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n}+f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{x}{x^2-1} \\ f_1(x) &= \frac{x}{x^2-1} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{x\,y^2}{x^2-1} + \frac{x}{x^2-1} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{xw(x)}{x^2-1} + \frac{x}{x^2-1} \\ w' &= \frac{2xw}{x^2-1} + \frac{2x}{x^2-1} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2x}{x^2-1} \\ q(x) &= \frac{2x}{x^2-1} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2xw(x)}{x^2-1} = \frac{2x}{x^2-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2x}{x^2-1}dx} \\ &= e^{-\ln(x-1)-\ln(x+1)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{x^2-1}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{2x}{x^2-1}\right) \\ \frac{d}{dx}\left(\frac{w}{x^2-1}\right) &= \left(\frac{1}{x^2-1}\right)\left(\frac{2x}{x^2-1}\right) \\ d\left(\frac{w}{x^2-1}\right) &= \left(\frac{2x}{(x^2-1)^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2-1} &= \int \frac{2x}{(x^2-1)^2} dx \\ \frac{w}{x^2-1} &= -\frac{1}{x^2-1} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2-1}$ results in

$$w(x) = -1 + c_1(x^2-1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -1 + c_1(x^2-1)$$

Which is simplified to

$$y^2 = c_1 x^2 - c_1 - 1$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{c_1 x^2 - c_1 - 1} \\ y(x) &= -\sqrt{c_1 x^2 - c_1 - 1}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1 x^2 - c_1 - 1} \tag{1}$$

$$y = -\sqrt{c_1 x^2 - c_1 - 1} \tag{2}$$

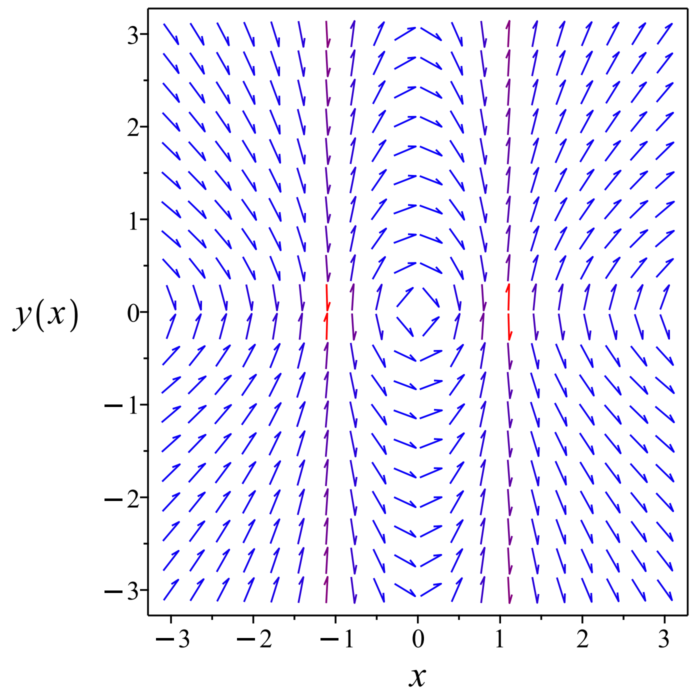


Figure 9: Slope field plot

Verification of solutions

$$y = \sqrt{c_1 x^2 - c_1 - 1}$$

Verified OK.

$$y = -\sqrt{c_1 x^2 - c_1 - 1}$$

Verified OK.

1.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{y^2+1}\right)\mathrm{d}y&=\left(\frac{x}{x^2-1}\right)\mathrm{d}x\\ \left(-\frac{x}{x^2-1}\right)\mathrm{d}x+\left(\frac{y}{y^2+1}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x}{x^2-1} \\
N(x,y) &= \frac{y}{y^2+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2-1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \quad (1) \\
\frac{\partial \phi}{\partial y} &= N \quad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2-1}\,\mathrm{d}x \\
\phi &= -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + f(y) \quad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{y^2+1}$. Therefore equation (4) becomes

$$\frac{y}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y}{y^2+1}\right)\mathrm{d}y$$

$$f(y) = \frac{\ln(y^2+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{\ln(y^2+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{\ln(y^2+1)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

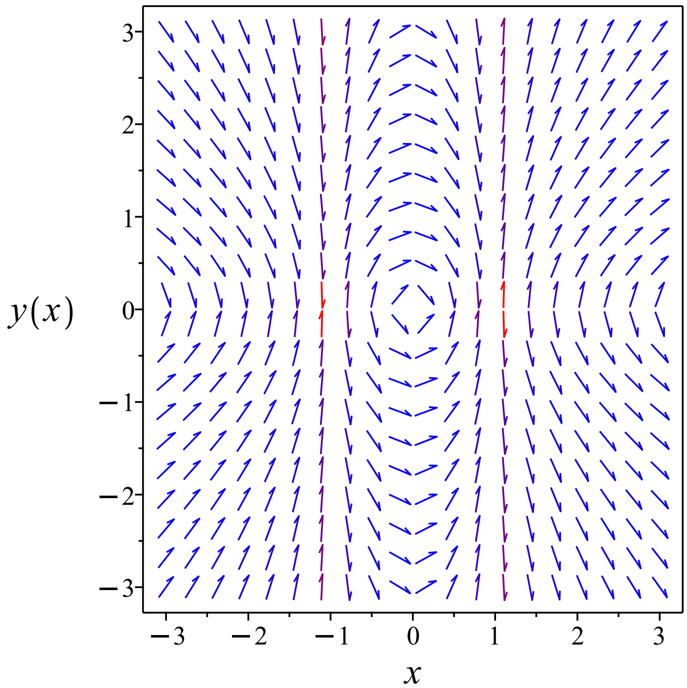


Figure 10: Slope field plot

Verification of solutions

$$-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}+\frac{\ln(1+y^2)}{2}=c_1$$

Verified OK.

1.2.6 Maple step by step solution

Let's solve

$xy^2+(y-x^2y)\,y'=-x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{yy'}{1+y^2}=\frac{x}{(x-1)(x+1)}$

- Integrate both sides with respect to x

  $\int\frac{yy'}{1+y^2}dx=\int\frac{x}{(x-1)(x+1)}dx+c_1$

- Evaluate integral

$$\frac{\ln(1+y^2)}{2} = \frac{\ln((x-1)(x+1))}{2} + c_1$$

- Solve for y

$$\left\{y = \sqrt{-1 + x^2 e^{2c_1} - e^{2c_1}}, y = -\sqrt{-1 + x^2 e^{2c_1} - e^{2c_1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve((x*y(x)^2+x)+(y(x)-x^2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{c_1 x^2 - c_1 - 1}$$
$$y(x) = -\sqrt{c_1 x^2 - c_1 - 1}$$

✓ Solution by Mathematica

Time used: 1.353 (sec). Leaf size: 61

```
DSolve[(x*y[x]^2+x)+(y[x]-x^2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-1 + e^{2c_1}(x^2 - 1)}$$
$$y(x) \to \sqrt{-1 + e^{2c_1}(x^2 - 1)}$$
$$y(x) \to -i$$
$$y(x) \to i$$

1.3 problem 3

Internal problem ID [1872]
Internal file name [`OUTPUT/1873_Sunday_June_05_2022_02_36_23_AM_43492413/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 + \left(x^2 + 1\right) y' = -1}$$

1.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-y^2 - 1}{x^2 + 1} \end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = -y^2 - 1$. Integrating both sides gives

$$\frac{1}{-y^2 - 1}\, dy = \frac{1}{x^2 + 1}\, dx$$

$$\int \frac{1}{-y^2 - 1} \, dy = \int \frac{1}{x^2 + 1} \, dx$$
$$-\arctan(y) = \arctan(x) + c_1$$

Which results in

$$y = -\tan(\arctan(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = -\tan(\arctan(x) + c_1) \tag{1}$$

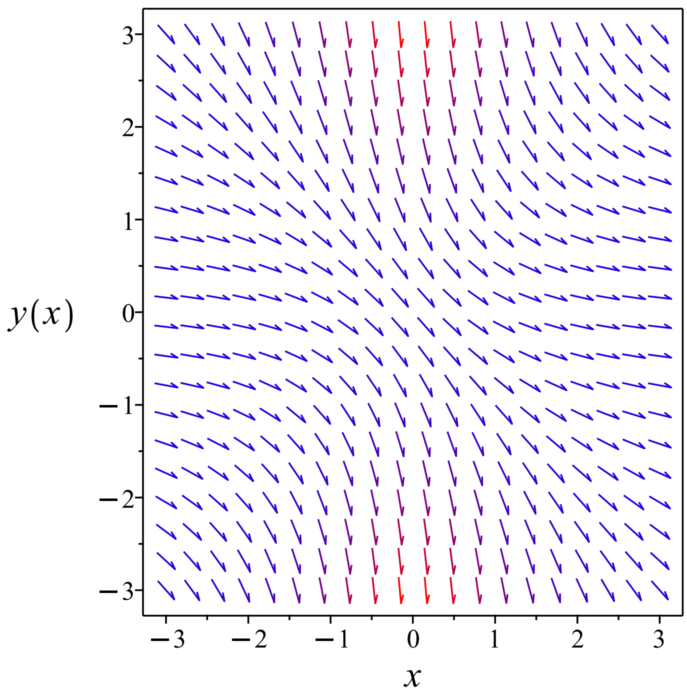


Figure 11: Slope field plot

Verification of solutions

$$y = -\tan(\arctan(x) + c_1)$$

Verified OK.

1.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y^2+1}{x^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f(\frac{y}{x})$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F(\frac{y}{x})$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x^2+1\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{x^2+1} dx\end{aligned}$$

Which results in

$$S = \arctan(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2+1}{x^2+1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\arctan(x) = -\arctan(y) + c_1$$

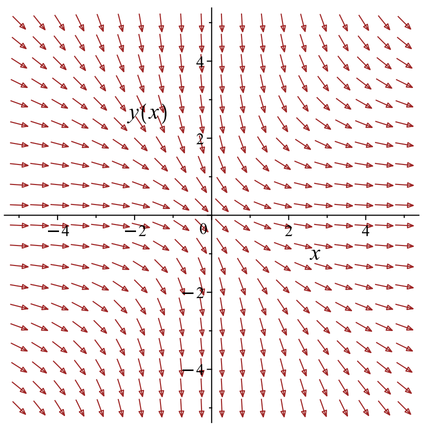
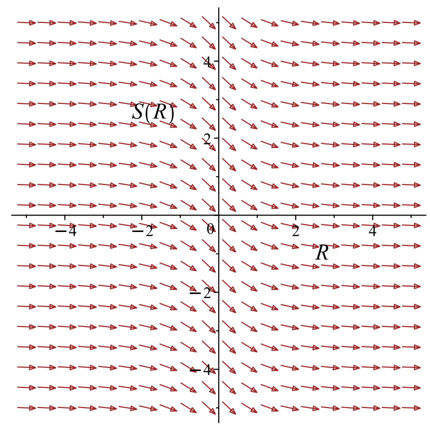
Which simplifies to

$$\arctan(x) = -\arctan(y) + c_1$$

Which gives

$$y = \tan(-\arctan(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2+1}{x^2+1}$ 	$R = y$ $S = \arctan(x)$	$\frac{dS}{dR} = -\frac{1}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = \tan(-\arctan(x) + c_1) \tag{1}$$

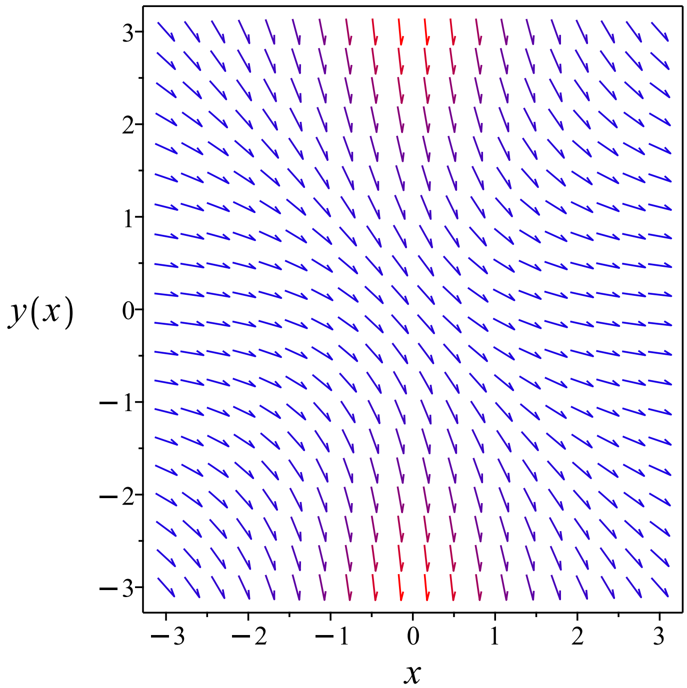


Figure 12: Slope field plot

Verification of solutions

$$y = \tan(-\arctan(x) + c_1)$$

Verified OK.

1.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{-y^2 - 1}\right) \mathrm{d}y = \left(\frac{1}{x^2 + 1}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x^2 + 1}\right) \mathrm{d}x + \left(\frac{1}{-y^2 - 1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x^2 + 1}$$
$$N(x, y) = \frac{1}{-y^2 - 1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2 + 1}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-y^2-1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x^2+1}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y^2-1}$. Therefore equation (4) becomes

$$\frac{1}{-y^2-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y^2+1}\right)\mathrm{d}y\\ f(y) &= -\arctan(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) - \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) - \arctan(y)$$

Summary

The solution(s) found are the following

$$-\arctan(x) - \arctan(y) = c_1 \tag{1}$$

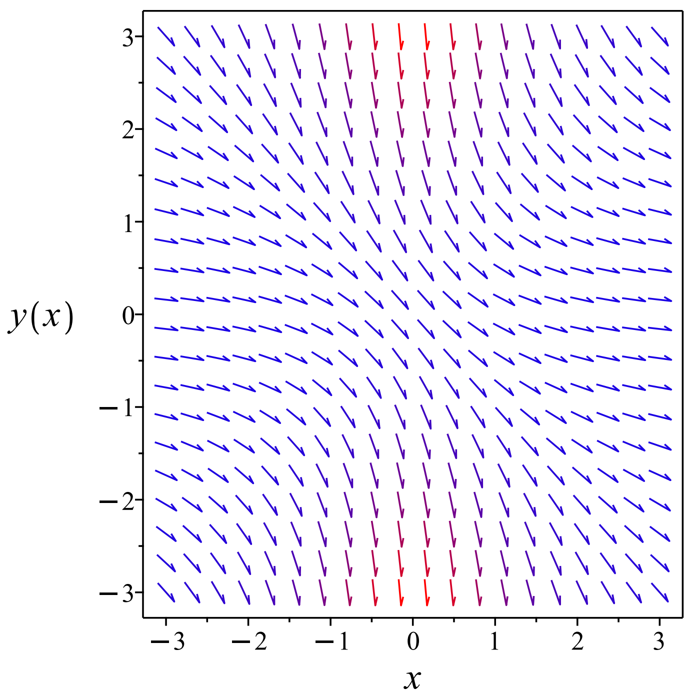


Figure 13: Slope field plot

Verification of solutions

$$-\arctan(x) - \arctan(y) = c_1$$

Verified OK.

1.3.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^2+1}{x^2+1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2}{x^2+1} - \frac{1}{x^2+1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{1}{x^2+1}$, $f_1(x) = 0$ and $f_2(x) = -\frac{1}{x^2+1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^2+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2x}{(x^2+1)^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\frac{1}{(x^2+1)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^2+1} - \frac{2xu'(x)}{(x^2+1)^2} - \frac{u(x)}{(x^2+1)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 x + c_2}{\sqrt{x^2+1}}$$

The above shows that

$$u'(x) = \frac{-c_2 x + c_1}{(x^2+1)^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = \frac{-c_2 x + c_1}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 - x}{c_3 x + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3 - x}{c_3 x + 1} \tag{1}$$

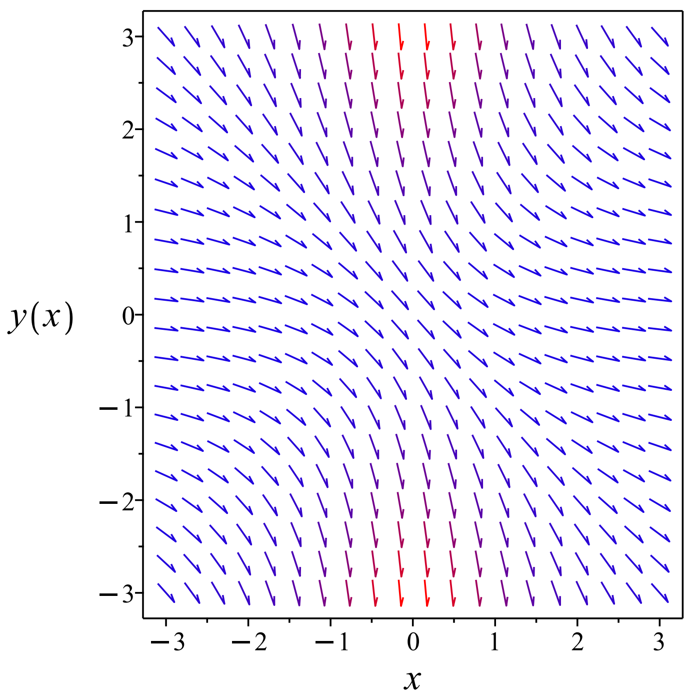


Figure 14: Slope field plot

Verification of solutions

$$y = \frac{c_3 - x}{c_3 x + 1}$$

Verified OK.

1.3.5 Maple step by step solution

Let's solve

$y^2 + (x^2 + 1) y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-1-y^2} = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-1-y^2} dx = \int \frac{1}{x^2+1} dx + c_1$

- Evaluate integral

  $-\arctan(y) = \arctan(x) + c_1$

- Solve for y

  $y = -\tan(\arctan(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve((1+y(x)^2)+(1+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\tan(\arctan(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.233 (sec). Leaf size: 29

```
DSolve[(1+y[x]^2)+(1+x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\tan(\arctan(x) - c_1)$$
$$y(x) \to -i$$
$$y(x) \to i$$

1.4 problem 4

Internal problem ID [1873]
Internal file name [`OUTPUT/1874_Sunday_June_05_2022_02_36_25_AM_87037268/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'x + y = 0}$$

1.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{y}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= -\frac{1}{x}\, dx \\
\int \frac{1}{y}\, dy &= \int -\frac{1}{x}\, dx \\
\ln(y) &= -\ln(x) + c_1 \\
y &= e^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

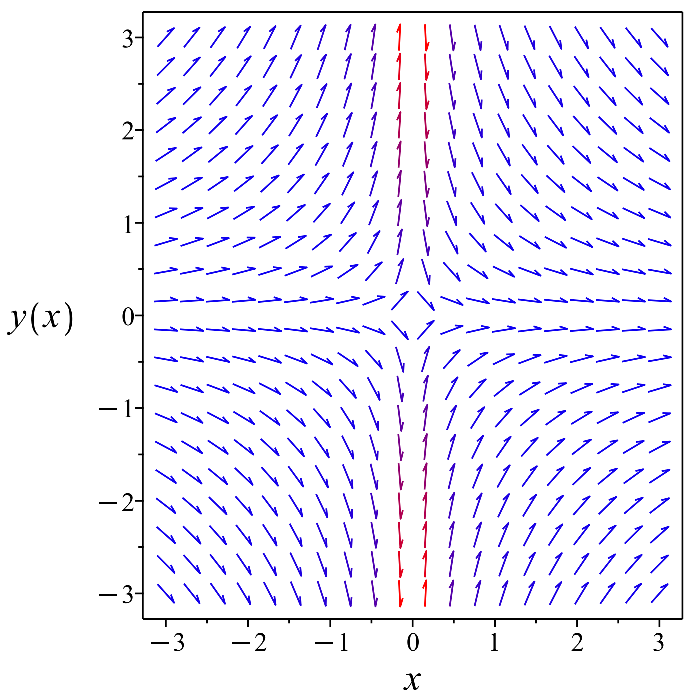


Figure 15: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

1.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx} \mu y &= 0 \\ \frac{d}{dx}(yx) &= 0 \end{aligned}$$

Integrating gives

$$yx = c_1$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{c_1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

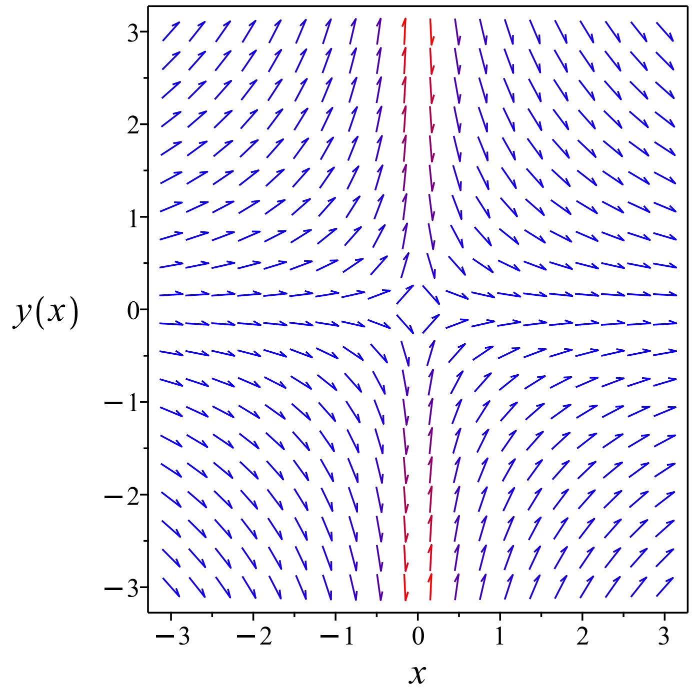


Figure 16: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{c_1}{x}$$

Verified OK.

1.4.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x + u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{2}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\
\ln(u) &= -2\ln(x) + c_2 \\
u &= \mathrm{e}^{-2\ln(x)+c_2} \\
&= \frac{c_2}{x^2}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{c_2}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2}{x} \tag{1}
$$

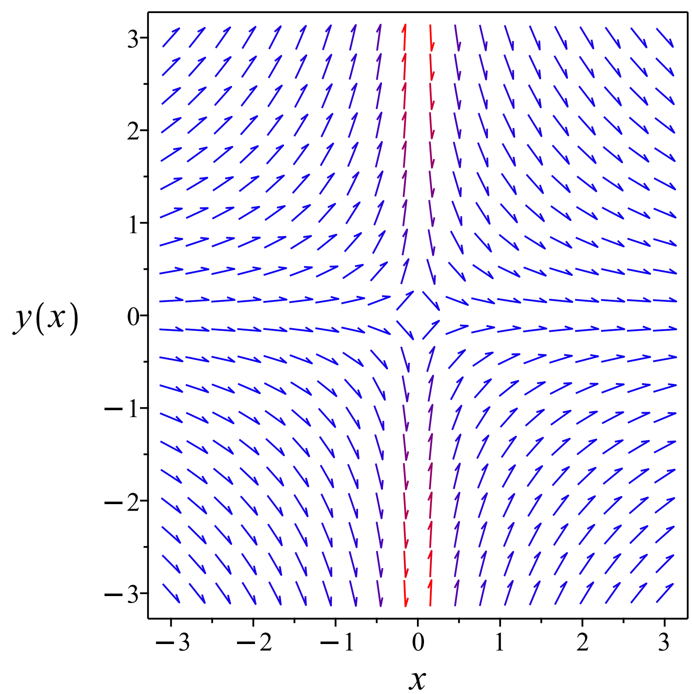


Figure 17: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x}$$

Verified OK.

1.4.4 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{y}{x} \tag{1}$$

Which becomes

$$0 = (-x)\,dy + (-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-y)\,dx = d(-yx)$$

Hence (2) becomes

$$0 = d(-yx)$$

Integrating both sides gives gives these solutions

$$y = \frac{c_1}{x} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_1 \tag{1}$$

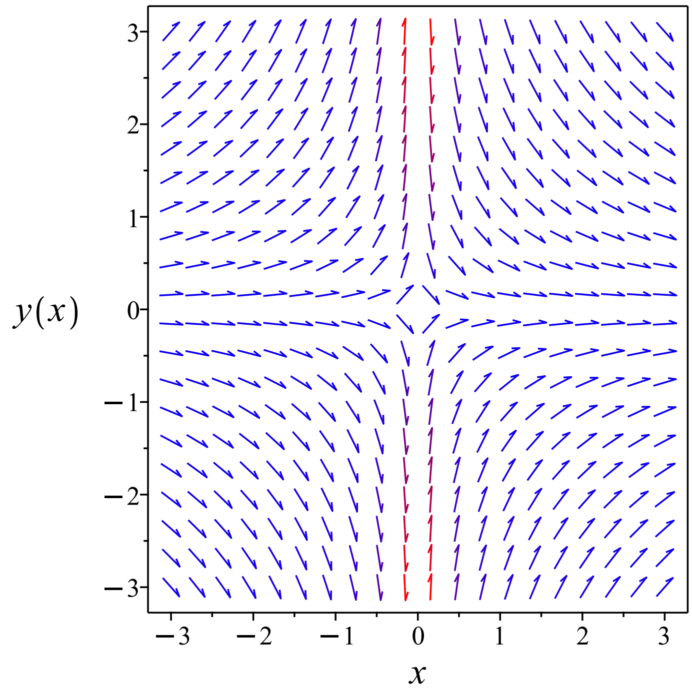


Figure 18: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x} + c_1$$

Verified OK.

1.4.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 10: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = yx$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = c_1$$

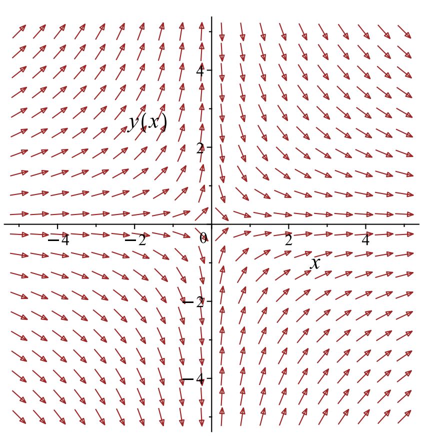
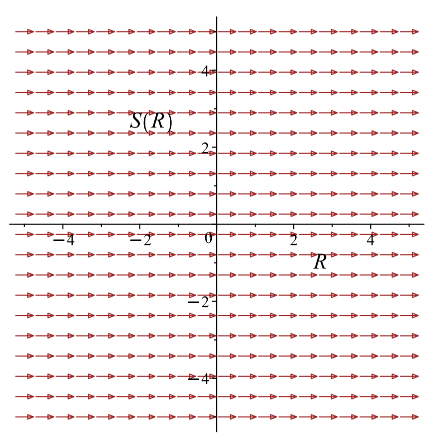
Which simplifies to

$$yx = c_1$$

Which gives

$$y = \frac{c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{x}$ 	$R = x$ $S = yx$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

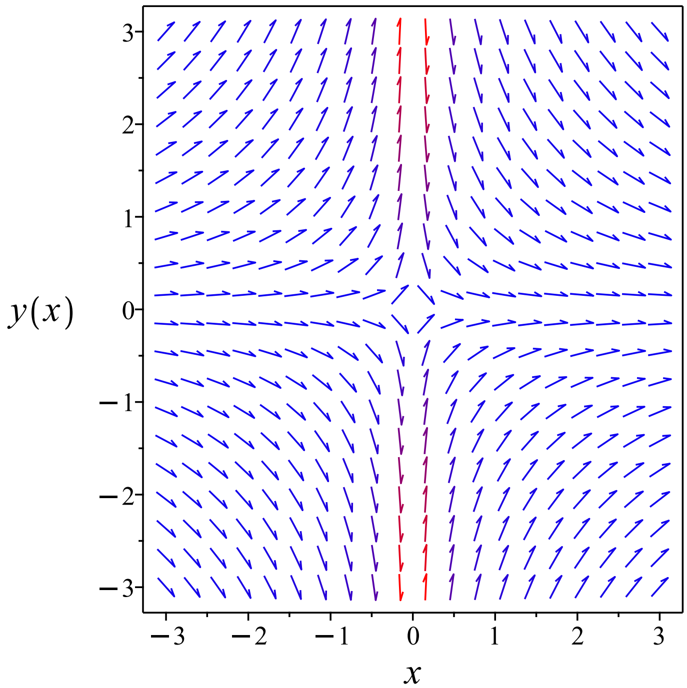


Figure 19: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

1.4.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= -\frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x \end{aligned}$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned} \int f'(y) \, \mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\ f(y) &= -\ln(y) + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y)$$

The solution becomes

$$y = \frac{e^{-c_1}}{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{e^{-c_1}}{x} \tag{1}$$

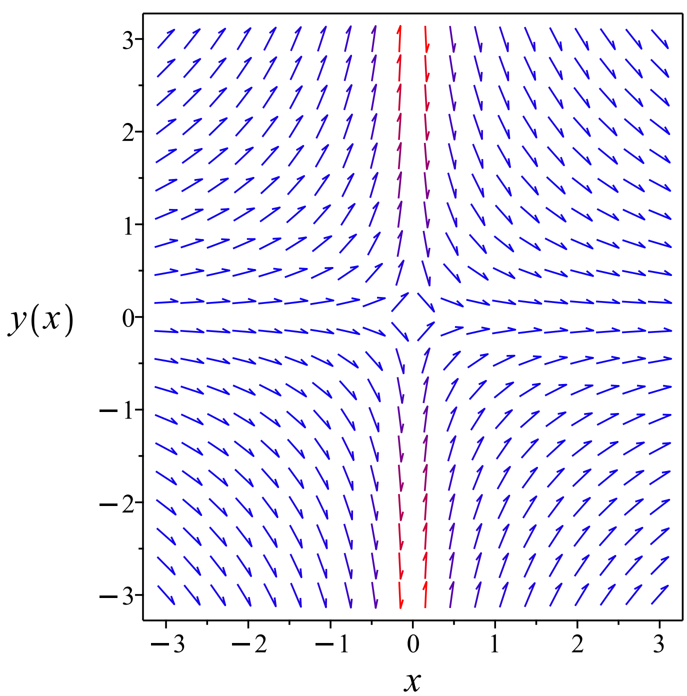


Figure 20: Slope field plot

Verification of solutions

$$y = \frac{e^{-c_1}}{x}$$

Verified OK.

1.4.7 Maple step by step solution

Let's solve

$y'x + y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (y'x + y)\, dx = \int 0 dx + c_1$

- Evaluate integral

  $yx = c_1$

- Solve for y

  $y = \frac{c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(y(x)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 16

```
DSolve[y[x]+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x}$$
$$y(x) \to 0$$

1.5 problem 5

Internal problem ID [1874]
Internal file name [`OUTPUT/1875_Sunday_June_05_2022_02_36_26_AM_3993850/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2yx = 0}$$

1.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= 2yx \end{aligned}$$

Where $f(x) = 2x$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= 2x\,dx \\
\int \frac{1}{y}\,dy &= \int 2x\,dx \\
\ln(y) &= x^2 + c_1 \\
y &= e^{x^2+c_1} \\
&= c_1 e^{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x^2} \tag{1}$$

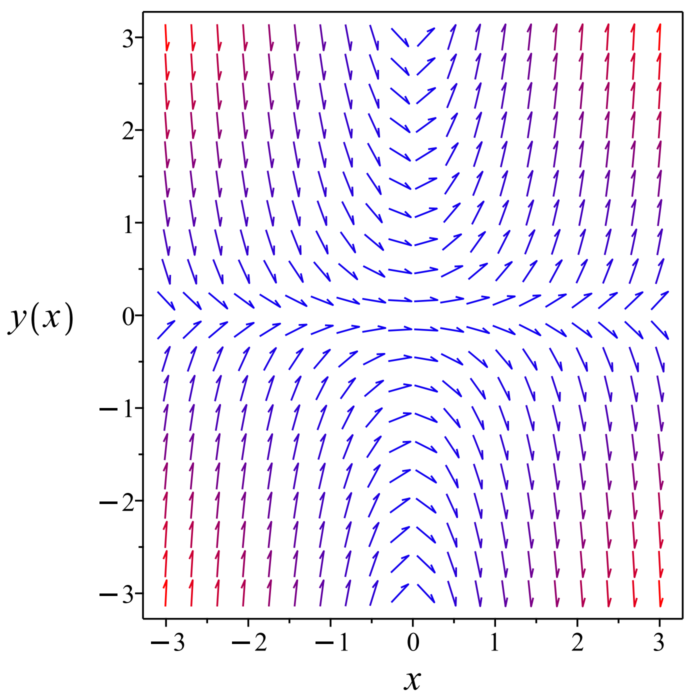


Figure 21: Slope field plot

Verification of solutions

$$y = c_1 e^{x^2}$$

Verified OK.

1.5.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - 2yx = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -2x dx} \\ &= e^{-x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(e^{-x^2} y\right) &= 0 \end{aligned}$$

Integrating gives

$$e^{-x^2} y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$y = c_1 e^{x^2}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x^2} \tag{1}$$

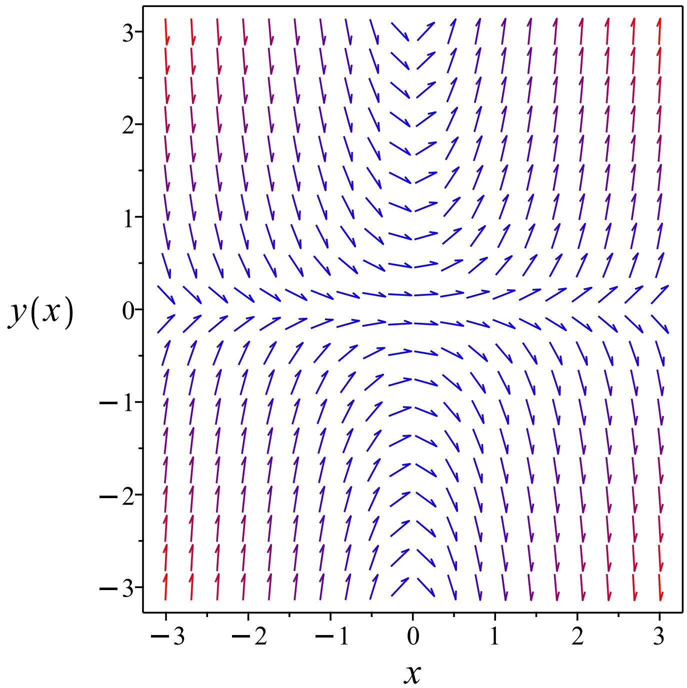


Figure 22: Slope field plot

Verification of solutions

$$y = c_1 e^{x^2}$$

Verified OK.

1.5.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - 2u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(2x^2-1)}{x} \end{aligned}$$

Where $f(x) = \frac{2x^2-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{2x^2-1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{2x^2-1}{x}\,dx \\
\ln(u) &= x^2 - \ln(x) + c_2 \\
u &= e^{x^2-\ln(x)+c_2} \\
&= c_2 e^{x^2-\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 e^{x^2}}{x}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= c_2 e^{x^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 e^{x^2} \tag{1}$$

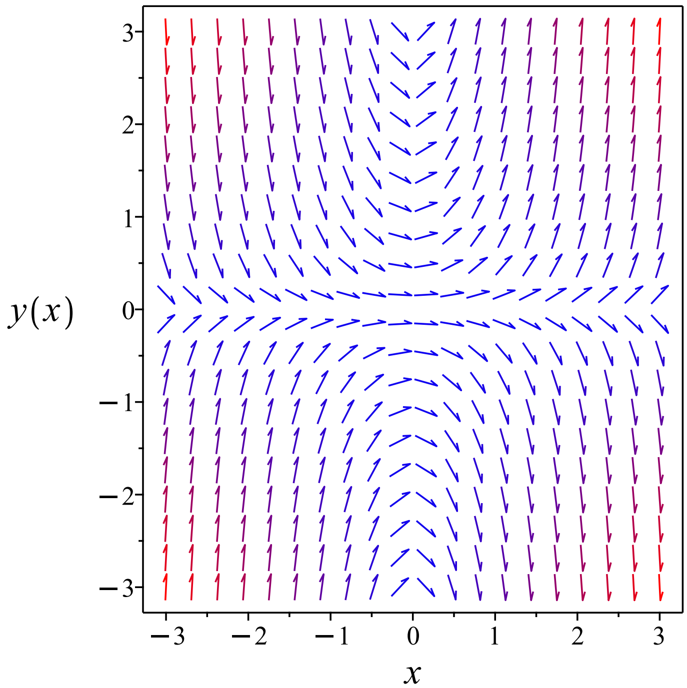


Figure 23: Slope field plot

Verification of solutions

$$y = c_2 e^{x^2}$$

Verified OK.

1.5.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2yx$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 13: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x^2}} dy \end{aligned}$$

Which results in

$$S = e^{-x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2yx$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -2x\, e^{-x^2} y \\ S_y &= e^{-x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x^2} y = c_1$$

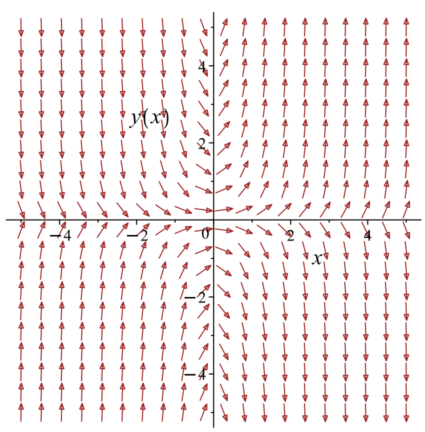
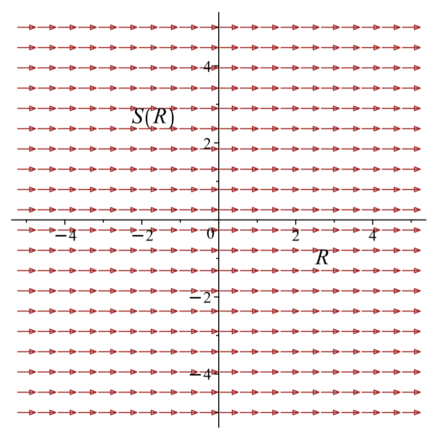
Which simplifies to

$$e^{-x^2} y = c_1$$

Which gives

$$y = c_1 e^{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2yx$ 	$R = x$ $S = e^{-x^2} y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 e^{x^2} \tag{1}$$

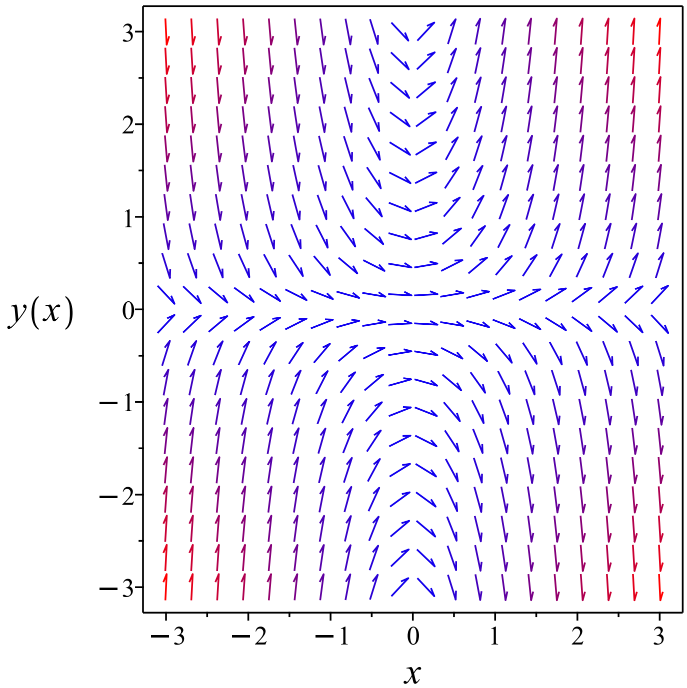


Figure 24: Slope field plot

Verification of solutions

$$y = c_1 e^{x^2}$$

Verified OK.

1.5.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2y}\right) \mathrm{d}y &= (x)\,\mathrm{d}x \\ (-x)\,\mathrm{d}x + \left(\frac{1}{2y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x \\ N(x, y) &= \frac{1}{2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x\,\mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y}$. Therefore equation (4) becomes

$$\frac{1}{2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, dy = \int \left(\frac{1}{2y}\right) dy$$
$$f(y) = \frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(y)}{2}$$

The solution becomes

$$y = e^{x^2+2c_1}$$

Summary

The solution(s) found are the following

$$y = e^{x^2+2c_1} \tag{1}$$

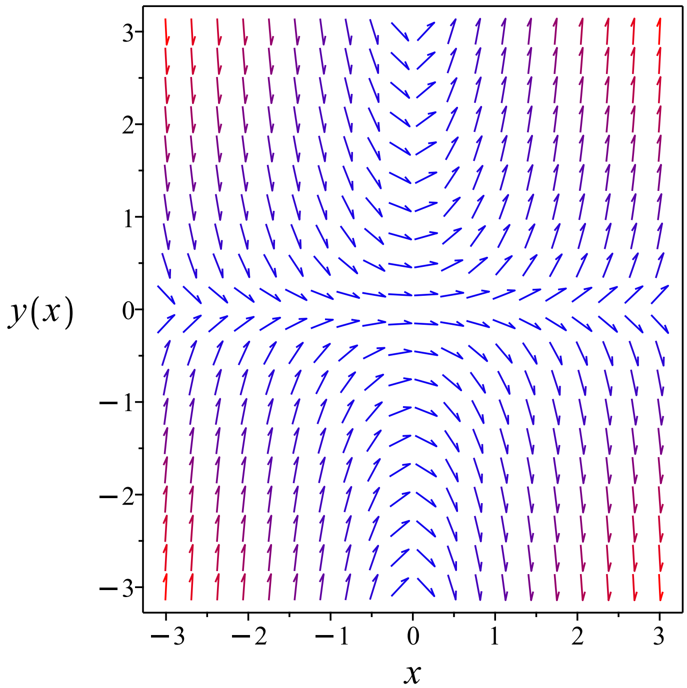


Figure 25: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{x^2+2c_1}$$

Verified OK.

1.5.6 Maple step by step solution

Let's solve

$y' - 2yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 2x dx + c_1$

- Evaluate integral

$$\ln(y) = x^2 + c_1$$

- Solve for y

$$y = e^{x^2 + c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=2*x*y(x),y(x), singsol=all)
```

$$y(x) = e^{x^2} c_1$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 18

```
DSolve[y'[x]==2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x^2}$$
$$y(x) \to 0$$

1.6 problem 6

Internal problem ID [1875]
Internal file name [`OUTPUT/1876_Sunday_June_05_2022_02_36_28_AM_66296928/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy^2 + \left(x^2y - y\right)y' = -x}$$

1.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -\frac{x(y^2+1)}{y\,(x^2-1)}
\end{aligned}$$

Where $f(x) = -\frac{x}{x^2-1}$ and $g(y) = \frac{y^2+1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{y}}\,dy = -\frac{x}{x^2-1}\,dx$$

$$\int \frac{1}{\frac{y^2+1}{y}}dy = \int -\frac{x}{x^2-1}dx$$

$$\frac{\ln(y^2+1)}{2} = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = \mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = c_2\mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}}$$

Which simplifies to

$$\sqrt{1+y^2} = \frac{c_2\mathrm{e}^{c_1}}{\sqrt{x-1}\,\sqrt{x+1}}$$

The solution is

$$\sqrt{1+y^2} = \frac{c_2\mathrm{e}^{c_1}}{\sqrt{x-1}\,\sqrt{x+1}}$$

Summary

The solution(s) found are the following

$$\sqrt{1+y^2} = \frac{c_2\mathrm{e}^{c_1}}{\sqrt{x-1}\,\sqrt{x+1}} \tag{1}$$

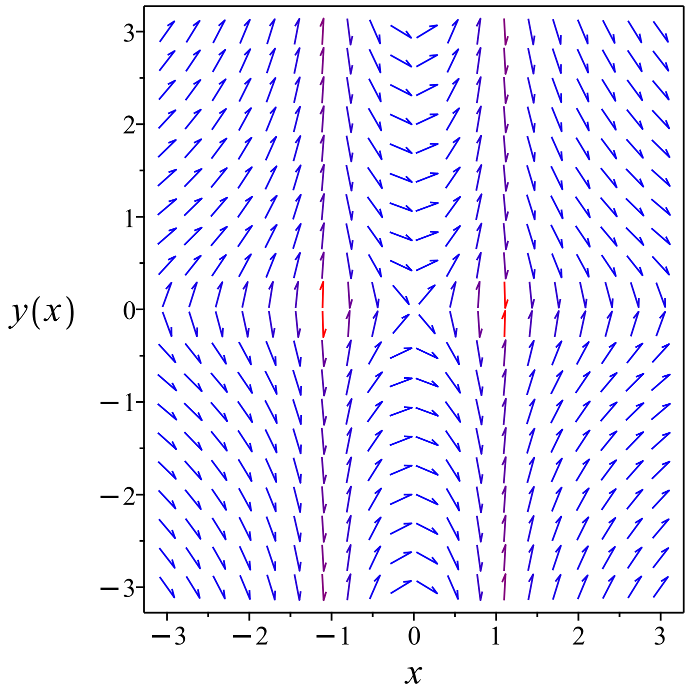


Figure 26: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} = \frac{c_2 e^{c_1}}{\sqrt{x-1}\,\sqrt{x+1}}$$

Verified OK.

1.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x(y^2+1)}{y\,(x^2-1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 16: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{x^2 - 1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x^2-1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x(y^2+1)}{y(x^2-1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{x}{x^2-1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R^2+1)}{2} + c_1 \tag{4}$$

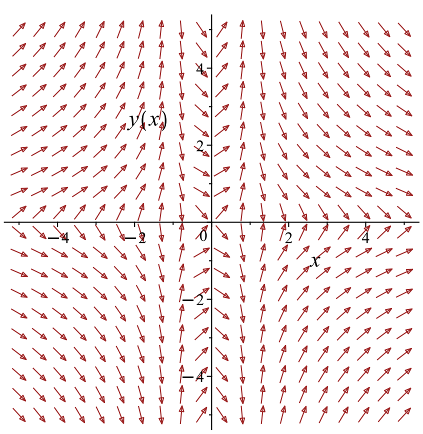
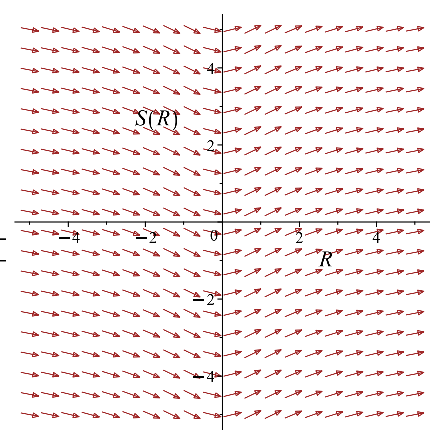
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln (x-1)}{2} - \frac{\ln (x+1)}{2} = \frac{\ln (1+y^2)}{2} + c_1$$

Which simplifies to

$$-\frac{\ln (x-1)}{2} - \frac{\ln (x+1)}{2} = \frac{\ln (1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x(y^2+1)}{y(x^2-1)}$ 	$R = y$ $S = -\frac{\ln (x-1)}{2} - \frac{\ln (x}{2}$	$\frac{dS}{dR} = \frac{R}{R^2+1}$ 

Summary

The solution(s) found are the following

$$-\frac{\ln (x-1)}{2} - \frac{\ln (x+1)}{2} = \frac{\ln (1+y^2)}{2} + c_1 \tag{1}$$

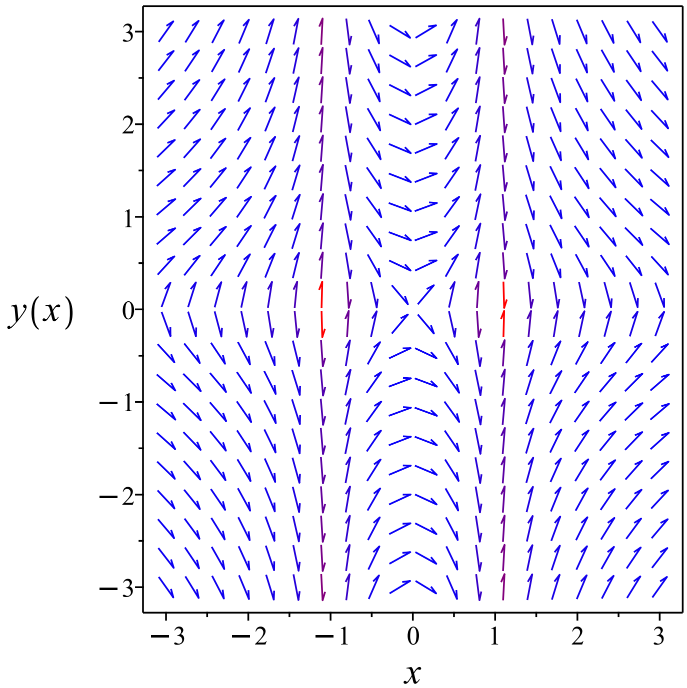


Figure 27: Slope field plot

Verification of solutions

$$-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}=\frac{\ln(1+y^2)}{2}+c_1$$

Verified OK.

1.6.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x(y^2+1)}{y(x^2-1)} \end{aligned}$$

This is a Bernoulli ODE.

$$y'=-\frac{x}{x^2-1}y-\frac{x}{x^2-1}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y'=f_0(x)y+f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n}=f_0(x)y^{1-n}+f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{x}{x^2 - 1} \\
f_1(x) &= -\frac{x}{x^2 - 1} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{x\,y^2}{x^2 - 1} - \frac{x}{x^2 - 1} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{xw(x)}{x^2 - 1} - \frac{x}{x^2 - 1} \\
w' &= -\frac{2xw}{x^2 - 1} - \frac{2x}{x^2 - 1}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2x}{x^2 - 1} \\
q(x) &= -\frac{2x}{x^2 - 1}
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{2xw(x)}{x^2 - 1} = -\frac{2x}{x^2 - 1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2x}{x^2-1}dx} \\ &= e^{\ln(x-1)+\ln(x+1)}\end{aligned}$$

Which simplifies to

$$\mu = x^2 - 1$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{2x}{x^2-1}\right) \\ \frac{d}{dx}\left(\left(x^2-1\right)w\right) &= \left(x^2-1\right)\left(-\frac{2x}{x^2-1}\right) \\ d\left(\left(x^2-1\right)w\right) &= (-2x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\left(x^2-1\right)w &= \int -2x\, dx \\ \left(x^2-1\right)w &= -x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2 - 1$ results in

$$w(x) = -\frac{x^2}{x^2-1} + \frac{c_1}{x^2-1}$$

which simplifies to

$$w(x) = \frac{-x^2 + c_1}{x^2 - 1}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{-x^2 + c_1}{x^2 - 1}$$

Solving for y gives

$$y(x) = \frac{\sqrt{(x^2-1)(-x^2+c_1)}}{x^2-1}$$
$$y(x) = -\frac{\sqrt{(x^2-1)(-x^2+c_1)}}{x^2-1}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{(x^2-1)(-x^2+c_1)}}{x^2-1} \tag{1}$$
$$y = -\frac{\sqrt{(x^2-1)(-x^2+c_1)}}{x^2-1} \tag{2}$$

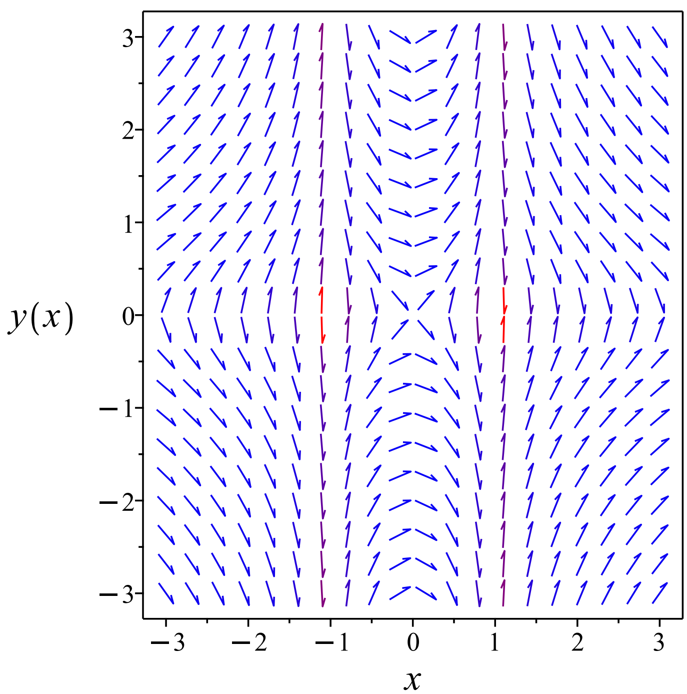


Figure 28: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{(x^2 - 1)(-x^2 + c_1)}}{x^2 - 1}$$

Verified OK.

$$y = -\frac{\sqrt{(x^2 - 1)(-x^2 + c_1)}}{x^2 - 1}$$

Verified OK.

1.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{y}{y^2+1}\right)\mathrm{d}y = \left(\frac{x}{x^2-1}\right)\mathrm{d}x$$

$$\left(-\frac{x}{x^2-1}\right)\mathrm{d}x + \left(-\frac{y}{y^2+1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{x}{x^2-1} \\ N(x,y) &= -\frac{y}{y^2+1} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2-1}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y}{y^2+1}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{x}{x^2-1}\,\mathrm{d}x$$
$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{y^2+1}$. Therefore equation (4) becomes

$$-\frac{y}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{y}{y^2+1}\right)\mathrm{d}y$$
$$f(y) = -\frac{\ln(y^2+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} - \frac{\ln(y^2+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} - \frac{\ln(y^2+1)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} - \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

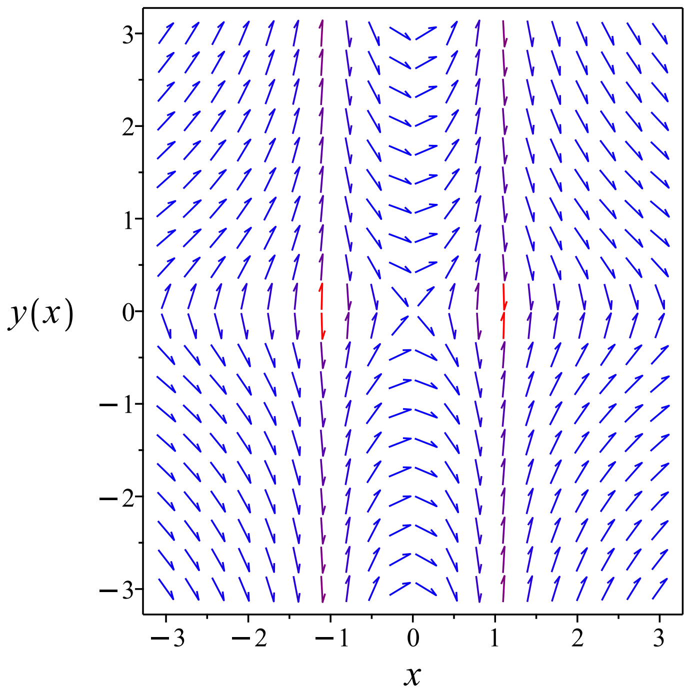


Figure 29: Slope field plot

Verification of solutions

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} - \frac{\ln(1+y^2)}{2} = c_1$$

Verified OK.

1.6.5 Maple step by step solution

Let's solve

$xy^2 + (x^2y - y)\, y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (xy^2 + (x^2y - y)\, y')\, dx = \int -x dx + c_1$

- Evaluate integral

  $\frac{y^2(x-1)(x+1)}{2} = -\frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{(x^2-1)(-x^2+2c_1)}}{x^2-1}, y = -\frac{\sqrt{(x^2-1)(-x^2+2c_1)}}{x^2-1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 54

```
dsolve((x*y(x)^2+x)+(x^2*y(x)-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(x^2 - 1)\,(-x^2 + c_1)}}{x^2 - 1}$$
$$y(x) = -\frac{\sqrt{(x^2 - 1)\,(-x^2 + c_1)}}{x^2 - 1}$$

✓ Solution by Mathematica

Time used: 0.369 (sec). Leaf size: 133

```
DSolve[(x*y[x]^2+x)+(x^2*y[x]-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-x^2+1-e^{2c_1}}}{\sqrt{x^2-1}}$$
$$y(x) \to \frac{\sqrt{-x^2+1-e^{2c_1}}}{\sqrt{x^2-1}}$$
$$y(x) \to -i$$
$$y(x) \to i$$
$$y(x) \to -\frac{\sqrt{1-x^2}}{\sqrt{x^2-1}}$$
$$y(x) \to \frac{\sqrt{1-x^2}}{\sqrt{x^2-1}}$$

1.7 problem 7

Internal problem ID [1876]
Internal file name [`OUTPUT/1877_Sunday_June_05_2022_02_36_30_AM_57437860/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\sqrt{1-y^2}\, y' = -\sqrt{1-x^2}}$$

1.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\sqrt{1-x^2}}{\sqrt{-y^2+1}} \end{aligned}$$

Where $f(x) = -\sqrt{1-x^2}$ and $g(y) = \frac{1}{\sqrt{-y^2+1}}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\sqrt{-y^2+1}}}\, dy = -\sqrt{1-x^2}\, dx$$

$$\int \frac{1}{\frac{1}{\sqrt{-y^2+1}}}\, dy = \int -\sqrt{1-x^2}\, dx$$

$$\frac{y\sqrt{-y^2+1}}{2} + \frac{\arcsin(y)}{2} = -\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} + c_1$$

The solution is

$$\frac{\sqrt{1-y^2}\, y}{2} + \frac{\arcsin(y)}{2} + \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin(x)}{2} - c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{1-y^2}\, y}{2} + \frac{\arcsin(y)}{2} + \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin(x)}{2} - c_1 = 0 \tag{1}$$

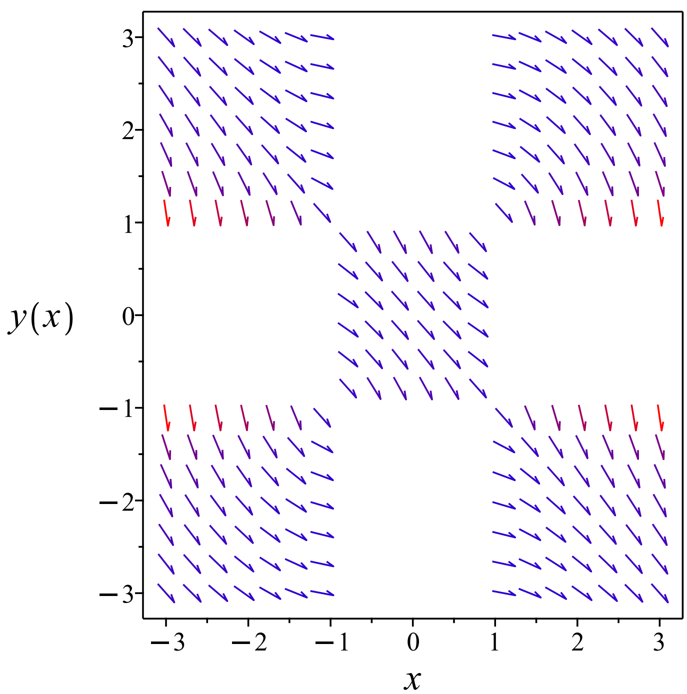


Figure 30: Slope field plot

Verification of solutions

$$\frac{\sqrt{1-y^2}\, y}{2} + \frac{\arcsin(y)}{2} + \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin(x)}{2} - c_1 = 0$$

Verified OK.

1.7.2 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}} \tag{1}$$

Which becomes

$$\left(\sqrt{-y^2+1}\right) dy = \left(-\sqrt{1-x^2}\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(-\sqrt{1-x^2}\right) dx = d\left(-\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2}\right)$$

Hence (2) becomes

$$\left(\sqrt{-y^2+1}\right) dy = d\left(-\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2}\right)$$

Integrating both sides gives gives the solution as

$$\frac{\sqrt{1-y^2}\,y}{2} + \frac{\arcsin(y)}{2} = -\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} + c_1$$

Summary
The solution(s) found are the following

$$\frac{\sqrt{1-y^2}\,y}{2} + \frac{\arcsin(y)}{2} = -\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} + c_1 \tag{1}$$

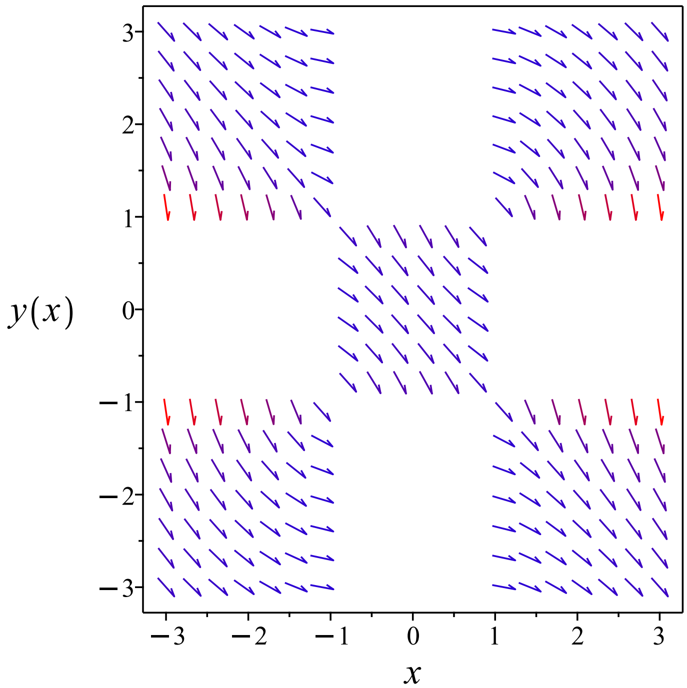


Figure 31: Slope field plot

Verification of solutions

$$\frac{\sqrt{1-y^2}\,y}{2}+\frac{\arcsin(y)}{2}=-\frac{x\sqrt{1-x^2}}{2}-\frac{\arcsin(x)}{2}+c_1$$

Verified OK.

1.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=-\frac{\sqrt{1-x^2}}{\sqrt{-y^2+1}}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 19: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{1}{\sqrt{1 - x^2}} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{\sqrt{1-x^2}}} dx \end{aligned}$$

Which results in

$$S = -\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sqrt{1-x^2}}{\sqrt{-y^2+1}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\sqrt{1-x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sqrt{-y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sqrt{-R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R\sqrt{-R^2+1}}{2} + \frac{\arcsin(R)}{2} + c_1 \tag{4}$$

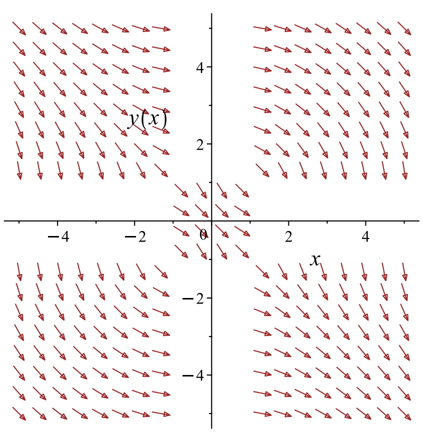
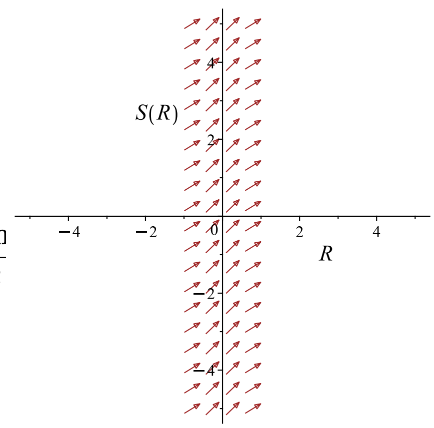
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} = \frac{\sqrt{1-y^2}\, y}{2} + \frac{\arcsin(y)}{2} + c_1$$

Which simplifies to

$$-\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} = \frac{\sqrt{1-y^2}\, y}{2} + \frac{\arcsin(y)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sqrt{1-x^2}}{\sqrt{-y^2+1}}$ 	$R = y$ $S = -\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin}{2}$	$\frac{dS}{dR} = \sqrt{-R^2+1}$ 

Summary

The solution(s) found are the following

$$-\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} = \frac{\sqrt{1-y^2}\, y}{2} + \frac{\arcsin(y)}{2} + c_1 \tag{1}$$

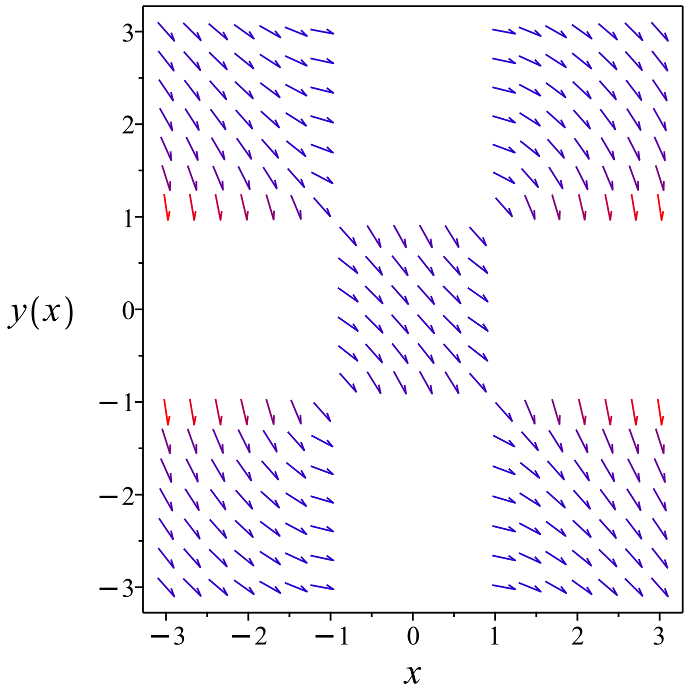


Figure 32: Slope field plot

Verification of solutions

$$-\frac{x\sqrt{1-x^2}}{2}-\frac{\arcsin\left(x\right)}{2}=\frac{\sqrt{1-y^2}\,y}{2}+\frac{\arcsin\left(y\right)}{2}+c_1$$

Verified OK.

1.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\sqrt{-y^2+1}\right)\mathrm{d}y = \left(\sqrt{1-x^2}\right)\mathrm{d}x$$
$$\left(-\sqrt{1-x^2}\right)\mathrm{d}x + \left(-\sqrt{-y^2+1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\sqrt{1-x^2}$$
$$N(x,y) = -\sqrt{-y^2+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\sqrt{1-x^2}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\sqrt{-y^2+1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\sqrt{1-x^2}\,\mathrm{d}x$$
$$\phi = -\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\sqrt{-y^2+1}$. Therefore equation (4) becomes

$$-\sqrt{-y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\sqrt{-y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\sqrt{-y^2+1}\right)\mathrm{d}y$$
$$f(y) = -\frac{y\sqrt{-y^2+1}}{2} - \frac{\arcsin(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} - \frac{y\sqrt{-y^2+1}}{2} - \frac{\arcsin(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} - \frac{y\sqrt{-y^2+1}}{2} - \frac{\arcsin(y)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\sqrt{1-y^2}\,y}{2} - \frac{\arcsin(y)}{2} - \frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} = c_1 \tag{1}$$

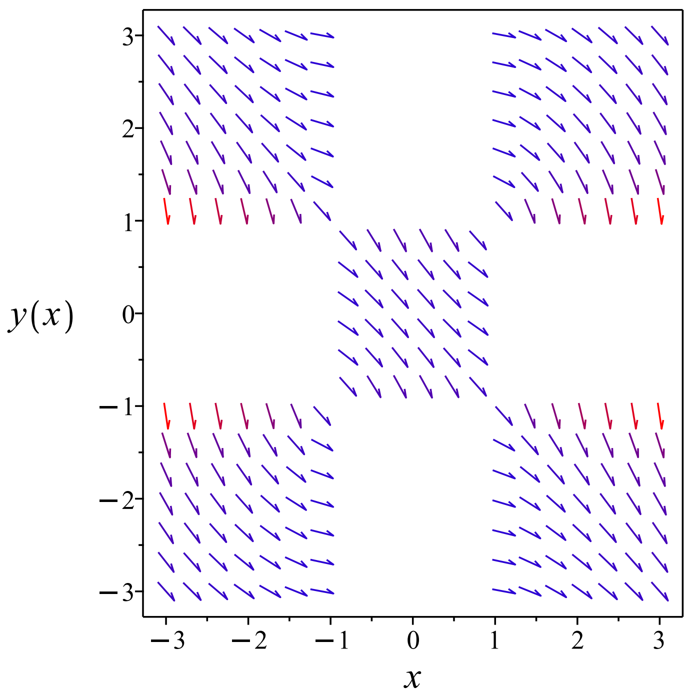


Figure 33: Slope field plot

Verification of solutions

$$-\frac{\sqrt{1-y^2}\,y}{2} - \frac{\arcsin(y)}{2} - \frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} = c_1$$

Verified OK.

1.7.5 Maple step by step solution

Let's solve

$\sqrt{1-y^2}\, y' = -\sqrt{1-x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \sqrt{1-y^2}\, y' dx = \int -\sqrt{1-x^2} dx + c_1$

- Evaluate integral

  $\frac{\sqrt{1-y^2}\, y}{2} + \frac{\arcsin(y)}{2} = -\frac{x\sqrt{1-x^2}}{2} - \frac{\arcsin(x)}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(sqrt(1-x^2)+sqrt(1-y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$c_1 + x\sqrt{-x^2+1} + \arcsin(x) + y(x)\sqrt{1-y(x)^2} + \arcsin(y(x)) = 0$$

✓ Solution by Mathematica

Time used: 0.626 (sec). Leaf size: 85

```
DSolve[Sqrt[1-x^2]+Sqrt[1-y[x]^2]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{1}{2}\#1\sqrt{1-\#1^2} - \arctan\left(\frac{\sqrt{1-\#1^2}}{\#1+1}\right)\&\right]\left[\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right) - \frac{1}{2}\sqrt{1-x^2}x + c_1\right]$$

1.8 problem 8

Internal problem ID [1877]
Internal file name [`OUTPUT/1878_Sunday_June_05_2022_02_36_33_AM_39155536/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x+1)\,y'+y=1}$$

1.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-y+1}{x+1} \end{aligned}$$

Where $f(x) = \frac{1}{x+1}$ and $g(y) = -y + 1$. Integrating both sides gives

$$\frac{1}{-y+1}\,dy = \frac{1}{x+1}\,dx$$

$$\int \frac{1}{-y+1}\,dy = \int \frac{1}{x+1}\,dx$$

$$-\ln(y-1) = \ln(x+1) + c_1$$

Raising both side to exponential gives

$$\frac{1}{y-1} = \mathrm{e}^{\ln(x+1)+c_1}$$

Which simplifies to

$$\frac{1}{y-1} = c_2(x+1)$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_2\mathrm{e}^{\ln(x+1)+c_1} + 1\right)\mathrm{e}^{-c_1}}{c_2\,(x+1)} \tag{1}$$

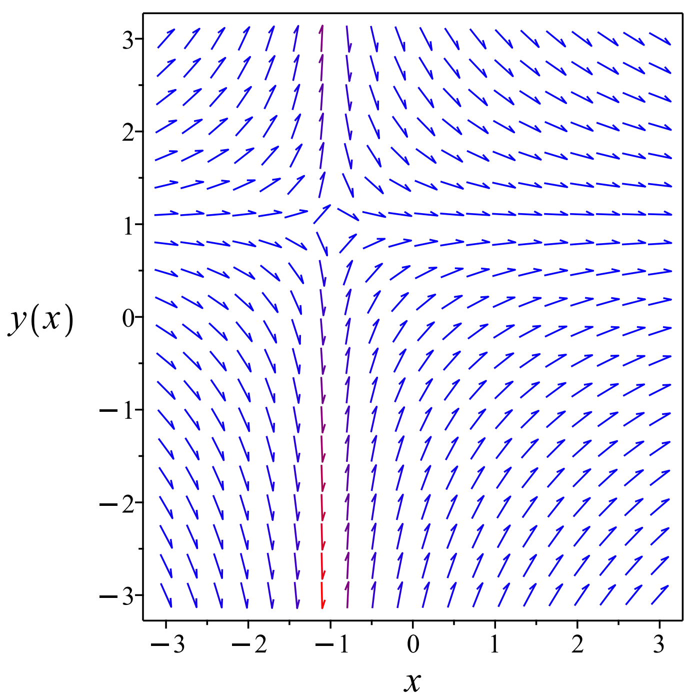


Figure 34: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\left(c_2 \mathrm{e}^{\ln(x+1)+c_1} + 1\right) \mathrm{e}^{-c_1}}{c_2 (x+1)}$$

Verified OK.

1.8.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x+1} \\ q(x) &= \frac{1}{x+1} \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x+1} = \frac{1}{x+1}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{x+1} dx} \\ &= x+1 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{x+1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}((x+1)\, y) &= (x+1)\left(\frac{1}{x+1}\right) \\ \mathrm{d}((x+1)\, y) &= \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} (x+1)\, y &= \int \mathrm{d}x \\ (x+1)\, y &= x + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x + 1$ results in

$$y = \frac{x}{x+1} + \frac{c_1}{x+1}$$

which simplifies to

$$y = \frac{x + c_1}{x + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x + c_1}{x + 1} \tag{1}$$

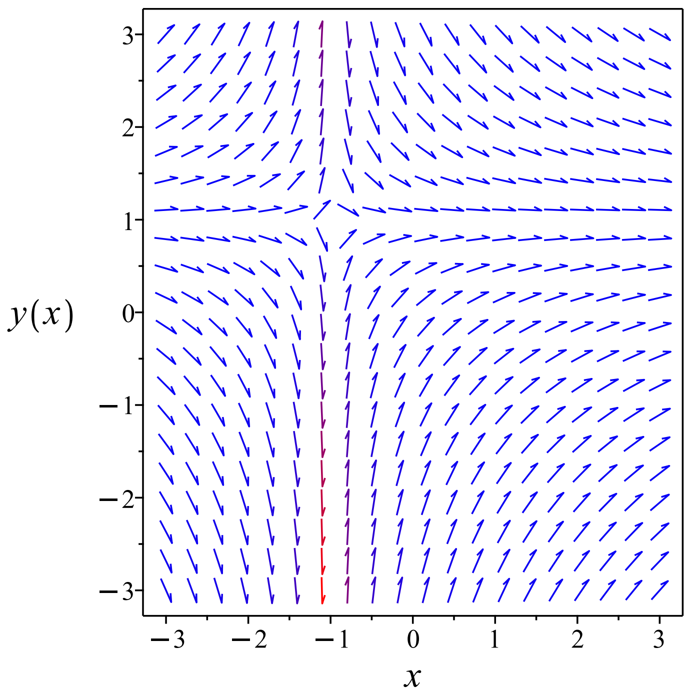


Figure 35: Slope field plot

Verification of solutions

$$y = \frac{x + c_1}{x + 1}$$

Verified OK.

1.8.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{1-y}{x+1} \tag{1}$$

Which becomes

$$0 = (-x-1)\,dy + (-y+1)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x-1)\,dy + (-y+1)\,dx = d(-(y-1)\,x - y)$$

Hence (2) becomes

$$0 = d(-(y-1)\,x - y)$$

Integrating both sides gives gives these solutions

$$y = \frac{x+c_1}{x+1} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{x+c_1}{x+1} + c_1 \tag{1}$$

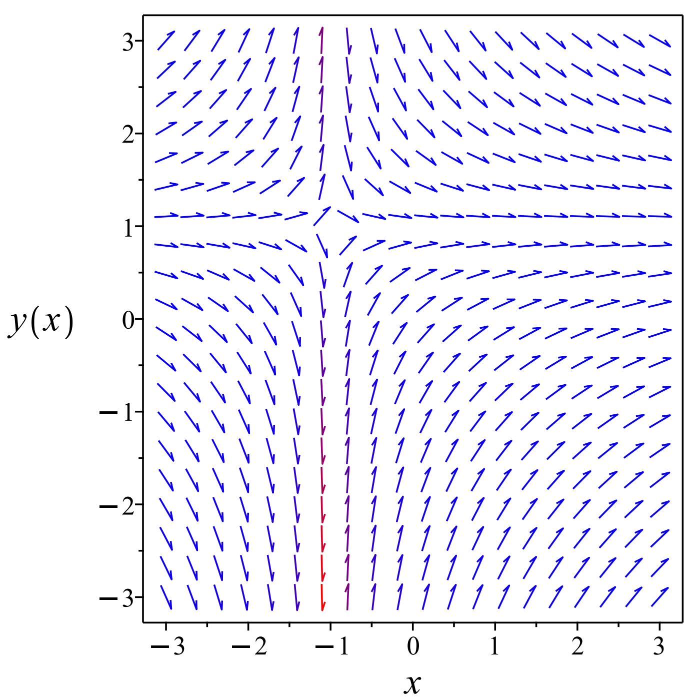


Figure 36: Slope field plot

Verification of solutions

$$y = \frac{x + c_1}{x + 1} + c_1$$

Verified OK.

1.8.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX} Y(X) = -\frac{Y(X) + y_0 - 1}{X + x_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= -1 \\ y_0 &= 1 \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX} Y(X) = -\frac{Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X}X + u = -u$$
$$\frac{\mathrm{d}u}{\mathrm{d}X} = -\frac{2u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) + \frac{2u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X + 2u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{2u}{X} \end{aligned}$$

Where $f(X) = -\frac{2}{X}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{X}\,dX \\ \int \frac{1}{u}\,du &= \int -\frac{2}{X}\,dX \\ \ln(u) &= -2\ln(X) + c_2 \\ u &= \mathrm{e}^{-2\ln(X)+c_2} \\ &= \frac{c_2}{X^2} \end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \frac{c_2}{X}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{c_2}{X}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x + x_0 \end{aligned}$$

Or

$$Y = y + 1$$
$$X = x - 1$$

Then the solution in y becomes

$$y - 1 = \frac{c_2}{x + 1}$$

Summary

The solution(s) found are the following

$$y - 1 = \frac{c_2}{x + 1} \tag{1}$$

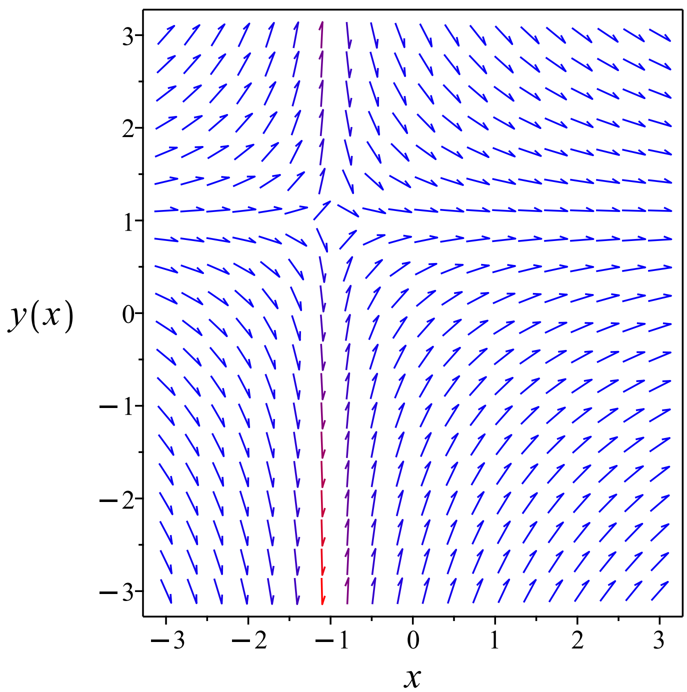


Figure 37: Slope field plot

Verification of solutions

$$y - 1 = \frac{c_2}{x + 1}$$

Verified OK.

1.8.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y-1}{x+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 22: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \frac{1}{x+1}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{1}{x+1}} dy\end{aligned}$$

Which results in

$$S = (x+1)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y-1}{x+1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \\
S_y &= x + 1
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
(x + 1) y = x + c_1
$$

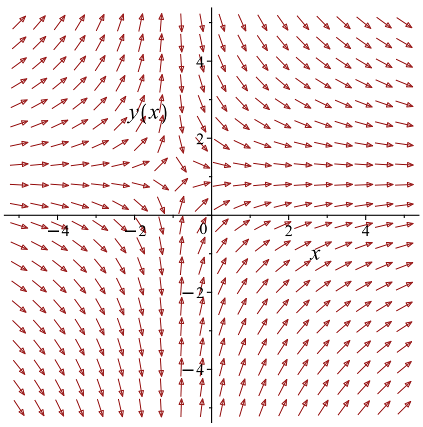
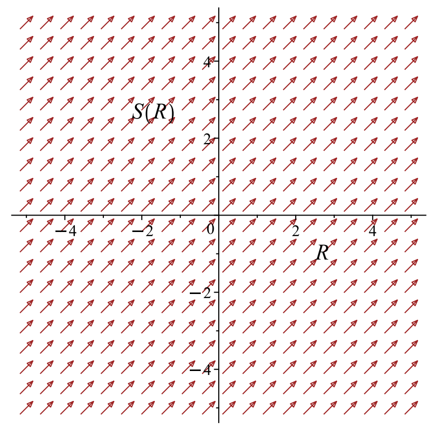
Which simplifies to

$$
(x + 1) y = x + c_1
$$

Which gives

$$
y = \frac{x + c_1}{x + 1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-1}{x+1}$  	$R = x$ $S = (x+1)\,y$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = \frac{x + c_1}{x + 1} \tag{1}$$

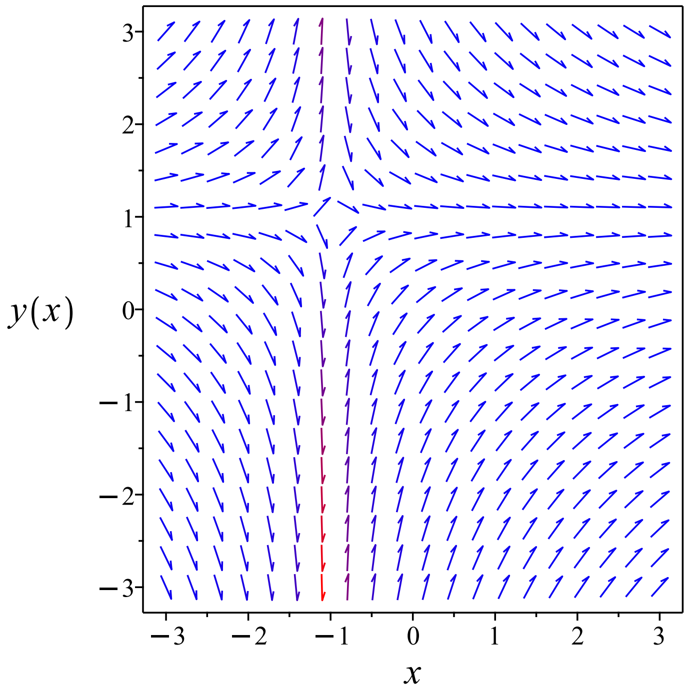


Figure 38: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x + c_1}{x + 1}$$

Verified OK.

1.8.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-y+1}\right)\mathrm{d}y &= \left(\frac{1}{x+1}\right)\mathrm{d}x \\ \left(-\frac{1}{x+1}\right)\mathrm{d}x + \left(\frac{1}{-y+1}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x+1} \\ N(x, y) &= \frac{1}{-y+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x+1}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-y+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x+1}\,\mathrm{d}x\\ \phi &= -\ln\left(x+1\right) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y+1}$. Therefore equation (4) becomes

$$\frac{1}{-y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y-1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y-1}\right)\mathrm{d}y\\ f(y) &= -\ln\left(y-1\right) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+1) - \ln(y-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+1) - \ln(y-1)$$

The solution becomes

$$y = \frac{(x\,e^{c_1} + e^{c_1} + 1)\,e^{-c_1}}{x+1}$$

Summary

The solution(s) found are the following

$$y = \frac{(x\,e^{c_1} + e^{c_1} + 1)\,e^{-c_1}}{x+1} \tag{1}$$

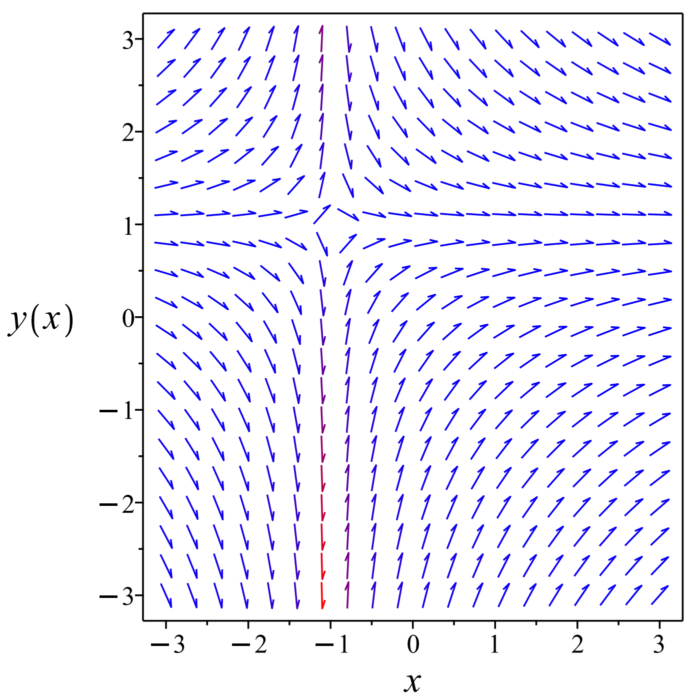


Figure 39: Slope field plot

Verification of solutions

$$y = \frac{(x\,e^{c_1} + e^{c_1} + 1)\,e^{-c_1}}{x + 1}$$

Verified OK.

1.8.7 Maple step by step solution

Let's solve

$(x + 1)\,y' + y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int ((x + 1)\,y' + y)\,dx = \int 1dx + c_1$

- Evaluate integral

  $(x + 1)\,y = x + c_1$

- Solve for y

  $y = \frac{x+c_1}{x+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve((1+x)*diff(y(x),x)-(1-y(x))=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 + x}{x + 1}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 20

```
DSolve[(1+x)*y'[x]-(1-y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x + c_1}{x + 1}$$
$$y(x) \to 1$$

1.9 problem 9

Internal problem ID [1878]
Internal file name [`OUTPUT/1879_Sunday_June_05_2022_02_36_35_AM_68472485/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' \tan(x) - y = 1}$$

1.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y+1}{\tan(x)} \end{aligned}$$

Where $f(x) = \frac{1}{\tan(x)}$ and $g(y) = y + 1$. Integrating both sides gives

$$\frac{1}{y+1}\, dy = \frac{1}{\tan(x)}\, dx$$

$$\int \frac{1}{y+1}dy = \int \frac{1}{\tan(x)}dx$$
$$\ln(y+1) = \ln(\sin(x)) + c_1$$

Raising both side to exponential gives

$$y+1 = e^{\ln(\sin(x))+c_1}$$

Which simplifies to

$$y+1 = c_2 \sin(x)$$

Which simplifies to

$$y = c_2 \sin(x) e^{c_1} - 1$$

Summary

The solution(s) found are the following

$$y = c_2 \sin(x) e^{c_1} - 1 \tag{1}$$

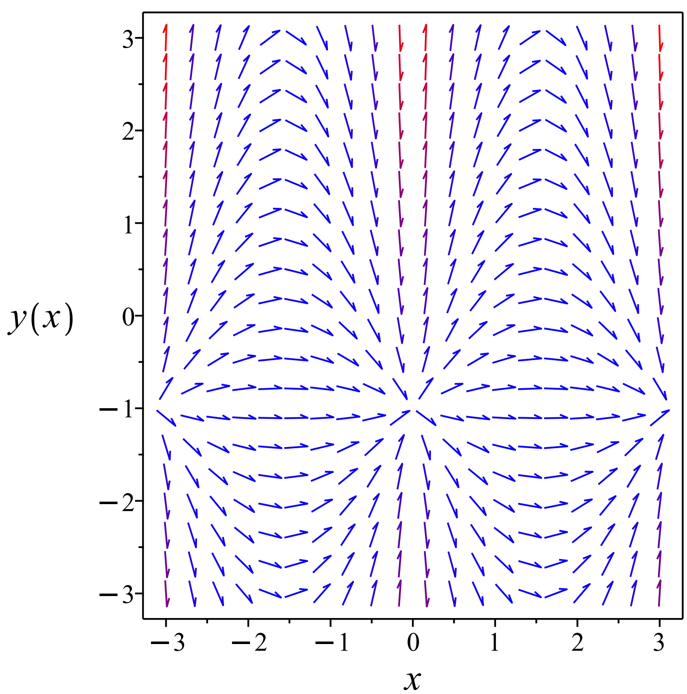


Figure 40: Slope field plot

Verification of solutions

$$y = c_2 \sin(x) e^{c_1} - 1$$

Verified OK.

1.9.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\cot(x) \\ q(x) &= \cot(x) \end{aligned}$$

Hence the ode is

$$y' - y\cot(x) = \cot(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\cot(x)dx} \\ &= \frac{1}{\sin(x)} \end{aligned}$$

Which simplifies to

$$\mu = \csc(x)$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(\cot(x)) \\ \frac{d}{dx}(\csc(x)\, y) &= (\csc(x))(\cot(x)) \\ d(\csc(x)\, y) &= (\csc(x)\cot(x))\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \csc(x)\, y &= \int \csc(x)\cot(x)\, dx \\ \csc(x)\, y &= -\csc(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \csc(x)$ results in

$$y = -\sin(x)\csc(x) + \sin(x)\, c_1$$

which simplifies to

$$y = \sin(x)\, c_1 - 1$$

Summary

The solution(s) found are the following

$$y = \sin(x) c_1 - 1 \tag{1}$$

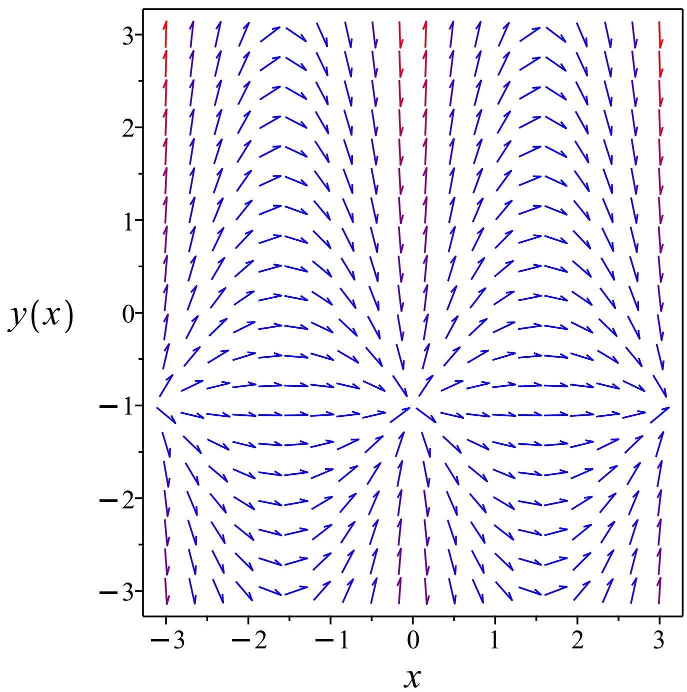


Figure 41: Slope field plot

Verification of solutions

$$y = \sin(x) c_1 - 1$$

Verified OK.

1.9.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y+1}{\tan(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 25: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \sin(x)
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sin(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sin(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+1}{\tan(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\csc(x)\cot(x)\,y \\ S_y &= \csc(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \csc(x)\cot(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \csc(R)\cot(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\csc(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\csc(x)\, y = -\csc(x) + c_1$$

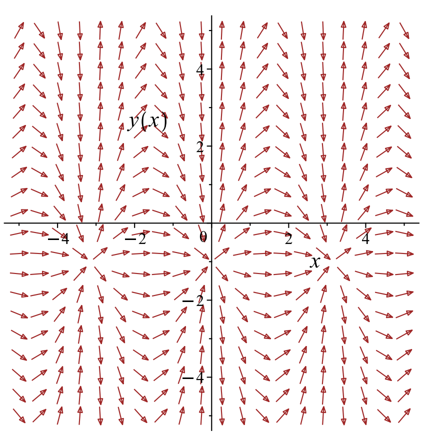
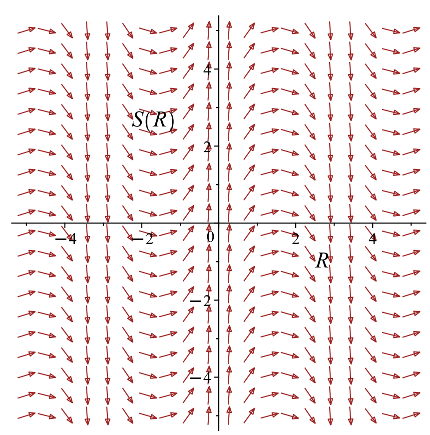
Which simplifies to

$$\csc(x)\, y = -\csc(x) + c_1$$

Which gives

$$y = -\frac{\csc(x) - c_1}{\csc(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+1}{\tan(x)}$ 	$R = x$ $S = \csc(x)\, y$	$\frac{dS}{dR} = \csc(R)\cot(R)$ 

Summary

The solution(s) found are the following

$$y = -\frac{\csc(x) - c_1}{\csc(x)} \tag{1}$$

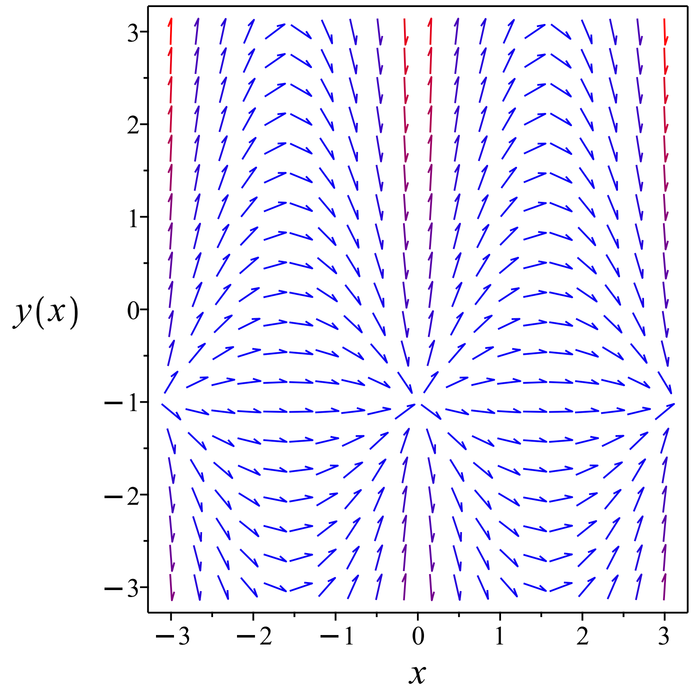


Figure 42: Slope field plot

Verification of solutions

$$y = -\frac{\csc(x) - c_1}{\csc(x)}$$

Verified OK.

1.9.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y+1}\right) \mathrm{d}y &= \left(\frac{1}{\tan (x)}\right) \mathrm{d}x \\
\left(-\frac{1}{\tan (x)}\right) \mathrm{d}x + \left(\frac{1}{y+1}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{\tan (x)} \\
N(x, y) &= \frac{1}{y+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{\tan (x)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y+1}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{\tan(x)} \, \mathrm{d}x\\
\phi &= -\ln(\sin(x)) + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y+1}$. Therefore equation (4) becomes

$$
\frac{1}{y+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y+1}\right) \mathrm{d}y\\
f(y) &= \ln(y+1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(x)) + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(x)) + \ln(y+1)$$

The solution becomes

$$y = \sin(x)\,e^{c_1} - 1$$

Summary

The solution(s) found are the following

$$y = \sin(x)\,e^{c_1} - 1 \tag{1}$$

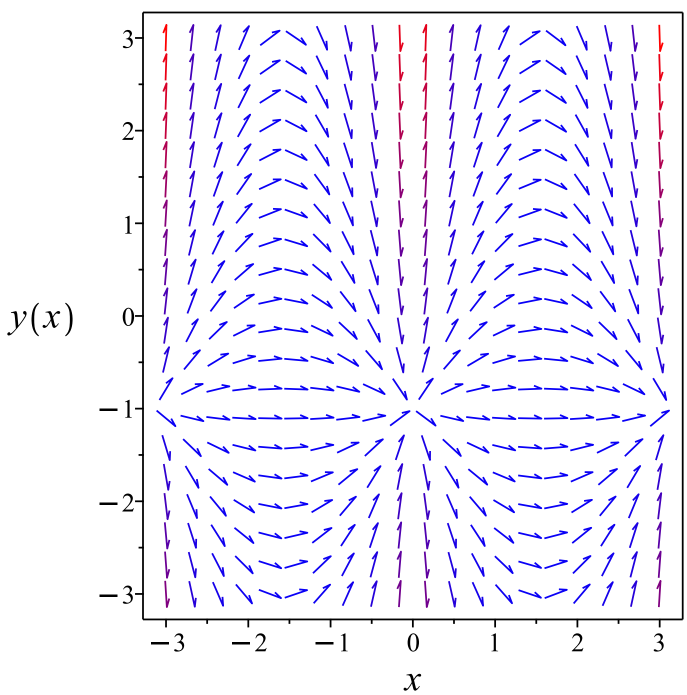


Figure 43: Slope field plot

Verification of solutions

$$y = \sin(x)\,e^{c_1} - 1$$

Verified OK.

1.9.5 Maple step by step solution

Let's solve

$y' \tan(x) - y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y} = \frac{1}{\tan(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y} dx = \int \frac{1}{\tan(x)} dx + c_1$

- Evaluate integral

  $\ln(1+y) = \ln(\sin(x)) + c_1$

- Solve for y

  $y = \sin(x) e^{c_1} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)*tan(x)-y(x)=1,y(x), singsol=all)
```

$$y(x) = c_1 \sin(x) - 1$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 17

```
DSolve[y'[x]*Tan[x]-y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 + c_1 \sin(x)$$
$$y(x) \to -1$$

1.10 problem 10

Internal problem ID [1879]
Internal file name [`OUTPUT/1880_Sunday_June_05_2022_02_36_37_AM_8113241/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y + \cot(x)\, y' = -3}$$

1.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-y-3}{\cot(x)} \end{aligned}$$

Where $f(x) = \frac{1}{\cot(x)}$ and $g(y) = -y - 3$. Integrating both sides gives

$$\frac{1}{-y-3}\, dy = \frac{1}{\cot(x)}\, dx$$

$$\int \frac{1}{-y-3}\,dy = \int \frac{1}{\cot(x)}\,dx$$
$$-\ln(3+y) = -\ln(\cos(x)) + c_1$$

Raising both side to exponential gives

$$\frac{1}{3+y} = e^{-\ln(\cos(x))+c_1}$$

Which simplifies to

$$\frac{1}{3+y} = \frac{c_2}{\cos(x)}$$

Which simplifies to

$$y = -\frac{\left(\frac{3c_2 e^{c_1}}{\cos(x)} - 1\right) e^{-c_1} \cos(x)}{c_2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\frac{3c_2 e^{c_1}}{\cos(x)} - 1\right) e^{-c_1} \cos(x)}{c_2} \tag{1}$$

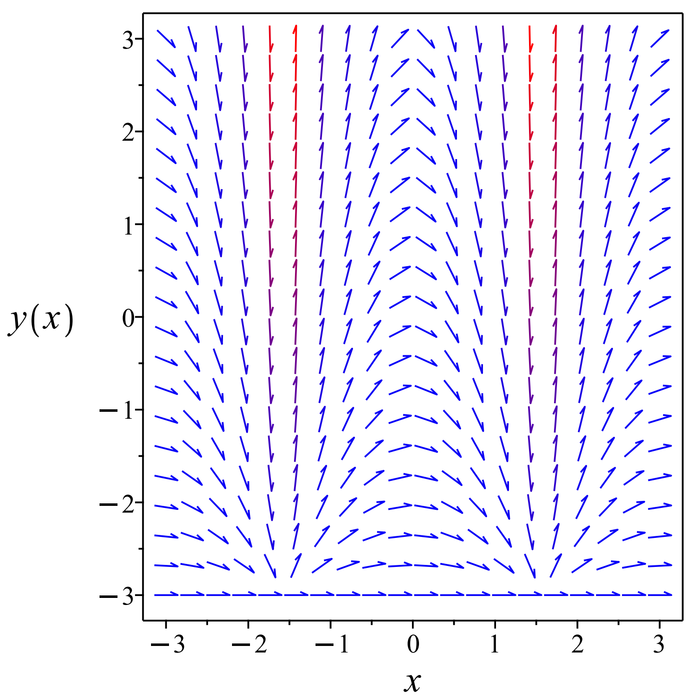


Figure 44: Slope field plot

Verification of solutions

$$y = -\frac{\left(\frac{3c_2 e^{c_1}}{\cos(x)} - 1\right) e^{-c_1} \cos(x)}{c_2}$$

Verified OK.

1.10.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= -3\tan(x) \end{aligned}$$

Hence the ode is

$$y' + \tan(x)\, y = -3\tan(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \tan(x) dx} \\ &= \frac{1}{\cos(x)} \end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(-3\tan(x)) \\ \frac{d}{dx}(\sec(x)\, y) &= (\sec(x))(-3\tan(x)) \\ d(\sec(x)\, y) &= (-3\tan(x)\sec(x))\; dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \sec(x)\, y &= \int -3\tan(x)\sec(x)\; dx \\ \sec(x)\, y &= -3\sec(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = -3\cos(x)\sec(x) + c_1\cos(x)$$

which simplifies to

$$y = c_1\cos(x) - 3$$

Summary

The solution(s) found are the following

$$y = c_1\cos(x) - 3 \tag{1}$$

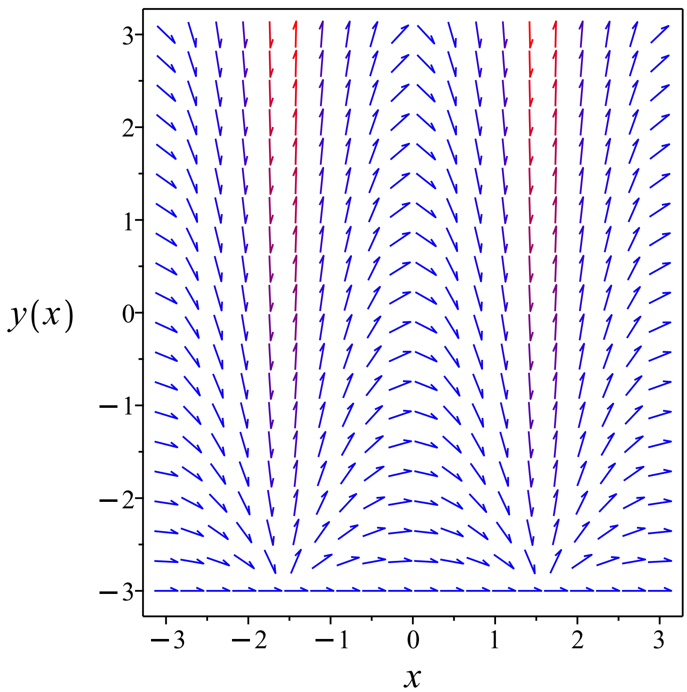


Figure 45: Slope field plot

Verification of solutions

$$y = c_1\cos(x) - 3$$

Verified OK.

1.10.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3+y}{\cot(x)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 28: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \cos(x)
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\cos(x)} dy
\end{aligned}
$$

Which results in

$$
S = \frac{y}{\cos(x)}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{3+y}{\cot(x)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \tan(x)\sec(x)\, y\\
S_y &= \sec(x)
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -3\tan(x)\sec(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -3\tan(R)\sec(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -3\sec(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sec(x)\, y = -3\sec(x) + c_1$$

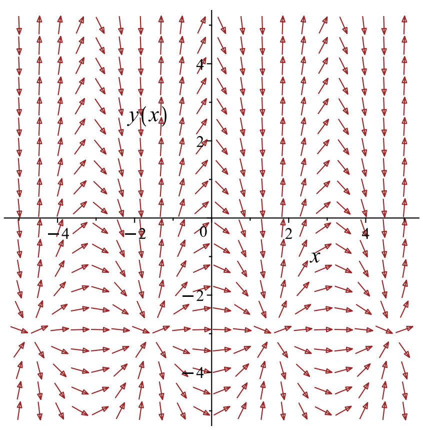
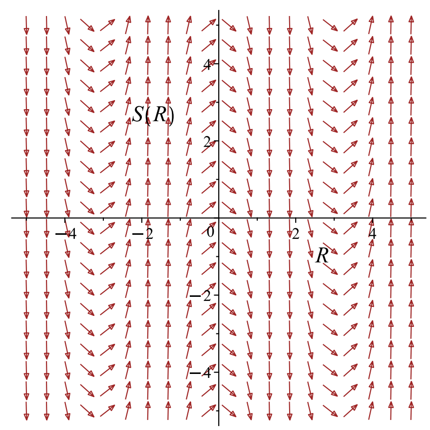
Which simplifies to

$$\sec(x)\, y = -3\sec(x) + c_1$$

Which gives

$$y = -\frac{3\sec(x) - c_1}{\sec(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3+y}{\cot(x)}$ 	$R = x$ $S = \sec(x)\, y$	$\frac{dS}{dR} = -3\tan(R)\sec(R)$ 

Summary

The solution(s) found are the following

$$y = -\frac{3\sec(x) - c_1}{\sec(x)} \tag{1}$$

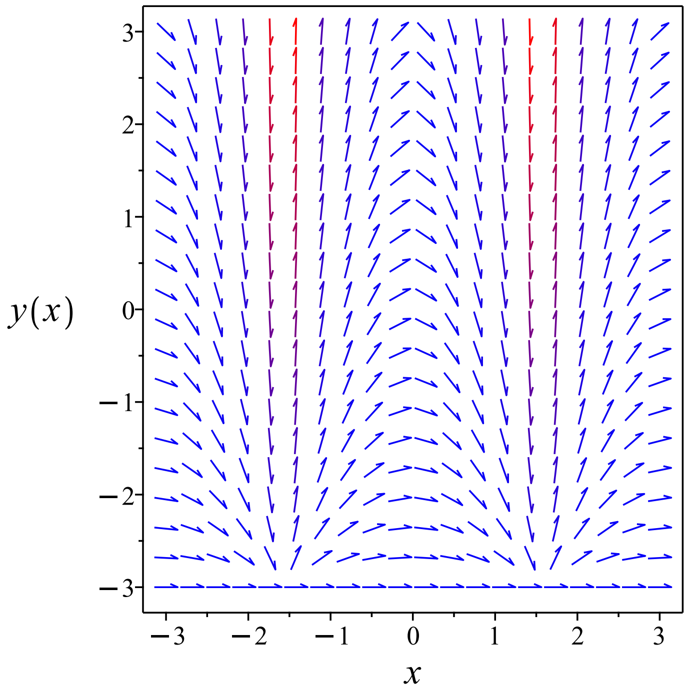


Figure 46: Slope field plot

Verification of solutions

$$y = -\frac{3\sec(x) - c_1}{\sec(x)}$$

Verified OK.

1.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{-y-3}\right) \mathrm{d}y &= \left(\frac{1}{\cot(x)}\right) \mathrm{d}x \\
\left(-\frac{1}{\cot(x)}\right) \mathrm{d}x + \left(\frac{1}{-y-3}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{\cot(x)} \\
N(x, y) &= \frac{1}{-y-3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{\cot(x)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-y-3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{\cot(x)} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \ln(\cos(x)) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y-3}$. Therefore equation (4) becomes

$$
\frac{1}{-y-3} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{3+y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{1}{3+y}\right) \mathrm{d}y \\
f(y) &= -\ln(3+y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(\cos(x)) - \ln(3 + y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(\cos(x)) - \ln(3 + y)$$

The solution becomes

$$y = -(3\,e^{c_1} - \cos(x))\,e^{-c_1}$$

Summary
The solution(s) found are the following

$$y = -(3\,e^{c_1} - \cos(x))\,e^{-c_1} \tag{1}$$

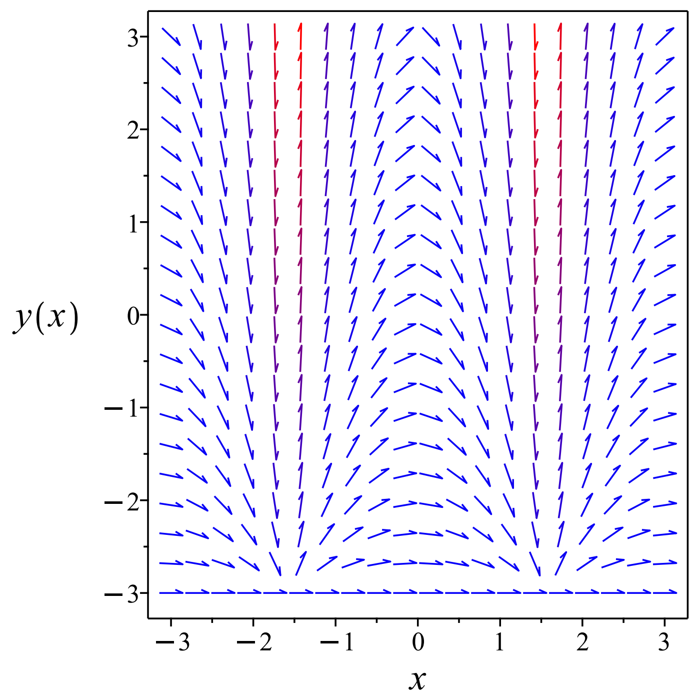


Figure 47: Slope field plot

Verification of solutions

$$y = -(3\,e^{c_1} - \cos(x))\,e^{-c_1}$$

Verified OK.

1.10.5 Maple step by step solution

Let's solve

$y + \cot(x)\, y' = -3$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y-3} = \frac{1}{\cot(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y-3} dx = \int \frac{1}{\cot(x)} dx + c_1$

- Evaluate integral

  $-\ln(-y-3) = -\ln(\cos(x)) + c_1$

- Solve for y

  $y = -\frac{3\,e^{c_1} + \cos(x)}{e^{c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve((y(x)+3)+cot(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \cos(x)\, c_1 - 3$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 17

```
DSolve[(y[x]+3)+Cot[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3 + c_1 \cos(x)$$
$$y(x) \to -3$$

1.11 problem 11

Internal problem ID [1880]
Internal file name [`OUTPUT/1881_Sunday_June_05_2022_02_36_39_AM_65049930/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{x}{y} = 0}$$

1.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x}{y} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = x\,dx$$

$$\int \frac{1}{\frac{1}{y}}\, dy = \int x\, dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{x^2 + 2c_1}$$

$$y = -\sqrt{x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{x^2 + 2c_1} \tag{2}$$

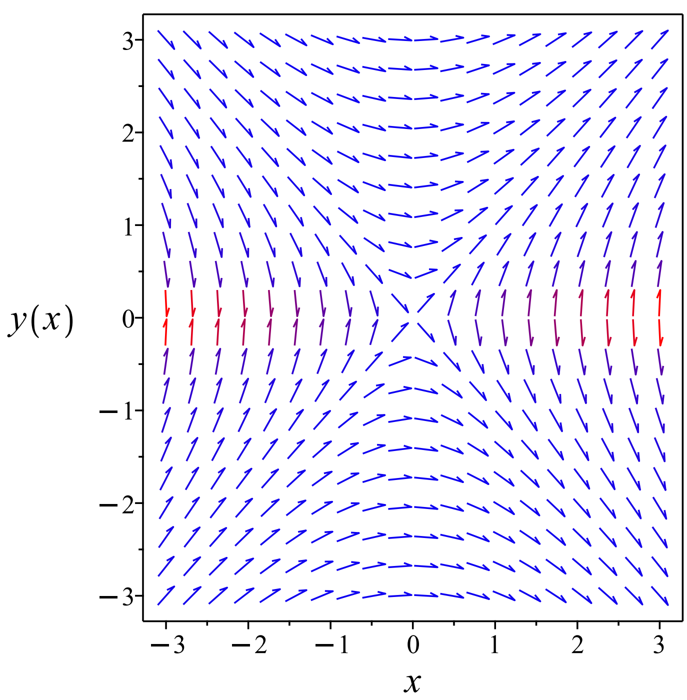


Figure 48: Slope field plot

Verification of solutions

$$y = \sqrt{x^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1}$$

Verified OK.

1.11.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{1}{u(x)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2-1}{xu} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2-1}{u}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} = -\ln(x) + c_2$$

The above can be written as

$$\begin{aligned} \left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) &= -\ln(x) + 2c_2 \\ \ln(u-1) + \ln(u+1) &= (2)(-\ln(x) + 2c_2) \\ &= -2\ln(x) + 4c_2 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-2\ln(x)+2c_2}$$

Which simplifies to

$$
\begin{aligned}
u^2 - 1 &= \frac{2c_2}{x^2} \\
&= \frac{c_3}{x^2}
\end{aligned}
$$

The solution is

$$
u(x)^2 - 1 = \frac{c_3}{x^2}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y^2}{x^2} - 1 &= \frac{c_3}{x^2} \\
\frac{y^2}{x^2} - 1 &= \frac{c_3}{x^2}
\end{aligned}
$$

Which simplifies to

$$
-(x - y)(x + y) = c_3
$$

<u>Summary</u>

The solution(s) found are the following

$$
-(x - y)(x + y) = c_3 \tag{1}
$$

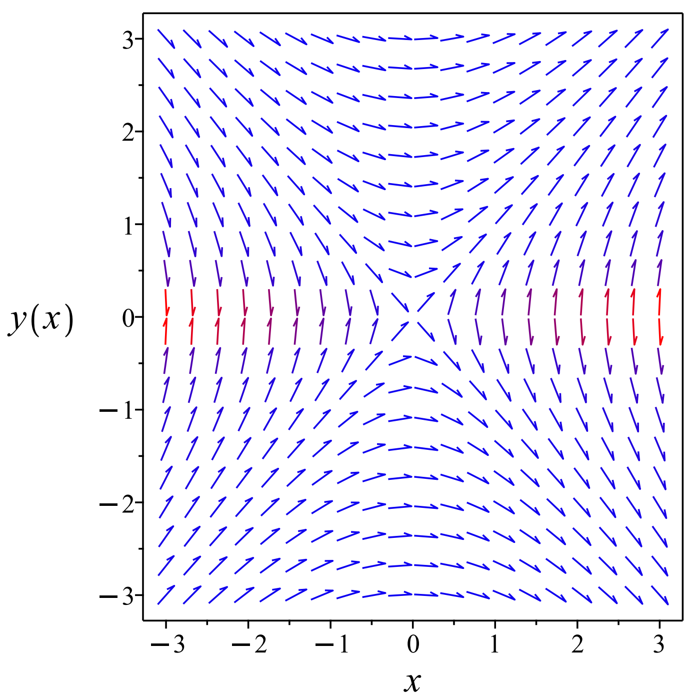


Figure 49: Slope field plot

Verification of solutions

$$-(x-y)(x+y)=c_3$$

Verified OK.

1.11.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x}{y} \tag{1}$$

Which becomes

$$(y)\, dy = (x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(x)\, dx = d\left(\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y)\, dy = d\left(\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{x^2+2c_1}+c_1$$
$$y = -\sqrt{x^2+2c_1}+c_1$$

Summary

The solution(s) found are the following

$$y = \sqrt{x^2+2c_1}+c_1 \tag{1}$$
$$y = -\sqrt{x^2+2c_1}+c_1 \tag{2}$$

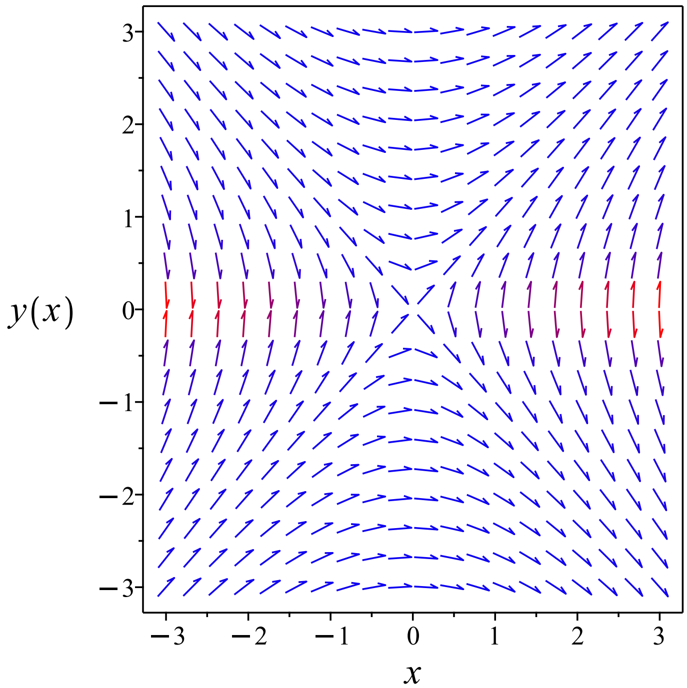


Figure 50: Slope field plot

Verification of solutions

$$y = \sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

1.11.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 31: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \frac{1}{x} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

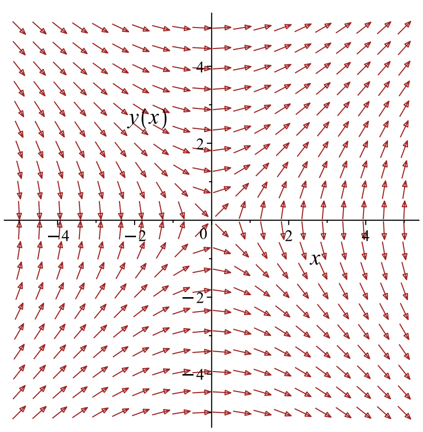
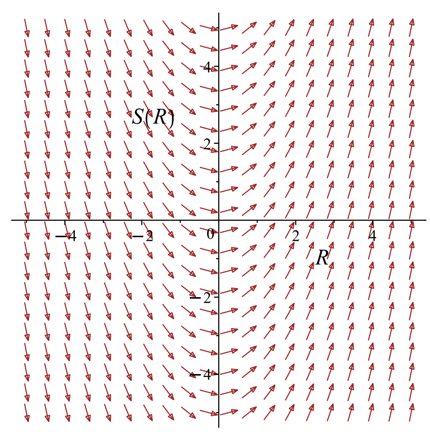
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{y}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = R$ 

Summary
The solution(s) found are the following

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

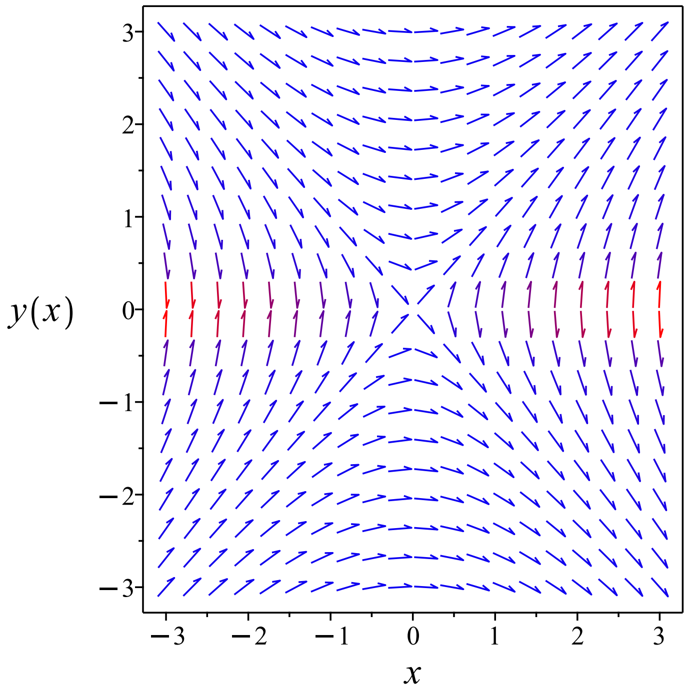


Figure 51: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

1.11.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y)\,\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + (y)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x\\ N(x, y) &= y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y) \, \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^2}{2}$$

Summary
The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

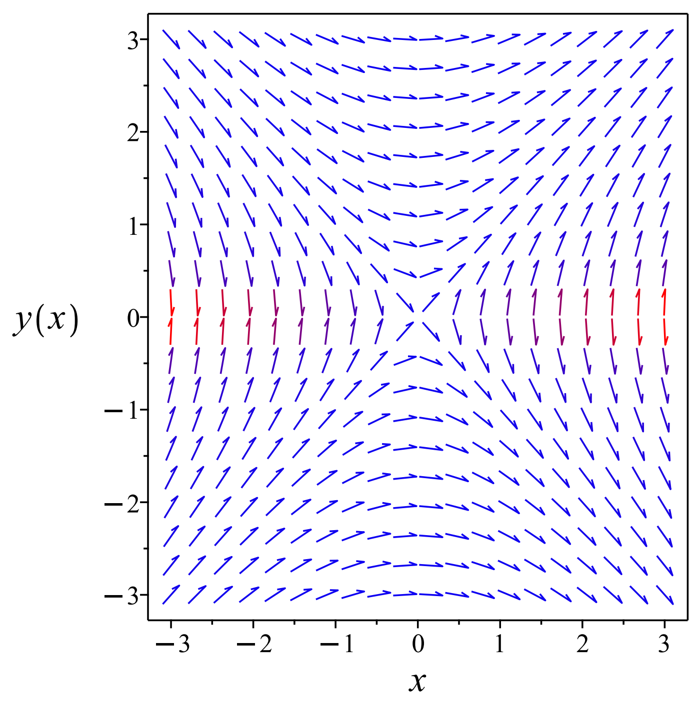


Figure 52: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

1.11.6 Maple step by step solution

Let's solve

$y' - \frac{x}{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = x$

- Integrate both sides with respect to x

  $\int yy'dx = \int xdx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{x^2}{2} + c_1$

- Solve for y

  $\{y = \sqrt{x^2 + 2c_1}, y = -\sqrt{x^2 + 2c_1}\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)=x/y(x),y(x), singsol=all)
```

$$y(x) = \sqrt{x^2 + c_1}$$
$$y(x) = -\sqrt{x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 35

```
DSolve[y'[x]==x/y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to -\sqrt{x^2 + 2c_1}
$$
$$
y(x) \to \sqrt{x^2 + 2c_1}
$$

1.12 problem 12

Internal problem ID [1881]
Internal file name [`OUTPUT/1882_Sunday_June_05_2022_02_36_41_AM_32393311/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x' = 1 - \sin(2t)}$$

1.12.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} x &= \int 1 - \sin(2t)\, \mathrm{d}t \\ &= t + \frac{\cos(2t)}{2} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$x = t + \frac{\cos(2t)}{2} + c_1 \tag{1}$$

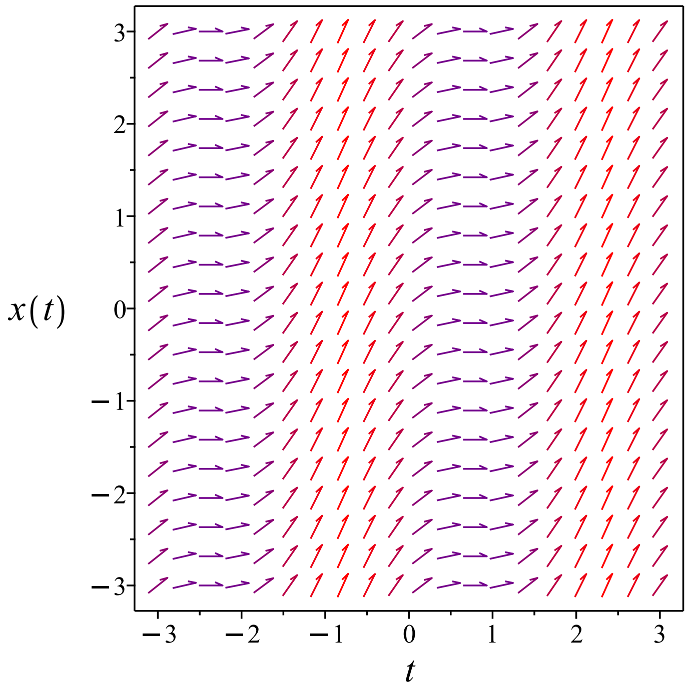


Figure 53: Slope field plot

Verification of solutions

$$x = t + \frac{\cos(2t)}{2} + c_1$$

Verified OK.

1.12.2 Maple step by step solution

Let's solve

$x' = 1 - \sin(2t)$

- Highest derivative means the order of the ODE is 1

  x'

- Integrate both sides with respect to t

  $\int x' dt = \int (1 - \sin(2t)) dt + c_1$

- Evaluate integral

  $x = t + \frac{\cos(2t)}{2} + c_1$

- Solve for x

$$x = t + \frac{\cos(2t)}{2} + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve(diff(x(t),t)=1-sin(2*t),x(t), singsol=all)
```

$$x(t) = \frac{\cos(2t)}{2} + t + c_1$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 17

```
DSolve[x'[t]==1-Sin[2*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t + \frac{1}{2}\cos(2t) + c_1$$

1.13 problem 13

Internal problem ID [1882]
Internal file name [`OUTPUT/1883_Sunday_June_05_2022_02_36_43_AM_65564060/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'x + y - y^2 = 0}$$

1.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y(y-1)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y(y-1)$. Integrating both sides gives

$$\frac{1}{y\,(y-1)}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{y(y-1)}\,dy = \int \frac{1}{x}\,dx$$

$$-\ln(y) + \ln(y-1) = \ln(x) + c_1$$

Raising both side to exponential gives

$$e^{-\ln(y)+\ln(y-1)} = e^{\ln(x)+c_1}$$

Which simplifies to

$$\frac{y-1}{y} = c_2 x$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{c_2 x - 1} \tag{1}$$

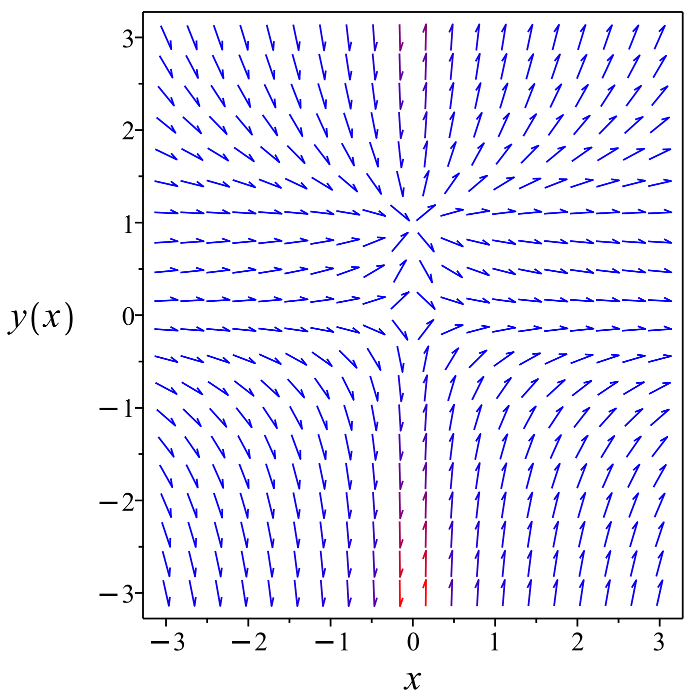


Figure 54: Slope field plot

Verification of solutions

$$y = -\frac{1}{c_2 x - 1}$$

Verified OK.

1.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(y-1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 35: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = x$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$S = \int \frac{1}{\xi} dx$$
$$= \int \frac{1}{x} dx$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(y-1)}{x}$$

Evaluating all the partial derivatives gives

$$R_x = 0$$
$$R_y = 1$$
$$S_x = \frac{1}{x}$$
$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y(y-1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(R-1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + \ln(R-1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = -\ln(y) + \ln(y-1) + c_1$$

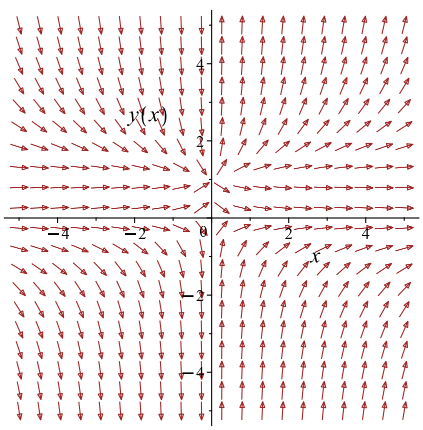
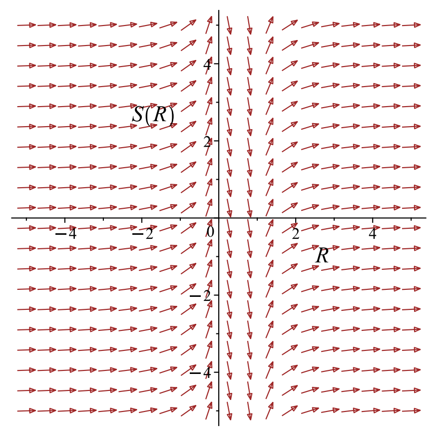
Which simplifies to

$$\ln(x) = -\ln(y) + \ln(y-1) + c_1$$

Which gives

$$y = \frac{e^{c_1}}{e^{c_1} - x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(y-1)}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{R(R-1)}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{c_1}}{e^{c_1} - x} \tag{1}$$

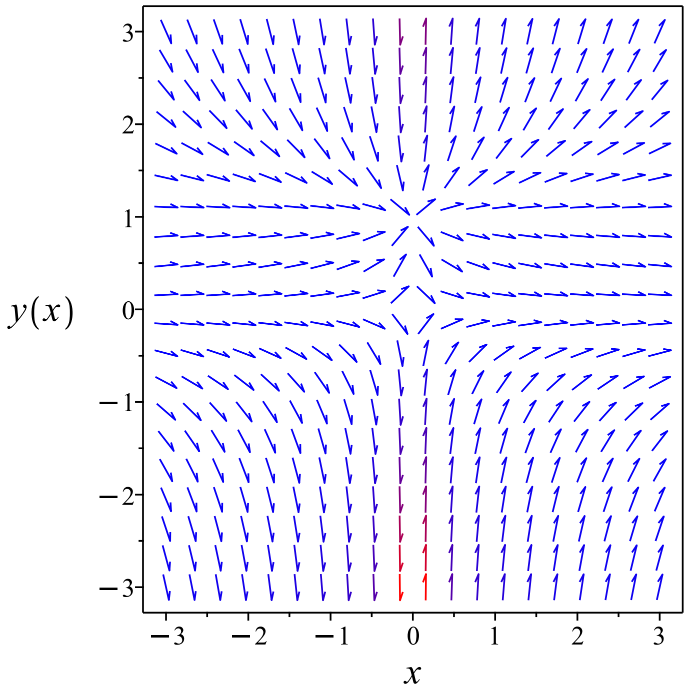


Figure 55: Slope field plot

Verification of solutions

$$y = \frac{e^{c_1}}{e^{c_1} - x}$$

Verified OK.

1.13.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(y-1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{1}{x}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= \frac{1}{x} \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y' \frac{1}{y^2} = -\frac{1}{yx} + \frac{1}{x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= -\frac{w(x)}{x} + \frac{1}{x} \\
w' &= \frac{w}{x} - \frac{1}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -\frac{1}{x}
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -\frac{1}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{1}{x}\right) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(-\frac{1}{x}\right) \\ d\left(\frac{w}{x}\right) &= \left(-\frac{1}{x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int -\frac{1}{x^2}\, dx \\ \frac{w}{x} &= \frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x + 1$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = c_1 x + 1$$

Or

$$y = \frac{1}{c_1 x + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{c_1 x + 1} \tag{1}$$

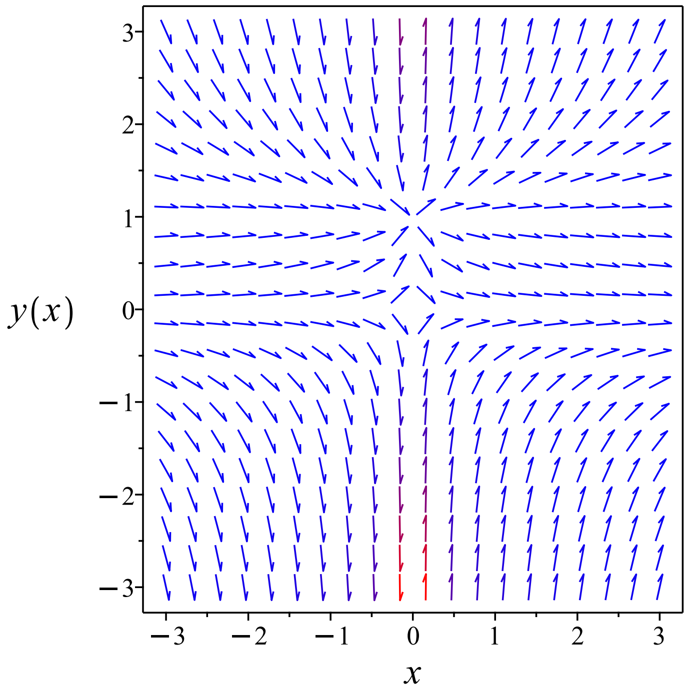


Figure 56: Slope field plot

Verification of solutions

$$y = \frac{1}{c_1 x + 1}$$

Verified OK.

1.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y\,(y-1)}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{y\,(y-1)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{1}{y\,(y-1)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y\,(y-1)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x}\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y(y-1)}$. Therefore equation (4) becomes

$$
\frac{1}{y\,(y-1)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y\,(y-1)}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int \left(\frac{1}{y\,(y-1)}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + \ln(y-1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y) + \ln(y - 1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y) + \ln(y - 1)$$

The solution becomes

$$y = -\frac{1}{-1 + x\,e^{c_1}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{1}{-1 + x\,e^{c_1}} \tag{1}$$

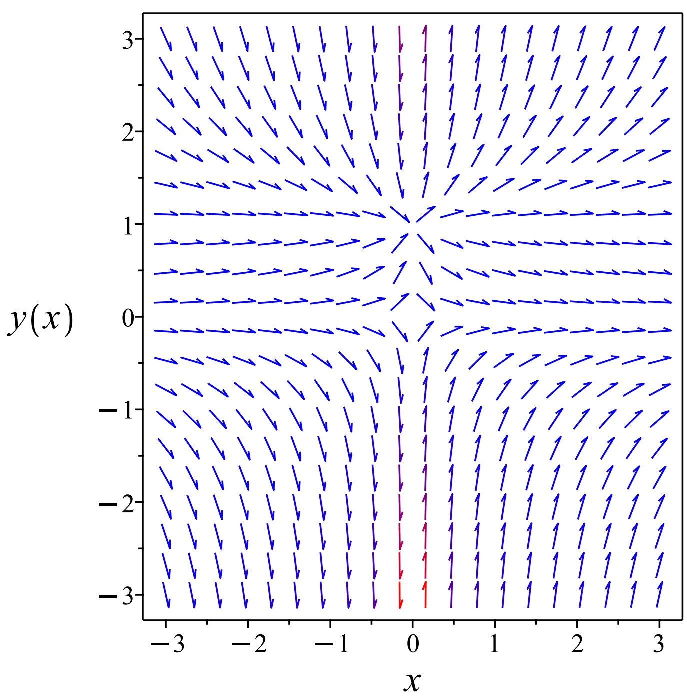


Figure 57: Slope field plot

Verification of solutions

$$y = -\frac{1}{-1 + x\,\mathrm{e}^{c_1}}$$

Verified OK.

1.13.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(y-1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x} - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= -\frac{1}{x^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} + \frac{2u'(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_2}{x\left(c_1 + \frac{c_2}{x}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{c_3 x + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{c_3 x + 1} \tag{1}$$

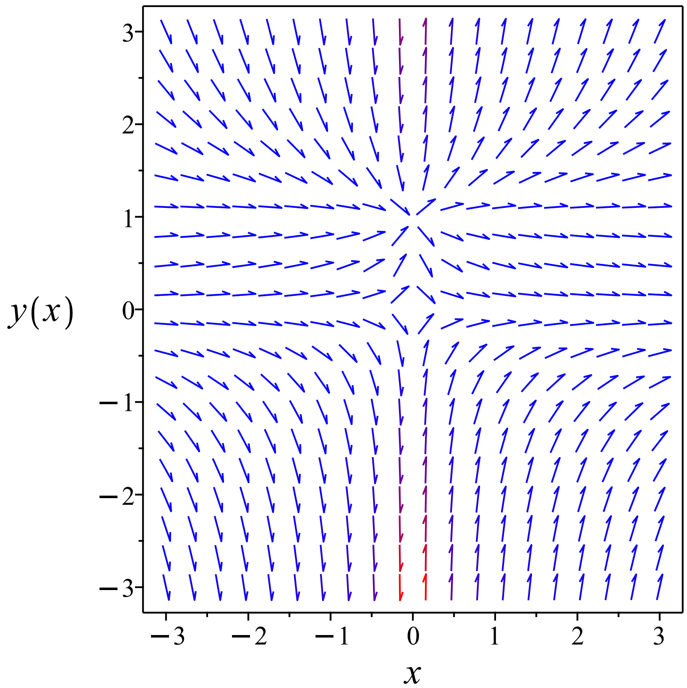


Figure 58: Slope field plot

Verification of solutions

$$y = \frac{1}{c_3 x + 1}$$

Verified OK.

1.13.6 Maple step by step solution

Let's solve

$y' x + y - y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2 - y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2 - y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

$$-\ln(y)+\ln(y-1)=\ln(x)+c_1$$

- Solve for y

$$y=-\frac{1}{-1+x\,e^{c_1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x*diff(y(x),x)+y(x)=y(x)^2,y(x), singsol=all)
```

$$y(x)=\frac{1}{c_1x+1}$$

✓ Solution by Mathematica

Time used: 0.274 (sec). Leaf size: 25

```
DSolve[x*y'[x]+y[x]==y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{1+e^{c_1}x}$$
$$y(x)\to 0$$
$$y(x)\to 1$$

1.14 problem 14

Internal problem ID [1883]
Internal file name [`OUTPUT/1884_Sunday_June_05_2022_02_36_45_AM_10569256/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\sin(x)\cos(y)^2 + y'\cos(x)^2 = 0}$$

1.14.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\sin(x)\cos(y)^2}{\cos(x)^2} \end{aligned}$$

Where $f(x) = -\frac{\sin(x)}{\cos(x)^2}$ and $g(y) = \cos(y)^2$. Integrating both sides gives

$$\frac{1}{\cos(y)^2}\, dy = -\frac{\sin(x)}{\cos(x)^2}\, dx$$
$$\int \frac{1}{\cos(y)^2}\, dy = \int -\frac{\sin(x)}{\cos(x)^2}\, dx$$

$$\tan(y) = -\frac{1}{\cos(x)} + c_1$$

Which results in

$$y = \arctan\left(\frac{c_1 \cos(x) - 1}{\cos(x)}\right)$$

Summary

The solution(s) found are the following

$$y = \arctan\left(\frac{c_1 \cos(x) - 1}{\cos(x)}\right) \tag{1}$$

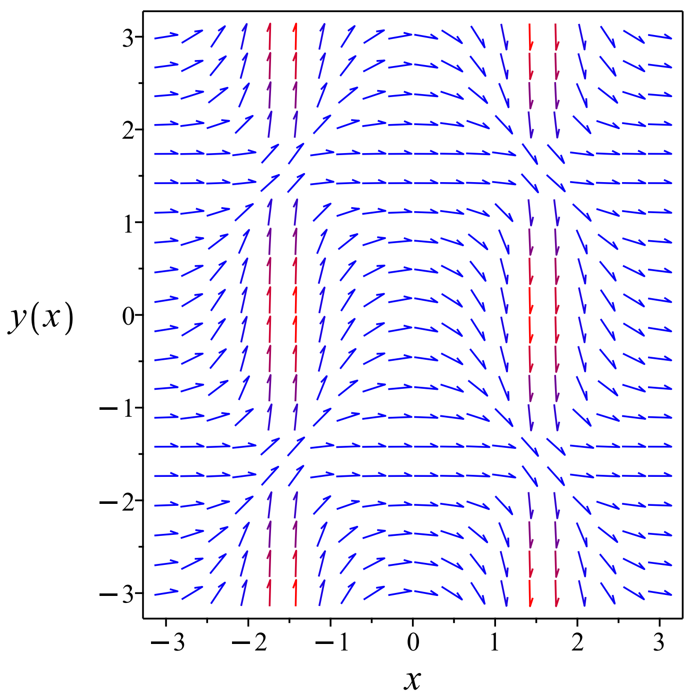


Figure 59: Slope field plot

Verification of solutions

$$y = \arctan\left(\frac{c_1 \cos(x) - 1}{\cos(x)}\right)$$

Verified OK.

1.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sin(x)\cos(y)^2}{\cos(x)^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 38: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{\cos(x)^2}{\sin(x)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{\cos(x)^2}{\sin(x)}} dx \end{aligned}$$

Which results in

$$S = -\frac{1}{\cos(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sin(x)\cos(y)^2}{\cos(x)^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\tan(x)\sec(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \sec(y)^2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \sec(R)^2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \tan(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\sec(x) = \tan(y) + c_1
$$

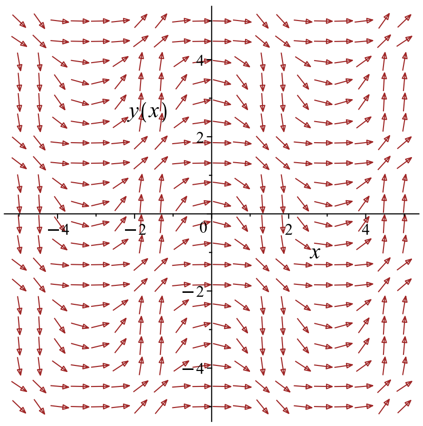
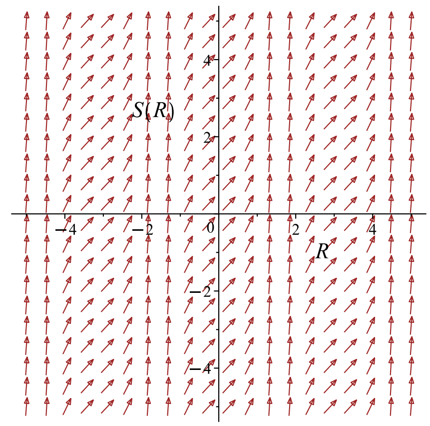
Which simplifies to

$$
-\sec(x) = \tan(y) + c_1
$$

Which gives

$$
y = -\arctan(\sec(x) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(x)\cos(y)^2}{\cos(x)^2}$ 	$R = y$ $S = -\sec(x)$	$\frac{dS}{dR} = \sec(R)^2$ 

Summary

The solution(s) found are the following

$$y = -\arctan(\sec(x) + c_1) \tag{1}$$

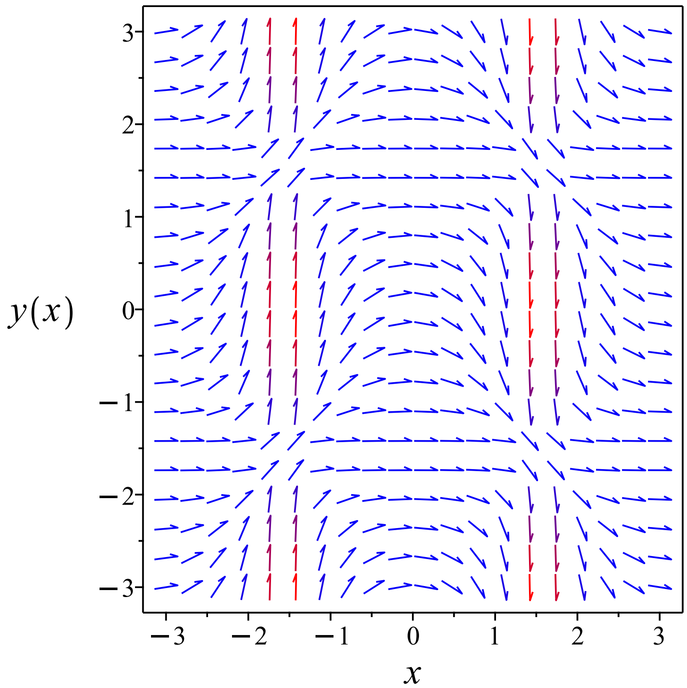


Figure 60: Slope field plot

Verification of solutions

$$y = -\arctan(\sec(x) + c_1)$$

Verified OK.

1.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{\cos(y)^2}\right)\mathrm{d}y &= \left(\frac{\sin(x)}{\cos(x)^2}\right)\mathrm{d}x \\ \left(-\frac{\sin(x)}{\cos(x)^2}\right)\mathrm{d}x + \left(-\frac{1}{\cos(y)^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{\sin(x)}{\cos(x)^2} \\ N(x, y) &= -\frac{1}{\cos(y)^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{\cos(x)^2}\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{\cos(y)^2}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int -\frac{\sin(x)}{\cos(x)^2} \, dx$$
$$\phi = -\sec(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{\cos(y)^2}$. Therefore equation (4) becomes

$$-\frac{1}{\cos(y)^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{\cos(y)^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(-\sec(y)^2\right) dy$$
$$f(y) = -\tan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sec(x) - \tan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sec(x) - \tan(y)$$

Summary

The solution(s) found are the following

$$-\sec(x) - \tan(y) = c_1 \tag{1}$$

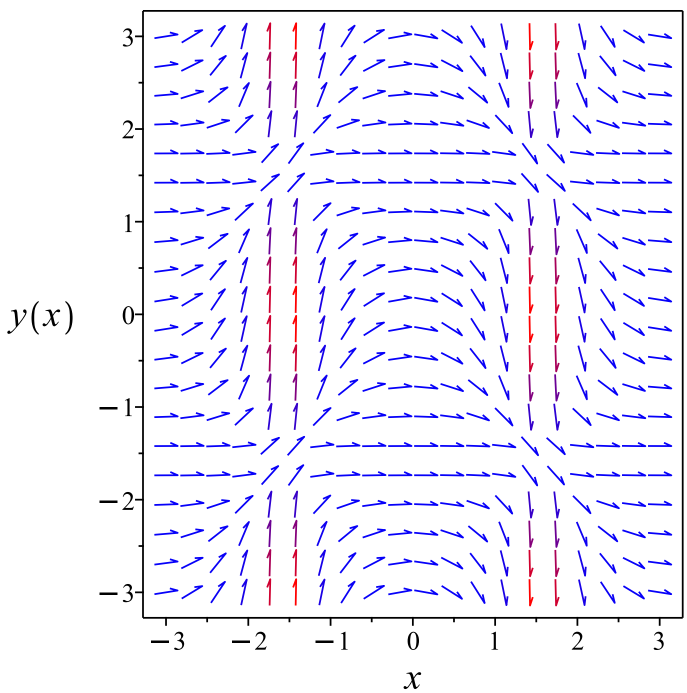


Figure 61: Slope field plot

Verification of solutions

$$-\sec(x) - \tan(y) = c_1$$

Verified OK.

1.14.4 Maple step by step solution

Let's solve

$\sin(x)\cos(y)^2 + y'\cos(x)^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(y)^2} = -\frac{\sin(x)}{\cos(x)^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\cos(y)^2}dx = \int -\frac{\sin(x)}{\cos(x)^2}dx + c_1$

- Evaluate integral

  $\tan(y) = -\frac{1}{\cos(x)} + c_1$

- Solve for y

  $y = \arctan\left(\frac{c_1\cos(x)-1}{\cos(x)}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(sin(x)*cos(y(x))^2+cos(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\arctan(\sec(x) + c_1)$$

✓ Solution by Mathematica

Time used: 1.616 (sec). Leaf size: 31

```
DSolve[Sin[x]*Cos[y[x]]^2+Cos[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arctan(-\sec(x) + c_1)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

1.15 problem 15

Internal problem ID [1884]
Internal file name [`OUTPUT/1885_Sunday_June_05_2022_02_36_47_AM_11007949/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sec(x)\cos(y)^2 - \cos(x)\sin(y)\,y' = 0}$$

1.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{\sec(x)\cos(y)\cot(y)}{\cos(x)}
\end{aligned}$$

Where $f(x) = \frac{\sec(x)}{\cos(x)}$ and $g(y) = \cos(y)\cot(y)$. Integrating both sides gives

$$\frac{1}{\cos(y)\cot(y)}\,dy = \frac{\sec(x)}{\cos(x)}\,dx$$

$$\int \frac{1}{\cos(y)\cot(y)}\,dy = \int \frac{\sec(x)}{\cos(x)}\,dx$$

$$\frac{1}{\cos(y)} = \tan(x) + c_1$$

Which results in

$$y = \arccos\left(\frac{1}{\tan(x) + c_1}\right)$$

Summary

The solution(s) found are the following

$$y = \arccos\left(\frac{1}{\tan(x) + c_1}\right) \tag{1}$$

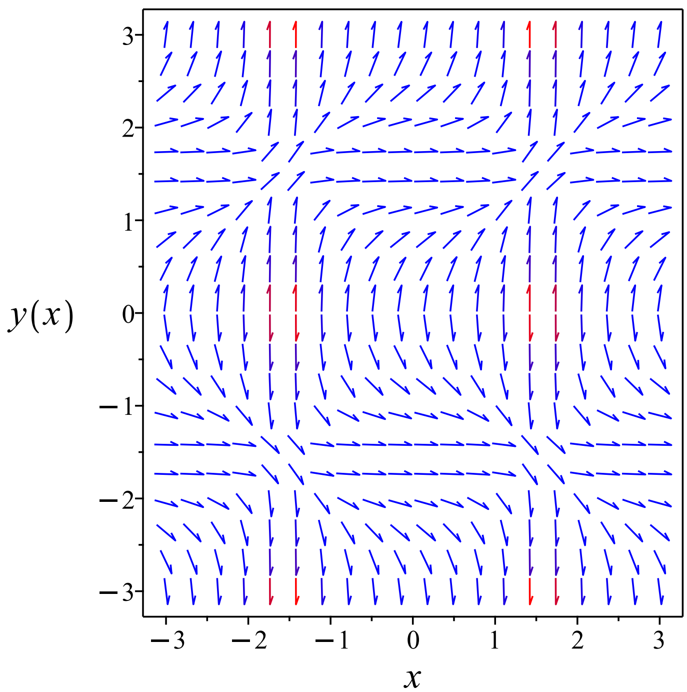


Figure 62: Slope field plot

Verification of solutions

$$y = \arccos\left(\frac{1}{\tan(x) + c_1}\right)$$

Verified OK.

1.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sec(x)\cos(y)^2}{\cos(x)\sin(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 41: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{\cos(x)}{\sec(x)} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{\cos(x)}{\sec(x)}} dx
\end{aligned}
$$

Which results in

$$
S = \tan(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\sec(x)\cos(y)^2}{\cos(x)\sin(y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \sec(x)^2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan(y)\sec(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R)\sec(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sec(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\tan(x) = \sec(y) + c_1$$

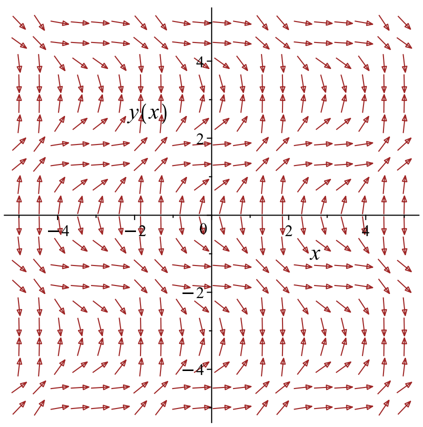
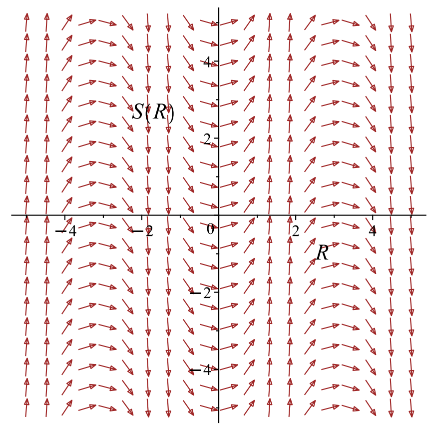
Which simplifies to

$$\tan(x) = \sec(y) + c_1$$

Which gives

$$y = \pi - \operatorname{arcsec}(-\tan(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sec(x)\cos(y)^2}{\cos(x)\sin(y)}$ 	$R = y$ $S = \tan(x)$	$\frac{dS}{dR} = \tan(R)\sec(R)$ 

Summary

The solution(s) found are the following

$$y = \pi - \operatorname{arcsec}(-\tan(x) + c_1) \tag{1}$$

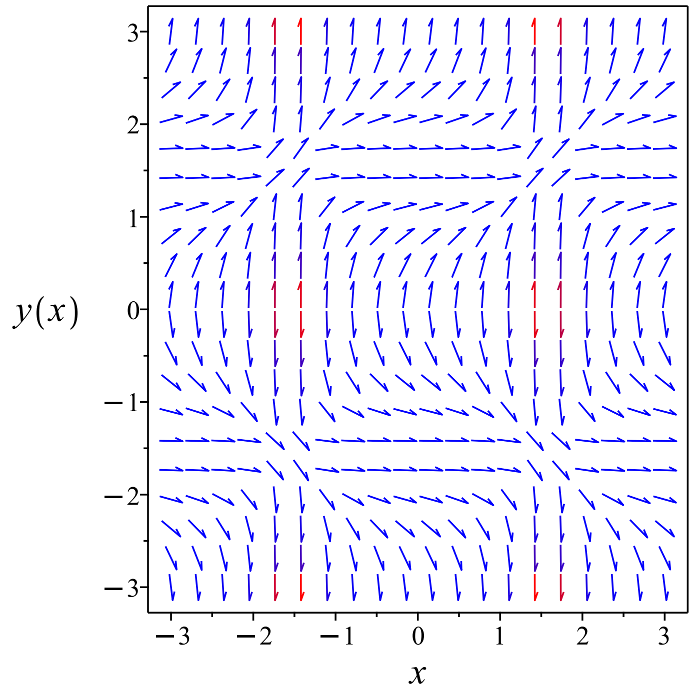


Figure 63: Slope field plot

Verification of solutions

$$y = \pi - \operatorname{arcsec}\left(-\tan\left(x\right) + c_1\right)$$

Verified OK.

1.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{\sin(y)}{\cos(y)^2}\right)\mathrm{d}y = \left(\frac{\sec(x)}{\cos(x)}\right)\mathrm{d}x$$
$$\left(-\frac{\sec(x)}{\cos(x)}\right)\mathrm{d}x + \left(\frac{\sin(y)}{\cos(y)^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\sec(x)}{\cos(x)}$$
$$N(x, y) = \frac{\sin(y)}{\cos(y)^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sec(x)}{\cos(x)}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\sin(y)}{\cos(y)^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{\sec(x)}{\cos(x)} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\tan(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sin(y)}{\cos(y)^2}$. Therefore equation (4) becomes

$$
\frac{\sin(y)}{\cos(y)^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{\sin(y)}{\cos(y)^2} \\
&= \tan(y)\sec(y)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\tan(y)\sec(y))\,\mathrm{d}y$$
$$f(y) = \sec(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\tan(x) + \sec(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\tan(x) + \sec(y)$$

The solution becomes

$$y = \operatorname{arcsec}(\tan(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \operatorname{arcsec}(\tan(x) + c_1) \tag{1}$$

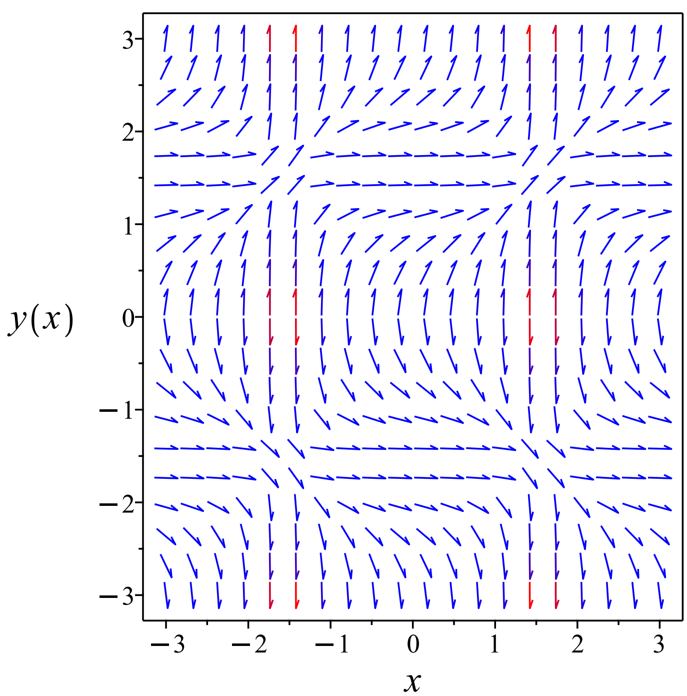


Figure 64: Slope field plot

Verification of solutions

$$y = \operatorname{arcsec}(\tan(x) + c_1)$$

Verified OK.

1.15.4 Maple step by step solution

Let's solve

$\sec(x)\cos(y)^2 - \cos(x)\sin(y)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \sin(y)}{\cos(y)^2} = \frac{\sec(x)}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \frac{y' \sin(y)}{\cos(y)^2} dx = \int \frac{\sec(x)}{\cos(x)} dx + c_1$

- Evaluate integral

  $\frac{1}{\cos(y)} = \tan(x) + c_1$

- Solve for y

  $y = \arccos\left(\frac{1}{\tan(x)+c_1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve(sec(x)*cos(y(x))^2=cos(x)*sin(y(x))*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{1}{\tan(x) + c_1}\right)$$

✓ Solution by Mathematica

Time used: 0.83 (sec). Leaf size: 45

```
DSolve[Sec[x]*Cos[y[x]]^2==Cos[x]*Sin[y[x]]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sec^{-1}(\tan(x) + 2c_1)$$
$$y(x) \to \sec^{-1}(\tan(x) + 2c_1)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

1.16 problem 16

Internal problem ID [1885]
Internal file name [`OUTPUT/1886_Sunday_June_05_2022_02_36_51_AM_14722568/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x + y - xy(y' - 1) = 0}$$

1.16.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(x+1)\,y}{(y-1)\,x} \end{aligned}$$

Where $f(x) = \frac{x+1}{x}$ and $g(y) = \frac{y}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y-1}}\,dy = \frac{x+1}{x}\,dx$$

$$\int \frac{1}{\frac{y}{y-1}}\,dy = \int \frac{x+1}{x}\,dx$$

$$y - \ln(y) = x + \ln(x) + c_1$$

Which results in

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x-c_1}}{x}\right)$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x-c_1}}{x}\right)$$

gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right)$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right) \tag{1}$$

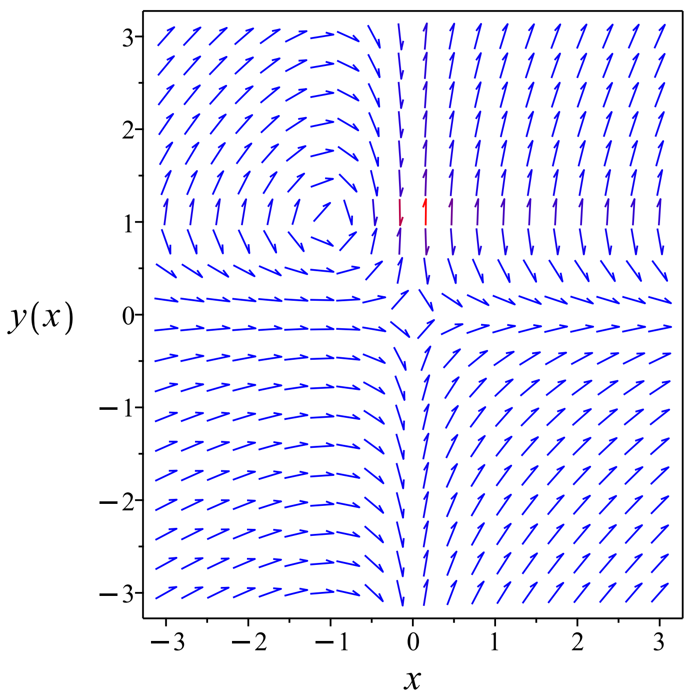


Figure 65: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right)$$

Verified OK.

1.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{(x+1)\,y}{(y-1)\,x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 44: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{x}{x+1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x}{x+1}} dx \end{aligned}$$

Which results in

$$S = x + \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{(x+1)\, y}{(y-1)\, x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 1 + \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y-1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R-1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x + \ln(x) = y - \ln(y) + c_1$$

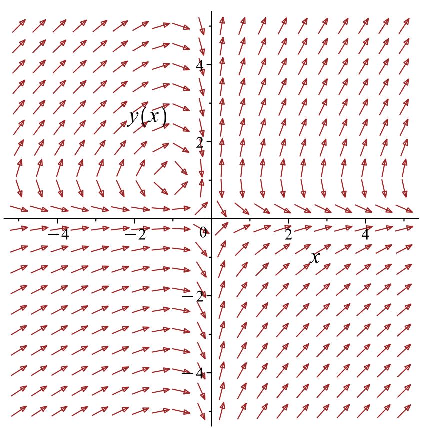
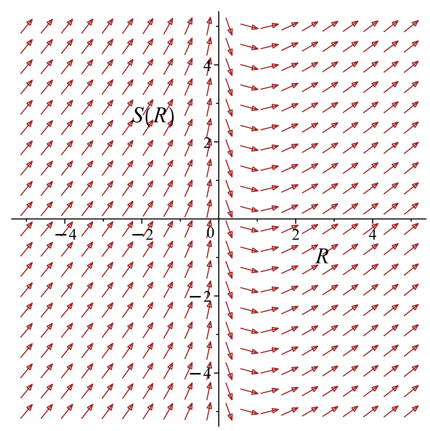
Which simplifies to

$$x + \ln(x) = y - \ln(y) + c_1$$

Which gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(x+1)y}{(y-1)x}$ 	$R = y$ $S = x + \ln(x)$	$\frac{dS}{dR} = \frac{R-1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right) \tag{1}$$

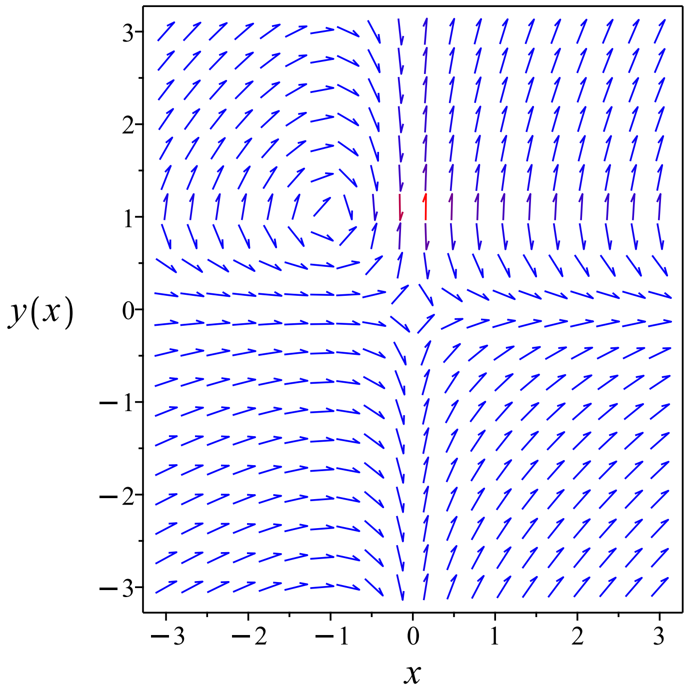


Figure 66: Slope field plot

Verification of solutions

$$y = -\text{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right)$$

Verified OK.

1.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y-1}{y}\right)\mathrm{d}y &= \left(\frac{x+1}{x}\right)\mathrm{d}x\\ \left(-\frac{x+1}{x}\right)\mathrm{d}x + \left(\frac{y-1}{y}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{x+1}{x}\\ N(x,y) &= \frac{y-1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+1}{x}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y-1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x+1}{x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -x - \ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{y-1}{y}$. Therefore equation (4) becomes

$$
\frac{y-1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y-1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{y-1}{y}\right)\mathrm{d}y \\
f(y) &= y - \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \ln(x) + y - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \ln(x) + y - \ln(y)$$

The solution becomes

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x-c_1}}{x}\right)$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x-c_1}}{x}\right) \tag{1}$$

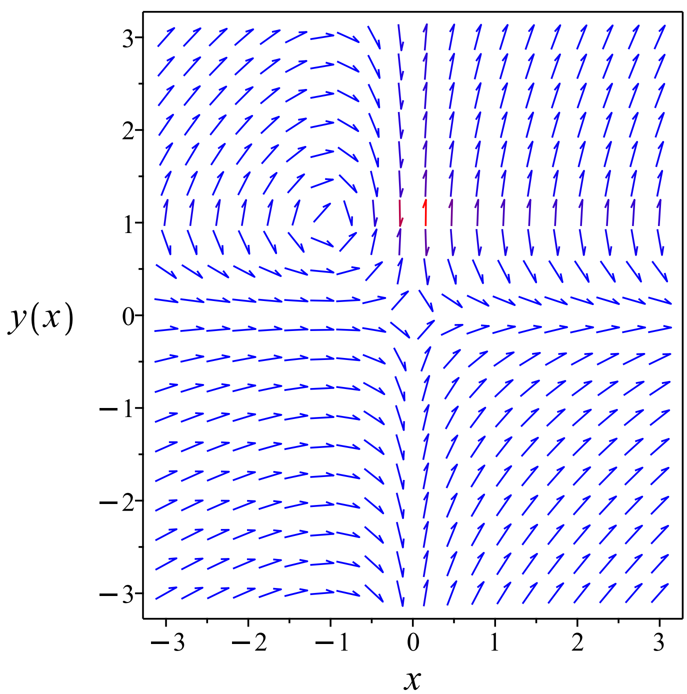


Figure 67: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x-c_1}}{x}\right)$$

Verified OK.

1.16.4 Maple step by step solution

Let's solve

$y'x + y - xy(y'-1) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y-1)}{y} = \frac{x+1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'(y-1)}{y} dx = \int \frac{x+1}{x} dx + c_1$

- Evaluate integral

  $y - \ln(y) = x + \ln(x) + c_1$

- Solve for y

  $y = -\mathit{LambertW}\left(-\frac{e^{-x-c_1}}{x}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(y(x)+x*diff(y(x),x)=x*y(x)*(diff(y(x),x)-1),y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right)$$

✓ Solution by Mathematica

Time used: 3.823 (sec). Leaf size: 28

```
DSolve[y[x]+x*y'[x]==x*y[x]*(y'[x]-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -W\left(-\frac{e^{-x-c_1}}{x}\right)$$
$$y(x) \to 0$$

1.17 problem 17

Internal problem ID [1886]
Internal file name [`OUTPUT/1887_Sunday_June_05_2022_02_36_53_AM_89753556/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{yx + \sqrt{x^2+1}\, y' = 0}$$

1.17.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{yx}{\sqrt{x^2+1}} \end{aligned}$$

Where $f(x) = -\frac{x}{\sqrt{x^2+1}}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= -\frac{x}{\sqrt{x^2+1}}\, dx \\
\int \frac{1}{y}\, dy &= \int -\frac{x}{\sqrt{x^2+1}}\, dx \\
\ln(y) &= -\sqrt{x^2+1} + c_1 \\
y &= \mathrm{e}^{-\sqrt{x^2+1}+c_1} \\
&= c_1 \mathrm{e}^{-\sqrt{x^2+1}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\sqrt{x^2+1}} \tag{1}
$$

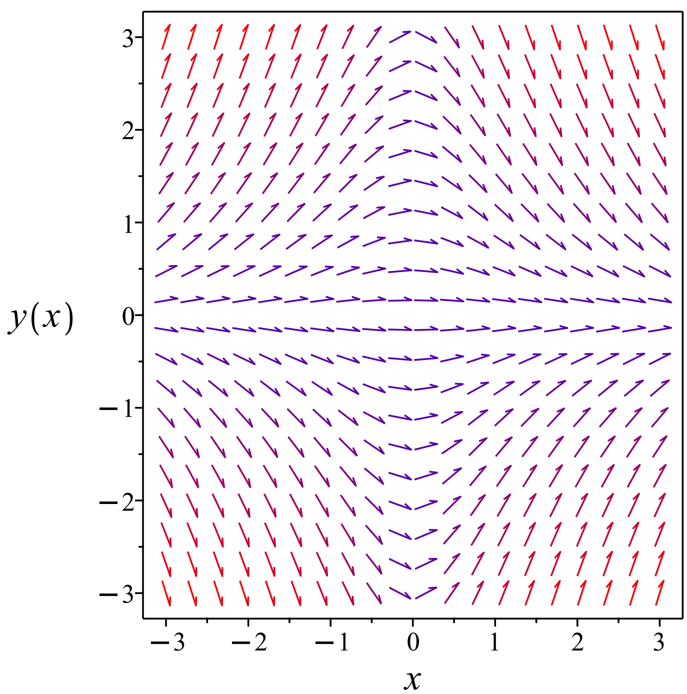


Figure 68: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\sqrt{x^2+1}}
$$

Verified OK.

1.17.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{x}{\sqrt{x^2+1}}\\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' + \frac{yx}{\sqrt{x^2+1}} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{x}{\sqrt{x^2+1}}dx}\\ &= \mathrm{e}^{\sqrt{x^2+1}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sqrt{x^2+1}}y\right) &= 0\end{aligned}$$

Integrating gives

$$\mathrm{e}^{\sqrt{x^2+1}}y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\sqrt{x^2+1}}$ results in

$$y = c_1\mathrm{e}^{-\sqrt{x^2+1}}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-\sqrt{x^2+1}} \tag{1}$$

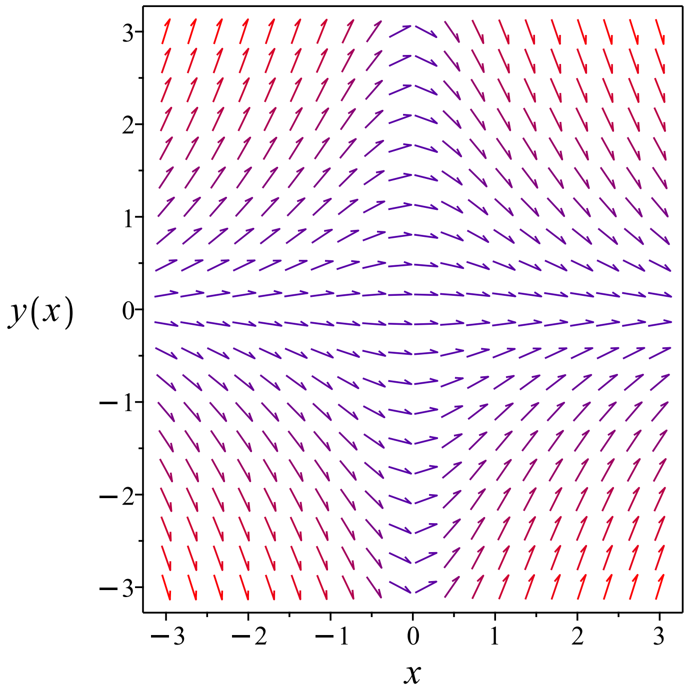


Figure 69: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\sqrt{x^2+1}}$$

Verified OK.

1.17.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x^2 + \sqrt{x^2+1}\, (u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u\left(x^2+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}\, x} \end{aligned}$$

Where $f(x) = -\frac{x^2+\sqrt{x^2+1}}{\sqrt{x^2+1}\,x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{x^2+\sqrt{x^2+1}}{\sqrt{x^2+1}\,x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{x^2+\sqrt{x^2+1}}{\sqrt{x^2+1}\,x}\,dx \\
\ln(u) &= -\sqrt{x^2+1} - \ln(x) + c_2 \\
u &= \mathrm{e}^{-\sqrt{x^2+1}-\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{-\sqrt{x^2+1}-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{-\sqrt{x^2+1}}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= c_2 \mathrm{e}^{-\sqrt{x^2+1}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 \mathrm{e}^{-\sqrt{x^2+1}} \tag{1}
$$

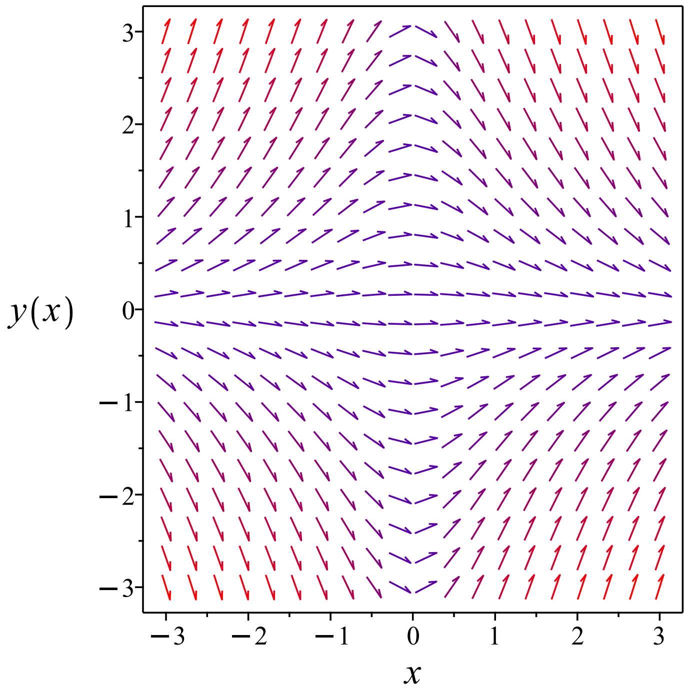


Figure 70: Slope field plot

Verification of solutions

$$y = c_2 \mathrm{e}^{-\sqrt{x^2+1}}$$

Verified OK.

1.17.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{yx}{\sqrt{x^2+1}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 47: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-\sqrt{x^2+1}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\sqrt{x^2+1}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\sqrt{x^2+1}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{yx}{\sqrt{x^2+1}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x\,\mathrm{e}^{\sqrt{x^2+1}} y}{\sqrt{x^2+1}} \\ S_y &= \mathrm{e}^{\sqrt{x^2+1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\sqrt{x^2+1}} y = c_1$$

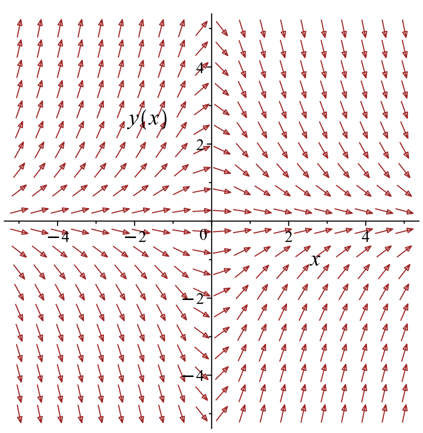
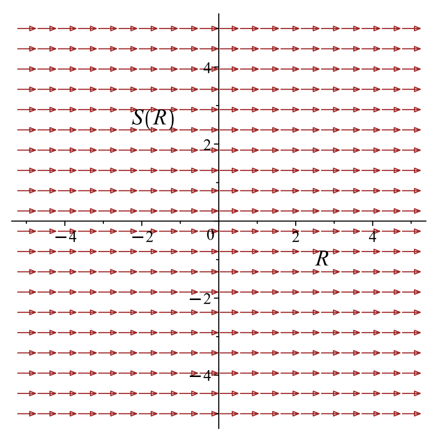
Which simplifies to

$$e^{\sqrt{x^2+1}} y = c_1$$

Which gives

$$y = c_1 e^{-\sqrt{x^2+1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = -\frac{yx}{\sqrt{x^2+1}}$

</td><td>$R = x$
$S = e^{\sqrt{x^2+1}} y$</td><td>$\frac{dS}{dR} = 0$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = c_1 e^{-\sqrt{x^2+1}} \tag{1}$$

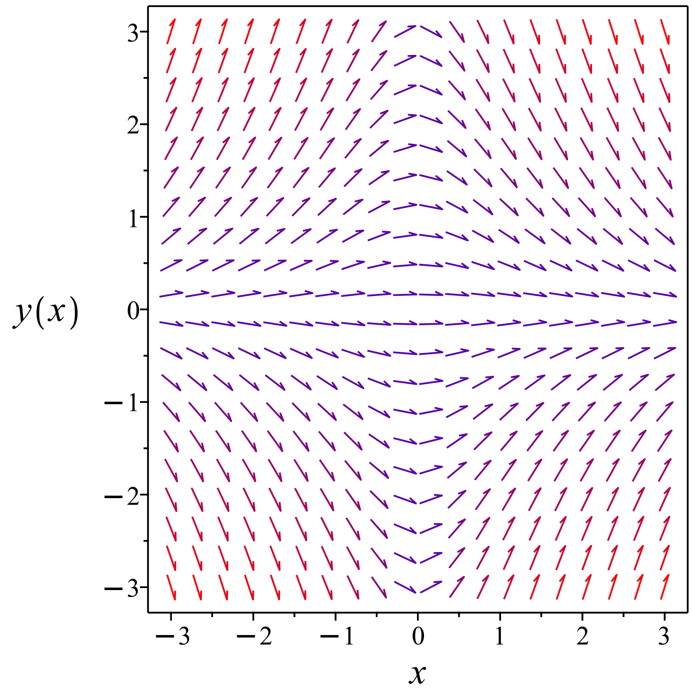


Figure 71: Slope field plot

Verification of solutions

$$y = c_1 e^{-\sqrt{x^2+1}}$$

Verified OK.

1.17.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{x}{\sqrt{x^2+1}}\right) \mathrm{d}x \\ \left(-\frac{x}{\sqrt{x^2+1}}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{x}{\sqrt{x^2+1}} \\ N(x, y) &= -\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{\sqrt{x^2+1}}\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{x}{\sqrt{x^2+1}} \, \mathrm{d}x$$
$$\phi = -\sqrt{x^2+1} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sqrt{x^2+1} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sqrt{x^2+1} - \ln(y)$$

The solution becomes

$$y = e^{-\sqrt{x^2+1}-c_1}$$

Summary

The solution(s) found are the following

$$y = e^{-\sqrt{x^2+1}-c_1} \tag{1}$$

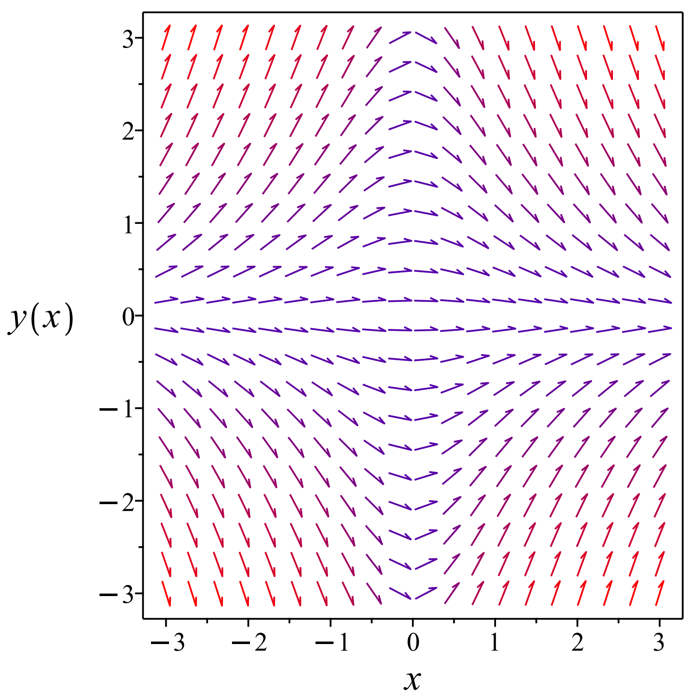


Figure 72: Slope field plot

Verification of solutions

$$y = e^{-\sqrt{x^2+1}-c_1}$$

Verified OK.

1.17.6 Maple step by step solution

Let's solve

$yx+\sqrt{x^2+1}\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y}=-\frac{x}{\sqrt{x^2+1}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx=\int -\frac{x}{\sqrt{x^2+1}}dx+c_1$

- Evaluate integral

  $\ln(y)=-\sqrt{x^2+1}+c_1$

- Solve for y

  $y=\mathrm{e}^{-\sqrt{x^2+1}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*y(x)+sqrt(1+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=c_1\mathrm{e}^{-\sqrt{x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 26

```
DSolve[x*y[x]+Sqrt[1+x^2]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\sqrt{x^2+1}}$$
$$y(x) \to 0$$

1.18 problem 18

Internal problem ID [1887]
Internal file name [`OUTPUT/1888_Sunday_June_05_2022_02_36_55_AM_83942862/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y - yx - y'x^2 = 0}$$

1.18.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(x-1)}{x^2} \end{aligned}$$

Where $f(x) = -\frac{x-1}{x^2}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -\frac{x-1}{x^2}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{x-1}{x^2}\,dx \\
\ln(y) &= -\ln(x) - \frac{1}{x} + c_1 \\
y &= e^{-\ln(x) - \frac{1}{x} + c_1} \\
&= c_1 e^{-\ln(x) - \frac{1}{x}}
\end{aligned}$$

Which simplifies to

$$y = \frac{c_1 e^{-\frac{1}{x}}}{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 e^{-\frac{1}{x}}}{x} \tag{1}$$

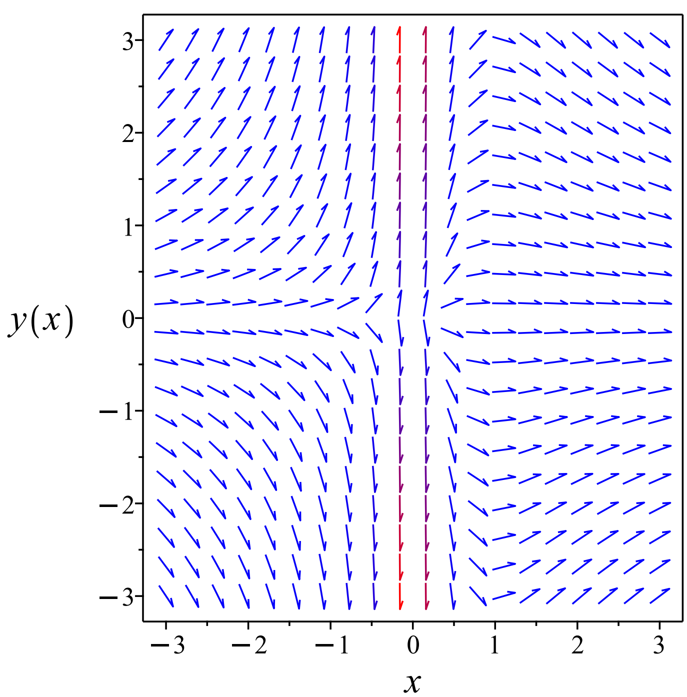


Figure 73: Slope field plot

Verification of solutions

$$y = \frac{c_1 e^{-\frac{1}{x}}}{x}$$

Verified OK.

1.18.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1-x}{x^2}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{(1-x)\, y}{x^2} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1-x}{x^2}dx}$$
$$= e^{\ln(x)+\frac{1}{x}}$$

Which simplifies to

$$\mu = e^{\frac{1}{x}} x$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(e^{\frac{1}{x}} x y\right) = 0$$

Integrating gives

$$e^{\frac{1}{x}} x y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{1}{x}} x$ results in

$$y = \frac{c_1 e^{-\frac{1}{x}}}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 e^{-\frac{1}{x}}}{x} \tag{1}$$

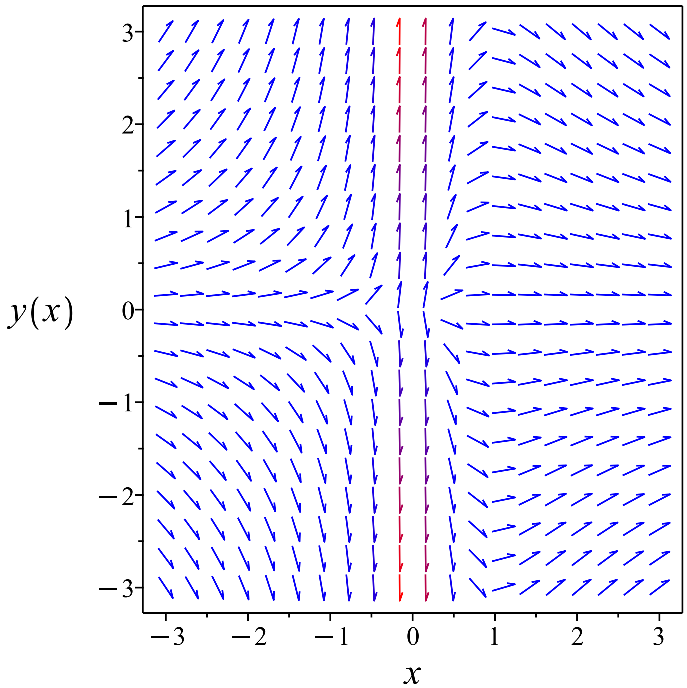


Figure 74: Slope field plot

Verification of solutions

$$y = \frac{c_1 e^{-\frac{1}{x}}}{x}$$

Verified OK.

1.18.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x - u(x)\,x^2 - (u'(x)\,x + u(x))\,x^2 = 0$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(2x-1)}{x^2}
\end{aligned}
$$

Where $f(x) = -\frac{2x-1}{x^2}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2x-1}{x^2}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2x-1}{x^2}\,dx \\
\ln(u) &= -2\ln(x) - \frac{1}{x} + c_2 \\
u &= \mathrm{e}^{-2\ln(x)-\frac{1}{x}+c_2} \\
&= c_2 \mathrm{e}^{-2\ln(x)-\frac{1}{x}}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{-\frac{1}{x}}}{x^2}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{c_2 \mathrm{e}^{-\frac{1}{x}}}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2 \mathrm{e}^{-\frac{1}{x}}}{x} \tag{1}
$$

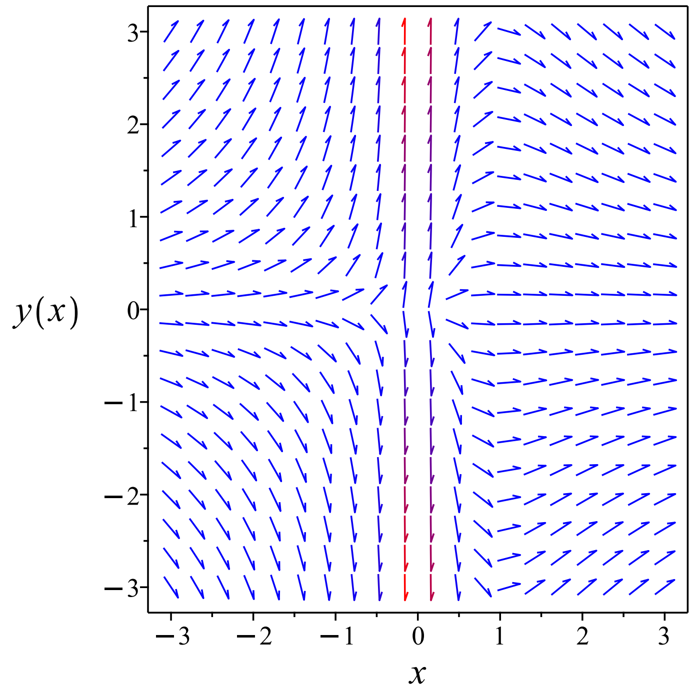


Figure 75: Slope field plot

Verification of solutions

$$y = \frac{c_2 e^{-\frac{1}{x}}}{x}$$

Verified OK.

1.18.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(x-1)}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 50: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-\ln(x)-\frac{1}{x}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\ln(x)-\frac{1}{x}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{1}{x}} x y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(x-1)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^{\frac{1}{x}} y(x-1)}{x} \\ S_y &= e^{\frac{1}{x}} x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\frac{1}{x}} x y = c_1$$

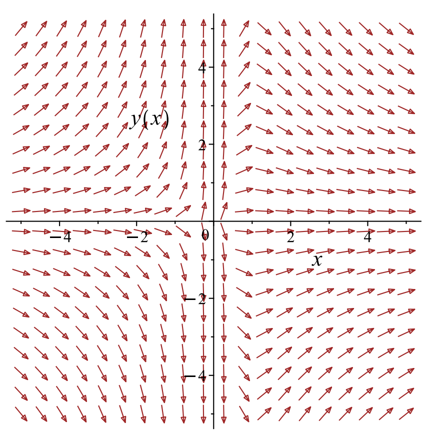
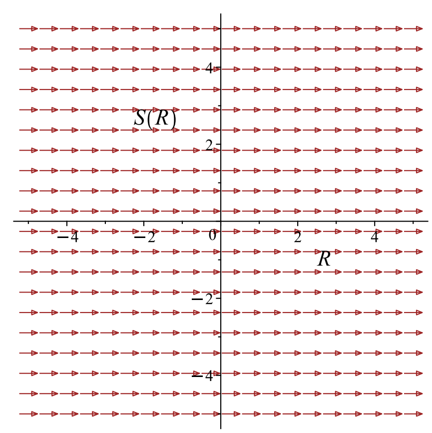
Which simplifies to

$$e^{\frac{1}{x}} x y = c_1$$

Which gives

$$y = \frac{c_1 e^{-\frac{1}{x}}}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(x-1)}{x^2}$  	$R = x$ $S = e^{\frac{1}{x}} x y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-\frac{1}{x}}}{x} \tag{1}$$

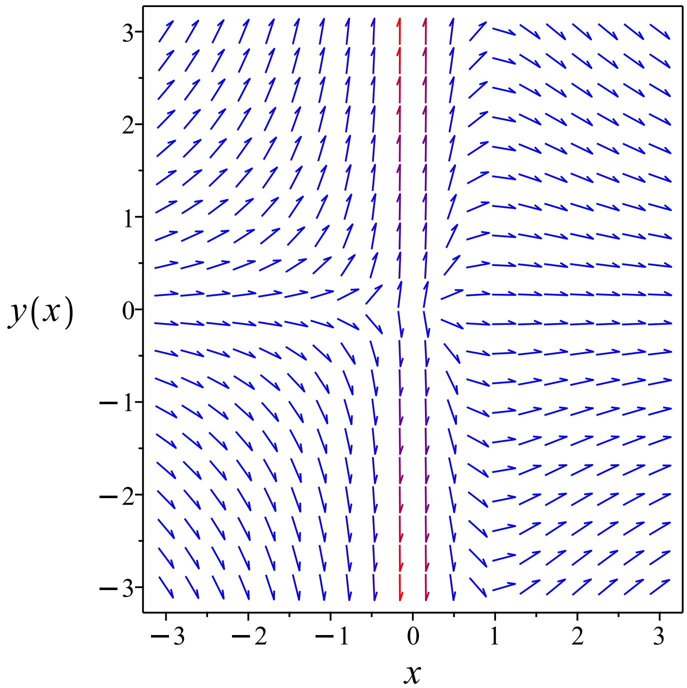


Figure 76: Slope field plot

Verification of solutions

$$y = \frac{c_1 e^{-\frac{1}{x}}}{x}$$

Verified OK.

1.18.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y = \left(\frac{x-1}{x^2}\right)\mathrm{d}x$$
$$\left(-\frac{x-1}{x^2}\right)\mathrm{d}x + \left(-\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x-1}{x^2}$$
$$N(x, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x-1}{x^2}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{x-1}{x^2} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) - \frac{1}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{1}{x} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{1}{x} - \ln(y)$$

The solution becomes

$$y = e^{-\frac{x\ln(x)+c_1x+1}{x}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x\ln(x)+c_1x+1}{x}} \tag{1}$$

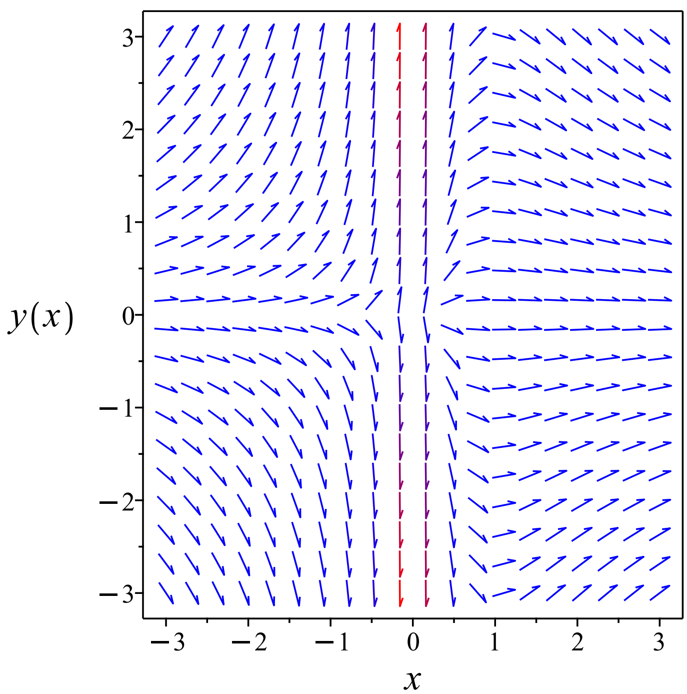


Figure 77: Slope field plot

Verification of solutions

$$y = e^{-\frac{x \ln(x) + c_1 x + 1}{x}}$$

Verified OK.

1.18.6 Maple step by step solution

Let's solve

$y - yx - y'x^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{x-1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{x-1}{x^2} dx + c_1$

- Evaluate integral

  $\ln(y) = -\ln(x) - \frac{1}{x} + c_1$

- Solve for y

  $y = e^{-\frac{x \ln(x) - c_1 x + 1}{x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(y(x)=x*y(x)+x^2*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{-\frac{1}{x}}}{x}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 23

```
DSolve[y[x]==x*y[x]+x^2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 e^{-1/x}}{x}$$
$$y(x) \to 0$$

1.19 problem 19

Internal problem ID [1888]
Internal file name [`OUTPUT/1889_Sunday_June_05_2022_02_36_56_AM_70414219/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\cos(x)^2 \cot(y) y' = -\tan(x)\sin(x)^2}$$

1.19.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\tan(x)\sin(x)^2\tan(y)}{\cos(x)^2} \end{aligned}$$

Where $f(x) = -\frac{\tan(x)\sin(x)^2}{\cos(x)^2}$ and $g(y) = \tan(y)$. Integrating both sides gives

$$\frac{1}{\tan(y)}\,dy = -\frac{\tan(x)\sin(x)^2}{\cos(x)^2}\,dx$$

$$\int \frac{1}{\tan(y)}\,dy = \int -\frac{\tan(x)\sin(x)^2}{\cos(x)^2}\,dx$$

$$\ln(\sin(y)) = -\frac{\tan(x)^2}{2} - \ln(\cos(x)) + c_1$$

Raising both side to exponential gives

$$\sin(y) = e^{-\frac{\tan(x)^2}{2} - \ln(\cos(x)) + c_1}$$

Which simplifies to

$$\sin(y) = c_2 e^{-\frac{\tan(x)^2}{2} - \ln(\cos(x))}$$

Summary

The solution(s) found are the following

$$y = \arcsin\left(\frac{c_2 e^{-\frac{\tan(x)^2}{2} + c_1}}{\cos(x)}\right) \tag{1}$$

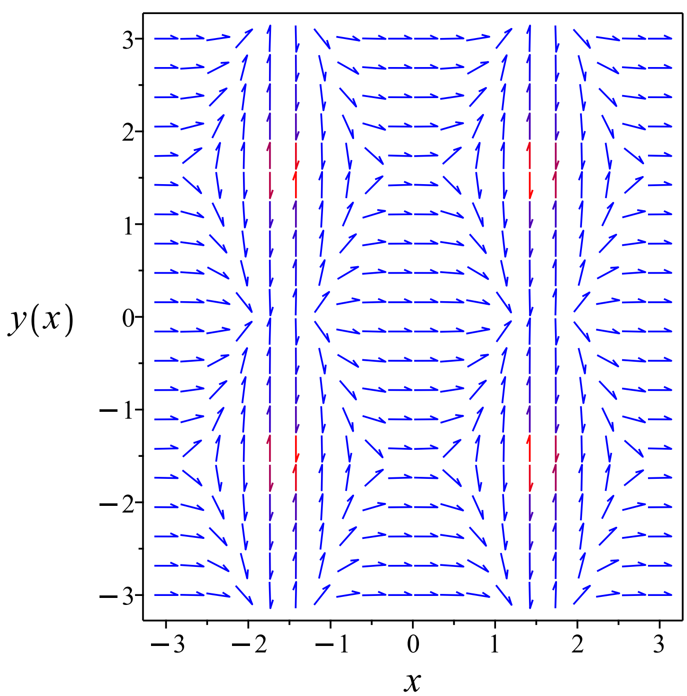


Figure 78: Slope field plot

Verification of solutions

$$y = \arcsin\left(\frac{c_2 e^{-\frac{\tan(x)^2}{2}+c_1}}{\cos(x)}\right)$$

Verified OK.

1.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\tan(x)\sin(x)^2}{\cos(x)^2\cot(y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 53: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{\cos(x)^2}{\tan(x)\sin(x)^2} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{\cos(x)^2}{\tan(x)\sin(x)^2}} dx \end{aligned}$$

Which results in

$$S = -\frac{\tan(x)^2}{2} - \ln(\cos(x))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\tan(x)\sin(x)^2}{\cos(x)^2 \cot(y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\tan(x)^3 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cot(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cot(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(\sin(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\tan(x)^2}{2} - \ln(\cos(x)) = \ln(\sin(y)) + c_1$$

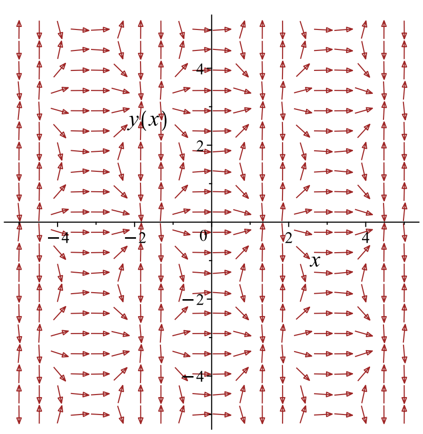
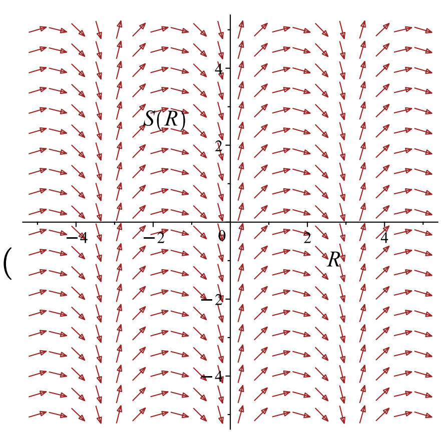
Which simplifies to

$$-\frac{\tan(x)^2}{2} - \ln(\cos(x)) = \ln(\sin(y)) + c_1$$

Which gives

$$y = \arcsin\left(\mathrm{e}^{-\frac{2\ln(\cos(x))\sin(x)^2+2c_1\sin(x)^2-\sin(x)^2-2\ln(\cos(x))-2c_1}{2\left(\sin(x)^2-1\right)}}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\tan(x)\sin(x)^2}{\cos(x)^2\cot(y)}$ 	$R = y$ $S = -\frac{\tan(x)^2}{2} - \ln(\cos($	$\frac{dS}{dR} = \cot(R)$ 

Summary

The solution(s) found are the following

$$y = \arcsin\left(\mathrm{e}^{-\frac{2\ln(\cos(x))\sin(x)^2+2c_1\sin(x)^2-\sin(x)^2-2\ln(\cos(x))-2c_1}{2(\sin(x)^2-1)}}\right) \tag{1}$$

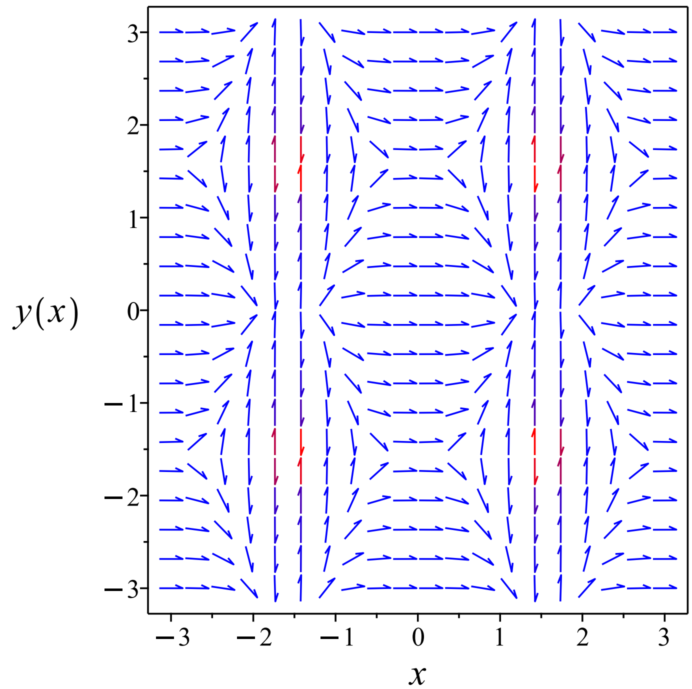


Figure 79: Slope field plot

Verification of solutions

$$y = \arcsin\left(\mathrm{e}^{-\frac{2\ln(\cos(x))\sin(x)^2+2c_1\sin(x)^2-\sin(x)^2-2\ln(\cos(x))-2c_1}{2(\sin(x)^2-1)}}\right)$$

Verified OK.

1.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-\cot(y))\,\mathrm{d}y = \left(\frac{\tan(x)\sin(x)^2}{\cos(x)^2}\right)\mathrm{d}x$$

$$\left(-\frac{\tan(x)\sin(x)^2}{\cos(x)^2}\right)\mathrm{d}x + (-\cot(y))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{\tan(x)\sin(x)^2}{\cos(x)^2} \\ N(x,y) &= -\cot(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\tan(x)\sin(x)^2}{\cos(x)^2}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\cot(y))\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{\tan(x)\sin(x)^2}{\cos(x)^2} \mathrm{d}x$$

$$\phi = -\frac{\tan(x)^2}{2} - \ln(\cos(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\cot(y)$. Therefore equation (4) becomes

$$-\cot(y) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\cot(y)$$

Integrating the above w.r.t y gives

$$\int f'(y)\,dy = \int (-\cot(y))\,dy$$
$$f(y) = -\ln(\sin(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\tan(x)^2}{2} - \ln(\cos(x)) - \ln(\sin(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\tan(x)^2}{2} - \ln(\cos(x)) - \ln(\sin(y))$$

<u>Summary</u>
The solution(s) found are the following

$$-\frac{\tan(x)^2}{2} - \ln(\cos(x)) - \ln(\sin(y)) = c_1 \tag{1}$$

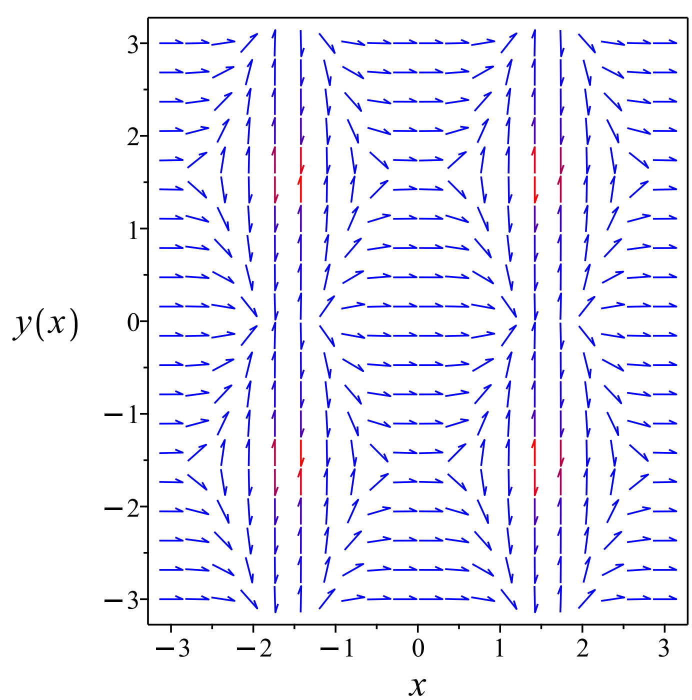


Figure 80: Slope field plot

Verification of solutions

$$-\frac{\tan(x)^{2}}{2}-\ln(\cos(x))-\ln(\sin(y))=c_{1}$$

Verified OK.

1.19.4 Maple step by step solution

Let's solve

$\cos(x)^{2}\cot(y)\,y'=-\tan(x)\sin(x)^{2}$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'\cot(y)=-\frac{\tan(x)\sin(x)^{2}}{\cos(x)^{2}}$

- Integrate both sides with respect to x

  $\int y'\cot(y)\,dx=\int-\frac{\tan(x)\sin(x)^{2}}{\cos(x)^{2}}dx+c_{1}$

- Evaluate integral

  $\ln(\sin(y))=-\frac{\tan(x)^{2}}{2}-\ln(\cos(x))+c_{1}$

- Solve for y

  $y=\arcsin\left(\mathrm{e}^{-\frac{2\ln(\cos(x))\sin(x)^{2}-2c_{1}\sin(x)^{2}-\sin(x)^{2}-2\ln(\cos(x))+2c_{1}}{2\left(\sin(x)^{2}-1\right)}}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 40

```
dsolve(tan(x)*sin(x)^2+cos(x)^2*cot(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arcsin\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{1+\cos(2x)}}\,\mathrm{e}^{\frac{-1+\cos(2x)}{2+2\cos(2x)}}}{c_1}\right)$$

✓ Solution by Mathematica

Time used: 17.888 (sec). Leaf size: 24

```
DSolve[Tan[x]*Sin[x]^2+Cos[x]^2*Cot[y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(\frac{1}{8}c_1 e^{-\frac{1}{2}\sec^2(x)}\sec(x)\right)$$

1.20 problem 20

Internal problem ID [1889]
Internal file name [`OUTPUT/1890_Sunday_June_05_2022_02_37_02_AM_76275324/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 + yy' + y'x^2y = 1}$$

1.20.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y^2-1}{(x^2+1)\,y} \end{aligned}$$

Where $f(x) = -\frac{1}{x^2+1}$ and $g(y) = \frac{y^2-1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2-1}{y}}\,dy = -\frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{\frac{y^2-1}{y}}\, dy = \int -\frac{1}{x^2+1}\, dx$$

$$\frac{\ln (y-1)}{2} + \frac{\ln (y+1)}{2} = -\arctan (x) + c_1$$

The above can be written as

$$\left(\frac{1}{2}\right) (\ln (y-1) + \ln (y+1)) = -\arctan (x) + 2c_1$$

$$\begin{aligned}
\ln (y-1) + \ln (y+1) &= (2) (-\arctan (x) + 2c_1) \\
&= -2\arctan (x) + 4c_1
\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(y-1)+\ln(y+1)} = \mathrm{e}^{-2\arctan(x)+2c_1}$$

Which simplifies to

$$\begin{aligned}
y^2 - 1 &= 2c_1 \mathrm{e}^{-2\arctan(x)} \\
&= c_2 \mathrm{e}^{-2\arctan(x)}
\end{aligned}$$

The solution is

$$-1 + y^2 = c_2 \mathrm{e}^{-2\arctan(x)}$$

Summary

The solution(s) found are the following

$$-1 + y^2 = c_2 \mathrm{e}^{-2\arctan(x)} \tag{1}$$

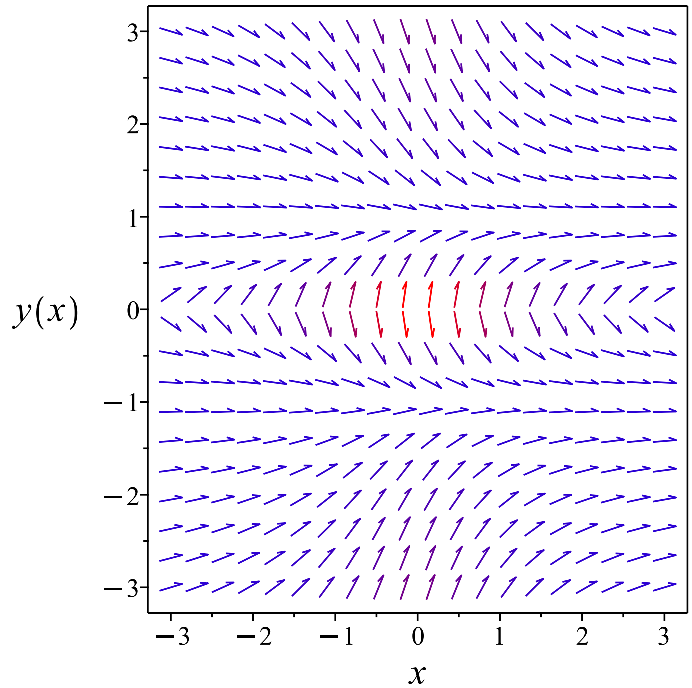


Figure 81: Slope field plot

Verification of solutions

$$-1 + y^2 = c_2 \mathrm{e}^{-2\arctan(x)}$$

Verified OK.

1.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y^2 - 1}{(x^2 + 1)\, y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 56: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -x^2 - 1 \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x^2 - 1} dx \end{aligned}$$

Which results in

$$S = -\arctan(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2 - 1}{(x^2 + 1) y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{1}{x^2 + 1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R-1)}{2} + \frac{\ln (R+1)}{2} + c_1 \tag{4}$$

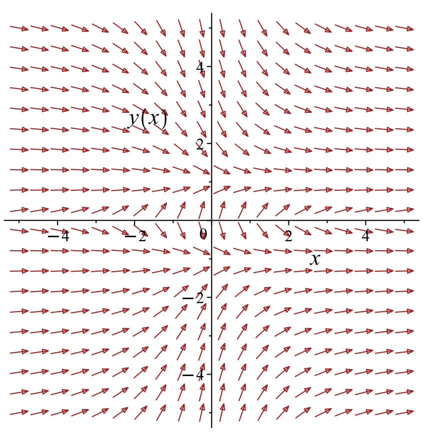
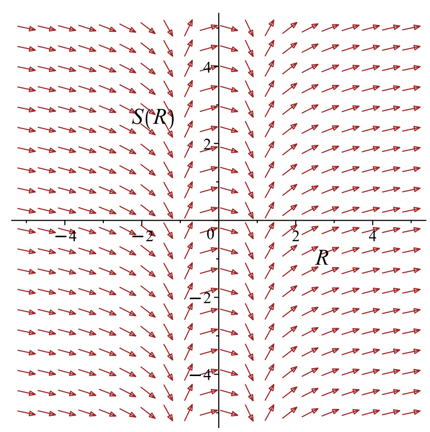
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\arctan (x) = \frac{\ln (y-1)}{2} + \frac{\ln (1+y)}{2} + c_1$$

Which simplifies to

$$-\arctan (x) = \frac{\ln (y-1)}{2} + \frac{\ln (1+y)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2-1}{(x^2+1)y}$ 	$R = y$ $S = -\arctan (x)$	$\frac{dS}{dR} = \frac{R}{R^2-1}$ 

Summary

The solution(s) found are the following

$$-\arctan (x) = \frac{\ln (y-1)}{2} + \frac{\ln (1+y)}{2} + c_1 \tag{1}$$

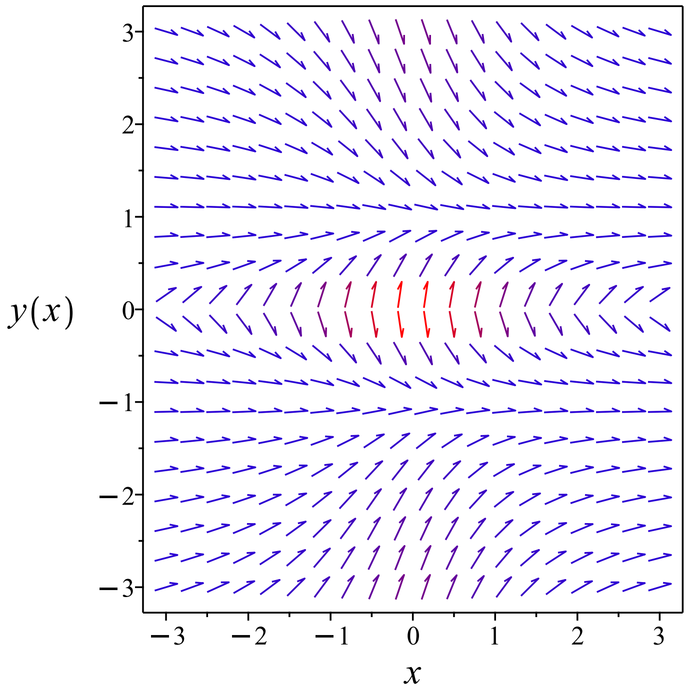


Figure 82: Slope field plot

Verification of solutions

$$-\arctan(x) = \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} + c_1$$

Verified OK.

1.20.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^2-1}{(x^2+1)\,y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x^2+1}y + \frac{1}{x^2+1}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x^2+1} \\ f_1(x) &= \frac{1}{x^2+1} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{x^2+1} + \frac{1}{x^2+1} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{w(x)}{x^2+1} + \frac{1}{x^2+1} \\ w' &= -\frac{2w}{x^2+1} + \frac{2}{x^2+1} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x^2+1} \\ q(x) &= \frac{2}{x^2+1} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x^2+1} = \frac{2}{x^2+1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x^2+1}dx} \\ &= e^{2\arctan(x)}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{2}{x^2+1}\right) \\ \frac{d}{dx}\left(e^{2\arctan(x)}w\right) &= \left(e^{2\arctan(x)}\right)\left(\frac{2}{x^2+1}\right) \\ d\left(e^{2\arctan(x)}w\right) &= \left(\frac{2\,e^{2\arctan(x)}}{x^2+1}\right)dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2\arctan(x)}w &= \int \frac{2\,e^{2\arctan(x)}}{x^2+1}\,dx \\ e^{2\arctan(x)}w &= e^{2\arctan(x)} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2\arctan(x)}$ results in

$$w(x) = e^{-2\arctan(x)}e^{2\arctan(x)} + c_1 e^{-2\arctan(x)}$$

which simplifies to

$$w(x) = 1 + c_1 e^{-2\arctan(x)}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = 1 + c_1 e^{-2\arctan(x)}$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{1 + c_1 e^{-2\arctan(x)}} \\ y(x) &= -\sqrt{1 + c_1 e^{-2\arctan(x)}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{1 + c_1 e^{-2 \arctan(x)}} \tag{1}$$
$$y = -\sqrt{1 + c_1 e^{-2 \arctan(x)}} \tag{2}$$

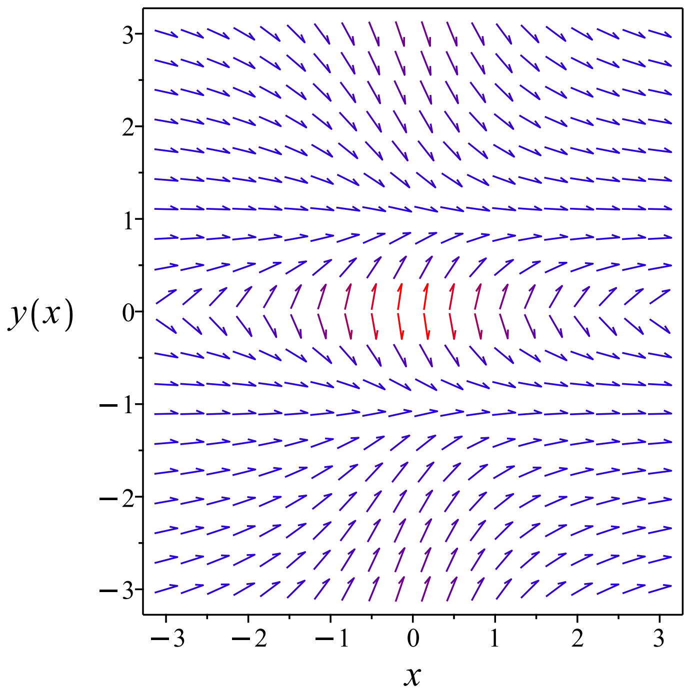


Figure 83: Slope field plot

Verification of solutions

$$y = \sqrt{1 + c_1 e^{-2 \arctan(x)}}$$

Verified OK.

$$y = -\sqrt{1 + c_1 e^{-2 \arctan(x)}}$$

Verified OK.

1.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{y}{y^2-1}\right)\mathrm{d}y&=\left(\frac{1}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2+1}\right)\mathrm{d}x+\left(-\frac{y}{y^2-1}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{1}{x^2+1} \\ N(x,y) &= -\frac{y}{y^2-1} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2+1}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y}{y^2-1}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x^2+1}\, \mathrm{d}x \end{aligned}$$

$$\phi = -\arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{y^2-1}$. Therefore equation (4) becomes

$$-\frac{y}{y^2-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y}{y^2-1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{y}{y^2-1}\right) \mathrm{d}y$$

$$f(y) = -\frac{\ln (y-1)}{2} - \frac{\ln (y+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan (x) - \frac{\ln (y-1)}{2} - \frac{\ln (y+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan (x) - \frac{\ln (y-1)}{2} - \frac{\ln (y+1)}{2}$$

Summary

The solution(s) found are the following

$$-\arctan (x) - \frac{\ln (y-1)}{2} - \frac{\ln (1+y)}{2} = c_1 \tag{1}$$

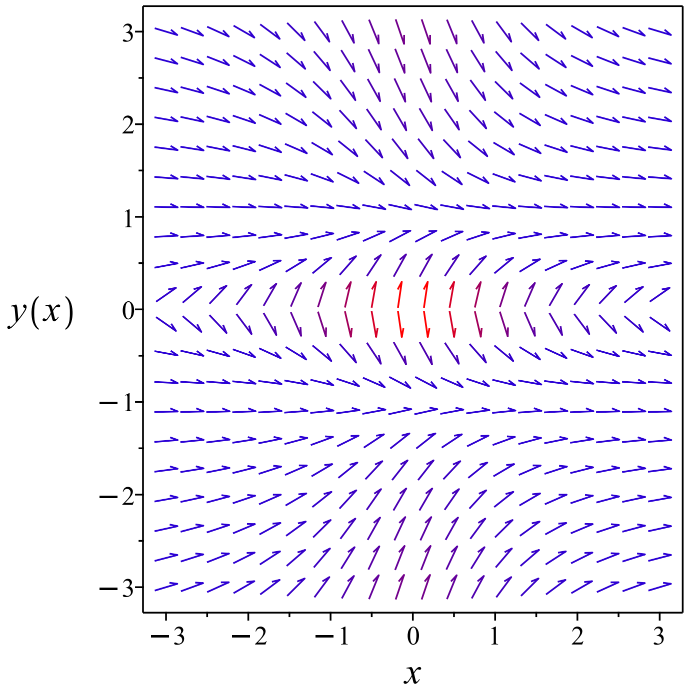


Figure 84: Slope field plot

Verification of solutions

$$-\arctan(x) - \frac{\ln(y-1)}{2} - \frac{\ln(1+y)}{2} = c_1$$

Verified OK.

1.20.5 Maple step by step solution

Let's solve

$y^2 + yy' + y'x^2y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{(y-1)(1+y)} = -\frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{(y-1)(1+y)}dx = \int -\frac{1}{x^2+1}dx + c_1$

- Evaluate integral

$$\frac{\ln((y-1)(1+y))}{2} = -\arctan(x) + c_1$$

- Solve for y

$$\left\{y = \sqrt{1+\mathrm{e}^{-2\arctan(x)+2c_1}}, y = -\sqrt{1+\mathrm{e}^{-2\arctan(x)+2c_1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(y(x)^2+y(x)*diff(y(x),x)+x^2*y(x)*diff(y(x),x)-1=0,y(x), singsol=all)
```

$$y(x) = \sqrt{\mathrm{e}^{-2\arctan(x)}c_1 + 1}$$
$$y(x) = -\sqrt{\mathrm{e}^{-2\arctan(x)}c_1 + 1}$$

✓ Solution by Mathematica

Time used: 0.923 (sec). Leaf size: 55

```
DSolve[y[x]^2+y[x]*y'[x]+x^2*y[x]*y'[x]-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{1+e^{-2\arctan(x)+2c_1}}$$
$$y(x) \to \sqrt{1+e^{-2\arctan(x)+2c_1}}$$
$$y(x) \to -1$$
$$y(x) \to 1$$

1.21 problem 21

Internal problem ID [1890]
Internal file name [`OUTPUT/1891_Sunday_June_05_2022_02_37_52_AM_36612744/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{y}{x} = 0}$$

With initial conditions

$$[y(1) = 3]$$

1.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

1.21.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{y}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y}\,dy &= \frac{1}{x}\,dx\\ \int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx\\ \ln(y) &= \ln(x) + c_1\\ y &= e^{\ln(x)+c_1}\\ &= c_1 x\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

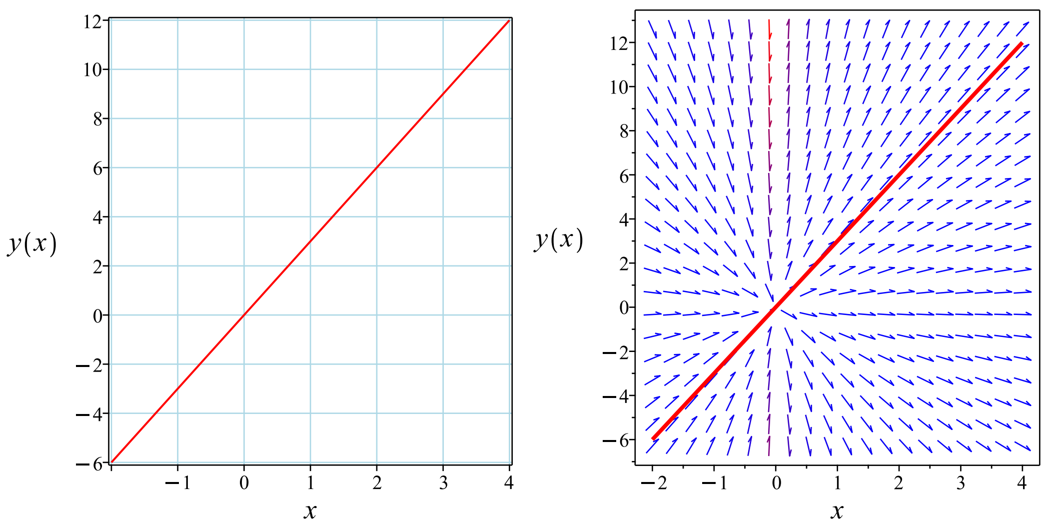
Substituting c_1 found above in the general solution gives

$$y = 3x$$

Summary

The solution(s) found are the following

$$y = 3x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3x$$

Verified OK.

1.21.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = 0$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

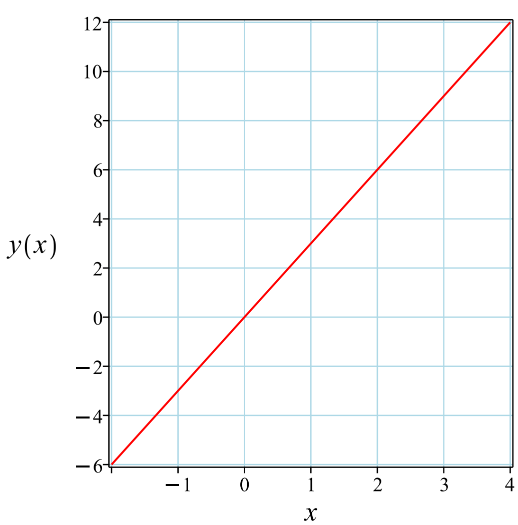
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

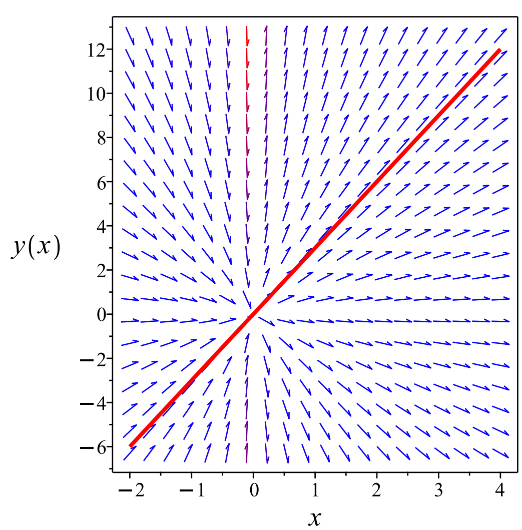
$$y = 3x$$

<u>Summary</u>
The solution(s) found are the following

$$y = 3x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3x$$

Verified OK.

1.21.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0\,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 x \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_2$$

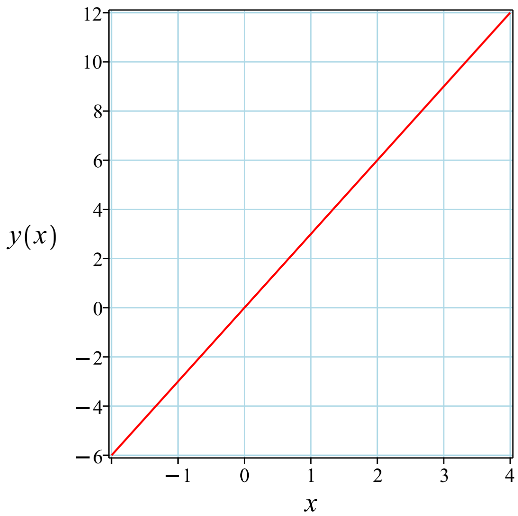
$$c_2 = 3$$

Substituting c_2 found above in the general solution gives

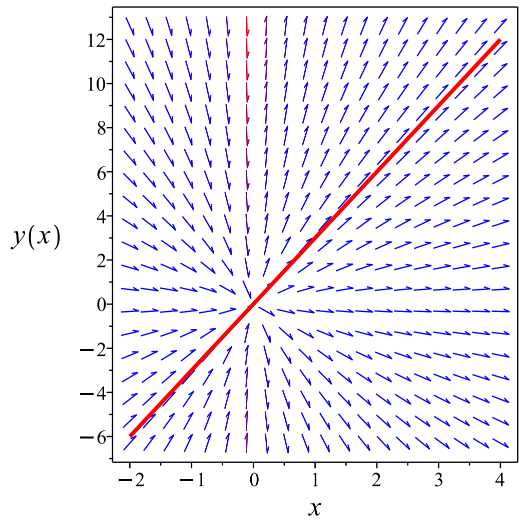
$$y = 3x$$

Summary
The solution(s) found are the following

$$y = 3x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3x$$

Verified OK.

1.21.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 59: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

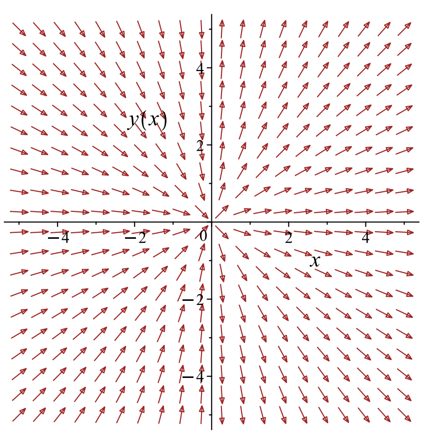
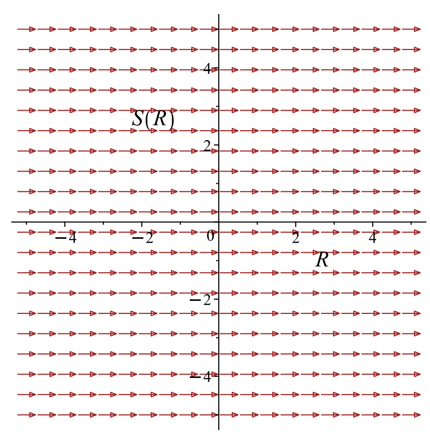
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = \frac{y}{x}$

</td><td>$R = x$
$S = \frac{y}{x}$</td><td>$\frac{dS}{dR} = 0$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

$$c_1 = 3$$

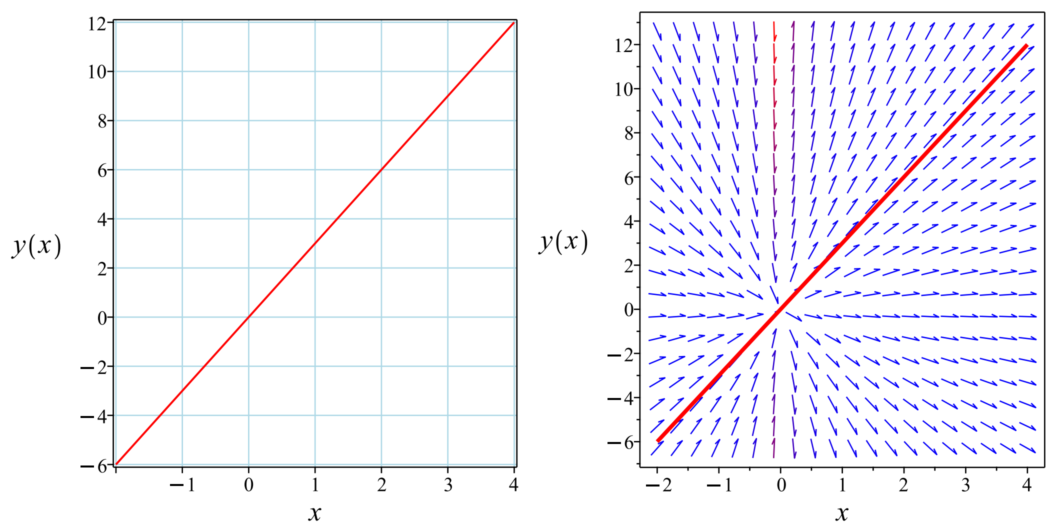
Substituting c_1 found above in the general solution gives

$$y = 3x$$

Summary

The solution(s) found are the following

$$y = 3x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3x$$

Verified OK.

1.21.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x+\left(\frac{1}{y}\right)\mathrm{d}y&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x}\\ N(x,y)&=\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &=0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x} \, \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = x\,\mathrm{e}^{c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = \mathrm{e}^{c_1}$$

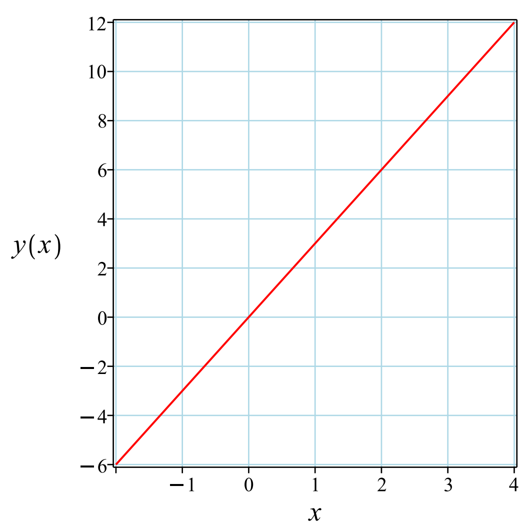
$$c_1 = \ln(3)$$

Substituting c_1 found above in the general solution gives

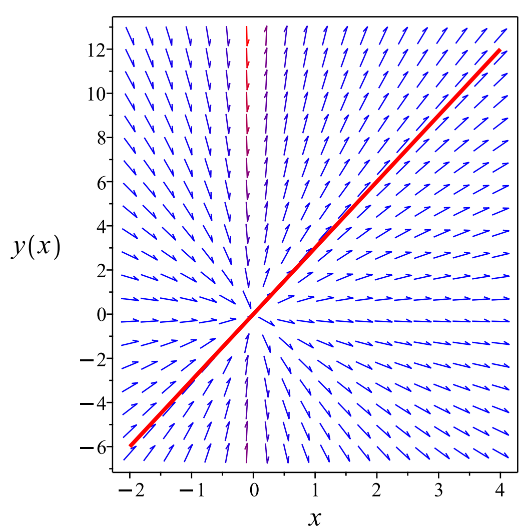
$$y = 3x$$

Summary
The solution(s) found are the following

$$y = 3x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3x$$

Verified OK.

1.21.7 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = 0, y(1) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = x\,\mathrm{e}^{c_1}$

- Use initial condition $y(1) = 3$

  $3 = \mathrm{e}^{c_1}$

- Solve for c_1

  $c_1 = \ln(3)$

- Substitute $c_1 = \ln(3)$ into general solution and simplify

  $y = 3x$

- Solution to the IVP

  $y = 3x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve([diff(y(x),x)=y(x)/x,y(1) = 3],y(x), singsol=all)
```

$$y(x) = 3x$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 8

```
DSolve[{y'[x]==y[x]/x,y[1]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x$$

1.22 problem 22

Internal problem ID [1891]
Internal file name [`OUTPUT/1892_Sunday_June_05_2022_02_37_53_AM_85352970/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2y + y'x = 0}$$

With initial conditions

$$[y(2) = 1]$$

1.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x} = 0$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. Hence solution exists and is unique.

1.22.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{2y}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= -\frac{2}{x}\, dx \\ \int \frac{1}{y}\, dy &= \int -\frac{2}{x}\, dx \\ \ln(y) &= -2\ln(x) + c_1 \\ y &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4}$$

$$c_1 = 4$$

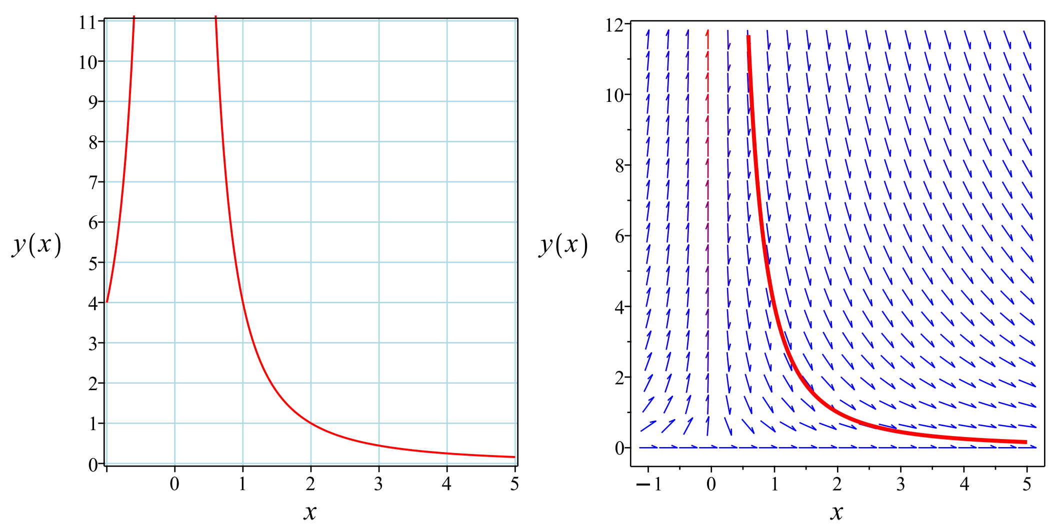
Substituting c_1 found above in the general solution gives

$$y = \frac{4}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

1.22.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$

$$\frac{d}{dx}\left(y\,x^2\right) = 0$$

Integrating gives

$$y\,x^2 = c_1$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4}$$

$$c_1 = 4$$

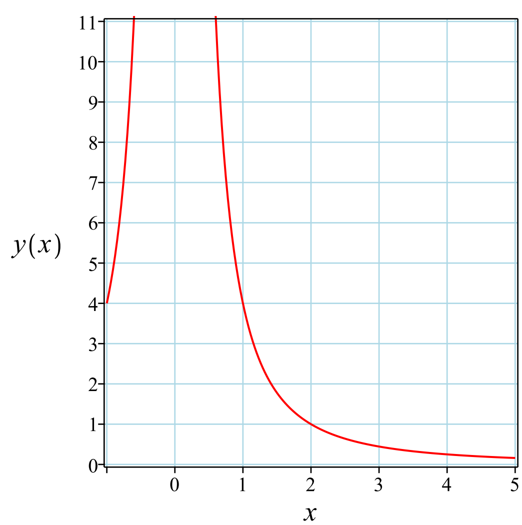
Substituting c_1 found above in the general solution gives

$$y = \frac{4}{x^2}$$

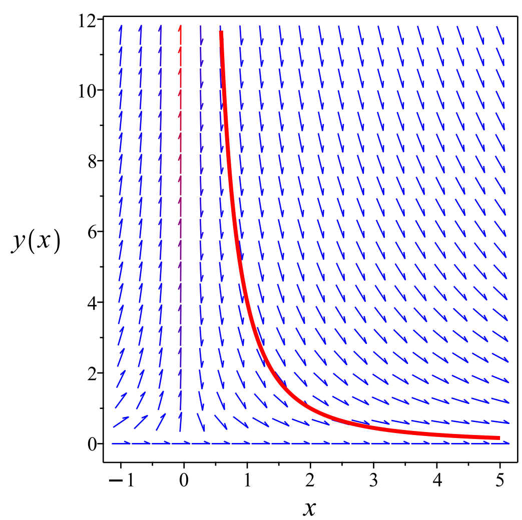
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

1.22.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2u(x)\,x + (u'(x)\,x + u(x))\,x = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{3u}{x}
\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{3}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\
\ln(u) &= -3\ln(x) + c_2 \\
u &= \mathrm{e}^{-3\ln(x)+c_2} \\
&= \frac{c_2}{x^3}
\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{c_2}{x^2}
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_2}{4}$$

$$c_2 = 4$$

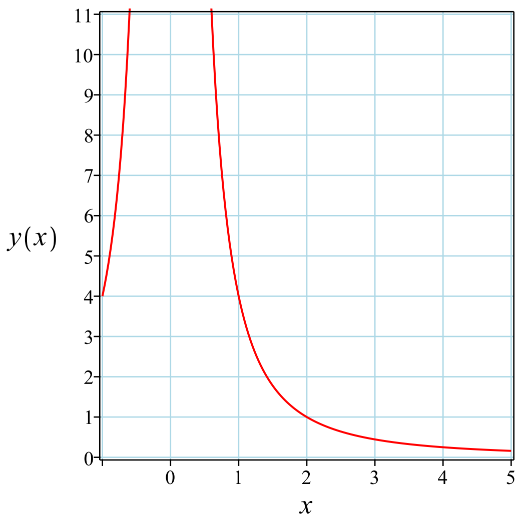
Substituting c_2 found above in the general solution gives

$$y = \frac{4}{x^2}$$

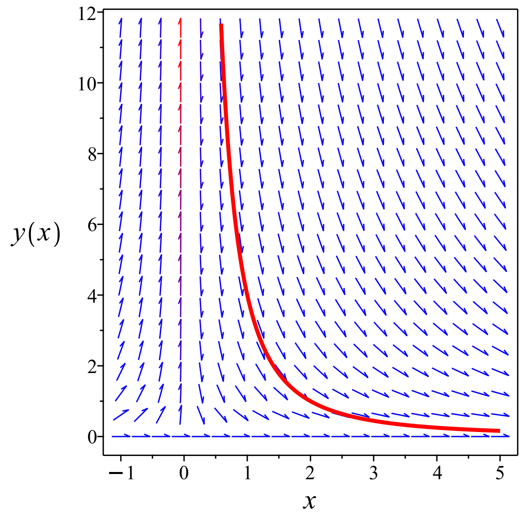
Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

1.22.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 62: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned} S&=\int \frac{1}{\eta}dy\\ &=\int \frac{1}{\frac{1}{x^2}}dy \end{aligned}$$

Which results in

$$S=y\,x^2$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=-\frac{2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x&=1\\ R_y&=0\\ S_x&=2yx\\ S_y&=x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 y = c_1$$

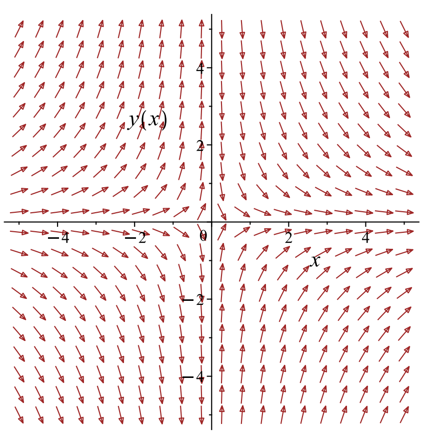
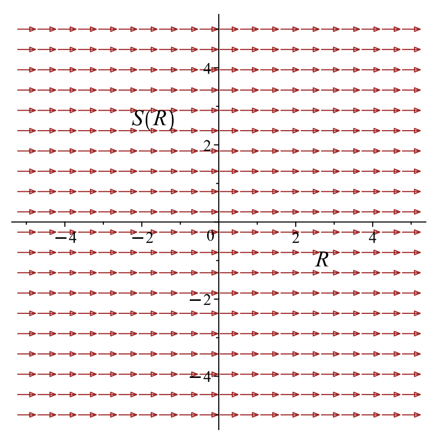
Which simplifies to

$$x^2 y = c_1$$

Which gives

$$y = \frac{c_1}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y}{x}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4}$$

$$c_1 = 4$$

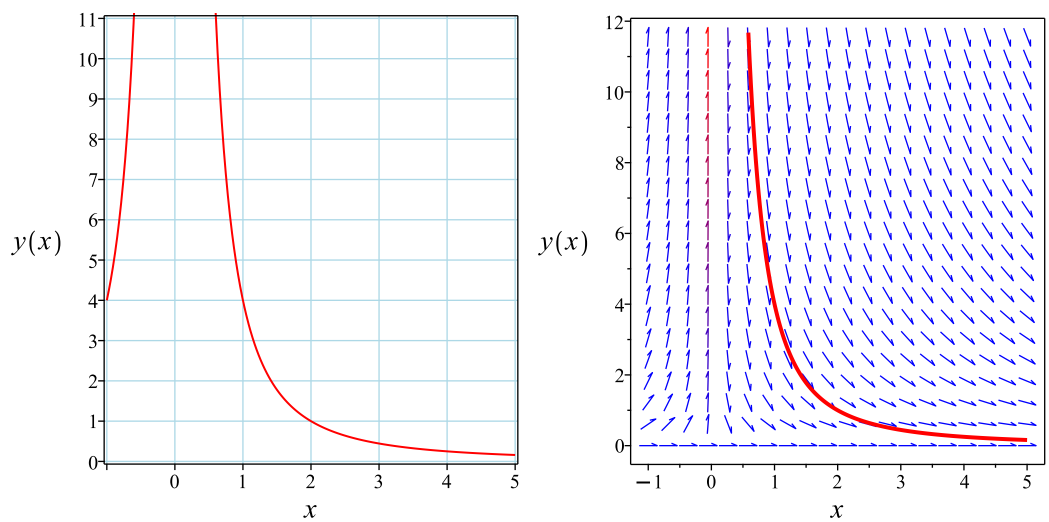
Substituting c_1 found above in the general solution gives

$$y = \frac{4}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

1.22.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{2y}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x \\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{1}{2y}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x} \\ N(x,y) &= -\frac{1}{2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y}$. Therefore equation (4) becomes

$$
-\frac{1}{2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{2y}\right)\mathrm{d}y$$

$$f(y) = -\frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(y)}{2}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-2c_1}}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\mathrm{e}^{-2c_1}}{4}$$

$$c_1 = -\ln(2)$$

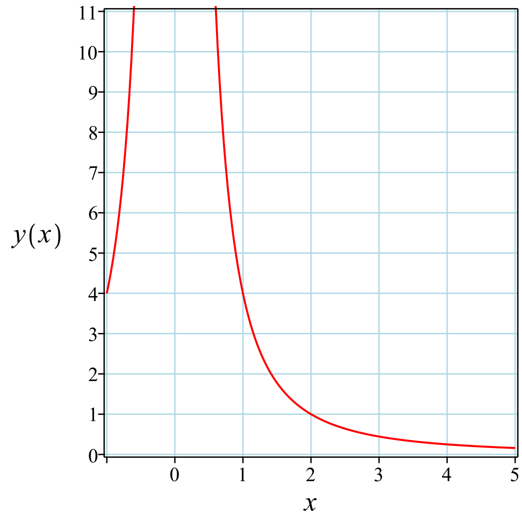
Substituting c_1 found above in the general solution gives

$$y = \frac{4}{x^2}$$

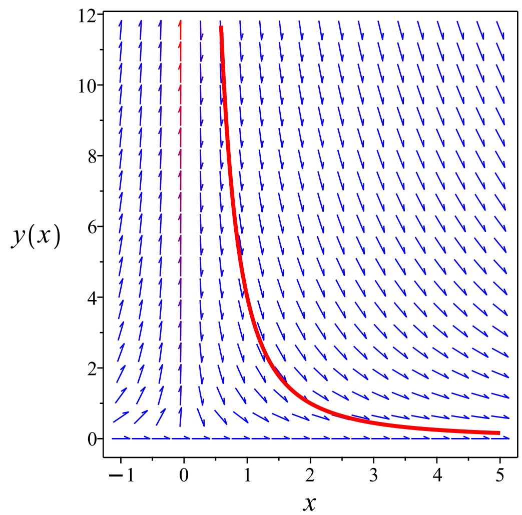
Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{4}{x^2}$$

Verified OK.

1.22.7 Maple step by step solution

Let's solve

$[2y + y'x = 0, y(2) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{2}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{2}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = -2\ln(x) + c_1$

- Solve for y

  $y = \frac{e^{c_1}}{x^2}$

- Use initial condition $y(2) = 1$

  $1 = \frac{e^{c_1}}{4}$
- Solve for c_1

  $c_1 = 2\ln(2)$
- Substitute $c_1 = 2\ln(2)$ into general solution and simplify

  $y = \frac{4}{x^2}$
- Solution to the IVP

  $y = \frac{4}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 9

```
dsolve([x*diff(y(x),x)+2*y(x)=0,y(2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{4}{x^2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 10

```
DSolve[{x*y'[x]+2*y[x]==0,y[2]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4}{x^2}$$

1.23 problem 23

Internal problem ID [1892]
Internal file name [`OUTPUT/1893_Sunday_June_05_2022_02_37_55_AM_38195050/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\cos(y)\sin(x)+\cos(x)\sin(y)\,y'=0}$$

With initial conditions

$$[y(0)=0]$$

1.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{\cos(y)\sin(x)}{\cos(x)\sin(y)} \end{aligned}$$

$f(x,y)$ is not defined at $y=0$ therefore existence and uniqueness theorem do not apply.

1.23.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\sin(x)\cot(y)}{\cos(x)} \end{aligned}$$

Where $f(x) = -\frac{\sin(x)}{\cos(x)}$ and $g(y) = \cot(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cot(y)}\,dy &= -\frac{\sin(x)}{\cos(x)}\,dx \\ \int \frac{1}{\cot(y)}\,dy &= \int -\frac{\sin(x)}{\cos(x)}\,dx \\ -\ln(\cos(y)) &= \ln(\cos(x)) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = e^{\ln(\cos(x))+c_1}$$

Which simplifies to

$$\sec(y) = c_2\cos(x)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\pi}{2} - \arcsin\left(\frac{e^{-c_1}}{c_2}\right)$$

$$c_1 = -\ln(c_2)$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{\cos(x)}\right)$$

Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{\cos(x)}\right) \tag{1}$$

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{\cos(x)}\right)$$

Verified OK.

1.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{\cos(y)\sin(x)}{\cos(x)\sin(y)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 65: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{\cos(x)}{\sin(x)} \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{\cos(x)}{\sin(x)}} dx \end{aligned}$$

Which results in

$$S = \ln(\cos(x))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\cos(y)\sin(x)}{\cos(x)\sin(y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\tan(x) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(\cos(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(\cos(x)) = -\ln(\cos(y)) + c_1$$

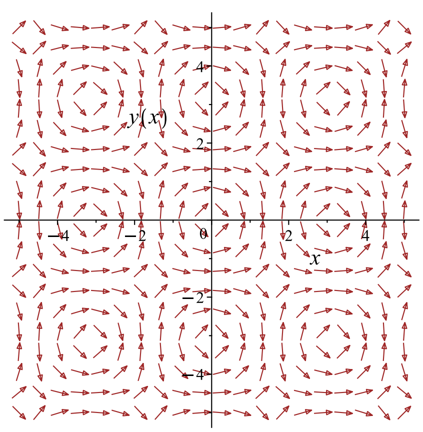
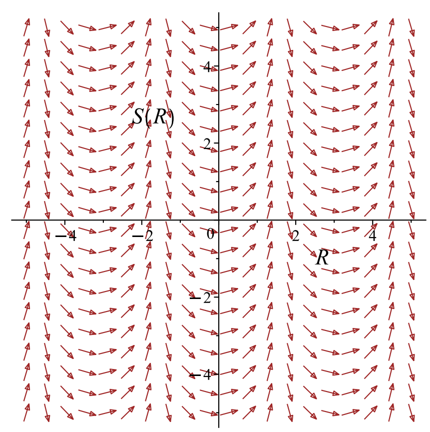
Which simplifies to

$$\ln(\cos(x)) = -\ln(\cos(y)) + c_1$$

Which gives

$$y = \arccos\left(\frac{e^{c_1}}{\cos(x)}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\cos(y)\sin(x)}{\cos(x)\sin(y)}$  	$R = y$ $S = \ln(\cos(x))$	$\frac{dS}{dR} = \tan(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\pi}{2} - \arcsin(e^{c_1})$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{\cos(x)}\right)$$

Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{\cos(x)}\right) \tag{1}$$

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{1}{\cos(x)}\right)$$

Verified OK.

1.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y &= \left(\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x\\ \left(-\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x+\left(-\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{\sin(x)}{\cos(x)}\\ N(x,y) &= -\frac{\sin(y)}{\cos(y)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{\cos(x)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\sin(y)}{\cos(y)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{\sin(x)}{\cos(x)} \, \mathrm{d}x \\
\phi &= \ln(\cos(x)) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\sin(y)}{\cos(y)}$. Therefore equation (4) becomes

$$
-\frac{\sin(y)}{\cos(y)} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -\frac{\sin(y)}{\cos(y)} \\
&= -\tan(y)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (-\tan(y)) \, \mathrm{d}y \\
f(y) &= \ln(\cos(y)) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \ln(\cos(x)) + \ln(\cos(y)) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \ln(\cos(x)) + \ln(\cos(y))
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\ln(\cos(x)) + \ln(\cos(y)) = 0$$

Solving for y from the above gives

$$y = \arccos(\sec(x))$$

Summary
The solution(s) found are the following

$$y = \arccos(\sec(x)) \tag{1}$$

Verification of solutions

$$y = \arccos(\sec(x))$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple
Time used: 0.219 (sec). Leaf size: 15

```
dsolve([sin(x)*cos(y(x))+cos(x)*sin(y(x))*diff(y(x),x)=0,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\pi}{2} - \arcsin(\sec(x))$$
$$y(x) = -\frac{\pi}{2} + \arcsin(\sec(x))$$

✓ Solution by Mathematica

Time used: 6.227 (sec). Leaf size: 17

```
DSolve[{Sin[x]*Cos[y[x]]+Cos[x]*Sin[y[x]]*y'[x]==0,y[0]==0},y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\arccos(\sec(x))$$
$$y(x) \to \arccos(\sec(x))$$

1.24 problem 24

Internal problem ID [1893]
Internal file name [`OUTPUT/1894_Sunday_June_05_2022_02_37_58_AM_31349066/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x^2 + y^2 = 0}$$

With initial conditions

$$[y(3) = 1]$$

1.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{y^2}{x^2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 3$ is inside this domain. The y domain of $f(x, y)$ when $x = 3$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y^2}{x^2}\right) \\ &= -\frac{2y}{x^2} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 3$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 3$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.24.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{y^2}{x^2} \end{aligned}$$

Where $f(x) = -\frac{1}{x^2}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\, dy = -\frac{1}{x^2}\, dx$$
$$\int \frac{1}{y^2}\, dy = \int -\frac{1}{x^2}\, dx$$
$$-\frac{1}{y} = \frac{1}{x} + c_1$$

Which results in

$$y = -\frac{x}{c_1 x + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{3}{3c_1 + 1}$$

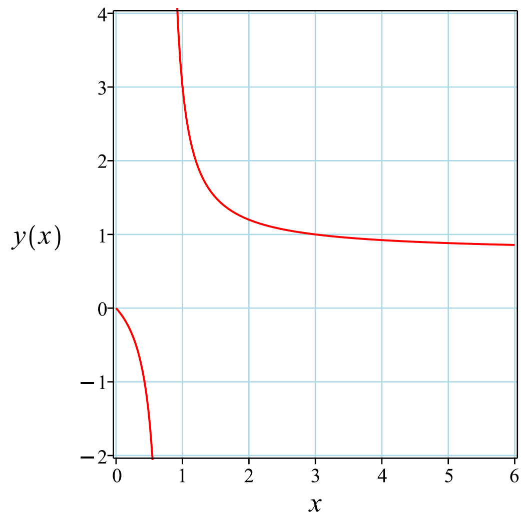
$$c_1 = -\frac{4}{3}$$

Substituting c_1 found above in the general solution gives

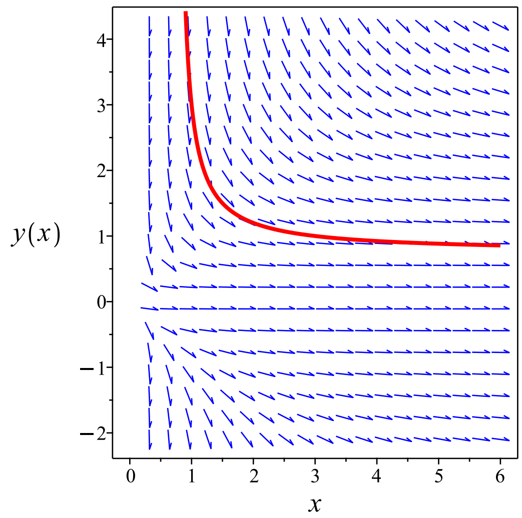
$$y = \frac{3x}{4x - 3}$$

Summary
The solution(s) found are the following

$$y = \frac{3x}{4x - 3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3x}{4x - 3}$$

Verified OK.

1.24.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 + u(x)^2\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u+1)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u(u+1)$. Integrating both sides gives

$$\frac{1}{u\,(u+1)}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{u\,(u+1)}\,du = \int -\frac{1}{x}\,dx$$

$$\ln\,(u) - \ln\,(u+1) = -\ln\,(x) + c_2$$

Raising both side to exponential gives

$$e^{\ln(u)-\ln(u+1)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u}{u+1} = \frac{c_3}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= -\frac{x c_3}{c_3 - x} \end{aligned}$$

Initial conditions are used to solve for c_3. Substituting $x = 3$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{3c_3}{c_3 - 3}$$

$$c_3 = \frac{3}{4}$$

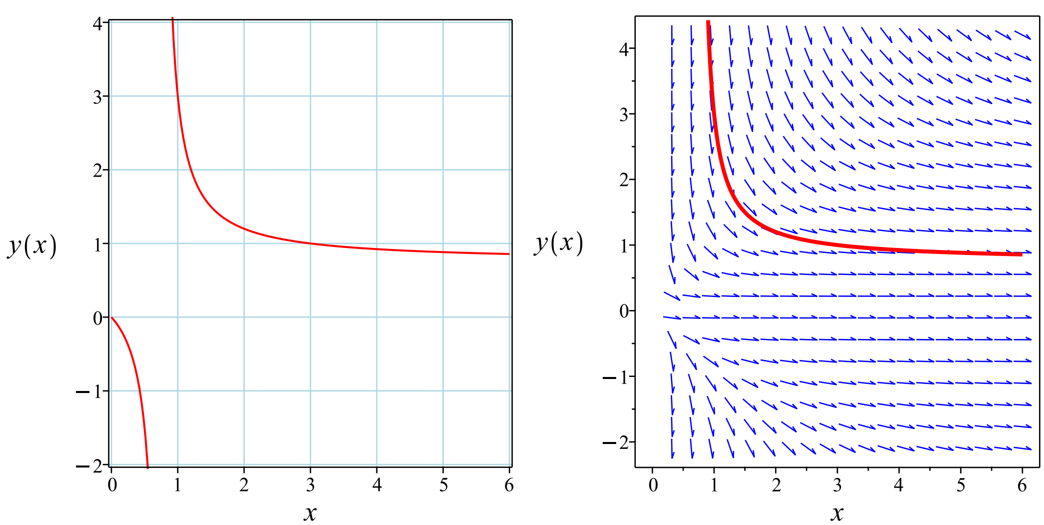
Substituting c_3 found above in the general solution gives

$$y = \frac{3x}{4x - 3}$$

Summary

The solution(s) found are the following

$$y = \frac{3x}{4x - 3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \frac{3x}{4x - 3}
$$

Verified OK.

1.24.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y^2}{x^2} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 67: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -x^2 \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x^2} dx \end{aligned}$$

Which results in

$$S = \frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{1}{x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{x} = -\frac{1}{y} + c_1$$

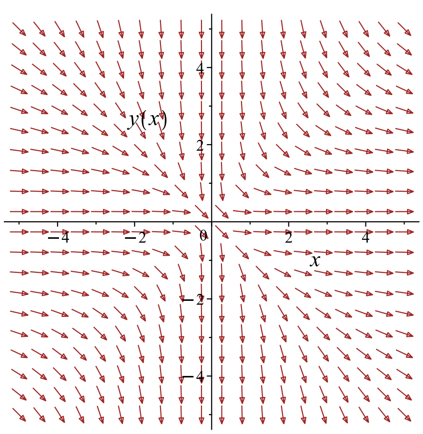
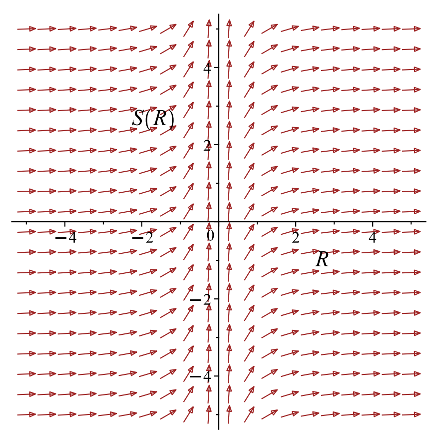
Which simplifies to

$$\frac{1}{x} = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{x}{c_1 x - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2}{x^2}$ 	$R = y$ $S = \frac{1}{x}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{3}{3c_1 - 1}$$

$$c_1 = \frac{4}{3}$$

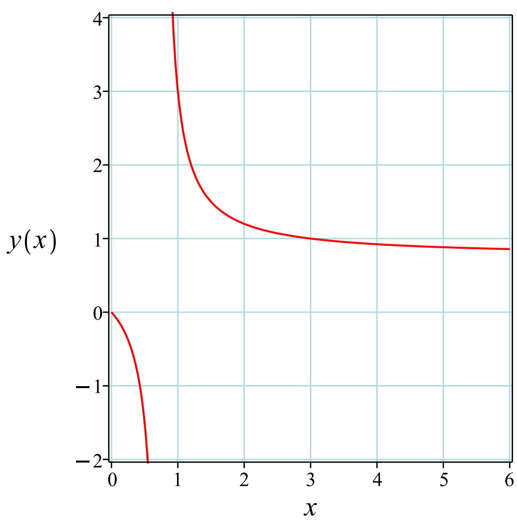
Substituting c_1 found above in the general solution gives

$$y = \frac{3x}{4x - 3}$$

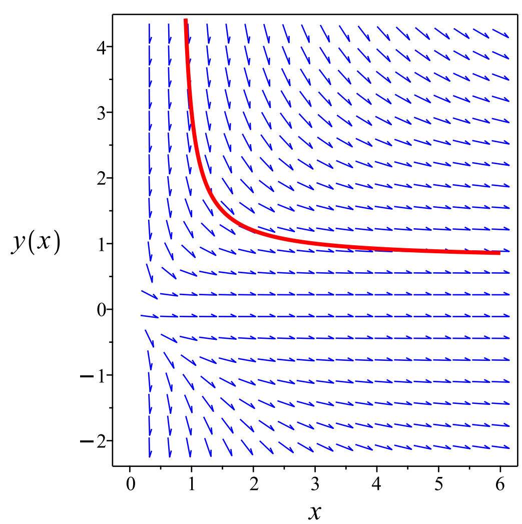
Summary

The solution(s) found are the following

$$y = \frac{3x}{4x - 3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3x}{4x - 3}$$

Verified OK.

1.24.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\\frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y^2}\right)\mathrm{d}y&=\left(\frac{1}{x^2}\right)\mathrm{d}x\\\left(-\frac{1}{x^2}\right)\mathrm{d}x+\left(-\frac{1}{y^2}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x^2} \\
N(x,y) &= -\frac{1}{y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1}{x^2} \mathrm{d}x
\end{aligned}
$$

$$\phi = \frac{1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^2}$. Therefore equation (4) becomes

$$-\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y^2}\right) \mathrm{d}y$$

$$f(y) = \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x} + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x} + \frac{1}{y}$$

The solution becomes

$$y = \frac{x}{c_1 x - 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{3}{3c_1 - 1}$$

$$c_1 = \frac{4}{3}$$

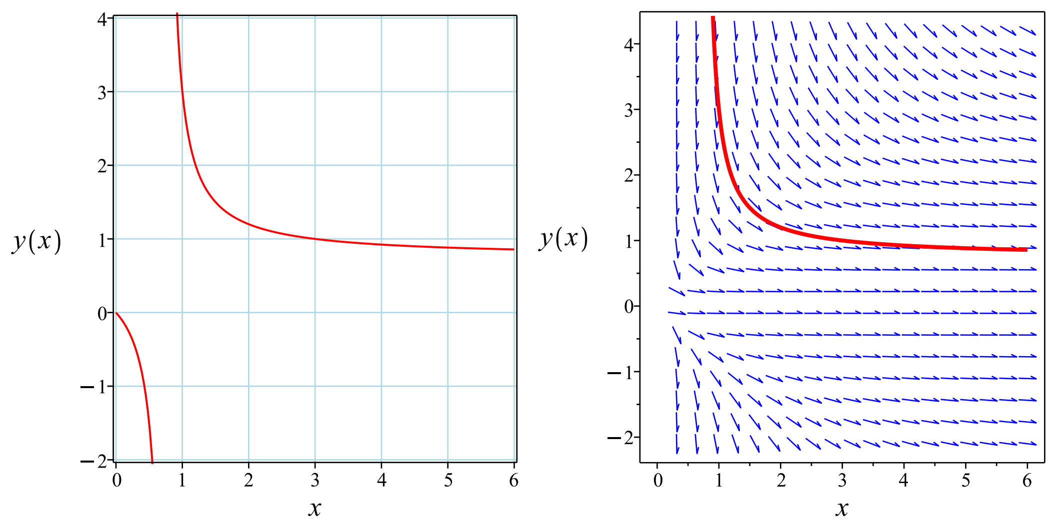
Substituting c_1 found above in the general solution gives

$$y = \frac{3x}{4x - 3}$$

Summary

The solution(s) found are the following

$$y = \frac{3x}{4x - 3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{3x}{4x - 3}$$

Verified OK.

1.24.6 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = -\frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2}{x^3} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^2} - \frac{2u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{c_1 + \frac{c_2}{x}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x}{c_3 x + 1}$$

Initial conditions are used to solve for c_3. Substituting $x = 3$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{3}{3c_3 + 1}$$

$$c_3 = -\frac{4}{3}$$

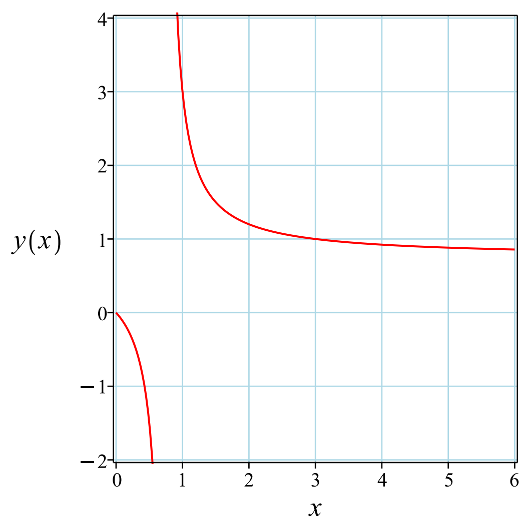
Substituting c_3 found above in the general solution gives

$$y = \frac{3x}{4x - 3}$$

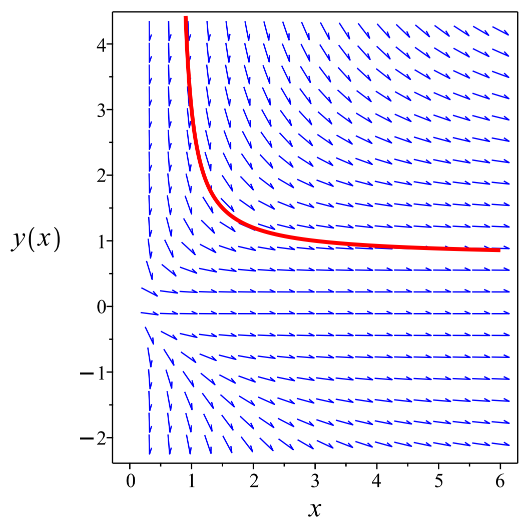
Summary

The solution(s) found are the following

$$y = \frac{3x}{4x - 3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3x}{4x - 3}$$

Verified OK.

1.24.7 Maple step by step solution

Let's solve

$[y'x^2+y^2=0, y(3)=1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2}=-\frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2}dx=\int -\frac{1}{x^2}dx+c_1$

- Evaluate integral

  $-\frac{1}{y}=\frac{1}{x}+c_1$

- Solve for y

  $y=-\frac{x}{c_1x+1}$

- Use initial condition $y(3)=1$

  $1=-\frac{3}{3c_1+1}$

- Solve for c_1

  $c_1=-\frac{4}{3}$

- Substitute $c_1=-\frac{4}{3}$ into general solution and simplify

  $y=\frac{3x}{4x-3}$

- Solution to the IVP

  $y=\frac{3x}{4x-3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([x^2*diff(y(x),x)+y(x)^2=0,y(3) = 1],y(x), singsol=all)
```

$$y(x) = \frac{3x}{4x-3}$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 15

```
DSolve[{x^2*y'[x]+y[x]^2==0,y[3]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3x}{4x-3}$$

1.25 problem 25

Internal problem ID [1894]
Internal file name [`OUTPUT/1895_Sunday_June_05_2022_02_38_00_AM_29794131/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - e^{y} = 0}$$

With initial conditions

$$[y(0) = 0]$$

1.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= e^{y} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(e^{y}) \\ &= e^{y} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.25.2 Solving as quadrature ode

Integrating both sides gives

$$\int e^{-y} dy = x + c_1$$
$$-e^{-y} = x + c_1$$

Solving for y gives these solutions

$$y_1 = \ln\left(-\frac{1}{x + c_1}\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln\left(-\frac{1}{c_1}\right)$$

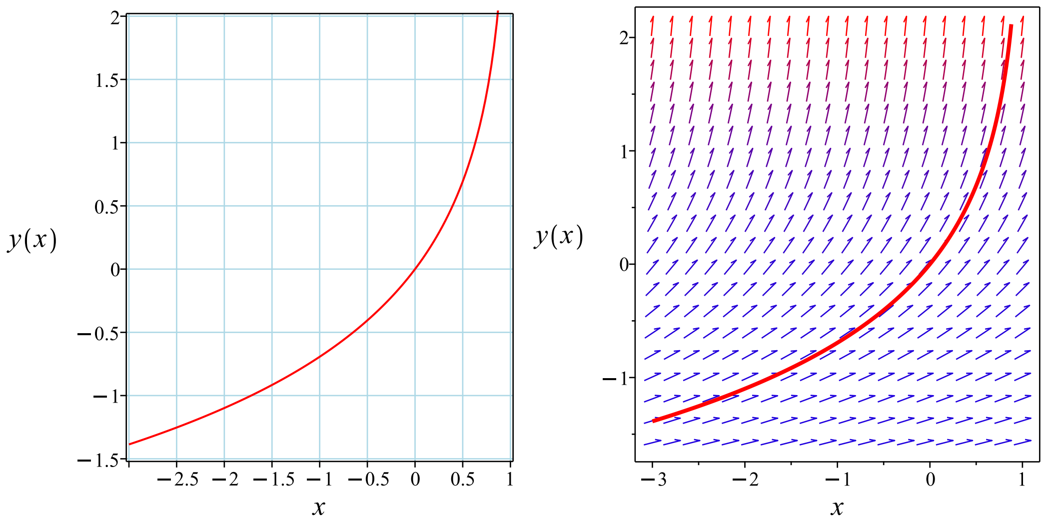
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$y = \ln\left(-\frac{1}{x - 1}\right)$$

Summary
The solution(s) found are the following

$$y = \ln\left(-\frac{1}{x - 1}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \ln\left(-\frac{1}{x-1}\right)$$

Verified OK.

1.25.3 Maple step by step solution

Let's solve

$[y' - e^y = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{e^y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{e^y} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{1}{e^y} = x + c_1$

- Solve for y

$$y = \ln\left(-\frac{1}{x + c_1}\right)$$

- Use initial condition $y(0) = 0$

$$0 = \ln\left(-\frac{1}{c_1}\right)$$

- Solve for c_1

$$c_1 = -1$$

- Substitute $c_1 = -1$ into general solution and simplify

$$y = \ln\left(-\frac{1}{x - 1}\right)$$

- Solution to the IVP

$$y = \ln\left(-\frac{1}{x - 1}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 12

```
dsolve([diff(y(x),x)=exp(y(x)),y(0) = 0],y(x), singsol=all)
```

$$y(x) = -\ln(1 - x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 13

```
DSolve[{y'[x]==Exp[y[x]],y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log(1 - x)$$

1.26 problem 26

Internal problem ID [1895]
Internal file name [`OUTPUT/1896_Sunday_June_05_2022_02_38_03_AM_24856868/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{e^{y}(y'+1)=1}$$

With initial conditions

$$[y(0)=1]$$

1.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -e^{-y}(-1+e^{y}) \end{aligned}$$

The y domain of $f(x,y)$ when $x=0$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-e^{-y}(-1+e^{y})\right) \\ &= -1+e^{-y}(-1+e^{y}) \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.26.2 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{e^y}{-1+e^y} dy = \int dx$$
$$-\ln(-1+e^y) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{-1+e^y} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{-1+e^y} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \ln\left(\frac{c_2+1}{c_2}\right)$$

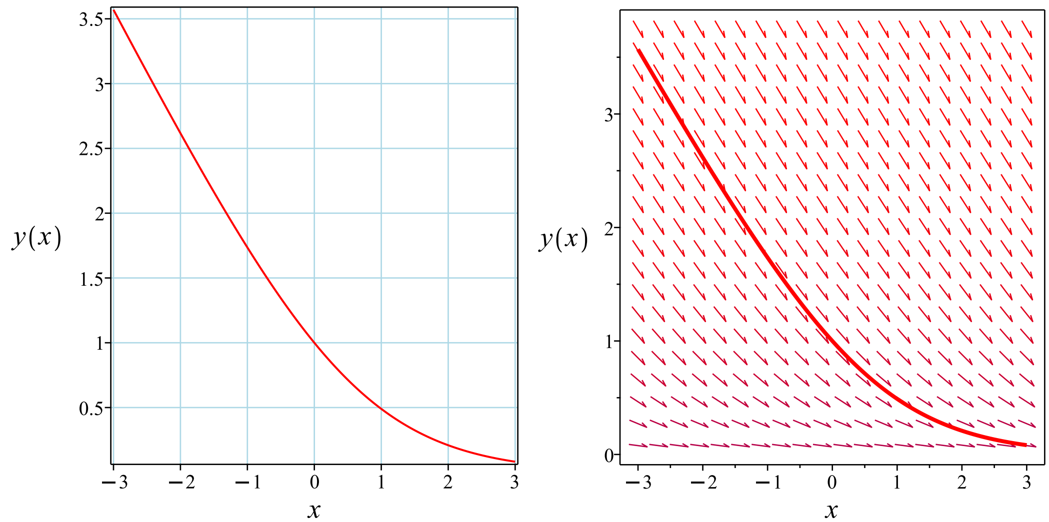
$$c_2 = \frac{1}{-1+e}$$

Substituting c_2 found above in the general solution gives

$$y = \ln(e + e^x - 1) - x$$

Summary
The solution(s) found are the following

$$y = \ln(e + e^x - 1) - x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \ln(e + e^x - 1) - x$$

Verified OK.

1.26.3 Maple step by step solution

Let's solve

$[e^y(y' + 1) = 1, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' e^y}{e^y - 1} = -1$

- Integrate both sides with respect to x

  $\int \frac{y' e^y}{e^y - 1} dx = \int (-1)\, dx + c_1$

- Evaluate integral

  $\ln(e^y - 1) = -x + c_1$

- Solve for y

  $y = \ln(e^{-x + c_1} + 1)$

- Use initial condition $y(0) = 1$

  $1 = \ln(e^{c_1} + 1)$

- Solve for c_1

  $c_1 = \ln(-1 + e)$

- Substitute $c_1 = \ln(-1 + e)$ into general solution and simplify

  $y = \ln(e^{1-x} - e^{-x} + 1)$

- Solution to the IVP

  $y = \ln(e^{1-x} - e^{-x} + 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 32

```
dsolve([exp(y(x))*(diff(y(x),x)+1)=1,y(0) = 1],y(x), singsol=all)
```

$$y(x) = -x + \ln(-e^x - e + 1) - i\pi$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 18

```
DSolve[{Exp[y[x]]*(y'[x]+1)==1,y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(e^{-x}(e^x - 1 + e)\right)$$

1.27 problem 27

Internal problem ID [1896]
Internal file name [`OUTPUT/1897_Sunday_June_05_2022_02_38_06_AM_64108406/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 - \frac{y'}{x^3 (x-1)} = -1}$$

With initial conditions

$$[y(2) = 0]$$

1.27.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= x^3(y^2+1)(x-1) \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x,y)$ when $x = 2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x^3(y^2+1)(x-1)\right) \\ &= 2x^3y(x-1) \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.27.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x^3(y^2+1)(x-1) \end{aligned}$$

Where $f(x) = x^3(x-1)$ and $g(y) = y^2+1$. Integrating both sides gives

$$\frac{1}{y^2+1}\,dy = x^3(x-1)\;dx$$

$$\int \frac{1}{y^2+1}\,dy = \int x^3(x-1)\;dx$$

$$\arctan(y) = \frac{1}{5}x^5 - \frac{1}{4}x^4 + c_1$$

Which results in

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \tan\left(\frac{12}{5} + c_1\right)$$

$$c_1 = -\frac{12}{5}$$

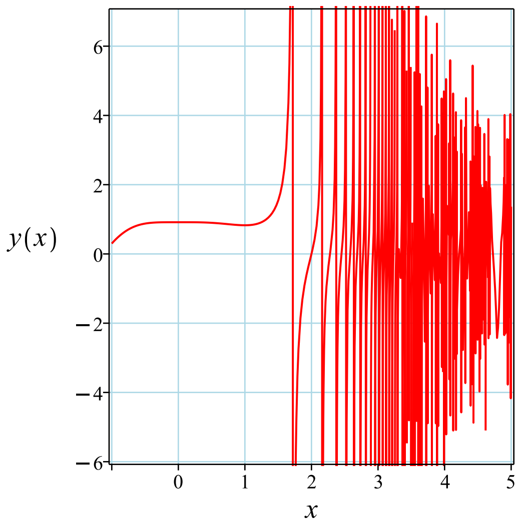
Substituting c_1 found above in the general solution gives

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right)$$

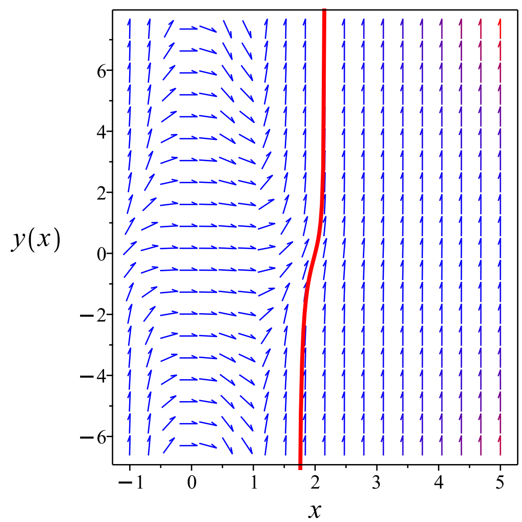
Summary

The solution(s) found are the following

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right)$$

Verified OK.

1.27.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x^3(y^2+1)(x-1)$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 72: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x^3 (x-1)} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x^3(x-1)}} dx \end{aligned}$$

Which results in

$$S = \frac{1}{5}x^5 - \frac{1}{4}x^4$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^3 \left(y^2 + 1\right) (x - 1)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x^3(x - 1) \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{5}x^5 - \frac{1}{4}x^4 = \arctan(y) + c_1$$

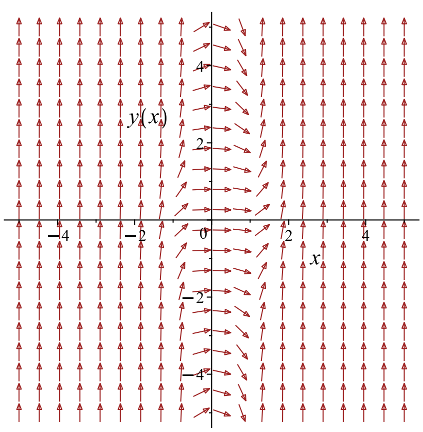
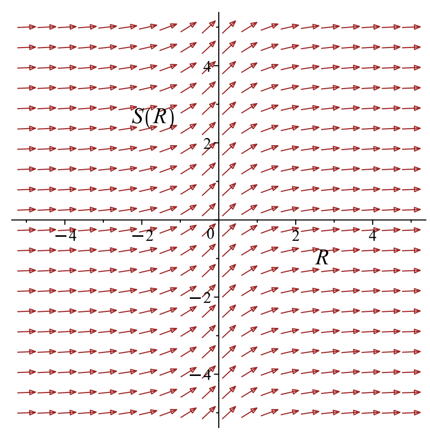
Which simplifies to

$$\frac{1}{5}x^5 - \frac{1}{4}x^4 = \arctan(y) + c_1$$

Which gives

$$y = -\tan\left(-\frac{1}{5}x^5 + \frac{1}{4}x^4 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^3(y^2+1)(x-1)$  	$R = y$ $S = \frac{1}{5}x^5 - \frac{1}{4}x^4$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\tan\left(-\frac{12}{5} + c_1\right)$$

$$c_1 = \frac{12}{5}$$

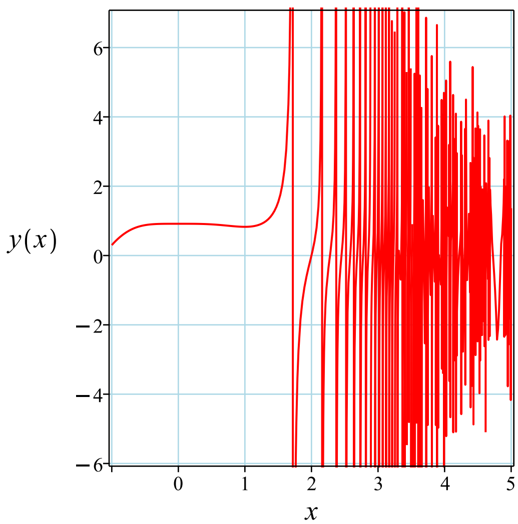
Substituting c_1 found above in the general solution gives

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right)$$

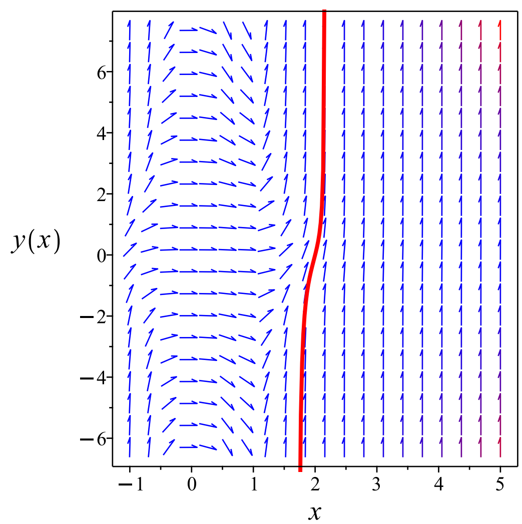
Summary

The solution(s) found are the following

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right)$$

Verified OK.

1.27.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2+1}\right)\mathrm{d}y &= \left(x^3(x-1)\right)\mathrm{d}x\\ \left(-x^3(x-1)\right)\mathrm{d}x + \left(\frac{1}{y^2+1}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x^3(x-1) \\
N(x,y) &= \frac{1}{y^2+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3(x-1)\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1) \\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x^3(x-1)\,\mathrm{d}x \\
\phi &= -\frac{1}{5}x^5 + \frac{1}{4}x^4 + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2+1}\right) \mathrm{d}y$$
$$f(y) = \arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^5}{5} + \frac{x^4}{4} + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^5}{5} + \frac{x^4}{4} + \arctan(y)$$

The solution becomes

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \tan\left(\frac{12}{5} + c_1\right)$$

$$c_1 = -\frac{12}{5}$$

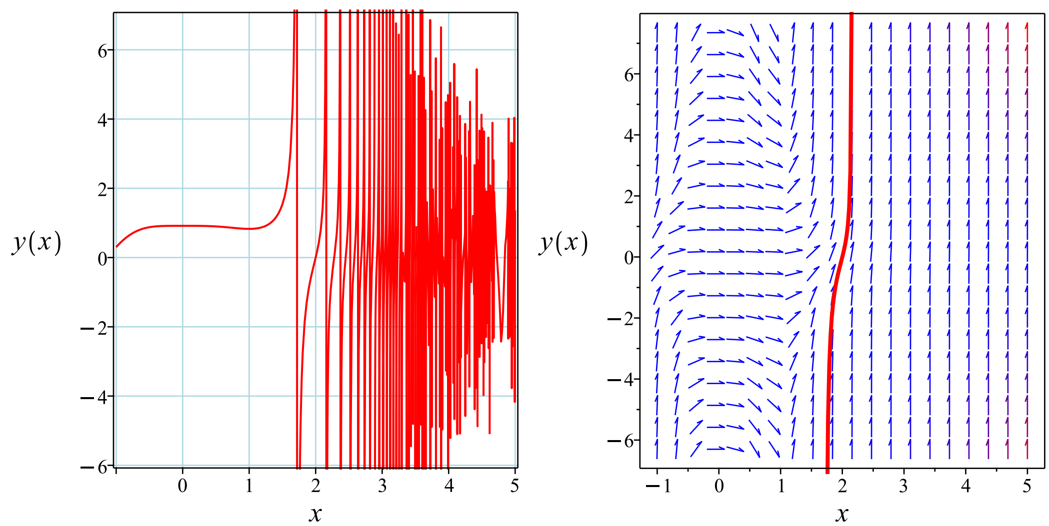
Substituting c_1 found above in the general solution gives

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right)$$

Verified OK.

1.27.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x^3(y^2 + 1)(x - 1) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^4y^2 - x^3y^2 + x^4 - x^3$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^3(x-1)$, $f_1(x) = 0$ and $f_2(x) = x^3(x-1)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x^3 (x-1) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 3x^2(x-1) + x^3 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^9(x-1)^3 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x^3(x-1) u''(x) - \left(3x^2(x-1) + x^3\right) u'(x) + x^9(x-1)^3 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right) + c_2 \cos\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right)$$

The above shows that

$$u'(x) = x^3\left(c_1 \cos\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right) - c_2 \sin\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right)\right)(x-1)$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 \cos\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right) - c_2 \sin\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right)}{c_1 \sin\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right) + c_2 \cos\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \cos\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right) + \sin\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right)}{c_3 \sin\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right) + \cos\left(\frac{1}{5}x^5 - \frac{1}{4}x^4\right)}$$

Initial conditions are used to solve for c_3. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-c_3 \cos\left(\frac{12}{5}\right) + \sin\left(\frac{12}{5}\right)}{c_3 \sin\left(\frac{12}{5}\right) + \cos\left(\frac{12}{5}\right)}$$

$$c_3 = \frac{\sin\left(\frac{12}{5}\right)}{\cos\left(\frac{12}{5}\right)}$$

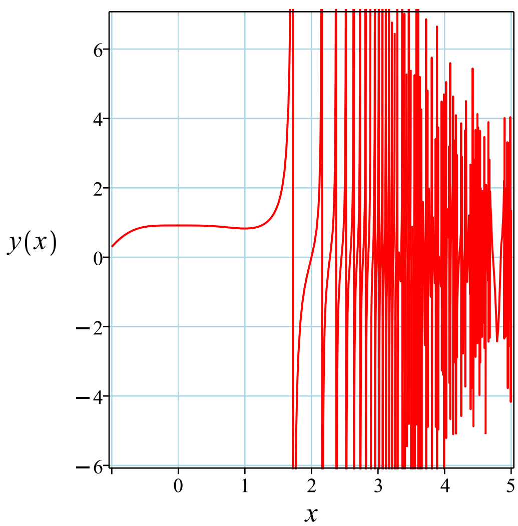
Substituting c_3 found above in the general solution gives

$$y = \frac{-\cos\left(\frac{x^4(4x-5)}{20}\right)\tan\left(\frac{12}{5}\right) + \sin\left(\frac{x^4(4x-5)}{20}\right)}{\sin\left(\frac{x^4(4x-5)}{20}\right)\tan\left(\frac{12}{5}\right) + \cos\left(\frac{x^4(4x-5)}{20}\right)}$$

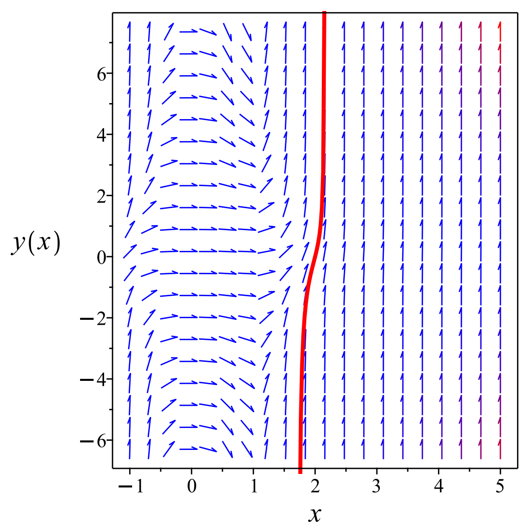
Summary

The solution(s) found are the following

$$y = \frac{-\cos\left(\frac{x^4(4x-5)}{20}\right)\tan\left(\frac{12}{5}\right) + \sin\left(\frac{x^4(4x-5)}{20}\right)}{\sin\left(\frac{x^4(4x-5)}{20}\right)\tan\left(\frac{12}{5}\right) + \cos\left(\frac{x^4(4x-5)}{20}\right)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-\cos\left(\frac{x^4(4x-5)}{20}\right)\tan\left(\frac{12}{5}\right)+\sin\left(\frac{x^4(4x-5)}{20}\right)}{\sin\left(\frac{x^4(4x-5)}{20}\right)\tan\left(\frac{12}{5}\right)+\cos\left(\frac{x^4(4x-5)}{20}\right)}$$

Verified OK.

1.27.6 Maple step by step solution

Let's solve

$\left[y^2 - \frac{y'}{x^3(x-1)} = -1, y(2) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-1-y^2} = -x^3(x-1)$

- Integrate both sides with respect to x

  $\int \frac{y'}{-1-y^2}dx = \int -x^3(x-1)\,dx + c_1$

- Evaluate integral

  $-\arctan(y) = -\frac{1}{5}x^5 + \frac{1}{4}x^4 + c_1$

- Solve for y

  $y = -\tan\left(-\frac{1}{5}x^5 + \frac{1}{4}x^4 + c_1\right)$

- Use initial condition $y(2) = 0$

  $0 = -\tan\left(-\frac{12}{5} + c_1\right)$

- Solve for c_1

  $c_1 = \frac{12}{5}$

- Substitute $c_1 = \frac{12}{5}$ into general solution and simplify

  $y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right)$

- Solution to the IVP

  $y = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 17

```
dsolve([(1+y(x)^2)=diff(y(x),x)/(x^3*(x-1)),y(2) = 0],y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{1}{5}x^5 - \frac{1}{4}x^4 - \frac{12}{5}\right)$$

✓ Solution by Mathematica

Time used: 0.353 (sec). Leaf size: 22

```
DSolve[{(1+y[x]^2)==y'[x]/(x^3*(x-1)),y[2]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan\left(\frac{1}{20}(4x^5 - 5x^4 - 48)\right)$$

1.28 problem 28

Internal problem ID [1897]
Internal file name [`OUTPUT/1898_Sunday_June_05_2022_02_38_09_AM_33704393/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind"**

Maple gives the following as the ode type

`[_rational, _Abel]`

Unable to solve or complete the solution.

$$\boxed{3y'x - y^3 - 2y = -x^2}$$

With initial conditions

$$[y(1) = 1]$$

1.28.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y^3 - x^2 + 2y}{3x} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^3 - x^2 + 2y}{3x}\right) \\ &= \frac{3y^2 + 2}{3x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.28.2 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = \frac{y^3}{3x} + \frac{2y}{3x} - \frac{x}{3} \tag{1}$$

Therefore

$$\begin{aligned}f_0(x) &= -\frac{x}{3} \\ f_1(x) &= \frac{2}{3x} \\ f_2(x) &= 0 \\ f_3(x) &= \frac{1}{3x}\end{aligned}$$

Since $f_2(x) = 0$ then we check the Abel invariant to see if it depends on x or not. The Abel invariant is given by

$$-\frac{f_1^3}{f_0^2 f_3}$$

Which when evaluating gives

$$0$$

Since the Abel invariant does not depend on x then this ode can be solved directly.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 7

```
dsolve([x^2+3*x*diff(y(x),x)=y(x)^3+2*y(x),y(1) = 1],y(x), singsol=all)
```

$$y(x) = x^{\frac{2}{3}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x^2+3*x*y'[x]==y[x]^3+2*y[x],y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

{}

1.29 problem 29

Internal problem ID [1898]
Internal file name [`OUTPUT/1899_Sunday_June_05_2022_02_38_12_AM_57529266/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2+x+1)\,y'-y^2-2y=5}$$

With initial conditions

$$[y(1)=1]$$

1.29.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y^2+2y+5}{x^2+x+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^2 + 2y + 5}{x^2 + x + 1}\right) \\ &= \frac{2y + 2}{x^2 + x + 1}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

1.29.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y^2 + 2y + 5}{x^2 + x + 1}\end{aligned}$$

Where $f(x) = \frac{1}{x^2+x+1}$ and $g(y) = y^2 + 2y + 5$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2 + 2y + 5}\,dy &= \frac{1}{x^2 + x + 1}\,dx \\ \int \frac{1}{y^2 + 2y + 5}\,dy &= \int \frac{1}{x^2 + x + 1}\,dx \\ \frac{\arctan\left(\frac{y}{2} + \frac{1}{2}\right)}{2} &= \frac{2\sqrt{3}\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + c_1\end{aligned}$$

Which results in

$$y = -1 + 2\tan\left(\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + 2c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -1 + 2\tan\left(\frac{4\pi\sqrt{3}}{9} + 2c_1\right)$$

$$c_1 = -\frac{2\pi\sqrt{3}}{9} + \frac{\pi}{8}$$

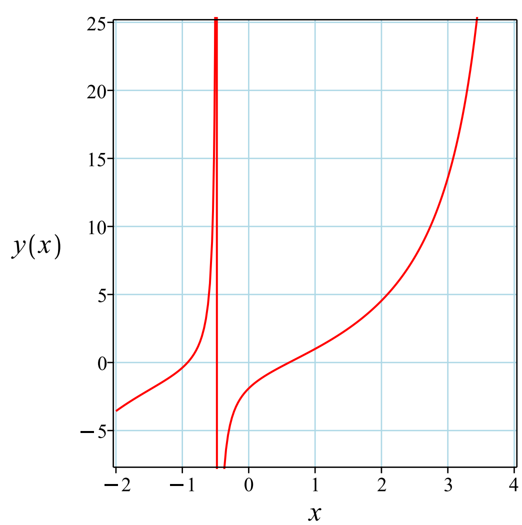
Substituting c_1 found above in the general solution gives

$$y = -1 + 2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + \frac{4\pi\sqrt{3}}{9} + \frac{\pi}{4}\right)$$

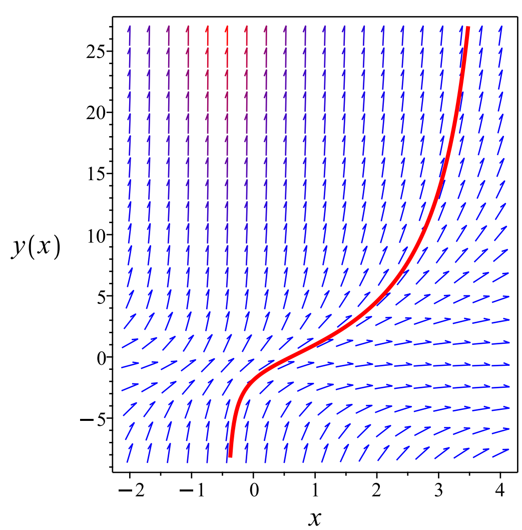
<u>Summary</u>

The solution(s) found are the following

$$y = -1 + 2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + \frac{4\pi\sqrt{3}}{9} + \frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -1 + 2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + \frac{4\pi\sqrt{3}}{9} + \frac{\pi}{4}\right)$$

Verified OK.

1.29.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \frac{y^2 + 2y + 5}{x^2 + x + 1}\\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 75: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= x^2 + x + 1 \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^2 + x + 1} dx \end{aligned}$$

Which results in

$$S = \frac{2\sqrt{3}\, \arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2 + 2y + 5}{x^2 + x + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x^2 + x + 1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2 + 2y + 5} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + 2R + 5}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\arctan\left(\frac{R}{2} + \frac{1}{2}\right)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} = \frac{\arctan\left(\frac{y}{2} + \frac{1}{2}\right)}{2} + c_1$$

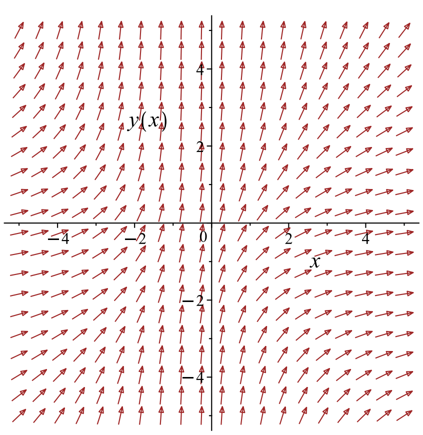
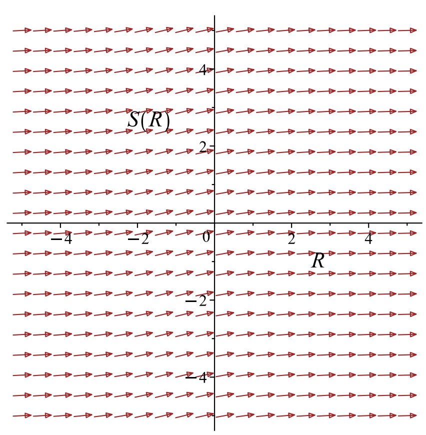
Which simplifies to

$$\frac{2\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} = \frac{\arctan\left(\frac{y}{2} + \frac{1}{2}\right)}{2} + c_1$$

Which gives

$$y = -1 - 2\tan\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + 2c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+2y+5}{x^2+x+1}$ 	$R = y$ $S = \frac{2\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3}$	$\frac{dS}{dR} = \frac{1}{R^2+2R+5}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -1 - 2\tan\left(-\frac{4\pi\sqrt{3}}{9} + 2c_1\right)$$

$$c_1 = \frac{2\pi\sqrt{3}}{9} - \frac{\pi}{8}$$

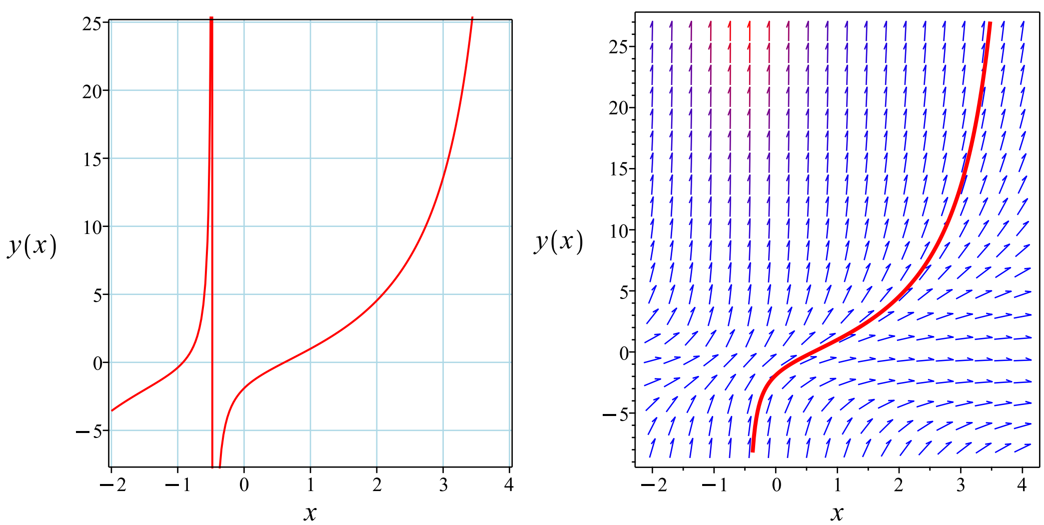
Substituting c_1 found above in the general solution gives

$$y = -1 + 2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + \frac{4\pi\sqrt{3}}{9} + \frac{\pi}{4}\right)$$

Summary

The solution(s) found are the following

$$y = -1 + 2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + \frac{4\pi\sqrt{3}}{9} + \frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -1 + 2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + \frac{4\pi\sqrt{3}}{9} + \frac{\pi}{4}\right)$$

Verified OK.

1.29.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2+2y+5}\right)\mathrm{d}y&=\left(\frac{1}{x^2+x+1}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2+x+1}\right)\mathrm{d}x+\left(\frac{1}{y^2+2y+5}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x^2+x+1} \\
N(x,y) &= \frac{1}{y^2+2y+5}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2+x+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+2y+5}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x^2+x+1} \, \mathrm{d}x
\end{aligned}
$$

$$\phi = -\frac{2\sqrt{3}\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+2y+5}$. Therefore equation (4) becomes

$$\frac{1}{y^2 + 2y + 5} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2 + 2y + 5}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2 + 2y + 5}\right) \mathrm{d}y$$

$$f(y) = \frac{\arctan\left(\frac{y}{2} + \frac{1}{2}\right)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{2\sqrt{3}\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + \frac{\arctan\left(\frac{y}{2} + \frac{1}{2}\right)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{2\sqrt{3}\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + \frac{\arctan\left(\frac{y}{2} + \frac{1}{2}\right)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{2\pi\sqrt{3}}{9} + \frac{\pi}{8} = c_1$$

$$c_1 = -\frac{(16\sqrt{3}-9)\,\pi}{72}$$

Substituting c_1 found above in the general solution gives

$$-\frac{2\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3}+\frac{\arctan\left(\frac{y}{2}+\frac{1}{2}\right)}{2} = -\frac{2\pi\sqrt{3}}{9}+\frac{\pi}{8}$$

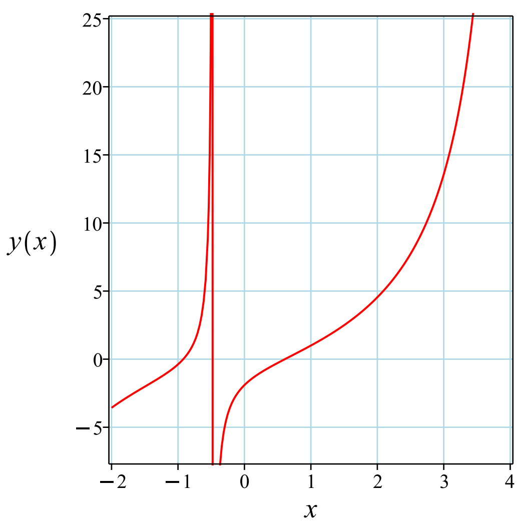
Solving for y from the above gives

$$y = -1+2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3}+\frac{4\pi\sqrt{3}}{9}+\frac{\pi}{4}\right)$$

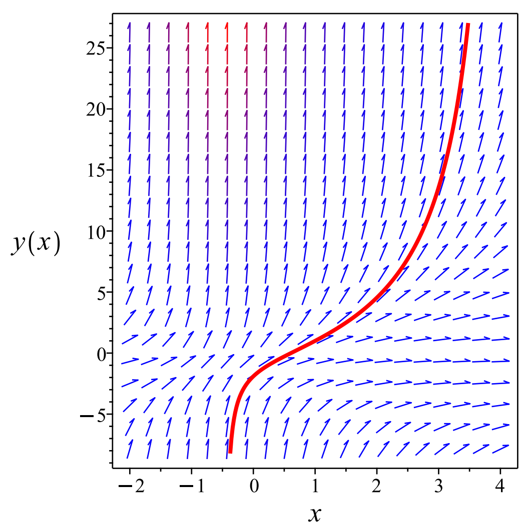
Summary

The solution(s) found are the following

$$y = -1+2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3}+\frac{4\pi\sqrt{3}}{9}+\frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -1+2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3}+\frac{4\pi\sqrt{3}}{9}+\frac{\pi}{4}\right)$$

Verified OK.

1.29.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2+2y+5}{x^2+x+1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^2+x+1} + \frac{2y}{x^2+x+1} + \frac{5}{x^2+x+1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{5}{x^2+x+1}$, $f_1(x) = \frac{2}{x^2+x+1}$ and $f_2(x) = \frac{1}{x^2+x+1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2+x+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1+2x}{\left(x^2+x+1\right)^2} \\ f_1 f_2 &= \frac{2}{\left(x^2+x+1\right)^2} \\ f_2^2 f_0 &= \frac{5}{\left(x^2+x+1\right)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2+x+1} - \left(-\frac{1+2x}{\left(x^2+x+1\right)^2} + \frac{2}{\left(x^2+x+1\right)^2}\right) u'(x) + \frac{5u(x)}{\left(x^2+x+1\right)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1\left(i\sqrt{3}-2x-1\right)^{\left(\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} \left(i\sqrt{3}+2x+1\right)^{\left(-\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}} + c_2\left(i\sqrt{3}-2x-1\right)^{\left(-\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} \left(i\sqrt{3}+2x+1\right)^{\left(\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}}$$

The above shows that

$$u'(x) = \frac{(1-2i)\left(i\sqrt{3}+2x+1\right)^{\left(\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}} c_2\left(i\sqrt{3}-2x-1\right)^{\left(-\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} + (1+2i)\left(i\sqrt{3}+2x+1\right)^{\left(-\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}} c_1\left(i\sqrt{3}-\right.}{x^2+x+1}$$

Using the above in (1) gives the solution

$$y = -\frac{(1-2i)\left(i\sqrt{3}+2x+1\right)^{\left(\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}} c_2\left(i\sqrt{3}-2x-1\right)^{\left(-\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} + (1+2i)\left(i\sqrt{3}+2x+1\right)^{\left(-\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}} c_1\left(i\sqrt{\right.}{c_1\left(i\sqrt{3}-2x-1\right)^{\left(\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} \left(i\sqrt{3}+2x+1\right)^{\left(-\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}} + c_2\left(i\sqrt{3}-2x-1\right)^{\left(-\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} \left(i\sqrt{3}+2x\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-1+2i)\left(i\sqrt{3}+2x+1\right)^{\left(\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}} \left(i\sqrt{3}-2x-1\right)^{\left(-\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} + (-1-2i)\left(i\sqrt{3}+2x+1\right)^{\left(-\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}} c_3\left(i\sqrt{\right.}{\left(i\sqrt{3}+2x+1\right)^{\left(-\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}} c_3\left(i\sqrt{3}-2x-1\right)^{\left(\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} + \left(i\sqrt{3}-2x-1\right)^{\left(-\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} \left(i\sqrt{3}+2x+1\right.}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{2i\left(e^{\frac{\sqrt{3}(i\pi+6\ln(2)+3\ln(3))}{6}}\right)^{\frac{4}{3}} e^{-\frac{\sqrt{3}(5i\pi+6\ln(2)+3\ln(3))}{9}} - \left(e^{\frac{\sqrt{3}(i\pi+6\ln(2)+3\ln(3))}{6}}\right)^{\frac{4}{3}} e^{-\frac{\sqrt{3}(5i\pi+6\ln(2)+3\ln(3))}{9}} - 2ic_3 e^{\frac{\sqrt{3}(5i\pi+6}{}}}{\left(e^{\frac{\sqrt{3}(i\pi+6\ln(2)+3\ln(3))}{6}}\right)^{\frac{4}{3}} e^{-\frac{\sqrt{3}(5i\pi+6\ln(2)+3\ln(3))}{9}} + c_3 e^{\frac{\sqrt{3}(5i\pi+6\ln(2)+3\ln(3))}{9}}}$$

$$c_3 = i\left(e^{\frac{\sqrt{3}(i\pi+6\ln(2)+3\ln(3))}{6}}\right)^{\frac{4}{3}} \left(e^{-\frac{\sqrt{3}(5i\pi+6\ln(2)+3\ln(3))}{9}}\right)^2$$

Substituting c_3 found above in the general solution gives

$$y = \frac{-i\left(i\sqrt{3}-2x-1\right)^{\left(\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} \left(i\sqrt{3}+2x+1\right)^{\left(-\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}} \left(e^{\frac{\sqrt{3}(i\pi+6\ln(2)+3\ln(3))}{6}}\right)^{\frac{4}{3}} e^{-\frac{2\sqrt{3}(5i\pi+6\ln(2)+3\ln(3))}{9}} + 2\left(i\sqrt{\right.}{i\left(i\sqrt{3}-2x-1\right)^{\left(\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}} \left(\right.}$$

Summary
The solution(s) found are the following

$$y = \frac{-i\left(i\sqrt{3}-2x-1\right)^{\left(\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}}\left(i\sqrt{3}+2x+1\right)^{\left(-\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}}\left(\mathrm{e}^{\frac{\sqrt{3}\,(i\pi+6\ln(2)+3\ln(3))}{6}}\right)^{\frac{4}{3}}\mathrm{e}^{-\frac{2\sqrt{3}\,(5i\pi+6\ln(2)+3\ln(3))}{9}}+2\left(i\sqrt{3}-\right.}{i\left(i\sqrt{3}-2x-1\right)^{\left(\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}}\left(i\sqrt{\right.}} \quad (1)$$

Verification of solutions

$$y = \frac{-i\left(i\sqrt{3}-2x-1\right)^{\left(\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}}\left(i\sqrt{3}+2x+1\right)^{\left(-\frac{2}{3}+\frac{i}{3}\right)\sqrt{3}}\left(\mathrm{e}^{\frac{\sqrt{3}\,(i\pi+6\ln(2)+3\ln(3))}{6}}\right)^{\frac{4}{3}}\mathrm{e}^{-\frac{2\sqrt{3}\,(5i\pi+6\ln(2)+3\ln(3))}{9}}+2\left(i\sqrt{3}-\right.}{i\left(i\sqrt{3}-2x-1\right)^{\left(\frac{2}{3}-\frac{i}{3}\right)\sqrt{3}}\left(i\sqrt{\right.}}$$

Warning, solution could not be verified

1.29.6 Maple step by step solution

Let's solve

$[(x^2+x+1)\,y'-y^2-2y=5, y(1)=1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2+2y+5}=\frac{1}{x^2+x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2+2y+5}dx=\int \frac{1}{x^2+x+1}dx+c_1$

- Evaluate integral

  $\frac{\arctan\left(\frac{y}{2}+\frac{1}{2}\right)}{2}=\frac{2\sqrt{3}\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3}+c_1$

- Solve for y

  $y=-1+2\tan\left(\frac{4\sqrt{3}\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3}+2c_1\right)$

- Use initial condition $y(1)=1$

  $1=-1+2\tan\left(\frac{4\pi\sqrt{3}}{9}+2c_1\right)$

- Solve for c_1

$$c_1 = -\frac{2\pi\sqrt{3}}{9} + \frac{\pi}{8}$$

- Substitute $c_1 = -\frac{2\pi\sqrt{3}}{9} + \frac{\pi}{8}$ into general solution and simplify

$$y = -1 + 2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + \frac{4\pi\sqrt{3}}{9} + \frac{\pi}{4}\right)$$

- Solution to the IVP

$$y = -1 + 2\cot\left(-\frac{4\sqrt{3}\,\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)}{3} + \frac{4\pi\sqrt{3}}{9} + \frac{\pi}{4}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 35

```
dsolve([(x^2+x+1)*diff(y(x),x)=y(x)^2+2*y(x)+5,y(1) = 1],y(x), singsol=all)
```

$$y(x) = -1 + 2\cot\left(\frac{4\sqrt{3}\,\pi}{9} - \frac{4\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(2x+1)}{3}\right)}{3} + \frac{\pi}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.905 (sec). Leaf size: 44

```
DSolve[{(x^2+x+1)*y'[x]==y[x]^2+2*y[x]+5,y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\tan\left(\frac{4\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}} + \frac{1}{36}\left(9 - 16\sqrt{3}\right)\pi\right) - 1$$

1.30 problem 30

Internal problem ID [1899]
Internal file name [`OUTPUT/1900_Sunday_June_05_2022_02_38_14_AM_28591331/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 5, page 21
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2 - 2x - 8)\, y' - y^2 - y = -2}$$

With initial conditions

$$[y(0) = 0]$$

1.30.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y^2 + y - 2}{x^2 - 2x - 8} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty \leq x < -2, -2 < x < 4, 4 < x \leq \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^2 + y - 2}{x^2 - 2x - 8}\right) \\ &= \frac{1 + 2y}{x^2 - 2x - 8}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty \leq x < -2, -2 < x < 4, 4 < x \leq \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

1.30.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y^2 + y - 2}{x^2 - 2x - 8}\end{aligned}$$

Where $f(x) = \frac{1}{x^2 - 2x - 8}$ and $g(y) = y^2 + y - 2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2 + y - 2}\,dy &= \frac{1}{x^2 - 2x - 8}\,dx \\ \int \frac{1}{y^2 + y - 2}\,dy &= \int \frac{1}{x^2 - 2x - 8}\,dx \\ \frac{\ln(y - 1)}{3} - \frac{\ln(2 + y)}{3} &= \frac{\ln(x - 4)}{6} - \frac{\ln(2 + x)}{6} + c_1\end{aligned}$$

The above can be written as

$$\left(\frac{1}{3}\right)(\ln (y-1)-\ln (2+y)) = \frac{\ln (x-4)}{6}-\frac{\ln (2+x)}{6}+2c_1$$
$$\ln (y-1)-\ln (2+y) = (3)\left(\frac{\ln (x-4)}{6}-\frac{\ln (2+x)}{6}+2c_1\right)$$
$$= \frac{\ln (x-4)}{2}-\frac{\ln (2+x)}{2}+6c_1$$

Raising both side to exponential gives

$$e^{\ln(y-1)-\ln(2+y)} = e^{\frac{\ln(x-4)}{2}-\frac{\ln(2+x)}{2}+3c_1}$$

Which simplifies to

$$\frac{y-1}{2+y} = 3c_1 e^{\frac{\ln(x-4)}{2}-\frac{\ln(2+x)}{2}}$$
$$= c_2 e^{\frac{\ln(x-4)}{2}-\frac{\ln(2+x)}{2}}$$

Which can be simplified to become

$$y = -\frac{\frac{2c_2\sqrt{x-4}}{\sqrt{2+x}}+1}{-1+\frac{c_2\sqrt{x-4}}{\sqrt{2+x}}}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{4ic_2+\sqrt{2}}{-2ic_2+\sqrt{2}}$$

$$c_2 = \frac{i\sqrt{2}}{4}$$

Substituting c_2 found above in the general solution gives

$$y = \frac{-2i\sqrt{2}\,\sqrt{x-4}-4\sqrt{2+x}}{i\sqrt{2}\,\sqrt{x-4}-4\sqrt{2+x}}$$

Summary

The solution(s) found are the following

$$y = \frac{-2i\sqrt{2}\,\sqrt{x-4}-4\sqrt{2+x}}{i\sqrt{2}\,\sqrt{x-4}-4\sqrt{2+x}} \tag{1}$$

Verification of solutions

$$y = \frac{-2i\sqrt{2}\,\sqrt{x-4}-4\sqrt{2+x}}{i\sqrt{2}\,\sqrt{x-4}-4\sqrt{2+x}}$$

Verified OK.

1.30.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2 + y - 2}{x^2 - 2x - 8}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 78: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}
\xi(x,y) &= x^2 - 2x - 8 \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x^2 - 2x - 8} dx
\end{aligned}$$

Which results in

$$S = \frac{\ln(x-4)}{6} - \frac{\ln(2+x)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2 + y - 2}{x^2 - 2x - 8}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{(2+x)(x-4)} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2+y-2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2+R-2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\ln(R-1)}{3} - \frac{\ln(R+2)}{3} + c_1 \tag{4}
$$

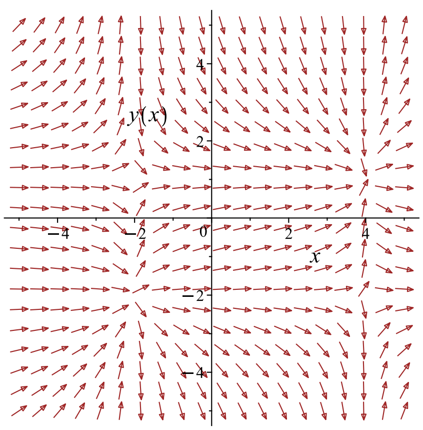
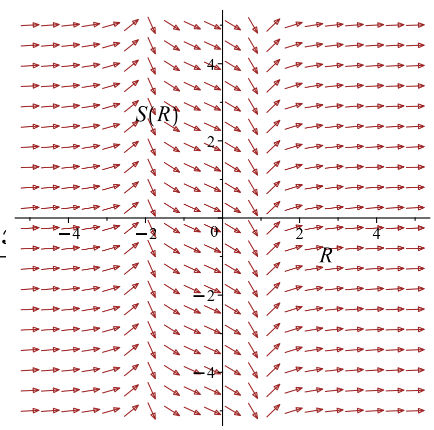
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(x-4)}{6} - \frac{\ln(2+x)}{6} = \frac{\ln(y-1)}{3} - \frac{\ln(2+y)}{3} + c_1
$$

Which simplifies to

$$
\frac{\ln(x-4)}{6} - \frac{\ln(2+x)}{6} = \frac{\ln(y-1)}{3} - \frac{\ln(2+y)}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+y-2}{x^2-2x-8}$ 	$R = y$ $S = \frac{\ln(x-4)}{6} - \frac{\ln(2+}{6}$	$\frac{dS}{dR} = \frac{1}{R^2+R-2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln(2)}{6} + \frac{i\pi}{6} = \frac{i\pi}{3} - \frac{\ln(2)}{3} + c_1$$

$$c_1 = -\frac{i\pi}{6} + \frac{\ln(2)}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(x-4)}{6} - \frac{\ln(2+x)}{6} = \frac{\ln(y-1)}{3} - \frac{\ln(2+y)}{3} - \frac{i\pi}{6} + \frac{\ln(2)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln(x-4)}{6} - \frac{\ln(2+x)}{6} = \frac{\ln(y-1)}{3} - \frac{\ln(2+y)}{3} - \frac{i\pi}{6} + \frac{\ln(2)}{2} \tag{1}$$

Verification of solutions

$$\frac{\ln(x-4)}{6} - \frac{\ln(2+x)}{6} = \frac{\ln(y-1)}{3} - \frac{\ln(2+y)}{3} - \frac{i\pi}{6} + \frac{\ln(2)}{2}$$

Verified OK.

1.30.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2+y-2}\right)\mathrm{d}y&=\left(\frac{1}{x^2-2x-8}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2-2x-8}\right)\mathrm{d}x+\left(\frac{1}{y^2+y-2}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x^2-2x-8} \\
N(x,y) &= \frac{1}{y^2+y-2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2-2x-8}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+y-2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x^2-2x-8}\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(x-4)}{6} + \frac{\ln(2+x)}{6} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+y-2}$. Therefore equation (4) becomes

$$\frac{1}{y^2+y-2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+y-2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{1}{y^2+y-2}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(y-1)}{3} - \frac{\ln(2+y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-4)}{6} + \frac{\ln(2+x)}{6} + \frac{\ln(y-1)}{3} - \frac{\ln(2+y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-4)}{6} + \frac{\ln(2+x)}{6} + \frac{\ln(y-1)}{3} - \frac{\ln(2+y)}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{i\pi}{6} - \frac{\ln(2)}{2} = c_1$$

$$c_1 = \frac{i\pi}{6} - \frac{\ln(2)}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{\ln(x-4)}{6} + \frac{\ln(2+x)}{6} + \frac{\ln(y-1)}{3} - \frac{\ln(2+y)}{3} = \frac{i\pi}{6} - \frac{\ln(2)}{2}$$

Summary
The solution(s) found are the following

$$
-\frac{\ln (x-4)}{6}+\frac{\ln (2+x)}{6}+\frac{\ln (y-1)}{3}-\frac{\ln (2+y)}{3}=\frac{i\pi}{6}-\frac{\ln (2)}{2} \tag{1}
$$

Verification of solutions

$$
-\frac{\ln (x-4)}{6}+\frac{\ln (2+x)}{6}+\frac{\ln (y-1)}{3}-\frac{\ln (2+y)}{3}=\frac{i\pi}{6}-\frac{\ln (2)}{2}
$$

Verified OK.

1.30.5 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y)\\
&= \frac{y^2+y-2}{x^2-2x-8}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = \frac{y^2}{x^2-2x-8}+\frac{y}{x^2-2x-8}-\frac{2}{x^2-2x-8}
$$

With Riccati ODE standard form

$$
y' = f_0(x)+f_1(x)y+f_2(x)y^2
$$

Shows that $f_0(x)=-\frac{2}{x^2-2x-8}$, $f_1(x)=\frac{1}{x^2-2x-8}$ and $f_2(x)=\frac{1}{x^2-2x-8}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u}\\
&= \frac{-u'}{\frac{u}{x^2-2x-8}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2u''(x)-(f_2'+f_1f_2)\,u'(x)+f_2^2f_0u(x)=0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2x-2}{(x^2-2x-8)^2}\\
f_1f_2 &= \frac{1}{(x^2-2x-8)^2}\\
f_2^2f_0 &= -\frac{2}{(x^2-2x-8)^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2-2x-8} - \left(-\frac{2x-2}{(x^2-2x-8)^2} + \frac{1}{(x^2-2x-8)^2}\right) u'(x) - \frac{2u(x)}{(x^2-2x-8)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \left(\frac{2+x}{x-4}\right)^{\frac{1}{6}} + c_2 \left(\frac{x-4}{2+x}\right)^{\frac{1}{3}}$$

The above shows that

$$u'(x) = -\frac{c_1}{\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}} (x-4)^2} + \frac{2c_2}{\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}} (2+x)^2}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\frac{c_1}{\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}(x-4)^2} + \frac{2c_2}{\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}(2+x)^2}\right)(x^2-2x-8)}{c_1 \left(\frac{2+x}{x-4}\right)^{\frac{1}{6}} + c_2 \left(\frac{x-4}{2+x}\right)^{\frac{1}{3}}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\left(-\frac{c_3}{\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}(x-4)^2} + \frac{2}{\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}(2+x)^2}\right)(x^2-2x-8)}{c_3 \left(\frac{2+x}{x-4}\right)^{\frac{1}{6}} + \left(\frac{x-4}{2+x}\right)^{\frac{1}{3}}}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{ic_3 2^{\frac{5}{6}} + c_3 2^{\frac{5}{6}}\sqrt{3} - 4i\sqrt{3}\, 2^{\frac{1}{3}} - 4\, 2^{\frac{1}{3}}}{ic_3 2^{\frac{5}{6}} + c_3 2^{\frac{5}{6}}\sqrt{3} + 2i\sqrt{3}\, 2^{\frac{1}{3}} + 2\, 2^{\frac{1}{3}}}$$

$$c_3 = \frac{2\sqrt{2}\,(1 + i\sqrt{3})}{\sqrt{3} + i}$$

Substituting c_3 found above in the general solution gives

$$y = \frac{8i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}\,x-2\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x^2+16i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}x+16\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x-32i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}+2i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}}{i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}x^2+\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x^2+2i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}\,x^2-8i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}x-8\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x+8i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}}$$

Summary
The solution(s) found are the following

$$y = \frac{8i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}\,x-2\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x^2+16i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}x+16\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x-32i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}+2i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}\,x^2}{i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}x^2+\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x^2+2i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}\,x^2-8i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}x-8\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x+8i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}\,x} \tag{1}$$

Verification of solutions

$$y = \frac{8i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}\,x-2\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x^2+16i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}x+16\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x-32i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}+2i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}\,x^2}{i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}x^2+\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x^2+2i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}\,x^2-8i\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}x-8\left(\frac{2+x}{x-4}\right)^{\frac{5}{6}}\sqrt{3}\,x+8i\sqrt{2}\left(\frac{x-4}{2+x}\right)^{\frac{2}{3}}\sqrt{3}\,x}$$

Verified OK. `{positive}`

1.30.6 Maple step by step solution

Let's solve

$[(x^2-2x-8)\,y'-y^2-y=-2, y(0)=0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2+y-2}=\frac{1}{x^2-2x-8}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2+y-2}dx=\int \frac{1}{x^2-2x-8}dx+c_1$

- Evaluate integral

  $\frac{\ln(y-1)}{3}-\frac{\ln(2+y)}{3}=\frac{\ln(x-4)}{6}-\frac{\ln(2+x)}{6}+c_1$

- Solve for y

  $\left\{y=-\frac{3\left(x\,\mathrm{e}^{6c_1}-4\,\mathrm{e}^{6c_1}-\sqrt{x^2\mathrm{e}^{6c_1}-2x\,\mathrm{e}^{6c_1}-8\,\mathrm{e}^{6c_1}}\right)}{x\,\mathrm{e}^{6c_1}-x-4\,\mathrm{e}^{6c_1}-2}+1, y=-\frac{3\left(x\,\mathrm{e}^{6c_1}-4\,\mathrm{e}^{6c_1}+\sqrt{x^2\mathrm{e}^{6c_1}-2x\,\mathrm{e}^{6c_1}-8\,\mathrm{e}^{6c_1}}\right)}{x\,\mathrm{e}^{6c_1}-x-4\,\mathrm{e}^{6c_1}-2}+1\right\}$

- Use initial condition $y(0)=0$

$$0 = -\frac{3\left(-4\,\mathrm{e}^{6c_1} - \sqrt{-8\,\mathrm{e}^{6c_1}}\right)}{-4\,\mathrm{e}^{6c_1} - 2} + 1$$

- Solve for c_1

$$c_1 = \frac{\mathrm{I}\pi}{6} - \frac{\ln(2)}{2}$$

- Substitute $c_1 = \frac{\mathrm{I}\pi}{6} - \frac{\ln(2)}{2}$ into general solution and simplify

$$y = \frac{2x+8-2\sqrt{-2x^2+4x+16}}{3x+4}$$

- Use initial condition $y(0) = 0$

$$0 = -\frac{3\left(-4\,\mathrm{e}^{6c_1} + \sqrt{-8\,\mathrm{e}^{6c_1}}\right)}{-4\,\mathrm{e}^{6c_1} - 2} + 1$$

- Solution does not satisfy initial condition
- Solution to the IVP

$$y = \frac{2x+8-2\sqrt{-2x^2+4x+16}}{3x+4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 31

```
dsolve([(x^2-2*x-8)*diff(y(x),x)=y(x)^2+y(x)-2,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{2x + 8 - 2\sqrt{-2x^2 + 4x + 16}}{3x + 4}$$

✓ Solution by Mathematica

Time used: 3.813 (sec). Leaf size: 48

```
DSolve[{(x^2-2*x-8)*y'[x]==y[x]^2+y[x]-2,y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4\sqrt{x+2} - 2\sqrt{8-2x}}{\sqrt{8-2x} + 4\sqrt{x+2}}$$

2 Exercise 6, page 25

2.1 problem 1

Internal problem ID [1900]
Internal file name [`OUTPUT/1901_Sunday_June_05_2022_02_38_18_AM_45819561/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y'x + y = -x}$$

2.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 1$$

Hence the ode is

$$y' - \frac{y}{x} = 1$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \frac{1}{x} \\
\mathrm{d}\left(\frac{y}{x}\right) &= \frac{1}{x}\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \frac{1}{x}\,\mathrm{d}x \\
\frac{y}{x} &= \ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = c_1 x + x \ln(x)
$$

which simplifies to

$$
y = x(\ln(x) + c_1)
$$

Summary

The solution(s) found are the following

$$
y = x(\ln(x) + c_1) \tag{1}
$$

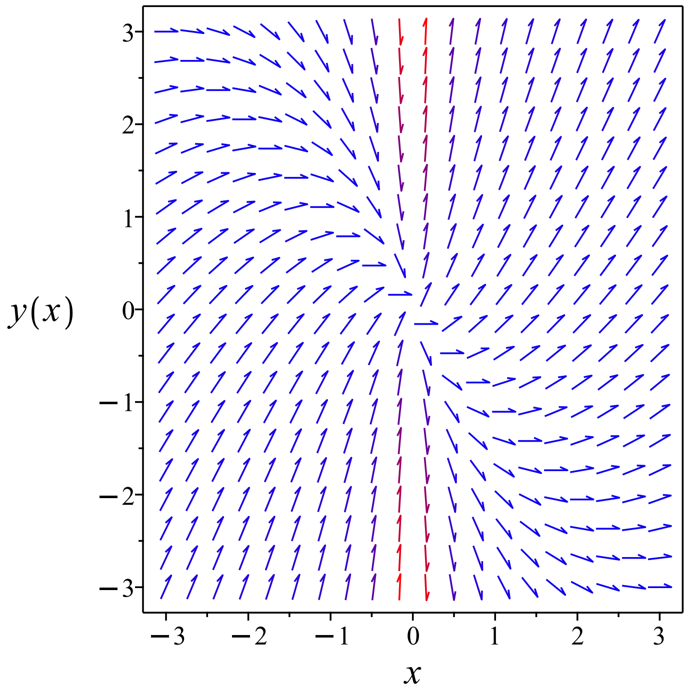


Figure 109: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_1)$$

Verified OK.

2.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-(u'(x)\, x + u(x))\, x + u(x)\, x = -x$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{1}{x}\, \mathrm{d}x \\ &= \ln(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(\ln(x) + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(\ln(x) + c_2) \tag{1}$$

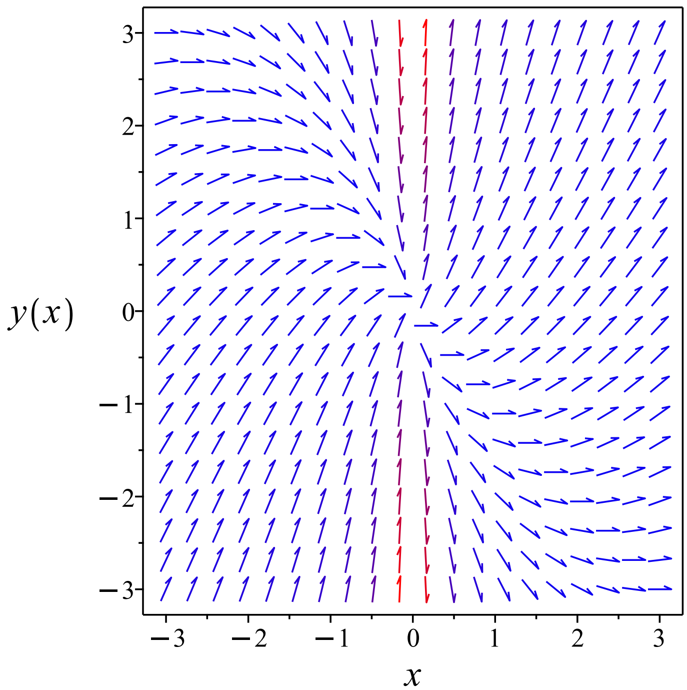


Figure 110: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_2)$$

Verified OK.

2.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 81: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x+y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \ln(x) + c_1$$

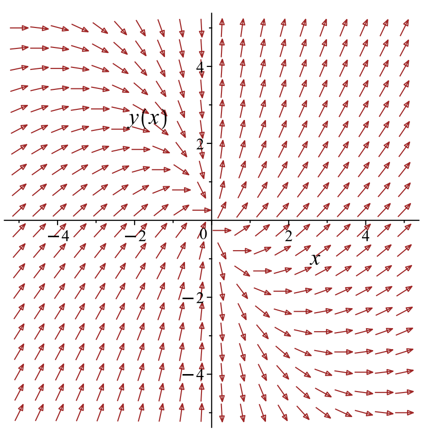
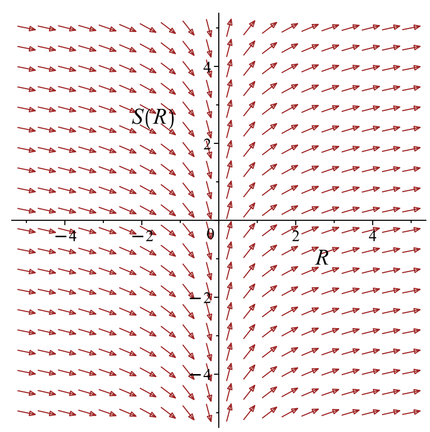
Which simplifies to

$$\frac{y}{x} = \ln(x) + c_1$$

Which gives

$$y = x(\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = x(\ln(x) + c_1) \tag{1}$$

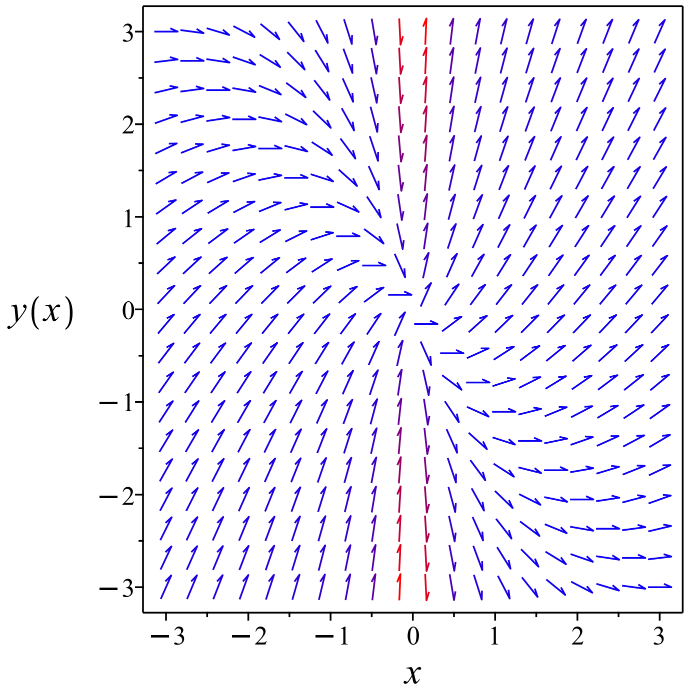


Figure 111: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_1)$$

Verified OK.

2.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-x) \, \mathrm{d}y &= (-x - y) \, \mathrm{d}x \\
(x + y) \, \mathrm{d}x + (-x) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x + y \\
N(x, y) &= -x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{x}((1) - (-1))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}(x+y)\\ &= \frac{x+y}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}(-x)\\ &= -\frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{x+y}{x^2}\right) + \left(-\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x+y}{x^2} \, \mathrm{d}x$$
$$\phi = \ln{(x)} - \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{x}$. Therefore equation (4) becomes

$$-\frac{1}{x} = -\frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln{(x)} - \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln{(x)} - \frac{y}{x}$$

The solution becomes

$$y = x(\ln(x) - c_1)$$

Summary

The solution(s) found are the following

$$y = x(\ln(x) - c_1) \tag{1}$$

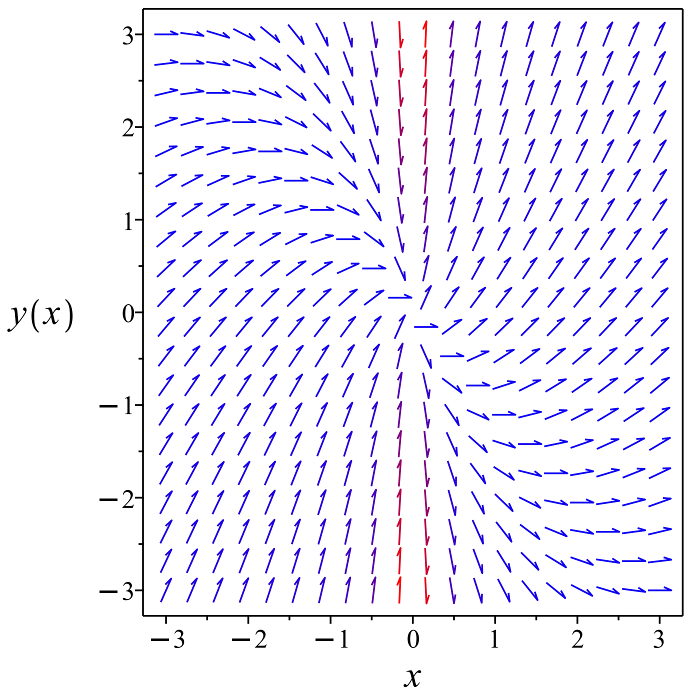


Figure 112: Slope field plot

Verification of solutions

$$y = x(\ln(x) - c_1)$$

Verified OK.

2.1.5 Maple step by step solution

Let's solve

$-y'x + y = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 1 + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{1}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(x+y(x)=x*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = (\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 12

```
DSolve[x+y[x]==x*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\log(x) + c_1)$$

2.2 problem 2

Internal problem ID [1901]
Internal file name [`OUTPUT/1902_Sunday_June_05_2022_02_38_21_AM_1244660/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(x+y)\,y'-y=-x}$$

2.2.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(x+u(x)\,x)\,(u'(x)\,x+u(x))-u(x)\,x=-x$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{u^2+1}{x\,(u+1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u+1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2+1}{u+1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u^2+1)}{2} + \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} + \arctan(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} + \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} + \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} + \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

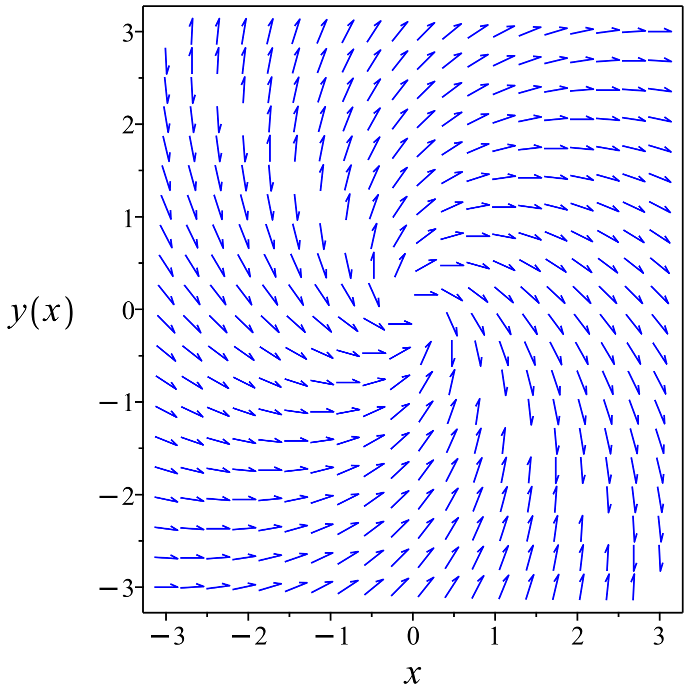


Figure 113: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}+\arctan\left(\frac{y}{x}\right)+\ln\left(x\right)-c_2=0$$

Verified OK.

2.2.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-x+y}{x+y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 + \frac{(-x+y)(b_3-a_2)}{x+y} - \frac{(-x+y)^2 a_3}{(x+y)^2}\\&- \left(-\frac{1}{x+y} - \frac{-x+y}{(x+y)^2}\right)(xa_2 + ya_3 + a_1)\\&- \left(\frac{1}{x+y} - \frac{-x+y}{(x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 - x^2b_2 - x^2b_3 + 2xya_2 + 2xya_3 + 2xyb_2 - 2xyb_3 - y^2a_2 + y^2a_3 + y^2b_2 + y^2b_3 - 2xb_1 + 2ya_1}{(x+y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}&x^2a_2 - x^2a_3 - x^2b_2 - x^2b_3 + 2xya_2 + 2xya_3 + 2xyb_2\\&- 2xyb_3 - y^2a_2 + y^2a_3 + y^2b_2 + y^2b_3 - 2xb_1 + 2ya_1 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&a_2v_1^2 + 2a_2v_1v_2 - a_2v_2^2 - a_3v_1^2 + 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2\\&+ 2b_2v_1v_2 + b_2v_2^2 - b_3v_1^2 - 2b_3v_1v_2 + b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(a_2 - a_3 - b_2 - b_3) v_1^2 + (2a_2 + 2a_3 + 2b_2 - 2b_3) v_1 v_2 \\ &- 2b_1 v_1 + (-a_2 + a_3 + b_2 + b_3) v_2^2 + 2a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}2a_1 &= 0 \\ -2b_1 &= 0 \\ -a_2 + a_3 + b_2 + b_3 &= 0 \\ a_2 - a_3 - b_2 - b_3 &= 0 \\ 2a_2 + 2a_3 + 2b_2 - 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x \\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(\frac{-x + y}{x + y}\right)(x) \\ &= \frac{x^2 + y^2}{x + y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+y^2}{x+y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln (x^2 + y^2)}{2} + \arctan \left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x + y}{x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x - y}{x^2 + y^2} \\ S_y &= \frac{x + y}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

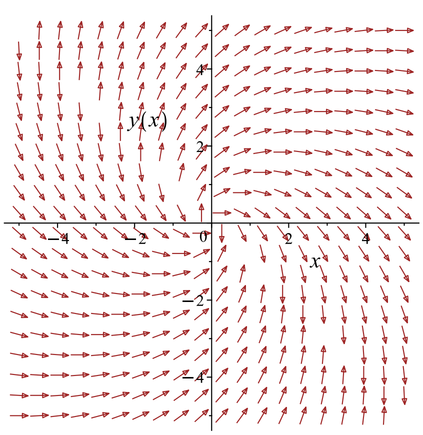
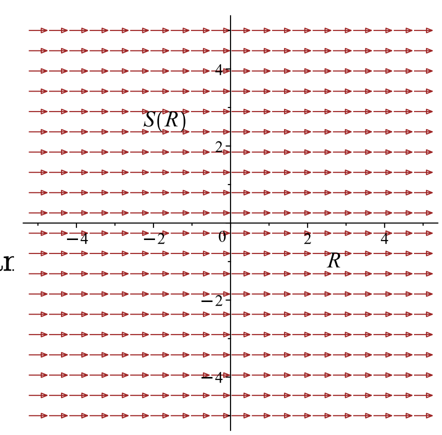
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2 + y^2)}{2} + \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2 + y^2)}{2} + \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+y}{x+y}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} + \arctan$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2}+\arctan\left(\frac{y}{x}\right)=c_1 \tag{1}$$

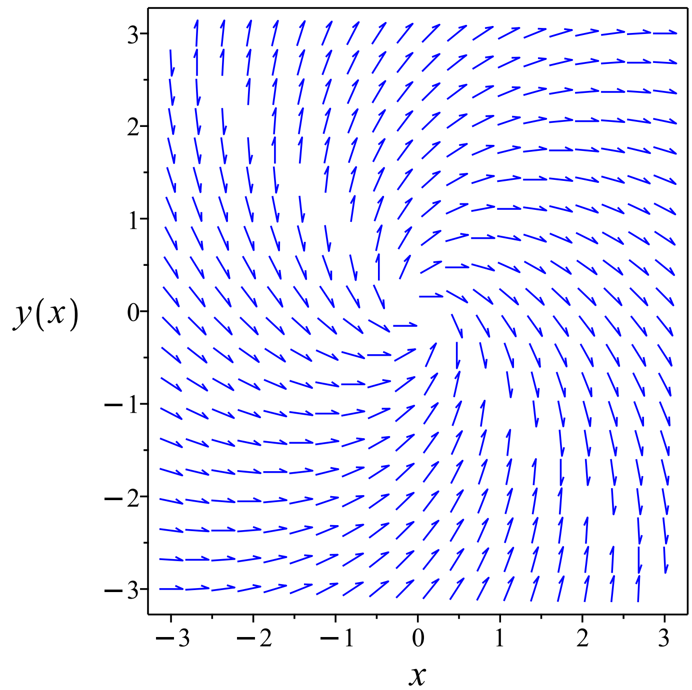


Figure 114: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2}+\arctan\left(\frac{y}{x}\right)=c_1$$

Verified OK.

2.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x+y)\,\mathrm{d}y &= (-x+y)\,\mathrm{d}x\\ (x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x-y\\ N(x,y) &= x+y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x-y)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x+y)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = x - y$ and $N = x + y$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{x-y}{x^2+y^2}\\ N &= \frac{x+y}{x^2+y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{x+y}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{x-y}{x^2+y^2}\right)\mathrm{d}x$$
$$\left(\frac{x-y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{x+y}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{x-y}{x^2+y^2}$$
$$N(x,y) = \frac{x+y}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x-y}{x^2+y^2}\right)\\ &= \frac{-x^2-2yx+y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x+y}{x^2+y^2}\right)\\ &= \frac{-x^2-2yx+y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{x-y}{x^2+y^2} \mathrm{d}x \\
\phi &= \frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{x}{y}\right) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{y}{x^2+y^2} + \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \qquad (4) \\
&= \frac{x+y}{x^2+y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x+y}{x^2+y^2}$. Therefore equation (4) becomes

$$
\frac{x+y}{x^2+y^2} = \frac{x+y}{x^2+y^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{x}{y}\right) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{x}{y}\right)
$$

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{x}{y}\right) = c_1 \tag{1}$$

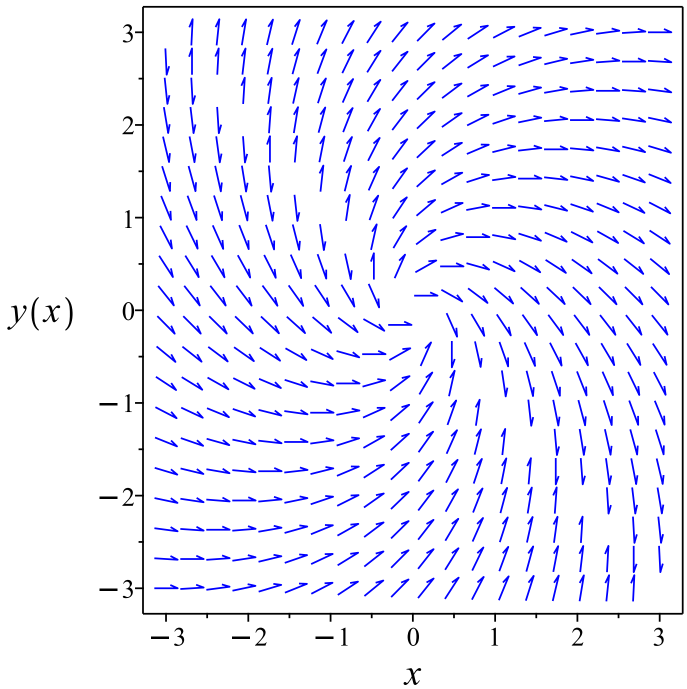


Figure 115: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{x}{y}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve((x+y(x))*diff(y(x),x)+x=y(x),y(x), singsol=all)
```

$$y(x) = \tan\left(\mathrm{RootOf}\left(2\_Z + \ln\left(\sec\left(\_Z\right)^{2}\right) + 2\ln\left(x\right) + 2c_{1}\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 34

```
DSolve[(x+y[x])*y'[x]+x==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\arctan\left(\frac{y(x)}{x}\right) + \frac{1}{2}\log\left(\frac{y(x)^2}{x^2} + 1\right) = -\log(x) + c_1, y(x)\right]$$

2.3 problem 3

Internal problem ID [1902]
Internal file name [`OUTPUT/1903_Sunday_June_05_2022_02_38_23_AM_99736484/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{-y + y'x - \sqrt{yx} = 0}$$

2.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y + \sqrt{yx}}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(y+\sqrt{yx})(b_3-a_2)}{x} - \frac{(y+\sqrt{yx})^2 a_3}{x^2} - \left(\frac{y}{2\sqrt{yx}\,x} - \frac{y+\sqrt{yx}}{x^2}\right)(xa_2+ya_3+a_1) - \frac{\left(1+\frac{x}{2\sqrt{yx}}\right)(xb_2+yb_3+b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2(yx)^{\frac{3}{2}} a_3 - x^2yb_3 + 3x\,y^2a_3 + x^3b_2 + x^2ya_2 - xya_1 + 2\sqrt{yx}\,xb_1 - 2\sqrt{yx}\,ya_1 + x^2b_1}{2\sqrt{yx}\,x^2} = 0$$

Setting the numerator to zero gives

$$-2(yx)^{\frac{3}{2}} a_3 - x^3b_2 - x^2ya_2 + x^2yb_3 - 3x\,y^2a_3 - 2\sqrt{yx}\,xb_1 + 2\sqrt{yx}\,ya_1 - x^2b_1 + xya_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-x^3b_2 - x^2ya_2 + x^2yb_3 - 2yx\sqrt{yx}\,a_3 - 3x\,y^2a_3 - x^2b_1 - 2\sqrt{yx}\,xb_1 + xya_1 + 2\sqrt{yx}\,ya_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{yx}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{yx} = v_3\}$$

The above PDE (6E) now becomes

$$-v_1^2v_2a_2 - 3v_1v_2^2a_3 - 2v_2v_1v_3a_3 - v_1^3b_2 + v_1^2v_2b_3 + v_1v_2a_1 + 2v_3v_2a_1 - v_1^2b_1 - 2v_3v_1b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-v_1^3 b_2 + (b_3 - a_2)\, v_1^2 v_2 - v_1^2 b_1 - 3 v_1 v_2^2 a_3 - 2 v_2 v_1 v_3 a_3 + v_1 v_2 a_1 - 2 v_3 v_1 b_1 + 2 v_3 v_2 a_1 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ 2a_1 &= 0 \\ -3a_3 &= 0 \\ -2a_3 &= 0 \\ -2b_1 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{y + \sqrt{yx}}{x}\right)(x) \\ &= -\sqrt{yx} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\sqrt{yx}} dy \end{aligned}$$

Which results in

$$S = -\frac{2y}{\sqrt{yx}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + \sqrt{yx}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\sqrt{y}}{x^{\frac{3}{2}}} \\ S_y &= -\frac{1}{\sqrt{y}\,\sqrt{x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sqrt{yx}}{\sqrt{y}\, x^{\frac{3}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln (R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2\sqrt{y}}{\sqrt{x}} = -\ln (x) + c_1$$

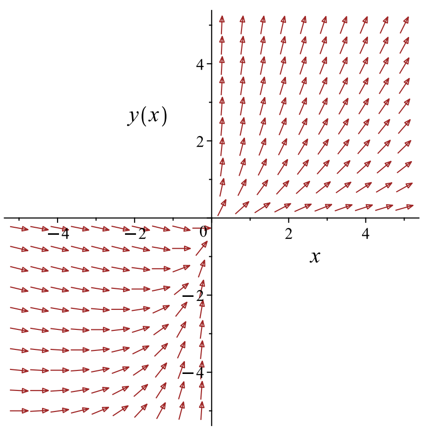
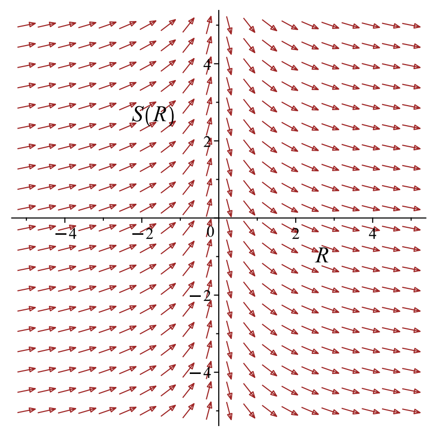
Which simplifies to

$$-\frac{2\sqrt{y}}{\sqrt{x}} = -\ln (x) + c_1$$

Which gives

$$y = \frac{x \ln (x)^2}{4} - \frac{x c_1 \ln (x)}{2} + \frac{c_1^2 x}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{yx}}{x}$ 	$R = x$ $S = -\frac{2\sqrt{y}}{\sqrt{x}}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x \ln(x)^2}{4} - \frac{x c_1 \ln(x)}{2} + \frac{c_1^2 x}{4} \tag{1}$$

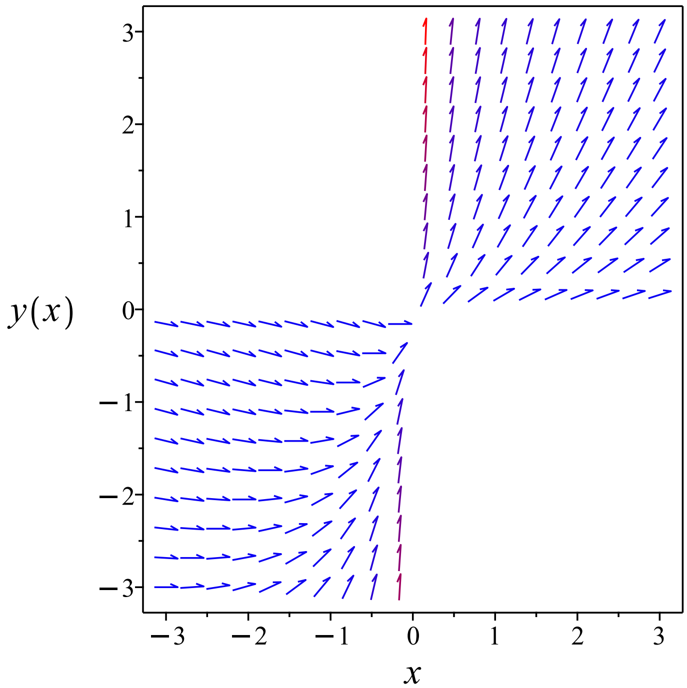


Figure 116: Slope field plot

Verification of solutions

$$y = \frac{x \ln(x)^2}{4} - \frac{x c_1 \ln(x)}{2} + \frac{c_1^2 x}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x*diff(y(x),x)-y(x)=sqrt(x*y(x)),y(x), singsol=all)
```

$$-\frac{y(x)}{\sqrt{xy\left(x\right)}}+\frac{\ln\left(x\right)}{2}-c_{1}=0$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 17

```
DSolve[x*y'[x]-y[x]==Sqrt[x*y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{4}x(\log(x)+c_1)^2$$

2.4 problem 4

Internal problem ID [1903]
Internal file name [`OUTPUT/1904_Sunday_June_05_2022_02_38_28_AM_8586825/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2x - y}{x + 4y} = 0}$$

2.4.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{2x - u(x)\,x}{x + 4u\,(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{2(2u^2 + u - 1)}{x\,(4u + 1)}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{2u^2+u-1}{4u+1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+u-1}{4u+1}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{2u^2+u-1}{4u+1}}\,du = \int -\frac{2}{x}\,dx$$
$$\ln\left(2u^2+u-1\right) = -2\ln(x) + c_2$$

Raising both side to exponential gives

$$2u^2+u-1 = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$2u^2+u-1 = \frac{c_3}{x^2}$$

Which simplifies to

$$2u(x)^2 + u(x) - 1 = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

The solution is

$$2u(x)^2 + u(x) - 1 = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{2y^2}{x^2} + \frac{y}{x} - 1 = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$
$$-\frac{(x+y)(x-2y)}{x^2} = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

Which simplifies to

$$-(x+y)(x-2y) = c_3\mathrm{e}^{c_2}$$

Summary

The solution(s) found are the following

$$-(x+y)(x-2y) = c_3\mathrm{e}^{c_2} \tag{1}$$

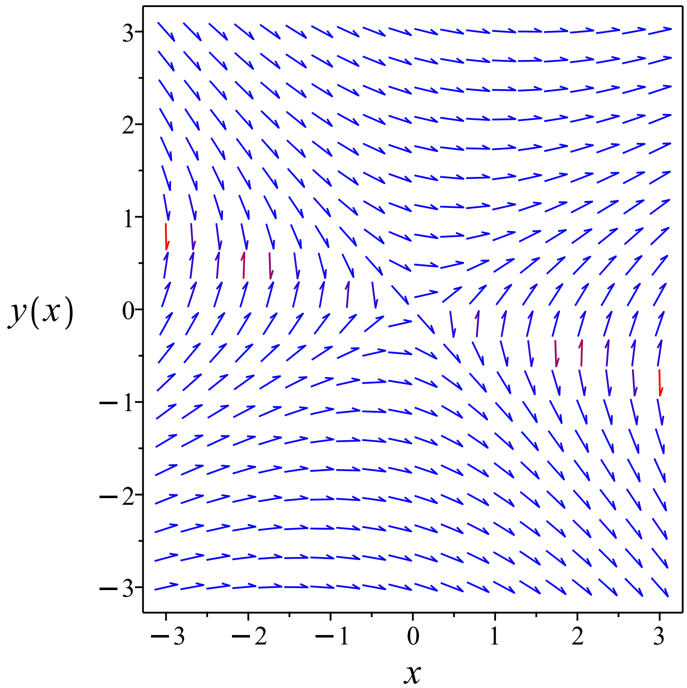


Figure 117: Slope field plot

Verification of solutions

$$-(x+y)(x-2y) = c_3 e^{c_2}$$

Verified OK.

2.4.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x-y}{x+4y} \tag{1}$$

Which becomes

$$(4y)\,dy = (-x)\,dy + (2x-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (2x-y)\,dx = d\left(x^2 - yx\right)$$

Hence (2) becomes

$$(4y)\,dy = d\left(x^2 - yx\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{x}{4} + \frac{\sqrt{9x^2 + 8c_1}}{4} + c_1$$
$$y = -\frac{x}{4} - \frac{\sqrt{9x^2 + 8c_1}}{4} + c_1$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{4} + \frac{\sqrt{9x^2 + 8c_1}}{4} + c_1 \tag{1}$$
$$y = -\frac{x}{4} - \frac{\sqrt{9x^2 + 8c_1}}{4} + c_1 \tag{2}$$

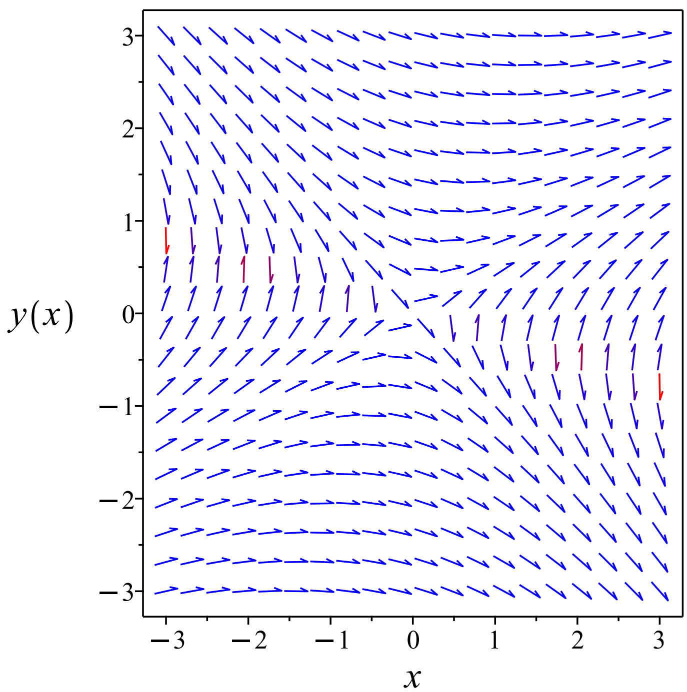


Figure 118: Slope field plot

Verification of solutions

$$y = -\frac{x}{4} + \frac{\sqrt{9x^2 + 8c_1}}{4} + c_1$$

Verified OK.

$$y = -\frac{x}{4} - \frac{\sqrt{9x^2 + 8c_1}}{4} + c_1$$

Verified OK.

2.4.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-2x + y}{x + 4y} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(-2x + y)(b_3 - a_2)}{x + 4y} - \frac{(-2x + y)^2 a_3}{(x + 4y)^2} \\ & - \left(\frac{2}{x + 4y} + \frac{-2x + y}{(x + 4y)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(-\frac{1}{x + 4y} + \frac{-8x + 4y}{(x + 4y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$
-\frac{2x^2a_2 + 4x^2a_3 - 10x^2b_2 - 2x^2b_3 + 16xya_2 - 4xya_3 - 8xyb_2 - 16xyb_3 - 4y^2a_2 + 10y^2a_3 - 16y^2b_2 + 4y^2}{(x+4y)^2}
$$
$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2x^2a_2 - 4x^2a_3 + 10x^2b_2 + 2x^2b_3 - 16xya_2 + 4xya_3 + 8xyb_2 \\
&+ 16xyb_3 + 4y^2a_2 - 10y^2a_3 + 16y^2b_2 - 4y^2b_3 + 9xb_1 - 9ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2a_2v_1^2 - 16a_2v_1v_2 + 4a_2v_2^2 - 4a_3v_1^2 + 4a_3v_1v_2 - 10a_3v_2^2 + 10b_2v_1^2 \\
&+ 8b_2v_1v_2 + 16b_2v_2^2 + 2b_3v_1^2 + 16b_3v_1v_2 - 4b_3v_2^2 - 9a_1v_2 + 9b_1v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-2a_2 - 4a_3 + 10b_2 + 2b_3)\, v_1^2 + (-16a_2 + 4a_3 + 8b_2 + 16b_3)\, v_1v_2 \\
&+ 9b_1v_1 + (4a_2 - 10a_3 + 16b_2 - 4b_3)\, v_2^2 - 9a_1v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-9a_1 &= 0 \\
9b_1 &= 0 \\
-16a_2 + 4a_3 + 8b_2 + 16b_3 &= 0 \\
-2a_2 - 4a_3 + 10b_2 + 2b_3 &= 0 \\
4a_2 - 10a_3 + 16b_2 - 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_2 + b_3 \\
a_3 &= 2b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{-2x + y}{x + 4y}\right)(x) \\
&= \frac{-2x^2 + 2yx + 4y^2}{x + 4y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x^2+2yx+4y^2}{x+4y}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln(-x^2 + yx + 2y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-2x + y}{x + 4y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x - y}{2(x + y)(x - 2y)} \\
S_y &= \frac{-x - 4y}{2(x + y)(x - 2y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

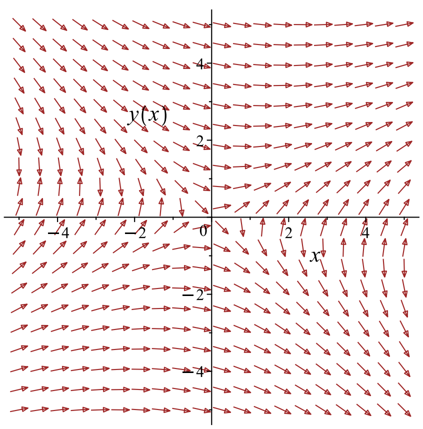
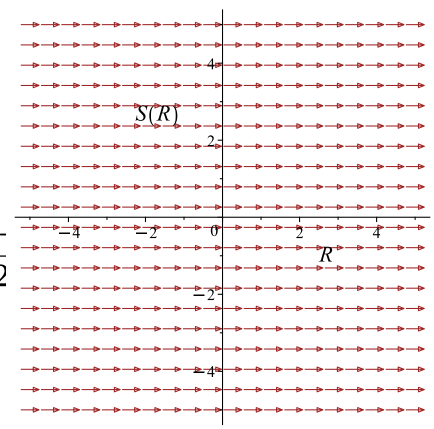
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(-x - y)}{2} + \frac{\ln(x - 2y)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(-x-y)}{2} + \frac{\ln(x-2y)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2x+y}{x+4y}$ 	$R = x$ $S = \frac{\ln(-x-y)}{2} + \frac{\ln(x -}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(-x-y)}{2} + \frac{\ln(x-2y)}{2} = c_1 \tag{1}$$

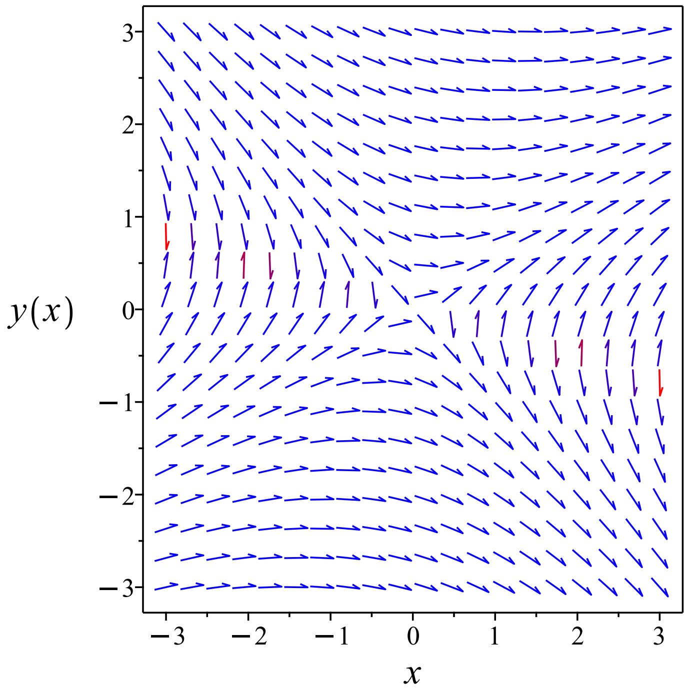


Figure 119: Slope field plot

Verification of solutions

$$\frac{\ln(-x-y)}{2} + \frac{\ln(x-2y)}{2} = c_1$$

Verified OK.

2.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x+4y)\,\mathrm{d}y = (2x-y)\,\mathrm{d}x$$
$$(-2x+y)\,\mathrm{d}x + (x+4y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -2x+y$$
$$N(x,y) = x+4y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x+y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x+4y)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2x + y \, \mathrm{d}x$$
$$\phi = -x(x - y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x + 4y$. Therefore equation (4) becomes

$$x + 4y = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (4y) \, \mathrm{d}y$$
$$f(y) = 2y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x(x - y) + 2y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x(x-y) + 2y^2$$

<u>Summary</u>
The solution(s) found are the following

$$-x(x-y) + 2y^2 = c_1 \tag{1}$$

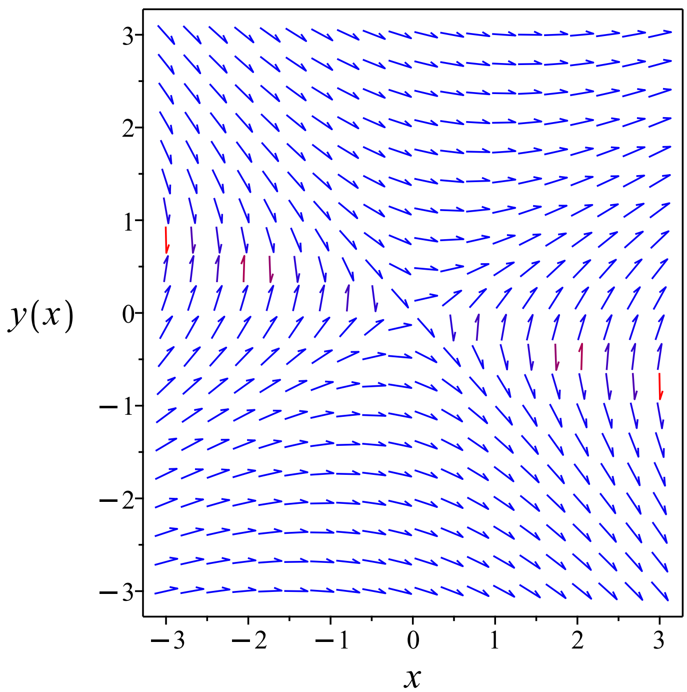


Figure 120: Slope field plot

<u>Verification of solutions</u>

$$-x(x-y) + 2y^2 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 53

```
dsolve(diff(y(x),x)=(2*x-y(x))/(x+4*y(x)),y(x), singsol=all)
```

$$y(x) = \frac{-c_1 x - \sqrt{9c_1^2 x^2 + 8}}{4c_1}$$
$$y(x) = \frac{-c_1 x + \sqrt{9c_1^2 x^2 + 8}}{4c_1}$$

✓ Solution by Mathematica

Time used: 0.472 (sec). Leaf size: 101

```
DSolve[y'[x]==(2*x-y[x])/(x+4*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-x - \sqrt{9x^2 + 8e^{c_1}}\right)$$
$$y(x) \to \frac{1}{4}\left(-x + \sqrt{9x^2 + 8e^{c_1}}\right)$$
$$y(x) \to \frac{1}{4}\left(-3\sqrt{x^2} - x\right)$$
$$y(x) \to \frac{1}{4}\left(3\sqrt{x^2} - x\right)$$

2.5 problem 5

Internal problem ID [1904]
Internal file name [`OUTPUT/1905_Sunday_June_05_2022_02_38_31_AM_44048697/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{-y + y'x - \sqrt{x^2 - y^2} = 0}$$

2.5.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y + \sqrt{x^2 - y^2}}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(y+\sqrt{x^2-y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{x^2-y^2}\right)^2 a_3}{x^2} - \left(\frac{1}{\sqrt{x^2-y^2}} - \frac{y+\sqrt{x^2-y^2}}{x^2}\right)(xa_2+ya_3+a_1) - \frac{\left(1-\frac{y}{\sqrt{x^2-y^2}}\right)(xb_2+yb_3+b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(x^2-y^2)^{\frac{3}{2}} a_3 + x^3a_2 - x^3b_3 + 2x^2ya_3 - x^2yb_2 - y^3a_3 + \sqrt{x^2-y^2}\,xb_1 - \sqrt{x^2-y^2}\,ya_1 - xyb_1 + y^2a_1}{\sqrt{x^2-y^2}\,x^2} = 0$$

Setting the numerator to zero gives

$$-(x^2-y^2)^{\frac{3}{2}} a_3 - x^3a_2 + x^3b_3 - 2x^2ya_3 + x^2yb_2 + y^3a_3 - \sqrt{x^2-y^2}\,xb_1 + \sqrt{x^2-y^2}\,ya_1 + xyb_1 - y^2a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-(x^2-y^2)^{\frac{3}{2}} a_3 + (x^2-y^2)\,xb_3 - (x^2-y^2)\,ya_3 - x^3a_2 - x^2ya_3 + x^2yb_2 + x\,y^2b_3 + (x^2-y^2)\,a_1 - \sqrt{x^2-y^2}\,xb_1 + \sqrt{x^2-y^2}\,ya_1 - x^2a_1 + xyb_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-x^3a_2 + x^3b_3 - x^2\sqrt{x^2-y^2}\,a_3 - 2x^2ya_3 + x^2yb_2 + \sqrt{x^2-y^2}\,y^2a_3 + y^3a_3 - \sqrt{x^2-y^2}\,xb_1 + xyb_1 + \sqrt{x^2-y^2}\,ya_1 - y^2a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2-y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2 - y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$-v_1^3 a_2 - 2v_1^2 v_2 a_3 - v_1^2 v_3 a_3 + v_2^3 a_3 + v_3 v_2^2 a_3 + v_1^2 v_2 b_2 + v_1^3 b_3 - v_2^2 a_1 + v_3 v_2 a_1 + v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$(b_3 - a_2) v_1^3 + (-2a_3 + b_2) v_1^2 v_2 - v_1^2 v_3 a_3 + v_1 v_2 b_1 - v_3 v_1 b_1 + v_2^3 a_3 + v_3 v_2^2 a_3 - v_2^2 a_1 + v_3 v_2 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2a_3 + b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y+\sqrt{x^2-y^2}}{x}\right)(x) \\
&= -\sqrt{x^2-y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{x^2-y^2}} dy
\end{aligned}
$$

Which results in

$$
S = -\arctan\left(\frac{y}{\sqrt{x^2-y^2}}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + \sqrt{x^2 - y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{\sqrt{x^2 - y^2}\, x} \\ S_y &= -\frac{1}{\sqrt{x^2 - y^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\arctan\left(\frac{y}{\sqrt{x^2 - y^2}}\right) = -\ln(x) + c_1$$

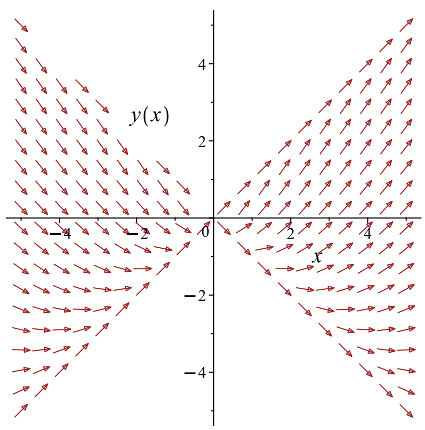
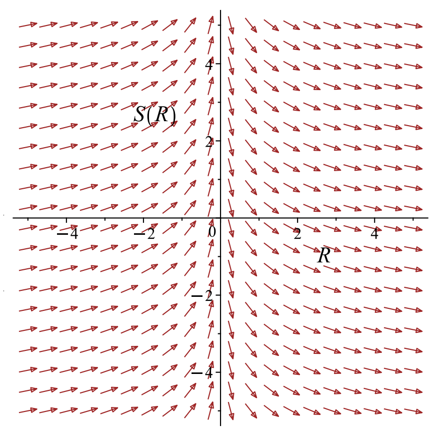
Which simplifies to

$$-\arctan\left(\frac{y}{\sqrt{x^2 - y^2}}\right) = -\ln(x) + c_1$$

Which gives

$$y = -\tan(-\ln(x) + c_1)\sqrt{\frac{x^2}{\tan(-\ln(x) + c_1)^2 + 1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{x^2-y^2}}{x}$ 	$R = x$ $S = -\arctan\left(\frac{y}{\sqrt{x^2-y^2}}\right)$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\tan\left(-\ln\left(x\right) + c_1\right)\sqrt{\frac{x^2}{\tan\left(-\ln\left(x\right) + c_1\right)^2 + 1}} \tag{1}$$

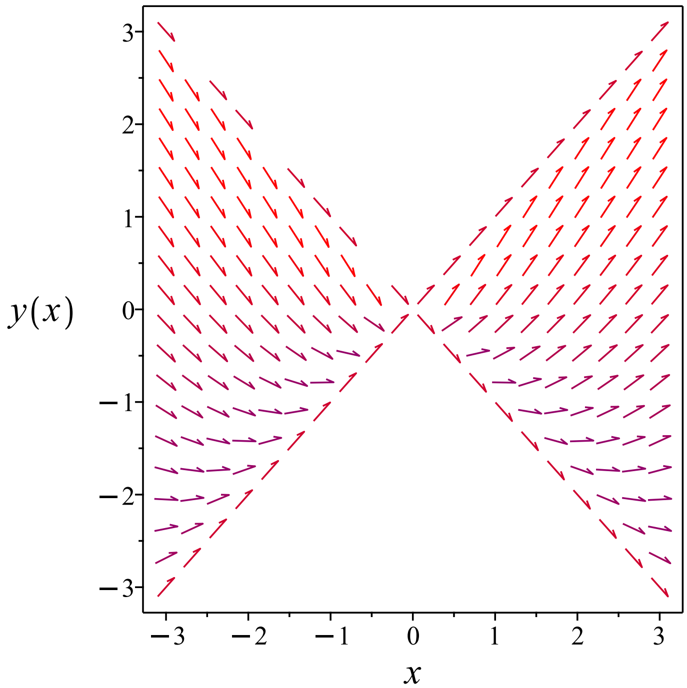


Figure 121: Slope field plot

Verification of solutions

$$y = -\tan\left(-\ln\left(x\right)+c_1\right)\sqrt{\frac{x^2}{\tan\left(-\ln\left(x\right)+c_1\right)^2+1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x)-y(x)=sqrt(x^2-y(x)^2),y(x), singsol=all)
```

$$-\arctan\left(\frac{y(x)}{\sqrt{x^2-y(x)^2}}\right)+\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.269 (sec). Leaf size: 18

```
DSolve[x*y'[x]-y[x]==Sqrt[x^2-y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -x\cosh(i\log(x)+c_1)$$

2.6 problem 6

Internal problem ID [1905]
Internal file name [`OUTPUT/1906_Sunday_June_05_2022_02_38_35_AM_24173088/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{yy' - 2y = -x}$$

2.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x(u'(x)\,x + u(x)) - 2u(x)\,x = -x$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{(u-1)^2}{ux}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{(u-1)^2}{u}$. Integrating both sides gives

$$\frac{1}{\frac{(u-1)^2}{u}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{(u-1)^2}{u}}\,du = \int -\frac{1}{x}\,dx$$

$$\ln\left(u-1\right) - \frac{1}{u-1} = -\ln\left(x\right) + c_2$$

The solution is

$$\ln\left(u(x)-1\right) - \frac{1}{u\left(x\right)-1} + \ln\left(x\right) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{y}{x}-1\right) - \frac{1}{\frac{y}{x}-1} + \ln\left(x\right) - c_2 = 0$$

$$\ln\left(\frac{-x+y}{x}\right) + \frac{x}{x-y} + \ln\left(x\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{-x+y}{x}\right) + \frac{x}{x-y} + \ln\left(x\right) - c_2 = 0 \tag{1}$$

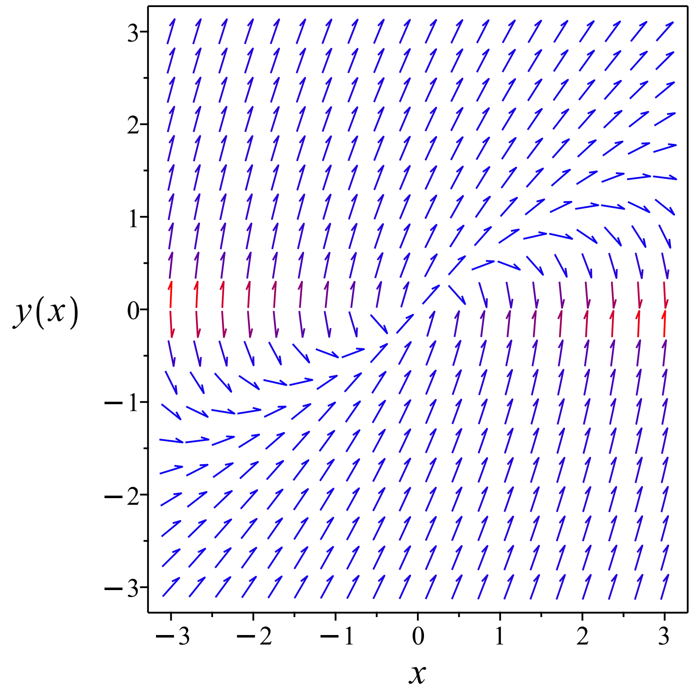


Figure 122: Slope field plot

Verification of solutions

$$\ln\left(\frac{-x+y}{x}\right)+\frac{x}{x-y}+\ln(x)-c_2=0$$

Verified OK.

2.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-x+2y}{y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-x+2y)(b_3-a_2)}{y} - \frac{(-x+2y)^2 a_3}{y^2} + \frac{xa_2 + ya_3 + a_1}{y} - \left(\frac{2}{y} - \frac{-x+2y}{y^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_3 + x^2b_2 - 2xya_2 - 4xya_3 + 2xyb_3 + 2y^2a_2 + 3y^2a_3 - b_2y^2 - 2y^2b_3 + xb_1 - ya_1}{y^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_3 - x^2b_2 + 2xya_2 + 4xya_3 - 2xyb_3 - 2y^2a_2 - 3y^2a_3 + b_2y^2 + 2y^2b_3 - xb_1 + ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 - 3a_3v_2^2 - b_2v_1^2 + b_2v_2^2 - 2b_3v_1v_2 + 2b_3v_2^2 + a_1v_2 - b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_3-b_2)\,v_1^2+(2a_2+4a_3-2b_3)\,v_1v_2-b_1v_1+(-2a_2-3a_3+b_2+2b_3)\,v_2^2+a_1v_2=0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0\\ -b_1 &= 0\\ -a_3-b_2 &= 0\\ 2a_2+4a_3-2b_3 &= 0\\ -2a_2-3a_3+b_2+2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0\\ a_2 &= 2b_2+b_3\\ a_3 &= -b_2\\ b_1 &= 0\\ b_2 &= b_2\\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x\\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi=0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta-\omega(x,y)\,\xi\\ &= y-\left(\frac{-x+2y}{y}\right)(x)\\ &= \frac{x^2-2yx+y^2}{y}\\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2 - 2yx + y^2}{y}} dy \end{aligned}$$

Which results in

$$S = \ln(-x + y) - \frac{x}{-x + y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x + 2y}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x - 2y}{(x - y)^2} \\ S_y &= \frac{y}{(x - y)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x-y)\ln(-x+y)+x}{x-y} = c_1$$

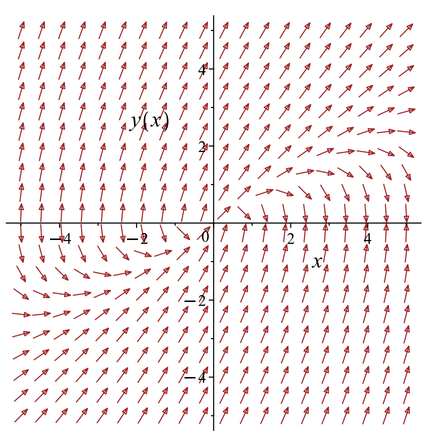
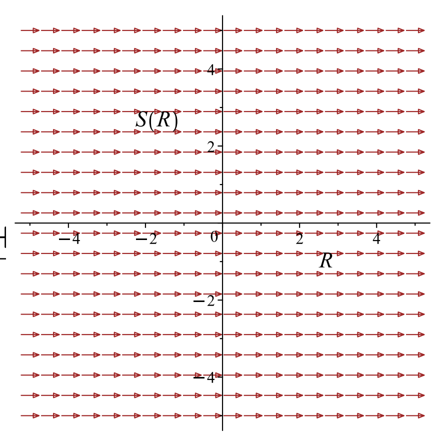
Which simplifies to

$$\frac{(x-y)\ln(-x+y)+x}{x-y} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(x\,\mathrm{e}^{-c_1}\right)+c_1} + x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+2y}{y}$ 	$R = x$ $S = \frac{(x-y)\ln(-x+y)+}{x-y}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = e^{\operatorname{LambertW}(x\,e^{-c_1})+c_1} + x \tag{1}$$

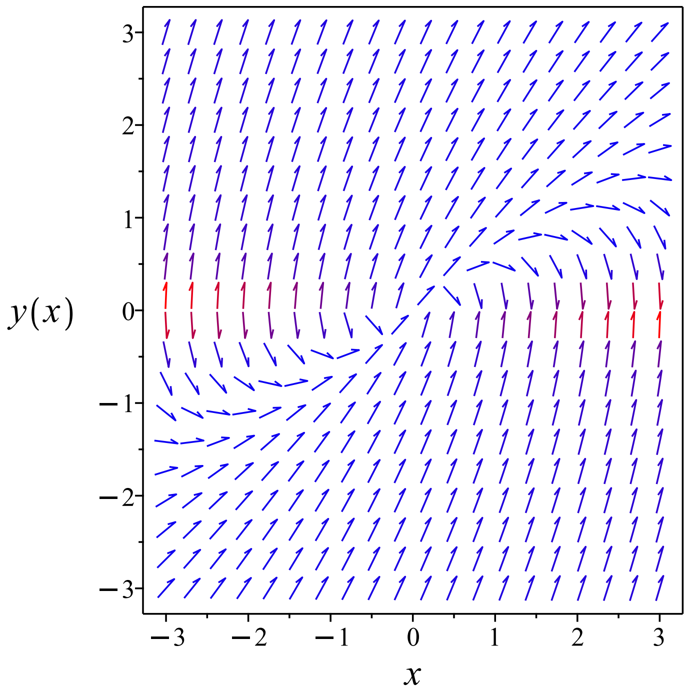


Figure 123: Slope field plot

Verification of solutions

$$y = e^{\operatorname{LambertW}(x\,e^{-c_1})+c_1} + x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(x+y(x)*diff(y(x),x)=2*y(x),y(x), singsol=all)
```

$$y(x) = \frac{x(\text{LambertW}(c_1 x) + 1)}{\text{LambertW}(c_1 x)}$$

✓ Solution by Mathematica

Time used: 0.118 (sec). Leaf size: 33

```
DSolve[x+y[x]*y'[x]==2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\log\left(\frac{y(x)}{x} - 1\right) - \frac{1}{\frac{y(x)}{x} - 1} = -\log(x) + c_1, y(x)\right]$$

2.7 problem 7

Internal problem ID [1906]
Internal file name [`OUTPUT/1907_Sunday_June_05_2022_02_38_39_AM_45525428/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y'x - y + \sqrt{-x^2 + y^2} = 0}$$

2.7.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y - \sqrt{-x^2 + y^2}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(y-\sqrt{-x^2+y^2}\right)(b_3-a_2)}{x} - \frac{\left(y-\sqrt{-x^2+y^2}\right)^2 a_3}{x^2} \\ &- \left(\frac{1}{\sqrt{-x^2+y^2}} - \frac{y-\sqrt{-x^2+y^2}}{x^2}\right)(xa_2+ya_3+a_1) \\ &- \frac{\left(1-\frac{y}{\sqrt{-x^2+y^2}}\right)(xb_2+yb_3+b_1)}{x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\left(-x^2+y^2\right)^{\frac{3}{2}} a_3 + x^3a_2 - x^3b_3 + 2x^2ya_3 - x^2yb_2 - y^3a_3 + \sqrt{-x^2+y^2}\,xb_1 - \sqrt{-x^2+y^2}\,ya_1 - xyb_1 + y^2}{\sqrt{-x^2+y^2}\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-\left(-x^2+y^2\right)^{\frac{3}{2}} a_3 - x^3a_2 + x^3b_3 - 2x^2ya_3 + x^2yb_2 + y^3a_3 \\ &- \sqrt{-x^2+y^2}\,xb_1 + \sqrt{-x^2+y^2}\,ya_1 + xyb_1 - y^2a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-\left(-x^2+y^2\right)^{\frac{3}{2}} a_3 - \left(-x^2+y^2\right)xb_3 + \left(-x^2+y^2\right)ya_3 - x^3a_2 - x^2ya_3 + x^2yb_2 \\ &+ x\,y^2b_3 - \left(-x^2+y^2\right)a_1 - \sqrt{-x^2+y^2}\,xb_1 + \sqrt{-x^2+y^2}\,ya_1 - x^2a_1 + xyb_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &-x^3a_2 + x^3b_3 + x^2\sqrt{-x^2+y^2}\,a_3 - 2x^2ya_3 + x^2yb_2 - \sqrt{-x^2+y^2}\,y^2a_3 \\ &+ y^3a_3 - \sqrt{-x^2+y^2}\,xb_1 + xyb_1 + \sqrt{-x^2+y^2}\,ya_1 - y^2a_1 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$-v_1^3 a_2 - 2v_1^2 v_2 a_3 + v_1^2 v_3 a_3 + v_2^3 a_3 - v_3 v_2^2 a_3 + v_1^2 v_2 b_2 + v_1^3 b_3 - v_2^2 a_1 + v_3 v_2 a_1 + v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$(b_3 - a_2) v_1^3 + (-2a_3 + b_2) v_1^2 v_2 + v_1^2 v_3 a_3 + v_1 v_2 b_1 - v_3 v_1 b_1 + v_2^3 a_3 - v_3 v_2^2 a_3 - v_2^2 a_1 + v_3 v_2 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2a_3 + b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{y - \sqrt{-x^2+y^2}}{x}\right)(x) \\ &= \sqrt{-x^2+y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{-x^2+y^2}} dy\end{aligned}$$

Which results in

$$S = \ln\left(\sqrt{-x^2+y^2} + y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y - \sqrt{-x^2 + y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{\sqrt{-x^2 + y^2}\,\left(\sqrt{-x^2 + y^2} + y\right)} \\ S_y &= \frac{1}{\sqrt{-x^2 + y^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(\sqrt{-x^2 + y^2} + y\right) = c_1$$

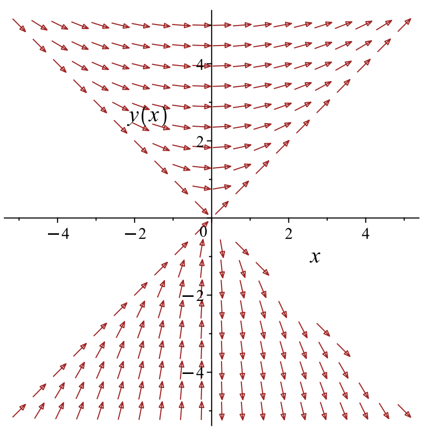
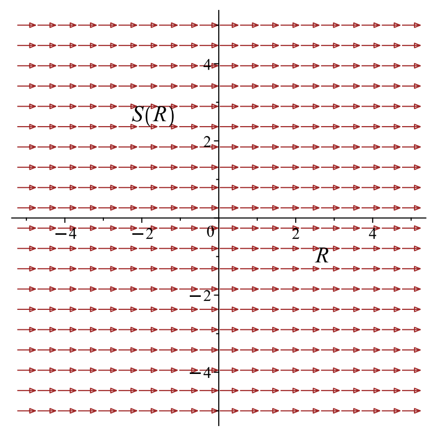
Which simplifies to

$$\ln\left(\sqrt{-x^2 + y^2} + y\right) = c_1$$

Which gives

$$y = \frac{\left(e^{2c_1} + x^2\right) e^{-c_1}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y-\sqrt{-x^2+y^2}}{x}$
</td><td>$R = x$
$S = \ln\left(\sqrt{-x^2+y^2}+y\right)$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{\left(e^{2c_1} + x^2\right) e^{-c_1}}{2} \tag{1}$$

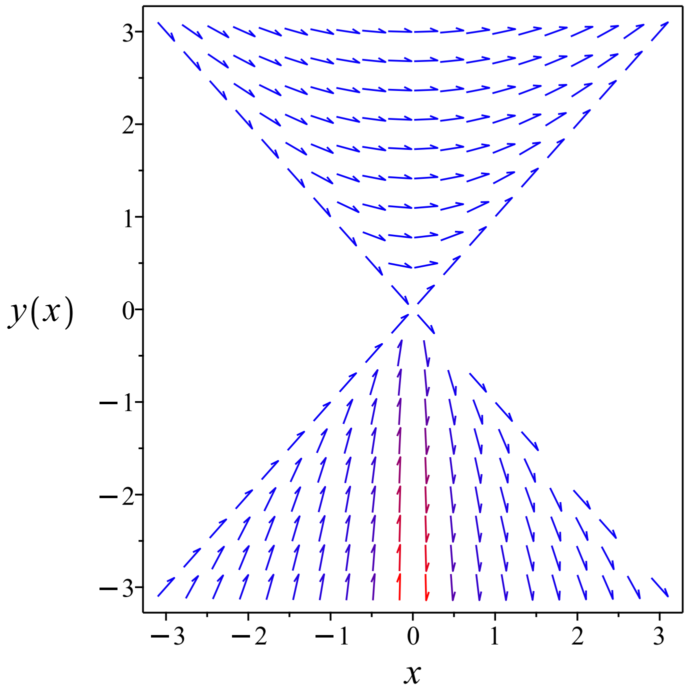


Figure 124: Slope field plot

Verification of solutions

$$y = \frac{\left(e^{2c_1} + x^2\right) e^{-c_1}}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x*diff(y(x),x)-y(x)+sqrt(y(x)^2-x^2)=0,y(x), singsol=all)
```

$$y(x)+\sqrt{y(x)^{2}-x^{2}}-c_{1}=0$$

✓ Solution by Mathematica

Time used: 0.362 (sec). Leaf size: 16

```
DSolve[x*y'[x]-y[x]+Sqrt[y[x]^2-x^2]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -x\cosh(-\log(x)+c_{1})$$

2.8 problem 8

Internal problem ID [1907]
Internal file name [`OUTPUT/1908_Sunday_June_05_2022_02_38_42_AM_35374353/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - y'xy = -x^2}$$

2.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 - (u'(x)\,x + u(x))\,x^2u(x) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{1}{ux} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{1}{u}}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{1}{u}}\,du = \int \frac{1}{x}\,dx$$
$$\frac{u^2}{2} = \ln(x) + c_2$$

The solution is

$$\frac{u(x)^2}{2} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{2x^2} - \ln(x) - c_2 = 0$$
$$\frac{y^2}{2x^2} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2x^2} - \ln(x) - c_2 = 0 \tag{1}$$

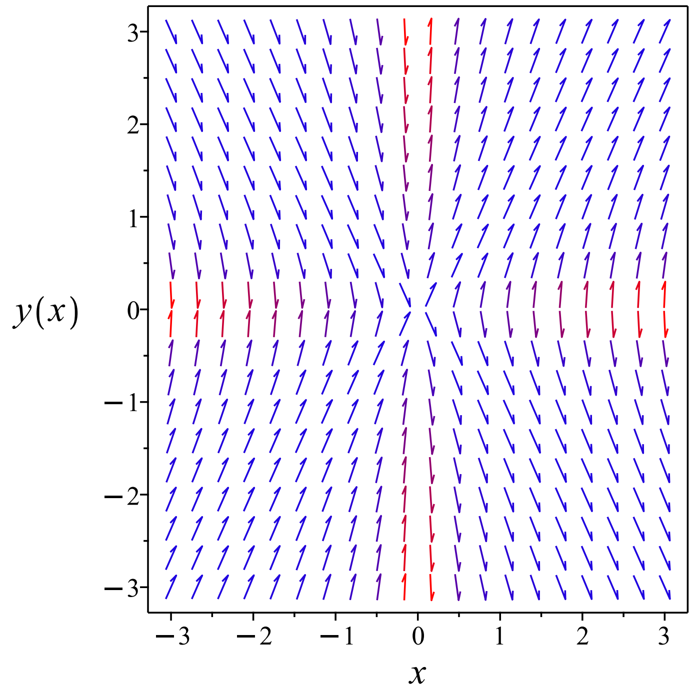


Figure 125: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} - \ln(x) - c_2 = 0$$

Verified OK.

2.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y^2}{xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 84: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x^2}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + y^2}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{x^3} \\ S_y &= \frac{y}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

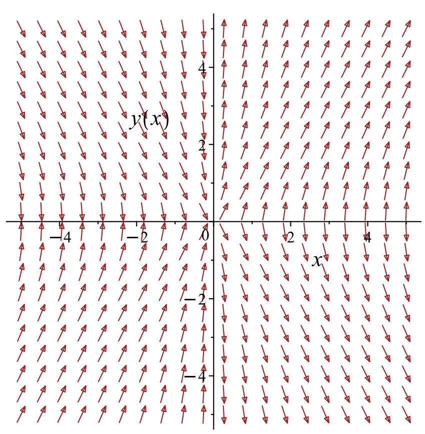
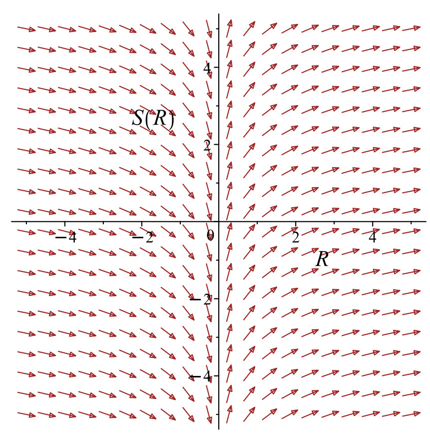
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^2} = \ln(x) + c_1$$

Which simplifies to

$$\frac{y^2}{2x^2} = \ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$ 	$R = x$ $S = \frac{y^2}{2x^2}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x^2} = \ln(x) + c_1 \tag{1}$$

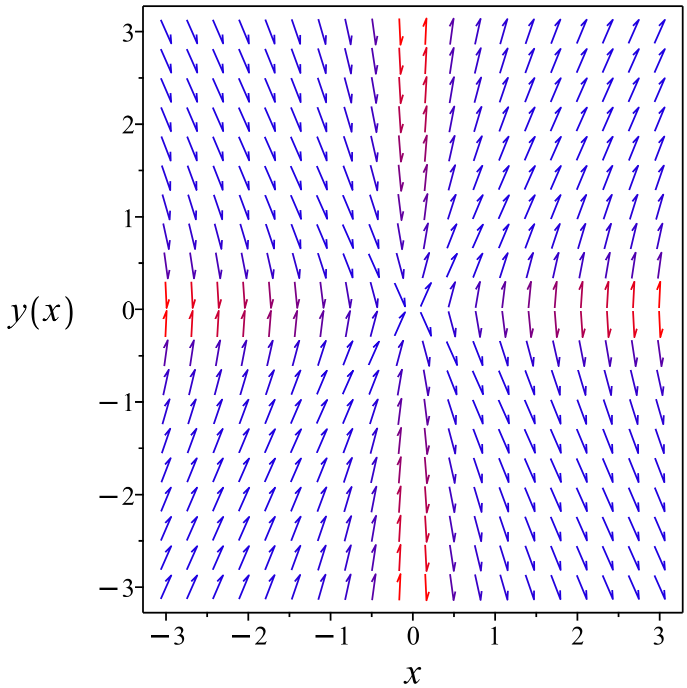


Figure 126: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} = \ln(x) + c_1$$

Verified OK.

2.8.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + y^2}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{x} \\
f_1(x) &= x \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{y^2}{x} + x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{w(x)}{x} + x \\
w' &= \frac{2w}{x} + 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= 2x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = 2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(2x) \\ \frac{d}{dx}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)(2x) \\ d\left(\frac{w}{x^2}\right) &= \left(\frac{2}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int \frac{2}{x}\, dx \\ \frac{w}{x^2} &= 2\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = 2\ln(x)\, x^2 + c_1 x^2$$

which simplifies to

$$w(x) = x^2(2\ln(x) + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x^2(2\ln(x) + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{2\ln(x) + c_1}\, x \\ y(x) &= -\sqrt{2\ln(x) + c_1}\, x\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2\ln(x) + c_1}\, x \tag{1}$$

$$y = -\sqrt{2\ln(x) + c_1}\, x \tag{2}$$

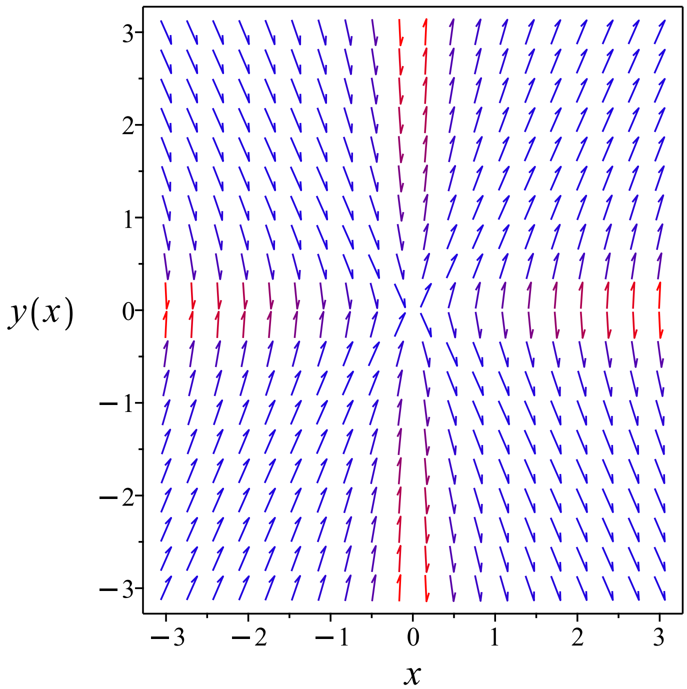


Figure 127: Slope field plot

Verification of solutions

$$y = \sqrt{2\ln(x) + c_1}\, x$$

Verified OK.

$$y = -\sqrt{2\ln(x) + c_1}\, x$$

Verified OK.

2.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-yx)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x\\ \left(x^2 + y^2\right)\mathrm{d}x + (-yx)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^2 + y^2\\ N(x,y) &= -yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2+y^2\right)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-yx)\\ &= -y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{yx}((2y)-(-y))\\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{3}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)}\\ &= \frac{1}{x^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^3}\left(x^2+y^2\right)\\ &= \frac{x^2+y^2}{x^3}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3}(-yx) \\
&= -\frac{y}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2+y^2}{x^3}\right) + \left(-\frac{y}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2+y^2}{x^3}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \ln(x) - \frac{y^2}{2x^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{y}{x^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{x^2}$. Therefore equation (4) becomes

$$
-\frac{y}{x^2} = -\frac{y}{x^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \frac{y^2}{2x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{y^2}{2x^2}$$

<u>Summary</u>

The solution(s) found are the following

$$-\frac{y^2}{2x^2} + \ln(x) = c_1 \tag{1}$$

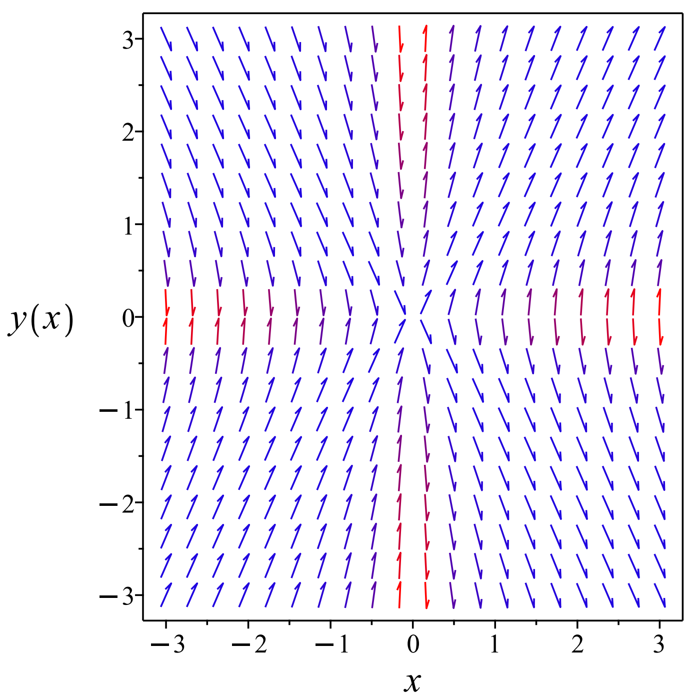


Figure 128: Slope field plot

Verification of solutions

$$-\frac{y^2}{2x^2} + \ln(x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve((x^2+y(x)^2)=x*y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \sqrt{2\ln(x) + c_1}\, x$$
$$y(x) = -\sqrt{2\ln(x) + c_1}\, x$$

✓ Solution by Mathematica

Time used: 0.17 (sec). Leaf size: 36

```
DSolve[(x^2+y[x]^2)==x*y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\sqrt{2\log(x) + c_1}$$
$$y(x) \to x\sqrt{2\log(x) + c_1}$$

2.9 problem 9

Internal problem ID [1908]
Internal file name [`OUTPUT/1909_Sunday_June_05_2022_02_38_45_AM_61774885/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{(-x^2 + yx)\, y' - y^2 = 0}$$

2.9.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$(-x^2 + u(x)\, x^2)\, (u'(x)\, x + u(x)) - u(x)^2\, x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u}{x\,(u-1)} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u}{u-1}}\, du = \frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u}{u-1}}\, du = \int \frac{1}{x}\, dx$$

$$u - \ln(u) = \ln(x) + c_2$$

The solution is

$$u(x) - \ln(u(x)) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

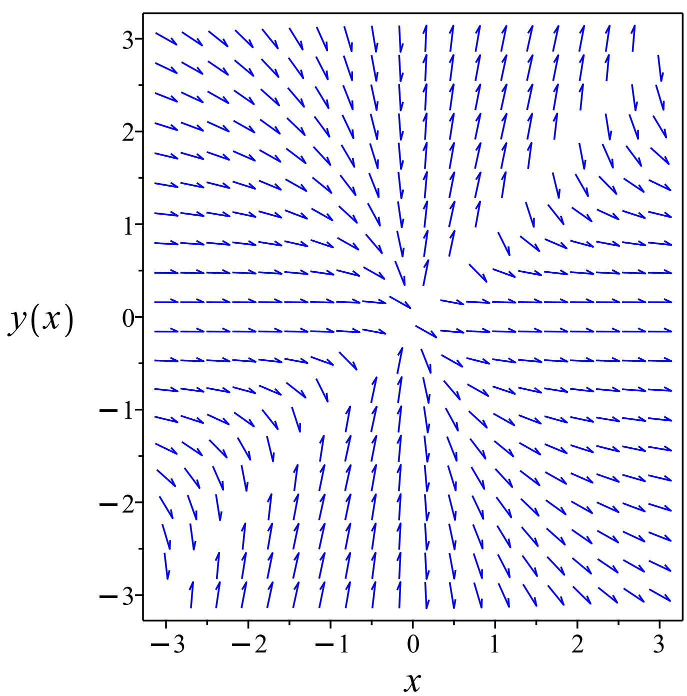


Figure 129: Slope field plot

Verification of solutions

$$\frac{y}{x}-\ln\left(\frac{y}{x}\right)-\ln\left(x\right)-c_2=0$$

Verified OK.

2.9.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{y^2}{x\left(-x+y\right)}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2+\frac{y^2(b_3-a_2)}{x\left(-x+y\right)}-\frac{y^4a_3}{x^2\left(-x+y\right)^2}\\&-\left(-\frac{y^2}{x^2\left(-x+y\right)}+\frac{y^2}{x\left(-x+y\right)^2}\right)\left(xa_2+ya_3+a_1\right)\\&-\left(\frac{2y}{x\left(-x+y\right)}-\frac{y^2}{x\left(-x+y\right)^2}\right)\left(xb_2+yb_3+b_1\right)=0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^4b_2-x^2y^2a_2+x^2y^2b_3-2x\,y^3a_3+2x^2yb_1-2x\,y^2a_1-x\,y^2b_1+y^3a_1}{x^2\left(x-y\right)^2}=0$$

Setting the numerator to zero gives

$$x^4 b_2 - x^2 y^2 a_2 + x^2 y^2 b_3 - 2x\,y^3 a_3 + 2x^2 y b_1 - 2x\,y^2 a_1 - x\,y^2 b_1 + y^3 a_1 = 0 \quad \text{(6E)}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_1^2 v_2^2 - 2a_3 v_1 v_2^3 + b_2 v_1^4 + b_3 v_1^2 v_2^2 - 2a_1 v_1 v_2^2 + a_1 v_2^3 + 2b_1 v_1^2 v_2 - b_1 v_1 v_2^2 = 0 \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2 v_1^4 + (b_3 - a_2)\, v_1^2 v_2^2 + 2b_1 v_1^2 v_2 - 2a_3 v_1 v_2^3 + (-2a_1 - b_1)\, v_1 v_2^2 + a_1 v_2^3 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
b_2 &= 0 \\
-2a_3 &= 0 \\
2b_1 &= 0 \\
-2a_1 - b_1 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y^2}{x\,(-x+y)}\right)(x) \\
&= \frac{yx}{x-y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{yx}{x-y}} dy
\end{aligned}$$

Which results in

$$S = \ln(y) - \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2}{x(-x + y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x^2} \\ S_y &= \frac{x - y}{xy} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x \ln(y) - y}{x} = c_1$$

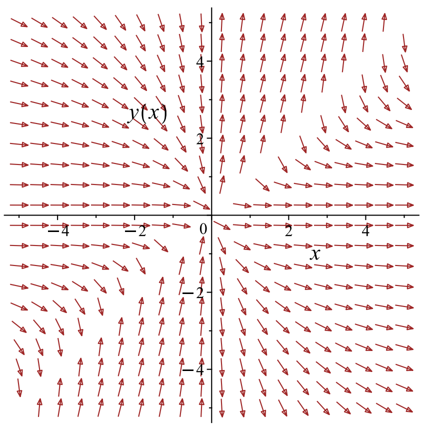
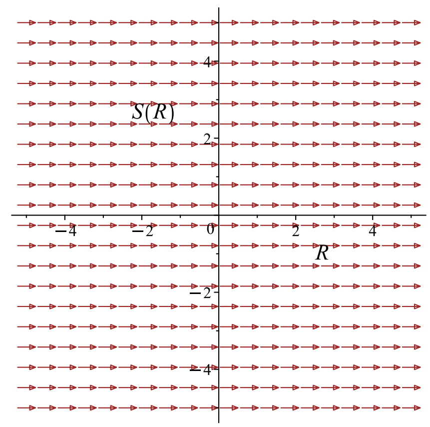
Which simplifies to

$$\frac{x \ln(y) - y}{x} = c_1$$

Which gives

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right) + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x(-x+y)}$ 	$R = x$ $S = \frac{x \ln(y) - y}{x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right)+c_1} \tag{1}$$

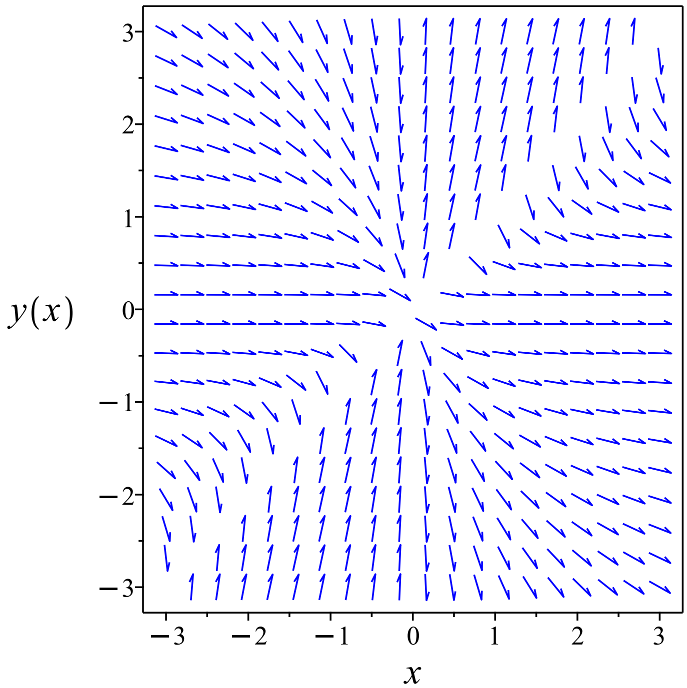


Figure 130: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)+c_1}$$

Verified OK.

2.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2 + yx\right) \mathrm{d}y &= \left(y^2\right) \mathrm{d}x\\ \left(-y^2\right) \mathrm{d}x + \left(-x^2 + yx\right) \mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y^2\\ N(x,y) &= -x^2 + yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y^2\right)\\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2 + yx\right)\\ &= -2x + y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = -y^2$ and $N = -x^2 + yx$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= -\frac{y}{x^2} \\
N &= \frac{-x^2 + yx}{x^2 y}
\end{aligned}
$$

To solve an ode of the form

$$
M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$\left(\frac{-x^2 + yx}{y\,x^2}\right) \mathrm{d}y = \left(\frac{y}{x^2}\right) \mathrm{d}x$$

$$\left(-\frac{y}{x^2}\right) \mathrm{d}x + \left(\frac{-x^2 + yx}{y\,x^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{y}{x^2} \\ N(x,y) &= \frac{-x^2 + yx}{y\,x^2} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x^2}\right) \\ &= -\frac{1}{x^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x^2 + yx}{y\,x^2}\right) \\ &= -\frac{1}{x^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{y}{x^2} \,\mathrm{d}x \end{aligned}$$

$$\phi = \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^2+yx}{y\,x^2}$. Therefore equation (4) becomes

$$\frac{-x^2+yx}{y\,x^2} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{x} - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)-c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)-c_1} \tag{1}$$

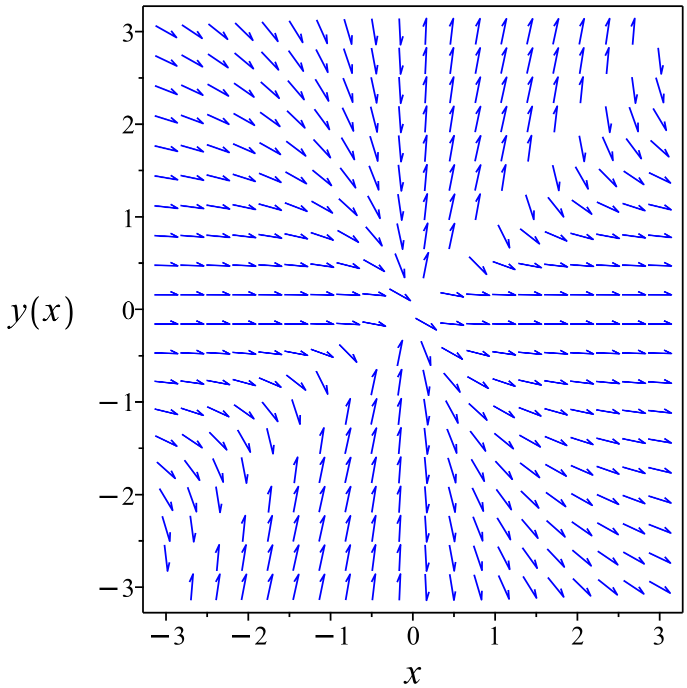


Figure 131: Slope field plot

Verification of solutions

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)-c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve((x*y(x)-x^2)*diff(y(x),x)-y(x)^2=0,y(x), singsol=all)
```

$$y(x) = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

✓ Solution by Mathematica

Time used: 2.229 (sec). Leaf size: 25

```
DSolve[(x*y[x]-x^2)*y'[x]-y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -xW\left(-\frac{e^{-c_1}}{x}\right)$$
$$y(x) \to 0$$

2.10 problem 10

Internal problem ID [1909]
Internal file name [`OUTPUT/1910_Sunday_June_05_2022_02_38_49_AM_52704044/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y'x + y - 2\sqrt{yx} = 0}$$

2.10.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y - 2\sqrt{yx}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(y-2\sqrt{yx})\,(b_3-a_2)}{x} - \frac{(y-2\sqrt{yx})^2\, a_3}{x^2} \\ & - \left(\frac{y}{\sqrt{yx}\,x} + \frac{y-2\sqrt{yx}}{x^2}\right)(xa_2+ya_3+a_1) + \frac{\left(1-\frac{x}{\sqrt{yx}}\right)(xb_2+yb_3+b_1)}{x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4(yx)^{\frac{3}{2}}\, a_3 - x^2yb_3 - 5x\,y^2a_3 - 2b_2\sqrt{yx}\,x^2 + 2\sqrt{yx}\,y^2a_3 + x^3b_2 + x^2ya_2 - xya_1 - \sqrt{yx}\,xb_1 + \sqrt{yx}\,ya_1 +}{\sqrt{yx}\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -4(yx)^{\frac{3}{2}}\, a_3 + 2b_2\sqrt{yx}\,x^2 - 2\sqrt{yx}\,y^2a_3 - x^3b_2 - x^2ya_2 \\ & + x^2yb_3 + 5x\,y^2a_3 + \sqrt{yx}\,xb_1 - \sqrt{yx}\,ya_1 - x^2b_1 + xya_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} & -x^3b_2 + 2b_2\sqrt{yx}\,x^2 - x^2ya_2 + x^2yb_3 - 4yx\sqrt{yx}\,a_3 + 5x\,y^2a_3 \\ & - 2\sqrt{yx}\,y^2a_3 - x^2b_1 + \sqrt{yx}\,xb_1 + xya_1 - \sqrt{yx}\,ya_1 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{yx}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{yx} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -v_1^2v_2a_2 + 5v_1v_2^2a_3 - 4v_2v_1v_3a_3 - 2v_3v_2^2a_3 - v_1^3b_2 \\ & + 2b_2v_3v_1^2 + v_1^2v_2b_3 + v_1v_2a_1 - v_3v_2a_1 - v_1^2b_1 + v_3v_1b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-v_1^3 b_2 + (b_3 - a_2) v_1^2 v_2 + 2b_2 v_3 v_1^2 - v_1^2 b_1 + 5v_1 v_2^2 a_3 \\
&- 4v_2 v_1 v_3 a_3 + v_1 v_2 a_1 + v_3 v_1 b_1 - 2v_3 v_2^2 a_3 - v_3 v_2 a_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
-a_1 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
5a_3 &= 0 \\
-b_1 &= 0 \\
-b_2 &= 0 \\
2b_2 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y - 2\sqrt{yx}}{x}\right)(x) \\ &= 2y - 2\sqrt{yx} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{2y - 2\sqrt{yx}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(-x+y)}{2} - \frac{\ln\left(x + \sqrt{yx}\right)}{2} + \frac{\ln\left(\sqrt{yx} - x\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y - 2\sqrt{yx}}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}\,(2x-2y)}\\
S_y &= -\frac{\sqrt{x}+\sqrt{y}}{\sqrt{y}\,(2x-2y)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{y}\,x-2\sqrt{x}\,\sqrt{yx}+\sqrt{x}\,y\right)}{x^{\frac{3}{2}}\sqrt{y}\,(2x-2y)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{2R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}
$$

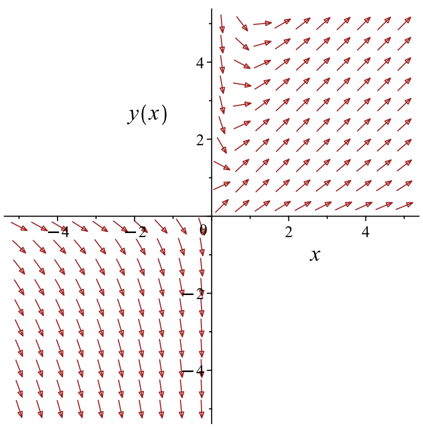
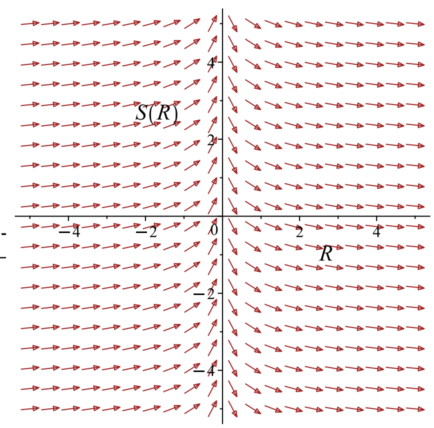
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(-x+y)}{2} - \frac{\ln\left(x+\sqrt{y}\,\sqrt{x}\right)}{2} + \frac{\ln\left(\sqrt{y}\,\sqrt{x}-x\right)}{2} = -\frac{\ln(x)}{2} + c_1
$$

Which simplifies to

$$
\frac{\ln(-x+y)}{2} - \frac{\ln\left(x+\sqrt{y}\,\sqrt{x}\right)}{2} + \frac{\ln\left(\sqrt{y}\,\sqrt{x}-x\right)}{2} = -\frac{\ln(x)}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-2\sqrt{yx}}{x}$ 	$R = x$ $S = \frac{\ln(-x+y)}{2} - \frac{\ln(x \cdot}{}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$\frac{\ln(-x+y)}{2} - \frac{\ln\left(x+\sqrt{y}\,\sqrt{x}\right)}{2} + \frac{\ln\left(\sqrt{y}\,\sqrt{x}-x\right)}{2} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

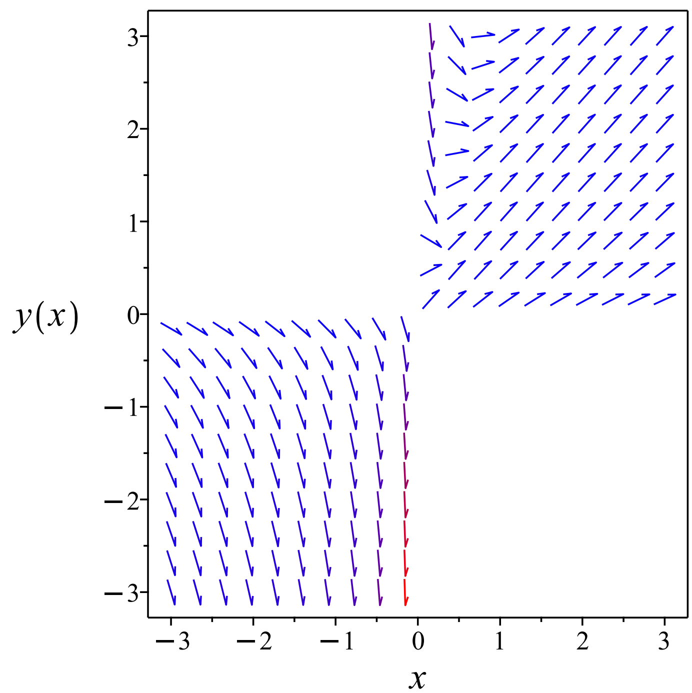


Figure 132: Slope field plot

Verification of solutions

$$\frac{\ln(-x+y)}{2} - \frac{\ln\left(x+\sqrt{y}\,\sqrt{x}\right)}{2} + \frac{\ln\left(\sqrt{y}\,\sqrt{x}-x\right)}{2} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

2.10.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = \left(-y + 2\sqrt{yx}\right)\mathrm{d}x$$
$$\left(y - 2\sqrt{yx}\right)\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = y - 2\sqrt{yx}$$
$$N(x,y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - 2\sqrt{yx}\right)\\ &= 1 - \frac{x}{\sqrt{yx}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}\left(\left(1 - \frac{x}{\sqrt{yx}}\right) - (1)\right) \\
&= -\frac{1}{\sqrt{yx}}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y - 2\sqrt{yx}}\left((1) - \left(1 - \frac{x}{\sqrt{yx}}\right)\right) \\
&= \frac{x}{\left(y - 2\sqrt{yx}\right)\sqrt{yx}}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - \left(1 - \frac{x}{\sqrt{yx}}\right)}{x\left(y - 2\sqrt{yx}\right) - y\left(x\right)} \\
&= -\frac{1}{2yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{1}{2t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{1}{2t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(t)}{2}} \\ &= \frac{1}{\sqrt{t}}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{\sqrt{yx}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{\sqrt{yx}}(y - 2\sqrt{yx}) \\ &= \frac{y - 2\sqrt{yx}}{\sqrt{yx}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{\sqrt{yx}}(x) \\ &= \frac{x}{\sqrt{yx}}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{y - 2\sqrt{yx}}{\sqrt{yx}}\right) + \left(\frac{x}{\sqrt{yx}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{y - 2\sqrt{yx}}{\sqrt{yx}} \, \mathrm{d}x$$

$$\phi = -2x + 2\sqrt{yx} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{\sqrt{yx}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{\sqrt{yx}}$. Therefore equation (4) becomes

$$\frac{x}{\sqrt{yx}} = \frac{x}{\sqrt{yx}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -2x + 2\sqrt{yx} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2x + 2\sqrt{yx}$$

The solution becomes

$$y = \frac{c_1^2 + 4c_1 x + 4x^2}{4x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1^2 + 4c_1 x + 4x^2}{4x} \tag{1}$$

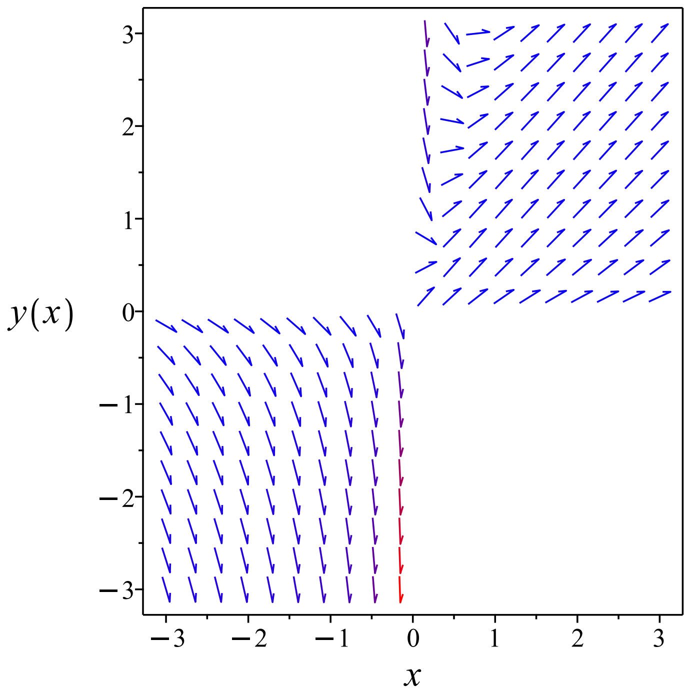


Figure 133: Slope field plot

Verification of solutions

$$y = \frac{c_1^2 + 4c_1 x + 4x^2}{4x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 71

```
dsolve(x*diff(y(x),x)+y(x)=2*sqrt(x*y(x)),y(x), singsol=all)
```

$$\frac{y(x)\,c_1x^2-\sqrt{xy\,(x)}\,y(x)\,c_1x-c_1x^3+\sqrt{xy\,(x)}\,c_1x^2+x+\sqrt{xy\,(x)}}{(-x+y\,(x))\left(\sqrt{xy\,(x)}-x\right)x}=0$$

✓ Solution by Mathematica

Time used: 0.209 (sec). Leaf size: 26

```
DSolve[x*y'[x]+y[x]==2*Sqrt[x*y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{\left(x+e^{\frac{c_1}{2}}\right)^2}{x}$$
$$y(x)\to x$$

2.11 problem 11

Internal problem ID [1910]
Internal file name [`OUTPUT/1911_Sunday_June_05_2022_02_38_52_AM_97128288/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y + (x - y)\, y' = -x}$$

2.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x + (x - u(x)\, x)\, (u'(x)\, x + u(x)) = -x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^2 - 2u - 1}{(u - 1)\, x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-2u-1}{u-1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2-2u-1}{u-1}}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{\frac{u^2-2u-1}{u-1}}\, du &= \int -\frac{1}{x}\, dx \\
\frac{\ln\left(u^2-2u-1\right)}{2} &= -\ln\left(x\right) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\sqrt{u^2-2u-1} = \mathrm{e}^{-\ln(x)+c_2}
$$

Which simplifies to

$$
\sqrt{u^2-2u-1} = \frac{c_3}{x}
$$

Which simplifies to

$$
\sqrt{u\left(x\right)^2-2u\left(x\right)-1} = \frac{c_3 \mathrm{e}^{c_2}}{x}
$$

The solution is

$$
\sqrt{u\left(x\right)^2-2u\left(x\right)-1} = \frac{c_3 \mathrm{e}^{c_2}}{x}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\sqrt{\frac{y^2}{x^2}-\frac{2y}{x}-1} &= \frac{c_3 \mathrm{e}^{c_2}}{x} \\
\sqrt{\frac{y^2-2yx-x^2}{x^2}} &= \frac{c_3 \mathrm{e}^{c_2}}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\sqrt{\frac{y^2-2yx-x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}
$$

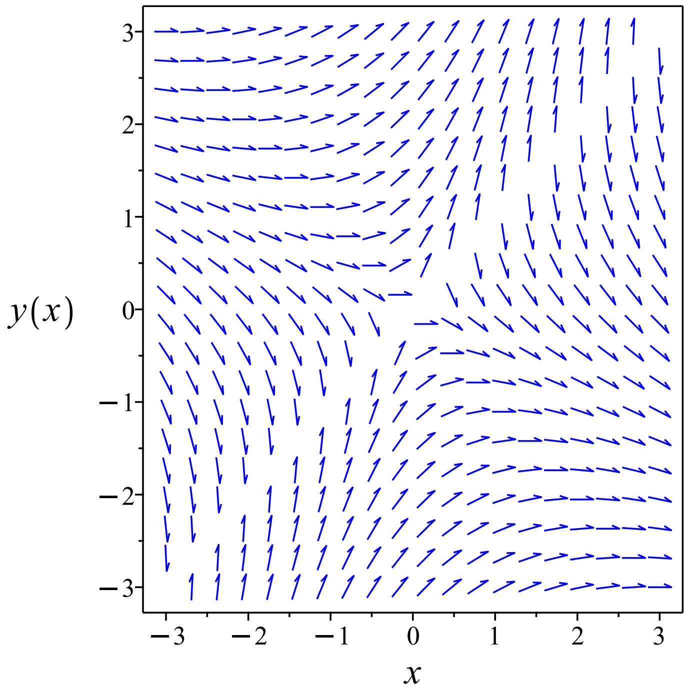


Figure 134: Slope field plot

Verification of solutions

$$\sqrt{\frac{y^2-2yx-x^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

2.11.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-x-y}{x-y} \tag{1}$$

Which becomes

$$(-y)\,dy = (-x)\,dy + (-x-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-x-y)\,dx = d\left(-\frac{1}{2}x^2 - yx\right)$$

Hence (2) becomes

$$(-y)\,dy = d\left(-\frac{1}{2}x^2 - yx\right)$$

Integrating both sides gives gives these solutions

$$y = x + \sqrt{2x^2 - 2c_1} + c_1$$
$$y = x - \sqrt{2x^2 - 2c_1} + c_1$$

Summary

The solution(s) found are the following

$$y = x + \sqrt{2x^2 - 2c_1} + c_1 \tag{1}$$
$$y = x - \sqrt{2x^2 - 2c_1} + c_1 \tag{2}$$

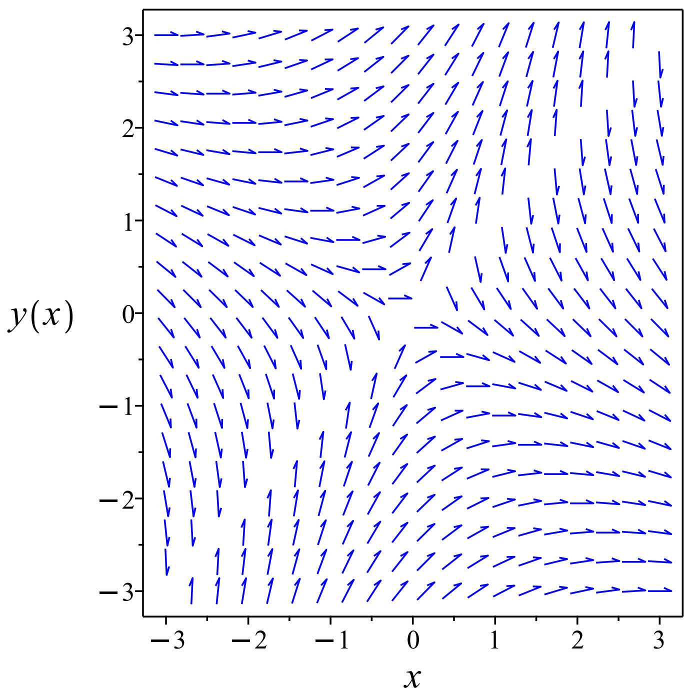


Figure 135: Slope field plot

Verification of solutions

$$y = x + \sqrt{2x^2 - 2c_1} + c_1$$

Verified OK.

$$y = x - \sqrt{2x^2 - 2c_1} + c_1$$

Verified OK.

2.11.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x+y}{-x+y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
b_2 + \frac{(x+y)(b_3 - a_2)}{-x+y} - \frac{(x+y)^2 a_3}{(-x+y)^2} - \left(\frac{1}{-x+y} + \frac{x+y}{(-x+y)^2}\right)(x a_2 + y a_3 + a_1) - \left(\frac{1}{-x+y} - \frac{x+y}{(-x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{x^2 a_2 - x^2 a_3 + 3x^2 b_2 - x^2 b_3 - 2xy a_2 - 2xy a_3 - 2xy b_2 + 2xy b_3 - y^2 a_2 - 3y^2 a_3 + y^2 b_2 + y^2 b_3 + 2x b_1 - 2ya}{(x-y)^2} = 0
$$

Setting the numerator to zero gives

$$
x^2 a_2 - x^2 a_3 + 3x^2 b_2 - x^2 b_3 - 2xy a_2 - 2xy a_3 - 2xy b_2 + 2xy b_3 - y^2 a_2 - 3y^2 a_3 + y^2 b_2 + y^2 b_3 + 2x b_1 - 2y a_1 = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &a_2v_1^2 - 2a_2v_1v_2 - a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - 3a_3v_2^2 + 3b_2v_1^2 \\ &- 2b_2v_1v_2 + b_2v_2^2 - b_3v_1^2 + 2b_3v_1v_2 + b_3v_2^2 - 2a_1v_2 + 2b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(a_2 - a_3 + 3b_2 - b_3)\, v_1^2 + (-2a_2 - 2a_3 - 2b_2 + 2b_3)\, v_1v_2 \\ &+ 2b_1v_1 + (-a_2 - 3a_3 + b_2 + b_3)\, v_2^2 - 2a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ 2b_1 &= 0 \\ -2a_2 - 2a_3 - 2b_2 + 2b_3 &= 0 \\ -a_2 - 3a_3 + b_2 + b_3 &= 0 \\ a_2 - a_3 + 3b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -2b_2 + b_3 \\ a_3 &= b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x+y}{-x+y}\right)(x) \\
&= \frac{x^2 + 2yx - y^2}{x - y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2+2yx-y^2}{x-y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2 - 2yx + y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + y}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + y}{x^2 + 2yx - y^2} \\ S_y &= \frac{x - y}{x^2 + 2yx - y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 - 2yx - x^2)}{2} = c_1$$

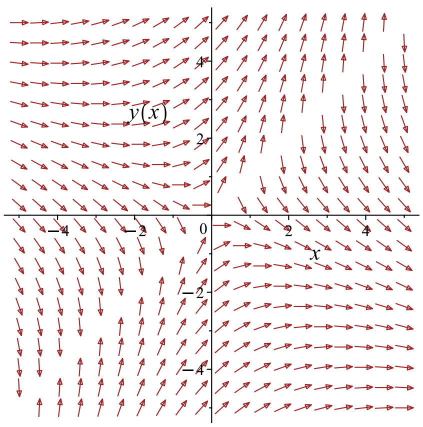
Which simplifies to

$$\frac{\ln(y^2 - 2yx - x^2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{x+y}{-x+y}$



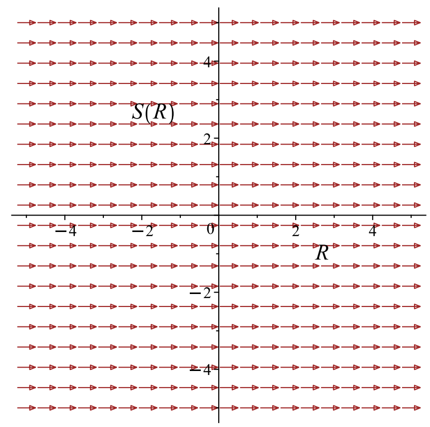
</td><td>

$R = x$

$S = \frac{\ln(-x^2 - 2yx + y^2)}{2}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(y^2 - 2yx - x^2)}{2} = c_1 \tag{1}$$

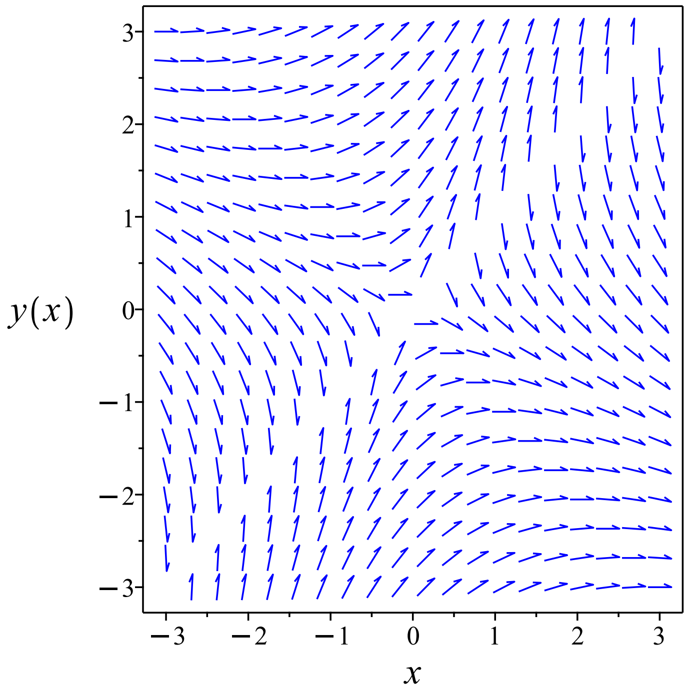


Figure 136: Slope field plot

Verification of solutions

$$\frac{\ln (y^2 - 2yx - x^2)}{2} = c_1$$

Verified OK.

2.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x - y)\,\mathrm{d}y &= (-x - y)\,\mathrm{d}x \\ (x + y)\,\mathrm{d}x + (x - y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= x + y \\ N(x,y) &= x - y \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x - y) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int x + y \, dx$$

$$\phi = \frac{x(x + 2y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x - y$. Therefore equation (4) becomes

$$x - y = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (-y) \, dy$$

$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x+2y)}{2} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x+2y)}{2} - \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$\frac{x(x+2y)}{2} - \frac{y^2}{2} = c_1 \tag{1}$$

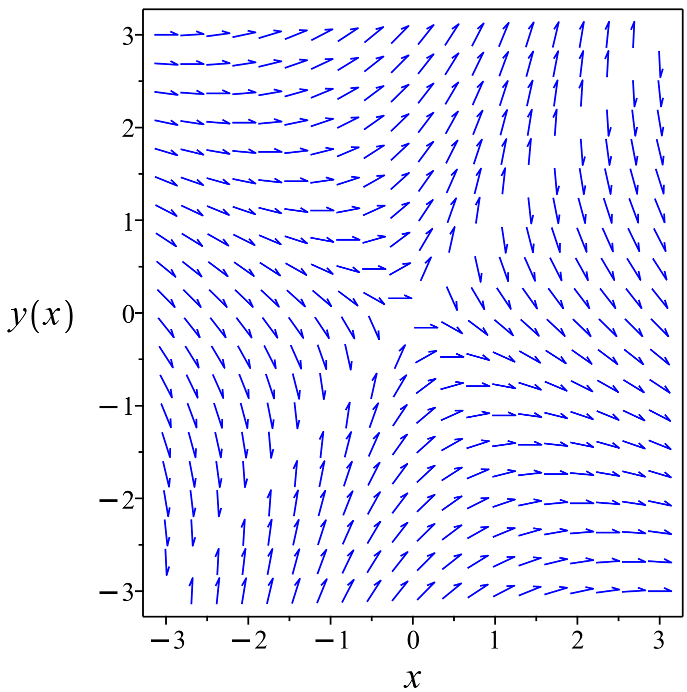


Figure 137: Slope field plot

Verification of solutions

$$\frac{x(x+2y)}{2} - \frac{y^2}{2} = c_1$$

Verified OK.

2.11.5 Maple step by step solution

Let's solve

$y + (x - y)\, y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- ◦ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- ◦ Evaluate derivatives

  $1 = 1$

- ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (x + y)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{x^2}{2} + yx + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x - y = x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -y$

- Solve for $f_1(y)$

  $f_1(y) = -\frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x,y) = \frac{1}{2}x^2 + yx - \frac{1}{2}y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{2}x^2 + yx - \frac{1}{2}y^2 = c_1$

- Solve for y

  $\left\{y = x - \sqrt{2x^2 - 2c_1}, y = x + \sqrt{2x^2 - 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 49

```
dsolve((x+y(x))+(x-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - \sqrt{2c_1^2 x^2 + 1}}{c_1}$$
$$y(x) = \frac{c_1 x + \sqrt{2c_1^2 x^2 + 1}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.489 (sec). Leaf size: 86

```
DSolve[(x+y[x])+(x-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - \sqrt{2x^2 + e^{2c_1}}$$
$$y(x) \to x + \sqrt{2x^2 + e^{2c_1}}$$
$$y(x) \to x - \sqrt{2}\sqrt{x^2}$$
$$y(x) \to \sqrt{2}\sqrt{x^2} + x$$

2.12 problem 12

Internal problem ID [1911]
Internal file name [`OUTPUT/1912_Sunday_June_05_2022_02_38_55_AM_65409539/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y(y^2-yx+x^2)+xy'(x^2+yx+y^2)=0}$$

2.12.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x\left(u(x)^2\,x^2-u(x)\,x^2+x^2\right)+x(u'(x)\,x+u(x))\left(x^2+u(x)\,x^2+u(x)^2\,x^2\right)=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u(u^2+1)}{x\,(u^2+u+1)} \end{aligned}$$

Where $f(x)=-\frac{2}{x}$ and $g(u)=\frac{u(u^2+1)}{u^2+u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+1)}{u^2+u+1}}\,du=-\frac{2}{x}\,dx$$

$$\int \frac{1}{\frac{u(u^2+1)}{u^2+u+1}}\, du = \int -\frac{2}{x}\, dx$$

$$\ln(u) + \arctan(u) = -2\ln(x) + c_2$$

The solution is

$$\ln(u(x)) + \arctan(u(x)) + 2\ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{y}{x}\right) + \arctan\left(\frac{y}{x}\right) + 2\ln(x) - c_2 = 0$$
$$\ln\left(\frac{y}{x}\right) + \arctan\left(\frac{y}{x}\right) + 2\ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{y}{x}\right) + \arctan\left(\frac{y}{x}\right) + 2\ln(x) - c_2 = 0 \tag{1}$$

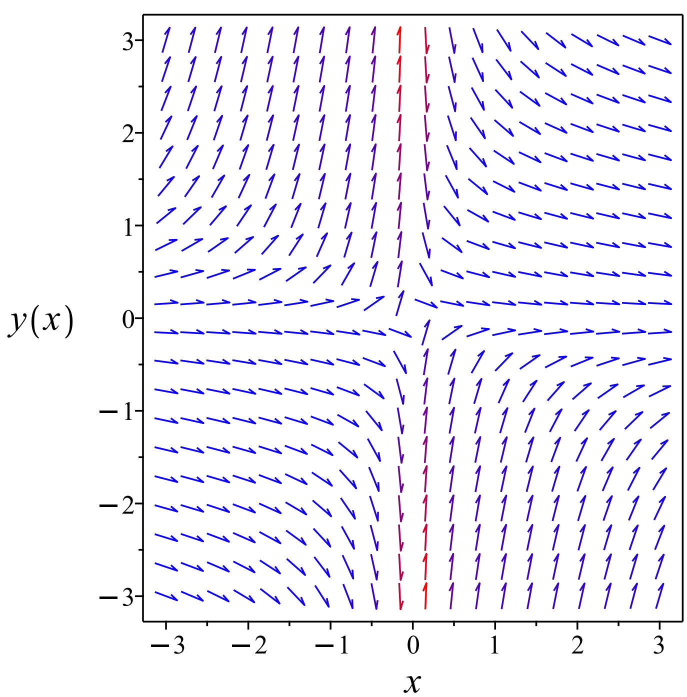


Figure 138: Slope field plot

Verification of solutions

$$\ln\left(\frac{y}{x}\right)+\arctan\left(\frac{y}{x}\right)+2\ln(x)-c_2=0$$

Verified OK.

2.12.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y(x^2-yx+y^2)}{x\,(x^2+yx+y^2)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2-\frac{y(x^2-yx+y^2)\,(b_3-a_2)}{x\,(x^2+yx+y^2)}-\frac{y^2(x^2-yx+y^2)^2\,a_3}{x^2\,(x^2+yx+y^2)^2}\\&-\left(-\frac{y(2x-y)}{x\,(x^2+yx+y^2)}+\frac{y(x^2-yx+y^2)}{x^2\,(x^2+yx+y^2)}\right.\\&\left.+\frac{y(x^2-yx+y^2)\,(2x+y)}{x\,(x^2+yx+y^2)^2}\right)(xa_2+ya_3+a_1)-\left(-\frac{x^2-yx+y^2}{(x^2+yx+y^2)\,x}\right.\\&\left.-\frac{y(-x+2y)}{x\,(x^2+yx+y^2)}+\frac{y(x^2-yx+y^2)\,(x+2y)}{x\,(x^2+yx+y^2)^2}\right)(xb_2+yb_3+b_1)=0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^6b_2 + 2x^4y^2a_2 - 2x^4y^2a_3 + 4x^4y^2b_2 - 2x^4y^2b_3 + 4x^3y^3a_3 + 4x^3y^3b_2 - 2x^2y^4a_2 - 4x^2y^4a_3 + 2x^2y^4b_2 + 2\cdots}{x^2\left(x^2 + \cdots\right.}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&2x^6b_2 + 2x^4y^2a_2 - 2x^4y^2a_3 + 4x^4y^2b_2 - 2x^4y^2b_3 + 4x^3y^3a_3 + 4x^3y^3b_2 \\
&- 2x^2y^4a_2 - 4x^2y^4a_3 + 2x^2y^4b_2 + 2x^2y^4b_3 - 2y^6a_3 + x^5b_1 - x^4ya_1 - 2x^4yb_1 \\
&+ 2x^3y^2a_1 + x^3y^2b_1 - x^2y^3a_1 + 2x^2y^3b_1 - 2x\,y^4a_1 + x\,y^4b_1 - y^5a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&2a_2v_1^4v_2^2 - 2a_2v_1^2v_2^4 - 2a_3v_1^4v_2^2 + 4a_3v_1^3v_2^3 - 4a_3v_1^2v_2^4 - 2a_3v_2^6 + 2b_2v_1^6 + 4b_2v_1^4v_2^2 \\
&+ 4b_2v_1^3v_2^3 + 2b_2v_1^2v_2^4 - 2b_3v_1^4v_2^2 + 2b_3v_1^2v_2^4 - a_1v_1^4v_2 + 2a_1v_1^3v_2^2 - a_1v_1^2v_2^3 \\
&- 2a_1v_1v_2^4 - a_1v_2^5 + b_1v_1^5 - 2b_1v_1^4v_2 + b_1v_1^3v_2^2 + 2b_1v_1^2v_2^3 + b_1v_1v_2^4 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&2b_2v_1^6 + b_1v_1^5 + (2a_2 - 2a_3 + 4b_2 - 2b_3)\,v_1^4v_2^2 + (-a_1 - 2b_1)\,v_1^4v_2 \\
&+ (4a_3 + 4b_2)\,v_1^3v_2^3 + (2a_1 + b_1)\,v_1^3v_2^2 + (-2a_2 - 4a_3 + 2b_2 + 2b_3)\,v_1^2v_2^4 \\
&+ (-a_1 + 2b_1)\,v_1^2v_2^3 + (-2a_1 + b_1)\,v_1v_2^4 - 2a_3v_2^6 - a_1v_2^5 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_1 &= 0\\
-a_1 &= 0\\
-2a_3 &= 0\\
2b_2 &= 0\\
-2a_1 + b_1 &= 0\\
-a_1 - 2b_1 &= 0\\
-a_1 + 2b_1 &= 0\\
2a_1 + b_1 &= 0\\
4a_3 + 4b_2 &= 0\\
-2a_2 - 4a_3 + 2b_2 + 2b_3 &= 0\\
2a_2 - 2a_3 + 4b_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= y - \left(-\frac{y(x^2 - yx + y^2)}{x\,(x^2 + yx + y^2)}\right)(x)\\
&= \frac{2y\,x^2 + 2y^3}{x^2 + yx + y^2}\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2y\,x^2+2y^3}{x^2+yx+y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln (y)}{2} + \frac{\arctan \left(\frac{y}{x}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(x^2 - yx + y^2)}{x\,(x^2 + yx + y^2)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{2x^2 + 2y^2} \\ S_y &= \frac{x^2 + yx + y^2}{2y\,(x^2 + y^2)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

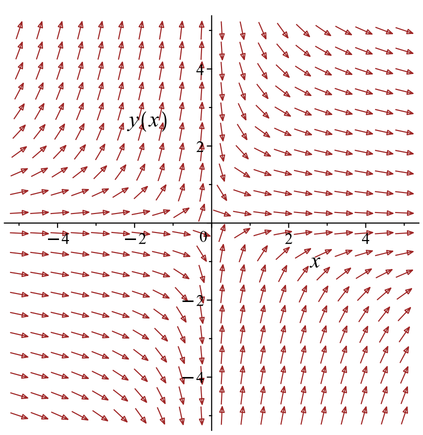
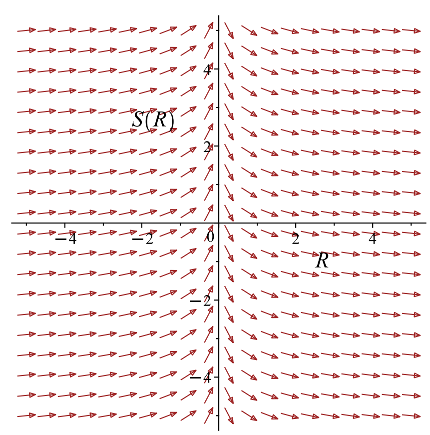
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{2} + \frac{\arctan\left(\frac{y}{x}\right)}{2} = -\frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$\frac{\ln(y)}{2} + \frac{\arctan\left(\frac{y}{x}\right)}{2} = -\frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(x^2-yx+y^2)}{x(x^2+yx+y^2)}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\arctan\left(\frac{y}{x}\right)}{2}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary
The solution(s) found are the following

$$\frac{\ln (y)}{2}+\frac{\arctan \left(\frac{y}{x}\right)}{2}=-\frac{\ln (x)}{2}+c_1 \tag{1}$$

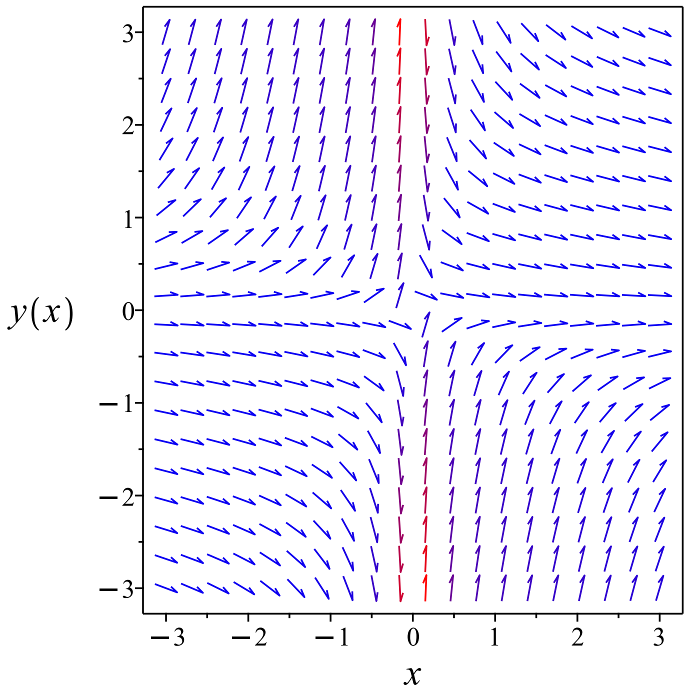


Figure 139: Slope field plot

Verification of solutions

$$\frac{\ln (y)}{2}+\frac{\arctan \left(\frac{y}{x}\right)}{2}=-\frac{\ln (x)}{2}+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(y(x)*(x^2-x*y(x)+y(x)^2)+x*diff(y(x),x)*(x^2+x*y(x)+y(x)^2)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\mathrm{RootOf}\left(\ln\left(\tan\left(\_Z\right)\right) + \_Z + 2\ln\left(x\right) + 2c_1\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 26

```
DSolve[y[x]*(x^2-x*y[x]+y[x]^2)+x*y'[x]*(x^2+x*y[x]+y[x]^2)==0,y[x],x,IncludeSingularSolutio
```

$$\mathrm{Solve}\left[\arctan\left(\frac{y(x)}{x}\right) + \log\left(\frac{y(x)}{x}\right) = -2\log(x) + c_1, y(x)\right]$$

2.13 problem 13

Internal problem ID [1912]
Internal file name [`OUTPUT/1913_Sunday_June_05_2022_02_38_57_AM_80584866/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y'x - y - x\sin\left(\frac{y}{x}\right) = 0}$$

2.13.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \sin\left(\frac{y}{x}\right) + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$
\begin{aligned}
\frac{du}{dx}x+u &= u+g(x)\, f(bu)^{\frac{n}{m}} \\
u' &= \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \qquad (2)
\end{aligned}
$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= \sin\left(\frac{y}{x}\right)
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{\sin(u(x))}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{\sin(u)}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sin(u)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\sin(u)}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\sin(u)}\, du &= \int \frac{1}{x}\, dx \\
\ln(\csc(u) - \cot(u)) &= \ln(x) + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\csc(u) - \cot(u) = e^{\ln(x)+c_1}
$$

Which simplifies to

$$
\csc(u) - \cot(u) = c_2 x
$$

Therefore the solution is

$$
\begin{aligned}
y &= ux \\
&= x \arctan\left(\frac{2c_2 x\, e^{c_1}}{e^{2c_1} c_2^2 x^2 + 1}, -\frac{e^{2c_1} c_2^2 x^2 - 1}{e^{2c_1} c_2^2 x^2 + 1}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x \arctan\left(\frac{2c_2 x\, e^{c_1}}{e^{2c_1} c_2^2 x^2 + 1}, -\frac{e^{2c_1} c_2^2 x^2 - 1}{e^{2c_1} c_2^2 x^2 + 1}\right) \tag{1}
$$

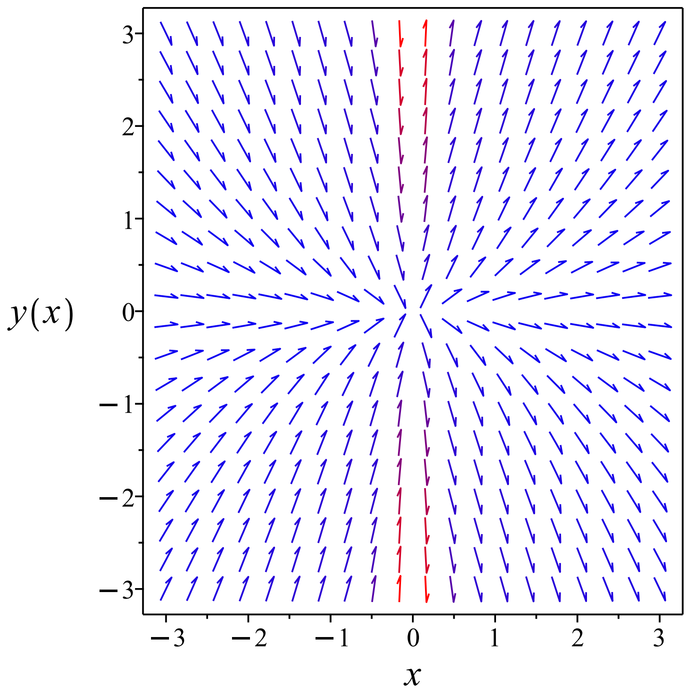


Figure 140: Slope field plot

Verification of solutions

$$
y = x \arctan\left(\frac{2c_2 x\, e^{c_1}}{e^{2c_1} c_2^2 x^2 + 1}, -\frac{e^{2c_1} c_2^2 x^2 - 1}{e^{2c_1} c_2^2 x^2 + 1}\right)
$$

Verified OK.

2.13.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x - u(x)\,x - x\sin(u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\sin(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \sin(u)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\sin(u)}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{\sin(u)}\,du &= \int \frac{1}{x}\,dx \\ \ln(\csc(u) - \cot(u)) &= \ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\csc(u) - \cot(u) = e^{\ln(x)+c_2}$$

Which simplifies to

$$\csc(u) - \cot(u) = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\arctan\left(\frac{2c_3 e^{c_2} x}{e^{2c_2}c_3^2x^2+1}, -\frac{e^{2c_2}c_3^2x^2-1}{e^{2c_2}c_3^2x^2+1}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\arctan\left(\frac{2c_3 e^{c_2} x}{e^{2c_2}c_3^2x^2+1}, -\frac{e^{2c_2}c_3^2x^2-1}{e^{2c_2}c_3^2x^2+1}\right) \tag{1}$$

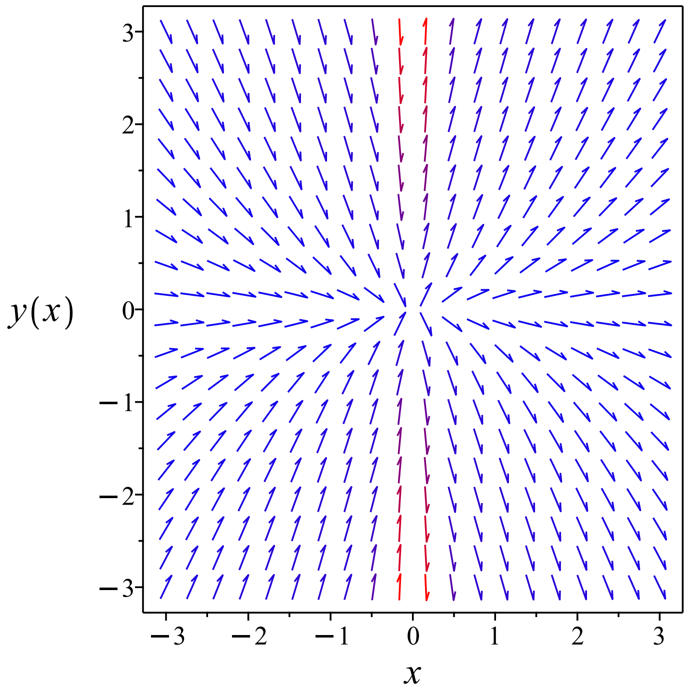


Figure 141: Slope field plot

Verification of solutions

$$
y = x \arctan\left(\frac{2c_3 e^{c_2} x}{e^{2c_2} c_3^2 x^2 + 1}, -\frac{e^{2c_2} c_3^2 x^2 - 1}{e^{2c_2} c_3^2 x^2 + 1}\right)
$$

Verified OK.

2.13.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y + x \sin\left(\frac{y}{x}\right)}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 87: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x^2 \\ \eta(x,y) &= yx\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{yx}{x^2}\\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + x\sin\left(\frac{y}{x}\right)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2}\\ R_y &= \frac{1}{x}\\ S_x &= \frac{1}{x^2}\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\csc\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\csc(R)\, S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1(\csc(R) + \cot(R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1\left(\csc\left(\frac{y}{x}\right) + \cot\left(\frac{y}{x}\right)\right)$$

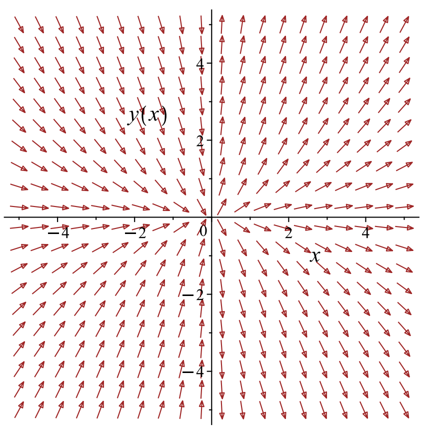
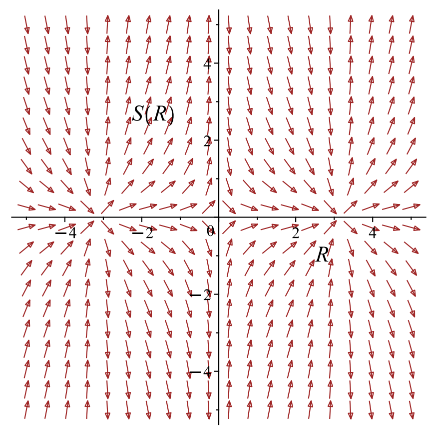
Which simplifies to

$$-\frac{1}{x} = c_1\left(\csc\left(\frac{y}{x}\right) + \cot\left(\frac{y}{x}\right)\right)$$

Which gives

$$y = \arctan\left(-\frac{2c_1 x}{c_1^2 x^2 + 1}, -\frac{c_1^2 x^2 - 1}{c_1^2 x^2 + 1}\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y + x\sin\left(\frac{y}{x}\right)}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -\csc(R)\, S(R)$ 

Summary

The solution(s) found are the following

$$y = \arctan\left(-\frac{2c_1 x}{c_1^2 x^2 + 1}, -\frac{c_1^2 x^2 - 1}{c_1^2 x^2 + 1}\right) x \tag{1}$$

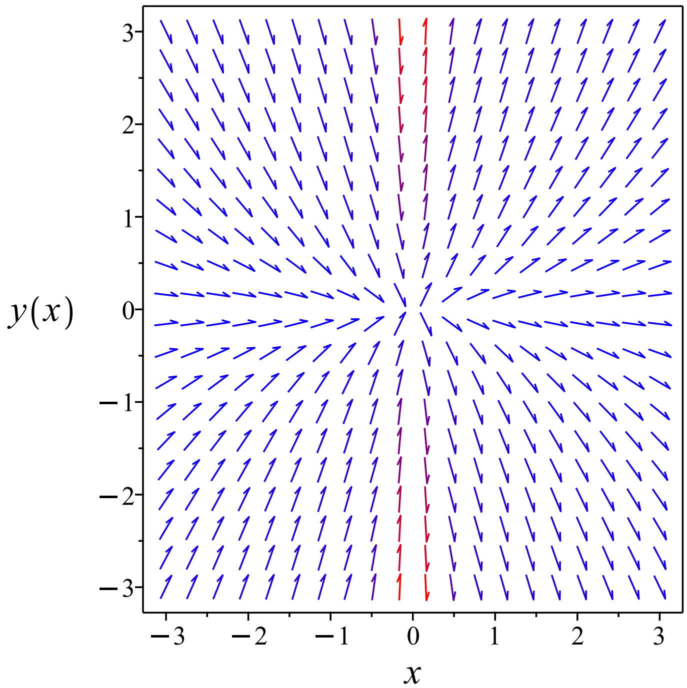


Figure 142: Slope field plot

Verification of solutions

$$y = \arctan\left(-\frac{2c_1 x}{c_1^2 x^2 + 1}, -\frac{c_1^2 x^2 - 1}{c_1^2 x^2 + 1}\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 44

```
dsolve(diff(y(x),x)*x-y(x)-x*sin(y(x)/x)=0,y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{2xc_1}{c_1^2x^2+1}, \frac{-c_1^2x^2+1}{c_1^2x^2+1}\right)x$$

✓ Solution by Mathematica

Time used: 0.341 (sec). Leaf size: 52

```
DSolve[y'[x]*x-y[x]-x*Sin[y[x]/x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \arccos(-\tanh(\log(x)+c_1))$$
$$y(x) \to x \arccos(-\tanh(\log(x)+c_1))$$
$$y(x) \to 0$$
$$y(x) \to -\pi x$$
$$y(x) \to \pi x$$

2.14 problem 14

Internal problem ID [1913]
Internal file name [`OUTPUT/1914_Sunday_June_05_2022_02_39_01_AM_15420033/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - \frac{y}{x} - \cosh\left(\frac{y}{x}\right) = 0}$$

2.14.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \frac{y}{x} + \cosh\left(\frac{y}{x}\right) \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$
\begin{aligned}
\frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\
u' &= \frac{1}{x} g(x)\, f(bu)^{\frac{n}{m}} \qquad (2)
\end{aligned}
$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= \cosh\left(\frac{y}{x}\right)
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{\cosh(u(x))}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{\cosh(u)}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \cosh(u)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\cosh(u)}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\cosh(u)}\, du &= \int \frac{1}{x}\, dx \\
2\arctan(e^{u}) &= \ln(x) + c_1
\end{aligned}
$$

The solution is

$$
2\arctan\left(e^{u(x)}\right) - \ln(x) - c_1 = 0
$$

Therefore the solution is found using $y = ux$. Hence

$$
2\arctan\left(e^{\frac{y}{x}}\right) - \ln(x) - c_1 = 0
$$

Summary

The solution(s) found are the following

$$2\arctan\left(\mathrm{e}^{\frac{y}{x}}\right)-\ln\left(x\right)-c_1=0 \tag{1}$$

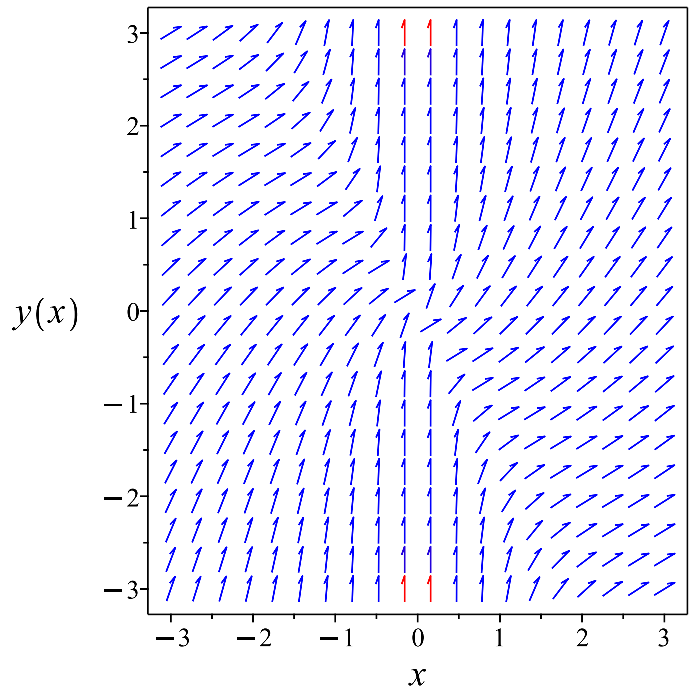


Figure 143: Slope field plot

Verification of solutions

$$2\arctan\left(\mathrm{e}^{\frac{y}{x}}\right)-\ln\left(x\right)-c_1=0$$

Verified OK.

2.14.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x-\cosh\left(u(x)\right)=0$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=\frac{\cosh\left(u\right)}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \cosh(u)$. Integrating both sides gives

$$\frac{1}{\cosh(u)}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\cosh(u)}\,du = \int \frac{1}{x}\,dx$$

$$2\arctan(e^{u}) = \ln(x) + c_2$$

The solution is

$$2\arctan\left(e^{u(x)}\right) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$2\arctan\left(e^{\frac{y}{x}}\right) - \ln(x) - c_2 = 0$$

$$2\arctan\left(e^{\frac{y}{x}}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$2\arctan\left(e^{\frac{y}{x}}\right) - \ln(x) - c_2 = 0 \tag{1}$$

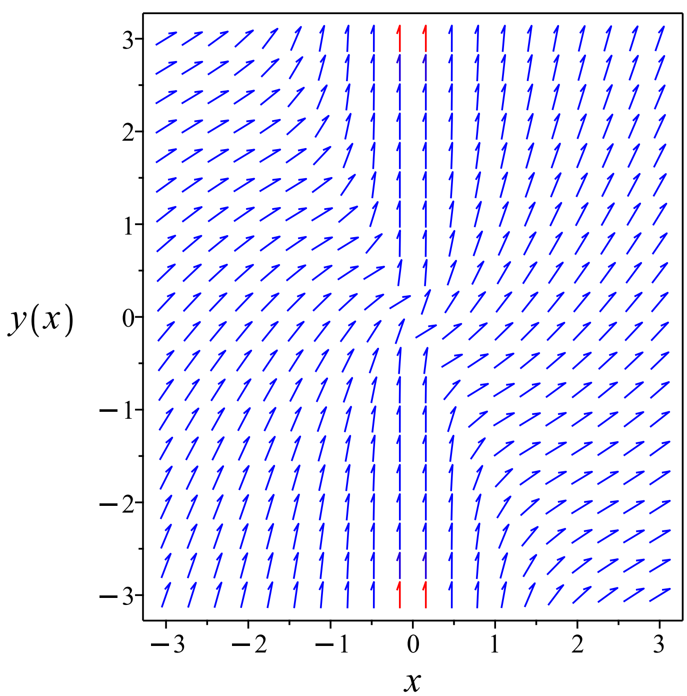


Figure 144: Slope field plot

Verification of solutions

$$2 \arctan \left(\mathrm{e}^{\frac{y}{x}}\right) - \ln (x) - c_2 = 0$$

Verified OK.

2.14.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + \cosh\left(\frac{y}{x}\right) x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 89: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x^2\\ \eta(x,y) &= yx\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{yx}{x^2} = \frac{y}{x}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$dS = \frac{dx}{\xi} = \frac{dx}{x^2}$$

Integrating gives

$$S = \int \frac{dx}{T} = -\frac{1}{x}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + \cosh\left(\frac{y}{x}\right) x}{x}$$

Evaluating all the partial derivatives gives

$$R_x = -\frac{y}{x^2}$$
$$R_y = \frac{1}{x}$$
$$S_x = \frac{1}{x^2}$$
$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\operatorname{sech}\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\operatorname{sech}(R)\, S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \mathrm{e}^{-\arctan(\sinh(R))} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 \mathrm{e}^{-\arctan\left(\sinh\left(\frac{y}{x}\right)\right)}$$

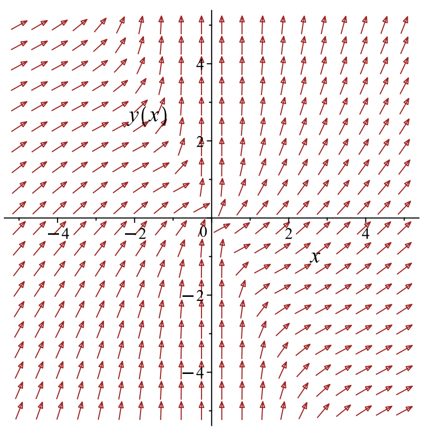
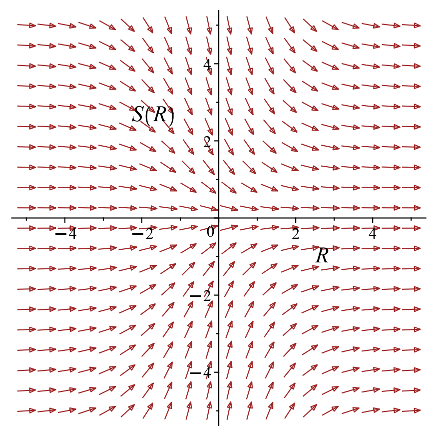
Which simplifies to

$$-\frac{1}{x} = c_1 \mathrm{e}^{-\arctan\left(\sinh\left(\frac{y}{x}\right)\right)}$$

Which gives

$$y = -\operatorname{arcsinh}\left(\tan\left(\ln\left(-\frac{1}{c_1 x}\right)\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\cosh\left(\frac{y}{x}\right)x}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -\operatorname{sech}(R)\, S(R)$ 

Summary

The solution(s) found are the following

$$y = -\operatorname{arcsinh}\left(\tan\left(\ln\left(-\frac{1}{c_1 x}\right)\right)\right) x \tag{1}$$

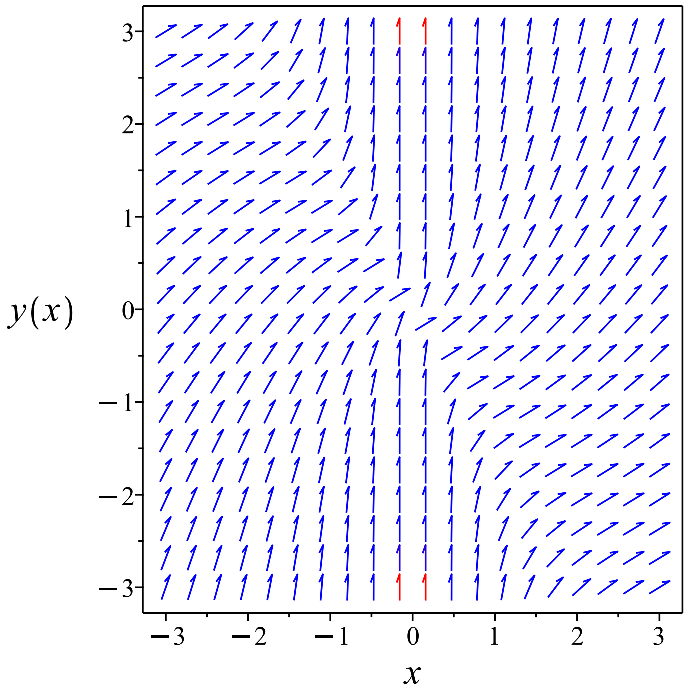


Figure 145: Slope field plot

Verification of solutions

$$y = -\operatorname{arcsinh}\left(\tan\left(\ln\left(-\frac{1}{c_1 x}\right)\right)\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=y(x)/x+cosh(y(x)/x),y(x), singsol=all)
```

$$y(x) = \ln\left(\tan\left(\frac{\ln(x)}{2} + \frac{c_1}{2}\right)\right) x$$

✓ Solution by Mathematica

Time used: 1.62 (sec). Leaf size: 15

```
DSolve[y'[x]==y[x]/x+Cosh[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\text{arcsinh}(\cot(\log(x) + c_1))$$

2.15 problem 15

Internal problem ID [1914]
Internal file name [`OUTPUT/1915_Sunday_June_05_2022_02_39_06_AM_601352/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - 2y'xy = -x^2}$$

With initial conditions

$$[y(-1) = 0]$$

2.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{x^2 + y^2}{2xy} \end{aligned}$$

$f(x, y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

2.15.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 - 2(u'(x)\,x + u(x))\,x^2u(x) = -x^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^2-1}{2ux}\end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2-1}{u}}\,du &= -\frac{1}{2x}\,dx\\ \int \frac{1}{\frac{u^2-1}{u}}\,du &= \int -\frac{1}{2x}\,dx\\ \frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} &= -\frac{\ln(x)}{2} + c_2\end{aligned}$$

The above can be written as

$$\begin{aligned}\left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) &= -\frac{\ln(x)}{2} + 2c_2\\ \ln(u-1) + \ln(u+1) &= (2)\left(-\frac{\ln(x)}{2} + 2c_2\right)\\ &= -\ln(x) + 4c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-\ln(x)+2c_2}$$

Which simplifies to

$$\begin{aligned}u^2 - 1 &= \frac{2c_2}{x}\\ &= \frac{c_3}{x}\end{aligned}$$

The solution is

$$u(x)^2 - 1 = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$
$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$

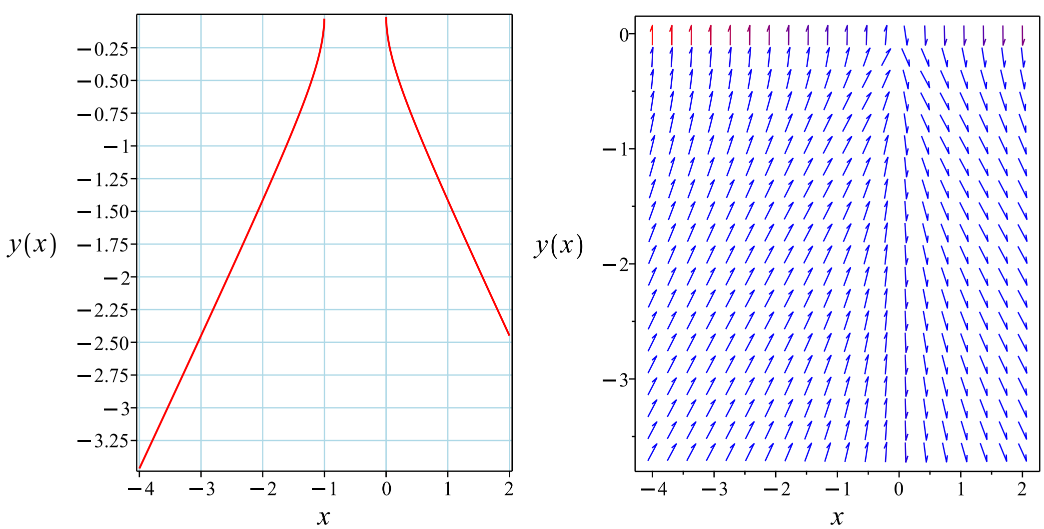
Substituting initial conditions and solving for c_3 gives $c_3 = 1$. Hence the solution becomes Solving for y from the above gives

$$y = \sqrt{x(x+1)}$$
$$y = -\sqrt{x(x+1)}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x(x+1)} \tag{1}$$
$$y = -\sqrt{x(x+1)} \tag{2}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sqrt{x(x+1)}$$

Verified OK.

$$y = -\sqrt{x(x+1)}$$

Verified OK.

2.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x^2+y^2}{2xy}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 91: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x}{y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{y^2}{2x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{x^2 + y^2}{2xy}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^2}{2x^2} \\
S_y &= \frac{y}{x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

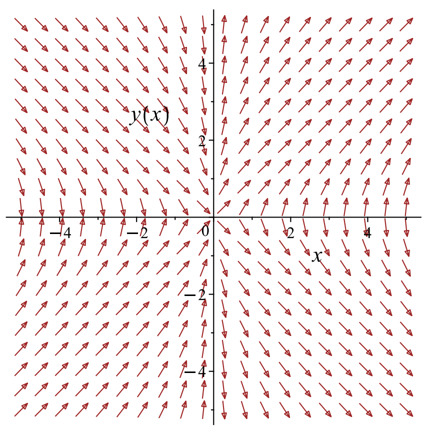
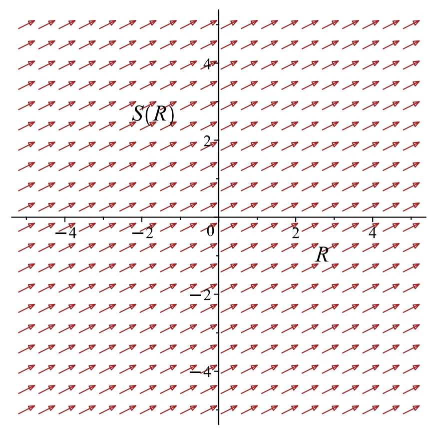
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{y^2}{2x} = \frac{x}{2} + \frac{1}{2}$$

The above simplifies to

$$-x^2 + y^2 - x = 0$$

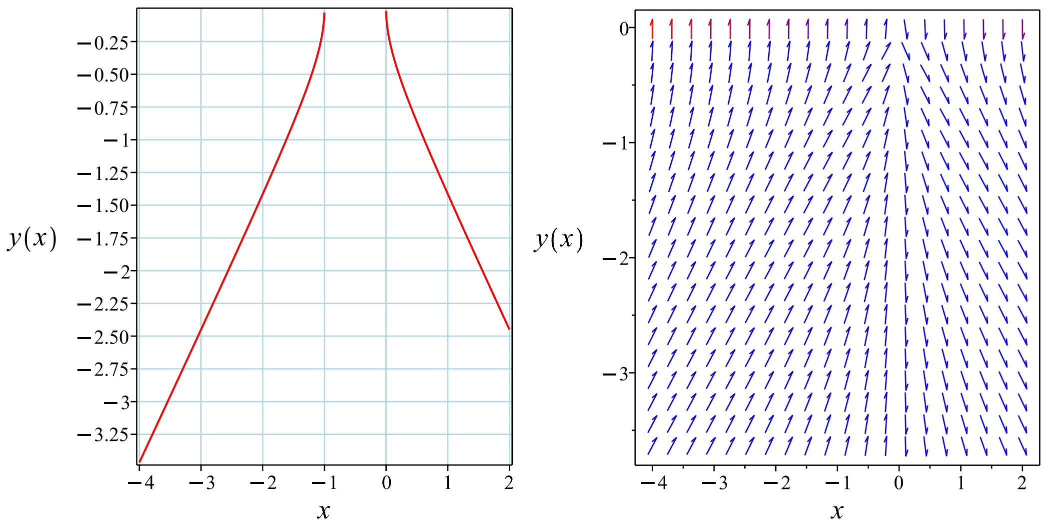
Solving for y from the above gives

$$y = \sqrt{x\,(x+1)}$$
$$y = -\sqrt{x\,(x+1)}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x(x+1)} \tag{1}$$
$$y = -\sqrt{x(x+1)} \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{x(x+1)}$$

Verified OK.

$$y = -\sqrt{x(x+1)}$$

Verified OK.

2.15.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2+y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{2x} \\ f_1(x) &= \frac{x}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{2x} + \frac{x}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{2x} + \frac{x}{2} \\ w' &= \frac{w}{x} + x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{w(x)}{x} = x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(x) \\
\frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(x) \\
d\left(\frac{w}{x}\right) &= dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x} &= \int dx \\
\frac{w}{x} &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
w(x) = c_1 x + x^2
$$

which simplifies to

$$
w(x) = x(x + c_1)
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$
y^2 = x(x + c_1)
$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = 1 - c_1
$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y^2 = x(x+1)$$

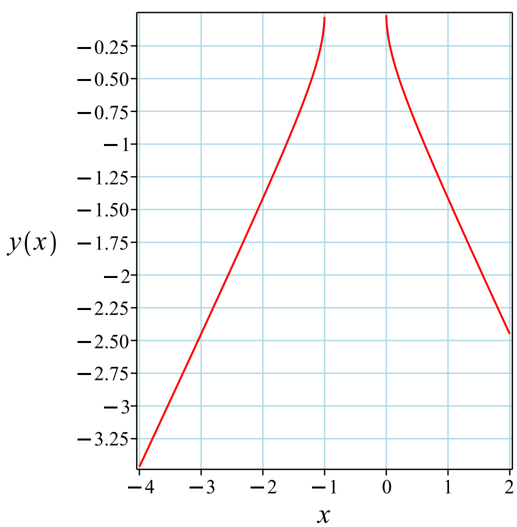
Solving for y from the above gives

$$y = \sqrt{x(x+1)}$$
$$y = -\sqrt{x(x+1)}$$

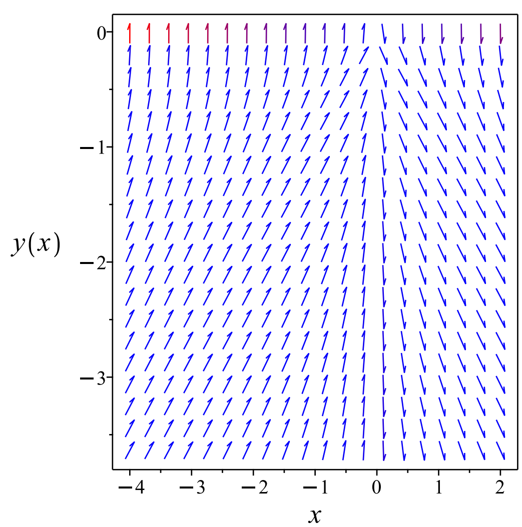
Summary

The solution(s) found are the following

$$y = \sqrt{x(x+1)} \tag{1}$$
$$y = -\sqrt{x(x+1)} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{x(x+1)}$$

Verified OK.

$$y = -\sqrt{x(x+1)}$$

Verified OK.

2.15.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2yx)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x\\ \left(x^2 + y^2\right)\mathrm{d}x + (-2yx)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^2 + y^2\\ N(x,y) &= -2yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2+y^2\right)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2yx)\\ &= -2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{2yx}((2y) - (-2y))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(x^2+y^2\right)\\ &= \frac{x^2+y^2}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}(-2yx)\\ &= -\frac{2y}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{x^2+y^2}{x^2}\right) + \left(-\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2+y^2}{x^2}\,\mathrm{d}x\\ \phi &= x - \frac{y^2}{x} + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{y^2}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$x - \frac{y^2}{x} = -1$$

The above simplifies to

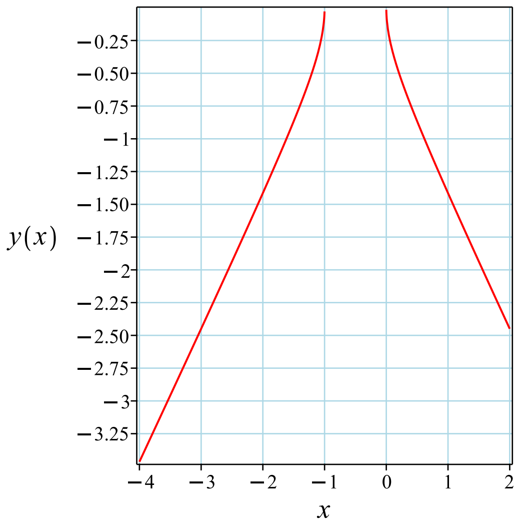
$$x^2 - y^2 + x = 0$$

Solving for y from the above gives

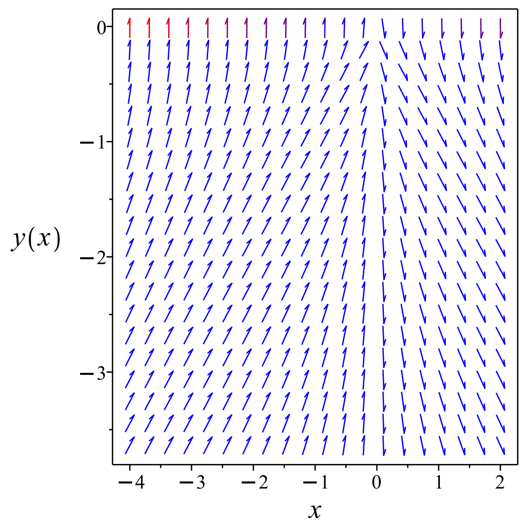
$$y = \sqrt{x(x+1)}$$
$$y = -\sqrt{x(x+1)}$$

Summary
The solution(s) found are the following

$$y = \sqrt{x(x+1)} \tag{1}$$
$$y = -\sqrt{x(x+1)} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{x\,(x+1)}$$

Verified OK.

$$y = -\sqrt{x\,(x+1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 23

```
dsolve([(x^2+y(x)^2)=2*x*y(x)*diff(y(x),x),y(-1) = 0],y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= \sqrt{(x+1)\,x} \\
y(x) &= -\sqrt{(x+1)\,x}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.197 (sec). Leaf size: 36

```
DSolve[{(x^2+y[x]^2)==2*x*y[x]*y'[x],y[-1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\sqrt{x}\sqrt{x+1} \\
y(x) &\to \sqrt{x}\sqrt{x+1}
\end{aligned}
$$

2.16 problem 16

Internal problem ID [1915]
Internal file name [`OUTPUT/1916_Sunday_June_05_2022_02_39_09_AM_8389209/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{\left(\frac{x}{y}+\frac{y}{x}\right) y'=-1}$$

2.16.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\left(\frac{1}{u(x)}+u(x)\right)(u'(x)\,x+u(x))=-1$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u(u^2+2)}{x\,(u^2+1)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2+2)}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+2)}{u^2+1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u(u^2+2)}{u^2+1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u)}{2} + \frac{\ln(u^2+2)}{4} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln(u)}{2}+\frac{\ln\left(u^2+2\right)}{4}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u}\,(u^2+2)^{\frac{1}{4}} = \frac{c_3}{x}$$

The solution is

$$\sqrt{u(x)}\,\left(u(x)^2+2\right)^{\frac{1}{4}} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y}{x}}\left(\frac{y^2}{x^2}+2\right)^{\frac{1}{4}} = \frac{c_3}{x}$$

$$\sqrt{\frac{y}{x}}\left(\frac{y^2+2x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y}{x}}\left(\frac{y^2+2x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3}{x} \tag{1}$$

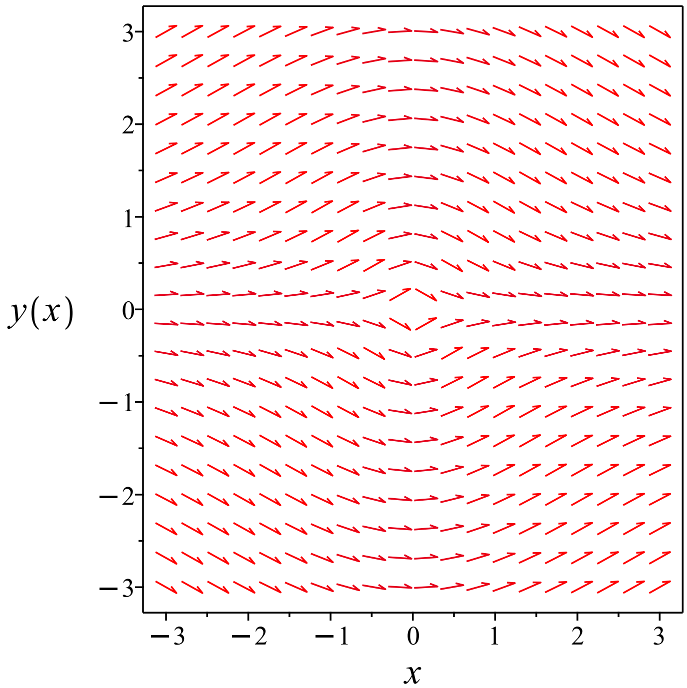


Figure 150: Slope field plot

Verification of solutions

$$\sqrt{\frac{y}{x}}\left(\frac{y^2+2x^2}{x^2}\right)^{\frac{1}{4}}=\frac{c_3}{x}$$

Verified OK.

2.16.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{yx}{x^2+y^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{yx(b_3 - a_2)}{x^2 + y^2} - \frac{y^2x^2a_3}{(x^2 + y^2)^2} - \left(-\frac{y}{x^2 + y^2} + \frac{2y\,x^2}{(x^2 + y^2)^2}\right)(xa_2 + ya_3 + a_1) \\ - \left(-\frac{x}{x^2 + y^2} + \frac{2y^2x}{(x^2 + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^4b_2 - 2y^2x^2a_3 + x^2y^2b_2 + 2x\,y^3a_2 - 2x\,y^3b_3 + y^4a_3 + y^4b_2 + x^3b_1 - x^2ya_1 - x\,y^2b_1 + y^3a_1}{(x^2 + y^2)^2} \\ = 0$$

Setting the numerator to zero gives

$$2x^4b_2 - 2y^2x^2a_3 + x^2y^2b_2 + 2x\,y^3a_2 - 2x\,y^3b_3 \\ + y^4a_3 + y^4b_2 + x^3b_1 - x^2ya_1 - x\,y^2b_1 + y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1v_2^3 - 2a_3v_1^2v_2^2 + a_3v_2^4 + 2b_2v_1^4 + b_2v_1^2v_2^2 + b_2v_2^4 \\ - 2b_3v_1v_2^3 - a_1v_1^2v_2 + a_1v_2^3 + b_1v_1^3 - b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&2b_2 v_1^4 + b_1 v_1^3 + (-2a_3 + b_2) v_1^2 v_2^2 - a_1 v_1^2 v_2 \\
&+ (2a_2 - 2b_3) v_1 v_2^3 - b_1 v_1 v_2^2 + (a_3 + b_2) v_2^4 + a_1 v_2^3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
-a_1 &= 0 \\
-b_1 &= 0 \\
2b_2 &= 0 \\
2a_2 - 2b_3 &= 0 \\
-2a_3 + b_2 &= 0 \\
a_3 + b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{yx}{x^2 + y^2}\right)(x) \\
&= \frac{2y\,x^2 + y^3}{x^2 + y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2y\,x^2+y^3}{x^2+y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{2} + \frac{\ln(2x^2+y^2)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{yx}{x^2+y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{2x^2+y^2} \\ S_y &= \frac{x^2+y^2}{2y\,x^2+y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

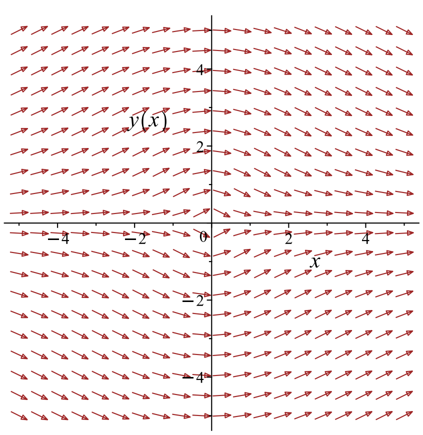
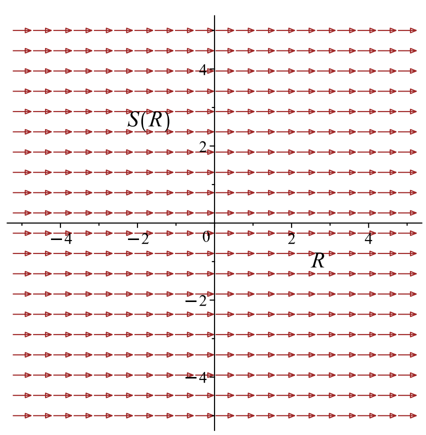
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{2} + \frac{\ln(y^2 + 2x^2)}{4} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{2} + \frac{\ln(y^2 + 2x^2)}{4} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{yx}{x^2+y^2}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(2x^2 + y^2)}{4}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(y)}{2} + \frac{\ln(y^2 + 2x^2)}{4} = c_1 \tag{1}$$

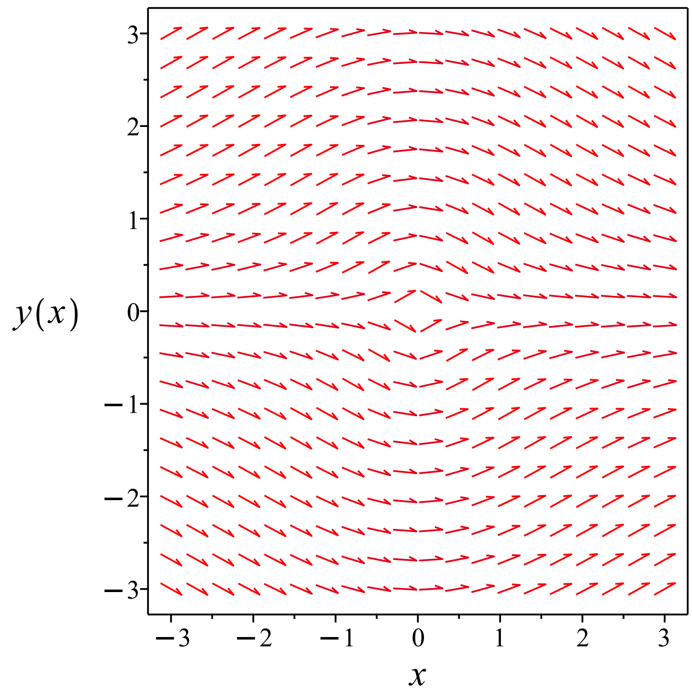


Figure 151: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{2} + \frac{\ln(y^2 + 2x^2)}{4} = c_1$$

Verified OK.

2.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d} x+N(x, y)\, \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(x^2+y^2\right) \mathrm{d} y &= (-y x)\, \mathrm{d} x \\
(y x)\, \mathrm{d} x+\left(x^2+y^2\right) \mathrm{d} y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= y x \\
N(x, y) &= x^2+y^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y x) \\
&= x
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2+y^2}((x)-(2x))\\ &= -\frac{x}{x^2+y^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\\ &= \frac{1}{yx}((2x)-(x))\\ &= \frac{1}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int \frac{1}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(y)}\\ &= y\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= y(yx)\\ &= x\,y^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= y(x^2+y^2) \\ &= y(x^2+y^2)\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (x\,y^2) + (y(x^2+y^2))\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int x\,y^2\,\mathrm{d}x \\ \phi &= \frac{x^2y^2}{2} + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = y\,x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = y(x^2+y^2)$. Therefore equation (4) becomes

$$y(x^2+y^2) = y\,x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^3)\,\mathrm{d}y$$

$$f(y) = \frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{2}x^2y^2 + \frac{1}{4}y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{2}x^2y^2 + \frac{1}{4}y^4$$

Summary
The solution(s) found are the following

$$\frac{x^2y^2}{2} + \frac{y^4}{4} = c_1 \tag{1}$$

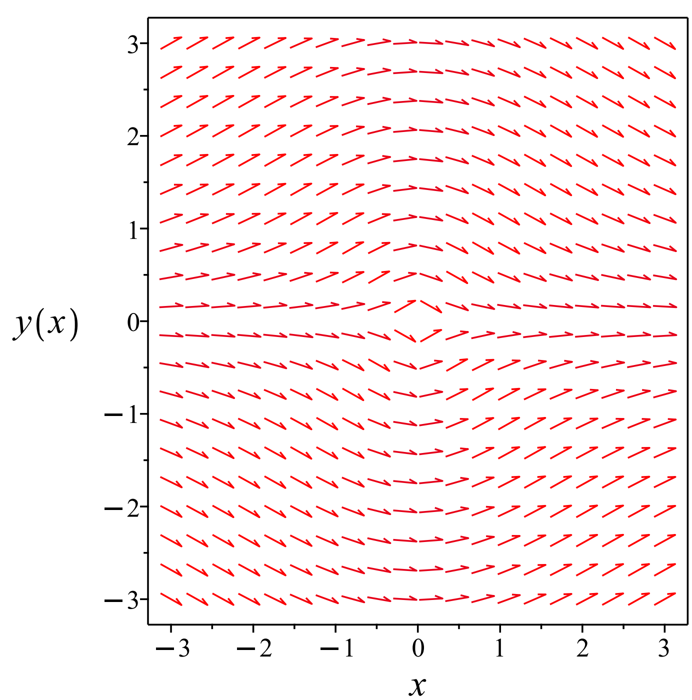


Figure 152: Slope field plot

Verification of solutions

$$\frac{x^2y^2}{2}+\frac{y^4}{4}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.453 (sec). Leaf size: 221

```
dsolve((x/y(x)+y(x)/x)*diff(y(x),x)+1=0,y(x), singsol=all)
```

$$y(x)=\frac{\sqrt{x^2c_1\left(c_1x^2+\sqrt{c_1^2x^4+1}\right)}}{x\left(c_1x^2+\sqrt{c_1^2x^4+1}\right)c_1}$$

$$y(x)=\frac{\sqrt{x^2c_1\left(c_1x^2-\sqrt{c_1^2x^4+1}\right)}}{x\left(c_1x^2-\sqrt{c_1^2x^4+1}\right)c_1}$$

$$y(x)=-\frac{\sqrt{x^2c_1\left(c_1x^2+\sqrt{c_1^2x^4+1}\right)}}{x\left(c_1x^2+\sqrt{c_1^2x^4+1}\right)c_1}$$

$$y(x)=\frac{\sqrt{x^2c_1\left(c_1x^2-\sqrt{c_1^2x^4+1}\right)}}{x\left(-c_1x^2+\sqrt{c_1^2x^4+1}\right)c_1}$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 121

```
DSolve[(x/y[x]+y[x]/x)*y'[x]+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2 - \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to \sqrt{-x^2 - \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to -\sqrt{-x^2 + \sqrt{x^4 + e^{4c_1}}}$$
$$y(x) \to \sqrt{-x^2 + \sqrt{x^4 + e^{4c_1}}}$$

2.17 problem 17

Internal problem ID [1916]
Internal file name [`OUTPUT/1917_Sunday_June_05_2022_02_39_14_AM_69688976/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{e^{\frac{y}{x}}x + y - y'x = 0}$$

With initial conditions

$$[y(1) = 0]$$

2.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{e^{\frac{y}{x}}x + y}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\mathrm{e}^{\frac{y}{x}} x + y}{x}\right) \\ &= \frac{\mathrm{e}^{\frac{y}{x}} + 1}{x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.17.2 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \mathrm{e}^{\frac{y}{x}} + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x) f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x) x$ then

$$\frac{dy}{dx} = \frac{du}{dx} x + u$$

Hence the given ode becomes

$$\frac{du}{dx} x + u = u + g(x) f(bu)^{\frac{n}{m}}$$

$$u' = \frac{1}{x} g(x) f(bu)^{\frac{n}{m}} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= e^{\frac{y}{x}}
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{e^{u(x)}}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{e^{u}}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = e^{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{e^{u}}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{e^{u}}\, du &= \int \frac{1}{x}\, dx \\
-e^{-u} &= \ln(x) + c_1
\end{aligned}
$$

The solution is

$$
-e^{-u(x)} - \ln(x) - c_1 = 0
$$

Therefore the solution is found using $y = ux$. Hence

$$
-e^{-\frac{y}{x}} - \ln(x) - c_1 = 0
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
-1 - c_1 = 0
$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$-e^{-\frac{y}{x}} - \ln(x) + 1 = 0$$

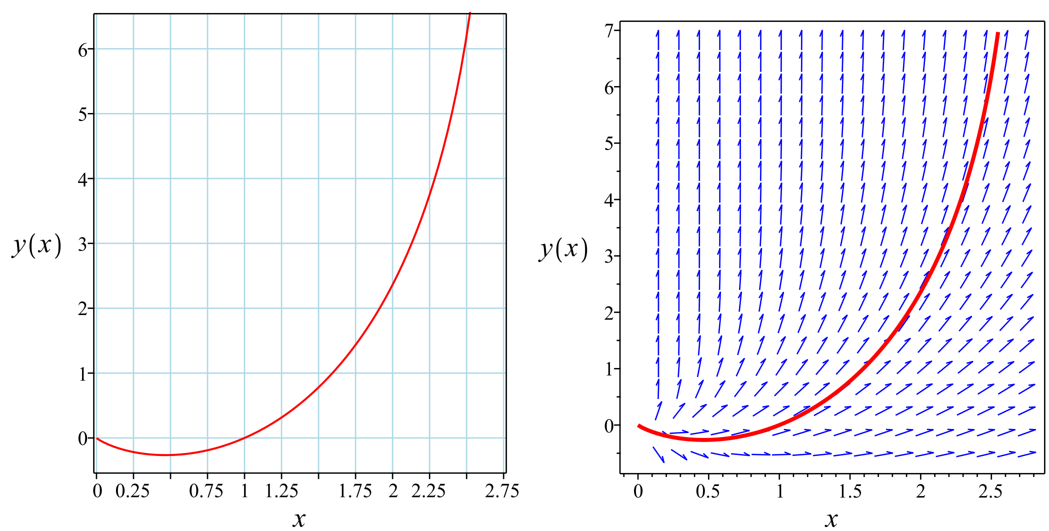
Solving for y from the above gives

$$y = -\ln(-\ln(x) + 1)\, x$$

Summary

The solution(s) found are the following

$$y = -\ln(-\ln(x) + 1)\, x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\ln(-\ln(x) + 1)\, x$$

Verified OK.

2.17.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$e^{u(x)}x + u(x)\,x - (u'(x)\,x + u(x))\,x = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{e^{u}}{x}
\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = e^{u}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{e^{u}}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{e^{u}}\,du &= \int \frac{1}{x}\,dx \\
-e^{-u} &= \ln(x) + c_2
\end{aligned}$$

The solution is

$$-e^{-u(x)} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\begin{aligned}
-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0 \\
-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0
\end{aligned}$$

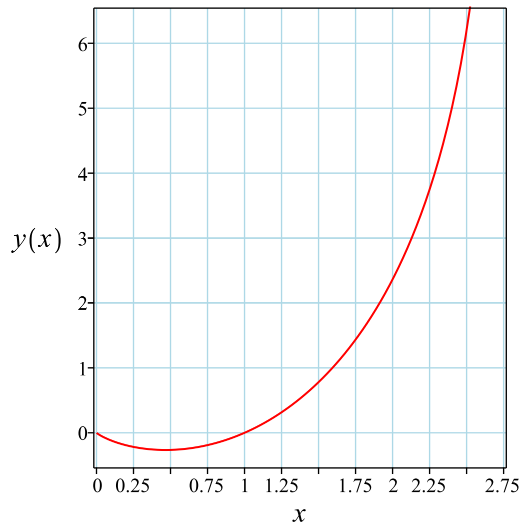
Substituting initial conditions and solving for c_2 gives $c_2 = -1$. Hence the solution becomes Solving for y from the above gives

$$y = -\ln(-\ln(x) + 1)\,x$$

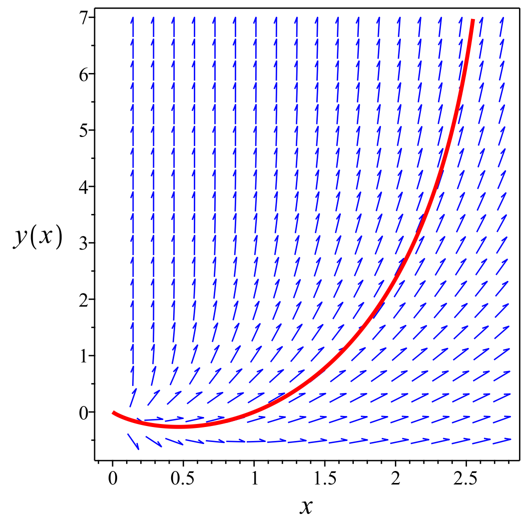
Summary

The solution(s) found are the following

$$y = -\ln(-\ln(x) + 1)\,x \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\ln(-\ln(x) + 1)\, x$$

Verified OK.

2.17.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{e^{\frac{y}{x}} x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 93: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= x^2 \\ \eta(x, y) &= yx \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{yx}{x^2} \\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{e^{\frac{y}{x}} x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x^2} \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\frac{y}{x}}}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -S(R)\, e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{e^{-R}} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{e^{-\frac{y}{x}}}$$

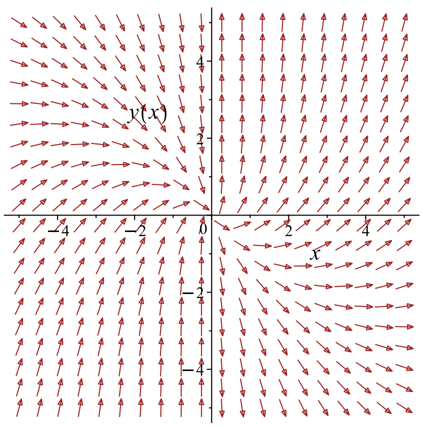
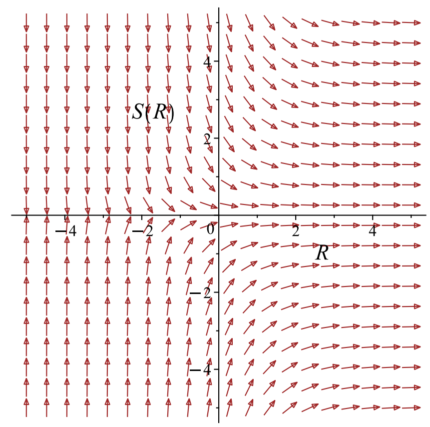
Which simplifies to

$$-\frac{1}{x} = c_1 e^{e^{-\frac{y}{x}}}$$

Which gives

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{e^{\frac{y}{x}} x + y}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -S(R)\, e^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln\left(\ln\left(-\frac{1}{c_1}\right)\right)$$

$$c_1 = -e^{-1}$$

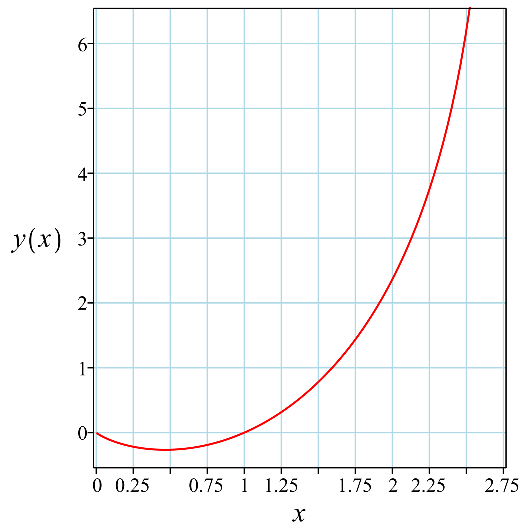
Substituting c_1 found above in the general solution gives

$$y = -\ln\left(1 + \ln\left(\frac{1}{x}\right)\right) x$$

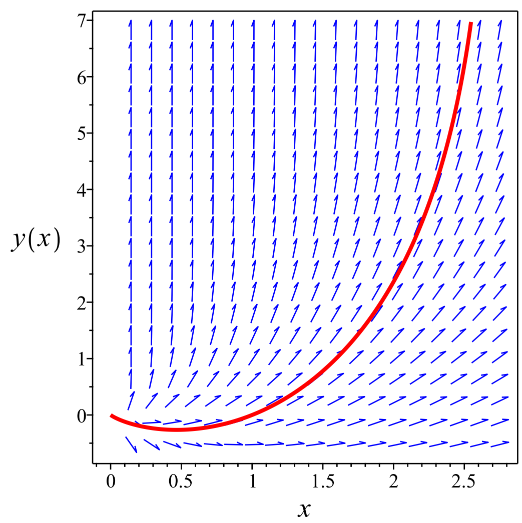
Summary

The solution(s) found are the following

$$y = -\ln\left(1 + \ln\left(\frac{1}{x}\right)\right) x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\ln\left(1+\ln\left(\frac{1}{x}\right)\right)x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 15

```
dsolve([x*exp(y(x)/x)+y(x)=x*diff(y(x),x),y(1) = 0],y(x), singsol=all)
```

$$y(x) = \ln\left(-\frac{1}{-1+\ln(x)}\right)x$$

✓ Solution by Mathematica

Time used: 0.319 (sec). Leaf size: 15

```
DSolve[{x*Exp[y[x]/x]+y[x]==x*y'[x],y[1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\log(1-\log(x))$$

2.18 problem 18

Internal problem ID [1917]
Internal file name [`OUTPUT/1918_Sunday_June_05_2022_02_39_17_AM_14314897/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x+y}{x-y} = 0}$$

With initial conditions

$$[y(1) = 0]$$

2.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{x+y}{-x+y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+y}{-x+y}\right) \\ &= -\frac{1}{-x+y} + \frac{x+y}{(-x+y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.18.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x + u(x)\,x}{x - u(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{x(u-1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan\left(u(x)\right) + \ln\left(x\right) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln\left(x\right) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln\left(x\right) - c_2 = 0$$

Substituting initial conditions and solving for c_2 gives $c_2 = 0$. Hence the solution be-

Summary

The solution(s) found are the following

comes

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln\left(x\right) = 0 \tag{1}$$

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln\left(x\right) = 0$$

Verified OK.

2.18.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x+y}{-x+y}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(x+y)(b_3-a_2)}{-x+y} - \frac{(x+y)^2 a_3}{(-x+y)^2} - \left(-\frac{1}{-x+y} - \frac{x+y}{(-x+y)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{1}{-x+y} + \frac{x+y}{(-x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 + 2xb_1 - 2ya}{(x-y)^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 - 2xyb_3 + y^2a_2 + y^2a_3 + y^2b_2 - y^2b_3 - 2xb_1 + 2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 - b_2 + b_3) v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3) v_1 v_2 \\ - 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3) v_2^2 + 2a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -2b_1 &= 0 \\ -a_2 - a_3 - b_2 + b_3 &= 0 \\ a_2 + a_3 + b_2 - b_3 &= 0 \\ 2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(-\frac{x + y}{-x + y}\right)(x) \\ &= \frac{-x^2 - y^2}{x - y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2 - y^2}{x - y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + y}{x^2 + y^2} \\ S_y &= \frac{-x + y}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

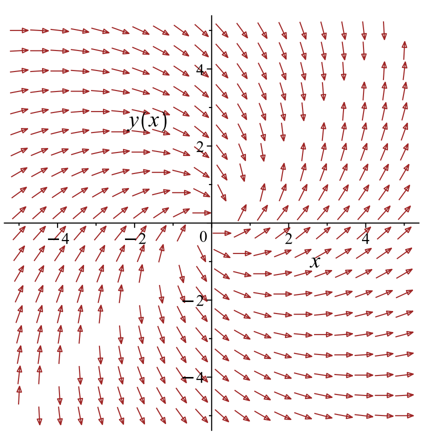
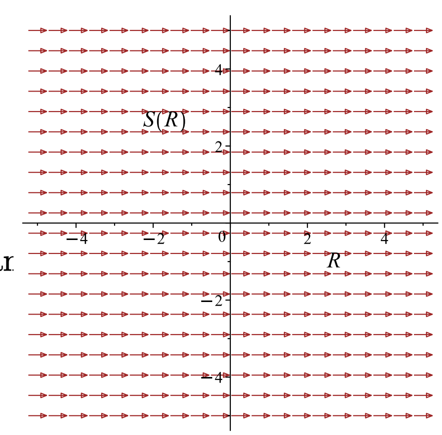
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{x+y}{-x+y}$
</td><td>$R = x$
$S = \frac{\ln(x^2+y^2)}{2} - \arctan$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = 0 \tag{1}$$

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = 0$$

Verified OK.

2.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(-x + y)\,\mathrm{d}y &= (-x - y)\,\mathrm{d}x \\
(x + y)\,\mathrm{d}x + (-x + y)\,\mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= x + y \\
N(x, y) &= -x + y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y) \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x + y) \\
&= -1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = x + y$ and $N = -x + y$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{x+y}{x^2+y^2}$$
$$N = \frac{-x+y}{x^2+y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{x+y}{x^2+y^2}\right)\mathrm{d}x$$
$$\left(\frac{x+y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{-x+y}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= \frac{x+y}{x^2+y^2} \\
N(x,y) &= \frac{-x+y}{x^2+y^2}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y}{x^2+y^2}\right) \\
&= \frac{x^2-2yx-y^2}{(x^2+y^2)^2}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+y}{x^2+y^2}\right) \\
&= \frac{x^2-2yx-y^2}{(x^2+y^2)^2}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x+y}{x^2+y^2}\,\mathrm{d}x \\
\phi &= \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \tag{4}$$
$$= \frac{-x+y}{x^2+y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+y}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{-x+y}{x^2+y^2} = \frac{-x+y}{x^2+y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration. Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 23

```
dsolve([diff(y(x),x)=(x+y(x))/(x-y(x)),y(1) = 0],y(x), singsol=all)
```

$$y(x) = \tan\left(\text{RootOf}\left(2\_Z - \ln\left(\sec\left(\_Z\right)^2\right) - 2\ln\left(x\right)\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 33

```
DSolve[{y'[x]==(x+y[x])/(x-y[x]),y[1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = -\log(x), y(x)\right]$$

2.19 problem 19

Internal problem ID [1918]
Internal file name [`OUTPUT/1919_Sunday_June_05_2022_02_39_20_AM_93947769/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - \frac{y}{x} - \tan\left(\frac{y}{x}\right) = 0}$$

With initial conditions

$$[y(6) = \pi]$$

2.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{\tan\left(\frac{y}{x}\right)x + y}{x} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \pi$ is

$$\left\{-\infty \le x < 0, 0 < x < \frac{2}{1+2\_Z104}, \frac{2}{1+2\_Z104} < x \le \infty\right\}$$

But the point $x_0 = 6$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

2.19.2 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned} \frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\ u' &= \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \end{aligned} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$\begin{aligned} g(x) &= 1 \\ b &= 1 \\ f\left(\frac{bx}{y}\right) &= \tan\left(\frac{y}{x}\right) \end{aligned}$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x) = \frac{\tan(u(x))}{x}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{\tan(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\tan(u)}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\tan(u)}\, du &= \int \frac{1}{x}\, dx \\
\ln(\sin(u)) &= \ln(x) + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\sin(u) = \mathrm{e}^{\ln(x)+c_1}
$$

Which simplifies to

$$
\sin(u) = c_2 x
$$

Therefore the solution is

$$
\begin{aligned}
y &= ux \\
&= x \arcsin\left(c_2 x\, \mathrm{e}^{c_1}\right)
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $x = 6$ and $y = \pi$ in the above solution gives an equation to solve for the constant of integration.

$$
\pi = 6 \arcsin\left(6 c_2 \mathrm{e}^{c_1}\right)
$$

$$
c_1 = \ln\left(\frac{1}{12 c_2}\right)
$$

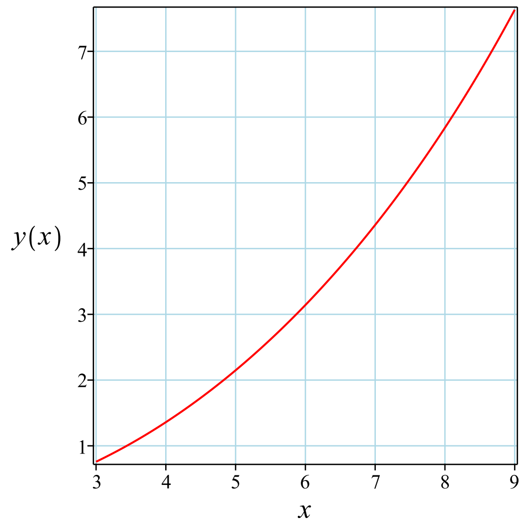
Substituting c_1 found above in the general solution gives

$$
y = \arcsin\left(\frac{x}{12}\right) x
$$

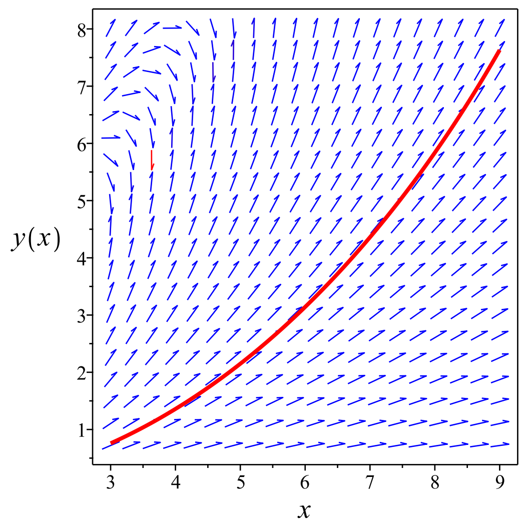
Summary

The solution(s) found are the following

$$
y = \arcsin\left(\frac{x}{12}\right) x \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \arcsin\left(\frac{x}{12}\right) x$$

Verified OK.

2.19.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x - \tan(u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\tan(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\frac{1}{\tan(u)}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\tan(u)}\,du = \int \frac{1}{x}\,dx$$

$$\ln(\sin(u)) = \ln(x) + c_2$$

Raising both side to exponential gives

$$\sin(u) = e^{\ln(x)+c_2}$$

Which simplifies to

$$\sin(u) = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x \arcsin(c_3 e^{c_2} x) \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 6$ and $y = \pi$ in the above solution gives an equation to solve for the constant of integration.

$$\pi = 6 \arcsin(6 c_3 e^{c_2})$$

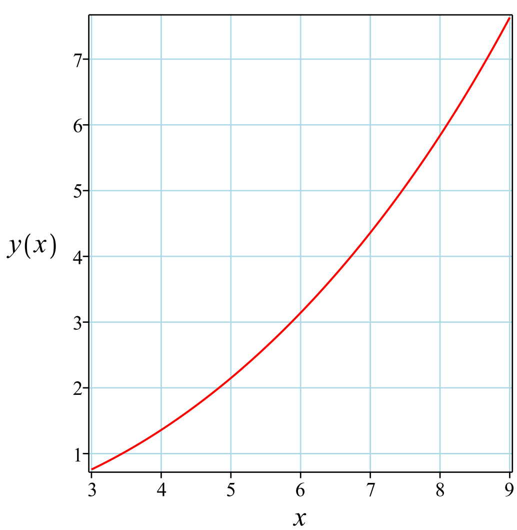
$$c_2 = \ln\left(\frac{1}{12 c_3}\right)$$

Substituting c_2 found above in the general solution gives

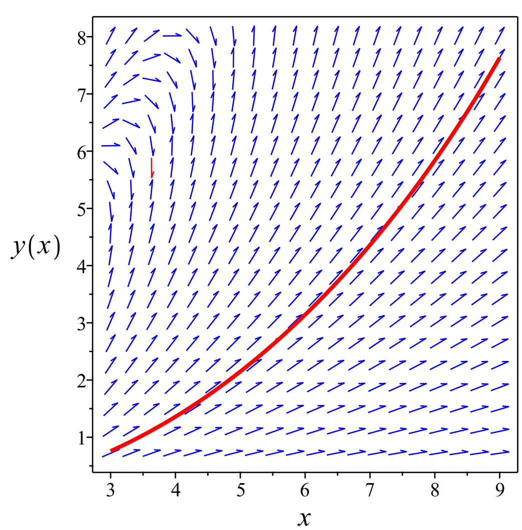
$$y = \arcsin\left(\frac{x}{12}\right) x$$

Summary
The solution(s) found are the following

$$y = \arcsin\left(\frac{x}{12}\right) x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \arcsin\left(\frac{x}{12}\right) x$$

Verified OK.

2.19.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\tan\left(\frac{y}{x}\right) x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 95: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= yx
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{yx}{x^2}\\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\tan\left(\frac{y}{x}\right) x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2}\\ R_y &= \frac{1}{x}\\ S_x &= \frac{1}{x^2}\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cot\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\cot(R)\, S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{c_1}{\sin(R)} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = \frac{c_1}{\sin\left(\frac{y}{x}\right)}$$

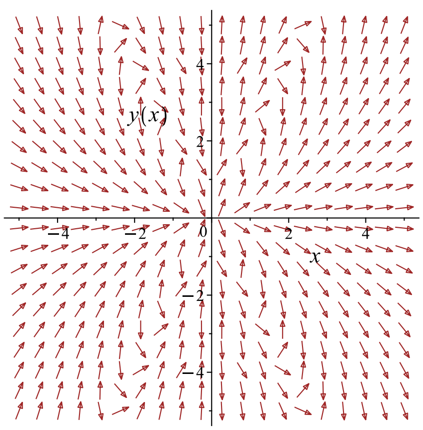
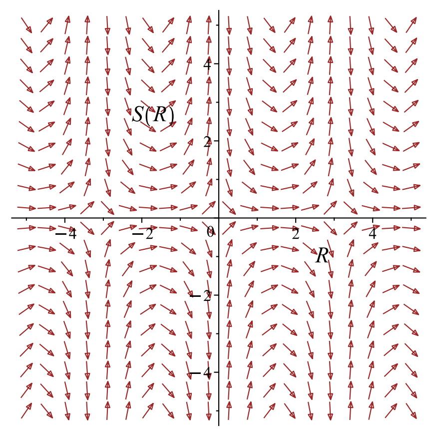
Which simplifies to

$$-\frac{1}{x} = \frac{c_1}{\sin\left(\frac{y}{x}\right)}$$

Which gives

$$y = -\arcsin(c_1 x)\, x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\tan\left(\frac{y}{x}\right)x+y}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -\cot(R)\, S(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = 6$ and $y = \pi$ in the above solution gives an equation to solve for the constant of integration.

$$\pi = -6 \arcsin(6c_1)$$

$$c_1 = -\frac{1}{12}$$

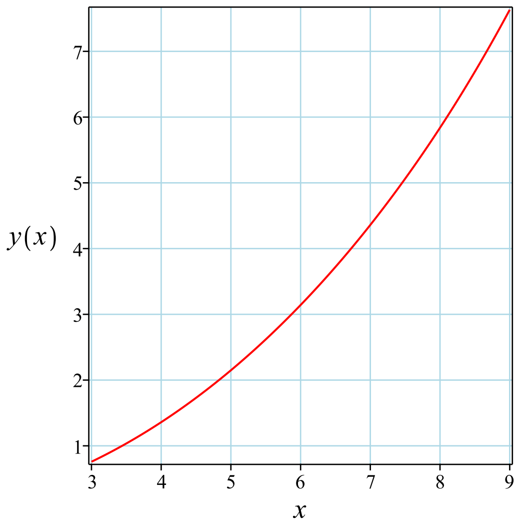
Substituting c_1 found above in the general solution gives

$$y = \arcsin\left(\frac{x}{12}\right) x$$

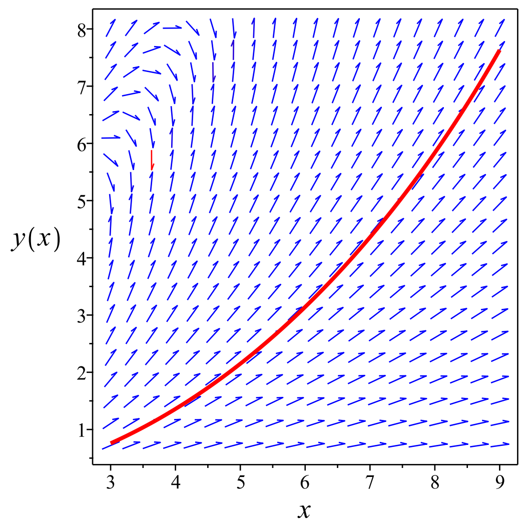
<u>Summary</u>

The solution(s) found are the following

$$y = \arcsin\left(\frac{x}{12}\right) x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \arcsin\left(\frac{x}{12}\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)=y(x)/x+tan(y(x)/x),y(6) = Pi],y(x), singsol=all)
```

$$y(x) = \arcsin\left(\frac{x}{12}\right) x$$

✓ Solution by Mathematica

Time used: 4.285 (sec). Leaf size: 13

```
DSolve[{y'[x]==y[x]/x+Tan[y[x]/x],y[6]==Pi},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin\left(\frac{x}{12}\right)$$

2.20 problem 20

Internal problem ID [1919]
Internal file name [`OUTPUT/1920_Sunday_June_05_2022_02_39_23_AM_43430877/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first__order__ode__lie__symmetry__calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{(3yx - 2x^2)\, y' - 2y^2 + yx = 0}$$

With initial conditions

$$[y(1) = -1]$$

2.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y(-x+2y)}{x\,(3y-2x)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = -1$ is

$$\left\{-\infty \leq x < 0, 0 < x < -\frac{3}{2}, -\frac{3}{2} < x \leq \infty\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\left\{y < \frac{2}{3} \vee \frac{2}{3} < y\right\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(-x+2y)}{x(3y-2x)}\right) \\ &= \frac{-x+2y}{x(3y-2x)} + \frac{2y}{x(3y-2x)} - \frac{3y(-x+2y)}{x(3y-2x)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\left\{-\infty \leq x < 0, 0 < x < -\frac{3}{2}, -\frac{3}{2} < x \leq \infty\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < \frac{2}{3} \vee \frac{2}{3} < y\right\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

2.20.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\left(3u(x)\,x^2 - 2x^2\right)\left(u'(x)\,x + u(x)\right) - 2u(x)^2\,x^2 + u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u(u-1)}{x(3u-2)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u-1)}{3u-2}$. Integrating both sides gives

$$\frac{1}{\frac{u(u-1)}{3u-2}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u(u-1)}{3u-2}}\,du = \int -\frac{1}{x}\,dx$$

$$2\ln(u) + \ln(u-1) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u)+\ln(u-1)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$u^2(u-1) = \frac{c_3}{x}$$

The solution is

$$u(x)^2\,(u(x)-1) = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2\left(\frac{y}{x}-1\right)}{x^2} = \frac{c_3}{x}$$
$$\frac{y^2(-x+y)}{x^3} = \frac{c_3}{x}$$

Which simplifies to

$$-\frac{y^2(x-y)}{x^2} = c_3$$

Substituting initial conditions and solving for c_3 gives $c_3 = -2$. Hence the solution be-
Summary
The solution(s) found are the following
comes

$$-\frac{y^2(x-y)}{x^2} = -2 \tag{1}$$

Verification of solutions

$$-\frac{y^2(x-y)}{x^2} = -2$$

Verified OK.

2.20.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(-x+2y)}{x\,(3y-2x)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{y(-x+2y)(b_3 - a_2)}{x(3y-2x)} - \frac{y^2(-x+2y)^2 a_3}{x^2(3y-2x)^2} \\ & - \left(-\frac{y}{x(3y-2x)} - \frac{y(-x+2y)}{x^2(3y-2x)} + \frac{2y(-x+2y)}{x(3y-2x)^2} \right)(x a_2 + y a_3 + a_1) \\ & - \left(\frac{-x+2y}{x(3y-2x)} + \frac{2y}{x(3y-2x)} - \frac{3y(-x+2y)}{x(3y-2x)^2} \right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^4 b_2 - 4x^3 y b_2 - x^2 y^2 a_2 + x^2 y^2 a_3 + 3x^2 y^2 b_2 + x^2 y^2 b_3 - 4x\, y^3 a_3 + 2y^4 a_3 - 2x^3 b_1 + 2x^2 y a_1 + 8x^2 y b_1 - 8x}{x^2(-3y+2x)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 2x^4 b_2 - 4x^3 y b_2 - x^2 y^2 a_2 + x^2 y^2 a_3 + 3x^2 y^2 b_2 + x^2 y^2 b_3 - 4x\, y^3 a_3 \\ & + 2y^4 a_3 - 2x^3 b_1 + 2x^2 y a_1 + 8x^2 y b_1 - 8x\, y^2 a_1 - 6x\, y^2 b_1 + 6y^3 a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_1^2 v_2^2 + a_3 v_1^2 v_2^2 - 4a_3 v_1 v_2^3 + 2a_3 v_2^4 + 2b_2 v_1^4 - 4b_2 v_1^3 v_2 + 3b_2 v_1^2 v_2^2 + b_3 v_1^2 v_2^2 + 2a_1 v_1^2 v_2 - 8a_1 v_1 v_2^2 + 6a_1 v_2^3 - 2b_1 v_1^3 + 8b_1 v_1^2 v_2 - 6b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2 v_1^4 - 4b_2 v_1^3 v_2 - 2b_1 v_1^3 + (-a_2 + a_3 + 3b_2 + b_3) v_1^2 v_2^2 + (2a_1 + 8b_1) v_1^2 v_2 - 4a_3 v_1 v_2^3 + (-8a_1 - 6b_1) v_1 v_2^2 + 2a_3 v_2^4 + 6a_1 v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
6a_1 &= 0\\
-4a_3 &= 0\\
2a_3 &= 0\\
-2b_1 &= 0\\
-4b_2 &= 0\\
2b_2 &= 0\\
-8a_1 - 6b_1 &= 0\\
2a_1 + 8b_1 &= 0\\
-a_2 + a_3 + 3b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{y(-x+2y)}{x\,(3y-2x)}\right)(x) \\ &= \frac{yx - y^2}{-3y+2x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{yx-y^2}{-3y+2x}} dy\end{aligned}$$

Which results in

$$S = 2\ln(y) + \ln(-x+y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(-x+2y)}{x(3y-2x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x-y} \\ S_y &= \frac{2}{y} + \frac{1}{-x+y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\ln(R) + c_1 \tag{4}$$

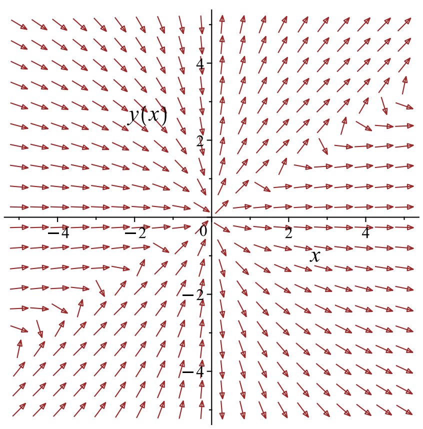
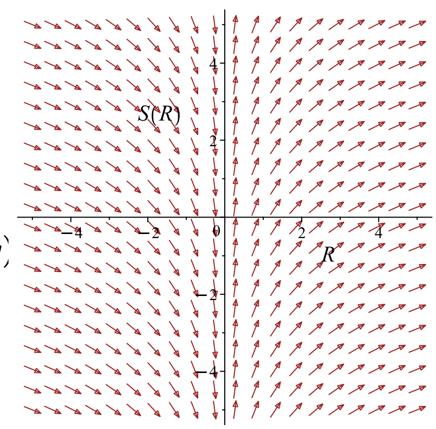
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\ln(y) + \ln(-x+y) = 2\ln(x) + c_1$$

Which simplifies to

$$2\ln(y) + \ln(-x+y) = 2\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(-x+2y)}{x(3y-2x)}$ 	$R = x$ $S = 2\ln(y) + \ln(-x + y)$	$\frac{dS}{dR} = \frac{2}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$3i\pi + \ln(2) = c_1$$

$$c_1 = 3i\pi + \ln(2)$$

Substituting c_1 found above in the general solution gives

$$2\ln(y) + \ln(-x + y) = 2\ln(x) + 3i\pi + \ln(2)$$

Summary

The solution(s) found are the following

$$2\ln(y) + \ln(-x + y) = 2\ln(x) + 3i\pi + \ln(2) \tag{1}$$

Verification of solutions

$$2\ln(y) + \ln(-x + y) = 2\ln(x) + 3i\pi + \ln(2)$$

Verified OK.

2.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-2x^2 + 3yx\right)\mathrm{d}y &= \left(-yx + 2y^2\right)\mathrm{d}x\\ \left(yx - 2y^2\right)\mathrm{d}x + \left(-2x^2 + 3yx\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= yx - 2y^2\\ N(x,y) &= -2x^2 + 3yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(yx - 2y^2) \\ &= x - 4y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2x^2 + 3yx) \\ &= -4x + 3y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{y}{x^3}$ is an integrating factor. Therefore by multiplying $M = -2y^2 + yx$ and $N = 3yx - 2x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{y(-2y^2 + yx)}{x^3} \\ N &= \frac{y(3yx - 2x^2)}{x^3}\end{aligned}$$

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y(-2x^2 + 3yx)}{x^3}\right)\mathrm{d}y = \left(-\frac{y(yx - 2y^2)}{x^3}\right)\mathrm{d}x$$

$$\left(\frac{y(yx - 2y^2)}{x^3}\right)\mathrm{d}x + \left(\frac{y(-2x^2 + 3yx)}{x^3}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \frac{y(yx - 2y^2)}{x^3} \\ N(x, y) &= \frac{y(-2x^2 + 3yx)}{x^3} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(yx - 2y^2)}{x^3}\right) \\ &= \frac{2y(x - 3y)}{x^3} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y(-2x^2 + 3yx)}{x^3}\right) \\ &= \frac{2y(x - 3y)}{x^3} \end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1)\\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{y(yx-2y^2)}{x^3} \mathrm{d}x\\
\phi &= -\frac{y^2(x-y)}{x^2} + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\frac{2y(x-y)}{x^2} + \frac{y^2}{x^2} + f'(y) && (4)\\
&= \frac{-2yx+3y^2}{x^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{y(-2x^2+3yx)}{x^3}$. Therefore equation (4) becomes

$$
\frac{y(-2x^2+3yx)}{x^3} = \frac{-2yx+3y^2}{x^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{y^2(x-y)}{x^2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y^2(x-y)}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$-\frac{y^2(x-y)}{x^2} = -2$$

The above simplifies to

$$-x\,y^2 + y^3 + 2x^2 = 0$$

Summary

The solution(s) found are the following

$$-xy^2 + y^3 + 2x^2 = 0 \tag{1}$$

Verification of solutions

$$-xy^2 + y^3 + 2x^2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.39 (sec). Leaf size: 116

```
dsolve([(3*x*y(x)-2*x^2)*diff(y(x),x)=2*y(x)^2-x*y(x),y(1) = -1],y(x), singsol=all)
```

$$y(x) = \frac{\left(-27x^2+x^3+3\sqrt{3}\,\sqrt{-x^4\,(2x-27)}\right)^{\frac{1}{3}}\,(i\sqrt{3}-1)}{6} - \frac{\left(i\sqrt{3}\,x+x-2\left(-27x^2+x^3+3\sqrt{3}\,\sqrt{-x^4\,(2x-27)}\right)^{\frac{1}{3}}\right)x}{6\left(-27x^2+x^3+3\sqrt{3}\,\sqrt{-x^4\,(2x-27)}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 60.327 (sec). Leaf size: 140

```
DSolve[{(3*x*y[x]-2*x^2)*y'[x]==2*y[x]^2-x*y[x],y[1]==-1},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{\left(\sqrt[3]{3\sqrt{3}\sqrt{-x^4(2x-27)}+x^3-27x^2}-x\right)\left(i\left(\sqrt{3}+i\right)\sqrt[3]{3\sqrt{3}\sqrt{-x^4(2x-27)}+x^3-27x^2}+i\sqrt{3}x+\right.}{6\sqrt[3]{3\sqrt{3}\sqrt{-x^4(2x-27)}+x^3-27x^2}}$$

2.21 problem 21

Internal problem ID [1920]
Internal file name [`OUTPUT/1921_Sunday_June_05_2022_02_39_28_AM_5298386/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y'-\frac{y}{x-k\sqrt{x^2+y^2}}=0}$$

2.21.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y}{k\sqrt{x^2+y^2}-x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(b_3 - a_2)}{k\sqrt{x^2+y^2} - x} - \frac{y^2 a_3}{\left(k\sqrt{x^2+y^2} - x\right)^2} - \frac{y\left(\frac{kx}{\sqrt{x^2+y^2}} - 1\right)(x a_2 + y a_3 + a_1)}{\left(k\sqrt{x^2+y^2} - x\right)^2} - \left(-\frac{1}{k\sqrt{x^2+y^2} - x} + \frac{y^2 k}{\left(k\sqrt{x^2+y^2} - x\right)^2 \sqrt{x^2+y^2}}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{\left(x^2+y^2\right)^{\frac{3}{2}} k^2 b_2 - k\,x^3 b_2 - kx\,y^2 a_3 - 2kx\,y^2 b_2 + k\,y^3 a_2 - k\,y^3 b_3 + k\,x^2 b_1 - kxy a_1 - \sqrt{x^2+y^2}\,x b_1 + \sqrt{x^2}}{\left(k\sqrt{x^2+y^2} - x\right)^2 \sqrt{x^2+y^2}} = 0$$

Setting the numerator to zero gives

$$\left(x^2+y^2\right)^{\frac{3}{2}} k^2 b_2 - k\,x^3 b_2 - kx\,y^2 a_3 - 2kx\,y^2 b_2 + k\,y^3 a_2 - k\,y^3 b_3 + k\,x^2 b_1 - kxy a_1 - \sqrt{x^2+y^2}\,x b_1 + \sqrt{x^2+y^2}\,y a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$\left(x^2+y^2\right)^{\frac{3}{2}} k^2 b_2 - \left(x^2+y^2\right) kx b_2 + \left(x^2+y^2\right) ky a_2 - k\,x^2 y a_2 - kx\,y^2 a_3 - kx\,y^2 b_2 - k\,y^3 b_3 + \left(x^2+y^2\right) k b_1 - kxy a_1 - k\,y^2 b_1 - \sqrt{x^2+y^2}\,x b_1 + \sqrt{x^2+y^2}\,y a_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$k^2 x^2 \sqrt{x^2+y^2}\,b_2 + k^2 \sqrt{x^2+y^2}\,y^2 b_2 - k\,x^3 b_2 - kx\,y^2 a_3 - 2kx\,y^2 b_2 + k\,y^3 a_2 - k\,y^3 b_3 + k\,x^2 b_1 - kxy a_1 - \sqrt{x^2+y^2}\,x b_1 + \sqrt{x^2+y^2}\,y a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$k^2 v_1^2 v_3 b_2 + k^2 v_3 v_2^2 b_2 + k v_2^3 a_2 - k v_1 v_2^2 a_3 - k v_1^3 b_2 - 2k v_1 v_2^2 b_2 \\ - k v_2^3 b_3 - k v_1 v_2 a_1 + k v_1^2 b_1 + v_3 v_2 a_1 - v_3 v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-k v_1^3 b_2 + k^2 v_1^2 v_3 b_2 + k v_1^2 b_1 + (-k a_3 - 2k b_2) v_1 v_2^2 - k v_1 v_2 a_1 \\ - v_3 v_1 b_1 + (k a_2 - k b_3) v_2^3 + k^2 v_3 v_2^2 b_2 + v_3 v_2 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ k b_1 &= 0 \\ k^2 b_2 &= 0 \\ -b_1 &= 0 \\ -k a_1 &= 0 \\ -k b_2 &= 0 \\ k a_2 - k b_3 &= 0 \\ -k a_3 - 2k b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y}{k\sqrt{x^2+y^2}-x}\right)(x) \\
&= \frac{yk\sqrt{x^2+y^2}}{k\sqrt{x^2+y^2}-x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{yk\sqrt{x^2+y^2}}{k\sqrt{x^2+y^2}-x}} dy
\end{aligned}$$

Which results in

$$S = \ln(y) + \frac{x\ln\left(\frac{2x^2+2\sqrt{x^2}\,\sqrt{x^2+y^2}}{y}\right)}{k\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y}{k\sqrt{x^2+y^2}-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{\sqrt{x^2+y^2}+x}{k\sqrt{x^2+y^2}\,x} \\
S_y &= \frac{x(k-1)\sqrt{x^2+y^2}+(k-1)\,x^2+y^2k}{\sqrt{x^2+y^2}\,\left(\sqrt{x^2+y^2}+x\right)yk}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left((k-1)\,x^2+y^2k\right)\sqrt{x^2+y^2}+x(x^2+y^2)\,(k-1)}{\sqrt{x^2+y^2}\,kx\left(\sqrt{x^2+y^2}+x\right)\left(k\sqrt{x^2+y^2}-x\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{(k-1)\,R^2\sqrt{R^2}+R^3(k-1)}{\sqrt{R^2}\,kR\left(\sqrt{R^2}+R\right)\left(k\sqrt{R^2}-R\right)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{(-k+1)\ln(R)}{k\left(\operatorname{csgn}(R)-k\right)} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(x+\sqrt{x^2+y^2}\right)+(k-1)\ln(y)+\ln(2)+\ln(x)}{k} = \frac{(-k+1)\ln(x)}{k\left(\operatorname{csgn}(x)-k\right)} + c_1$$

Which simplifies to

$$\frac{\ln\left(x+\sqrt{x^2+y^2}\right)+(k-1)\ln(y)-c_1k+\ln(2)}{k}=0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(x+\sqrt{x^2+y^2}\right)+(k-1)\ln(y)-c_1k+\ln(2)}{k}=0 \tag{1}$$

Verification of solutions

$$\frac{\ln\left(x+\sqrt{x^2+y^2}\right)+(k-1)\ln(y)-c_1k+\ln(2)}{k}=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 32

```
dsolve(diff(y(x),x)=y(x)/(x-k*sqrt(x^2+y(x)^2)),y(x), singsol=all)
```

$$-c_1+y(x)^{k-1}\sqrt{x^2+y(x)^2}+xy(x)^{k-1}=0$$

✓ Solution by Mathematica

Time used: 0.22 (sec). Leaf size: 59

```
DSolve[y'[x]==y[x]/(x-k*Sqrt[x^2+y[x]^2]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\left((k-1)\log\left(\sqrt{\frac{y(x)^2}{x^2}+1}-1\right)+(k+1)\log\left(\sqrt{\frac{y(x)^2}{x^2}+1}+1\right)\right)=-k\log(x)+c_1,y(x)\right]$$

2.22 problem 22

Internal problem ID [1921]
Internal file name [OUTPUT/1922_Sunday_June_05_2022_02_39_31_AM_50898880/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y^2(yy' - x) = -x^3}$$

2.22.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2(u(x)\,x(u'(x)\,x + u(x)) - x) = -x^3$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^4 - u^2 + 1}{u^3 x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^4-u^2+1}{u^3}$. Integrating both sides gives

$$\frac{1}{\frac{u^4-u^2+1}{u^3}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^4-u^2+1}{u^3}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln \left(u^4-u^2+1\right)}{4}+\frac{\sqrt{3}\, \arctan \left(\frac{\left(2 u^2-1\right) \sqrt{3}}{3}\right)}{6} = -\ln \left(x\right)+c_2$$

The solution is

$$\frac{\ln \left(u(x)^4-u(x)^2+1\right)}{4}+\frac{\sqrt{3}\, \arctan \left(\frac{\left(2 u(x)^2-1\right) \sqrt{3}}{3}\right)}{6}+\ln \left(x\right)-c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln \left(\frac{y^4}{x^4}-\frac{y^2}{x^2}+1\right)}{4}+\frac{\sqrt{3}\, \arctan \left(\frac{\left(\frac{2 y^2}{x^2}-1\right) \sqrt{3}}{3}\right)}{6}+\ln \left(x\right)-c_2 = 0$$

$$\frac{\ln \left(\frac{y^4-x^2 y^2+x^4}{x^4}\right)}{4}-\frac{\sqrt{3}\, \arctan \left(\frac{\left(-2 y^2+x^2\right) \sqrt{3}}{3 x^2}\right)}{6}+\ln \left(x\right)-c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln \left(\frac{y^4-x^2 y^2+x^4}{x^4}\right)}{4}-\frac{\sqrt{3}\, \arctan \left(\frac{\left(-2 y^2+x^2\right) \sqrt{3}}{3 x^2}\right)}{6}+\ln \left(x\right)-c_2 = 0 \tag{1}$$

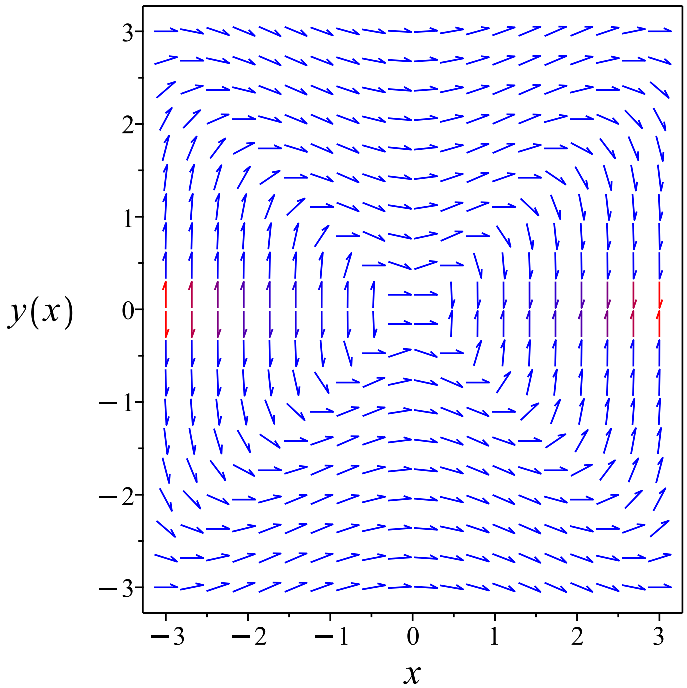


Figure 159: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^4-x^2y^2+x^4}{x^4}\right)}{4}-\frac{\sqrt{3}\,\arctan\left(\frac{(-2y^2+x^2)\sqrt{3}}{3x^2}\right)}{6}+\ln(x)-c_2=0$$

Verified OK.

2.22.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x(-x^2+y^2)}{y^3}$$

$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$

$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 &+ \frac{x(-x^2+y^2)(b_3-a_2)}{y^3} - \frac{x^2(-x^2+y^2)^2 a_3}{y^6} \\
&- \left(\frac{-x^2+y^2}{y^3} - \frac{2x^2}{y^3}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{2x}{y^2} - \frac{3x(-x^2+y^2)}{y^4}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^6a_3 - 2x^4y^2a_3 + 3x^4y^2b_2 - 4x^3y^3a_2 + 4x^3y^3b_3 - 2x^2y^4a_3 - x^2y^4b_2 + 2x\,y^5a_2 - 2x\,y^5b_3 + y^6a_3 - b_2y^6 +}{y^6}
$$
$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^6a_3 + 2x^4y^2a_3 - 3x^4y^2b_2 + 4x^3y^3a_2 - 4x^3y^3b_3 + 2x^2y^4a_3 + x^2y^4b_2 \\
&- 2x\,y^5a_2 + 2x\,y^5b_3 - y^6a_3 + b_2y^6 - 3x^3y^2b_1 + 3x^2y^3a_1 + x\,y^4b_1 - y^5a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4a_2v_1^3v_2^3 - 2a_2v_1v_2^5 - a_3v_1^6 + 2a_3v_1^4v_2^2 + 2a_3v_1^2v_2^4 - a_3v_2^6 - 3b_2v_1^4v_2^2 + b_2v_1^2v_2^4 \\
&+ b_2v_2^6 - 4b_3v_1^3v_2^3 + 2b_3v_1v_2^5 + 3a_1v_1^2v_2^3 - a_1v_2^5 - 3b_1v_1^3v_2^2 + b_1v_1v_2^4 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&-a_3 v_1^6 + (2a_3 - 3b_2)\, v_1^4 v_2^2 + (4a_2 - 4b_3)\, v_1^3 v_2^3 - 3b_1 v_1^3 v_2^2 + (2a_3 + b_2)\, v_1^2 v_2^4 \\ &+ 3a_1 v_1^2 v_2^3 + (-2a_2 + 2b_3)\, v_1 v_2^5 + b_1 v_1 v_2^4 + (-a_3 + b_2)\, v_2^6 - a_1 v_2^5 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ 3a_1 &= 0 \\ -a_3 &= 0 \\ -3b_1 &= 0 \\ -2a_2 + 2b_3 &= 0 \\ 4a_2 - 4b_3 &= 0 \\ -a_3 + b_2 &= 0 \\ 2a_3 - 3b_2 &= 0 \\ 2a_3 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x(-x^2+y^2)}{y^3}\right)(x) \\
&= \frac{x^4 - x^2y^2 + y^4}{y^3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^4-x^2y^2+y^4}{y^3}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(x^4 - x^2y^2 + y^4\right)}{4} + \frac{\sqrt{3}\,\arctan\left(\frac{(-x^2+2y^2)\sqrt{3}}{3x^2}\right)}{6}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x(-x^2+y^2)}{y^3}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x^3 - x\,y^2}{x^4 - x^2y^2 + y^4} \\ S_y &= \frac{y^3}{x^4 - x^2y^2 + y^4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

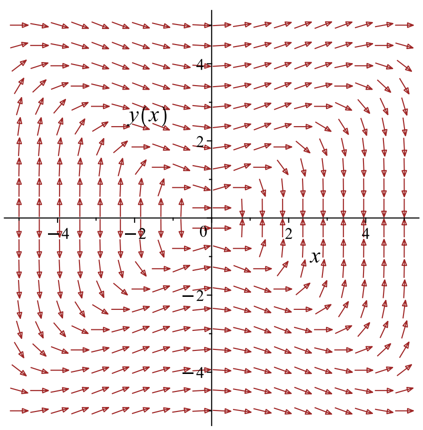
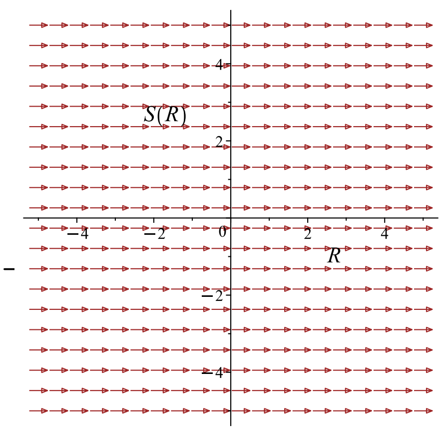
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(y^4 - x^2y^2 + x^4\right)}{4} - \frac{\sqrt{3}\,\arctan\left(\frac{(-2y^2+x^2)\sqrt{3}}{3x^2}\right)}{6} = c_1$$

Which simplifies to

$$\frac{\ln\left(y^4 - x^2y^2 + x^4\right)}{4} - \frac{\sqrt{3}\,\arctan\left(\frac{(-2y^2+x^2)\sqrt{3}}{3x^2}\right)}{6} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(-x^2+y^2)}{y^3}$ 	$R = x$ $S = \frac{\ln(x^4 - x^2y^2 + y^4)}{4}$ -	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^4 - x^2y^2 + x^4)}{4} - \frac{\sqrt{3}\arctan\left(\frac{(-2y^2+x^2)\sqrt{3}}{3x^2}\right)}{6} = c_1 \tag{1}$$

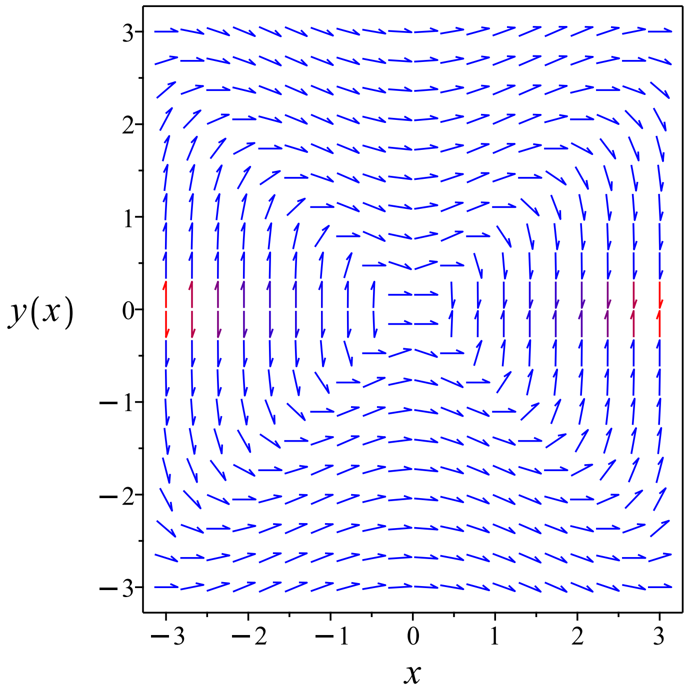


Figure 160: Slope field plot

Verification of solutions

$$\frac{\ln(y^4 - x^2y^2 + x^4)}{4} - \frac{\sqrt{3}\arctan\left(\frac{(-2y^2+x^2)\sqrt{3}}{3x^2}\right)}{6} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 56

```
dsolve(y(x)^2*(y(x)*diff(y(x),x)-x)+x^3=0,y(x), singsol=all)
```

$$y(x) = \operatorname{RootOf}\left(2\_Z^2 + \sqrt{3}\tan\left(\operatorname{RootOf}\left(\sqrt{3}\ln\left(\sec\left(\_Z\right)^2 x^4\right) + \sqrt{3}\ln\left(3\right) - 2\sqrt{3}\ln\left(2\right) + 4\sqrt{3}\,c_1 - 2\_Z\right)\right) - 1\right) x$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 63

```
DSolve[y[x]^2*(y[x]*y'[x]-x)+x^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\arctan\left(\frac{\frac{2y(x)^2}{x^2}-1}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{1}{4}\log\left(\frac{y(x)^4}{x^4} - \frac{y(x)^2}{x^2} + 1\right) = -\log(x) + c_1, y(x)\right]$$

2.23 problem 23

Internal problem ID [1922]
Internal file name [`OUTPUT/1923_Sunday_June_05_2022_02_39_35_AM_70017145/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 6, page 25
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - \frac{y}{x} - \tanh\left(\frac{y}{x}\right) = 0}$$

2.23.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \frac{y}{x} + \tanh\left(\frac{y}{x}\right) \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$
\begin{aligned}
\frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\
u' &= \frac{1}{x} g(x)\, f(bu)^{\frac{n}{m}}
\end{aligned} \tag{2}
$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= \tanh\left(\frac{y}{x}\right)
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{\tanh(u(x))}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{\tanh(u)}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \tanh(u)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\tanh(u)}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\tanh(u)}\, du &= \int \frac{1}{x}\, dx \\
\ln(\sinh(u)) &= \ln(x) + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\sinh(u) = e^{\ln(x) + c_1}
$$

Which simplifies to

$$
\sinh(u) = c_2 x
$$

Therefore the solution is

$$
\begin{aligned}
y &= ux \\
&= x \operatorname{arcsinh}\left(c_2 x \, e^{c_1}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x \operatorname{arcsinh}\left(c_2 x \, e^{c_1}\right) \tag{1}
$$

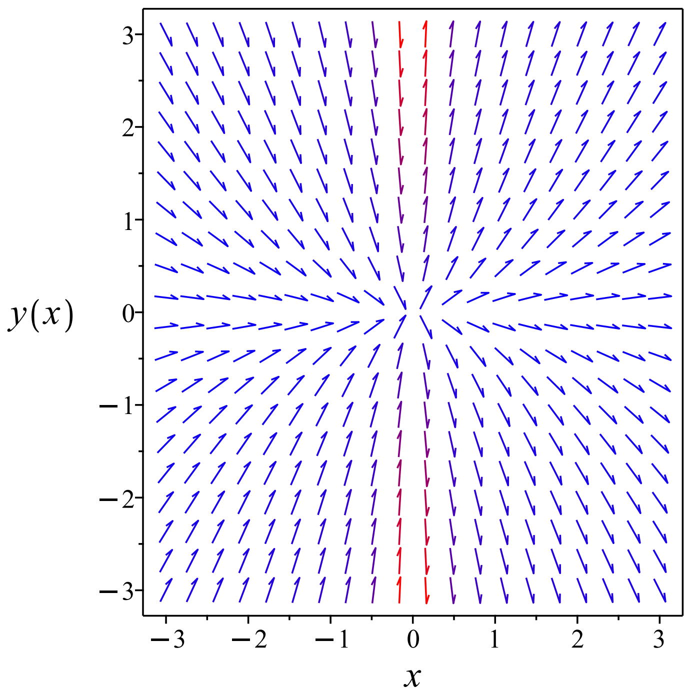


Figure 161: Slope field plot

Verification of solutions

$$
y = x \operatorname{arcsinh}\left(c_2 x \, e^{c_1}\right)
$$

Verified OK.

2.23.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x - \tanh(u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{\tanh(u)}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \tanh(u)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\tanh(u)}\,du &= \frac{1}{x}\,dx\\ \int \frac{1}{\tanh(u)}\,du &= \int \frac{1}{x}\,dx\\ \ln(\sinh(u)) &= \ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\sinh(u) = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\sinh(u) = c_3 x$$

Therefore the solution y is

$$\begin{aligned}y &= xu\\ &= x\ \mathrm{arcsinh}(c_3 \mathrm{e}^{c_2} x)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\ \mathrm{arcsinh}(c_3 \mathrm{e}^{c_2} x) \tag{1}$$

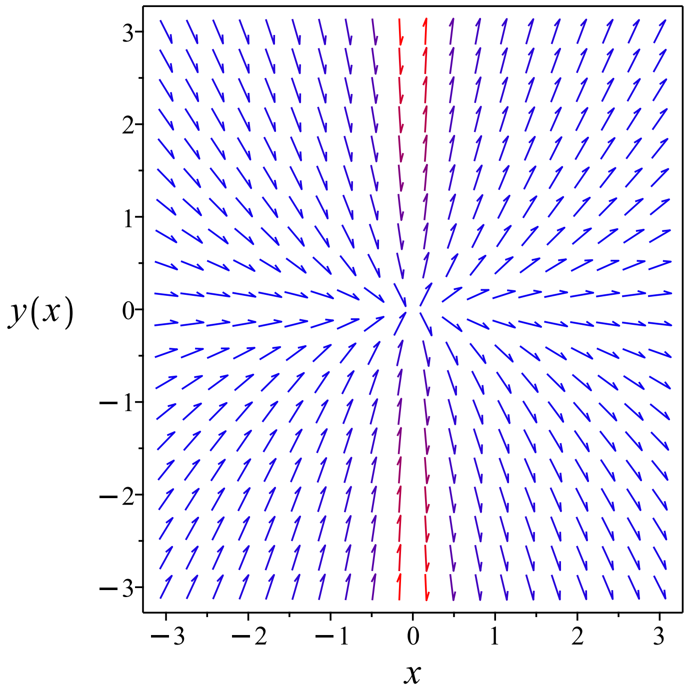


Figure 162: Slope field plot

Verification of solutions

$$y = x \operatorname{arcsinh}\left(c_3 e^{c_2} x\right)$$

Verified OK.

2.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\tanh\left(\frac{y}{x}\right) x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 97: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= yx
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{yx}{x^2} \\
&= \frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x^2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -\frac{1}{x}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\tanh\left(\frac{y}{x}\right) x + y}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{y}{x^2} \\
R_y &= \frac{1}{x} \\
S_x &= \frac{1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\coth\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\coth(R)\, S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{c_1}{\sinh(R)} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = \frac{c_1}{\sinh\left(\frac{y}{x}\right)}$$

Which simplifies to

$$-\frac{1}{x} = \frac{c_1}{\sinh\left(\frac{y}{x}\right)}$$

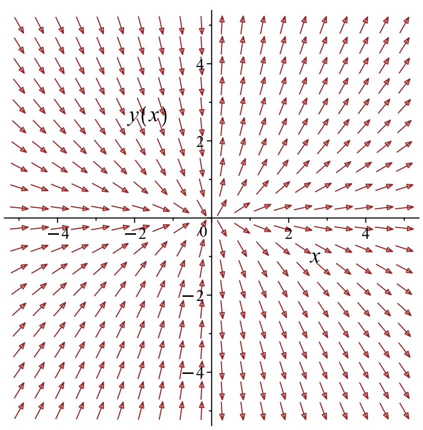
Which gives

$$y = -\operatorname{arcsinh}(c_1 x)\, x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{\tanh\left(\frac{y}{x}\right)x+y}{x}$

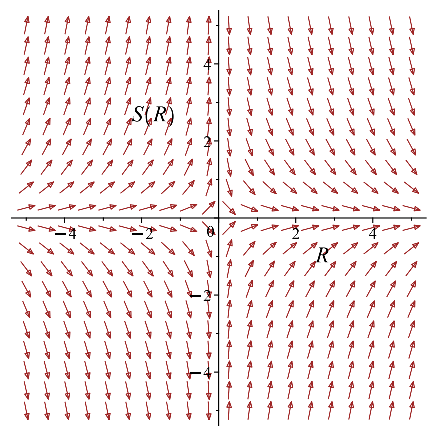


</td><td>

$$R = \frac{y}{x}$$
$$S = -\frac{1}{x}$$

</td><td>

$\frac{dS}{dR} = -\coth(R)\,S(R)$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\operatorname{arcsinh}(c_1 x)\, x \tag{1}$$

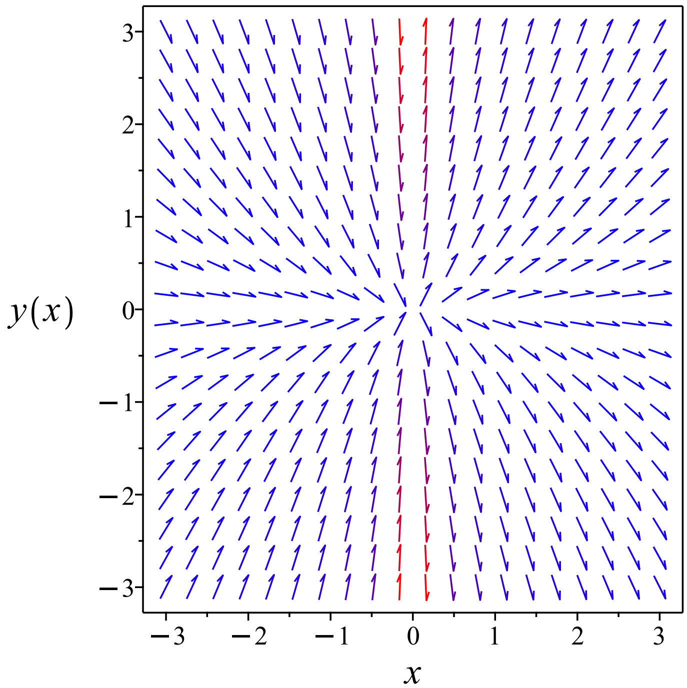


Figure 163: Slope field plot

Verification of solutions

$$y = -\operatorname{arcsinh}(c_1 x)\, x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 103

```
dsolve(diff(y(x),x)=y(x)/x+tanh(y(x)/x),y(x), singsol=all)
```

$$y(x) = \operatorname{arctanh}\left(\frac{-c_1x^2 + \sqrt{c_1x^2\left(c_1x^2 - 1\right)}}{-c_1x^2 + \sqrt{c_1x^2\left(c_1x^2 - 1\right)} + 1}\right)x$$
$$y(x) = \operatorname{arctanh}\left(\frac{c_1x^2 + \sqrt{c_1x^2\left(c_1x^2 - 1\right)}}{c_1x^2 - 1 + \sqrt{c_1x^2\left(c_1x^2 - 1\right)}}\right)x$$

✓ Solution by Mathematica

Time used: 2.284 (sec). Leaf size: 19

```
DSolve[y'[x]==y[x]/x+Tanh[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\operatorname{arcsinh}(e^{c_1}x)$$
$$y(x) \to 0$$

3 Exercise 7, page 28

3.1 problem 1

Internal problem ID [1923]
Internal file name [`OUTPUT/1923_Sunday_February_25_2024_06_37_40_AM_67136864/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y - (x - y + 2)\, y' = -x}$$

3.1.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{X + x_0 + Y(X) + y_0}{-X - x_0 + Y(X) + y_0 - 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{X + Y(X)}{-X + Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{X+Y}{-X+Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = X+Y$ and $N = X-Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u-1}{u-1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right) X u(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 1 = 0
$$

Or

$$
X(u(X) - 1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X)g(u) \\
&= -\frac{u^2+1}{X(u-1)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 1$$
$$X = x - 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{(y-1)^2}{(x+1)^2}+1\right)}{2} - \arctan\left(\frac{y-1}{x+1}\right) + \ln(x+1) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{(y-1)^2}{(x+1)^2}+1\right)}{2}-\arctan\left(\frac{y-1}{x+1}\right)+\ln(x+1)-c_2=0 \tag{1}$$

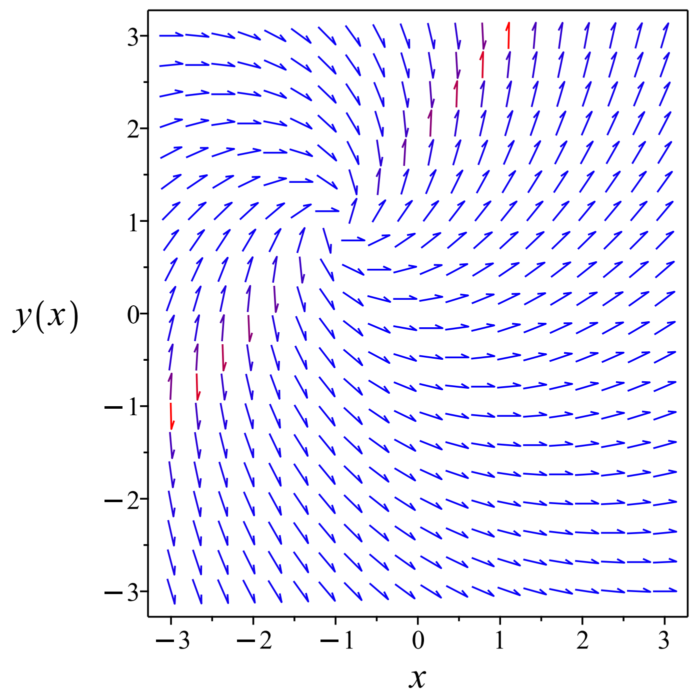


Figure 164: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{(y-1)^2}{(x+1)^2}+1\right)}{2}-\arctan\left(\frac{y-1}{x+1}\right)+\ln(x+1)-c_2=0$$

Verified OK.

3.1.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{x+y}{-x+y-2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(x+y)(b_3 - a_2)}{-x+y-2} - \frac{(x+y)^2 a_3}{(-x+y-2)^2} \\ &- \left(-\frac{1}{-x+y-2} - \frac{x+y}{(-x+y-2)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{1}{-x+y-2} + \frac{x+y}{(-x+y-2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2 a_2 + x^2 a_3 + x^2 b_2 - x^2 b_3 - 2xy a_2 + 2xy a_3 + 2xy b_2 + 2xy b_3 - y^2 a_2 - y^2 a_3 - y^2 b_2 + y^2 b_3 + 4x a_2 + 2xb}{(x-y+2)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^2 a_2 - x^2 a_3 - x^2 b_2 + x^2 b_3 + 2xy a_2 - 2xy a_3 - 2xy b_2 \\ &- 2xy b_3 + y^2 a_2 + y^2 a_3 + y^2 b_2 - y^2 b_3 - 4x a_2 - 2x b_1 + 2x b_2 \\ &+ 2x b_3 + 2y a_1 - 2y a_2 - 2y a_3 - 4y b_2 - 2a_1 - 2b_1 + 4b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 - 2b_2v_1v_2 \\
&+ b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 4a_2v_1 - 2a_2v_2 \\
&- 2a_3v_2 - 2b_1v_1 + 2b_2v_1 - 4b_2v_2 + 2b_3v_1 - 2a_1 - 2b_1 + 4b_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-a_2 - a_3 - b_2 + b_3)\, v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3)\, v_1v_2 + (-4a_2 - 2b_1 + 2b_2 + 2b_3)\, v_1 \\
&+ (a_2 + a_3 + b_2 - b_3)\, v_2^2 + (2a_1 - 2a_2 - 2a_3 - 4b_2)\, v_2 - 2a_1 - 2b_1 + 4b_2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_1 - 2b_1 + 4b_2 &= 0 \\
2a_1 - 2a_2 - 2a_3 - 4b_2 &= 0 \\
-4a_2 - 2b_1 + 2b_2 + 2b_3 &= 0 \\
-a_2 - a_3 - b_2 + b_3 &= 0 \\
a_2 + a_3 + b_2 - b_3 &= 0 \\
2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= b_2 + b_3 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= b_2 - b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x + 1$$
$$\eta = y - 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= y - 1 - \left(-\frac{x + y}{-x + y - 2}\right)(x + 1) \\ &= \frac{-x^2 - y^2 - 2x + 2y - 2}{x - y + 2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2 - y^2 - 2x + 2y - 2}{x - y + 2}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2 + 2x - 2y + 2)}{2} + \frac{2(-x - 1)\arctan\left(\frac{2y - 2}{2 + 2x}\right)}{2 + 2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{-x + y - 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + y}{x^2 + y^2 + 2x - 2y + 2} \\ S_y &= \frac{-x + y - 2}{x^2 + y^2 + 2x - 2y + 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

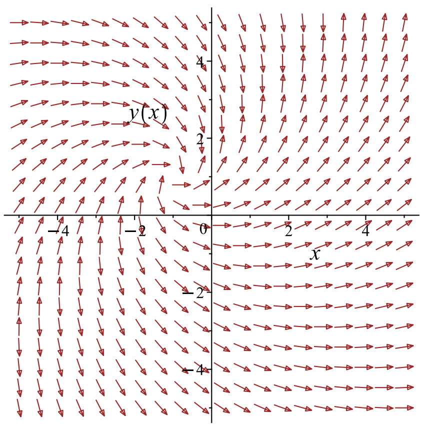
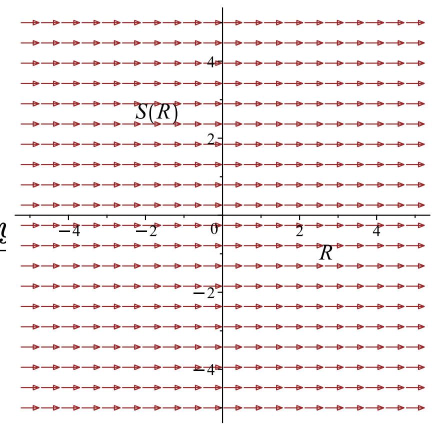
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 + x^2 - 2y + 2x + 2)}{2} - \arctan\left(\frac{y - 1}{x + 1}\right) = c_1$$

Which simplifies to

$$\frac{\ln(y^2 + x^2 - 2y + 2x + 2)}{2} - \arctan\left(\frac{y - 1}{x + 1}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{-x+y-2}$ 	$R = x$ $S = \frac{\ln(x^2 + y^2 + 2x - 2y}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2 + x^2 - 2y + 2x + 2)}{2} - \arctan\left(\frac{y-1}{x+1}\right) = c_1 \tag{1}$$

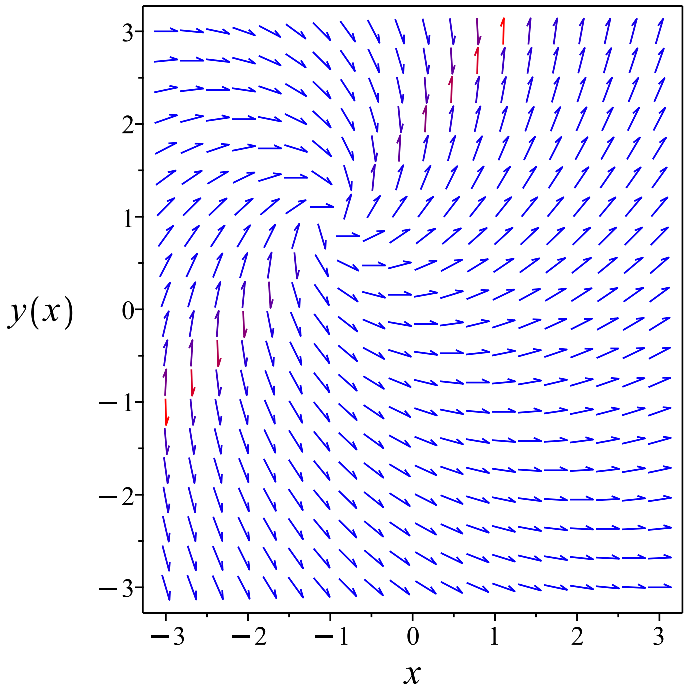


Figure 165: Slope field plot

<u>Verification of solutions</u>

$$\frac{\ln \left(y^{2}+x^{2}-2 y+2 x+2\right)}{2}-\arctan \left(\frac{y-1}{x+1}\right)=c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve((x+y(x))-(x-y(x)+2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 1 + \tan\left(\operatorname{RootOf}\left(2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x+1\right) + 2c_1\right)\right)\left(-x-1\right)$$

✓ Solution by Mathematica

Time used: 0.16 (sec). Leaf size: 56

```
DSolve[(x+y[x])-(x-y[x]+2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\arctan\left(\frac{y(x)+x}{-y(x)+x+2}\right) = \log\left(\frac{x^2+y(x)^2-2y(x)+2x+2}{2(x+1)^2}\right) + 2\log(x+1) + c_1, y(x)\right]$$

3.2 problem 2

Internal problem ID [1924]
Internal file name [`OUTPUT/1924_Sunday_February_25_2024_06_37_41_AM_44874972/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class C`], _dAlembert]
```

$$\boxed{(x-2y+2)\,y'=-x}$$

3.2.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=\frac{X+x_0}{-X-x_0+2Y\left(X\right)+2y_0-2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=0$$
$$y_0=1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=\frac{X}{-X+2Y\left(X\right)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{X}{-X+2Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -X$ and $N = X - 2Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{1}{2u-1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{1}{2u(X)-1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{\frac{1}{2u(X)-1} - u(X)}{X} = 0
$$

Or

$$
2\left(\frac{d}{dX} u(X)\right) X u(X) - \left(\frac{d}{dX} u(X)\right) X + 2u(X)^2 - u(X) - 1 = 0
$$

Or

$$
-1 + X(2u(X) - 1)\left(\frac{d}{dX} u(X)\right) + 2u(X)^2 - u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{2u^2 - u - 1}{X(2u-1)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{2u^2-u-1}{2u-1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2-u-1}{2u-1}}\,du = -\frac{1}{X}\,dX$$
$$\int \frac{1}{\frac{2u^2-u-1}{2u-1}}\,du = \int -\frac{1}{X}\,dX$$
$$\frac{\ln(u-1)}{3} + \frac{2\ln(2u+1)}{3} = -\ln(X) + c_2$$

The above can be written as

$$\begin{aligned}\frac{\ln(u-1)+2\ln(2u+1)}{3} &= -\ln(X) + c_2\\ \ln(u-1)+2\ln(2u+1) &= (3)\left(-\ln(X)+c_2\right)\\ &= -3\ln(X) + 3c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)+2\ln(2u+1)} = e^{-3\ln(X)+3c_2}$$

Which simplifies to

$$\begin{aligned}4u^3-3u-1 &= \frac{3c_2}{X^3}\\ &= \frac{c_3}{X^3}\end{aligned}$$

Which simplifies to

$$4u(X)^3 - 3u(X) - 1 = \frac{c_3 e^{3c_2}}{X^3}$$

The solution is

$$4u(X)^3 - 3u(X) - 1 = \frac{c_3 e^{3c_2}}{X^3}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{4Y(X)^3}{X^3} - \frac{3Y(X)}{X} - 1 = \frac{c_3 e^{3c_2}}{X^3}$$

Which simplifies to

$$-(X - Y(X))\left(X + 2Y(X)\right)^2 = c_3 e^{3c_2}$$

Using the solution for $Y(X)$

$$-(X - Y(X))(X + 2Y(X))^2 = c_3 e^{3c_2}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 1$$
$$X = x$$

Then the solution in y becomes

$$-(1 + x - y)(x + 2y - 2)^2 = c_3 e^{3c_2}$$

Summary

The solution(s) found are the following

$$-(1 + x - y)(x + 2y - 2)^2 = c_3 e^{3c_2} \tag{1}$$

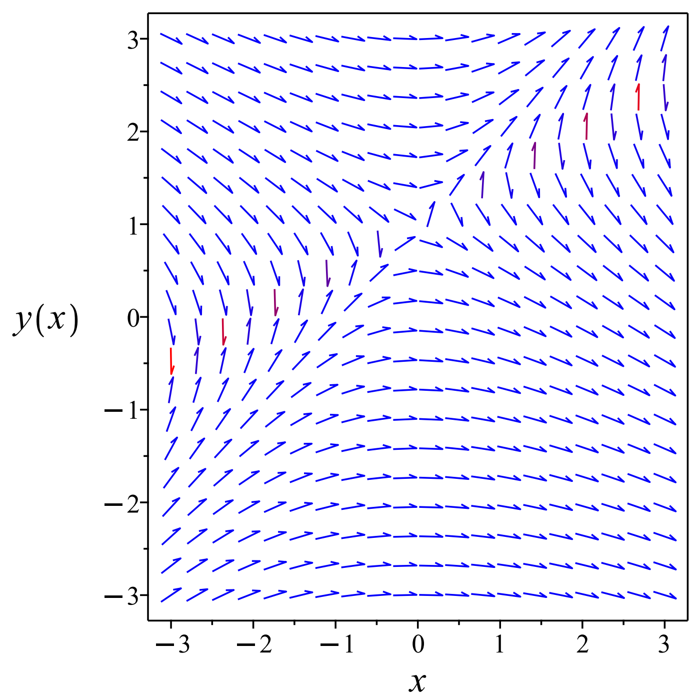


Figure 166: Slope field plot

Verification of solutions

$$
-(1+x-y)(x+2y-2)^2 = c_3 e^{3c_2}
$$

Verified OK.

3.2.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x}{-x+2y-2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{x(b_3 - a_2)}{-x+2y-2} - \frac{x^2 a_3}{(-x+2y-2)^2} \\
& - \left(\frac{1}{-x+2y-2} + \frac{x}{(-x+2y-2)^2}\right)(x a_2 + y a_3 + a_1) \\
& + \frac{2x(x b_2 + y b_3 + b_1)}{(-x+2y-2)^2} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{x^2 a_2 - x^2 a_3 + 3x^2 b_2 - x^2 b_3 - 4xy a_2 - 4xy b_2 + 4xy b_3 - 2y^2 a_3 + 4y^2 b_2 + 4x a_2 + 2x b_1 + 4x b_2 - 2b_3 x - 2y}{(x-2y+2)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& x^2a_2 - x^2a_3 + 3x^2b_2 - x^2b_3 - 4xya_2 - 4xyb_2 + 4xyb_3 - 2y^2a_3 + 4y^2b_2 \\
& + 4xa_2 + 2xb_1 + 4xb_2 - 2b_3x - 2ya_1 + 2ya_3 - 8yb_2 + 2a_1 + 4b_2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& a_2v_1^2 - 4a_2v_1v_2 - a_3v_1^2 - 2a_3v_2^2 + 3b_2v_1^2 - 4b_2v_1v_2 + 4b_2v_2^2 - b_3v_1^2 + 4b_3v_1v_2 \\
& - 2a_1v_2 + 4a_2v_1 + 2a_3v_2 + 2b_1v_1 + 4b_2v_1 - 8b_2v_2 - 2b_3v_1 + 2a_1 + 4b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
& (a_2 - a_3 + 3b_2 - b_3)\, v_1^2 + (-4a_2 - 4b_2 + 4b_3)\, v_1v_2 + (4a_2 + 2b_1 + 4b_2 - 2b_3)\, v_1 \\
& + (-2a_3 + 4b_2)\, v_2^2 + (-2a_1 + 2a_3 - 8b_2)\, v_2 + 2a_1 + 4b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 + 4b_2 &= 0 \\
-2a_3 + 4b_2 &= 0 \\
-2a_1 + 2a_3 - 8b_2 &= 0 \\
-4a_2 - 4b_2 + 4b_3 &= 0 \\
a_2 - a_3 + 3b_2 - b_3 &= 0 \\
4a_2 + 2b_1 + 4b_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -2b_2 \\
a_2 &= -b_2 + b_3 \\
a_3 &= 2b_2 \\
b_1 &= -b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y - 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - 1 - \left(\frac{x}{-x + 2y - 2}\right)(x) \\
&= \frac{x^2 + yx - 2y^2 - x + 4y - 2}{x - 2y + 2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2 + yx - 2y^2 - x + 4y - 2}{x - 2y + 2}} dy
\end{aligned}
$$

Which results in

$$S = \frac{2\ln(x+2y-2)}{3} + \frac{\ln(-1-x+y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{-x+2y-2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{(x+2y-2)(x-y+1)} \\
S_y &= \frac{x-2y+2}{(x+2y-2)(x-y+1)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

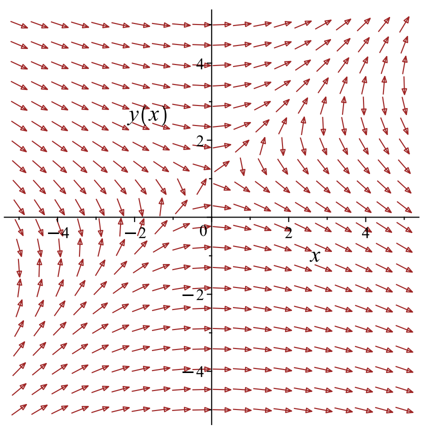
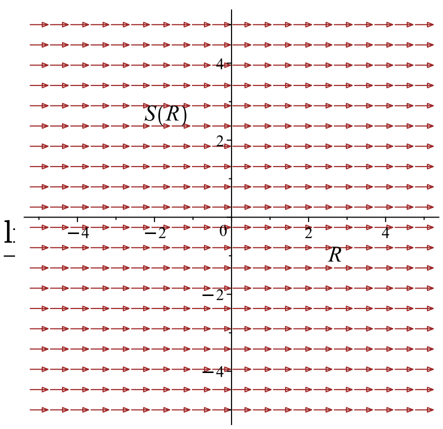
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(x+2y-2)}{3} + \frac{\ln(-1-x+y)}{3} = c_1$$

Which simplifies to

$$\frac{2\ln(x+2y-2)}{3}+\frac{\ln(-1-x+y)}{3}=c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=\frac{x}{-x+2y-2}$ 	$R = x$ $S = \frac{2\ln(x+2y-2)}{3} + \text{l}$	$\frac{dS}{dR}=0$ 

Summary

The solution(s) found are the following

$$\frac{2\ln(x+2y-2)}{3}+\frac{\ln(-1-x+y)}{3}=c_1 \tag{1}$$

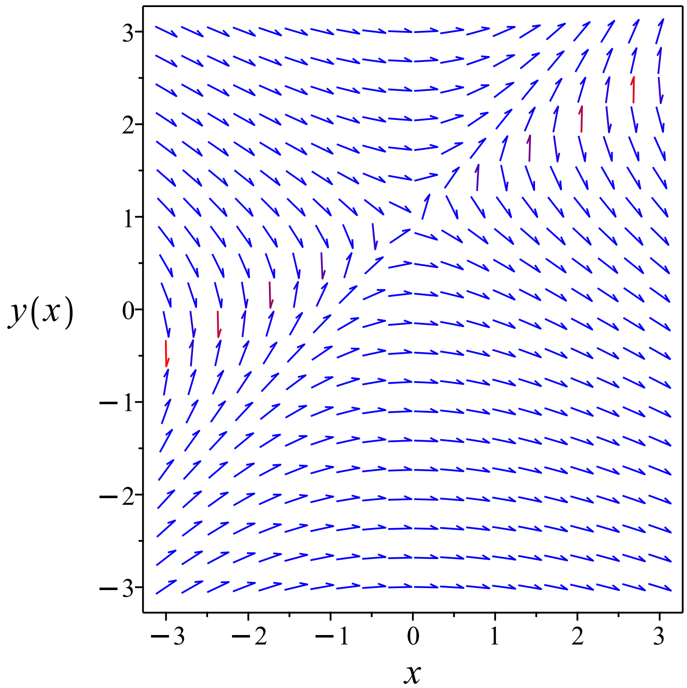


Figure 167: Slope field plot

Verification of solutions

$$\frac{2\ln\left(x+2y-2\right)}{3}+\frac{\ln\left(-1-x+y\right)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.219 (sec). Leaf size: 151

```
dsolve(x+(x-2*y(x)+2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{-2\left(2c_1x^3+2\sqrt{-2\left(c_1x^3-\frac{1}{2}\right)c_1^2x^6}\right)^{\frac{2}{3}}c_1x^2-\frac{\left(c_1x^3+\sqrt{-2\left(c_1x^3-\frac{1}{2}\right)c_1^2x^6}\right)\left(1+i\sqrt{3}\right)\left(2c_1x^3+2\sqrt{-2\left(c_1x^3-\frac{1}{2}\right)c_1^2x^6}\right)^{\frac{1}{3}}}{2}+}{2\left(2c_1x^3+2\sqrt{-2\left(c_1x^3-\frac{1}{2}\right)c_1^2x^6}\right)^{\frac{2}{3}}x^2c_1}$$

✓ Solution by Mathematica

Time used: 60.036 (sec). Leaf size: 445

```
DSolve[x+(x-2*y[x]+2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x+2}{2} - \frac{1}{2\text{Root}\left[\#1^6\left(16x^6+16e^{12c_1}\right)-24\#1^4x^4+8\#1^3x^3+9\#1^2x^2-6\#1x+1\&,1\right]}$$

$$y(x) \to \frac{x+2}{2} - \frac{1}{2\text{Root}\left[\#1^6\left(16x^6+16e^{12c_1}\right)-24\#1^4x^4+8\#1^3x^3+9\#1^2x^2-6\#1x+1\&,2\right]}$$

$$y(x) \to \frac{x+2}{2} - \frac{1}{2\text{Root}\left[\#1^6\left(16x^6+16e^{12c_1}\right)-24\#1^4x^4+8\#1^3x^3+9\#1^2x^2-6\#1x+1\&,3\right]}$$

$$y(x) \to \frac{x+2}{2} - \frac{1}{2\text{Root}\left[\#1^6\left(16x^6+16e^{12c_1}\right)-24\#1^4x^4+8\#1^3x^3+9\#1^2x^2-6\#1x+1\&,4\right]}$$

$$y(x) \to \frac{x+2}{2} - \frac{1}{2\text{Root}\left[\#1^6\left(16x^6+16e^{12c_1}\right)-24\#1^4x^4+8\#1^3x^3+9\#1^2x^2-6\#1x+1\&,5\right]}$$

$$y(x) \to \frac{x+2}{2} - \frac{1}{2\text{Root}\left[\#1^6\left(16x^6+16e^{12c_1}\right)-24\#1^4x^4+8\#1^3x^3+9\#1^2x^2-6\#1x+1\&,6\right]}$$

3.3 problem 3

Internal problem ID [1925]
Internal file name [`OUTPUT/1925_Sunday_February_25_2024_06_37_44_AM_94584042/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(x+y)\,y'-y=-1-2x}$$

3.3.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=\frac{-2X-2x_0+Y(X)+y_0-1}{X+x_0+Y(X)+y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=-\frac{1}{3}$$
$$y_0=\frac{1}{3}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=\frac{-2X+Y(X)}{X+Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{-2X + Y}{X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -2X + Y$ and $N = X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{u - 2}{u + 1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)-2}{u(X)+1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX} u(X) - \frac{\frac{u(X)-2}{u(X)+1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX} u(X)\right) X u(X) + \left(\frac{d}{dX} u(X)\right) X + u(X)^2 + 2 = 0$$

Or

$$(u(X) + 1) X \left(\frac{d}{dX} u(X)\right) + u(X)^2 + 2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X) g(u) \\ &= -\frac{u^2 + 2}{(u + 1) X} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+2}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+2}{u+1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+2}{u+1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+2)}{2} + \frac{\sqrt{2}\arctan\left(\frac{\sqrt{2}\,u}{2}\right)}{2} = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+2\right)}{2} + \frac{\sqrt{2}\arctan\left(\frac{\sqrt{2}\,u(X)}{2}\right)}{2} + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+2\right)}{2} + \frac{\sqrt{2}\arctan\left(\frac{\sqrt{2}\,Y(X)}{2X}\right)}{2} + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+2\right)}{2} + \frac{\sqrt{2}\arctan\left(\frac{\sqrt{2}\,Y(X)}{2X}\right)}{2} + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + \frac{1}{3}$$
$$X = x - \frac{1}{3}$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{\left(y-\frac{1}{3}\right)^2}{\left(x+\frac{1}{3}\right)^2}+2\right)}{2} + \frac{\sqrt{2}\arctan\left(\frac{\sqrt{2}\left(y-\frac{1}{3}\right)}{2x+\frac{2}{3}}\right)}{2} + \ln\left(x+\frac{1}{3}\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{\left(y-\frac{1}{3}\right)^2}{\left(x+\frac{1}{3}\right)^2}+2\right)}{2}+\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,\left(y-\frac{1}{3}\right)}{2x+\frac{2}{3}}\right)}{2}+\ln\left(x+\frac{1}{3}\right)-c_2=0 \tag{1}$$

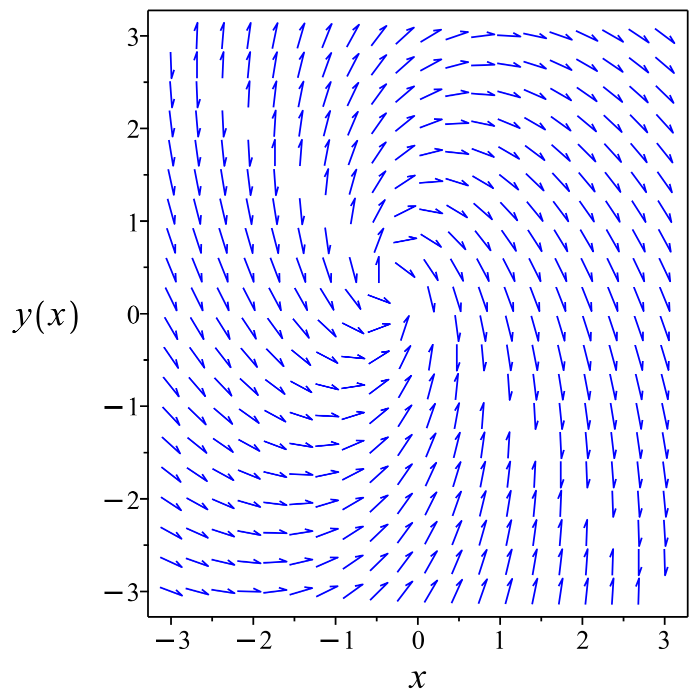


Figure 168: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{\left(y-\frac{1}{3}\right)^2}{\left(x+\frac{1}{3}\right)^2}+2\right)}{2}+\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,\left(y-\frac{1}{3}\right)}{2x+\frac{2}{3}}\right)}{2}+\ln\left(x+\frac{1}{3}\right)-c_2=0$$

Verified OK.

3.3.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-2x + y - 1}{x + y} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{(-2x + y - 1)(b_3 - a_2)}{x + y} - \frac{(-2x + y - 1)^2 a_3}{(x + y)^2} \\
& - \left(-\frac{2}{x + y} - \frac{-2x + y - 1}{(x + y)^2} \right) (x a_2 + y a_3 + a_1) \\
& - \left(\frac{1}{x + y} - \frac{-2x + y - 1}{(x + y)^2} \right) (x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x^2 a_2 - 4x^2 a_3 - 2x^2 b_2 - 2x^2 b_3 + 4xy a_2 + 4xy a_3 + 2xy b_2 - 4xy b_3 - y^2 a_2 + 2y^2 a_3 + y^2 b_2 + y^2 b_3 - 4x a_3 -}{(x + y)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 2x^2 a_2 - 4x^2 a_3 - 2x^2 b_2 - 2x^2 b_3 + 4xy a_2 + 4xy a_3 + 2xy b_2 \\
& - 4xy b_3 - y^2 a_2 + 2y^2 a_3 + y^2 b_2 + y^2 b_3 - 4x a_3 - 3x b_1 \\
& - x b_2 - x b_3 + 3y a_1 + y a_2 + y a_3 - 2y b_3 - a_1 - a_3 - b_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} 2a_2v_1^2 + 4a_2v_1v_2 - a_2v_2^2 - 4a_3v_1^2 + 4a_3v_1v_2 + 2a_3v_2^2 - 2b_2v_1^2 \\ + 2b_2v_1v_2 + b_2v_2^2 - 2b_3v_1^2 - 4b_3v_1v_2 + b_3v_2^2 + 3a_1v_2 + a_2v_2 \\ - 4a_3v_1 + a_3v_2 - 3b_1v_1 - b_2v_1 - b_3v_1 - 2b_3v_2 - a_1 - a_3 - b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} (2a_2 - 4a_3 - 2b_2 - 2b_3) v_1^2 + (4a_2 + 4a_3 + 2b_2 - 4b_3) v_1v_2 + (-4a_3 - 3b_1 - b_2 - b_3) v_1 \\ + (-a_2 + 2a_3 + b_2 + b_3) v_2^2 + (3a_1 + a_2 + a_3 - 2b_3) v_2 - a_1 - a_3 - b_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_1 - a_3 - b_1 &= 0 \\ 3a_1 + a_2 + a_3 - 2b_3 &= 0 \\ -a_2 + 2a_3 + b_2 + b_3 &= 0 \\ 2a_2 - 4a_3 - 2b_2 - 2b_3 &= 0 \\ 4a_2 + 4a_3 + 2b_2 - 4b_3 &= 0 \\ -4a_3 - 3b_1 - b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -a_3 - b_1 \\ a_2 &= -3b_1 - 2a_3 \\ a_3 &= a_3 \\ b_1 &= b_1 \\ b_2 &= -2a_3 \\ b_3 &= -3b_1 - 2a_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -3x - 1$$
$$\eta = -3y + 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= -3y + 1 - \left(\frac{-2x + y - 1}{x + y}\right)(-3x - 1) \\
&= \frac{-6x^2 - 3y^2 - 4x + 2y - 1}{x + y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-6x^2 - 3y^2 - 4x + 2y - 1}{x + y}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(6x^2 + 3y^2 + 4x - 2y + 1)}{6} + \frac{\left(-x - \frac{1}{3}\right)\sqrt{2}\, \arctan\left(\frac{(6y - 2)\sqrt{2}}{12x + 4}\right)}{6x + 2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-2x + y - 1}{x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-2x + y - 1}{6x^2 + 3y^2 + 4x - 2y + 1} \\
S_y &= \frac{-x - y}{6x^2 + 3y^2 + 4x - 2y + 1}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

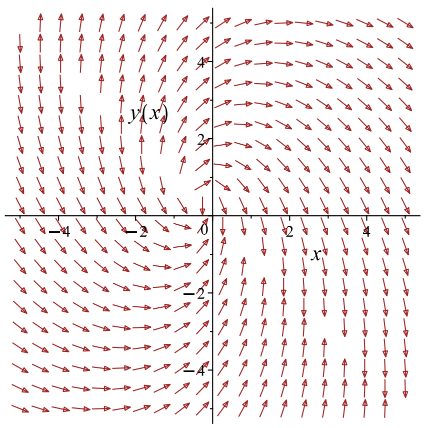
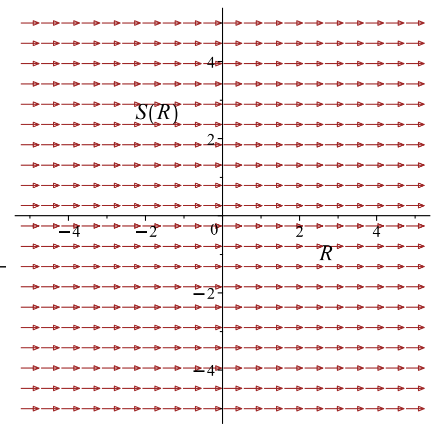
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(3y^2 + 6x^2 - 2y + 4x + 1)}{6} - \frac{\sqrt{2}\arctan\left(\frac{(3y-1)\sqrt{2}}{6x+2}\right)}{6} = c_1$$

Which simplifies to

$$-\frac{\ln \left(3 y^{2}+6 x^{2}-2 y+4 x+1\right)}{6}-\frac{\sqrt{2}\, \arctan \left(\frac{(3 y-1) \sqrt{2}}{6 x+2}\right)}{6}=c_{1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=\frac{-2x+y-1}{x+y}$ 	$R = x$ $S = -\frac{\ln\left(6x^{2}+3y^{2}+4x\right.}{6}$ $\left.2\right)$	$\frac{dS}{dR}=0$ 

Summary

The solution(s) found are the following

$$-\frac{\ln \left(3 y^{2}+6 x^{2}-2 y+4 x+1\right)}{6}-\frac{\sqrt{2}\, \arctan \left(\frac{(3 y-1) \sqrt{2}}{6 x+2}\right)}{6}=c_{1} \tag{1}$$

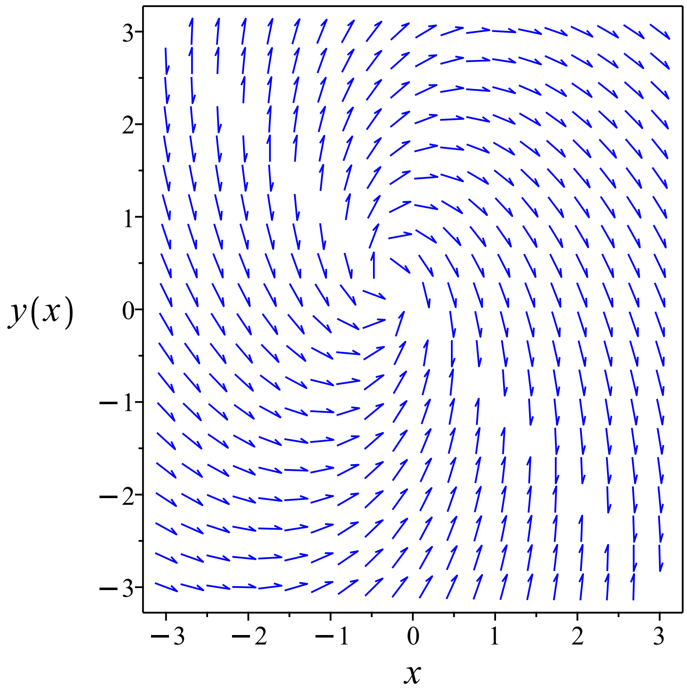


Figure 169: Slope field plot

Verification of solutions

$$-\frac{\ln\left(3y^2+6x^2-2y+4x+1\right)}{6}-\frac{\sqrt{2}\,\arctan\left(\frac{(3y-1)\sqrt{2}}{6x+2}\right)}{6}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.188 (sec). Leaf size: 51

```
dsolve((2*x-y(x)+1)+(x+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{3} + \frac{\tan\left(\operatorname{RootOf}\left(\sqrt{2}\,\ln\left(\sec\left(\_Z\right)^2 (3x+1)^2\right) + \sqrt{2}\,\ln(2) + 2\sqrt{2}\,c_1 - 2\_Z\right)\right)\sqrt{2}\,(-3x-1)}{3}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 77

```
DSolve[(2*x-y[x]+1)+(x+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\sqrt{2}\arctan\left(\frac{-y(x)+2x+1}{\sqrt{2}(y(x)+x)}\right) = 2\log\left(\frac{6x^2+3y(x)^2-2y(x)+4x+1}{(3x+1)^2}\right) + 4\log(3x+1)+3c_1, y(x)\right]$$

3.4 problem 4

Internal problem ID [1926]
Internal file name [`OUTPUT/1926_Sunday_February_25_2024_06_37_46_AM_40692390/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y + (x - 1 + y)\, y' = -x - 2}$$

3.4.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-X - x_0 + Y(X) + y_0 - 2}{X + x_0 - 1 + Y(X) + y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -\frac{1}{2}$$
$$y_0 = \frac{3}{2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-X + Y(X)}{X + Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= \frac{-X+Y}{X+Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = -X+Y$ and $N = X+Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u-1}{u+1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)-1}{u(X)+1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{u(X)-1}{u(X)+1} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right) X u(X) + \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 1 = 0
$$

Or

$$
(u(X)+1)\, X \left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{u^2+1}{(u+1)\, X}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u+1}} du = -\frac{1}{X} dX$$

$$\int \frac{1}{\frac{u^2+1}{u+1}} du = \int -\frac{1}{X} dX$$

$$\frac{\ln(u^2+1)}{2} + \arctan(u) = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+1\right)}{2} + \arctan(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} + \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} + \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + \frac{3}{2}$$
$$X = x - \frac{1}{2}$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{\left(y-\frac{3}{2}\right)^2}{\left(x+\frac{1}{2}\right)^2}+1\right)}{2} + \arctan\left(\frac{y-\frac{3}{2}}{x+\frac{1}{2}}\right) + \ln\left(x+\frac{1}{2}\right) - c_2 = 0$$

Summary
The solution(s) found are the following

$$\frac{\ln\left(\frac{\left(y-\frac{3}{2}\right)^{2}}{\left(x+\frac{1}{2}\right)^{2}}+1\right)}{2}+\arctan\left(\frac{y-\frac{3}{2}}{x+\frac{1}{2}}\right)+\ln\left(x+\frac{1}{2}\right)-c_2=0 \tag{1}$$

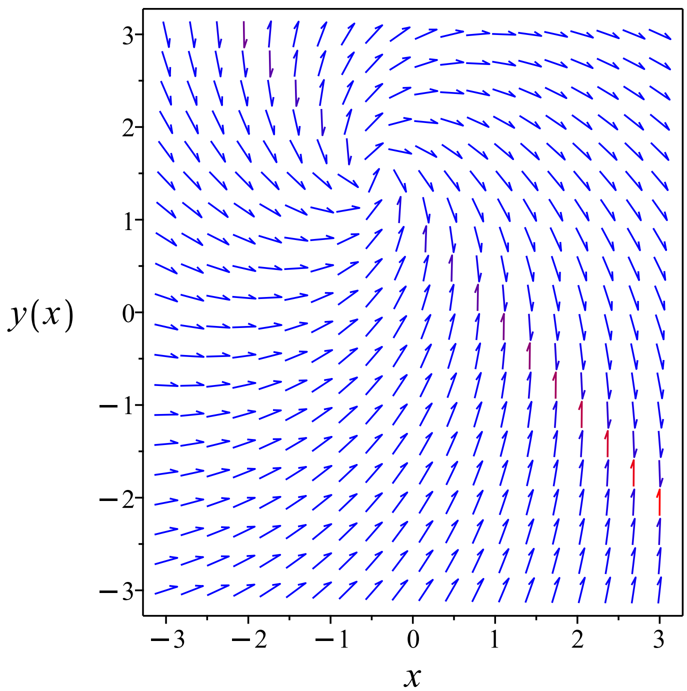


Figure 170: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{\left(y-\frac{3}{2}\right)^{2}}{\left(x+\frac{1}{2}\right)^{2}}+1\right)}{2}+\arctan\left(\frac{y-\frac{3}{2}}{x+\frac{1}{2}}\right)+\ln\left(x+\frac{1}{2}\right)-c_2=0$$

Verified OK.

3.4.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-x+y-2}{x-1+y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{(-x+y-2)(b_3-a_2)}{x-1+y} - \frac{(-x+y-2)^2 a_3}{(x-1+y)^2} \\
&- \left(-\frac{1}{x-1+y} - \frac{-x+y-2}{(x-1+y)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(\frac{1}{x-1+y} - \frac{-x+y-2}{(x-1+y)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{x^2 a_2 - x^2 a_3 - x^2 b_2 - x^2 b_3 + 2xy a_2 + 2xy a_3 + 2xy b_2 - 2xy b_3 - y^2 a_2 + y^2 a_3 + y^2 b_2 + y^2 b_3 - 2x a_2 - 4x a_3}{(x-1+y)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&x^2 a_2 - x^2 a_3 - x^2 b_2 - x^2 b_3 + 2xy a_2 + 2xy a_3 + 2xy b_2 - 2xy b_3 - y^2 a_2 \\
&+ y^2 a_3 + y^2 b_2 + y^2 b_3 - 2x a_2 - 4x a_3 - 2x b_1 - 3x b_2 - x b_3 + 2y a_1 \\
&+ 3y a_2 + y a_3 - 2y b_2 - 4y b_3 - 3a_1 - 2a_2 - 4a_3 - b_1 + b_2 + 2b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_2 v_1^2 + 2a_2 v_1 v_2 - a_2 v_2^2 - a_3 v_1^2 + 2a_3 v_1 v_2 + a_3 v_2^2 - b_2 v_1^2 + 2b_2 v_1 v_2 + b_2 v_2^2 \\ & - b_3 v_1^2 - 2b_3 v_1 v_2 + b_3 v_2^2 + 2a_1 v_2 - 2a_2 v_1 + 3a_2 v_2 - 4a_3 v_1 + a_3 v_2 - 2b_1 v_1 \\ & - 3b_2 v_1 - 2b_2 v_2 - b_3 v_1 - 4b_3 v_2 - 3a_1 - 2a_2 - 4a_3 - b_1 + b_2 + 2b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & (a_2 - a_3 - b_2 - b_3) v_1^2 + (2a_2 + 2a_3 + 2b_2 - 2b_3) v_1 v_2 \\ & + (-2a_2 - 4a_3 - 2b_1 - 3b_2 - b_3) v_1 + (-a_2 + a_3 + b_2 + b_3) v_2^2 \\ & + (2a_1 + 3a_2 + a_3 - 2b_2 - 4b_3) v_2 - 3a_1 - 2a_2 - 4a_3 - b_1 + b_2 + 2b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_2 + a_3 + b_2 + b_3 &= 0 \\ a_2 - a_3 - b_2 - b_3 &= 0 \\ 2a_2 + 2a_3 + 2b_2 - 2b_3 &= 0 \\ 2a_1 + 3a_2 + a_3 - 2b_2 - 4b_3 &= 0 \\ -2a_2 - 4a_3 - 2b_1 - 3b_2 - b_3 &= 0 \\ -3a_1 - 2a_2 - 4a_3 - b_1 + b_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 2a_1 - 3b_2 \\ a_3 &= -b_2 \\ b_1 &= -3a_1 + 5b_2 \\ b_2 &= b_2 \\ b_3 &= 2a_1 - 3b_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1 + 2x$$
$$\eta = -3 + 2y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= -3 + 2y - \left(\frac{-x + y - 2}{x - 1 + y}\right)(1 + 2x) \\ &= \frac{2x^2 + 2y^2 + 2x - 6y + 5}{x - 1 + y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x^2+2y^2+2x-6y+5}{x-1+y}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln\left(2x^2 + 2y^2 + 2x - 6y + 5\right)}{4} + \frac{2\left(x + \frac{1}{2}\right)\arctan\left(\frac{4y-6}{4x+2}\right)}{4x + 2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x+y-2}{x-1+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x-y+2}{2x^2+2y^2+2x-6y+5} \\ S_y &= \frac{x-1+y}{2x^2+2y^2+2x-6y+5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

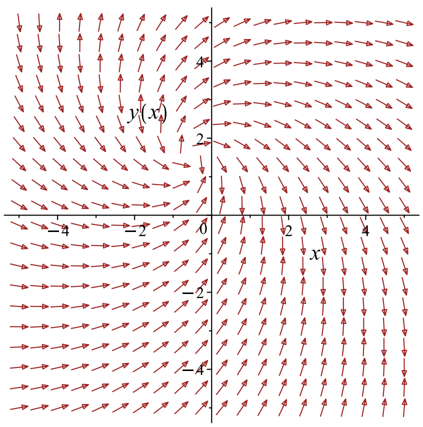
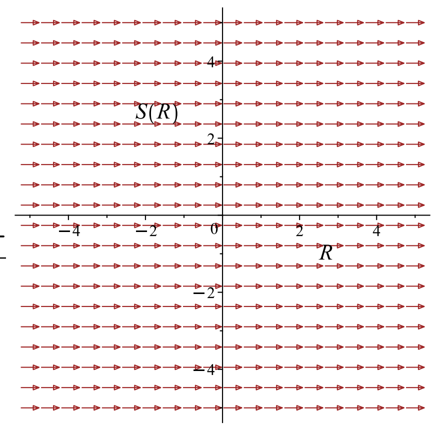
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2y^2+2x^2-6y+2x+5)}{4} + \frac{\arctan\left(\frac{-3+2y}{1+2x}\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(2y^2+2x^2-6y+2x+5)}{4} + \frac{\arctan\left(\frac{-3+2y}{1+2x}\right)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+y-2}{x-1+y}$ 	$R = x$ $S = \frac{\ln(2x^2 + 2y^2 + 2x -}{4}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(2y^2 + 2x^2 - 6y + 2x + 5)}{4} + \frac{\arctan\left(\frac{-3+2y}{1+2x}\right)}{2} = c_1 \tag{1}$$

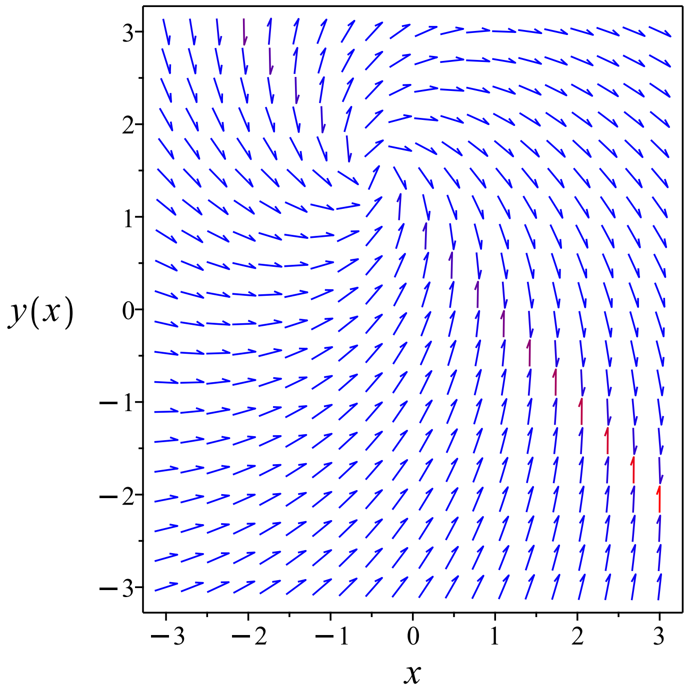


Figure 171: Slope field plot

Verification of solutions

$$\frac{\ln\left(2y^2+2x^2-6y+2x+5\right)}{4}+\frac{\arctan\left(\frac{-3+2y}{1+2x}\right)}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 34

```
dsolve((x-y(x)+2)+(x+y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3}{2} + \tan\left(\text{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(2x+1\right) + 2c_1\right)\right)\left(-\frac{1}{2} - x\right)$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 63

```
DSolve[(x-y[x]+2)+(x+y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\arctan\left(\frac{y(x)-x-2}{y(x)+x-1}\right) + \log\left(\frac{2x^2+2y(x)^2-6y(x)+2x+5}{(2x+1)^2}\right) + 2\log(2x+1) + c_1 = 0, y(x)\right]$$

3.5 problem 5

Internal problem ID [1927]
Internal file name [`OUTPUT/1927_Sunday_February_25_2024_06_37_47_AM_8254222/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{-y+(1-x+y)\,y'=-x}$$

3.5.1 Solving as differentialType ode

Writing the ode as

$$y'=\frac{-x+y}{1-x+y} \tag{1}$$

Which becomes

$$(-1-y)\,dy=(-x)\,dy+(x-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy+(x-y)\,dx=d\left(\frac{1}{2}x^2-yx\right)$$

Hence (2) becomes

$$(-1-y)\,dy = d\left(\frac{1}{2}x^2 - yx\right)$$

Integrating both sides gives gives these solutions

$$y = x - 1 + \sqrt{-2c_1 - 2x + 1} + c_1$$
$$y = x - 1 - \sqrt{-2c_1 - 2x + 1} + c_1$$

Summary

The solution(s) found are the following

$$y = x - 1 + \sqrt{-2c_1 - 2x + 1} + c_1 \tag{1}$$
$$y = x - 1 - \sqrt{-2c_1 - 2x + 1} + c_1 \tag{2}$$

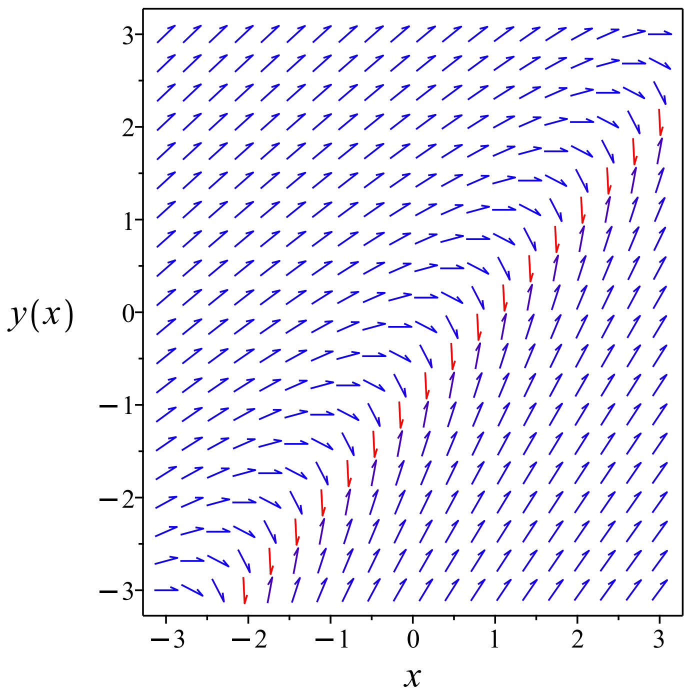


Figure 172: Slope field plot

Verification of solutions

$$y = x - 1 + \sqrt{-2c_1 - 2x + 1} + c_1$$

Verified OK.

$$y = x - 1 - \sqrt{-2c_1 - 2x + 1} + c_1$$

Verified OK.

3.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-x+y}{-x+y+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2 + ya_3 + a_1 \tag{1E}
$$

$$
\eta = xb_2 + yb_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{(-x+y)(b_3 - a_2)}{-x+y+1} - \frac{(-x+y)^2 a_3}{(-x+y+1)^2} \\
& - \left(-\frac{1}{-x+y+1} + \frac{-x+y}{(-x+y+1)^2}\right)(xa_2 + ya_3 + a_1) \\
& - \left(\frac{1}{-x+y+1} - \frac{-x+y}{(-x+y+1)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^2a_2 + x^2a_3 - x^2b_2 - x^2b_3 - 2xya_2 - 2xya_3 + 2xyb_2 + 2xyb_3 + y^2a_2 + y^2a_3 - y^2b_2 - y^2b_3 - 2xa_2 + 3xb}{(x-y-1)^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -x^2a_2 - x^2a_3 + x^2b_2 + x^2b_3 + 2xya_2 + 2xya_3 - 2xyb_2 - 2xyb_3 - y^2a_2 - y^2a_3 \\
& + y^2b_2 + y^2b_3 + 2xa_2 - 3xb_2 - xb_3 - ya_2 + ya_3 + 2yb_2 + a_1 - b_1 + b_2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^2 + 2a_2v_1v_2 - a_2v_2^2 - a_3v_1^2 + 2a_3v_1v_2 - a_3v_2^2 + b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 + b_3v_2^2 + 2a_2v_1 - a_2v_2 + a_3v_2 - 3b_2v_1 + 2b_2v_2 - b_3v_1 + a_1 - b_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 + b_2 + b_3) v_1^2 + (2a_2 + 2a_3 - 2b_2 - 2b_3) v_1v_2 + (2a_2 - 3b_2 - b_3) v_1 + (-a_2 - a_3 + b_2 + b_3) v_2^2 + (-a_2 + a_3 + 2b_2) v_2 + a_1 - b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 - b_1 + b_2 &= 0 \\
-a_2 + a_3 + 2b_2 &= 0 \\
2a_2 - 3b_2 - b_3 &= 0 \\
-a_2 - a_3 + b_2 + b_3 &= 0 \\
2a_2 + 2a_3 - 2b_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= b_1 + 2a_3 - b_3 \\
a_2 &= -3a_3 + 2b_3 \\
a_3 &= a_3 \\
b_1 &= b_1 \\
b_2 &= -2a_3 + b_3 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 - \left(\frac{-x+y}{-x+y+1}\right)(1) \\
&= -\frac{1}{x-y-1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{1}{x-y-1}} dy
\end{aligned}$$

Which results in

$$S = -yx + \frac{1}{2}y^2 + y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x + y}{-x + y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y \\ S_y &= -x + y + 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R^2}{2} + c_1 \tag{4}$$

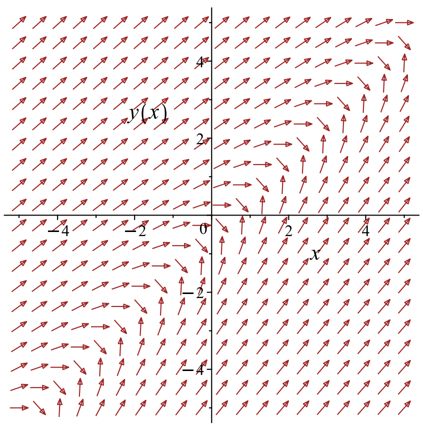
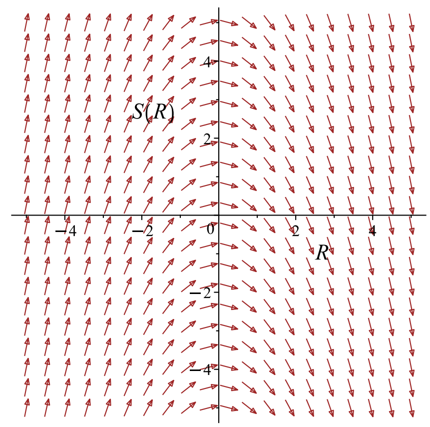
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{y(2x - y - 2)}{2} = -\frac{x^2}{2} + c_1$$

Which simplifies to

$$-\frac{y(2x - y - 2)}{2} = -\frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+y}{-x+y+1}$ 	$R = x$ $S = -\frac{y(2x-y-2)}{2}$	$\frac{dS}{dR} = -R$ 

Summary

The solution(s) found are the following

$$-\frac{y(2x-y-2)}{2} = -\frac{x^2}{2} + c_1 \tag{1}$$

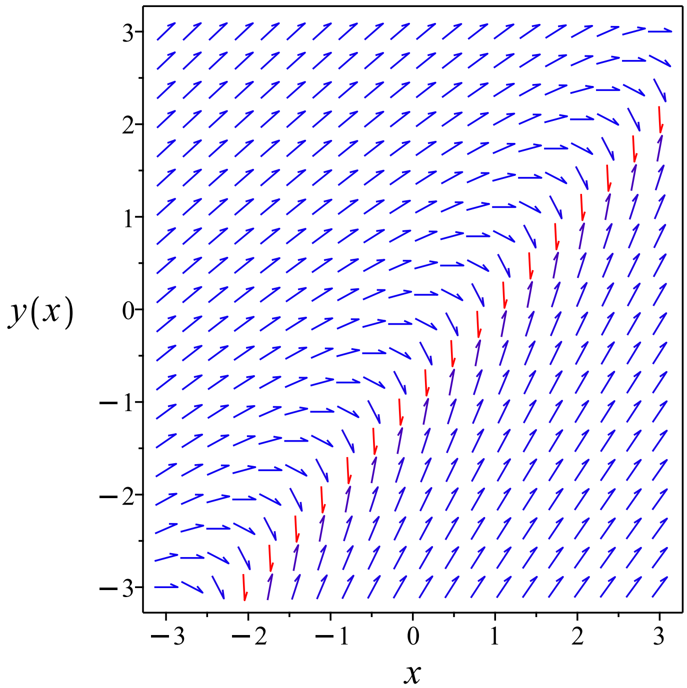


Figure 173: Slope field plot

Verification of solutions

$$-\frac{y(2x-y-2)}{2} = -\frac{x^2}{2} + c_1$$

Verified OK.

3.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x + y + 1)\,\mathrm{d}y &= (-x + y)\,\mathrm{d}x \\ (x - y)\,\mathrm{d}x + (-x + y + 1)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x - y \\ N(x, y) &= -x + y + 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x - y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x + y + 1) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x - y \, \mathrm{d}x$$

$$\phi = \frac{x(x - 2y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x + y + 1$. Therefore equation (4) becomes

$$-x + y + 1 = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y + 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y + 1) \, \mathrm{d}y$$

$$f(y) = \frac{1}{2}y^2 + y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x - 2y)}{2} + \frac{y^2}{2} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x-2y)}{2} + \frac{y^2}{2} + y$$

Summary

The solution(s) found are the following

$$\frac{x(x-2y)}{2} + \frac{y^2}{2} + y = c_1 \tag{1}$$

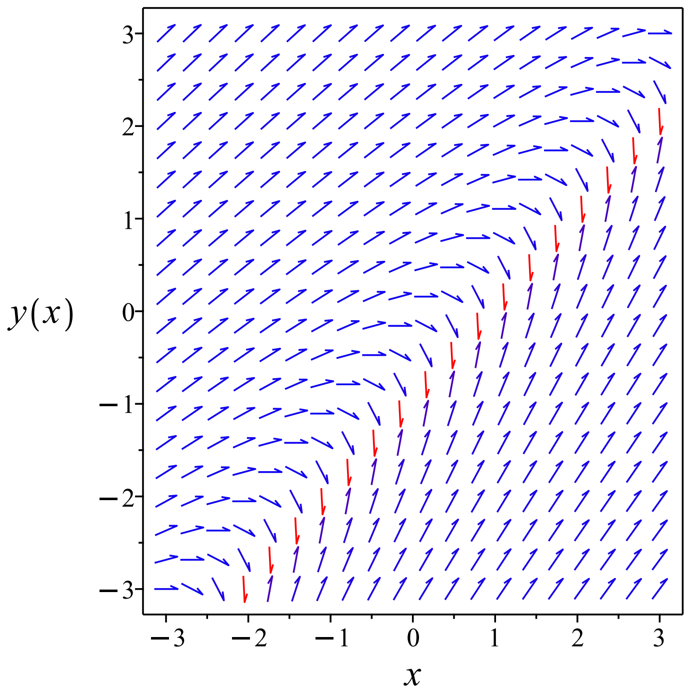


Figure 174: Slope field plot

Verification of solutions

$$\frac{x(x-2y)}{2} + \frac{y^2}{2} + y = c_1$$

Verified OK.

3.5.4 Maple step by step solution

Let's solve

$-y + (1 - x + y)\, y' = -x$

- Highest derivative means the order of the ODE is 1

y'

□ Check if ODE is exact

- ○ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- ○ Compute derivative of lhs

$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- ○ Evaluate derivatives

$-1 = -1$

- ○ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$\left[F(x, y) = c_1, M(x, y) = F'(x, y)\, , N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$F(x, y) = \int (x - y)\, dx + f_1(y)$

- Evaluate integral

$F(x, y) = \frac{x^2}{2} - yx + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

$N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

$-x + y + 1 = -x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

$\frac{d}{dy} f_1(y) = y + 1$

- Solve for $f_1(y)$

$f_1(y) = \frac{1}{2} y^2 + y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = \frac{1}{2}x^2 - yx + \frac{1}{2}y^2 + y$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{1}{2}x^2 - yx + \frac{1}{2}y^2 + y = c_1$$

- Solve for y

$$\left\{y = x - 1 - \sqrt{2c_1 - 2x + 1}, y = x - 1 + \sqrt{2c_1 - 2x + 1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve((x-y(x))+(y(x)-x+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x - 1 - \sqrt{2c_1 - 2x + 1}$$
$$y(x) = x - 1 + \sqrt{2c_1 - 2x + 1}$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 49

```
DSolve[(x-y[x])+(y[x]-x+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - i\sqrt{2x - 1 - c_1} - 1$$
$$y(x) \to x + i\sqrt{2x - 1 - c_1} - 1$$

3.6 problem 6

Internal problem ID [1928]
Internal file name [`OUTPUT/1928_Sunday_February_25_2024_06_37_48_AM_92229544/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x - 1 + y}{x - 1 - y} = 0}$$

3.6.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{X + x_0 - 1 + Y(X) + y_0}{1 - X - x_0 + Y(X) + y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{X + Y(X)}{-X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{X+Y}{-X+Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = X+Y$ and $N = X-Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u-1}{u-1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) X u(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 1 = 0$$

Or

$$(u(X)-1)\, X \left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2+1}{(u-1)\, X} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x + 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{y^2}{(x-1)^2}+1\right)}{2} - \arctan\left(\frac{y}{x-1}\right) + \ln(x-1) - c_2 = 0$$

Summary
The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{(x-1)^2}+1\right)}{2}-\arctan\left(\frac{y}{x-1}\right)+\ln(x-1)-c_2=0 \tag{1}$$

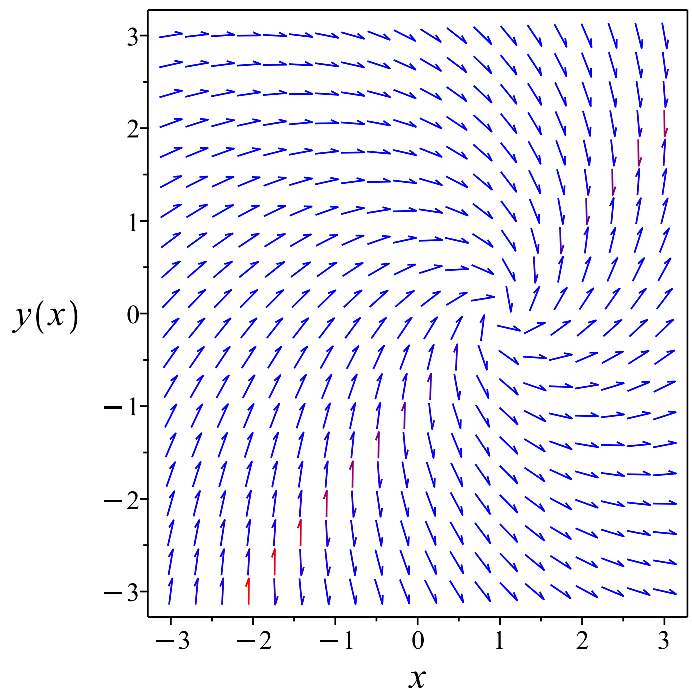


Figure 175: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{(x-1)^2}+1\right)}{2}-\arctan\left(\frac{y}{x-1}\right)+\ln(x-1)-c_2=0$$

Verified OK.

3.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{x-1+y}{-x+y+1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(x-1+y)(b_3-a_2)}{-x+y+1} - \frac{(x-1+y)^2 a_3}{(-x+y+1)^2} \\ &- \left(-\frac{1}{-x+y+1} - \frac{x-1+y}{(-x+y+1)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{1}{-x+y+1} + \frac{x-1+y}{(-x+y+1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2 a_2 + x^2 a_3 + x^2 b_2 - x^2 b_3 - 2xy a_2 + 2xy a_3 + 2xy b_2 + 2xy b_3 - y^2 a_2 - y^2 a_3 - y^2 b_2 + y^2 b_3 - 2x a_2 - 2x a}{(x-y-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^2 a_2 - x^2 a_3 - x^2 b_2 + x^2 b_3 + 2xy a_2 - 2xy a_3 - 2xy b_2 - 2xy b_3 \\ &+ y^2 a_2 + y^2 a_3 + y^2 b_2 - y^2 b_3 + 2x a_2 + 2x a_3 - 2x b_1 - 2x b_3 \\ &+ 2y a_1 + 2y a_3 + 2y b_2 + 2y b_3 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 - 2b_2v_1v_2 \\ &+ b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 + 2a_2v_1 + 2a_3v_1 + 2a_3v_2 \\ &- 2b_1v_1 + 2b_2v_2 - 2b_3v_1 + 2b_3v_2 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-a_2 - a_3 - b_2 + b_3)\,v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3)\,v_1v_2 + (2a_2 + 2a_3 - 2b_1 - 2b_3)\,v_1 \\ &+ (a_2 + a_3 + b_2 - b_3)\,v_2^2 + (2a_1 + 2a_3 + 2b_2 + 2b_3)\,v_2 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 + 2a_3 + 2b_2 + 2b_3 &= 0 \\ -a_2 - a_3 - b_2 + b_3 &= 0 \\ a_2 + a_3 + b_2 - b_3 &= 0 \\ 2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \\ 2a_2 + 2a_3 - 2b_1 - 2b_3 &= 0 \\ -a_2 - a_3 + 2b_1 + b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_3 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= -b_2 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -y$$
$$\eta = x - 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x - 1 - \left(-\frac{x-1+y}{-x+y+1}\right)(-y) \\
&= \frac{x^2+y^2-2x+1}{x-y-1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2+y^2-2x+1}{x-y-1}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(x^2+y^2-2x+1)}{2} + \frac{2(x-1)\arctan\left(\frac{2y}{2x-2}\right)}{2x-2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x - 1 + y}{-x + y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-x + 1 - y}{x^2 + y^2 - 2x + 1} \\ S_y &= \frac{x - y - 1}{x^2 + y^2 - 2x + 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

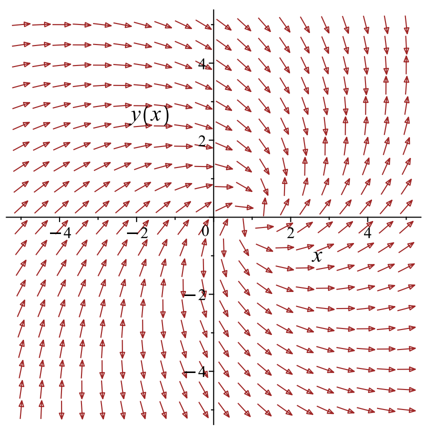
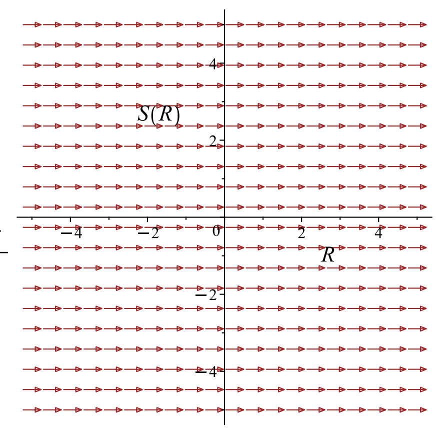
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y^2 + x^2 - 2x + 1)}{2} + \arctan\left(\frac{y}{x - 1}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y^2 + x^2 - 2x + 1)}{2} + \arctan\left(\frac{y}{x - 1}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x-1+y}{-x+y+1}$ 	$R = x$ $S = -\frac{\ln(x^2+y^2-2x+}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(y^2+x^2-2x+1)}{2} + \arctan\left(\frac{y}{x-1}\right) = c_1 \tag{1}$$

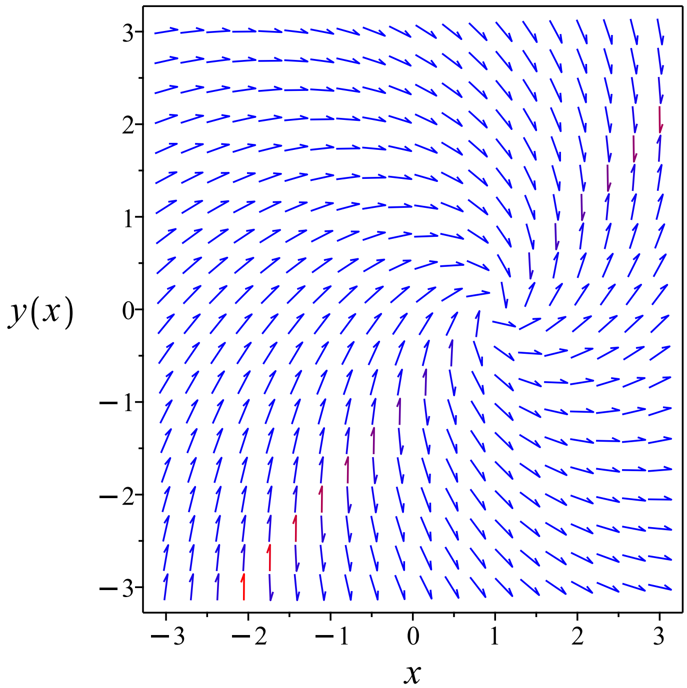


Figure 176: Slope field plot

Verification of solutions

$$-\frac{\ln \left(y^{2}+x^{2}-2x+1\right)}{2}+\arctan \left(\frac{y}{x-1}\right) = c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x)=(x+y(x)-1)/(x-y(x)-1),y(x), singsol=all)
```

$$y(x) = \tan\left(\mathrm{RootOf}\left(2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x-1\right) + 2c_1\right)\right)(1-x)$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 48

```
DSolve[y'[x]==(x+y[x]-1)/(x-y[x]-1),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\arctan\left(\frac{y(x)+x-1}{-y(x)+x-1}\right) = \log\left(\frac{1}{2}\left(\frac{y(x)^2}{(x-1)^2}+1\right)\right) + 2\log(x-1) + c_1, y(x)\right]$$

3.7 problem 7

Internal problem ID [1929]
Internal file name [`OUTPUT/1929_Sunday_February_25_2024_06_37_49_AM_93686659/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (2x + 2y - 1)\, y' = -x}$$

3.7.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x + y}{2x + 2y - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(x+y)(b_3-a_2)}{2x+2y-1} - \frac{(x+y)^2 a_3}{(2x+2y-1)^2} \\ &- \left(-\frac{1}{2x+2y-1} + \frac{2x+2y}{(2x+2y-1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{1}{2x+2y-1} + \frac{2x+2y}{(2x+2y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 + 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 - 2xa_2}{(2x+2y-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 + 2y^2a_2 - y^2a_3 \\ &+ 4y^2b_2 - 2y^2b_3 - 2xa_2 - 5xb_2 + xb_3 - ya_2 - ya_3 - 4yb_2 - a_1 - b_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &2a_2v_1^2 + 4a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - a_3v_2^2 + 4b_2v_1^2 + 8b_2v_1v_2 + 4b_2v_2^2 \\ &- 2b_3v_1^2 - 4b_3v_1v_2 - 2b_3v_2^2 - 2a_2v_1 - a_2v_2 - a_3v_2 - 5b_2v_1 - 4b_2v_2 + b_3v_1 - a_1 - b_1 \\ &+ b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(2a_2 - a_3 + 4b_2 - 2b_3) v_1^2 + (4a_2 - 2a_3 + 8b_2 - 4b_3) v_1 v_2 + (-2a_2 - 5b_2 + b_3) v_1 + (2a_2 - a_3 + 4b_2 - 2b_3) v_2^2 + (-a_2 - a_3 - 4b_2) v_2 - a_1 - b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_1 - b_1 + b_2 &= 0 \\
-2a_2 - 5b_2 + b_3 &= 0 \\
-a_2 - a_3 - 4b_2 &= 0 \\
2a_2 - a_3 + 4b_2 - 2b_3 &= 0 \\
4a_2 - 2a_3 + 8b_2 - 4b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -b_1 + b_2 \\
a_2 &= -2b_2 \\
a_3 &= -2b_2 \\
b_1 &= b_1 \\
b_2 &= b_2 \\
b_3 &= b_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -1 \\
\eta &= 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 1 - \left(-\frac{x + y}{2x + 2y - 1}\right)(-1) \\
&= \frac{x - 1 + y}{2x + 2y - 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x-1+y}{2x+2y-1}} dy \end{aligned}$$

Which results in

$$S = 2y + \ln (x - 1 + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{2x + 2y - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x - 1 + y} \\ S_y &= 2 + \frac{1}{x - 1 + y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2y + \ln(x - 1 + y) = -x + c_1$$

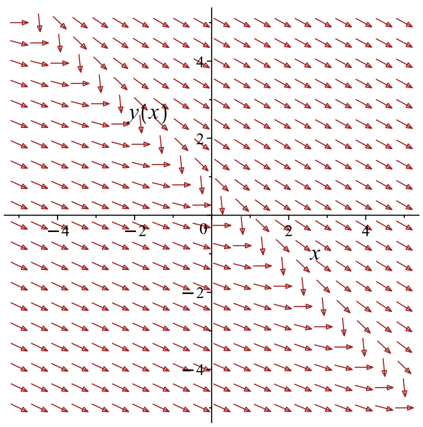
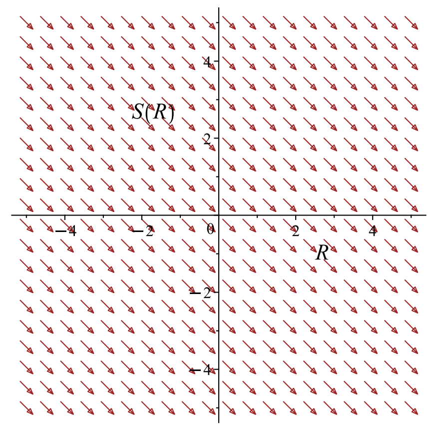
Which simplifies to

$$2y + \ln(x - 1 + y) = -x + c_1$$

Which gives

$$y = \frac{\text{LambertW}\left(2\,e^{x-2+c_1}\right)}{2} - x + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{2x+2y-1}$ 	$R = x$ $S = 2y + \ln(x - 1 + y)$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \frac{\operatorname{LambertW}\left(2\,e^{x-2+c_1}\right)}{2} - x + 1 \tag{1}$$

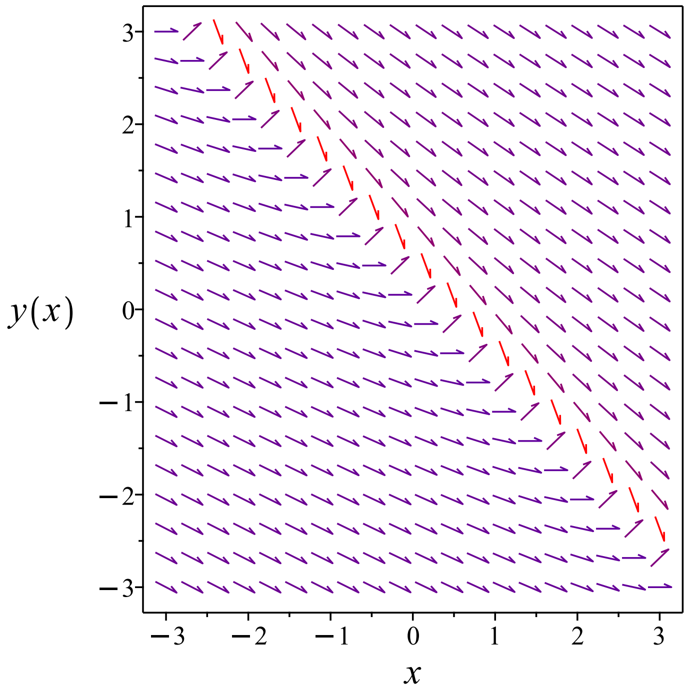


Figure 177: Slope field plot

Verification of solutions

$$y = \frac{\mathrm{LambertW}\left(2\,\mathrm{e}^{x-2+c_1}\right)}{2} - x + 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((x+y(x))+(2*x+2*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{LambertW}\left(2\,\mathrm{e}^{x-2-c_1}\right)}{2} - x + 1$$

✓ Solution by Mathematica

Time used: 4.381 (sec). Leaf size: 33

```
DSolve[(x+y[x])+(2*x+2*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}\left(W\left(-e^{x-1+c_1}\right) - 2x + 2\right)$$
$$y(x) \rightarrow 1 - x$$

3.8 problem 8

Internal problem ID [1930]
Internal file name [`OUTPUT/1930_Sunday_February_25_2024_06_37_50_AM_29384153/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y+(x-1-y)\,y'=-x-1}$$

3.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{-1-x+y}{-x+y+1} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2-\frac{(-1-x+y)(b_3-a_2)}{-x+y+1}-\frac{(-1-x+y)^2 a_3}{(-x+y+1)^2} \\ & -\left(\frac{1}{-x+y+1}-\frac{-1-x+y}{(-x+y+1)^2}\right)(x a_2+y a_3+a_1) \\ & -\left(-\frac{1}{-x+y+1}+\frac{-1-x+y}{(-x+y+1)^2}\right)(x b_2+y b_3+b_1)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 - 2xyb_2 + 2xyb_3 + y^2a_2 - y^2a_3 + y^2b_2 - y^2b_3 - 2xa_2 - 2xa_3}{(x-y-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & x^2 a_2-x^2 a_3+x^2 b_2-x^2 b_3-2 x y a_2+2 x y a_3-2 x y b_2+2 x y b_3+y^2 a_2-y^2 a_3 \\ & +y^2 b_2-y^2 b_3-2 x a_2-2 x a_3+2 y b_2+2 y b_3-2 a_1-a_2-a_3+2 b_1+b_2+b_3=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_2 v_1^2-2 a_2 v_1 v_2+a_2 v_2^2-a_3 v_1^2+2 a_3 v_1 v_2-a_3 v_2^2+b_2 v_1^2-2 b_2 v_1 v_2+b_2 v_2^2-b_3 v_1^2 \\ & +2 b_3 v_1 v_2-b_3 v_2^2-2 a_2 v_1-2 a_3 v_1+2 b_2 v_2+2 b_3 v_2-2 a_1-a_2-a_3+2 b_1+b_2+b_3=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(a_2 - a_3 + b_2 - b_3) v_1^2 + (-2a_2 + 2a_3 - 2b_2 + 2b_3) v_1 v_2 + (-2a_2 - 2a_3) v_1 \\ + (a_2 - a_3 + b_2 - b_3) v_2^2 + (2b_2 + 2b_3) v_2 - 2a_1 - a_2 - a_3 + 2b_1 + b_2 + b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-2a_2 - 2a_3 &= 0 \\ 2b_2 + 2b_3 &= 0 \\ -2a_2 + 2a_3 - 2b_2 + 2b_3 &= 0 \\ a_2 - a_3 + b_2 - b_3 &= 0 \\ -2a_1 - a_2 - a_3 + 2b_1 + b_2 + b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= b_1 \\ a_2 &= b_3 \\ a_3 &= -b_3 \\ b_1 &= b_1 \\ b_2 &= -b_3 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 1 \\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= 1 - \left(-\frac{-1 - x + y}{-x + y + 1}\right)(1) \\ &= \frac{2x - 2y}{x - y - 1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x-2y}{x-y-1}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{2} + \frac{\ln(-x + y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-1 - x + y}{-x + y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2x - 2y} \\ S_y &= \frac{x - y - 1}{2x - 2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{2} + \frac{\ln(-x+y)}{2} = -\frac{x}{2} + c_1$$

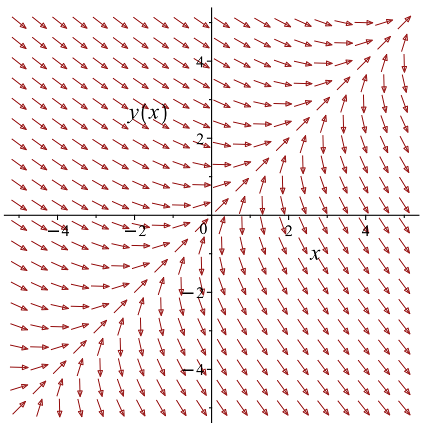
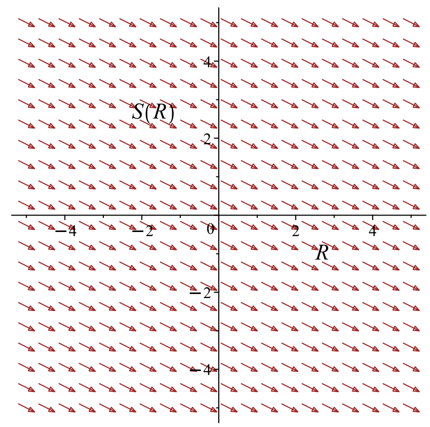
Which simplifies to

$$\frac{y}{2} + \frac{\ln(-x+y)}{2} = -\frac{x}{2} + c_1$$

Which gives

$$y = x + \text{LambertW}\left(\mathrm{e}^{-2x+2c_1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-1-x+y}{-x+y+1}$ 	$R = x$ $S = \frac{y}{2} + \frac{\ln(-x+y)}{2}$	$\frac{dS}{dR} = -\frac{1}{2}$ 

Summary

The solution(s) found are the following

$$y = x + \text{LambertW}\left(e^{-2x+2c_1}\right) \tag{1}$$

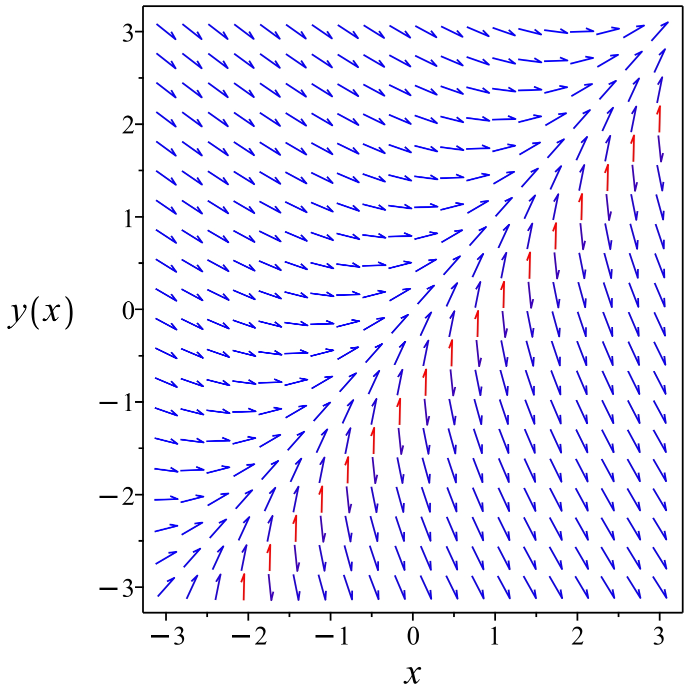


Figure 178: Slope field plot

Verification of solutions

$$y = x + \text{LambertW}\left(e^{-2x+2c_1}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve((x-y(x)+1)+(x-y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \text{LambertW}\left(e^{-2x}c_1\right) + x$$

✓ Solution by Mathematica

Time used: 3.902 (sec). Leaf size: 24

```
DSolve[(x-y[x]+1)+(x-y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + W\left(-e^{-2x-1+c_1}\right)$$
$$y(x) \to x$$

3.9 problem 9

Internal problem ID [1931]
Internal file name [`OUTPUT/1931_Sunday_February_25_2024_06_37_51_AM_45464710/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{2y+(3x+6y+3)\,y'=-x}$$

3.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{x+2y}{3\,(x+2y+1)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(x+2y)(b_3-a_2)}{3(x+2y+1)} - \frac{(x+2y)^2 a_3}{9(x+2y+1)^2} \\ &- \left(-\frac{1}{3(x+2y+1)} + \frac{x+2y}{3(x+2y+1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{2}{3(x+2y+1)} + \frac{\frac{2x}{3}+\frac{4y}{3}}{(x+2y+1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 + 12xya_2 - 4xya_3 + 36xyb_2 - 12xyb_3 + 12y^2a_2 - 4y^2a_3 + 36y^2b_2 - 12y^2b_3}{9(x+2y+1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 + 12xya_2 - 4xya_3 + 36xyb_2 \\ &- 12xyb_3 + 12y^2a_2 - 4y^2a_3 + 36y^2b_2 - 12y^2b_3 + 6xa_2 \\ &+ 24xb_2 - 3xb_3 + 6ya_2 + 3ya_3 + 36yb_2 + 3a_1 + 6b_1 + 9b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &3a_2v_1^2 + 12a_2v_1v_2 + 12a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - 4a_3v_2^2 + 9b_2v_1^2 \\ &+ 36b_2v_1v_2 + 36b_2v_2^2 - 3b_3v_1^2 - 12b_3v_1v_2 - 12b_3v_2^2 + 6a_2v_1 \\ &+ 6a_2v_2 + 3a_3v_2 + 24b_2v_1 + 36b_2v_2 - 3b_3v_1 + 3a_1 + 6b_1 + 9b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(3a_2 - a_3 + 9b_2 - 3b_3) v_1^2 + (12a_2 - 4a_3 + 36b_2 - 12b_3) v_1 v_2 + (6a_2 + 24b_2 - 3b_3) v_1 \\ + (12a_2 - 4a_3 + 36b_2 - 12b_3) v_2^2 + (6a_2 + 3a_3 + 36b_2) v_2 + 3a_1 + 6b_1 + 9b_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}3a_1 + 6b_1 + 9b_2 &= 0 \\ 6a_2 + 3a_3 + 36b_2 &= 0 \\ 6a_2 + 24b_2 - 3b_3 &= 0 \\ 3a_2 - a_3 + 9b_2 - 3b_3 &= 0 \\ 12a_2 - 4a_3 + 36b_2 - 12b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -2b_1 - 3b_2 \\ a_2 &= -3b_2 \\ a_3 &= -6b_2 \\ b_1 &= b_1 \\ b_2 &= b_2 \\ b_3 &= 2b_2\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -2 \\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= 1 - \left(-\frac{x + 2y}{3\,(x + 2y + 1)}\right)(-2) \\ &= \frac{x + 2y + 3}{3x + 6y + 3} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x+2y+3}{3x+6y+3}} dy \end{aligned}$$

Which results in

$$S = 3y - 3 \ln (x + 2y + 3)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + 2y}{3 (x + 2y + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3}{x + 2y + 3} \\ S_y &= 3 - \frac{6}{x + 2y + 3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$3y - 3\ln(x + 2y + 3) = -x + c_1$$

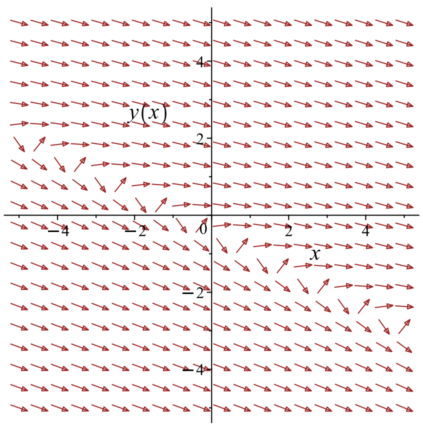
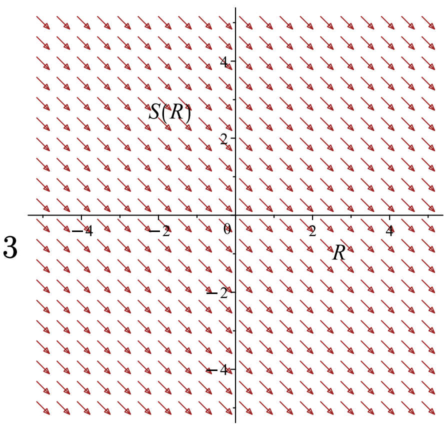
Which simplifies to

$$3y - 3\ln(x + 2y + 3) = -x + c_1$$

Which gives

$$y = -\frac{x}{2} - \text{LambertW}\left(-\frac{e^{-\frac{x}{6}-\frac{c_1}{3}-\frac{3}{2}}}{2}\right) - \frac{3}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+2y}{3(x+2y+1)}$ 	$R = x$ $S = 3y - 3\ln(x + 2y + 3$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = -\frac{x}{2} - \text{LambertW}\left(-\frac{\mathrm{e}^{-\frac{x}{6}-\frac{c_1}{3}-\frac{3}{2}}}{2}\right) - \frac{3}{2} \tag{1}$$

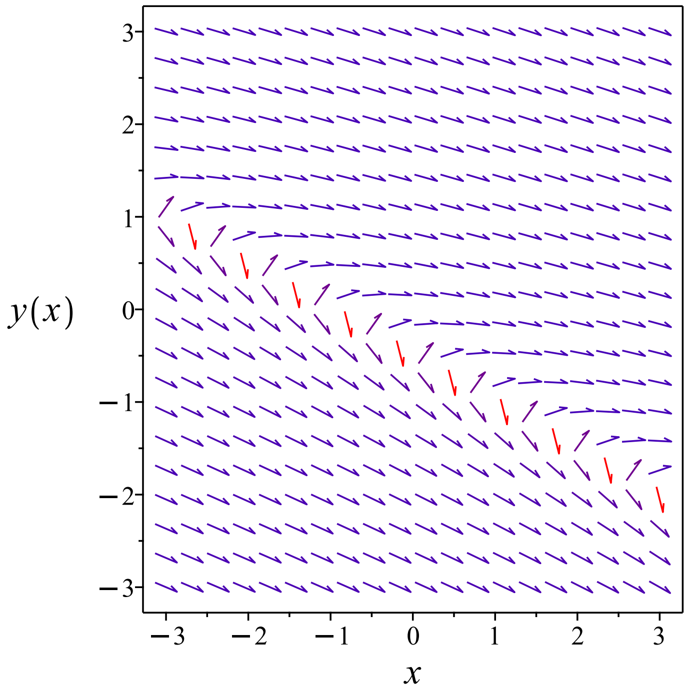


Figure 179: Slope field plot

Verification of solutions

$$y = -\frac{x}{2} - \mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{x}{6}-\frac{c_1}{3}-\frac{3}{2}}}{2}\right) - \frac{3}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 23

```
dsolve((x+2*y(x))+(3*x+6*y(x)+3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-\frac{e^{-\frac{x}{6}-\frac{3}{2}+\frac{c_1}{6}}}{2}\right) - \frac{3}{2} - \frac{x}{2}$$

✓ Solution by Mathematica

Time used: 4.116 (sec). Leaf size: 43

```
DSolve[(x+2*y[x])+(3*x+6*y[x]+3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-2W\left(-e^{-\frac{x}{6}-1+c_1}\right) - x - 3\right)$$
$$y(x) \to \frac{1}{2}(-x - 3)$$

3.10 problem 10

Internal problem ID [1932]
Internal file name [`OUTPUT/1932_Sunday_February_25_2024_06_37_52_AM_29327924/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{2y-(2x+y-1)\,y'=-x-2}$$

3.10.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=\frac{X+x_0+2Y(X)+2y_0+2}{2X+2x_0+Y(X)+y_0-1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=\frac{4}{3}$$
$$y_0=-\frac{5}{3}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=\frac{X+2Y(X)}{2X+Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{X+2Y}{2X+Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = X + 2Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{2u+1}{u+2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{2u(X)+1}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{2u(X)+1}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) X u(X) + 2\left(\frac{d}{dX}u(X)\right) X + u(X)^2 - 1 = 0$$

Or

$$(u(X)+2)\, X \left(\frac{d}{dX}u(X)\right) + u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2-1}{(u+2)\, X} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-1}{u+2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2-1}{u+2}} \, du &= -\frac{1}{X} \, dX \\
\int \frac{1}{\frac{u^2-1}{u+2}} \, du &= \int -\frac{1}{X} \, dX \\
\frac{3 \ln (u-1)}{2} - \frac{\ln (u+1)}{2} &= -\ln (X) + c_2
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\frac{3 \ln (u-1) - \ln (u+1)}{2} &= -\ln (X) + c_2 \\
3 \ln (u-1) - \ln (u+1) &= (2) (-\ln (X) + c_2) \\
&= -2 \ln (X) + 2c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{3 \ln(u-1) - \ln(u+1)} = \mathrm{e}^{-2 \ln(X) + 2c_2}
$$

Which simplifies to

$$
\begin{aligned}
\frac{(u-1)^3}{u+1} &= \frac{2c_2}{X^2} \\
&= \frac{c_3}{X^2}
\end{aligned}
$$

Which simplifies to

$$
\frac{(u(X)-1)^3}{u(X)+1} = \frac{c_3 \mathrm{e}^{2c_2}}{X^2}
$$

The solution is

$$
\frac{(u(X)-1)^3}{u(X)+1} = \frac{c_3 \mathrm{e}^{2c_2}}{X^2}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
\frac{\left(\frac{Y(X)}{X} - 1\right)^3}{\frac{Y(X)}{X} + 1} = \frac{c_3 \mathrm{e}^{2c_2}}{X^2}
$$

Which simplifies to

$$-\frac{\left(-Y(X)+X\right)^{3}}{Y\left(X\right)+X} = c_3 \mathrm{e}^{2c_2}$$

Using the solution for $Y(X)$

$$-\frac{\left(-Y(X)+X\right)^{3}}{Y\left(X\right)+X} = c_3 \mathrm{e}^{2c_2}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - \frac{5}{3}$$
$$X = x + \frac{4}{3}$$

Then the solution in y becomes

$$-\frac{\left(-y-3+x\right)^{3}}{y+\frac{1}{3}+x} = c_3 \mathrm{e}^{2c_2}$$

Summary

The solution(s) found are the following

$$-\frac{\left(-y-3+x\right)^{3}}{y+\frac{1}{3}+x} = c_3 \mathrm{e}^{2c_2} \tag{1}$$

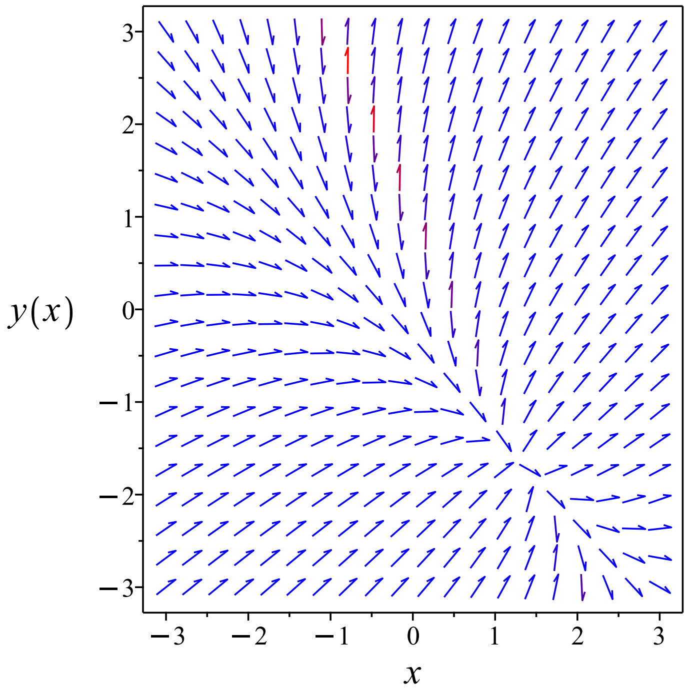


Figure 180: Slope field plot

Verification of solutions

$$-\frac{(-y-3+x)^3}{y+\frac{1}{3}+x} = c_3 e^{2c_2}$$

Verified OK.

3.10.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x+2y+2}{2x+y-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(x+2y+2)(b_3-a_2)}{2x+y-1} - \frac{(x+2y+2)^2 a_3}{(2x+y-1)^2} \\ &- \left(\frac{1}{2x+y-1} - \frac{2(x+2y+2)}{(2x+y-1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(\frac{2}{2x+y-1} - \frac{x+2y+2}{(2x+y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + x^2a_3 - x^2b_2 - 2x^2b_3 + 2xya_2 + 4xya_3 - 4xyb_2 - 2xyb_3 + 2y^2a_2 + y^2a_3 - y^2b_2 - 2y^2b_3 - 2xa_2 -}{(2x+y-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-2x^2a_2 - x^2a_3 + x^2b_2 + 2x^2b_3 - 2xya_2 - 4xya_3 + 4xyb_2 + 2xyb_3 \\ &- 2y^2a_2 - y^2a_3 + y^2b_2 + 2y^2b_3 + 2xa_2 - 4xa_3 - 3xb_1 + 3xb_3 + 3ya_1 \\ &- 3ya_3 - 2yb_2 + 4yb_3 + 5a_1 + 2a_2 - 4a_3 + 4b_1 + b_2 - 2b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-2a_2v_1^2 - 2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - a_3v_2^2 + b_2v_1^2 + 4b_2v_1v_2 \\ &+ b_2v_2^2 + 2b_3v_1^2 + 2b_3v_1v_2 + 2b_3v_2^2 + 3a_1v_2 + 2a_2v_1 - 4a_3v_1 - 3a_3v_2 \\ &- 3b_1v_1 - 2b_2v_2 + 3b_3v_1 + 4b_3v_2 + 5a_1 + 2a_2 - 4a_3 + 4b_1 + b_2 - 2b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-2a_2 - a_3 + b_2 + 2b_3)\, v_1^2 + (-2a_2 - 4a_3 + 4b_2 + 2b_3)\, v_1 v_2 \\
&+ (2a_2 - 4a_3 - 3b_1 + 3b_3)\, v_1 + (-2a_2 - a_3 + b_2 + 2b_3)\, v_2^2 \\
&+ (3a_1 - 3a_3 - 2b_2 + 4b_3)\, v_2 + 5a_1 + 2a_2 - 4a_3 + 4b_1 + b_2 - 2b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
3a_1 - 3a_3 - 2b_2 + 4b_3 &= 0 \\
-2a_2 - 4a_3 + 4b_2 + 2b_3 &= 0 \\
-2a_2 - a_3 + b_2 + 2b_3 &= 0 \\
2a_2 - 4a_3 - 3b_1 + 3b_3 &= 0 \\
5a_1 + 2a_2 - 4a_3 + 4b_1 + b_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= \frac{5a_3}{3} - \frac{4b_3}{3} \\
a_2 &= b_3 \\
a_3 &= a_3 \\
b_1 &= -\frac{4a_3}{3} + \frac{5b_3}{3} \\
b_2 &= a_3 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= \frac{5}{3} + y \\
\eta &= x - \frac{4}{3}
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x - \frac{4}{3} - \left(\frac{x+2y+2}{2x+y-1}\right)\left(\frac{5}{3}+y\right) \\
&= \frac{6x^2 - 6y^2 - 16x - 20y - 6}{6x+3y-3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{6x^2-6y^2-16x-20y-6}{6x+3y-3}} dy
\end{aligned}$$

Which results in

$$S = -\frac{3\ln(y+3-x)}{4} + \frac{\ln(3x+3y+1)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x+2y+2}{2x+y-1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-3x - 6y - 6}{6\left(x - 3 - y\right)\left(x + \frac{1}{3} + y\right)} \\
S_y &= \frac{6x + 3y - 3}{6\left(x - 3 - y\right)\left(x + \frac{1}{3} + y\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

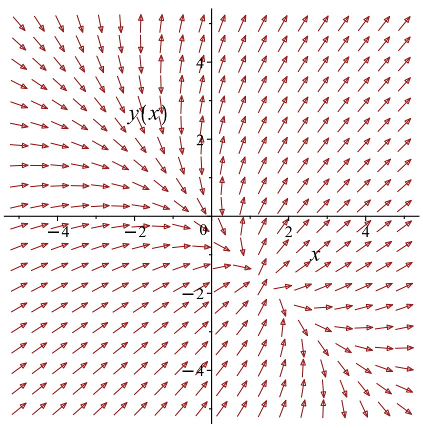
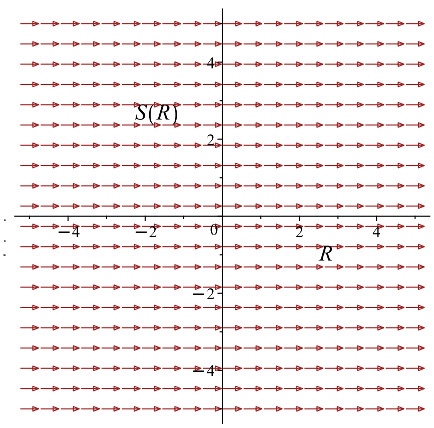
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{3\ln\left(y + 3 - x\right)}{4} + \frac{\ln\left(3x + 3y + 1\right)}{4} = c_1$$

Which simplifies to

$$-\frac{3\ln\left(y + 3 - x\right)}{4} + \frac{\ln\left(3x + 3y + 1\right)}{4} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+2y+2}{2x+y-1}$ 	$R = x$ $S = -\frac{3\ln(y+3-x)}{4} +$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\frac{3\ln(y+3-x)}{4} + \frac{\ln(3x+3y+1)}{4} = c_1 \tag{1}$$

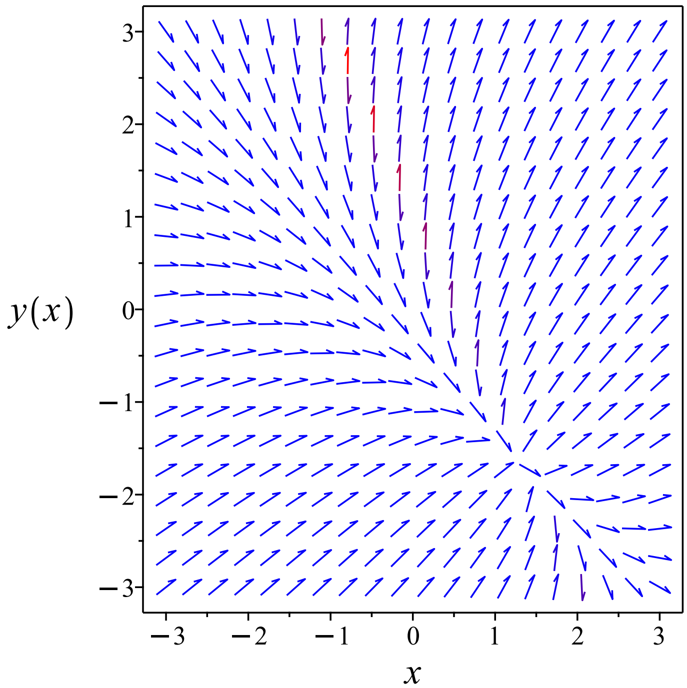


Figure 181: Slope field plot

<u>Verification of solutions</u>

$$-\frac{3\ln\left(y+3-x\right)}{4}+\frac{\ln\left(3x+3y+1\right)}{4}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 217

```
dsolve((x+2*y(x)+2)=(2*x+y(x)-1)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{(x-3)\left(i\sqrt{3}-1\right)\left(486\sqrt{\left(x-\frac{4}{3}\right)^2 c_1\left(-\frac{1}{243}+\left(x-\frac{4}{3}\right)^2 c_1\right)}+1-486\left(x-\frac{4}{3}\right)^2 c_1\right)^{\frac{2}{3}}+}{i\left(486\sqrt{\left(x-\frac{4}{3}\right)^2 c_1\left(-\frac{1}{243}+\left(x-\frac{4}{3}\right)^2 c_1\right)}+1-486\left(x-\frac{4}{3}\right)^2 c_1\right)^{\frac{2}{3}}\sqrt{3}-i\sqrt{3}-\left(486\sqrt{\left(x-\frac{4}{3}\right)^2 c_1\left(-\right.\right.}\right.}$$

✓ Solution by Mathematica

Time used: 60.167 (sec). Leaf size: 1687

```
DSolve[(x+2*y[x]+2)==(2*x+y[x]-1)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

3.11 problem 11

Internal problem ID [1933]
Internal file name [`OUTPUT/1933_Sunday_February_25_2024_06_37_54_AM_96757665/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y+(x-3y-5)\,y'=-3x-1}$$

With initial conditions

$$[y(0)=0]$$

3.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}y' &= f(x,y)\\ &= -\frac{-3x+y-1}{-x+3y+5}\end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{x<5\vee 5<x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\left\{y < -\frac{5}{3} \lor -\frac{5}{3} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-3x + y - 1}{-x + 3y + 5}\right) \\ &= -\frac{1}{-x + 3y + 5} + \frac{-9x + 3y - 3}{(-x + 3y + 5)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 5 \lor 5 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\left\{y < -\frac{5}{3} \lor -\frac{5}{3} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

3.11.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{-3X - 3x_0 + Y(X) + y_0 - 1}{-X - x_0 + 3Y(X) + 3y_0 + 5}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= -1 \\ y_0 &= -2\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{-3X + Y(X)}{-X + 3Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{-3X + Y}{-X + 3Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -3X + Y$ and $N = X - 3Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{-u + 3}{3u - 1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)+3}{3u(X)-1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{\frac{-u(X)+3}{3u(X)-1} - u(X)}{X} = 0
$$

Or

$$
3\left(\frac{d}{dX} u(X)\right) X u(X) - \left(\frac{d}{dX} u(X)\right) X + 3u(X)^2 - 3 = 0
$$

Or

$$
-3 + (3u(X) - 1) X \left(\frac{d}{dX} u(X)\right) + 3u(X)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{3(u^2 - 1)}{(3u - 1) X}
\end{aligned}
$$

Where $f(X) = -\frac{3}{X}$ and $g(u) = \frac{u^2-1}{3u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{3u-1}}\,du = -\frac{3}{X}\,dX$$

$$\int \frac{1}{\frac{u^2-1}{3u-1}}\,du = \int -\frac{3}{X}\,dX$$

$$\ln(u-1) + 2\ln(u+1) = -3\ln(X) + c_2$$

Raising both side to exponential gives

$$e^{\ln(u-1)+2\ln(u+1)} = e^{-3\ln(X)+c_2}$$

Which simplifies to

$$(u-1)(u+1)^2 = \frac{c_3}{X^3}$$

The solution is

$$(u(X)-1)(u(X)+1)^2 = \frac{c_3}{X^3}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\left(\frac{Y(X)}{X}-1\right)\left(\frac{Y(X)}{X}+1\right)^2 = \frac{c_3}{X^3}$$

Which simplifies to

$$-(-Y(X)+X)(Y(X)+X)^2 = c_3$$

Using the solution for $Y(X)$

$$-(-Y(X)+X)(Y(X)+X)^2 = c_3$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 2$$
$$X = x - 1$$

Then the solution in y becomes

$$-(x-1-y)(3+y+x)^2 = c_3$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$9 = c_3$$

$$c_3 = 9$$

Substituting c_3 found above in the general solution gives

$$-(x-y-1)(x+3+y)^2 = 9$$

Summary
The solution(s) found are the following

$$-(x-1-y)(3+y+x)^2 = 9 \tag{1}$$

Verification of solutions

$$-(x-1-y)(3+y+x)^2 = 9$$

Verified OK.

3.11.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-3x+y-1}{-x+3y+5} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(-3x+y-1)(b_3-a_2)}{-x+3y+5} - \frac{(-3x+y-1)^2 a_3}{(-x+3y+5)^2} \\
&- \left(\frac{3}{-x+3y+5} - \frac{-3x+y-1}{(-x+3y+5)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{1}{-x+3y+5} + \frac{-9x+3y-3}{(-x+3y+5)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - 9x^2a_3 + 9x^2b_2 - 3x^2b_3 - 18xya_2 + 6xya_3 - 6xyb_2 + 18xyb_3 + 3y^2a_2 - 9y^2a_3 + 9y^2b_2 - 3y^2b_3 - 3}{(x}$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&3x^2a_2 - 9x^2a_3 + 9x^2b_2 - 3x^2b_3 - 18xya_2 + 6xya_3 - 6xyb_2 + 18xyb_3 + 3y^2a_2 \\
&- 9y^2a_3 + 9y^2b_2 - 3y^2b_3 - 30xa_2 - 6xa_3 + 8xb_1 - 2xb_2 + 14xb_3 - 8ya_1 \\
&+ 2ya_2 - 14ya_3 + 30yb_2 + 6yb_3 - 16a_1 - 5a_2 - a_3 + 8b_1 + 25b_2 + 5b_3 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&3a_2v_1^2 - 18a_2v_1v_2 + 3a_2v_2^2 - 9a_3v_1^2 + 6a_3v_1v_2 - 9a_3v_2^2 + 9b_2v_1^2 - 6b_2v_1v_2 + 9b_2v_2^2 \\
&- 3b_3v_1^2 + 18b_3v_1v_2 - 3b_3v_2^2 - 8a_1v_2 - 30a_2v_1 + 2a_2v_2 - 6a_3v_1 - 14a_3v_2 + 8b_1v_1 \\
&- 2b_2v_1 + 30b_2v_2 + 14b_3v_1 + 6b_3v_2 - 16a_1 - 5a_2 - a_3 + 8b_1 + 25b_2 + 5b_3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(3a_2 - 9a_3 + 9b_2 - 3b_3)\, v_1^2 + (-18a_2 + 6a_3 - 6b_2 + 18b_3)\, v_1 v_2 \\
&+ (-30a_2 - 6a_3 + 8b_1 - 2b_2 + 14b_3)\, v_1 + (3a_2 - 9a_3 + 9b_2 - 3b_3)\, v_2^2 \\
&+ (-8a_1 + 2a_2 - 14a_3 + 30b_2 + 6b_3)\, v_2 - 16a_1 - 5a_2 - a_3 + 8b_1 + 25b_2 + 5b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-18a_2 + 6a_3 - 6b_2 + 18b_3 &= 0 \\
3a_2 - 9a_3 + 9b_2 - 3b_3 &= 0 \\
-8a_1 + 2a_2 - 14a_3 + 30b_2 + 6b_3 &= 0 \\
-30a_2 - 6a_3 + 8b_1 - 2b_2 + 14b_3 &= 0 \\
-16a_1 - 5a_2 - a_3 + 8b_1 + 25b_2 + 5b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 2b_2 + b_3 \\
a_2 &= b_3 \\
a_3 &= b_2 \\
b_1 &= b_2 + 2b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 2 + y \\
\eta &= x + 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= x + 1 - \left(-\frac{-3x + y - 1}{-x + 3y + 5}\right)(2 + y) \\
&= \frac{x^2 - y^2 + 2x - 4y - 3}{x - 3y - 5} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2 - y^2 + 2x - 4y - 3}{x - 3y - 5}} dy \end{aligned}$$

Which results in

$$S = \ln(-x + y + 1) + 2\ln(x + 3 + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-3x + y - 1}{-x + 3y + 5}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x - y - 1} + \frac{2}{x + 3 + y} \\ S_y &= \frac{1}{-x + y + 1} + \frac{2}{x + 3 + y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

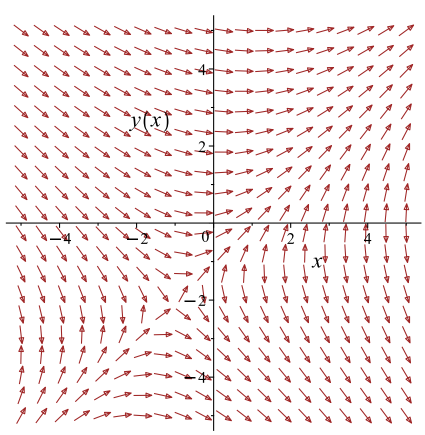
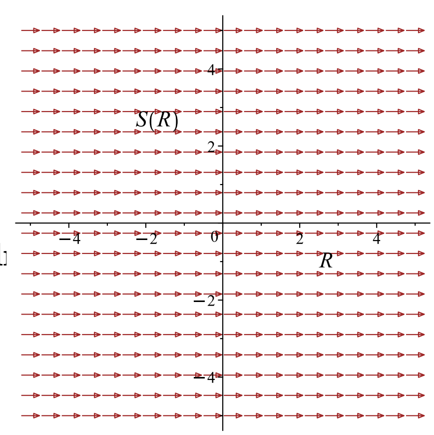
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(1 - x + y) + 2\ln(3 + y + x) = c_1$$

Which simplifies to

$$\ln(1 - x + y) + 2\ln(3 + y + x) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-3x+y-1}{-x+3y+5}$ 	$R = x$ $S = \ln(-x + y + 1) + 2\mathrm{l}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$2\ln(3) = c_1$$

$$c_1 = 2\ln(3)$$

Substituting c_1 found above in the general solution gives

$$\ln(-x+y+1) + 2\ln(x+3+y) = 2\ln(3)$$

Summary

The solution(s) found are the following

$$\ln(1-x+y) + 2\ln(3+y+x) = 2\ln(3) \tag{1}$$

Verification of solutions

$$\ln(1-x+y) + 2\ln(3+y+x) = 2\ln(3)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 3.516 (sec). Leaf size: 84

```
dsolve([(3*x-y(x)+1)+(x-3*y(x)-5)*diff(y(x),x)=0,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\left(-324+12\sqrt{96x^3+288x^2+288x+825}\right)^{\frac{4}{3}}-12\left(-324+12\sqrt{96x^3+288x^2+288x+825}\right)^{\frac{2}{3}}x-84\left(-3\right.}{36\left(-324+12\sqrt{96x^3+288x^2+288x+82\right.}}$$

✓ Solution by Mathematica

Time used: 60.756 (sec). Leaf size: 341

```
DSolve[{(3*x-y[x]+1)+(x-3*y[x]-5)*y'[x]==0,{y[0]==0}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{x\text{Root}\left[\#1^6(1024x^6+6144x^5+15360x^4+20480x^3+15360x^2+6144x-58025)+\#1^4(-384x^4-153\right.}{}$$

3.12 problem 12

Internal problem ID [1934]
Internal file name [`OUTPUT/1934_Sunday_February_25_2024_06_37_57_AM_37936287/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-3y + (2x - y + 5)\, y' = -6x - 6}$$

With initial conditions

$$[y(-1) = 1]$$

3.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{3(-2x + y - 2)}{-2x + y - 5} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{y < 3 \lor 3 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3(-2x + y - 2)}{-2x + y - 5}\right) \\ &= -\frac{3}{-2x + y - 5} + \frac{-6x + 3y - 6}{(-2x + y - 5)^2} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{y < 3 \lor 3 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.12.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{3(-2x + y - 2)}{-2x + y - 5} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &- \frac{3(-2x+y-2)(b_3-a_2)}{-2x+y-5} - \frac{9(-2x+y-2)^2 a_3}{(-2x+y-5)^2} \\ &- \left(\frac{6}{-2x+y-5} - \frac{6(-2x+y-2)}{(-2x+y-5)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{3}{-2x+y-5} + \frac{-6x+3y-6}{(-2x+y-5)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{12x^2a_2 - 36x^2a_3 + 4x^2b_2 - 12x^2b_3 - 12xya_2 + 36xya_3 - 4xyb_2 + 12xyb_3 + 3y^2a_2 - 9y^2a_3 + y^2b_2 - 3y^2b_3}{(2x} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &12x^2a_2 - 36x^2a_3 + 4x^2b_2 - 12x^2b_3 - 12xya_2 + 36xya_3 - 4xyb_2 + 12xyb_3 \\ &+ 3y^2a_2 - 9y^2a_3 + y^2b_2 - 3y^2b_3 + 60xa_2 - 72xa_3 + 11xb_2 - 42xb_3 - 21ya_2 \\ &+ 54ya_3 - 10yb_2 + 12yb_3 + 18a_1 + 30a_2 - 36a_3 - 9b_1 + 25b_2 - 30b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &12a_2v_1^2 - 12a_2v_1v_2 + 3a_2v_2^2 - 36a_3v_1^2 + 36a_3v_1v_2 - 9a_3v_2^2 + 4b_2v_1^2 - 4b_2v_1v_2 \\ &+ b_2v_2^2 - 12b_3v_1^2 + 12b_3v_1v_2 - 3b_3v_2^2 + 60a_2v_1 - 21a_2v_2 - 72a_3v_1 + 54a_3v_2 \\ &+ 11b_2v_1 - 10b_2v_2 - 42b_3v_1 + 12b_3v_2 + 18a_1 + 30a_2 - 36a_3 - 9b_1 + 25b_2 - 30b_3 \\ &= 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(12a_2 - 36a_3 + 4b_2 - 12b_3)\, v_1^2 + (-12a_2 + 36a_3 - 4b_2 + 12b_3)\, v_1 v_2 \\
&+ (60a_2 - 72a_3 + 11b_2 - 42b_3)\, v_1 + (3a_2 - 9a_3 + b_2 - 3b_3)\, v_2^2 \\
&+ (-21a_2 + 54a_3 - 10b_2 + 12b_3)\, v_2 + 18a_1 + 30a_2 - 36a_3 - 9b_1 + 25b_2 - 30b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-21a_2 + 54a_3 - 10b_2 + 12b_3 &= 0 \\
-12a_2 + 36a_3 - 4b_2 + 12b_3 &= 0 \\
3a_2 - 9a_3 + b_2 - 3b_3 &= 0 \\
12a_2 - 36a_3 + 4b_2 - 12b_3 &= 0 \\
60a_2 - 72a_3 + 11b_2 - 42b_3 &= 0 \\
18a_1 + 30a_2 - 36a_3 - 9b_1 + 25b_2 - 30b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= -2a_3 \\
a_3 &= a_3 \\
b_1 &= 2a_1 + 16a_3 \\
b_2 &= 6a_3 \\
b_3 &= -3a_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= 2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 2 - \left(-\frac{3(-2x + y - 2)}{-2x + y - 5}\right)(1) \\
&= \frac{10x - 5y + 16}{2x - y + 5} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{10x-5y+16}{2x-y+5}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{5} - \frac{9 \ln(-10x + 5y - 16)}{25}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3(-2x + y - 2)}{-2x + y - 5}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{18}{50x - 25y + 80} \\ S_y &= \frac{2x - y + 5}{10x - 5y + 16} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3}{5} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3}{5}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{3R}{5} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{5} - \frac{9\ln(-10x + 5y - 16)}{25} = -\frac{3x}{5} + c_1$$

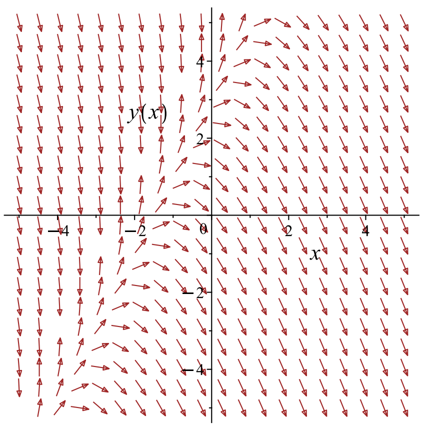
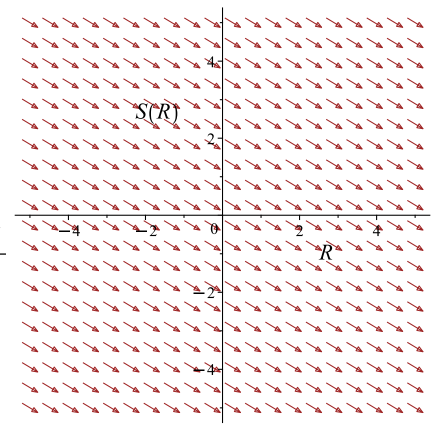
Which simplifies to

$$\frac{y}{5} - \frac{9\ln(-10x + 5y - 16)}{25} = -\frac{3x}{5} + c_1$$

Which gives

$$y = -\frac{9\,\mathrm{LambertW}\left(-\frac{e^{\frac{25x}{9} - \frac{25c_1}{9} + \frac{16}{9}}}{9}\right)}{5} + 2x + \frac{16}{5}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3(-2x+y-2)}{-2x+y-5}$ 	$R = x$ $S = \frac{y}{5} - \frac{9\ln(-10x+5y)}{25}$	$\frac{dS}{dR} = -\frac{3}{5}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{9\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-1-\frac{25c_1}{9}}}{9}\right)}{5} + \frac{6}{5}$$

$$c_1 = -\frac{9i\pi}{25} - \frac{2}{5}$$

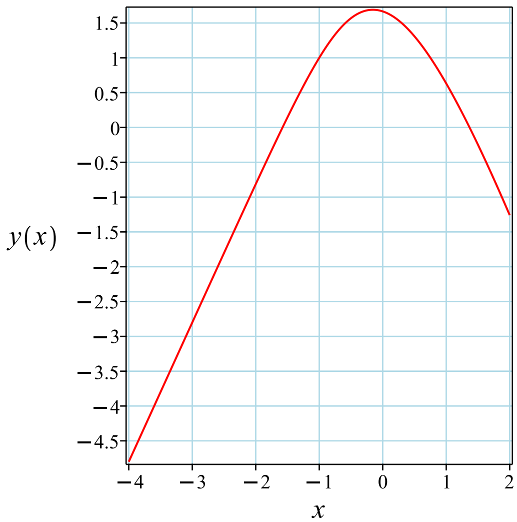
Substituting c_1 found above in the general solution gives

$$y = -\frac{9\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{\frac{25x}{9}+\frac{26}{9}}}{9}\right)}{5} + 2x + \frac{16}{5}$$

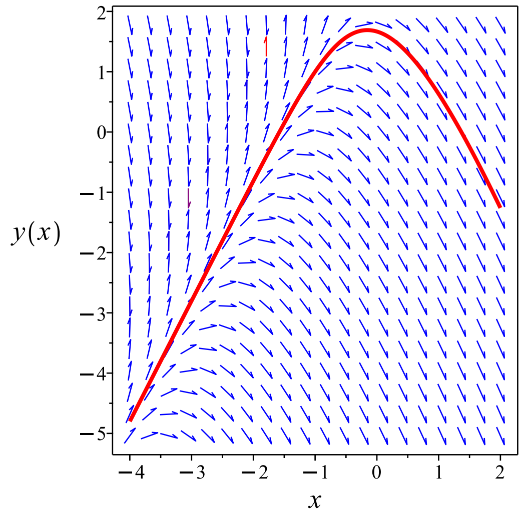
Summary

The solution(s) found are the following

$$y = -\frac{9\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{\frac{25x}{9}+\frac{26}{9}}}{9}\right)}{5} + 2x + \frac{16}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{9\,\mathrm{LambertW}\left(\frac{e^{\frac{25x}{9}+\frac{26}{9}}}{9}\right)}{5} + 2x + \frac{16}{5}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.25 (sec). Leaf size: 20

```
dsolve([3*(2*x-y(x)+2)+(2*x-y(x)+5)*diff(y(x),x)=0,y(-1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{16}{5} - \frac{9\,\mathrm{LambertW}\left(\frac{e^{\frac{25x}{9}+\frac{26}{9}}}{9}\right)}{5} + 2x$$

✓ Solution by Mathematica

Time used: 3.718 (sec). Leaf size: 32

```
DSolve[{3*(2*x-y[x]+2)+(2*x-y[x]+5)*y'[x]==0,{y[-1]==1}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{9}{5}W\left(\frac{1}{9}e^{\frac{25x}{9}+\frac{26}{9}}\right) + 2x + \frac{16}{5}$$

3.13 problem 13

Internal problem ID [1935]
Internal file name [`OUTPUT/1935_Sunday_February_25_2024_06_37_58_AM_51517481/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y + (-x + y)\, y' = -2x - 2}$$

With initial conditions

$$[y(0) = -2]$$

3.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{2x + 3y + 2}{-x + y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -2$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x+3y+2}{-x+y}\right)\\ &= -\frac{3}{-x+y} + \frac{2x+3y+2}{(-x+y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -2$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

3.13.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{2X + 2x_0 + 3Y(X) + 3y_0 + 2}{-X - x_0 + Y(X) + y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= -\frac{2}{5}\\ y_0 &= -\frac{2}{5}\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2X + 3Y(X)}{-X + Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= -\frac{2X+3Y}{-X+Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = 2X + 3Y$ and $N = X - Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-3u-2}{u-1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-3u(X)-2}{u(X)-1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{-3u(X)-2}{u(X)-1} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right)Xu(X) - \left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 2u(X) + 2 = 0
$$

Or

$$
(u(X)-1)\,X\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 2u(X) + 2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{u^2+2u+2}{(u-1)\,X}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+2u+2}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+2u+2}{u-1}} du = -\frac{1}{X} dX$$

$$\int \frac{1}{\frac{u^2+2u+2}{u-1}} du = \int -\frac{1}{X} dX$$

$$\frac{\ln(u^2+2u+2)}{2} - 2\arctan(u+1) = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2 + 2u(X) + 2\right)}{2} - 2\arctan(u(X)+1) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2} + \frac{2Y(X)}{X} + 2\right)}{2} - 2\arctan\left(\frac{Y(X)}{X} + 1\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2} + \frac{2Y(X)}{X} + 2\right)}{2} - 2\arctan\left(\frac{Y(X)+X}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - \frac{2}{5}$$
$$X = x - \frac{2}{5}$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{\left(y+\frac{2}{5}\right)^2}{\left(x+\frac{2}{5}\right)^2} + \frac{2y+\frac{4}{5}}{x+\frac{2}{5}} + 2\right)}{2} - 2\arctan\left(\frac{y+\frac{4}{5}+x}{x+\frac{2}{5}}\right) + \ln\left(x+\frac{2}{5}\right) - c_2 = 0$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3\ln(2)}{2} - \frac{\ln(5)}{2} + 2\arctan(3) - c_2 = 0$$

$$c_2 = \frac{3\ln(2)}{2} - \frac{\ln(5)}{2} + 2\arctan(3)$$

Substituting c_2 found above in the general solution gives

$$\frac{\ln\left(\frac{10x^2+(10y+12)x+5y^2+8y+4}{(5x+2)^2}\right)}{2} - 2\arctan\left(\frac{5y+4+5x}{5x+2}\right) + \ln(5x+2) - \frac{3\ln(2)}{2} - 2\arctan(3) = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{10x^2+(10y+12)x+5y^2+8y+4}{(5x+2)^2}\right)}{2} - 2\arctan\left(\frac{5y+4+5x}{5x+2}\right) + \ln(5x+2) - \frac{3\ln(2)}{2} - 2\arctan(3) = 0 \tag{1}$$

Verification of solutions

$$\frac{\ln\left(\frac{10x^2+(10y+12)x+5y^2+8y+4}{(5x+2)^2}\right)}{2} - 2\arctan\left(\frac{5y+4+5x}{5x+2}\right) + \ln(5x+2) - \frac{3\ln(2)}{2} - 2\arctan(3) = 0$$

Verified OK.

3.13.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2x+3y+2}{-x+y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(2x+3y+2)(b_3-a_2)}{-x+y} - \frac{(2x+3y+2)^2 a_3}{(-x+y)^2} \\ &- \left(-\frac{2}{-x+y} - \frac{2x+3y+2}{(-x+y)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{3}{-x+y} + \frac{2x+3y+2}{(-x+y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + 4x^2a_3 + 4x^2b_2 - 2x^2b_3 - 4xya_2 + 12xya_3 + 2xyb_2 + 4xyb_3 - 3y^2a_2 + 4y^2a_3 - y^2b_2 + 3y^2b_3 + 8}{(x-y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-2x^2a_2 - 4x^2a_3 - 4x^2b_2 + 2x^2b_3 + 4xya_2 - 12xya_3 - 2xyb_2 \\ &- 4xyb_3 + 3y^2a_2 - 4y^2a_3 + y^2b_2 - 3y^2b_3 - 8xa_3 - 5xb_1 - 2xb_2 \\ &+ 2xb_3 + 5ya_1 + 2ya_2 - 10ya_3 - 4yb_3 + 2a_1 - 4a_3 - 2b_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
-2a_2v_1^2 + 4a_2v_1v_2 + 3a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 - 4a_3v_2^2 - 4b_2v_1^2 \\
- 2b_2v_1v_2 + b_2v_2^2 + 2b_3v_1^2 - 4b_3v_1v_2 - 3b_3v_2^2 + 5a_1v_2 + 2a_2v_2 - 8a_3v_1 \\
- 10a_3v_2 - 5b_1v_1 - 2b_2v_1 + 2b_3v_1 - 4b_3v_2 + 2a_1 - 4a_3 - 2b_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
(-2a_2 - 4a_3 - 4b_2 + 2b_3)\, v_1^2 + (4a_2 - 12a_3 - 2b_2 - 4b_3)\, v_1v_2 \\
+ (-8a_3 - 5b_1 - 2b_2 + 2b_3)\, v_1 + (3a_2 - 4a_3 + b_2 - 3b_3)\, v_2^2 \\
+ (5a_1 + 2a_2 - 10a_3 - 4b_3)\, v_2 + 2a_1 - 4a_3 - 2b_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 - 4a_3 - 2b_1 &= 0 \\
5a_1 + 2a_2 - 10a_3 - 4b_3 &= 0 \\
-2a_2 - 4a_3 - 4b_2 + 2b_3 &= 0 \\
3a_2 - 4a_3 + b_2 - 3b_3 &= 0 \\
4a_2 - 12a_3 - 2b_2 - 4b_3 &= 0 \\
-8a_3 - 5b_1 - 2b_2 + 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 2a_3 + b_1 \\
a_2 &= \frac{5b_1}{2} + 4a_3 \\
a_3 &= a_3 \\
b_1 &= b_1 \\
b_2 &= -2a_3 \\
b_3 &= 2a_3 + \frac{5b_1}{2}
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = \frac{5x}{2} + 1$$
$$\eta = 1 + \frac{5y}{2}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 + \frac{5y}{2} - \left(-\frac{2x + 3y + 2}{-x + y}\right)\left(\frac{5x}{2} + 1\right) \\
&= \frac{-10x^2 - 10yx - 5y^2 - 12x - 8y - 4}{2x - 2y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-10x^2 - 10yx - 5y^2 - 12x - 8y - 4}{2x - 2y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(10x^2 + 10yx + 5y^2 + 12x + 8y + 4\right)}{5} + \frac{4\left(-2x - \frac{4}{5}\right) \arctan\left(\frac{10y + 10x + 8}{10x + 4}\right)}{10x + 4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x+3y+2}{-x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{6y+4+4x}{10x^2+(10y+12)\,x+5y^2+8y+4} \\
S_y &= \frac{-2x+2y}{10x^2+(10y+12)\,x+5y^2+8y+4}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

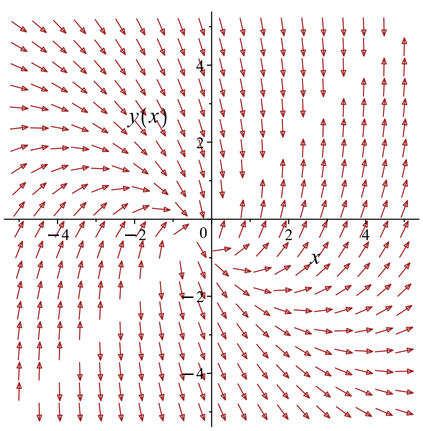
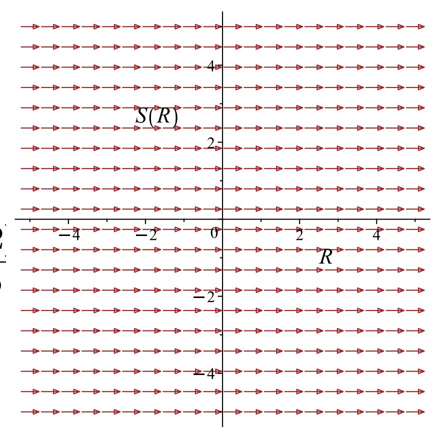
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(10x^2+(10y+12)\,x+5y^2+8y+4\right)}{5} - \frac{4\arctan\left(\frac{5y+4+5x}{5x+2}\right)}{5} = c_1$$

Which simplifies to

$$\frac{\ln\left(10x^2+(10y+12)x+5y^2+8y+4\right)}{5}-\frac{4\arctan\left(\frac{5y+4+5x}{5x+2}\right)}{5}=c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=-\frac{2x+3y+2}{-x+y}$ 	$R = x$ $S=\frac{\ln\left(10x^2+(10y+12\right.}{5}$ ... $\frac{5y+4+5x}{5x+2}$	$\frac{dS}{dR}=0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3\ln(2)}{5}+\frac{4\arctan(3)}{5}=c_1$$

$$c_1=\frac{3\ln(2)}{5}+\frac{4\arctan(3)}{5}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln\left(10x^2+(10y+12)x+5y^2+8y+4\right)}{5}-\frac{4\arctan\left(\frac{5y+4+5x}{5x+2}\right)}{5}=\frac{3\ln(2)}{5}+\frac{4\arctan(3)}{5}$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(10x^2+(10y+12)x+5y^2+8y+4\right)}{5}-\frac{4\arctan\left(\frac{5y+4+5x}{5x+2}\right)}{5}=\frac{3\ln(2)}{5}+\frac{4\arctan(3)}{5} \tag{1}$$

Verification of solutions

$$\frac{\ln\left(10x^2+(10y+12)\,x+5y^2+8y+4\right)}{5}-\frac{4\arctan\left(\frac{5y+4+5x}{5x+2}\right)}{5}=\frac{3\ln(2)}{5}+\frac{4\arctan(3)}{5}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✗ Solution by Maple

```
dsolve([(2*x+3*y(x)+2)+(y(x)-x)*diff(y(x),x)=0,y(0) = -2],y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 78

```
DSolve[{(2*x+3*y[x]+2)+(y[x]-x)*y'[x]==0,{y[0]==-2}},y[x],x,IncludeSingularSolutions -> True
```

$$\text{Solve}\left[32\arctan\left(\frac{2y(x)+3x+2}{x-y(x)}\right)=8\log\left(\frac{10x^2+10xy(x)+5y(x)^2+8y(x)+12x+4}{(5x+2)^2}\right)+16\log(5x+2)-8(\pi+3\log(2)),y(x)\right]$$

3.14 problem 14

Internal problem ID [1936]
Internal file name [`OUTPUT/1936_Sunday_February_25_2024_06_38_04_AM_85390140/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(2x+2y-1)\,y' = -x-4}$$

With initial conditions

$$[y(0) = 0]$$

3.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}y' &= f(x,y)\\ &= \frac{x+y+4}{2x+2y-1}\end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\left\{x<\frac{1}{2}\vee\frac{1}{2}<x\right\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\left\{y < \frac{1}{2} \vee \frac{1}{2} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y+4}{2x+2y-1}\right)\\ &= \frac{1}{2x+2y-1} - \frac{2(x+y+4)}{(2x+2y-1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\left\{x < \frac{1}{2} \vee \frac{1}{2} < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\left\{y < \frac{1}{2} \vee \frac{1}{2} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

3.14.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{x+y+4}{2x+2y-1}\\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(x+y+4)(b_3-a_2)}{2x+2y-1} - \frac{(x+y+4)^2 a_3}{(2x+2y-1)^2} - \left(\frac{1}{2x+2y-1} - \frac{2(x+y+4)}{(2x+2y-1)^2}\right)(xa_2+ya_3+a_1) - \left(\frac{1}{2x+2y-1} - \frac{2(x+y+4)}{(2x+2y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + x^2a_3 - 4x^2b_2 - 2x^2b_3 + 4xya_2 + 2xya_3 - 8xyb_2 - 4xyb_3 + 2y^2a_2 + y^2a_3 - 4y^2b_2 - 2y^2b_3 - 2xa}{(2x+2y-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$-2x^2a_2 - x^2a_3 + 4x^2b_2 + 2x^2b_3 - 4xya_2 - 2xya_3 + 8xyb_2 + 4xyb_3 - 2y^2a_2 - y^2a_3 + 4y^2b_2 + 2y^2b_3 + 2xa_2 - 8xa_3 + 5xb_2 + 7xb_3 - 7ya_2 + ya_3 - 4yb_2 + 16yb_3 + 9a_1 + 4a_2 - 16a_3 + 9b_1 + b_2 - 4b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2v_1^2 - 4a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - a_3v_2^2 + 4b_2v_1^2 + 8b_2v_1v_2 + 4b_2v_2^2 + 2b_3v_1^2 + 4b_3v_1v_2 + 2b_3v_2^2 + 2a_2v_1 - 7a_2v_2 - 8a_3v_1 + a_3v_2 + 5b_2v_1 - 4b_2v_2 + 7b_3v_1 + 16b_3v_2 + 9a_1 + 4a_2 - 16a_3 + 9b_1 + b_2 - 4b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-2a_2 - a_3 + 4b_2 + 2b_3) v_1^2 + (-4a_2 - 2a_3 + 8b_2 + 4b_3) v_1 v_2 \\ + (2a_2 - 8a_3 + 5b_2 + 7b_3) v_1 + (-2a_2 - a_3 + 4b_2 + 2b_3) v_2^2 \\ + (-7a_2 + a_3 - 4b_2 + 16b_3) v_2 + 9a_1 + 4a_2 - 16a_3 + 9b_1 + b_2 - 4b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-7a_2 + a_3 - 4b_2 + 16b_3 &= 0\\ -4a_2 - 2a_3 + 8b_2 + 4b_3 &= 0\\ -2a_2 - a_3 + 4b_2 + 2b_3 &= 0\\ 2a_2 - 8a_3 + 5b_2 + 7b_3 &= 0\\ 9a_1 + 4a_2 - 16a_3 + 9b_1 + b_2 - 4b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 3b_3 - b_1\\ a_2 &= 2b_3\\ a_3 &= 2b_3\\ b_1 &= b_1\\ b_2 &= b_3\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -1\\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi\\ &= 1 - \left(\frac{x + y + 4}{2x + 2y - 1}\right)(-1)\\ &= \frac{3x + 3y + 3}{2x + 2y - 1}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3x+3y+3}{2x+2y-1}} dy \end{aligned}$$

Which results in

$$S = \frac{2y}{3} - \ln(x + y + 1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + y + 4}{2x + 2y - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{x + y + 1} \\ S_y &= \frac{2}{3} - \frac{1}{x + y + 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2y}{3} - \ln(1 + x + y) = \frac{x}{3} + c_1$$

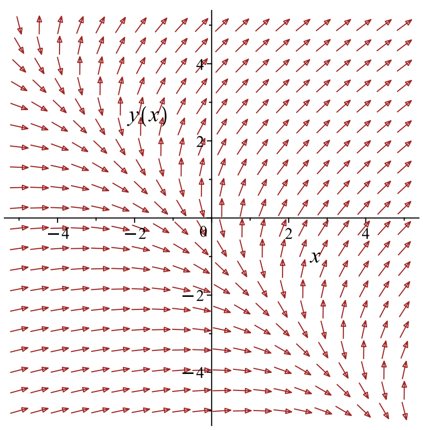
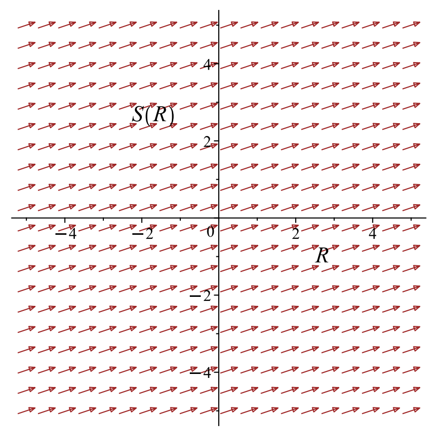
Which simplifies to

$$\frac{2y}{3} - \ln(1 + x + y) = \frac{x}{3} + c_1$$

Which gives

$$y = -\frac{3\,\mathrm{LambertW}\left(-\frac{2\,\mathrm{e}^{-x-c_1-\frac{2}{3}}}{3}\right)}{2} - x - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+y+4}{2x+2y-1}$ 	$R = x$ $S = \frac{2y}{3} - \ln(x+y+1)$	$\frac{dS}{dR} = \frac{1}{3}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{3\,\mathrm{LambertW}\left(-\frac{2\,\mathrm{e}^{-c_1-\frac{2}{3}}}{3}\right)}{2} - 1$$

$$c_1 = 0$$

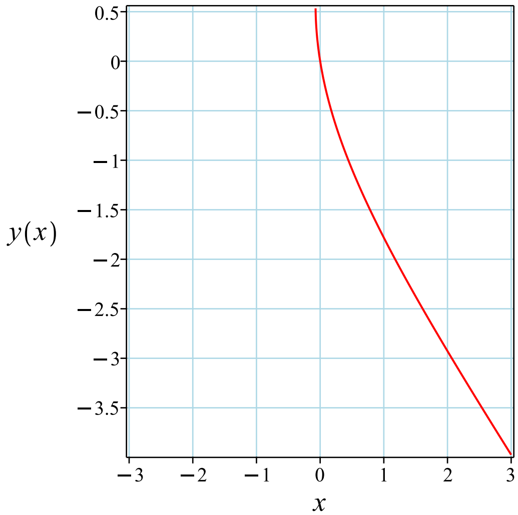
Substituting c_1 found above in the general solution gives

$$y = -\frac{3\,\mathrm{LambertW}\left(-\frac{2\,\mathrm{e}^{-x-\frac{2}{3}}}{3}\right)}{2} - x - 1$$

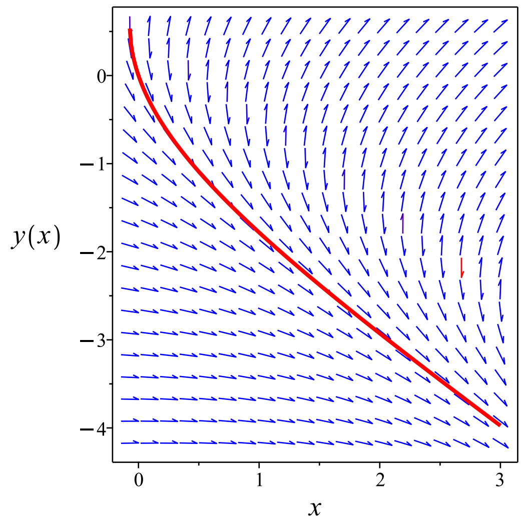
Summary

The solution(s) found are the following

$$y = -\frac{3\,\mathrm{LambertW}\left(-\frac{2\,\mathrm{e}^{-x-\frac{2}{3}}}{3}\right)}{2} - x - 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3\,\mathrm{LambertW}\left(-\frac{2\,e^{-x-\frac{2}{3}}}{3}\right)}{2} - x - 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 20

```
dsolve([(x+y(x)+4)=(2*x+2*y(x)-1)*diff(y(x),x),y(0) = 0],y(x), singsol=all)
```

$$y(x) = -x - \frac{3\,\mathrm{LambertW}\left(-\frac{2\,\mathrm{e}^{-x-\frac{2}{3}}}{3}\right)}{2} - 1$$

✓ Solution by Mathematica

Time used: 3.156 (sec). Leaf size: 28

```
DSolve[{(x+y[x]+4)==(2*x+2*y[x]-1)*y'[x],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3}{2}W\left(-\frac{2}{3}e^{-x-\frac{2}{3}}\right) - x - 1$$

3.15 problem 15

Internal problem ID [1937]
Internal file name [`OUTPUT/1937_Sunday_February_25_2024_06_38_06_AM_13303745/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y + (2x + 3y + 2)\, y' = 1 - 2x}$$

With initial conditions

$$[y(3) = 1]$$

3.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{2x + 3y - 1}{2x + 3y + 2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\left\{x < -\frac{5}{2} \vee -\frac{5}{2} < x\right\}$$

And the point $x_0 = 3$ is inside this domain. The y domain of $f(x,y)$ when $x = 3$ is

$$\left\{y < -\frac{8}{3} \vee -\frac{8}{3} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x + 3y - 1}{2x + 3y + 2}\right)\\ &= -\frac{3}{2x + 3y + 2} + \frac{6x + 9y - 3}{(2x + 3y + 2)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\left\{x < -\frac{5}{2} \vee -\frac{5}{2} < x\right\}$$

And the point $x_0 = 3$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 3$ is

$$\left\{y < -\frac{8}{3} \vee -\frac{8}{3} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.15.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{2x + 3y - 1}{2x + 3y + 2}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(2x+3y-1)(b_3-a_2)}{2x+3y+2} - \frac{(2x+3y-1)^2 a_3}{(2x+3y+2)^2} - \left(-\frac{2}{2x+3y+2} + \frac{4x+6y-2}{(2x+3y+2)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{3}{2x+3y+2} + \frac{6x+9y-3}{(2x+3y+2)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^2a_2 - 4x^2a_3 + 4x^2b_2 - 4x^2b_3 + 12xya_2 - 12xya_3 + 12xyb_2 - 12xyb_3 + 9y^2a_2 - 9y^2a_3 + 9y^2b_2 - 9y^2b_3 +}{(2x+3y+} = 0$$

Setting the numerator to zero gives

$$4x^2a_2 - 4x^2a_3 + 4x^2b_2 - 4x^2b_3 + 12xya_2 - 12xya_3 + 12xyb_2 - 12xyb_3 + 9y^2a_2 - 9y^2a_3 + 9y^2b_2 - 9y^2b_3 + 8xa_2 + 4xa_3 + 17xb_2 - 2xb_3 + 3ya_2 + 12ya_3 + 12yb_2 + 6yb_3 + 6a_1 - 2a_2 - a_3 + 9b_1 + 4b_2 + 2b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$4a_2v_1^2 + 12a_2v_1v_2 + 9a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 - 9a_3v_2^2 + 4b_2v_1^2 + 12b_2v_1v_2 + 9b_2v_2^2 - 4b_3v_1^2 - 12b_3v_1v_2 - 9b_3v_2^2 + 8a_2v_1 + 3a_2v_2 + 4a_3v_1 + 12a_3v_2 + 17b_2v_1 + 12b_2v_2 - 2b_3v_1 + 6b_3v_2 + 6a_1 - 2a_2 - a_3 + 9b_1 + 4b_2 + 2b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(4a_2 - 4a_3 + 4b_2 - 4b_3) v_1^2 + (12a_2 - 12a_3 + 12b_2 - 12b_3) v_1 v_2 \\ + (8a_2 + 4a_3 + 17b_2 - 2b_3) v_1 + (9a_2 - 9a_3 + 9b_2 - 9b_3) v_2^2 \\ + (3a_2 + 12a_3 + 12b_2 + 6b_3) v_2 + 6a_1 - 2a_2 - a_3 + 9b_1 + 4b_2 + 2b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}3a_2 + 12a_3 + 12b_2 + 6b_3 &= 0 \\ 4a_2 - 4a_3 + 4b_2 - 4b_3 &= 0 \\ 8a_2 + 4a_3 + 17b_2 - 2b_3 &= 0 \\ 9a_2 - 9a_3 + 9b_2 - 9b_3 &= 0 \\ 12a_2 - 12a_3 + 12b_2 - 12b_3 &= 0 \\ 6a_1 - 2a_2 - a_3 + 9b_1 + 4b_2 + 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -\frac{7b_3}{6} - \frac{3b_1}{2} \\ a_2 &= -\frac{2b_3}{3} \\ a_3 &= -b_3 \\ b_1 &= b_1 \\ b_2 &= \frac{2b_3}{3} \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -\frac{3}{2} \\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 1 - \left(-\frac{2x+3y-1}{2x+3y+2}\right)\left(-\frac{3}{2}\right) \\
&= \frac{-2x-3y+7}{6y+4+4x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x-3y+7}{6y+4+4x}} dy
\end{aligned}$$

Which results in

$$S = -2y - 6\ln(2x+3y-7)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x+3y-1}{2x+3y+2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{12}{2x+3y-7} \\ S_y &= -2-\frac{18}{2x+3y-7} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-2y - 6\ln(2x+3y-7) = 2x + c_1$$

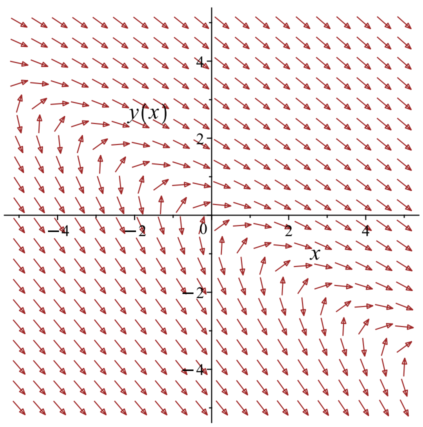
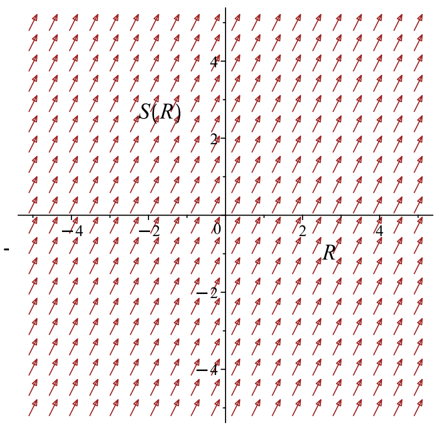
Which simplifies to

$$-2y - 6\ln(2x+3y-7) = 2x + c_1$$

Which gives

$$y = 3\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{x}{9}-\frac{c_1}{6}-\frac{7}{9}}}{9}\right) - \frac{2x}{3} + \frac{7}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+3y-1}{2x+3y+2}$ 	$R = x$ $S = -2y - 6\ln(2x + 3y$ -	$\frac{dS}{dR} = 2$ 

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 3\,\mathrm{LambertW}\left(\frac{e^{-\frac{10}{9}-\frac{c_1}{6}}}{9}\right) + \frac{1}{3}$$

$$c_1 = -8 - 6\ln(2)$$

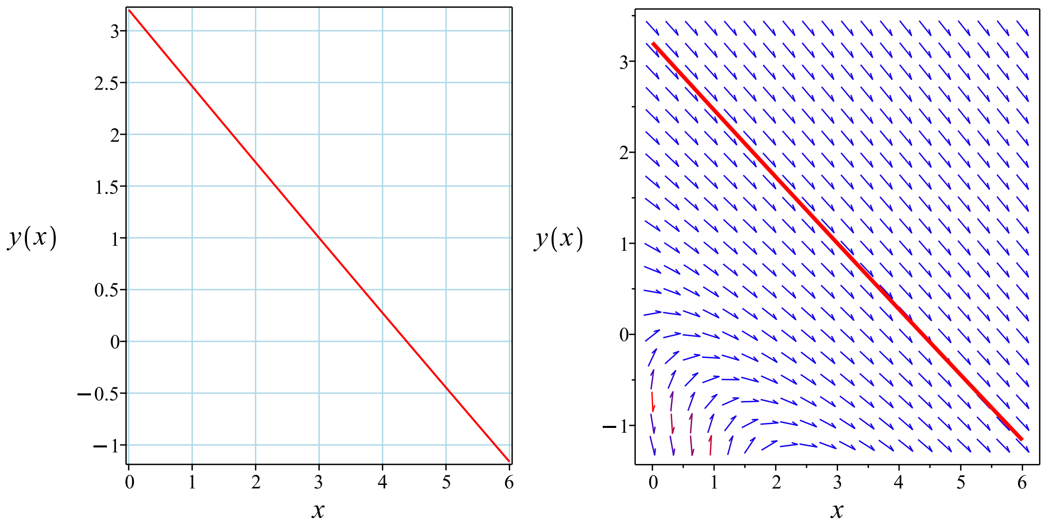
Substituting c_1 found above in the general solution gives

$$y = 3\,\mathrm{LambertW}\left(\frac{2\,e^{-\frac{x}{9}+\frac{5}{9}}}{9}\right) - \frac{2x}{3} + \frac{7}{3}$$

Summary

The solution(s) found are the following

$$y = 3\,\mathrm{LambertW}\left(\frac{2\,e^{-\frac{x}{9}+\frac{5}{9}}}{9}\right) - \frac{2x}{3} + \frac{7}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 3\,\text{LambertW}\left(\frac{2\,e^{-\frac{x}{9}+\frac{5}{9}}}{9}\right) - \frac{2x}{3} + \frac{7}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2/3, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 20

```
dsolve([(2*x+3*y(x)-1)+(2*x+3*y(x)+2)*diff(y(x),x)=0,y(3) = 1],y(x), singsol=all)
```

$$y(x) = \frac{7}{3} - \frac{2x}{3} + 3\,\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{5}{9}-\frac{x}{9}}}{9}\right)$$

✓ Solution by Mathematica

Time used: 4.691 (sec). Leaf size: 32

```
DSolve[{(2*x+3*y[x]-1)+(2*x+3*y[x]+2)*y'[x]==0,{y[3]==1}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{3}\left(9W\left(\frac{2}{9}e^{\frac{5}{9}-\frac{x}{9}}\right) - 2x + 7\right)$$

3.16 problem 16

Internal problem ID [1938]
Internal file name [`OUTPUT/1938_Sunday_February_25_2024_06_38_08_AM_48390682/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y+(x+2y+1)\,y'=-3x-2}$$

With initial conditions

$$[y(0)=0]$$

3.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{-3x+y-2}{x+2y+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{x<-1 \lor -1<x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\left\{y < -\frac{1}{2} \lor -\frac{1}{2} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-3x + y - 2}{x + 2y + 1}\right) \\ &= \frac{1}{x + 2y + 1} - \frac{2(-3x + y - 2)}{(x + 2y + 1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\left\{y < -\frac{1}{2} \lor -\frac{1}{2} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

3.16.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-3X - 3x_0 + Y(X) + y_0 - 2}{X + x_0 + 2Y(X) + 2y_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= -\frac{5}{7} \\ y_0 &= -\frac{1}{7}\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-3X + Y(X)}{X + 2Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{-3X + Y}{X + 2Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -3X + Y$ and $N = X + 2Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{u - 3}{2u + 1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)-3}{2u(X)+1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{\frac{u(X)-3}{2u(X)+1} - u(X)}{X} = 0
$$

Or

$$
2\left(\frac{d}{dX} u(X)\right) X u(X) + \left(\frac{d}{dX} u(X)\right) X + 2u(X)^2 + 3 = 0
$$

Or

$$
3 + (2u(X) + 1) X \left(\frac{d}{dX} u(X)\right) + 2u(X)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{2u^2 + 3}{(2u + 1) X}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{2u^2+3}{2u+1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+3}{2u+1}}\, du = -\frac{1}{X}\, dX$$

$$\int \frac{1}{\frac{2u^2+3}{2u+1}}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{\ln\left(2u^2+3\right)}{2} + \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,u}{3}\right)}{6} = -\ln\left(X\right) + c_2$$

The solution is

$$\frac{\ln\left(2u(X)^2+3\right)}{2} + \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,u(X)}{3}\right)}{6} + \ln\left(X\right) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{2Y(X)^2}{X^2}+3\right)}{2} + \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,Y(X)}{3X}\right)}{6} + \ln\left(X\right) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{2Y(X)^2}{X^2}+3\right)}{2} + \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,Y(X)}{3X}\right)}{6} + \ln\left(X\right) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - \frac{1}{7}$$
$$X = x - \frac{5}{7}$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{2\left(y+\frac{1}{7}\right)^2}{\left(x+\frac{5}{7}\right)^2}+3\right)}{2} + \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,\left(y+\frac{1}{7}\right)}{3x+\frac{15}{7}}\right)}{6} + \ln\left(x+\frac{5}{7}\right) - c_2 = 0$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\ln(7)}{2} + \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}}{15}\right)}{6} - c_2 + \frac{\ln(11)}{2} = 0$$

$$c_2 = -\frac{\ln(7)}{2} + \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}}{15}\right)}{6} + \frac{\ln(11)}{2}$$

Substituting c_2 found above in the general solution gives

$$\frac{\ln\left(\frac{21x^2+14y^2+30x+4y+11}{(5+7x)^2}\right)}{2} + \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,(7y+1)}{15+21x}\right)}{6} + \ln(5+7x) - \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}}{15}\right)}{6} - \frac{\ln(11)}{2} = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{21x^2+14y^2+30x+4y+11}{(5+7x)^2}\right)}{2} + \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,(7y+1)}{15+21x}\right)}{6} + \ln(5+7x) - \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}}{15}\right)}{6} - \frac{\ln(11)}{2} = 0 \tag{1}$$

Verification of solutions

$$\frac{\ln\left(\frac{21x^2+14y^2+30x+4y+11}{(5+7x)^2}\right)}{2} + \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,(7y+1)}{15+21x}\right)}{6} + \ln(5+7x) - \frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}}{15}\right)}{6} - \frac{\ln(11)}{2} = 0$$

Verified OK.

3.16.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-3x + y - 2}{x + 2y + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(-3x + y - 2)(b_3 - a_2)}{x + 2y + 1} - \frac{(-3x + y - 2)^2 a_3}{(x + 2y + 1)^2} \\ & - \left(-\frac{3}{x + 2y + 1} - \frac{-3x + y - 2}{(x + 2y + 1)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(\frac{1}{x + 2y + 1} - \frac{2(-3x + y - 2)}{(x + 2y + 1)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - 9x^2a_3 - 6x^2b_2 - 3x^2b_3 + 12xya_2 + 6xya_3 + 4xyb_2 - 12xyb_3 - 2y^2a_2 + 6y^2a_3 + 4y^2b_2 + 2y^2b_3 + 6}{(x + 2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 3x^2a_2 - 9x^2a_3 - 6x^2b_2 - 3x^2b_3 + 12xya_2 + 6xya_3 + 4xyb_2 - 12xyb_3 \\ & - 2y^2a_2 + 6y^2a_3 + 4y^2b_2 + 2y^2b_3 + 6xa_2 - 12xa_3 - 7xb_1 - 3xb_2 - 5xb_3 \\ & + 7ya_1 + 3ya_2 + 5ya_3 + 4yb_2 - 8yb_3 + a_1 + 2a_2 - 4a_3 - 5b_1 + b_2 - 2b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&3a_2v_1^2 + 12a_2v_1v_2 - 2a_2v_2^2 - 9a_3v_1^2 + 6a_3v_1v_2 + 6a_3v_2^2 - 6b_2v_1^2 + 4b_2v_1v_2 \\
&+ 4b_2v_2^2 - 3b_3v_1^2 - 12b_3v_1v_2 + 2b_3v_2^2 + 7a_1v_2 + 6a_2v_1 + 3a_2v_2 - 12a_3v_1 + 5a_3v_2 \\
&- 7b_1v_1 - 3b_2v_1 + 4b_2v_2 - 5b_3v_1 - 8b_3v_2 + a_1 + 2a_2 - 4a_3 - 5b_1 + b_2 - 2b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(3a_2 - 9a_3 - 6b_2 - 3b_3)\, v_1^2 + (12a_2 + 6a_3 + 4b_2 - 12b_3)\, v_1v_2 \\
&+ (6a_2 - 12a_3 - 7b_1 - 3b_2 - 5b_3)\, v_1 + (-2a_2 + 6a_3 + 4b_2 + 2b_3)\, v_2^2 \\
&+ (7a_1 + 3a_2 + 5a_3 + 4b_2 - 8b_3)\, v_2 + a_1 + 2a_2 - 4a_3 - 5b_1 + b_2 - 2b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_2 + 6a_3 + 4b_2 + 2b_3 &= 0 \\
3a_2 - 9a_3 - 6b_2 - 3b_3 &= 0 \\
12a_2 + 6a_3 + 4b_2 - 12b_3 &= 0 \\
7a_1 + 3a_2 + 5a_3 + 4b_2 - 8b_3 &= 0 \\
6a_2 - 12a_3 - 7b_1 - 3b_2 - 5b_3 &= 0 \\
a_1 + 2a_2 - 4a_3 - 5b_1 + b_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= b_3 \\
a_3 &= 7a_1 - 5b_3 \\
b_1 &= -\frac{15a_1}{2} + \frac{11b_3}{2} \\
b_2 &= -\frac{21a_1}{2} + \frac{15b_3}{2} \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x - 5y$$
$$\eta = \frac{11}{2} + \frac{15x}{2} + y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= \frac{11}{2} + \frac{15x}{2} + y - \left(\frac{-3x+y-2}{x+2y+1}\right)(x-5y) \\ &= \frac{21x^2+14y^2+30x+4y+11}{2x+4y+2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{21x^2+14y^2+30x+4y+11}{2x+4y+2}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(21x^2+14y^2+30x+4y+11)}{7} + \frac{\left(x+\frac{5}{7}\right)\sqrt{6}\,\arctan\left(\frac{(28y+4)\sqrt{6}}{60+84x}\right)}{15+21x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-3x + y - 2}{x + 2y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{4 + 6x - 2y}{21x^2 + 14y^2 + 30x + 4y + 11} \\ S_y &= \frac{2x + 4y + 2}{21x^2 + 14y^2 + 30x + 4y + 11} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

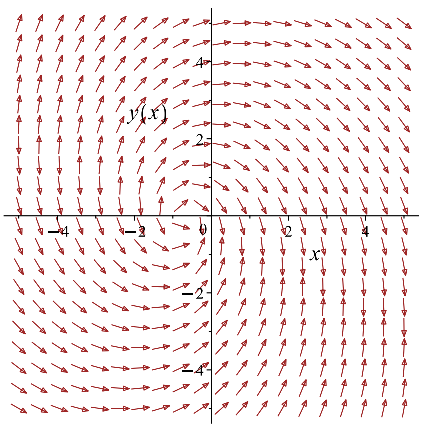
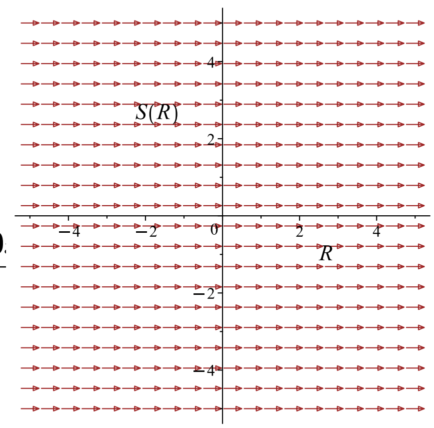
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(21x^2 + 14y^2 + 30x + 4y + 11)}{7} + \frac{\sqrt{6}\arctan\left(\frac{\sqrt{6}\,(7y+1)}{15+21x}\right)}{21} = c_1$$

Which simplifies to

$$\frac{\ln\left(21x^2+14y^2+30x+4y+11\right)}{7}+\frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,(7y+1)}{15+21x}\right)}{21}=c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=\frac{-3x+y-2}{x+2y+1}$ 	$R=x$ $S=\frac{\ln\left(21x^2+14y^2+30\ldots\right.}{7}$ $\ldots\frac{y+1)}{21x}\Big)$	$\frac{dS}{dR}=0$ 

Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln(11)}{7}+\frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}}{15}\right)}{21}=c_1$$

$$c_1=\frac{\ln(11)}{7}+\frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}}{15}\right)}{21}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln\left(21x^2+14y^2+30x+4y+11\right)}{7}+\frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}\,(7y+1)}{15+21x}\right)}{21}=\frac{\ln(11)}{7}+\frac{\sqrt{6}\,\arctan\left(\frac{\sqrt{6}}{15}\right)}{21}$$

Summary

The solution(s) found are the following

$$\frac{\ln(21x^2+14y^2+30x+4y+11)}{7}+\frac{\sqrt{6}\arctan\left(\frac{\sqrt{6}\,(7y+1)}{15+21x}\right)}{21} = \frac{\ln(11)}{7}+\frac{\sqrt{6}\arctan\left(\frac{\sqrt{6}}{15}\right)}{21} \tag{1}$$

Verification of solutions

$$\frac{\ln(21x^2+14y^2+30x+4y+11)}{7}+\frac{\sqrt{6}\arctan\left(\frac{\sqrt{6}\,(7y+1)}{15+21x}\right)}{21} = \frac{\ln(11)}{7}+\frac{\sqrt{6}\arctan\left(\frac{\sqrt{6}}{15}\right)}{21}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✗ Solution by Maple

```
dsolve([(3*x-y(x)+2)+(x+2*y(x)+1)*diff(y(x),x)=0,y(0) = 0],y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 0.186 (sec). Leaf size: 111

```
DSolve[{(3*x-y[x]+2)+(x+2*y[x]+1)*y'[x]==0,{y[0]==0}},y[x],x,IncludeSingularSolutions -> Tru
```

$$\text{Solve}\left[2\sqrt{6}\arctan\left(\frac{\sqrt{\frac{2}{3}}(-y(x)+3x+2)}{2y(x)+x+1}\right)=3\left(\frac{2}{3}\left(\sqrt{6}\arctan\left(2\sqrt{\frac{2}{3}}\right)+3\log\left(\frac{25}{22}\right)-6\log(5)\right)\right.\right.$$
$$\left.\left.+2\log\left(\frac{42x^2+28y(x)^2+8y(x)+60x+22}{(7x+5)^2}\right)+4\log(7x+5)\right),y(x)\right]$$

3.17 problem 17

Internal problem ID [1939]
Internal file name [`OUTPUT/1939_Sunday_February_25_2024_06_38_13_AM_90909664/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{2y - (x + 2y - 1)\, y' = -3x - 3}$$

With initial conditions

$$[y(-2) = 1]$$

3.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{3x + 2y + 3}{x + 2y - 1} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = -2$ is inside this domain. The y domain of $f(x, y)$ when $x = -2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3x + 2y + 3}{x + 2y - 1}\right) \\ &= \frac{2}{x + 2y - 1} - \frac{2(3x + 2y + 3)}{(x + 2y - 1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = -2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.17.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{3X + 3x_0 + 2Y(X) + 2y_0 + 3}{X + x_0 + 2Y(X) + 2y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= -2 \\ y_0 &= \frac{3}{2}\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{3X + 2Y(X)}{X + 2Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{3X+2Y}{X+2Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = 3X + 2Y$ and $N = X + 2Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{2u+3}{2u+1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{2u(X)+3}{2u(X)+1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{2u(X)+3}{2u(X)+1} - u(X)}{X} = 0$$

Or

$$2\left(\frac{d}{dX}u(X)\right)Xu(X) + \left(\frac{d}{dX}u(X)\right)X + 2u(X)^2 - u(X) - 3 = 0$$

Or

$$-3 + (2u(X)+1)\,X\left(\frac{d}{dX}u(X)\right) + 2u(X)^2 - u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{2u^2 - u - 3}{(2u+1)\,X} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{2u^2-u-3}{2u+1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{2u^2-u-3}{2u+1}}\,du &= -\frac{1}{X}\,dX \\
\int \frac{1}{\frac{2u^2-u-3}{2u+1}}\,du &= \int -\frac{1}{X}\,dX \\
\frac{4\ln(2u-3)}{5} + \frac{\ln(u+1)}{5} &= -\ln(X) + c_2
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\frac{4\ln(2u-3) + \ln(u+1)}{5} &= -\ln(X) + c_2 \\
4\ln(2u-3) + \ln(u+1) &= (5)(-\ln(X) + c_2) \\
&= -5\ln(X) + 5c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{4\ln(2u-3)+\ln(u+1)} = e^{-5\ln(X)+5c_2}
$$

Which simplifies to

$$
\begin{aligned}
(2u-3)^4(u+1) &= \frac{5c_2}{X^5} \\
&= \frac{c_3}{X^5}
\end{aligned}
$$

Which simplifies to

$$
u(X) = \operatorname{RootOf}\left(16\_Z^5 - 80\_Z^4 + 120\_Z^3 - \frac{c_3 e^{5c_2}}{X^5} - 135\_Z + 81\right)
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
Y(X) = X \operatorname{RootOf}\left(16\_Z^5 X^5 - 80\_Z^4 X^5 + 120\_Z^3 X^5 - c_3 e^{5c_2} - 135\_Z X^5 + 81X^5\right)
$$

Using the solution for $Y(X)$

$$
Y(X) = X \operatorname{RootOf}\left(16\_Z^5 X^5 - 80\_Z^4 X^5 + 120\_Z^3 X^5 - c_3 e^{5c_2} - 135\_Z X^5 + 81X^5\right)
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y + \frac{3}{2} \\
X &= -2 + x
\end{aligned}
$$

Then the solution in y becomes

$$
y - \frac{3}{2} = (2 + x) \operatorname{RootOf}\left(\left(16x^5 + 160x^4 + 640x^3 + 1280x^2 + 1280x + 512\right)\_Z^5 + \left(-80x^5 - 800x^4 - 320
$$

Unable to solve for constant of integration due to RootOf in solution.

Summary

The solution(s) found are the following

$$
\begin{aligned}
y - \frac{3}{2} = (2 + x) \operatorname{RootOf}\Big(&\left(16x^5 + 160x^4 + 640x^3 + 1280x^2 + 1280x + 512\right)\_Z^5 \\
&+ \left(-80x^5 - 800x^4 - 3200x^3 - 6400x^2 - 6400x - 2560\right)\_Z^4 \\
&+ \left(120x^5 + 1200x^4 + 4800x^3 + 9600x^2 + 9600x + 3840\right)\_Z^3 \\
&+ \left(-135x^5 - 1350x^4 - 5400x^3 - 10800x^2 - 10800x - 4320\right)\_Z - c_3 e^{5c_2} \\
&+ 81x^5 + 810x^4 + 3240x^3 + 6480x^2 + 6480x + 2592\Big)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y - \frac{3}{2} = (2 + x) \operatorname{RootOf}\Big(&\left(16x^5 + 160x^4 + 640x^3 + 1280x^2 + 1280x + 512\right)\_Z^5 \\
&+ \left(-80x^5 - 800x^4 - 3200x^3 - 6400x^2 - 6400x - 2560\right)\_Z^4 \\
&+ \left(120x^5 + 1200x^4 + 4800x^3 + 9600x^2 + 9600x + 3840\right)\_Z^3 \\
&+ \left(-135x^5 - 1350x^4 - 5400x^3 - 10800x^2 - 10800x - 4320\right)\_Z - c_3 e^{5c_2} \\
&+ 81x^5 + 810x^4 + 3240x^3 + 6480x^2 + 6480x + 2592\Big)
\end{aligned}
$$

Warning, solution could not be verified

3.17.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{3x+2y+3}{x+2y-1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{(3x+2y+3)(b_3-a_2)}{x+2y-1} - \frac{(3x+2y+3)^2 a_3}{(x+2y-1)^2} \\
&- \left(\frac{3}{x+2y-1} - \frac{3x+2y+3}{(x+2y-1)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(\frac{2}{x+2y-1} - \frac{2(3x+2y+3)}{(x+2y-1)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{3x^2a_2 + 9x^2a_3 - 5x^2b_2 - 3x^2b_3 + 12xya_2 + 12xya_3 - 4xyb_2 - 12xyb_3 + 4y^2a_2 + 8y^2a_3 - 4y^2b_2 - 4y^2b_3 \cdots}{(x+2 \cdots}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-3x^2a_2 - 9x^2a_3 + 5x^2b_2 + 3x^2b_3 - 12xya_2 - 12xya_3 + 4xyb_2 + 12xyb_3 \\
&- 4y^2a_2 - 8y^2a_3 + 4y^2b_2 + 4y^2b_3 + 6xa_2 - 18xa_3 + 4xb_1 + 6xb_2 - 4ya_1 \\
&- 4ya_2 - 6ya_3 - 4yb_2 + 12yb_3 + 6a_1 + 3a_2 - 9a_3 + 8b_1 + b_2 - 3b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-3a_2v_1^2 - 12a_2v_1v_2 - 4a_2v_2^2 - 9a_3v_1^2 - 12a_3v_1v_2 - 8a_3v_2^2 + 5b_2v_1^2 + 4b_2v_1v_2 + 4b_2v_2^2 + 3b_3v_1^2 + 12b_3v_1v_2 + 4b_3v_2^2 - 4a_1v_2 + 6a_2v_1 - 4a_2v_2 - 18a_3v_1 - 6a_3v_2 + 4b_1v_1 + 6b_2v_1 - 4b_2v_2 + 12b_3v_2 + 6a_1 + 3a_2 - 9a_3 + 8b_1 + b_2 - 3b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-3a_2 - 9a_3 + 5b_2 + 3b_3)\, v_1^2 + (-12a_2 - 12a_3 + 4b_2 + 12b_3)\, v_1v_2 + (6a_2 - 18a_3 + 4b_1 + 6b_2)\, v_1 + (-4a_2 - 8a_3 + 4b_2 + 4b_3)\, v_2^2 + (-4a_1 - 4a_2 - 6a_3 - 4b_2 + 12b_3)\, v_2 + 6a_1 + 3a_2 - 9a_3 + 8b_1 + b_2 - 3b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-12a_2 - 12a_3 + 4b_2 + 12b_3 &= 0\\
-4a_2 - 8a_3 + 4b_2 + 4b_3 &= 0\\
-3a_2 - 9a_3 + 5b_2 + 3b_3 &= 0\\
6a_2 - 18a_3 + 4b_1 + 6b_2 &= 0\\
-4a_1 - 4a_2 - 6a_3 - 4b_2 + 12b_3 &= 0\\
6a_1 + 3a_2 - 9a_3 + 8b_1 + b_2 - 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -\frac{5a_3}{2} + 2b_3 \\ a_2 &= -\frac{a_3}{2} + b_3 \\ a_3 &= a_3 \\ b_1 &= 3a_3 - \frac{3b_3}{2} \\ b_2 &= \frac{3a_3}{2} \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 2 + x \\ \eta &= -\frac{3}{2} + y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= -\frac{3}{2} + y - \left(\frac{3x+2y+3}{x+2y-1}\right)(2+x) \\ &= \frac{-6x^2 - 2yx + 4y^2 - 21x - 16y - 9}{2x + 4y - 2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-6x^2-2yx+4y^2-21x-16y-9}{2x+4y-2}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{4\ln\left(2y-3x-9\right)}{5} + \frac{\ln\left(2x+2y+1\right)}{5}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{3x+2y+3}{x+2y-1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{6x+4y+6}{\left(2x+2y+1\right)\left(3x-2y+9\right)} \\
S_y &= \frac{-2x-4y+2}{\left(2x+2y+1\right)\left(3x-2y+9\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

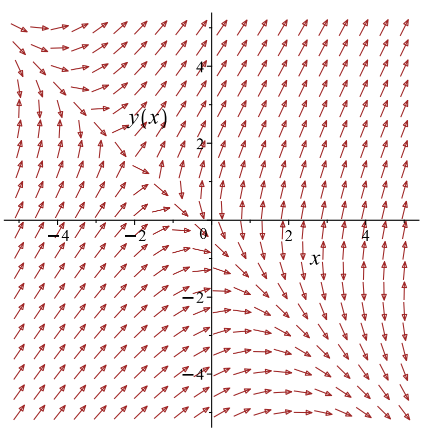
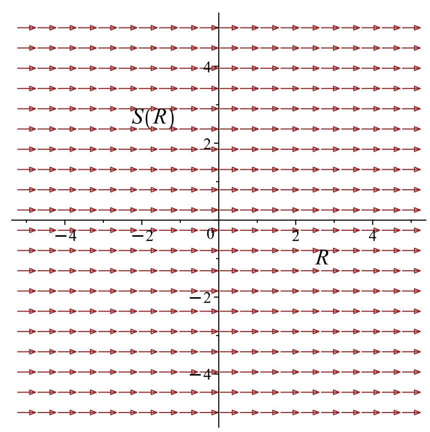
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{4 \ln (2y - 3x - 9)}{5} + \frac{\ln (2x + 2y + 1)}{5} = c_1$$

Which simplifies to

$$\frac{4 \ln (2y - 3x - 9)}{5} + \frac{\ln (2x + 2y + 1)}{5} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x+2y+3}{x+2y-1}$  	$R = x$ $S = \frac{4 \ln (2y - 3x - 9)}{5} +$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = -2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$i\pi = c_1$$

$$c_1 = i\pi$$

Substituting c_1 found above in the general solution gives

$$\frac{4\ln(2y-3x-9)}{5}+\frac{\ln(2x+2y+1)}{5}=i\pi$$

Summary

The solution(s) found are the following

$$\frac{4\ln(2y-3x-9)}{5}+\frac{\ln(2x+2y+1)}{5}=i\pi \tag{1}$$

Verification of solutions

$$\frac{4\ln(2y-3x-9)}{5}+\frac{\ln(2x+2y+1)}{5}=i\pi$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.968 (sec). Leaf size: 136

```
dsolve([(3*x+2*y(x)+3)-(x+2*y(x)-1)*diff(y(x),x)=0,y(-2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{(-2-x)\operatorname{RootOf}\left(-1+(x^5+10x^4+40x^3+80x^2+80x+32)\_Z^{25}+(-5x^5-50x^4-200x^3-400x^2\right.}{2} + \frac{3x}{2}+\frac{9}{2}$$

✓ Solution by Mathematica

Time used: 63.287 (sec). Leaf size: 850

```
DSolve[{(3*x+2*y[x]+3)-(x+2*y[x]-1)*y'[x]==0,{y[-2]==1}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \rightarrow \frac{-x\text{Root}\left[(65536x^{10} + 1310720x^9 + 11796480x^8 + 62914560x^7 + 220200960x^6 + 528482304x^5 + 880803\right.}{}$$

3.18 problem 18

Internal problem ID [1940]
Internal file name [`OUTPUT/1940_Sunday_February_25_2024_06_38_16_AM_20314154/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-2y + (1 - x + 2y)\, y' = -x - 3}$$

With initial conditions

$$[y(-4) = 2]$$

3.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{-x + 2y - 3}{1 - x + 2y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\{x < 5 \lor 5 < x\}$$

And the point $x_0 = -4$ is inside this domain. The y domain of $f(x, y)$ when $x = -4$ is

$$\left\{y < -\frac{5}{2} \vee -\frac{5}{2} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-x + 2y - 3}{1 - x + 2y}\right) \\ &= \frac{2}{1 - x + 2y} - \frac{2(-x + 2y - 3)}{(1 - x + 2y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{x < 5 \vee 5 < x\}$$

And the point $x_0 = -4$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -4$ is

$$\left\{y < -\frac{5}{2} \vee -\frac{5}{2} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

3.18.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{-x + 2y - 3}{1 - x + 2y} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-x+2y-3)(b_3-a_2)}{1-x+2y} - \frac{(-x+2y-3)^2 a_3}{(1-x+2y)^2} - \left(-\frac{1}{1-x+2y} + \frac{-x+2y-3}{(1-x+2y)^2}\right)(xa_2+ya_3+a_1) - \left(\frac{2}{1-x+2y} - \frac{2(-x+2y-3)}{(1-x+2y)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 - x^2b_2 - x^2b_3 - 4xya_2 - 4xya_3 + 4xyb_2 + 4xyb_3 + 4y^2a_2 + 4y^2a_3 - 4y^2b_2 - 4y^2b_3 - 2xa_2 -}{(-1+x-2y)}$$
$$= 0$$

Setting the numerator to zero gives

$$-x^2a_2 - x^2a_3 + x^2b_2 + x^2b_3 + 4xya_2 + 4xya_3 - 4xyb_2 - 4xyb_3 - 4y^2a_2 - 4y^2a_3 + 4y^2b_2 + 4y^2b_3 + 2xa_2 - 6xa_3 - 10xb_2 + 2xb_3 + 4ya_2 + 16ya_3 + 4yb_2 - 12yb_3 + 4a_1 + 3a_2 - 9a_3 - 8b_1 + b_2 - 3b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^2 + 4a_2v_1v_2 - 4a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 - 4a_3v_2^2 + b_2v_1^2 - 4b_2v_1v_2 + 4b_2v_2^2 + b_3v_1^2 - 4b_3v_1v_2 + 4b_3v_2^2 + 2a_2v_1 + 4a_2v_2 - 6a_3v_1 + 16a_3v_2 - 10b_2v_1 + 4b_2v_2 + 2b_3v_1 - 12b_3v_2 + 4a_1 + 3a_2 - 9a_3 - 8b_1 + b_2 - 3b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-a_2 - a_3 + b_2 + b_3) v_1^2 + (4a_2 + 4a_3 - 4b_2 - 4b_3) v_1 v_2 \\ &+ (2a_2 - 6a_3 - 10b_2 + 2b_3) v_1 + (-4a_2 - 4a_3 + 4b_2 + 4b_3) v_2^2 \\ &+ (4a_2 + 16a_3 + 4b_2 - 12b_3) v_2 + 4a_1 + 3a_2 - 9a_3 - 8b_1 + b_2 - 3b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-4a_2 - 4a_3 + 4b_2 + 4b_3 &= 0 \\ -a_2 - a_3 + b_2 + b_3 &= 0 \\ 2a_2 - 6a_3 - 10b_2 + 2b_3 &= 0 \\ 4a_2 + 4a_3 - 4b_2 - 4b_3 &= 0 \\ 4a_2 + 16a_3 + 4b_2 - 12b_3 &= 0 \\ 4a_1 + 3a_2 - 9a_3 - 8b_1 + b_2 - 3b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -7b_2 + 2b_1 \\ a_2 &= b_2 \\ a_3 &= -2b_2 \\ b_1 &= b_1 \\ b_2 &= b_2 \\ b_3 &= -2b_2\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 2 \\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= 1 - \left(\frac{-x + 2y - 3}{1 - x + 2y}\right)(2) \\ &= \frac{-x + 2y - 7}{-1 + x - 2y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x+2y-7}{-1+x-2y}} dy \end{aligned}$$

Which results in

$$S = -y - 4 \ln (-x + 2y - 7)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x + 2y - 3}{1 - x + 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{4}{x - 2y + 7} \\ S_y &= -1 + \frac{8}{x - 2y + 7} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-y - 4\ln(-x + 2y - 7) = -x + c_1$$

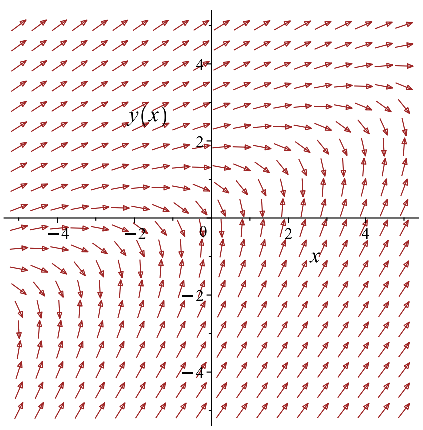
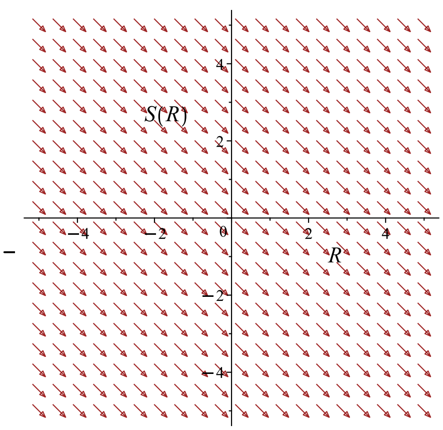
Which simplifies to

$$-y - 4\ln(-x + 2y - 7) = -x + c_1$$

Which gives

$$y = 4\,\mathrm{LambertW}\left(\frac{\mathrm{e}^{\frac{x}{8}-\frac{c_1}{4}-\frac{7}{8}}}{8}\right) + \frac{x}{2} + \frac{7}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+2y-3}{1-x+2y}$ 	$R = x$ $S = -y - 4\ln(-x + 2y -$	$\frac{dS}{dR} = -1$ 

Initial conditions are used to solve for c_1. Substituting $x = -4$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 4\,\mathrm{LambertW}\left(\frac{e^{-\frac{11}{8}-\frac{c_1}{4}}}{8}\right) + \frac{3}{2}$$

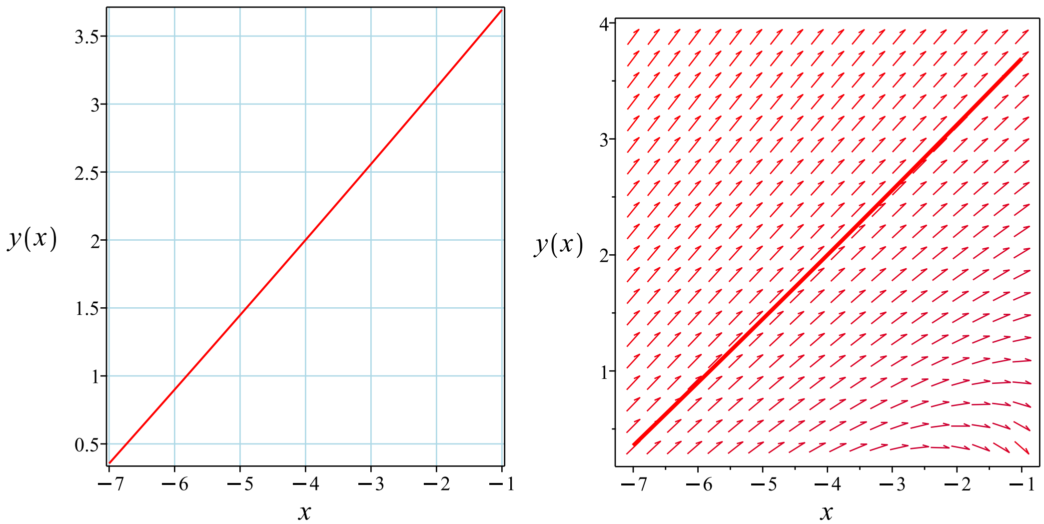
$$c_1 = -6$$

Substituting c_1 found above in the general solution gives

$$y = 4\,\mathrm{LambertW}\left(\frac{e^{\frac{x}{8}+\frac{5}{8}}}{8}\right) + \frac{x}{2} + \frac{7}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 4\,\mathrm{LambertW}\left(\frac{e^{\frac{x}{8}+\frac{5}{8}}}{8}\right) + \frac{x}{2} + \frac{7}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 4 \operatorname{LambertW}\left(\frac{e^{\frac{x}{8}+\frac{5}{8}}}{8}\right) + \frac{x}{2} + \frac{7}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 20

```
dsolve([(x-2*y(x)+3)+(1-x+2*y(x))*diff(y(x),x)=0,y(-4) = 2],y(x), singsol=all)
```

$$y(x) = \frac{7}{2} + \frac{x}{2} + 4\,\mathrm{LambertW}\left(\frac{e^{\frac{x}{8}+\frac{5}{8}}}{8}\right)$$

✓ Solution by Mathematica

Time used: 4.645 (sec). Leaf size: 28

```
DSolve[{(x-2*y[x]+3)+(1-x+2*y[x])*y'[x]==0,{y[-4]==2}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{2}\left(8W\left(\frac{1}{8}e^{\frac{x+5}{8}}\right) + x + 7\right)$$

3.19 problem 19

Internal problem ID [1941]
Internal file name [`OUTPUT/1941_Sunday_February_25_2024_06_38_18_AM_53765149/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (4x + 2y + 1)\, y' = -2x}$$

With initial conditions

$$\left[y\left(-\frac{1}{6}\right) = 0\right]$$

3.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{2x + y}{4x + 2y + 1} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\left\{x < -\frac{1}{4} \lor -\frac{1}{4} < x\right\}$$

And the point $x_0 = -\frac{1}{6}$ is inside this domain. The y domain of $f(x, y)$ when $x = -\frac{1}{6}$ is

$$\left\{y < -\frac{1}{6} \lor -\frac{1}{6} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x + y}{4x + 2y + 1}\right) \\ &= -\frac{1}{4x + 2y + 1} + \frac{4x + 2y}{(4x + 2y + 1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\left\{x < -\frac{1}{4} \lor -\frac{1}{4} < x\right\}$$

And the point $x_0 = -\frac{1}{6}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -\frac{1}{6}$ is

$$\left\{y < -\frac{1}{6} \lor -\frac{1}{6} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

3.19.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{2x + y}{4x + 2y + 1} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{(2x+y)(b_3-a_2)}{4x+2y+1} - \frac{(2x+y)^2 a_3}{(4x+2y+1)^2} \\
&- \left(-\frac{2}{4x+2y+1} + \frac{8x+4y}{(4x+2y+1)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{1}{4x+2y+1} + \frac{4x+2y}{(4x+2y+1)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{8x^2a_2 - 4x^2a_3 + 16x^2b_2 - 8x^2b_3 + 8xya_2 - 4xya_3 + 16xyb_2 - 8xyb_3 + 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 + 4x}{(4x+2y+1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&8x^2a_2 - 4x^2a_3 + 16x^2b_2 - 8x^2b_3 + 8xya_2 - 4xya_3 + 16xyb_2 - 8xyb_3 + 2y^2a_2 - y^2a_3 \\
&+ 4y^2b_2 - 2y^2b_3 + 4xa_2 + 9xb_2 - 2xb_3 + ya_2 + 2ya_3 + 4yb_2 + 2a_1 + b_1 + b_2 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&8a_2v_1^2 + 8a_2v_1v_2 + 2a_2v_2^2 - 4a_3v_1^2 - 4a_3v_1v_2 - a_3v_2^2 + 16b_2v_1^2 \\
&+ 16b_2v_1v_2 + 4b_2v_2^2 - 8b_3v_1^2 - 8b_3v_1v_2 - 2b_3v_2^2 + 4a_2v_1 \\
&+ a_2v_2 + 2a_3v_2 + 9b_2v_1 + 4b_2v_2 - 2b_3v_1 + 2a_1 + b_1 + b_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(8a_2-4a_3+16b_2-8b_3)\,v_1^2+(8a_2-4a_3+16b_2-8b_3)\,v_1v_2+(4a_2+9b_2-2b_3)\,v_1 \\ +(2a_2-a_3+4b_2-2b_3)\,v_2^2+(a_2+2a_3+4b_2)\,v_2+2a_1+b_1+b_2=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}2a_1+b_1+b_2&=0\\a_2+2a_3+4b_2&=0\\4a_2+9b_2-2b_3&=0\\2a_2-a_3+4b_2-2b_3&=0\\8a_2-4a_3+16b_2-8b_3&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1&=a_1\\a_2&=-2b_2\\a_3&=-b_2\\b_1&=-2a_1-b_2\\b_2&=b_2\\b_3&=\frac{b_2}{2}\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi&=1\\\eta&=-2\end{aligned}$$

Shifting is now applied to make $\xi=0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta&=\eta-\omega(x,y)\,\xi\\&=-2-\left(-\frac{2x+y}{4x+2y+1}\right)(1)\\&=\frac{-6x-3y-2}{4x+2y+1}\\\xi&=0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-6x-3y-2}{4x+2y+1}} dy \end{aligned}$$

Which results in

$$S = -\frac{2y}{3} + \frac{\ln (6x + 3y + 2)}{9}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x + y}{4x + 2y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{18x + 9y + 6} \\ S_y &= \frac{-4x - 2y - 1}{6x + 3y + 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2y}{3} + \frac{\ln(6x + 3y + 2)}{9} = \frac{x}{3} + c_1$$

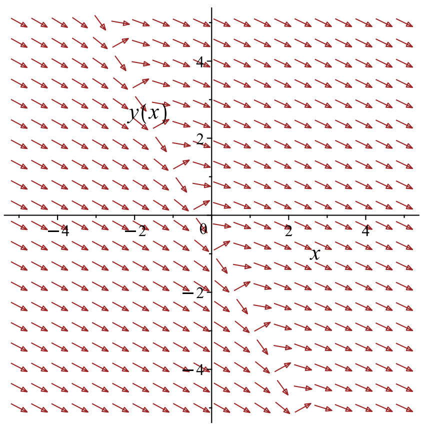
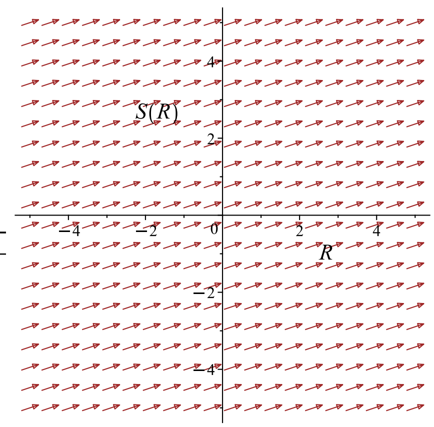
Which simplifies to

$$-\frac{2y}{3} + \frac{\ln(6x + 3y + 2)}{9} = \frac{x}{3} + c_1$$

Which gives

$$y = \frac{e^{-\operatorname{LambertW}\left(-2\,e^{-9x-4+9c_1}\right)-9x-4+9c_1}}{3} - 2x - \frac{2}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+y}{4x+2y+1}$ 	$R = x$ $S = -\frac{2y}{3} + \frac{\ln(6x + 3y +}{9}$	$\frac{dS}{dR} = \frac{1}{3}$ 

Initial conditions are used to solve for c_1. Substituting $x = -\frac{1}{6}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{\text{LambertW}\left(-2\,e^{-\frac{5}{2}+9c_1}\right)}{6} - \frac{1}{3}$$

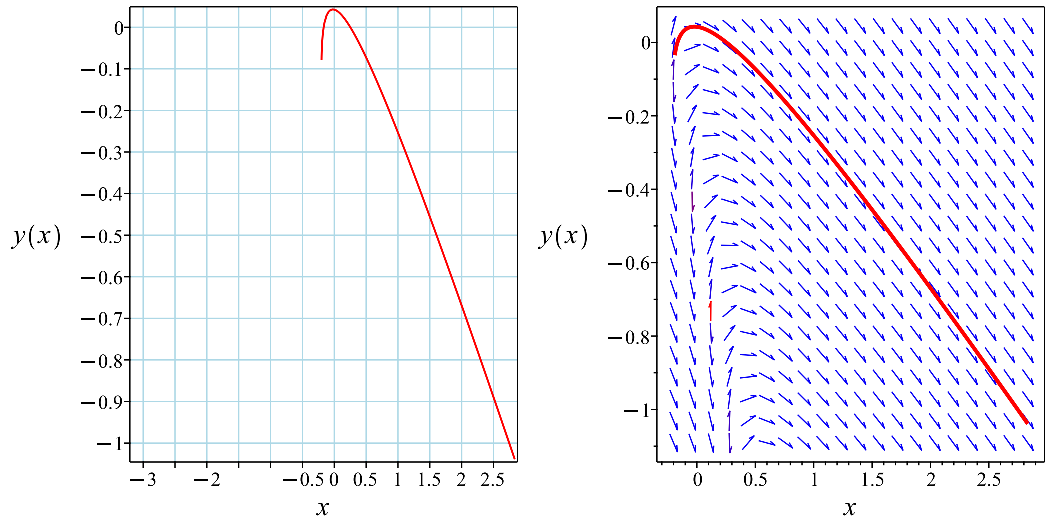
$$c_1 = \frac{1}{18}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\text{LambertW}\left(-2\,e^{-9x-\frac{7}{2}}\right)}{6} - 2x - \frac{2}{3}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\text{LambertW}\left(-1, -2\,e^{-9x-\frac{7}{2}}\right)}{6} - 2x - \frac{2}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{\operatorname{LambertW}\left(-1, -2\,\mathrm{e}^{-9x-\frac{7}{2}}\right)}{6} - 2x - \frac{2}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 21

```
dsolve([(2*x+y(x))+(4*x+2*y(x)+1)*diff(y(x),x)=0,y(-1/6) = 0],y(x), singsol=all)
```

$$y(x) = -\frac{\operatorname{LambertW}\left(-1, -2\,\mathrm{e}^{-9x-\frac{7}{2}}\right)}{6} - \frac{2}{3} - 2x$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(2*x+y[x])+(4*x+2*y[x]+1)*y'[x]==0,{y[-1/6]==0}},y[x],x,IncludeSingularSolutions ->
```

{}

3.20 problem 20

Internal problem ID [1942]
Internal file name [`OUTPUT/1942_Sunday_February_25_2024_06_38_20_AM_50652990/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 7, page 28
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (4x - 2y + 1)\, y' = -2x}$$

With initial conditions

$$\left[y\left(\frac{1}{2}\right) = 0\right]$$

3.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{2x + y}{-4x + 2y - 1} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\left\{x < -\frac{1}{4} \vee -\frac{1}{4} < x\right\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The y domain of $f(x,y)$ when $x = \frac{1}{2}$ is

$$\left\{y < \frac{3}{2} \lor \frac{3}{2} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x + y}{-4x + 2y - 1}\right)\\ &= \frac{1}{-4x + 2y - 1} - \frac{2(2x + y)}{(-4x + 2y - 1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\left\{x < -\frac{1}{4} \lor -\frac{1}{4} < x\right\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{1}{2}$ is

$$\left\{y < \frac{3}{2} \lor \frac{3}{2} < y\right\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

3.20.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2X + 2x_0 + Y(X) + y_0}{-4X - 4x_0 + 2Y(X) + 2y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= -\frac{1}{8}\\ y_0 &= \frac{1}{4}\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2X + Y(X)}{-4X + 2Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{2X + Y}{-4X + 2Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -2X - Y$ and $N = 4X - 2Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{2 + u}{2u - 4} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{2+u(X)}{2u(X)-4} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX} u(X) - \frac{\frac{2+u(X)}{2u(X)-4} - u(X)}{X} = 0$$

Or

$$2\left(\frac{d}{dX} u(X)\right) X u(X) - 4\left(\frac{d}{dX} u(X)\right) X + 2u(X)^2 - 5u(X) - 2 = 0$$

Or

$$2X(u(X) - 2)\left(\frac{d}{dX} u(X)\right) + 2u(X)^2 - 5u(X) - 2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X) g(u) \\ &= -\frac{2u^2 - 5u - 2}{2X (u - 2)} \end{aligned}$$

Where $f(X) = -\frac{1}{2X}$ and $g(u) = \frac{2u^2-5u-2}{u-2}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2-5u-2}{u-2}}\,du = -\frac{1}{2X}\,dX$$

$$\int \frac{1}{\frac{2u^2-5u-2}{u-2}}\,du = \int -\frac{1}{2X}\,dX$$

$$\frac{\ln(2u^2-5u-2)}{4} + \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(4u-5)\sqrt{41}}{41}\right)}{82} = -\frac{\ln(X)}{2} + c_2$$

The solution is

$$\frac{\ln\left(2u(X)^2-5u(X)-2\right)}{4} + \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(4u(X)-5)\sqrt{41}}{41}\right)}{82} + \frac{\ln(X)}{2} - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{2Y(X)^2}{X^2} - \frac{5Y(X)}{X} - 2\right)}{4} + \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{\left(\frac{4Y(X)}{X}-5\right)\sqrt{41}}{41}\right)}{82} + \frac{\ln(X)}{2} - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{2Y(X)^2}{X^2} - \frac{5Y(X)}{X} - 2\right)}{4} + \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(4Y(X)-5X)\sqrt{41}}{41X}\right)}{82} + \frac{\ln(X)}{2} - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + \frac{1}{4}$$
$$X = x - \frac{1}{8}$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{2\left(y-\frac{1}{4}\right)^2}{\left(x+\frac{1}{8}\right)^2} - \frac{5\left(y-\frac{1}{4}\right)}{x+\frac{1}{8}} - 2\right)}{4} + \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{\left(4y-\frac{13}{8}-5x\right)\sqrt{41}}{41x+\frac{41}{8}}\right)}{82} + \frac{\ln\left(x+\frac{1}{8}\right)}{2} - c_2 = 0$$

Initial conditions are used to solve for c_2. Substituting $x = \frac{1}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{3\ln(2)}{4} - \frac{3\sqrt{41}\,\operatorname{arccoth}\left(\frac{33\sqrt{41}}{205}\right)}{82} + \frac{3i\sqrt{41}\,\pi}{164} - c_2 = 0$$

$$c_2 = -\frac{3\ln(2)}{4} - \frac{3\sqrt{41}\,\operatorname{arccoth}\left(\frac{33\sqrt{41}}{205}\right)}{82} + \frac{3i\sqrt{41}\,\pi}{164}$$

Substituting c_2 found above in the general solution gives

$$\frac{\ln\left(\frac{-16x^2+(-40y+6)x+16y^2-13y+2}{(8x+1)^2}\right)}{4} - \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(-32y+13+40x)\sqrt{41}}{328x+41}\right)}{82} + \frac{\ln(8x+1)}{2} + \frac{3\sqrt{41}\,\operatorname{arccoth}\left(\frac{33\sqrt{4}}{205}\right.}{82}$$

Summary

The solution(s) found are the following

$$\begin{aligned}&\frac{\ln\left(\frac{-16x^2+(-40y+6)x+16y^2-13y+2}{(8x+1)^2}\right)}{4} - \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(-32y+13+40x)\sqrt{41}}{328x+41}\right)}{82}\\&+\frac{\ln(8x+1)}{2} + \frac{3\sqrt{41}\,\operatorname{arccoth}\left(\frac{33\sqrt{41}}{205}\right)}{82} - \frac{3i\sqrt{41}\,\pi}{164} = 0\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}&\frac{\ln\left(\frac{-16x^2+(-40y+6)x+16y^2-13y+2}{(8x+1)^2}\right)}{4} - \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(-32y+13+40x)\sqrt{41}}{328x+41}\right)}{82}\\&+\frac{\ln(8x+1)}{2} + \frac{3\sqrt{41}\,\operatorname{arccoth}\left(\frac{33\sqrt{41}}{205}\right)}{82} - \frac{3i\sqrt{41}\,\pi}{164} = 0\end{aligned}$$

Verified OK.

3.20.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{2x+y}{-4x+2y-1}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(2x+y)(b_3-a_2)}{-4x+2y-1} - \frac{(2x+y)^2 a_3}{(-4x+2y-1)^2} \\ &- \left(\frac{2}{-4x+2y-1} + \frac{8x+4y}{(-4x+2y-1)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(\frac{1}{-4x+2y-1} - \frac{2(2x+y)}{(-4x+2y-1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{8x^2a_2 - 4x^2a_3 + 24x^2b_2 - 8x^2b_3 - 8xya_2 - 4xya_3 - 16xyb_2 + 8xyb_3 - 2y^2a_2 - 9y^2a_3 + 4y^2b_2 + 2y^2b_3 + 4}{(4x-2y+1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &8x^2a_2 - 4x^2a_3 + 24x^2b_2 - 8x^2b_3 - 8xya_2 - 4xya_3 - 16xyb_2 \\ &+ 8xyb_3 - 2y^2a_2 - 9y^2a_3 + 4y^2b_2 + 2y^2b_3 + 4xa_2 + 8xb_1 + 9xb_2 \\ &- 2xb_3 - 8ya_1 + ya_2 + 2ya_3 - 4yb_2 + 2a_1 + b_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&8a_2v_1^2 - 8a_2v_1v_2 - 2a_2v_2^2 - 4a_3v_1^2 - 4a_3v_1v_2 - 9a_3v_2^2 + 24b_2v_1^2 \\
&- 16b_2v_1v_2 + 4b_2v_2^2 - 8b_3v_1^2 + 8b_3v_1v_2 + 2b_3v_2^2 - 8a_1v_2 + 4a_2v_1 \\
&+ a_2v_2 + 2a_3v_2 + 8b_1v_1 + 9b_2v_1 - 4b_2v_2 - 2b_3v_1 + 2a_1 + b_1 + b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(8a_2 - 4a_3 + 24b_2 - 8b_3)\, v_1^2 + (-8a_2 - 4a_3 - 16b_2 + 8b_3)\, v_1v_2 \\
&+ (4a_2 + 8b_1 + 9b_2 - 2b_3)\, v_1 + (-2a_2 - 9a_3 + 4b_2 + 2b_3)\, v_2^2 \\
&+ (-8a_1 + a_2 + 2a_3 - 4b_2)\, v_2 + 2a_1 + b_1 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 + b_1 + b_2 &= 0 \\
-8a_1 + a_2 + 2a_3 - 4b_2 &= 0 \\
-8a_2 - 4a_3 - 16b_2 + 8b_3 &= 0 \\
-2a_2 - 9a_3 + 4b_2 + 2b_3 &= 0 \\
4a_2 + 8b_1 + 9b_2 - 2b_3 &= 0 \\
8a_2 - 4a_3 + 24b_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 8a_1 + 2b_2 \\
a_3 &= b_2 \\
b_1 &= -2a_1 - b_2 \\
b_2 &= b_2 \\
b_3 &= 8a_1 + \frac{9b_2}{2}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x + y \\
\eta &= x - 1 + \frac{9y}{2}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x - 1 + \frac{9y}{2} - \left(\frac{2x+y}{-4x+2y-1}\right)(2x+y) \\
&= \frac{16x^2 + 40yx - 16y^2 - 6x + 13y - 2}{8x - 4y + 2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{16x^2+40yx-16y^2-6x+13y-2}{8x-4y+2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(-16x^2 - 40yx + 16y^2 + 6x - 13y + 2\right)}{8} - \frac{4\left(-\frac{3x}{2} - \frac{3}{16}\right)\sqrt{41}\,\operatorname{arctanh}\left(\frac{(32y-13-40x)\sqrt{41}}{328x+41}\right)}{41\,(8x+1)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x+y}{-4x+2y-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{4x + 2y}{16x^2 + (40y - 6)\,x - 16y^2 + 13y - 2} \\
S_y &= \frac{8x - 4y + 2}{16x^2 + (40y - 6)\,x - 16y^2 + 13y - 2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

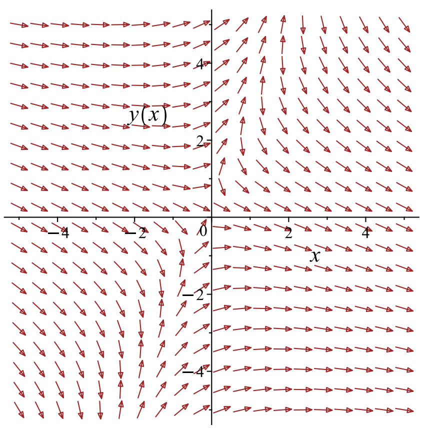
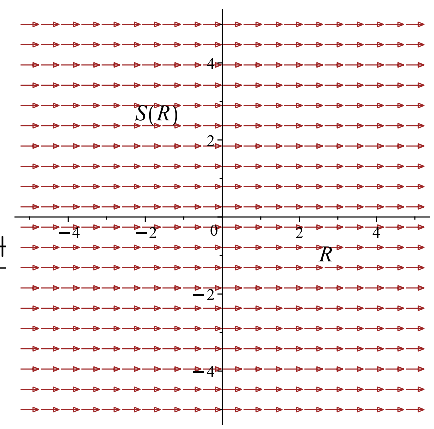
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(-16x^2 + (-40y + 6)\,x + 16y^2 - 13y + 2\right)}{8} - \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(-32y+13+40x)\sqrt{41}}{328x+41}\right)}{164} = c_1$$

Which simplifies to

$$\frac{\ln\left(-16x^2 + (-40y + 6)\,x + 16y^2 - 13y + 2\right)}{8} - \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(-32y+13+40x)\sqrt{41}}{328x+41}\right)}{164} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x+y}{-4x+2y-1}$ 	$R = x$ $S = \frac{\ln\left(-16x^2 + (-40y + \dots\right.}{\dots}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{3\sqrt{41}\,\operatorname{arccoth}\left(\frac{33\sqrt{41}}{205}\right)}{164} + \frac{3i\sqrt{41}\,\pi}{328} = c_1$$

$$c_1 = -\frac{3\sqrt{41}\,\operatorname{arccoth}\left(\frac{33\sqrt{41}}{205}\right)}{164} + \frac{3i\sqrt{41}\,\pi}{328}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln\left(-16x^2 + (-40y + 6)\,x + 16y^2 - 13y + 2\right)}{8} - \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(-32y+13+40x)\sqrt{41}}{328x+41}\right)}{164} = -\frac{3\sqrt{41}\,\operatorname{arccoth}\left(\frac{33\sqrt{\dots}}{20\dots}\right.}{164}\dots$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(-16x^2 + (-40y + 6)\,x + 16y^2 - 13y + 2\right)}{8} - \frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(-32y+13+40x)\sqrt{41}}{328x+41}\right)}{164} = \frac{3\sqrt{41}\left(i\pi - 2\operatorname{arccoth}\left(\frac{33\sqrt{41}}{205}\right)\right)}{328} \quad (1)$$

Verification of solutions

$$\frac{\ln\left(-16x^2+(-40y+6)\,x+16y^2-13y+2\right)}{8}-\frac{3\sqrt{41}\,\operatorname{arctanh}\left(\frac{(-32y+13+40x)\sqrt{41}}{328x+41}\right)}{164}=\frac{3\sqrt{41}\left(i\pi-2\,\operatorname{arccoth}\left(\frac{33\sqrt{41}}{205}\right)\right)}{328}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 13.969 (sec). Leaf size: 93

```
dsolve([(2*x+y(x))+(4*x-2*y(x)+1)*diff(y(x),x)=0,y(1/2) = 0],y(x), singsol=all)
```

$$y(x)=\operatorname{RootOf}\left(6\sqrt{41}\,\operatorname{arctanh}\left(\frac{(-13+32\_Z-40x)\sqrt{41}}{328x+41}\right)+41\ln\left(\frac{16\_Z^2-40x\_Z-16x^2-13\_Z+6x+2}{(8x+1)^2}\right)+82\ln(8x+1)+6\sqrt{41}\,\operatorname{arctanh}\left(\frac{33\sqrt{41}}{205}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.172 (sec). Leaf size: 128

```
DSolve[{(2*x+y[x])+(4*x-2*y[x]+1)*y'[x]==0,{y[1/2]==0}},y[x],x,IncludeSingularSolutions -> T
```

$$\text{Solve}\left[\frac{9}{656}\left(6\sqrt{41}\text{arctanh}\left(\frac{-\frac{2(8x+1)}{-2y(x)+4x+1}-3}{\sqrt{41}}\right)\right.\right.$$
$$\left.\left.+41\left(\log\left(\frac{2(16x^2-16y(x)^2+(40x+13)y(x)-6x-2)}{(8x+1)^2}\right)+2\log(8x+1)\right)\right)=\frac{1}{656}\left(-9\left(6\sqrt{41}\text{arctanh}\left(\right.\right.\right.$$
$$\left.+369i\pi\right),y(x)\Big]$$

4 Exercise 8, page 34

4.1 problem 1

Internal problem ID [1943]
Internal file name [`OUTPUT/1943_Sunday_February_25_2024_06_38_26_AM_61657721/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y + (x - 2y)\, y' = -x}$$

4.1.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x + (x - 2u(x)\, x)\, (u'(x)\, x + u(x)) = -x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^2 - 2u - 1}{(2u-1)\, x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2-2u-1}{2u-1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2-2u-1}{2u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{2u^2-2u-1}{2u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(2u^2-2u-1)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\sqrt{2u^2-2u-1} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{2u^2-2u-1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{2u(x)^2 - 2u(x) - 1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{2u(x)^2 - 2u(x) - 1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{2y^2}{x^2} - \frac{2y}{x} - 1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

$$\sqrt{\frac{2y^2 - 2yx - x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Which simplifies to

$$\sqrt{-\frac{-2y^2 + 2yx + x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{-\frac{-2y^2 + 2yx + x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}$$

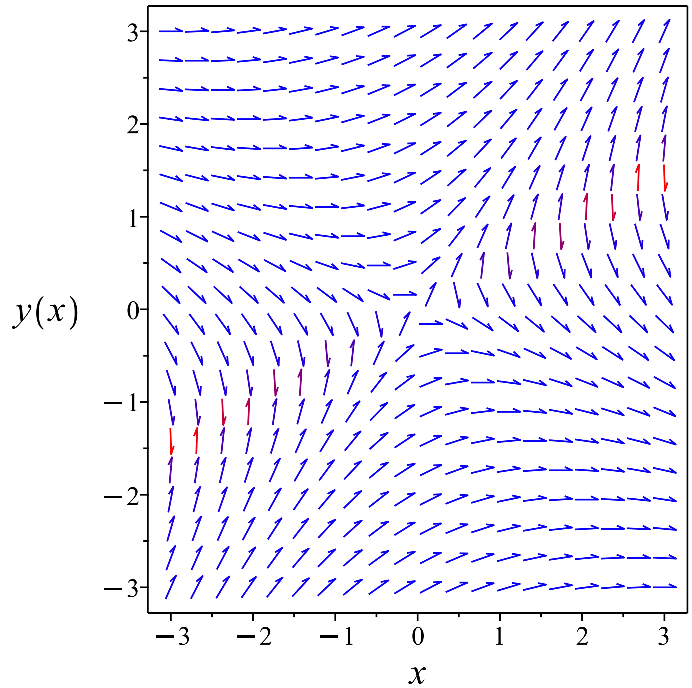


Figure 187: Slope field plot

Verification of solutions

$$\sqrt{-\frac{-2y^2+2yx+x^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

4.1.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-x-y}{x-2y} \tag{1}$$

Which becomes

$$(-2y)\,dy = (-x)\,dy + (-x-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-x-y)\,dx = d\left(-\frac{1}{2}x^2 - yx\right)$$

Hence (2) becomes

$$(-2y)\,dy = d\left(-\frac{1}{2}x^2 - yx\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x}{2} + \frac{\sqrt{3x^2 - 4c_1}}{2} + c_1$$
$$y = \frac{x}{2} - \frac{\sqrt{3x^2 - 4c_1}}{2} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{x}{2} + \frac{\sqrt{3x^2 - 4c_1}}{2} + c_1 \tag{1}$$
$$y = \frac{x}{2} - \frac{\sqrt{3x^2 - 4c_1}}{2} + c_1 \tag{2}$$

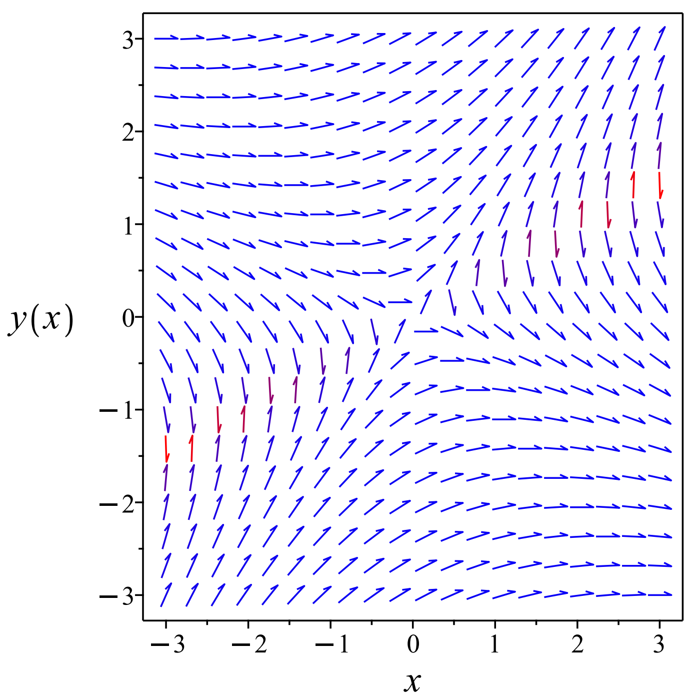


Figure 188: Slope field plot

Verification of solutions

$$y = \frac{x}{2} + \frac{\sqrt{3x^2 - 4c_1}}{2} + c_1$$

Verified OK.

$$y = \frac{x}{2} - \frac{\sqrt{3x^2 - 4c_1}}{2} + c_1$$

Verified OK.

4.1.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x + y}{-x + 2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(x + y)(b_3 - a_2)}{-x + 2y} - \frac{(x + y)^2 a_3}{(-x + 2y)^2} \\ & - \left(\frac{1}{-x + 2y} + \frac{x + y}{(-x + 2y)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(\frac{1}{-x + 2y} - \frac{2(x + y)}{(-x + 2y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 + 4x^2b_2 - x^2b_3 - 4xya_2 - 2xya_3 - 4xyb_2 + 4xyb_3 - 2y^2a_2 - 4y^2a_3 + 4y^2b_2 + 2y^2b_3 + 3xb_1 -}{(x-2y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &x^2a_2 - x^2a_3 + 4x^2b_2 - x^2b_3 - 4xya_2 - 2xya_3 - 4xyb_2 + 4xyb_3 \\ &- 2y^2a_2 - 4y^2a_3 + 4y^2b_2 + 2y^2b_3 + 3xb_1 - 3ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &a_2v_1^2 - 4a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - 4a_3v_2^2 + 4b_2v_1^2 \\ &- 4b_2v_1v_2 + 4b_2v_2^2 - b_3v_1^2 + 4b_3v_1v_2 + 2b_3v_2^2 - 3a_1v_2 + 3b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(a_2 - a_3 + 4b_2 - b_3)\,v_1^2 + (-4a_2 - 2a_3 - 4b_2 + 4b_3)\,v_1v_2 \\ &+ 3b_1v_1 + (-2a_2 - 4a_3 + 4b_2 + 2b_3)\,v_2^2 - 3a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -3a_1 &= 0 \\ 3b_1 &= 0 \\ -4a_2 - 2a_3 - 4b_2 + 4b_3 &= 0 \\ -2a_2 - 4a_3 + 4b_2 + 2b_3 &= 0 \\ a_2 - a_3 + 4b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -2b_2 + b_3 \\
a_3 &= 2b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x + y}{-x + 2y}\right)(x) \\
&= \frac{x^2 + 2yx - 2y^2}{x - 2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2 + 2yx - 2y^2}{x - 2y}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln(-x^2 - 2yx + 2y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + y}{-x + 2y}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = \frac{x + y}{x^2 + 2yx - 2y^2}$$
$$S_y = \frac{x - 2y}{x^2 + 2yx - 2y^2}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

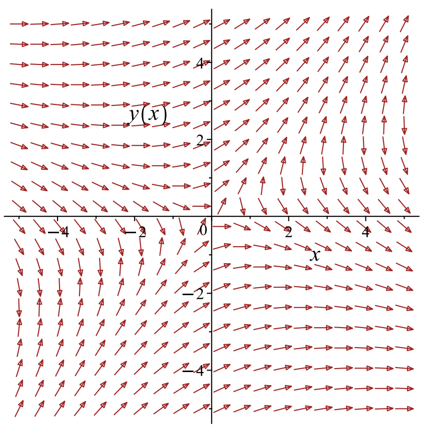
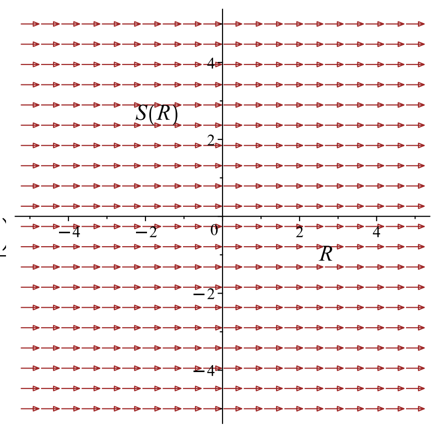
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2y^2 - 2yx - x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln \left(2 y^{2}-2 y x -x^{2}\right)}{2} = c_{1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+y}{-x+2y}$ 	$R = x$ $S = \frac{\ln \left(-x^{2}-2 y x +2 y^{2}\right)}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln \left(2 y^{2}-2 y x -x^{2}\right)}{2} = c_{1} \tag{1}$$

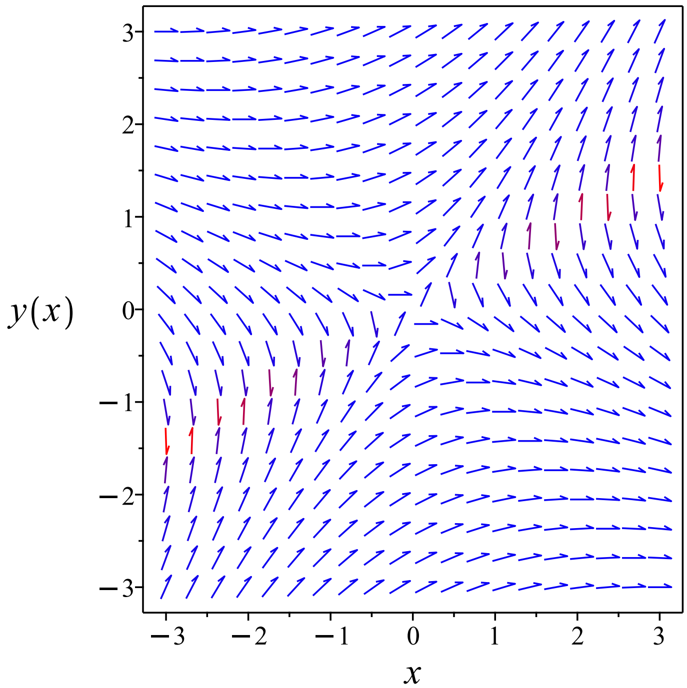


Figure 189: Slope field plot

Verification of solutions

$$\frac{\ln(2y^2 - 2yx - x^2)}{2} = c_1$$

Verified OK.

4.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x - 2y)\,\mathrm{d}y &= (-x - y)\,\mathrm{d}x \\ (x + y)\,\mathrm{d}x + (x - 2y)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x + y \\ N(x, y) &= x - 2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x - 2y) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M &(1)\\ \frac{\partial\phi}{\partial y}&=N &(2)\end{aligned}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int M\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int x+y\,\mathrm{d}x\\ \phi&=\frac{x(x+2y)}{2}+f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=x+f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=x-2y$. Therefore equation (4) becomes

$$x-2y=x+f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-2y$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y&=\int(-2y)\,\mathrm{d}y\\ f(y)&=-y^2+c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=\frac{x(x+2y)}{2}-y^2+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x+2y)}{2} - y^2$$

Summary

The solution(s) found are the following

$$\frac{x(x+2y)}{2} - y^2 = c_1 \tag{1}$$

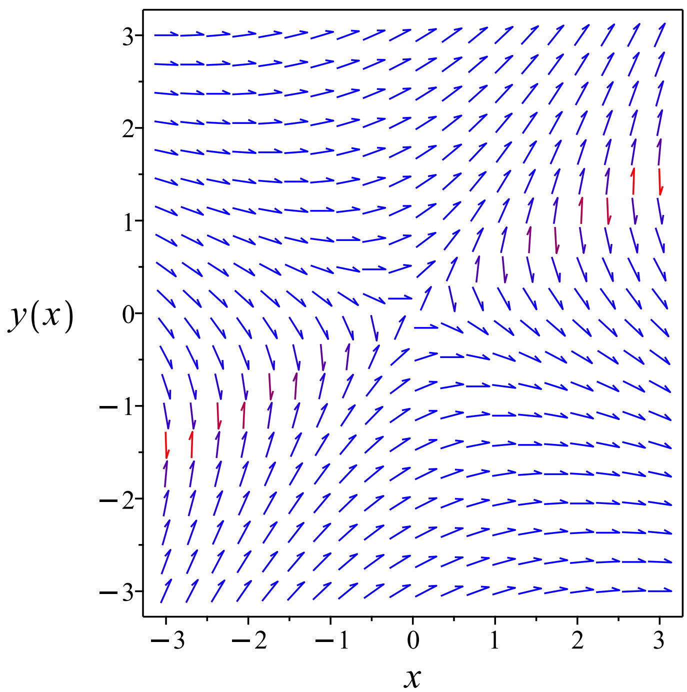


Figure 190: Slope field plot

Verification of solutions

$$\frac{x(x+2y)}{2} - y^2 = c_1$$

Verified OK.

4.1.5 Maple step by step solution

Let's solve

$y+(x-2y)\,y'=-x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

- ◦ Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- ◦ Evaluate derivatives

  $1=1$

- ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y)\,, N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (x+y)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{x^2}{2}+yx+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x-2y=x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-2y$

- Solve for $f_1(y)$

  $f_1(y)=-y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x,y) = \frac{1}{2}x^2 + yx - y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{2}x^2 + yx - y^2 = c_1$

- Solve for y

  $\left\{y = \frac{x}{2} - \frac{\sqrt{3x^2-4c_1}}{2}, y = \frac{x}{2} + \frac{\sqrt{3x^2-4c_1}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 51

```
dsolve((x+y(x))+(x-2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1x - \sqrt{3c_1^2x^2 + 2}}{2c_1}$$
$$y(x) = \frac{c_1x + \sqrt{3c_1^2x^2 + 2}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.535 (sec). Leaf size: 106

```
DSolve[(x+y[x])+(x-2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x - \sqrt{3x^2 - 2e^{2c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(x + \sqrt{3x^2 - 2e^{2c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(x - \sqrt{3}\sqrt{x^2}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{3}\sqrt{x^2} + x\right)$$

4.2 problem 2

Internal problem ID [1944]
Internal file name [`OUTPUT/1944_Sunday_February_25_2024_06_38_27_AM_34659944/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y + (x + 3y)\, y' = -3x}$$

4.2.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x + (x + 3u(x)\, x)\, (u'(x)\, x + u(x)) = -3x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u^2 + 2u + 3}{x\,(3u + 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{3u^2+2u+3}{3u+1}$. Integrating both sides gives

$$\frac{1}{\frac{3u^2+2u+3}{3u+1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{3u^2+2u+3}{3u+1}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln(3u^2+2u+3)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\sqrt{3u^2+2u+3} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{3u^2+2u+3} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{3u(x)^2+2u(x)+3} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{3u(x)^2+2u(x)+3} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{3y^2}{x^2}+\frac{2y}{x}+3} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

$$\sqrt{\frac{3y^2+2yx+3x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{3y^2+2yx+3x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}$$

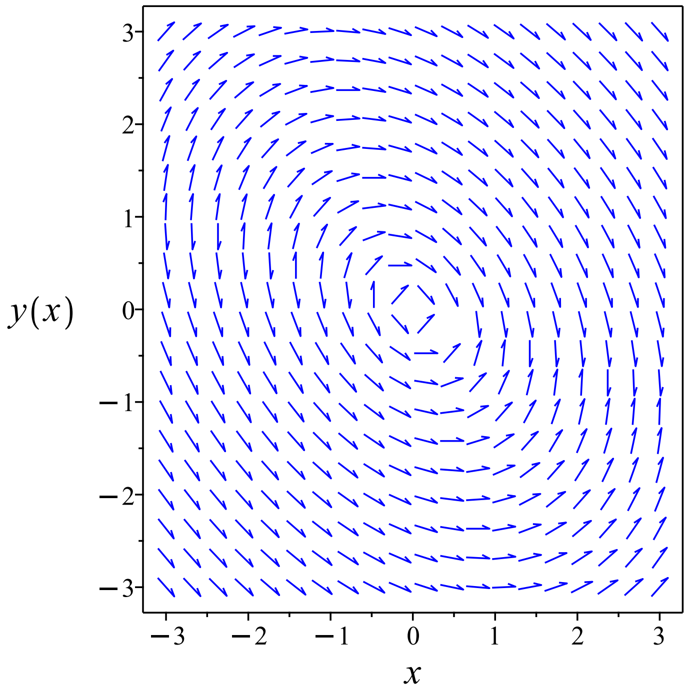


Figure 191: Slope field plot

Verification of solutions

$$\sqrt{\frac{3y^2 + 2yx + 3x^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

4.2.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-3x - y}{x + 3y} \tag{1}$$

Which becomes

$$(3y)\, dy = (-x)\, dy + (-3x - y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (-3x - y)\, dx = d\left(-\frac{3}{2}x^2 - yx\right)$$

Hence (2) becomes

$$(3y)\,dy = d\left(-\frac{3}{2}x^2 - yx\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{x}{3} + \frac{\sqrt{-8x^2 + 6c_1}}{3} + c_1$$
$$y = -\frac{x}{3} - \frac{\sqrt{-8x^2 + 6c_1}}{3} + c_1$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{3} + \frac{\sqrt{-8x^2 + 6c_1}}{3} + c_1 \tag{1}$$
$$y = -\frac{x}{3} - \frac{\sqrt{-8x^2 + 6c_1}}{3} + c_1 \tag{2}$$

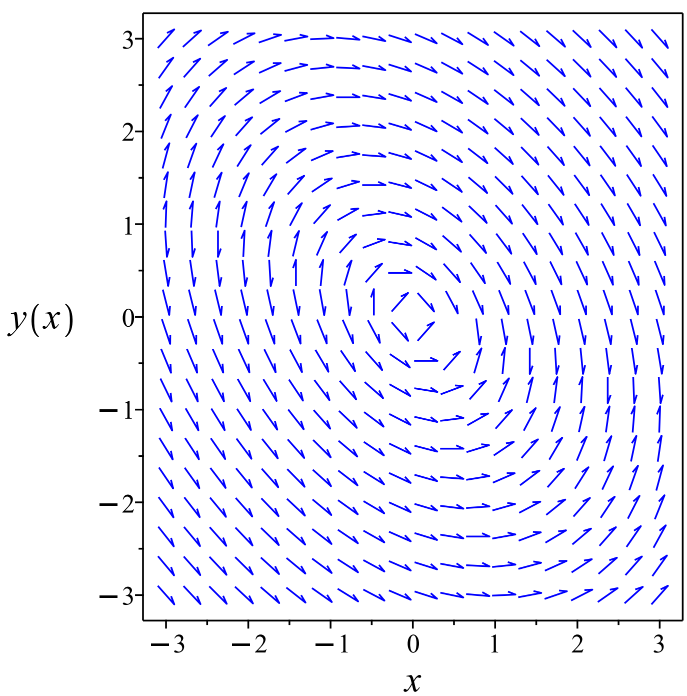


Figure 192: Slope field plot

Verification of solutions

$$y = -\frac{x}{3} + \frac{\sqrt{-8x^2 + 6c_1}}{3} + c_1$$

Verified OK.

$$y = -\frac{x}{3} - \frac{\sqrt{-8x^2 + 6c_1}}{3} + c_1$$

Verified OK.

4.2.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{3x + y}{x + 3y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(3x + y)(b_3 - a_2)}{x + 3y} - \frac{(3x + y)^2 a_3}{(x + 3y)^2} - \left(-\frac{3}{x + 3y} + \frac{3x + y}{(x + 3y)^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{1}{x + 3y} + \frac{9x + 3y}{(x + 3y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - 9x^2a_3 - 7x^2b_2 - 3x^2b_3 + 18xya_2 - 6xya_3 + 6xyb_2 - 18xyb_3 + 3y^2a_2 + 7y^2a_3 + 9y^2b_2 - 3y^2b_3 - 8}{(x+3y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &3x^2a_2 - 9x^2a_3 - 7x^2b_2 - 3x^2b_3 + 18xya_2 - 6xya_3 + 6xyb_2 \\ &- 18xyb_3 + 3y^2a_2 + 7y^2a_3 + 9y^2b_2 - 3y^2b_3 - 8xb_1 + 8ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &3a_2v_1^2 + 18a_2v_1v_2 + 3a_2v_2^2 - 9a_3v_1^2 - 6a_3v_1v_2 + 7a_3v_2^2 - 7b_2v_1^2 \\ &+ 6b_2v_1v_2 + 9b_2v_2^2 - 3b_3v_1^2 - 18b_3v_1v_2 - 3b_3v_2^2 + 8a_1v_2 - 8b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(3a_2 - 9a_3 - 7b_2 - 3b_3)\,v_1^2 + (18a_2 - 6a_3 + 6b_2 - 18b_3)\,v_1v_2 \\ &- 8b_1v_1 + (3a_2 + 7a_3 + 9b_2 - 3b_3)\,v_2^2 + 8a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 8a_1 &= 0 \\ -8b_1 &= 0 \\ 3a_2 - 9a_3 - 7b_2 - 3b_3 &= 0 \\ 3a_2 + 7a_3 + 9b_2 - 3b_3 &= 0 \\ 18a_2 - 6a_3 + 6b_2 - 18b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{2b_2}{3} + b_3 \\
a_3 &= -b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{3x+y}{x+3y}\right)(x) \\
&= \frac{3x^2+2yx+3y^2}{x+3y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2+2yx+3y^2}{x+3y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(3x^2 + 2yx + 3y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3x + y}{x + 3y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3x + y}{3x^2 + 2yx + 3y^2} \\ S_y &= \frac{x + 3y}{3x^2 + 2yx + 3y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

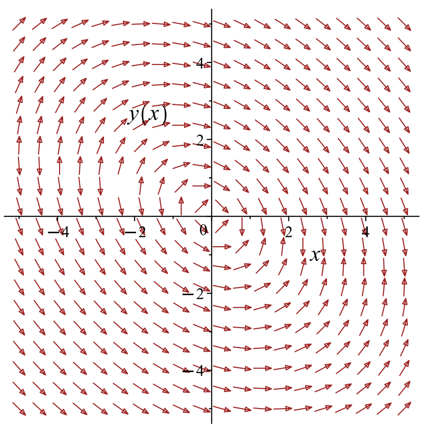
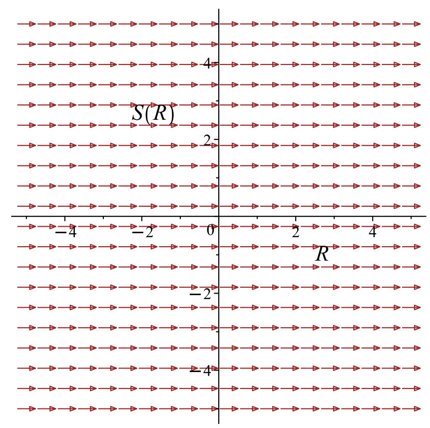
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(3y^2 + 2yx + 3x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln\left(3y^{2}+2yx+3x^{2}\right)}{2}=c_{1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=-\frac{3x+y}{x+3y}$ 	$R=x$ $S=\frac{\ln\left(3x^{2}+2yx+3y^{2}\right)}{2}$	$\frac{dS}{dR}=0$ 

Summary

The solution(s) found are the following

$$\frac{\ln\left(3y^{2}+2yx+3x^{2}\right)}{2}=c_{1} \tag{1}$$

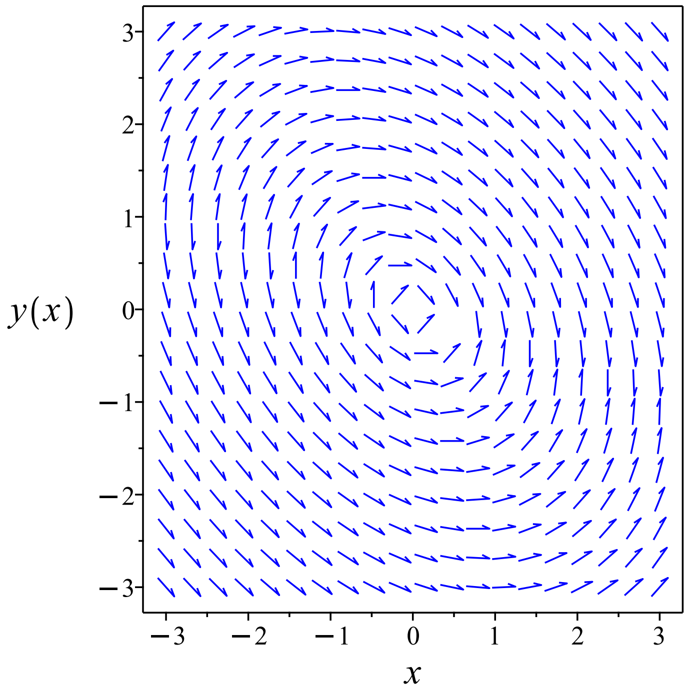


Figure 193: Slope field plot

Verification of solutions

$$\frac{\ln \left(3y^2 + 2yx + 3x^2\right)}{2} = c_1$$

Verified OK.

4.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x+3y)\,\mathrm{d}y &= (-3x-y)\,\mathrm{d}x\\ (3x+y)\,\mathrm{d}x + (x+3y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 3x+y$$
$$N(x,y) = x+3y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x+y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x+3y)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 3x + y \,\mathrm{d}x$$

$$\phi = \frac{x(3x + 2y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x + 3y$. Therefore equation (4) becomes

$$x + 3y = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (3y) \,\mathrm{d}y$$

$$f(y) = \frac{3y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(3x+2y)}{2} + \frac{3y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(3x+2y)}{2} + \frac{3y^2}{2}$$

Summary

The solution(s) found are the following

$$\frac{x(3x+2y)}{2} + \frac{3y^2}{2} = c_1 \tag{1}$$

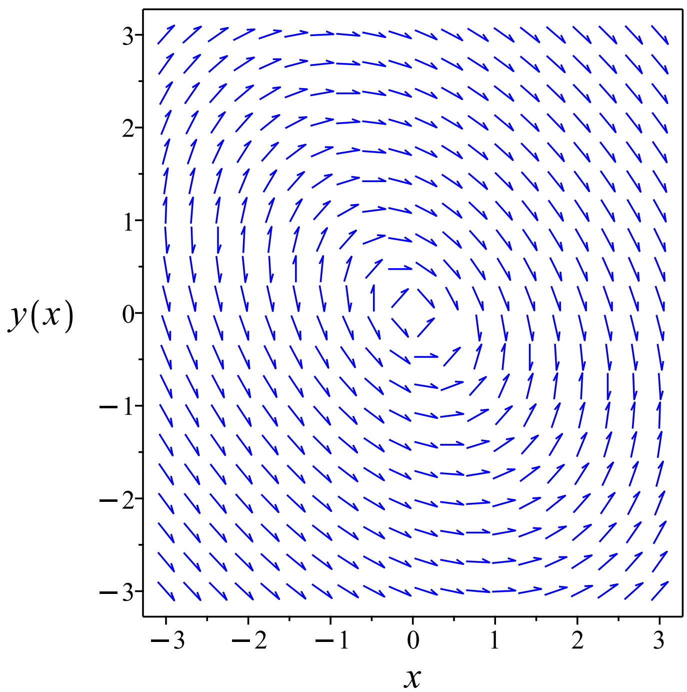


Figure 194: Slope field plot

Verification of solutions

$$\frac{x(3x+2y)}{2} + \frac{3y^2}{2} = c_1$$

Verified OK.

4.2.5 Maple step by step solution

Let's solve

$y + (x + 3y)\, y' = -3x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

  ◦ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

  $1 = 1$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (3x + y)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{3x^2}{2} + yx + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x + 3y = x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 3y$

- Solve for $f_1(y)$

  $f_1(y) = \frac{3y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x,y) = \frac{3}{2}x^2 + yx + \frac{3}{2}y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{3}{2}x^2 + yx + \frac{3}{2}y^2 = c_1$

- Solve for y

  $\left\{y = -\frac{x}{3} - \frac{\sqrt{-8x^2+6c_1}}{3}, y = -\frac{x}{3} + \frac{\sqrt{-8x^2+6c_1}}{3}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 53

```
dsolve((3*x+y(x))+(x+3*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-c_1x - \sqrt{-8c_1^2x^2 + 3}}{3c_1}$$
$$y(x) = \frac{-c_1x + \sqrt{-8c_1^2x^2 + 3}}{3c_1}$$

✓ Solution by Mathematica

Time used: 0.487 (sec). Leaf size: 119

```
DSolve[(3*x+y[x])+(x+3*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(-x - \sqrt{-8x^2 + 3e^{2c_1}}\right)$$
$$y(x) \to \frac{1}{3}\left(-x + \sqrt{-8x^2 + 3e^{2c_1}}\right)$$
$$y(x) \to \frac{1}{3}\left(-2\sqrt{2}\sqrt{-x^2} - x\right)$$
$$y(x) \to \frac{1}{3}\left(2\sqrt{2}\sqrt{-x^2} - x\right)$$

4.3 problem 3

Internal problem ID [1945]
Internal file name [`OUTPUT/1945_Sunday_February_25_2024_06_38_29_AM_69721577/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{b_1 y + (b_1 x + b_2 y + c_2)\, y' = -a_1 x - c_1}$$

4.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{a_1 x + b_1 y + c_1}{b_1 x + b_2 y + c_2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(a_1x + b_1y + c_1)(b_3 - a_2)}{b_1x + b_2y + c_2} - \frac{(a_1x + b_1y + c_1)^2 a_3}{(b_1x + b_2y + c_2)^2} - \left(-\frac{a_1}{b_1x + b_2y + c_2} + \frac{(a_1x + b_1y + c_1) b_1}{(b_1x + b_2y + c_2)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{b_1}{b_1x + b_2y + c_2} + \frac{(a_1x + b_1y + c_1) b_2}{(b_1x + b_2y + c_2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{a_1^2x^2a_3 - a_1b_1x^2a_2 + a_1b_1x^2b_3 + 2a_1b_1xya_3 + a_1b_2x^2b_2 - 2a_1b_2xya_2 + 2a_1b_2xyb_3 - a_1b_2y^2a_3 - 2b_1^2x^2b_2 +}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-a_1^2x^2a_3 + a_1b_1x^2a_2 - a_1b_1x^2b_3 - 2a_1b_1xya_3 - a_1b_2x^2b_2 + 2a_1b_2xya_2 \\
&- 2a_1b_2xyb_3 + a_1b_2y^2a_3 + 2b_1^2x^2b_2 - 2b_1^2y^2a_3 + 2b_1b_2xyb_2 + b_1b_2y^2a_2 \\
&- b_1b_2y^2b_3 + b_2^2y^2b_2 - a_1b_2xb_1 + a_1b_2ya_1 - 2a_1c_1xa_3 + 2a_1c_2xa_2 \\
&- a_1c_2xb_3 + a_1c_2ya_3 + b_1^2xb_1 - b_1^2ya_1 - b_1c_1xb_3 - 3b_1c_1ya_3 + 3b_1c_2xb_2 \\
&+ b_1c_2ya_2 - b_2c_1xb_2 + b_2c_1ya_2 - 2b_2c_1yb_3 + 2b_2c_2yb_2 + a_1c_2a_1 \\
&- b_1c_1a_1 + b_1c_2b_1 - b_2c_1b_1 - c_1^2a_3 + c_1c_2a_2 - c_1c_2b_3 + c_2^2b_2 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_1^2 a_3 v_1^2 + a_1 b_1 a_2 v_1^2 - 2a_1 b_1 a_3 v_1 v_2 - a_1 b_1 b_3 v_1^2 + 2a_1 b_2 a_2 v_1 v_2 + a_1 b_2 a_3 v_2^2 \\
&\quad - a_1 b_2 b_2 v_1^2 - 2a_1 b_2 b_3 v_1 v_2 - 2b_1^2 a_3 v_2^2 + 2b_1^2 b_2 v_1^2 + b_1 b_2 a_2 v_2^2 + 2b_1 b_2 b_2 v_1 v_2 \\
&\quad - b_1 b_2 b_3 v_2^2 + b_2^2 b_2 v_2^2 + a_1 b_2 a_1 v_2 - a_1 b_2 b_1 v_1 - 2a_1 c_1 a_3 v_1 + 2a_1 c_2 a_2 v_1 \\
&\quad + a_1 c_2 a_3 v_2 - a_1 c_2 b_3 v_1 - b_1^2 a_1 v_2 + b_1^2 b_1 v_1 - 3b_1 c_1 a_3 v_2 - b_1 c_1 b_3 v_1 \\
&\quad + b_1 c_2 a_2 v_2 + 3b_1 c_2 b_2 v_1 + b_2 c_1 a_2 v_2 - b_2 c_1 b_2 v_1 - 2b_2 c_1 b_3 v_2 + 2b_2 c_2 b_2 v_2 \\
&\quad + a_1 c_2 a_1 - b_1 c_1 a_1 + b_1 c_2 b_1 - b_2 c_1 b_1 - c_1^2 a_3 + c_1 c_2 a_2 - c_1 c_2 b_3 + c_2^2 b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-a_1^2 a_3 + a_1 b_1 a_2 - a_1 b_1 b_3 - a_1 b_2 b_2 + 2b_1^2 b_2\right) v_1^2 \\
&+ \left(-2a_1 b_1 a_3 + 2a_1 b_2 a_2 - 2a_1 b_2 b_3 + 2b_1 b_2 b_2\right) v_1 v_2 \\
&+ \left(-a_1 b_2 b_1 - 2a_1 c_1 a_3 + 2a_1 c_2 a_2 - a_1 c_2 b_3 + b_1^2 b_1 - b_1 c_1 b_3 + 3b_1 c_2 b_2 - b_2 c_1 b_2\right) v_1 \\
&+ \left(a_1 b_2 a_3 - 2b_1^2 a_3 + b_1 b_2 a_2 - b_1 b_2 b_3 + b_2^2 b_2\right) v_2^2 \\
&+ \left(a_1 b_2 a_1 + a_1 c_2 a_3 - b_1^2 a_1 - 3b_1 c_1 a_3 + b_1 c_2 a_2 + b_2 c_1 a_2 - 2b_2 c_1 b_3 + 2b_2 c_2 b_2\right) v_2 \\
&- c_1^2 a_3 + c_1 c_2 a_2 - c_1 c_2 b_3 - b_1 c_1 a_1 - b_2 c_1 b_1 + c_2^2 b_2 + a_1 c_2 a_1 + b_1 c_2 b_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 b_1 a_3 + 2a_1 b_2 a_2 - 2a_1 b_2 b_3 + 2b_1 b_2 b_2 &= 0 \\
a_1 b_2 a_3 - 2b_1^2 a_3 + b_1 b_2 a_2 - b_1 b_2 b_3 + b_2^2 b_2 &= 0 \\
-a_1^2 a_3 + a_1 b_1 a_2 - a_1 b_1 b_3 - a_1 b_2 b_2 + 2b_1^2 b_2 &= 0 \\
a_1 b_2 a_1 + a_1 c_2 a_3 - b_1^2 a_1 - 3b_1 c_1 a_3 + b_1 c_2 a_2 + b_2 c_1 a_2 - 2b_2 c_1 b_3 + 2b_2 c_2 b_2 &= 0 \\
-a_1 b_2 b_1 - 2a_1 c_1 a_3 + 2a_1 c_2 a_2 - a_1 c_2 b_3 + b_1^2 b_1 - b_1 c_1 b_3 + 3b_1 c_2 b_2 - b_2 c_1 b_2 &= 0 \\
a_1 c_2 a_1 - b_1 c_1 a_1 + b_1 c_2 b_1 - b_2 c_1 b_1 - c_1^2 a_3 + c_1 c_2 a_2 - c_1 c_2 b_3 + c_2^2 b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= \frac{-a_1 b_1 c_2 b_3 + a_1 b_2 c_1 b_3 - a_1 b_2 c_2 b_2 + 2b_1^2 c_2 b_2 - b_1 b_2 c_1 b_2}{a_1 \left(a_1 b_2 - b_1^2\right)} \\
a_2 &= \frac{a_1 b_3 - 2b_1 b_2}{a_1} \\
a_3 &= -\frac{b_2 b_2}{a_1} \\
b_1 &= -\frac{-a_1 c_2 b_3 + b_1 c_1 b_3 + b_1 c_2 b_2 - b_2 c_1 b_2}{a_1 b_2 - b_1^2} \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{a_1 b_2 x - x\, b_1^2 - b_1 c_2 + b_2 c_1}{a_1 b_2 - b_1^2} \\
\eta &= -\frac{-a_1 b_2 y + b_1^2 y - a_1 c_2 + b_1 c_1}{a_1 b_2 - b_1^2}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= -\frac{-a_1 b_2 y + b_1^2 y - a_1 c_2 + b_1 c_1}{a_1 b_2 - b_1^2} - \left(-\frac{a_1 x + b_1 y + c_1}{b_1 x + b_2 y + c_2}\right) \left(\frac{a_1 b_2 x - x\, b_1^2 - b_1 c_2 + b_2 c_1}{a_1 b_2 - b_1^2}\right) \\
&= \frac{a_1^2 b_2 x^2 - a_1 b_1^2 x^2 + 2a_1 b_1 b_2 x y + a_1 b_2^2 y^2 - 2b_1^3 x y - b_1^2 b_2 y^2 + 2a_1 b_2 c_1 x + 2a_1 b_2 c_2 y - 2b_1^2 c_1 x - 2b_1^2 c_2 y + a}{a_1 b_1 b_2 x + a_1 b_2^2 y - b_1^3 x - b_1^2 b_2 y + a_1 b_2 c_2 - b_1^2 c_2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{a_1^2 b_2 x^2 - a_1 b_1^2 x^2 + 2a_1 b_1 b_2 xy + a_1 b_2^2 y^2 - 2b_1^3 xy - b_1^2 b_2 y^2 + 2a_1 b_2 c_1 x + 2a_1 b_2 c_2 y - 2b_1^2 c_1 x - 2b_1^2 c_2 y + a_1 c_2^2 - 2b_1 c_1 c_2 + b_2 c_1^2}{a_1 b_1 b_2 x + a_1 b_2^2 y - b_1^3 x - b_1^2 b_2 y + a_1 b_2 c_2 - b_1^2 c_2}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln \left(a_1^2 b_2 x^2 - a_1 b_1^2 x^2 + 2a_1 b_1 b_2 xy + a_1 b_2^2 y^2 - 2b_1^3 xy - b_1^2 b_2 y^2 + 2a_1 b_2 c_1 x + 2a_1 b_2 c_2 y - 2b_1^2 c_1 x - 2b_1^2 c_2 y\right.}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{a_1 x + b_1 y + c_1}{b_1 x + b_2 y + c_2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{(a_1 b_2 - b_1^2)(a_1 x + b_1 y + c_1)}{a_1^2 b_2 x^2 + (-x^2 b_1^2 + 2xy b_1 b_2 + y^2 b_2^2 + (2c_1 x + 2c_2 y) b_2 + c_2^2) a_1 - 2(b_1 y + c_1)\left(x b_1^2 + \left(\frac{b_2 y}{2} + c_2\right) b_1 - \right.} \\ S_y &= \frac{(a_1 b_2 - b_1^2)(b_1 x + b_2 y + c_2)}{a_1^2 b_2 x^2 + (-x^2 b_1^2 + 2xy b_1 b_2 + y^2 b_2^2 + (2c_1 x + 2c_2 y) b_2 + c_2^2) a_1 - 2(b_1 y + c_1)\left(x b_1^2 + \left(\frac{b_2 y}{2} + c_2\right) b_1 - \right.} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln \left(a_1^2 b_2 x^2 + \left(-x^2 b_1^2 + 2xy b_1 b_2 + 2x c_1 b_2 + (b_2 y + c_2)^2\right) a_1 - 2(b_1 y + c_1)\left(x\, b_1^2 + \left(\frac{b_2 y}{2} + c_2\right) b_1 - \frac{b_2 c_1}{2}\right)\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln \left(a_1^2 b_2 x^2 + \left(-x^2 b_1^2 + 2xy b_1 b_2 + 2x c_1 b_2 + (b_2 y + c_2)^2\right) a_1 - 2(b_1 y + c_1)\left(x\, b_1^2 + \left(\frac{b_2 y}{2} + c_2\right) b_1 - \frac{b_2 c_1}{2}\right)\right)}{2} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln \left(a_1^2 b_2 x^2 + \left(-x^2 b_1^2 + 2xy b_1 b_2 + 2x c_1 b_2 + (b_2 y + c_2)^2\right) a_1 - 2(b_1 y + c_1)\left(x\, b_1^2 + \left(\frac{b_2 y}{2} + c_2\right) b_1 - \frac{b_2 c_1}{2}\right)\right)}{2} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln \left(a_1^2 b_2 x^2 + \left(-x^2 b_1^2 + 2xy b_1 b_2 + 2x c_1 b_2 + (b_2 y + c_2)^2\right) a_1 - 2(b_1 y + c_1)\left(x\, b_1^2 + \left(\frac{b_2 y}{2} + c_2\right) b_1 - \frac{b_2 c_1}{2}\right)\right)}{2} = c_1$$

Verified OK.

4.3.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(b_1x+b_2y+c_2)\,\mathrm{d}y&=(-a_1x-b_1y-c_1)\,\mathrm{d}x\\ (a_1x+b_1y+c_1)\,\mathrm{d}x+(b_1x+b_2y+c_2)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=a_1x+b_1y+c_1\\ N(x,y)&=b_1x+b_2y+c_2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(a_1x+b_1y+c_1)\\ &=b_1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(b_1 x + b_2 y + c_2)\\ &= b_1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int a_1 x + b_1 y + c_1\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{x(a_1 x + 2b_1 y + 2c_1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = b_1 x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = b_1 x + b_2 y + c_2$. Therefore equation (4) becomes

$$b_1 x + b_2 y + c_2 = b_1 x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = b_2 y + c_2$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int (b_2 y + c_2)\,\mathrm{d}y\\ f(y) &= \frac{1}{2} b_2 y^2 + c_2 y + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(a_1x + 2b_1y + 2c_1)}{2} + \frac{b_2y^2}{2} + c_2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(a_1x + 2b_1y + 2c_1)}{2} + \frac{b_2y^2}{2} + c_2y$$

Summary
The solution(s) found are the following

$$\frac{x(a_1x + 2b_1y + 2c_1)}{2} + \frac{b_2y^2}{2} + c_2y = c_1 \tag{1}$$

Verification of solutions

$$\frac{x(a_1x + 2b_1y + 2c_1)}{2} + \frac{b_2y^2}{2} + c_2y = c_1$$

Verified OK.

4.3.3 Maple step by step solution

Let's solve

$b_1y + (b_1x + b_2y + c_2)\, y' = -a_1x - c_1$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $b_1 = b_1$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $$\left[F(x,y)=c_3, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $$F(x,y)=\int (a_1x+b_1y+c_1)\,dx+f_1(y)$$

- Evaluate integral

  $$F(x,y)=\frac{a_1x^2}{2}+b_1xy+c_1x+f_1(y)$$

- Take derivative of $F(x,y)$ with respect to y

  $$N(x,y)=\frac{\partial}{\partial y}F(x,y)$$

- Compute derivative

  $$b_1x+b_2y+c_2=b_1x+\frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

  $$\frac{d}{dy}f_1(y)=b_2y+c_2$$

- Solve for $f_1(y)$

  $$f_1(y)=\frac{1}{2}b_2y^2+c_2y$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $$F(x,y)=\frac{1}{2}a_1x^2+b_1xy+c_1x+\frac{1}{2}b_2y^2+c_2y$$

- Substitute $F(x,y)$ into the solution of the ODE

  $$\frac{1}{2}a_1x^2+b_1xy+c_1x+\frac{1}{2}b_2y^2+c_2y=c_3$$

- Solve for y

  $$\left\{y=\frac{-b_1x+\sqrt{-a_1b_2x^2+x^2b_1^2-2xc_1b_2+2c_2b_1x+c_2^2+2c_3b_2}-c_2}{b_2}, y=-\frac{b_1x+\sqrt{-a_1b_2x^2+x^2b_1^2-2xc_1b_2+2c_2b_1x+c_2^2+2c_3b_2}+c_2}{b_2}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 80

```
dsolve((a__1*x+b__1*y(x)+c__1)+(b__1*x+b__2*y(x)+c__2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-\left(a_1 b_2 - b_1^2\right)\left(\left(a_1 x + c_1\right) b_2 - b_1^2 x - c_2 b_1\right)^2 e^{2c_1} + b_2}\, e^{-c_1} - \left(a_1 b_2 - b_1^2\right)\left(b_1 x + c_2\right)}{\left(a_1 b_2 - b_1^2\right) b_2}$$

✓ Solution by Mathematica

Time used: 16.604 (sec). Leaf size: 106

```
DSolve[(a1*x+b1*y[x]+c1)+(b1*x+b2*y[x]+c2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\frac{\sqrt{-x(\text{a1}x+2\text{c1})+\frac{(\text{b1}x+\text{c2})^2}{\text{b2}}+\text{b2}c_1}}{\sqrt{\frac{1}{\text{b2}}}} + \text{b1}x + \text{c2}}{\text{b2}}$$

$$y(x) \to -\frac{\text{b1}x+\text{c2}}{\text{b2}} + \sqrt{\frac{1}{\text{b2}}}\sqrt{-x(\text{a1}x+2\text{c1}) + \frac{(\text{b1}x+\text{c2})^2}{\text{b2}} + \text{b2}c_1}$$

4.4 problem 4

Internal problem ID [1946]
Internal file name [`OUTPUT/1946_Sunday_February_25_2024_06_38_31_AM_97653576/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
    [_Abel, `2nd type`, `class A`]]
```

$$\boxed{x(6yx+5)+\left(2x^{3}+3y\right)y'=0}$$

4.4.1 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{x(6yx+5)}{2x^{3}+3y} \tag{1}$$

Which becomes

$$(3y)\,dy = \left(-2x^{3}\right)dy + \left(-x(6yx+5)\right)dx \tag{2}$$

But the RHS is complete differential because

$$\left(-2x^{3}\right)dy + \left(-x(6yx+5)\right)dx = d\left(-2y\,x^{3}-\frac{5}{2}x^{2}\right)$$

Hence (2) becomes

$$(3y)\,dy = d\left(-2y\,x^{3}-\frac{5}{2}x^{2}\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{2x^3}{3} + \frac{\sqrt{4x^6 - 15x^2 + 6c_1}}{3} + c_1$$
$$y = -\frac{2x^3}{3} - \frac{\sqrt{4x^6 - 15x^2 + 6c_1}}{3} + c_1$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{2x^3}{3} + \frac{\sqrt{4x^6 - 15x^2 + 6c_1}}{3} + c_1 \tag{1}$$
$$y = -\frac{2x^3}{3} - \frac{\sqrt{4x^6 - 15x^2 + 6c_1}}{3} + c_1 \tag{2}$$

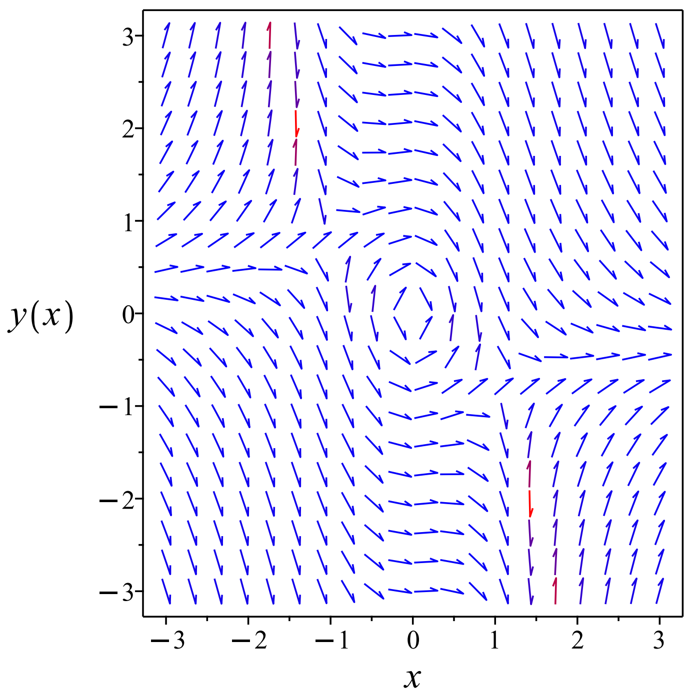


Figure 195: Slope field plot

Verification of solutions

$$y = -\frac{2x^3}{3} + \frac{\sqrt{4x^6 - 15x^2 + 6c_1}}{3} + c_1$$

Verified OK.

$$y = -\frac{2x^3}{3} - \frac{\sqrt{4x^6 - 15x^2 + 6c_1}}{3} + c_1$$

Verified OK.

4.4.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(2x^3+3y\right)\mathrm{d}y &= (-x(6yx+5))\,\mathrm{d}x\\ (x(6yx+5))\,\mathrm{d}x+\left(2x^3+3y\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x(6yx+5)\\ N(x,y) &= 2x^3+3y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x(6yx+5))\\ &= 6x^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x^3+3y\right)\\ &= 6x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int x(6yx+5)\,\mathrm{d}x\end{aligned}$$

$$\phi = 2y\,x^3+\frac{5}{2}x^2+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x^3 + 3y$. Therefore equation (4) becomes

$$2x^3 + 3y = 2x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (3y)\,\mathrm{d}y$$

$$f(y) = \frac{3y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 2y\,x^3 + \frac{5}{2}x^2 + \frac{3}{2}y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2y\,x^3 + \frac{5}{2}x^2 + \frac{3}{2}y^2$$

Summary

The solution(s) found are the following

$$2yx^3 + \frac{5x^2}{2} + \frac{3y^2}{2} = c_1 \tag{1}$$

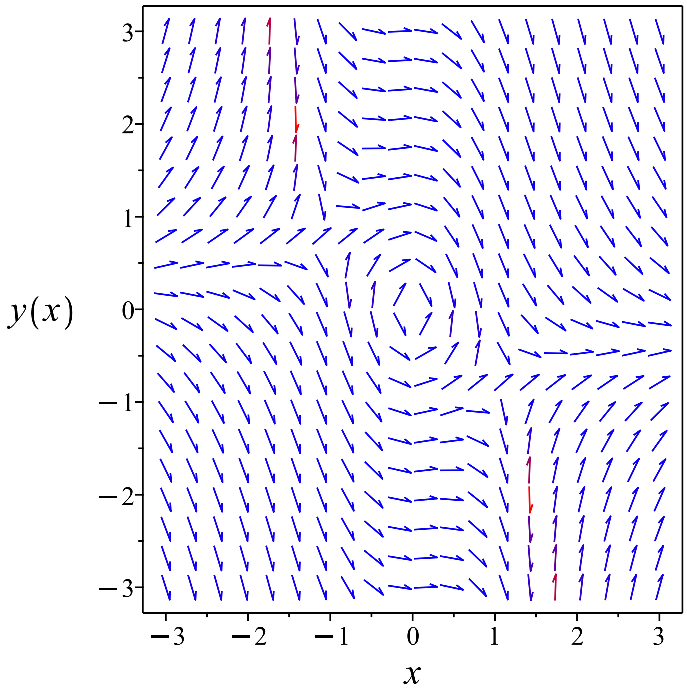


Figure 196: Slope field plot

<u>Verification of solutions</u>

$$2yx^3+\frac{5x^2}{2}+\frac{3y^2}{2}=c_1$$

Verified OK.

4.4.3 Maple step by step solution

Let's solve

$x(6yx+5)+(2x^3+3y)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $6x^2 = 6x^2$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int x(6yx+5)\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = 2y\,x^3 + \frac{5x^2}{2} + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $2x^3 + 3y = 2x^3 + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 3y$

- Solve for $f_1(y)$

  $f_1(y) = \frac{3y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = 2y\,x^3 + \frac{5}{2}x^2 + \frac{3}{2}y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $2y\,x^3 + \frac{5}{2}x^2 + \frac{3}{2}y^2 = c_1$

- Solve for y

  $\left\{y = -\frac{2x^3}{3} - \frac{\sqrt{4x^6-15x^2+6c_1}}{3}, y = -\frac{2x^3}{3} + \frac{\sqrt{4x^6-15x^2+6c_1}}{3}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(x*(6*x*y(x)+5)+(2*x^3+3*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2x^3}{3} - \frac{\sqrt{4x^6 - 15x^2 - 6c_1}}{3}$$
$$y(x) = -\frac{2x^3}{3} + \frac{\sqrt{4x^6 - 15x^2 - 6c_1}}{3}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 69

```
DSolve[x*(6*x*y[x]+5)+(2*x^3+3*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(-2x^3 - \sqrt{4x^6 - 15x^2 + 9c_1}\right)$$
$$y(x) \to \frac{1}{3}\left(-2x^3 + \sqrt{4x^6 - 15x^2 + 9c_1}\right)$$

4.5 problem 5

Internal problem ID [1947]
Internal file name [`OUTPUT/1947_Sunday_February_25_2024_06_38_32_AM_61071856/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{3x^2y + xy^2 + \left(x^3 + x^2y + \sin(y)\right) y' = -\mathrm{e}^x}$$

4.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^3 + y\,x^2 + \sin(y)\right)\mathrm{d}y &= \left(-3y\,x^2 - x\,y^2 - \mathrm{e}^x\right)\mathrm{d}x\\ \left(3y\,x^2 + x\,y^2 + \mathrm{e}^x\right)\mathrm{d}x + \left(x^3 + y\,x^2 + \sin(y)\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 3y\,x^2 + x\,y^2 + \mathrm{e}^x\\ N(x,y) &= x^3 + y\,x^2 + \sin(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y\,x^2 + x\,y^2 + \mathrm{e}^x\right)\\ &= x(3x + 2y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^3 + y\,x^2 + \sin(y)\right)\\ &= x(3x + 2y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 3y\,x^2 + x\,y^2 + \mathrm{e}^x \mathrm{d}x \\
\phi &= \frac{x^2 y^2}{2} + y\,x^3 + \mathrm{e}^x + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= x^3 + y\,x^2 + f'(y) \qquad (4) \\
&= x^2(x+y) + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 + y\,x^2 + \sin(y)$. Therefore equation (4) becomes

$$
x^3 + y\,x^2 + \sin(y) = x^2(x+y) + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \sin(y)
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int (\sin(y)) \mathrm{d}y \\
f(y) &= -\cos(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{x^2 y^2}{2} + y\,x^3 + \mathrm{e}^x - \cos(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{x^2 y^2}{2} + y\,x^3 + \mathrm{e}^x - \cos(y)
$$

Summary

The solution(s) found are the following

$$\frac{x^2y^2}{2} + yx^3 + e^x - \cos(y) = c_1 \tag{1}$$

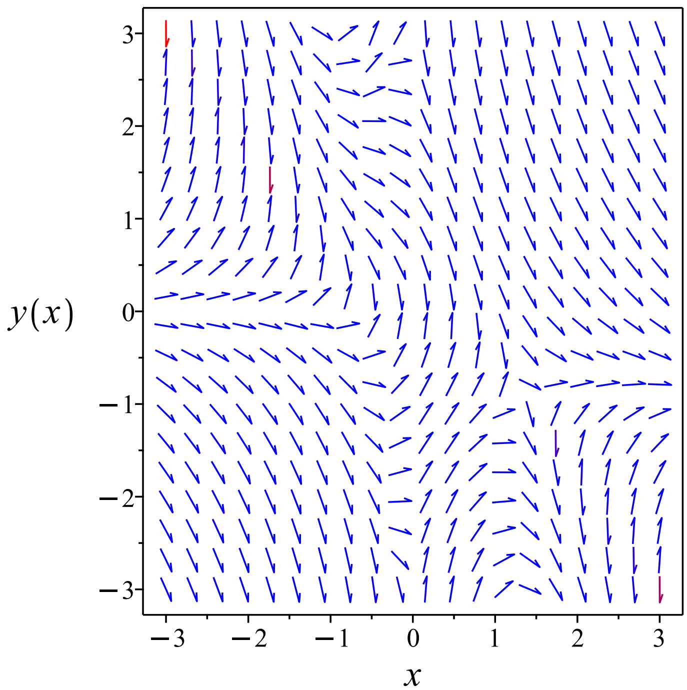


Figure 197: Slope field plot

Verification of solutions

$$\frac{x^2y^2}{2} + yx^3 + e^x - \cos(y) = c_1$$

Verified OK.

4.5.2 Maple step by step solution

Let's solve

$3x^2y + xy^2 + (x^3 + x^2y + \sin(y))\, y' = -e^x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $3x^2+2yx=3x^2+2yx$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (3y\,x^2+x\,y^2+\mathrm{e}^x)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{x^2y^2}{2}+y\,x^3+\mathrm{e}^x+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^3+y\,x^2+\sin(y)=y\,x^2+x^3+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=\sin(y)$

- Solve for $f_1(y)$

  $f_1(y)=-\cos(y)$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=\frac{x^2y^2}{2}+y\,x^3+\mathrm{e}^x-\cos(y)$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{x^2y^2}{2}+y\,x^3+\mathrm{e}^x-\cos(y)=c_1$

- Solve for y

  $y=\mathit{RootOf}\left(-x^2\_Z^2-2x^3\_Z-2\,\mathrm{e}^x+2\cos(\_Z)+2c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve((3*x^2*y(x)+x*y(x)^2+exp(x))+(x^3+x^2*y(x)+sin(y(x)))*diff(y(x),x)=0,y(x), singsol=al
```

$$\frac{x^2 y(x)^2}{2} + x^3 y(x) + \mathrm{e}^x - \cos(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.377 (sec). Leaf size: 32

```
DSolve[(3*x^2*y[x]+x*y[x]^2+Exp[x])+(x^3+x^2*y[x]+Sin[y[x]])*y'[x]==0,y[x],x,IncludeSingular
```

$$\text{Solve}\left[x^3 y(x) + \frac{1}{2}x^2 y(x)^2 - \cos(y(x)) + e^x = c_1, y(x)\right]$$

4.6 problem 6

Internal problem ID [1948]
Internal file name [`OUTPUT/1948_Sunday_February_25_2024_06_38_34_AM_18878808/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2yx-\left(x^{2}+y^{2}\right)y'=0}$$

4.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2u(x)\,x^{2}-\left(x^{2}+u(x)^{2}\,x^{2}\right)\left(u'(x)\,x+u(x)\right)=0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^{3}-u}{x\,(u^{2}+1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3-u}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3-u}{u^2+1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^3-u}{u^2+1}}\,du = \int -\frac{1}{x}\,dx$$

$$-\ln(u) + \ln(u+1) + \ln(u-1) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(u)+\ln(u+1)+\ln(u-1)} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u^2-1}{u} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)^2-1}{u(x)} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{x\left(\frac{y^2}{x^2}-1\right)}{y} = \frac{c_3}{x}$$

$$\frac{-x^2+y^2}{xy} = \frac{c_3}{x}$$

Which simplifies to

$$-\frac{(x-y)(x+y)}{y} = c_3$$

Summary

The solution(s) found are the following

$$-\frac{(x-y)(x+y)}{y} = c_3 \tag{1}$$

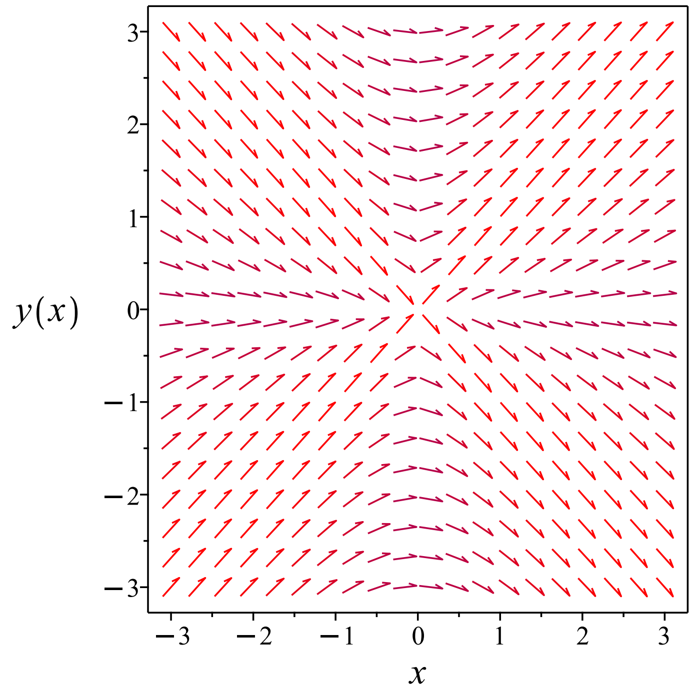


Figure 198: Slope field plot

Verification of solutions

$$-\frac{(x-y)(x+y)}{y} = c_3$$

Verified OK.

4.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{2yx}{x^2+y^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{2yx(b_3 - a_2)}{x^2 + y^2} - \frac{4y^2x^2a_3}{(x^2 + y^2)^2} - \left(\frac{2y}{x^2 + y^2} - \frac{4y\,x^2}{(x^2 + y^2)^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{2x}{x^2 + y^2} - \frac{4y^2x}{(x^2 + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4b_2 + 2y^2x^2a_3 - 4x^2y^2b_2 + 4x\,y^3a_2 - 4x\,y^3b_3 + 2y^4a_3 - y^4b_2 + 2x^3b_1 - 2x^2ya_1 - 2x\,y^2b_1 + 2y^3a_1}{(x^2 + y^2)^2} = 0$$

Setting the numerator to zero gives

$$-x^4b_2 - 2y^2x^2a_3 + 4x^2y^2b_2 - 4x\,y^3a_2 + 4x\,y^3b_3 - 2y^4a_3 + y^4b_2 - 2x^3b_1 + 2x^2ya_1 + 2x\,y^2b_1 - 2y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-4a_2v_1v_2^3 - 2a_3v_1^2v_2^2 - 2a_3v_2^4 - b_2v_1^4 + 4b_2v_1^2v_2^2 + b_2v_2^4 + 4b_3v_1v_2^3 + 2a_1v_1^2v_2 - 2a_1v_2^3 - 2b_1v_1^3 + 2b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-b_2 v_1^4 - 2b_1 v_1^3 + (-2a_3 + 4b_2)\, v_1^2 v_2^2 + 2a_1 v_1^2 v_2 \\
&+ (-4a_2 + 4b_3)\, v_1 v_2^3 + 2b_1 v_1 v_2^2 + (-2a_3 + b_2)\, v_2^4 - 2a_1 v_2^3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
2a_1 &= 0 \\
-2b_1 &= 0 \\
2b_1 &= 0 \\
-b_2 &= 0 \\
-4a_2 + 4b_3 &= 0 \\
-2a_3 + b_2 &= 0 \\
-2a_3 + 4b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{2yx}{x^2 + y^2}\right)(x) \\
&= \frac{-y\, x^2 + y^3}{x^2 + y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-y\,x^2+y^3}{x^2+y^2}} dy \end{aligned}$$

Which results in

$$S = \ln(x + y) - \ln(y) + \ln(-x + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2yx}{x^2 + y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x}{x^2 - y^2} \\ S_y &= \frac{1}{x + y} - \frac{1}{y} + \frac{1}{-x + y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

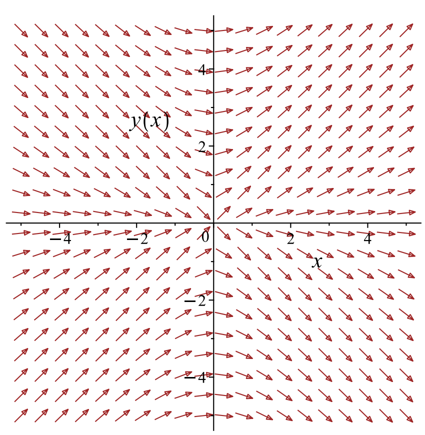
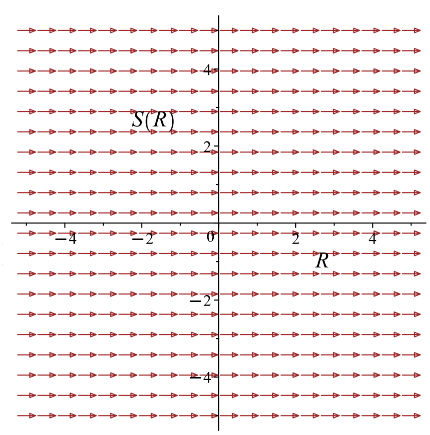
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x+y) - \ln(y) + \ln(-x+y) = c_1$$

Which simplifies to

$$\ln(x+y) - \ln(y) + \ln(-x+y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2yx}{x^2+y^2}$ 	$R = x$ $S = \ln(x+y) - \ln(y) +$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\ln(x+y)-\ln(y)+\ln(-x+y)=c_1 \tag{1}$$

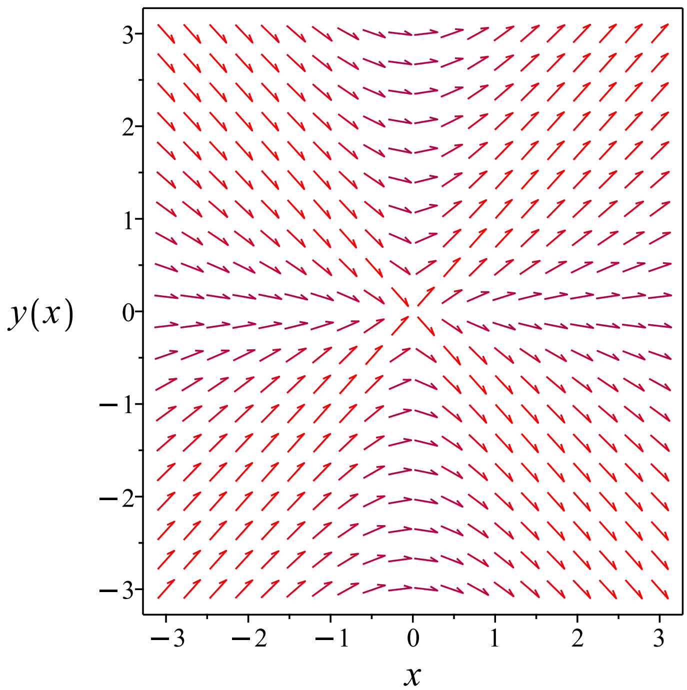


Figure 199: Slope field plot

Verification of solutions

$$\ln(x+y)-\ln(y)+\ln(-x+y)=c_1$$

Verified OK.

4.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2 - y^2\right)\mathrm{d}y &= (-2yx)\,\mathrm{d}x \\ (2yx)\,\mathrm{d}x + \left(-x^2 - y^2\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2yx \\ N(x,y) &= -x^2 - y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2yx) \\ &= 2x\end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2 - y^2) \\ &= -2x \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{x^2+y^2}((2x) - (-2x)) \\ &= -\frac{4x}{x^2+y^2} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{2yx}((-2x) - (2x)) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2}(2yx) \\ &= \frac{2x}{y} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}\left(-x^2-y^2\right) \\
&= \frac{-x^2-y^2}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2x}{y}\right)+\left(\frac{-x^2-y^2}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2x}{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2}{y}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{x^2}{y^2}+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^2-y^2}{y^2}$. Therefore equation (4) becomes

$$
\frac{-x^2-y^2}{y^2} = -\frac{x^2}{y^2}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-1)\,\mathrm{d}y$$
$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{y} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{y} - y$$

Summary
The solution(s) found are the following

$$\frac{x^2}{y} - y = c_1 \tag{1}$$

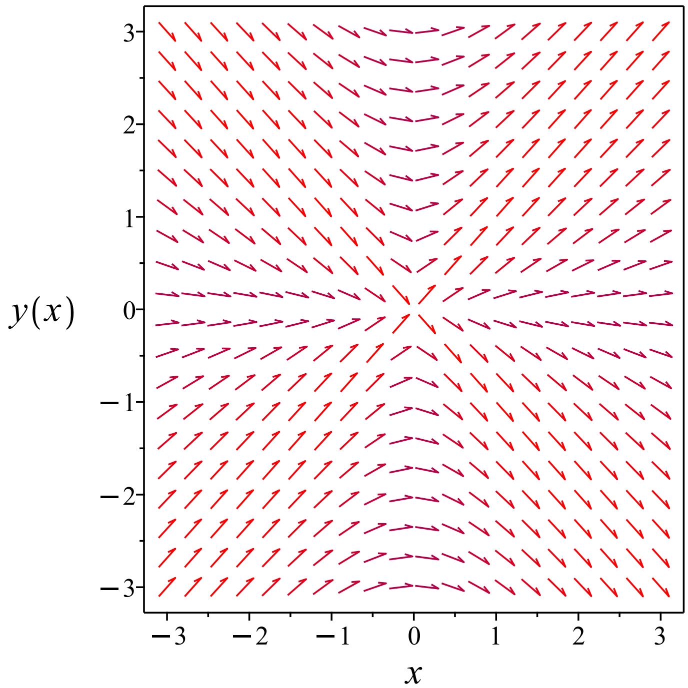


Figure 200: Slope field plot

Verification of solutions

$$\frac{x^2}{y} - y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
dsolve(2*x*y(x)-(x^2+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1 - \sqrt{4c_1^2x^2 + 1}}{2c_1}$$
$$y(x) = \frac{1 + \sqrt{4c_1^2x^2 + 1}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.923 (sec). Leaf size: 70

```
DSolve[2*x*y[x]-(x^2+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-\sqrt{4x^2 + e^{2c_1}} - e^{c_1}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{4x^2 + e^{2c_1}} - e^{c_1}\right)$$
$$y(x) \to 0$$

4.7 problem 7

Internal problem ID [1949]
Internal file name [`OUTPUT/1949_Sunday_February_25_2024_06_38_36_AM_96515593/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\cos(x)\, y - 2\sin(y) - (2x\cos(y) - \sin(x))\, y' = 0}$$

4.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2x\cos(y) + \sin(x))\,\mathrm{d}y &= (-\cos(x)\,y + 2\sin(y))\,\mathrm{d}x \\ (\cos(x)\,y - 2\sin(y))\,\mathrm{d}x + (-2x\cos(y) + \sin(x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \cos(x)\,y - 2\sin(y)\\ N(x,y) &= -2x\cos(y) + \sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\cos(x)\,y - 2\sin(y))\\ &= -2\cos(y) + \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2x\cos(y) + \sin(x))\\ &= -2\cos(y) + \cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \cos(x) y - 2\sin(y) \, \mathrm{d}x$$
$$\phi = \sin(x) y - 2x \sin(y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -2x \cos(y) + \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -2x \cos(y) + \sin(x)$. Therefore equation (4) becomes

$$-2x \cos(y) + \sin(x) = -2x \cos(y) + \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x) y - 2x \sin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x) y - 2x \sin(y)$$

Summary

The solution(s) found are the following

$$\sin(x) y - 2\sin(y) x = c_1 \tag{1}$$

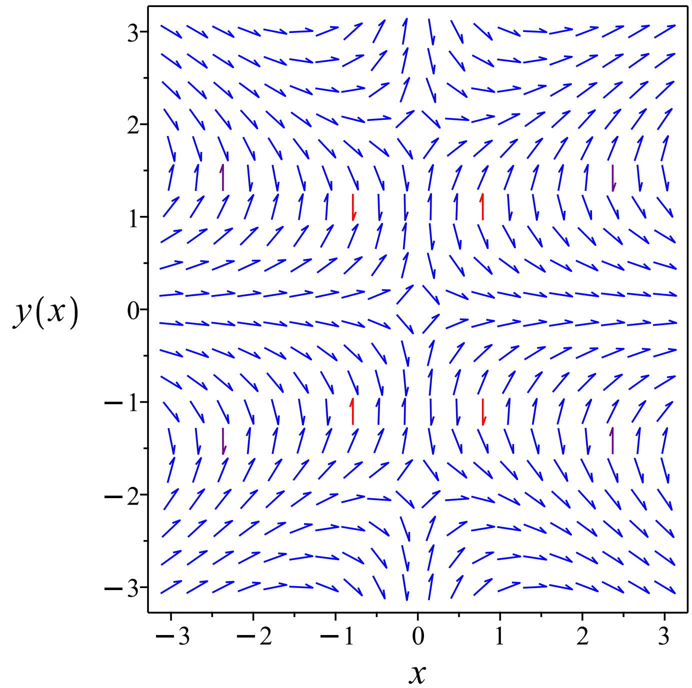


Figure 201: Slope field plot

Verification of solutions

$$\sin(x)\,y - 2\sin(y)\,x = c_1$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$\cos(x)\,y - 2\sin(y) - (2x\cos(y) - \sin(x))\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- ◦ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $-2\cos(y)+\cos(x) = -2\cos(y)+\cos(x)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (\cos(x)\,y - 2\sin(y))\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \sin(x)\,y - 2x\sin(y) + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-2x\cos(y)+\sin(x) = \sin(x) - 2x\cos(y) + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \sin(x)\,y - 2x\sin(y)$

- Substitute $F(x,y)$ into the solution of the ODE

  $\sin(x)\,y - 2x\sin(y) = c_1$

- Solve for y

  $y = \mathit{RootOf}(-\sin(x)\_Z + 2x\sin(\_Z) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve((y(x)*cos(x)-2*sin(y(x)))=(2*x*cos(y(x))-sin(x))*diff(y(x),x),y(x), singsol=all)
```

$$\sin(x)\, y(x) - 2x \sin(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.151 (sec). Leaf size: 19

```
DSolve[(y[x]*Cos[x]-2*Sin[y[x]])==(2*x*Cos[y[x]]-Sin[x])*y'[x],y[x],x,IncludeSingularSolutio
```

$$\text{Solve}[2x\sin(y(x)) - y(x)\sin(x) = c_1, y(x)]$$

4.8 problem 8

Internal problem ID [1950]
Internal file name [`OUTPUT/1950_Sunday_February_25_2024_06_38_43_AM_90028069/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{\frac{2yx-1}{y}+\frac{(x+3y)\,y'}{y^2}=0}$$

4.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{2u(x)\,x^2-1}{u\,(x)\,x}+\frac{(x+3u(x)\,x)\,(u'(x)\,x+u(x))}{u\,(x)^2\,x^2}=0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u^2(2x^2+3)}{x\,(3u+1)}
\end{aligned}$$

Where $f(x) = -\frac{2x^2+3}{x}$ and $g(u) = \frac{u^2}{3u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{3u+1}}\, du = -\frac{2x^2+3}{x}\, dx$$
$$\int \frac{1}{\frac{u^2}{3u+1}}\, du = \int -\frac{2x^2+3}{x}\, dx$$
$$3\ln(u) - \frac{1}{u} = -x^2 - 3\ln(x) + c_2$$

The solution is

$$3\ln(u(x)) - \frac{1}{u(x)} + x^2 + 3\ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$3\ln\left(\frac{y}{x}\right) - \frac{x}{y} + x^2 + 3\ln(x) - c_2 = 0$$
$$3\ln\left(\frac{y}{x}\right) - \frac{x}{y} + x^2 + 3\ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$3\ln\left(\frac{y}{x}\right) - \frac{x}{y} + x^2 + 3\ln(x) - c_2 = 0 \tag{1}$$

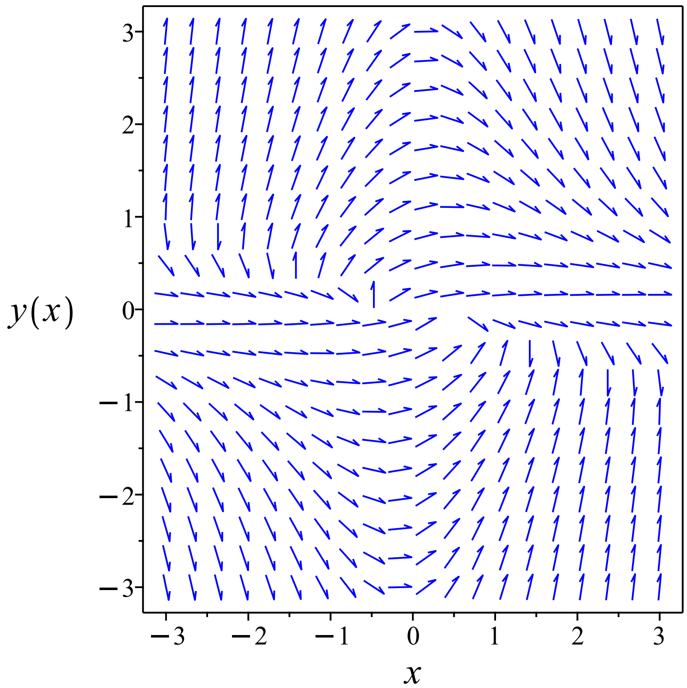


Figure 202: Slope field plot

Verification of solutions

$$3\ln\left(\frac{y}{x}\right) - \frac{x}{y} + x^2 + 3\ln(x) - c_2 = 0$$

Verified OK.

4.8.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{x + 3y}{y^2}\right) \mathrm{d}y = \left(-\frac{2yx - 1}{y}\right) \mathrm{d}x$$
$$\left(\frac{2yx - 1}{y}\right) \mathrm{d}x + \left(\frac{x + 3y}{y^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{2yx - 1}{y}$$
$$N(x, y) = \frac{x + 3y}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2yx - 1}{y}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x+3y}{y^2}\right) \\
&= \frac{1}{y^2}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2yx-1}{y}\,\mathrm{d}x
\end{aligned}$$

$$\phi = \frac{x(yx-1)}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2}{y} - \frac{x(yx-1)}{y^2} + f'(y) \tag{4}$$

$$= \frac{x}{y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x+3y}{y^2}$. Therefore equation (4) becomes

$$\frac{x+3y}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{3}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{3}{y}\right)\mathrm{d}y$$
$$f(y) = 3\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(yx-1)}{y} + 3\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(yx-1)}{y} + 3\ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(\frac{x\,\mathrm{e}^{\frac{x^2}{3}-\frac{c_1}{3}}}{3}\right)-\frac{x^2}{3}+\frac{c_1}{3}}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(\frac{x\,\mathrm{e}^{\frac{x^2}{3}-\frac{c_1}{3}}}{3}\right)-\frac{x^2}{3}+\frac{c_1}{3}} \tag{1}$$

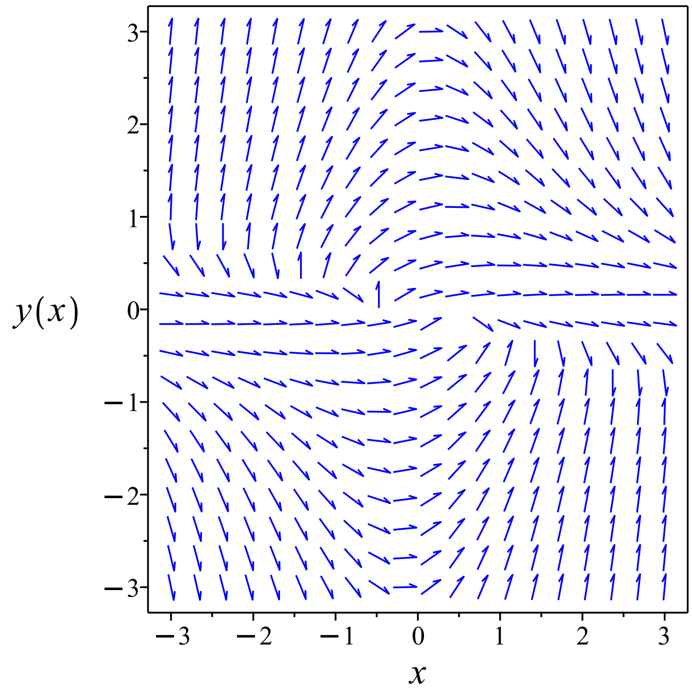


Figure 203: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}\left(\frac{x\, e^{\frac{x^2}{3}-\frac{c_1}{3}}}{3}\right)-\frac{x^2}{3}+\frac{c_1}{3}}$$

Verified OK.

4.8.3 Maple step by step solution

Let's solve

$\frac{2yx-1}{y}+\frac{(x+3y)y'}{y^2}=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

- ◦ Compute derivative of lhs

$$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$$

- Evaluate derivatives

$$\frac{2x}{y} - \frac{2yx-1}{y^2} = \frac{1}{y^2}$$

- Simplify

$$\frac{1}{y^2} = \frac{1}{y^2}$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$$\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$$F(x, y) = \int \frac{2yx-1}{y} dx + f_1(y)$$

- Evaluate integral

$$F(x, y) = \frac{y\, x^2 - x}{y} + f_1(y)$$

- Take derivative of $F(x, y)$ with respect to y

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

- Compute derivative

$$\frac{x+3y}{y^2} = -\frac{y\, x^2 - x}{y^2} + \frac{x^2}{y} + \frac{d}{dy} f_1(y)$$

- Isolate for $\frac{d}{dy} f_1(y)$

$$\frac{d}{dy} f_1(y) = \frac{x+3y}{y^2} + \frac{y\, x^2 - x}{y^2} - \frac{x^2}{y}$$

- Solve for $f_1(y)$

$$f_1(y) = 3 \ln(y)$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = \frac{y\, x^2 - x}{y} + 3 \ln(y)$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\frac{y\, x^2 - x}{y} + 3 \ln(y) = c_1$$

- Solve for y

$$y = e^{\mathit{LambertW}\left(\frac{x\, e^{\frac{x^2}{3} - \frac{c_1}{3}}}{3}\right) - \frac{x^2}{3} + \frac{c_1}{3}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 20

```
dsolve((2*x*y(x)-1)/y(x)+(x+3*y(x))/y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{3\,\mathrm{LambertW}\left(\frac{e^{\frac{x^2}{3}} c_1 x}{3}\right)}$$

✓ Solution by Mathematica

Time used: 3.032 (sec). Leaf size: 37

```
DSolve[(2*x*y[x]-1)/y[x]+(x+3*y[x])/y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{3W\left(\frac{1}{3}xe^{\frac{1}{3}(x^2-c_1)}\right)}$$
$$y(x) \to 0$$

4.9 problem 10

Internal problem ID [1951]
Internal file name [`OUTPUT/1951_Sunday_February_25_2024_06_38_44_AM_77772942/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{e^{x} y + e^{x} y' = 2x}$$

4.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 2x\,e^{-x} \end{aligned}$$

Hence the ode is

$$y' + y = 2x\,e^{-x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 1dx} \\
&= e^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(2x\,e^{-x}\right) \\
\frac{d}{dx}(e^{x} y) &= (e^{x})\left(2x\,e^{-x}\right) \\
d(e^{x} y) &= (2x)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{x} y &= \int 2x\,dx \\
e^{x} y &= x^{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$
y = x^{2}e^{-x} + c_1 e^{-x}
$$

which simplifies to

$$
y = e^{-x}\left(x^{2} + c_1\right)
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = e^{-x}\left(x^{2} + c_1\right) \tag{1}
$$

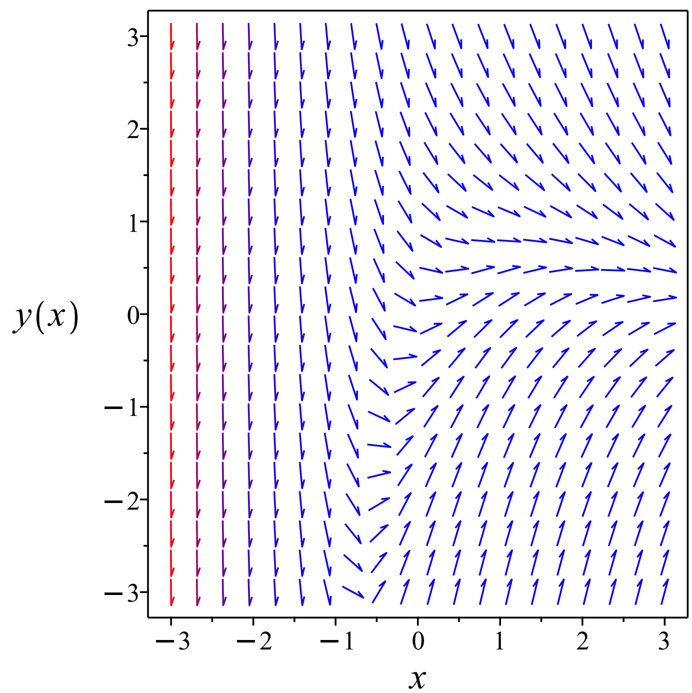


Figure 204: Slope field plot

Verification of solutions

$$y = e^{-x}(x^2 + c_1)$$

Verified OK.

4.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -(e^x y - 2x) e^{-x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 107: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\left(e^{x} y - 2x\right) e^{-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x} y \\ S_y &= e^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^x y = x^2 + c_1$$

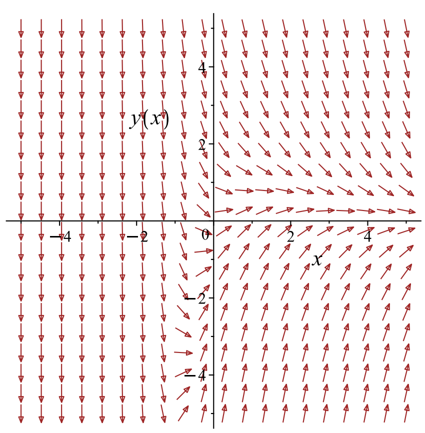
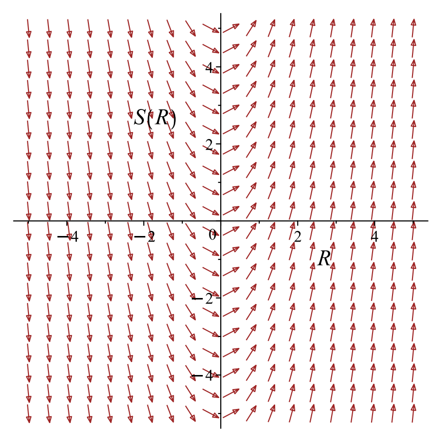
Which simplifies to

$$e^x y = x^2 + c_1$$

Which gives

$$y = e^{-x}\left(x^2 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -(e^x y - 2x)\, e^{-x}$

</td><td>$R = x$
$S = e^x y$</td><td>$\frac{dS}{dR} = 2R$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = e^{-x}\left(x^2 + c_1\right) \tag{1}$$

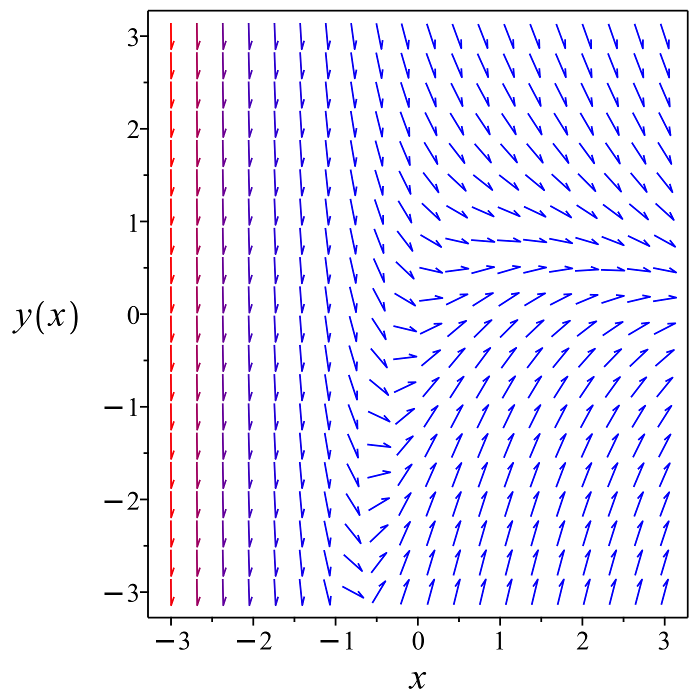


Figure 205: Slope field plot

Verification of solutions

$$y = e^{-x}(x^2 + c_1)$$

Verified OK.

4.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(\mathrm{e}^x)\, \mathrm{d}y &= (-\mathrm{e}^x y + 2x)\, \mathrm{d}x \\
(\mathrm{e}^x y - 2x)\, \mathrm{d}x + (\mathrm{e}^x)\, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \mathrm{e}^x y - 2x \\
N(x, y) &= \mathrm{e}^x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\mathrm{e}^x y - 2x) \\
&= \mathrm{e}^x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^x) \\
&= \mathrm{e}^x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x=\int M\, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x=\int \mathrm{e}^{x}y-2x\, \mathrm{d}x$$
$$\phi=\mathrm{e}^{x}y-x^{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=\mathrm{e}^{x}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x}=\mathrm{e}^{x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\mathrm{e}^{x}y-x^{2}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=\mathrm{e}^{x}y-x^{2}$$

The solution becomes

$$y = e^{-x}\left(x^2 + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-x}\left(x^2 + c_1\right) \tag{1}$$

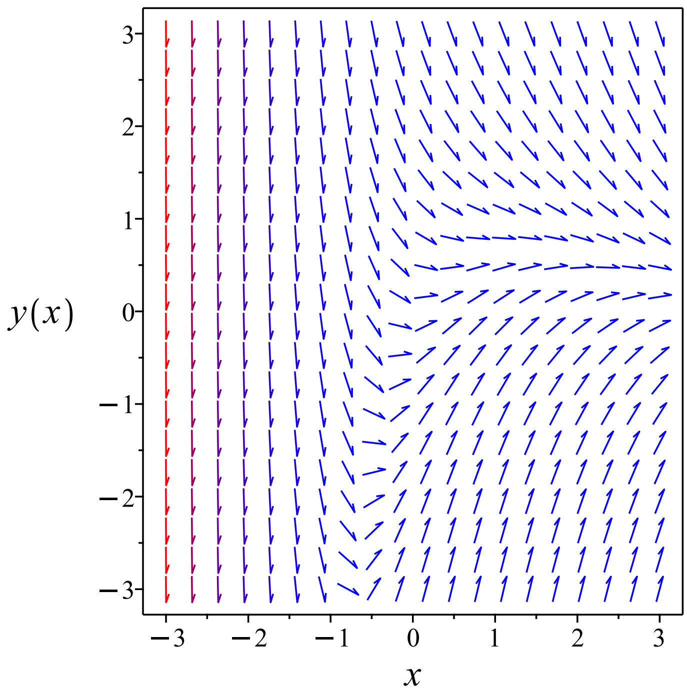


Figure 206: Slope field plot

Verification of solutions

$$y = e^{-x}\left(x^2 + c_1\right)$$

Verified OK.

4.9.4 Maple step by step solution

Let's solve

$e^x y + e^x y' = 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + \frac{2x}{e^x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = \frac{2x}{e^x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y) = \frac{2\mu(x)x}{e^x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' + y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$

- Solve to find the integrating factor

  $\mu(x) = (e^x)^2\,e^{-x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{2\mu(x)x}{e^x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{2\mu(x)x}{e^x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{2\mu(x)x}{e^x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = (e^x)^2\,e^{-x}$

  $y = \frac{\int 2x\,e^{-x} e^x dx + c_1}{(e^x)^2 e^{-x}}$

- Evaluate the integrals on the rhs

  $y = \frac{x^2 + c_1}{(e^x)^2 e^{-x}}$

- Simplify

$$y = e^{-x}(x^2 + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((y(x)*exp(x)-2*x)+exp(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (x^2 + c_1) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 17

```
DSolve[(y[x]*Exp[x]-2*x)+Exp[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(x^2 + c_1)$$

4.10 problem 11

Internal problem ID [1952]
Internal file name [`OUTPUT/1952_Sunday_February_25_2024_06_38_44_AM_15335326/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{3\sin(x)\,y-\cos(y)+(\sin(y)\,x-3\cos(x))\,y'=0}$$

4.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x\sin(y) - 3\cos(x))\,\mathrm{d}y &= (-3\sin(x)\,y + \cos(y))\,\mathrm{d}x\\ (3\sin(x)\,y - \cos(y))\,\mathrm{d}x + (x\sin(y) - 3\cos(x))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 3\sin(x)\,y - \cos(y)\\ N(x,y) &= x\sin(y) - 3\cos(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3\sin(x)\,y - \cos(y))\\ &= 3\sin(x) + \sin(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\sin(y) - 3\cos(x))\\ &= 3\sin(x) + \sin(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 3 \sin(x) y - \cos(y) \mathrm{d}x$$

$$\phi = -3 \cos(x) y - x \cos(y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x \sin(y) - 3 \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x \sin(y) - 3 \cos(x)$. Therefore equation (4) becomes

$$x \sin(y) - 3 \cos(x) = x \sin(y) - 3 \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -3 \cos(x) y - x \cos(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3 \cos(x) y - x \cos(y)$$

<u>Summary</u>

The solution(s) found are the following

$$-3 \cos(x) y - x \cos(y) = c_1 \tag{1}$$

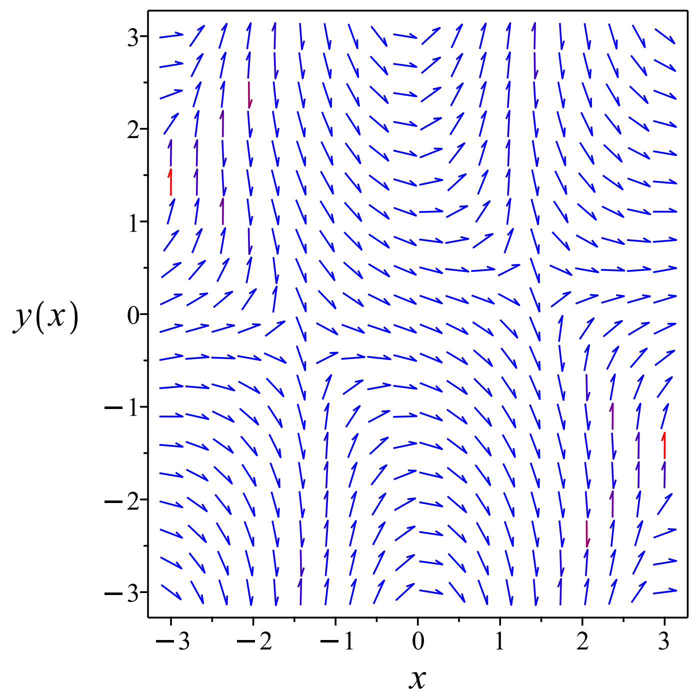


Figure 207: Slope field plot

Verification of solutions

$$-3\cos(x)\,y - x\cos(y) = c_1$$

Verified OK.

4.10.2 Maple step by step solution

Let's solve

$3\sin(x)\,y - \cos(y) + (\sin(y)\,x - 3\cos(x))\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $3\sin(x)+\sin(y)=3\sin(x)+\sin(y)$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(3\sin(x)\,y-\cos(y))\,dx+f_1(y)$
- Evaluate integral

  $F(x,y)=-3\cos(x)\,y-x\cos(y)+f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $x\sin(y)-3\cos(x)=-3\cos(x)+x\sin(y)+\frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=0$
- Solve for $f_1(y)$

  $f_1(y)=0$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=-3\cos(x)\,y-x\cos(y)$
- Substitute $F(x,y)$ into the solution of the ODE

  $-3\cos(x)\,y-x\cos(y)=c_1$
- Solve for y

  $y=\mathit{RootOf}(3\_Z\cos(x)+x\cos(\_Z)+c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((3*y(x)*sin(x)-cos(y(x)))+(x*sin(y(x))-3*cos(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$-3\cos(x)\, y(x) - x\cos(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 18

```
DSolve[(3*y[x]*Sin[x]-Cos[y[x]])+(x*Sin[y[x]]-3*Cos[x])*y'[x]==0,y[x],x,IncludeSingularSolut
```

$$\text{Solve}[x\cos(y(x)) + 3y(x)\cos(x) = c_1, y(x)]$$

4.11 problem 12

Internal problem ID [1953]
Internal file name [OUTPUT/1953_Sunday_February_25_2024_06_38_55_AM_13790226/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational]
```

Unable to solve or complete the solution.

$$\boxed{xy^2 + 2y + (2y^3 - x^2y + 2x)\, y' = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve((x*y(x)^2+2*y(x))+(2*y(x)^3-x^2*y(x)+2*x)*diff(y(x),x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x*y[x]^2+2*y[x])+(2*y[x]^3-x^2*y[x]+2*x)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

Not solved

4.12 problem 13

Internal problem ID [1954]
Internal file name [`OUTPUT/1954_Sunday_February_25_2024_06_38_56_AM_36226880/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{2}{y}-\frac{y}{x^{2}}+\left(\frac{1}{x}-\frac{2x}{y^{2}}\right)y'=0}$$

4.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\
\ln(y) &= \ln(x) + c_1 \\
y &= e^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x \tag{1}
$$

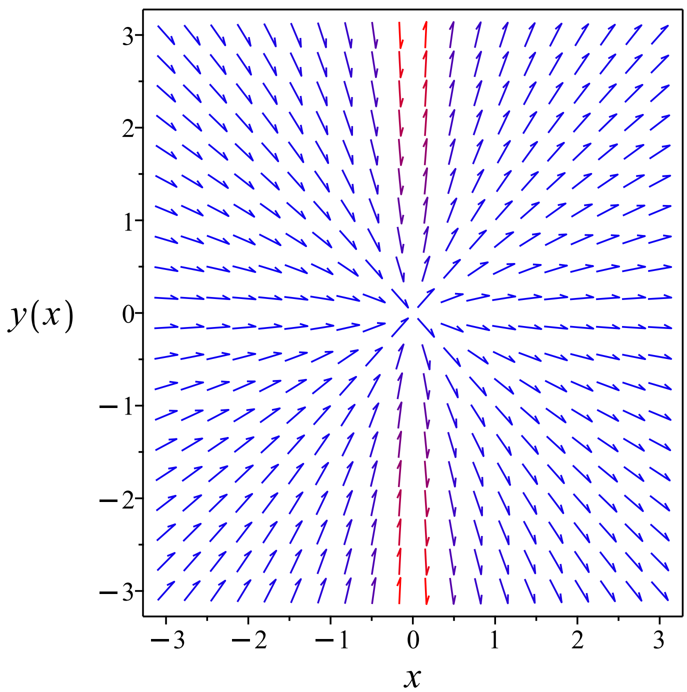


Figure 208: Slope field plot

Verification of solutions

$$
y = c_1 x
$$

Verified OK.

4.12.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

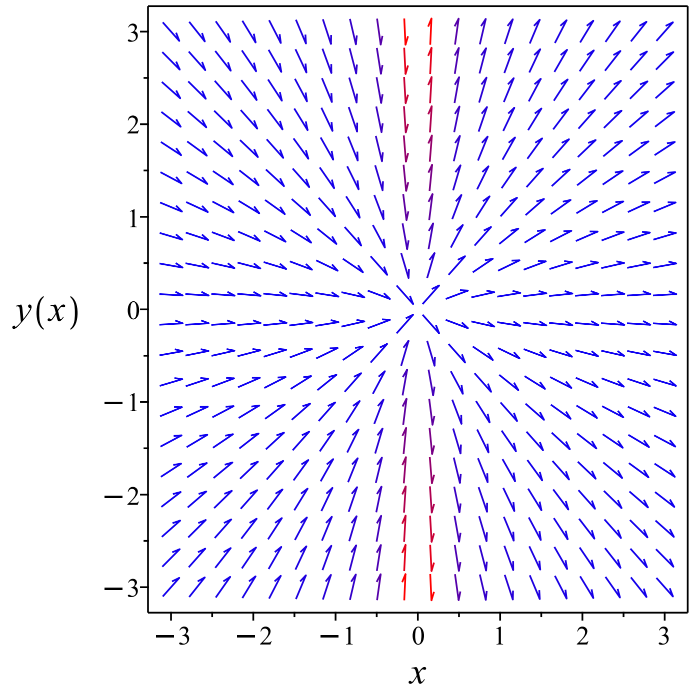


Figure 209: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

4.12.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$\frac{2}{u\left(x\right)x} - \frac{u(x)}{x} + \left(\frac{1}{x} - \frac{2}{x u\left(x\right)^{2}}\right)\left(u'(x)\, x + u(x)\right) = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0 \, \mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x \tag{1}$$

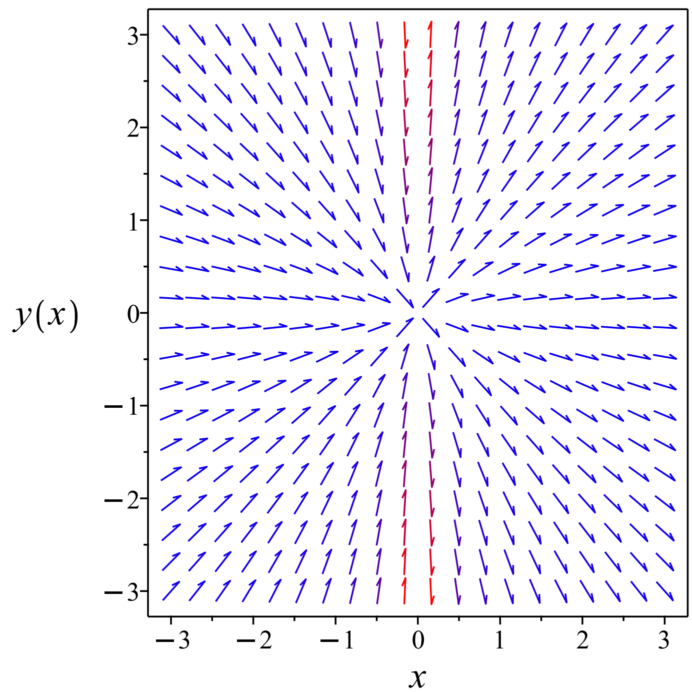


Figure 210: Slope field plot

Verification of solutions

$$y = c_2 x$$

Verified OK.

4.12.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 111: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

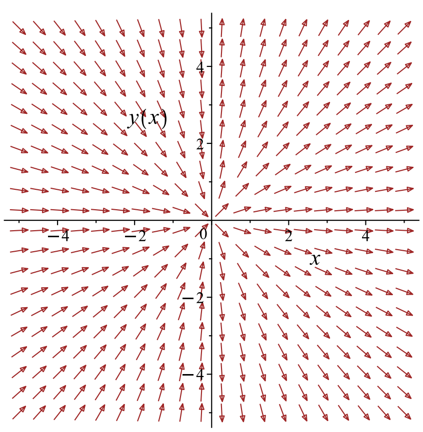
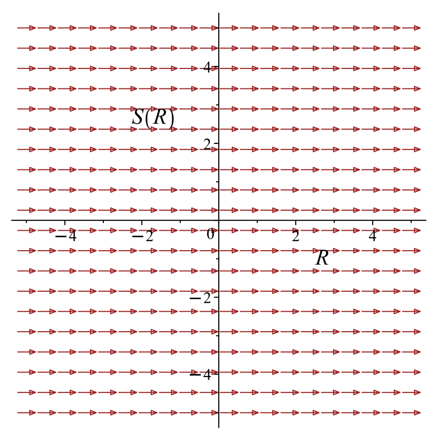
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

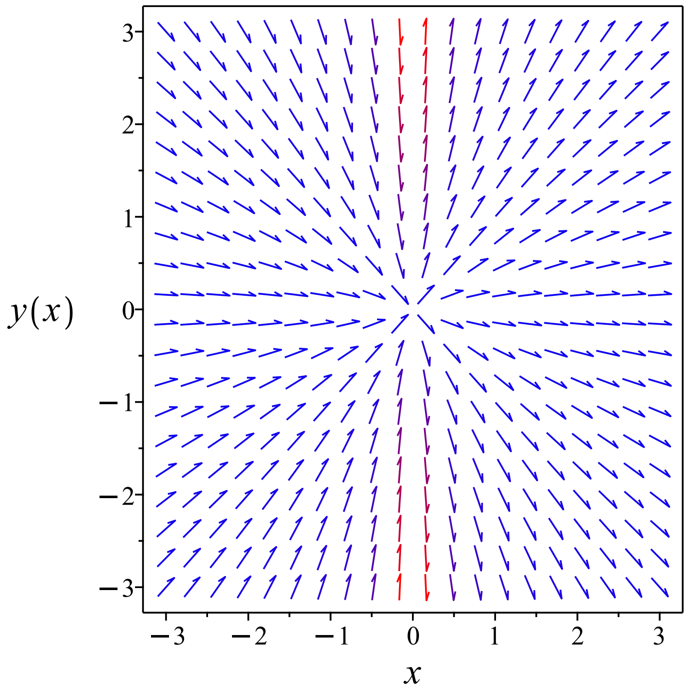


Figure 211: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

4.12.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= \frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y}\right)\mathrm{d}y\\ f(y) &= \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = x\,\mathrm{e}^{c_1}$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{c_1} \tag{1}$$

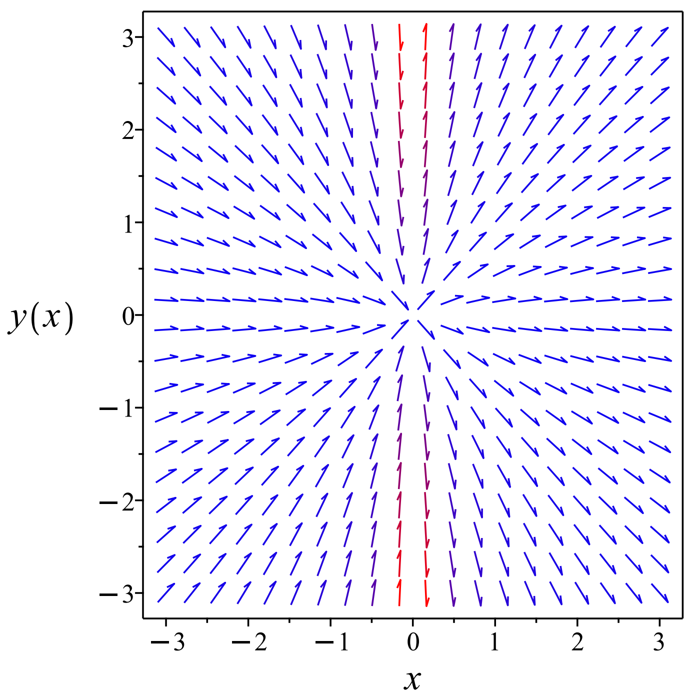


Figure 212: Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{c_1}$$

Verified OK.

4.12.6 Maple step by step solution

Let's solve

$\frac{2}{y}-\frac{y}{x^{2}}+\left(\frac{1}{x}-\frac{2x}{y^{2}}\right)y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(\frac{2}{y}-\frac{y}{x^{2}}+\left(\frac{1}{x}-\frac{2x}{y^{2}}\right)y'\right)dx=\int 0dx+c_1$

- Evaluate integral

  $-\frac{-y-\frac{2x^{2}}{y}}{x}=c_1$

- Solve for y

  $\left\{y=\left(\frac{c_1}{2}-\frac{\sqrt{c_1^{2}-8}}{2}\right)x, y=\left(\frac{c_1}{2}+\frac{\sqrt{c_1^{2}-8}}{2}\right)x\right\}$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
dsolve((2/y(x)-y(x)/x^2)+(1/x-2*x/y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{2}\, x$$
$$y(x) = -\sqrt{2}\, x$$
$$y(x) = -\frac{\left(c_1 + \sqrt{c_1^2 - 8}\right) x}{2}$$
$$y(x) = \frac{\left(-c_1 + \sqrt{c_1^2 - 8}\right) x}{2}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 55

```
DSolve[(2/y[x]-y[x]/x^2)+(1/x-2*x/y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}x$$
$$y(x) \to \sqrt{2}x$$
$$y(x) \to c_1 x$$
$$y(x) \to -\sqrt{2}x$$
$$y(x) \to \sqrt{2}x$$

4.13 problem 14

Internal problem ID [1955]
Internal file name [`OUTPUT/1955_Sunday_February_25_2024_06_38_57_AM_49652375/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{\frac{yx+1}{y}+\frac{(-x+2y)\,y'}{y^2}=0}$$

4.13.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{u(x)\,x^2+1}{u\,(x)\,x}+\frac{(-x+2u(x)\,x)\,(u'(x)\,x+u(x))}{u\,(x)^2\,x^2}=0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^2(x^2+2)}{x\,(2u-1)}\end{aligned}$$

Where $f(x) = -\frac{x^2+2}{x}$ and $g(u) = \frac{u^2}{2u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{2u-1}}\,du = -\frac{x^2+2}{x}\,dx$$
$$\int \frac{1}{\frac{u^2}{2u-1}}\,du = \int -\frac{x^2+2}{x}\,dx$$
$$2\ln(u) + \frac{1}{u} = -\frac{x^2}{2} - 2\ln(x) + c_2$$

The solution is

$$2\ln(u(x)) + \frac{1}{u(x)} + \frac{x^2}{2} + 2\ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$2\ln\left(\frac{y}{x}\right) + \frac{x}{y} + \frac{x^2}{2} + 2\ln(x) - c_2 = 0$$
$$2\ln\left(\frac{y}{x}\right) + \frac{x}{y} + \frac{x^2}{2} + 2\ln(x) - c_2 = 0$$

Summary
The solution(s) found are the following

$$2\ln\left(\frac{y}{x}\right) + \frac{x}{y} + \frac{x^2}{2} + 2\ln(x) - c_2 = 0 \tag{1}$$

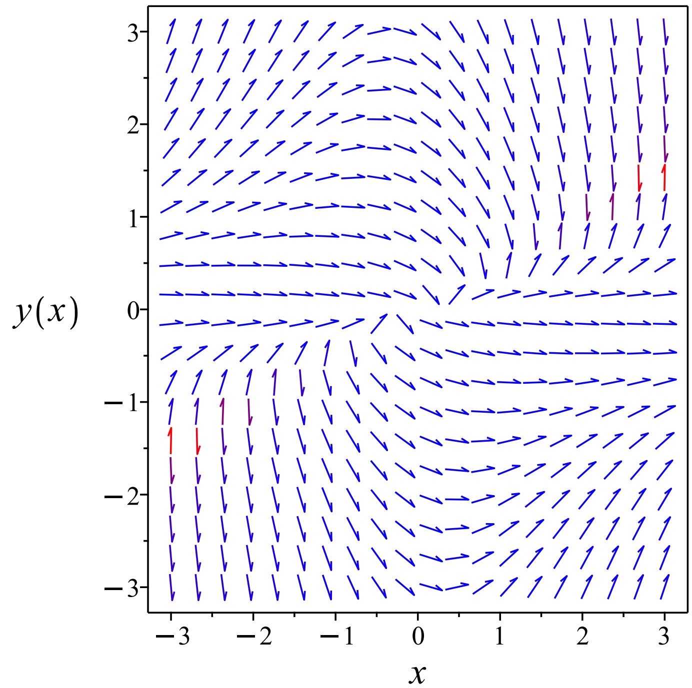


Figure 213: Slope field plot

Verification of solutions

$$2\ln\left(\frac{y}{x}\right)+\frac{x}{y}+\frac{x^{2}}{2}+2\ln\left(x\right)-c_{2}=0$$

Verified OK.

4.13.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{-x + 2y}{y^2}\right) \mathrm{d}y &= \left(-\frac{yx + 1}{y}\right) \mathrm{d}x \\
\left(\frac{yx + 1}{y}\right) \mathrm{d}x + \left(\frac{-x + 2y}{y^2}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \frac{yx + 1}{y} \\
N(x, y) &= \frac{-x + 2y}{y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{yx + 1}{y}\right) \\
&= -\frac{1}{y^2}
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-x+2y}{y^2}\right)$$
$$= -\frac{1}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{yx+1}{y}\,\mathrm{d}x$$
$$\phi = \frac{x(yx+2)}{2y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2}{2y} - \frac{x(yx+2)}{2y^2} + f'(y) \tag{4}$$
$$= -\frac{x}{y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+2y}{y^2}$. Therefore equation (4) becomes

$$\frac{-x+2y}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2}{y}\right) \mathrm{d}y$$
$$f(y) = 2\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(yx+2)}{2y} + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(yx+2)}{2y} + 2\ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-\frac{x\,\mathrm{e}^{\frac{x^2}{4}-\frac{c_1}{2}}}{2}\right)-\frac{x^2}{4}+\frac{c_1}{2}}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-\frac{x\,\mathrm{e}^{\frac{x^2}{4}-\frac{c_1}{2}}}{2}\right)-\frac{x^2}{4}+\frac{c_1}{2}} \tag{1}$$

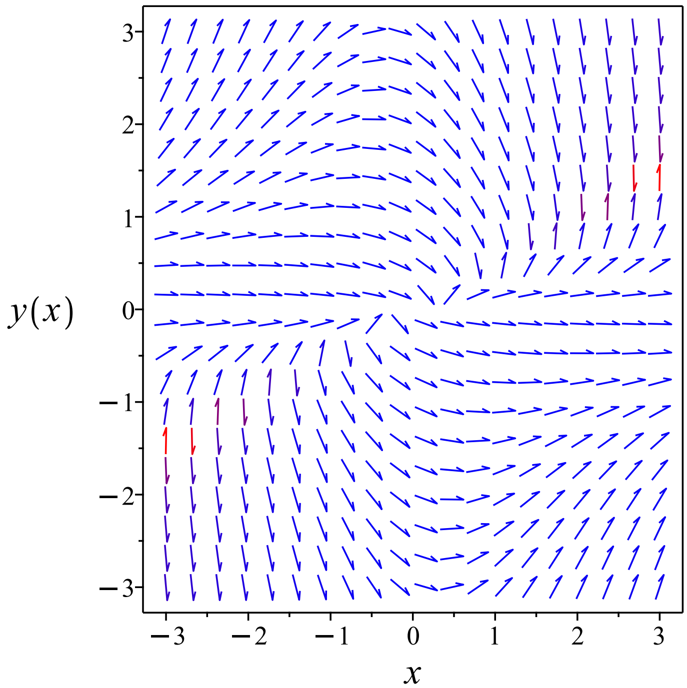


Figure 214: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-\frac{x\,\mathrm{e}^{\frac{x^2}{4}-\frac{c_1}{2}}}{2}\right)-\frac{x^2}{4}+\frac{c_1}{2}}$$

Verified OK.

4.13.3 Maple step by step solution

Let's solve

$\frac{yx+1}{y}+\frac{(-x+2y)y'}{y^2}=0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$
  - Compute derivative of lhs

$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $\frac{x}{y} - \frac{yx+1}{y^2} = -\frac{1}{y^2}$

- Simplify

  $-\frac{1}{y^2} = -\frac{1}{y^2}$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \frac{yx+1}{y} dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{\frac{1}{2} y x^2 + x}{y} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $\frac{-x+2y}{y^2} = -\frac{\frac{1}{2} y x^2 + x}{y^2} + \frac{x^2}{2y} + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = \frac{-x+2y}{y^2} + \frac{\frac{1}{2} y x^2 + x}{y^2} - \frac{x^2}{2y}$

- Solve for $f_1(y)$

  $f_1(y) = 2 \ln(y)$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \frac{\frac{1}{2} y x^2 + x}{y} + 2 \ln(y)$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{\frac{1}{2} y x^2 + x}{y} + 2 \ln(y) = c_1$

- Solve for y

  $y = e^{\mathit{LambertW}\left(-\frac{x \, e^{\frac{x^2}{4} - \frac{c_1}{2}}}{2}\right) - \frac{x^2}{4} + \frac{c_1}{2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 20

```
dsolve((x*y(x)+1)/y(x)+(2*y(x)-x)/y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{2\,\mathrm{LambertW}\left(-\frac{x c_1 \mathrm{e}^{\frac{x^2}{4}}}{2}\right)}$$

✓ Solution by Mathematica

Time used: 3.516 (sec). Leaf size: 37

```
DSolve[(x*y[x]+1)/y[x]+(2*y[x]-x)/y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{2W\left(-\frac{1}{2}xe^{\frac{1}{4}(x^2-2c_1)}\right)}$$
$$y(x) \to 0$$

4.14 problem 15

Internal problem ID [1956]
Internal file name [`OUTPUT/1956_Sunday_February_25_2024_06_38_58_AM_627272/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational, [_Abel, `2nd
   type`, `class B`]]
```

$$\boxed{\frac{y(2+y x^3)}{x^3}-\frac{(1-2y x^3)\, y'}{x^2}=0}$$

4.14.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(y\, x^3+2)}{x\,(2y\, x^3-1)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{y(y\,x^3+2)\,(b_3-a_2)}{x\,(2y\,x^3-1)} - \frac{y^2(y\,x^3+2)^2\,a_3}{x^2\,(2y\,x^3-1)^2} \\ & - \left(-\frac{3y^2x}{2y\,x^3-1} + \frac{y(y\,x^3+2)}{x^2\,(2y\,x^3-1)} + \frac{6y^2(y\,x^3+2)\,x}{(2y\,x^3-1)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{y\,x^3+2}{(2y\,x^3-1)\,x} - \frac{y\,x^2}{2y\,x^3-1} + \frac{2y(y\,x^3+2)\,x^2}{(2y\,x^3-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^8y^2b_2 - 3x^6y^4a_3 + 2x^7y^2b_1 - 2x^6y^3a_1 - 6x^5yb_2 - 15x^4y^2a_2 - 5x^4y^2b_3 - 22x^3y^3a_3 - 2x^4yb_1 - 18x^3y^2a_1}{(2y\,x^3-1)^2\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 6x^8y^2b_2 - 3x^6y^4a_3 + 2x^7y^2b_1 - 2x^6y^3a_1 - 6x^5yb_2 - 15x^4y^2a_2 - 5x^4y^2b_3 \\ & - 22x^3y^3a_3 - 2x^4yb_1 - 18x^3y^2a_1 - b_2x^2 - 2y^2a_3 - 2xb_1 + 2ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -3a_3v_1^6v_2^4 + 6b_2v_1^8v_2^2 - 2a_1v_1^6v_2^3 + 2b_1v_1^7v_2^2 - 15a_2v_1^4v_2^2 - 22a_3v_1^3v_2^3 - 6b_2v_1^5v_2 \\ & - 5b_3v_1^4v_2^2 - 18a_1v_1^3v_2^2 - 2b_1v_1^4v_2 - 2a_3v_2^2 - b_2v_1^2 + 2a_1v_2 - 2b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}6b_2v_1^8v_2^2 + 2b_1v_1^7v_2^2 - 3a_3v_1^6v_2^4 - 2a_1v_1^6v_2^3 - 6b_2v_1^5v_2 + (-15a_2 - 5b_3)\, v_1^4v_2^2 \\ - 2b_1v_1^4v_2 - 22a_3v_1^3v_2^3 - 18a_1v_1^3v_2^2 - b_2v_1^2 - 2b_1v_1 - 2a_3v_2^2 + 2a_1v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-18a_1 &= 0 \\
-2a_1 &= 0 \\
2a_1 &= 0 \\
-22a_3 &= 0 \\
-3a_3 &= 0 \\
-2a_3 &= 0 \\
-2b_1 &= 0 \\
2b_1 &= 0 \\
-6b_2 &= 0 \\
-b_2 &= 0 \\
6b_2 &= 0 \\
-15a_2 - 5b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -3a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= -3y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -3y - \left(-\frac{y(y\,x^3 + 2)}{x\,(2y\,x^3 - 1)}\right)(x) \\
&= \frac{-5x^3y^2 + 5y}{2y\,x^3 - 1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-5x^3y^2+5y}{2y\,x^3-1}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln\left(y(y\,x^3 - 1)\right)}{5}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{y(y\,x^3 + 2)}{x\,(2y\,x^3 - 1)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{3y\,x^2}{5y\,x^3-5} \\
S_y &= \frac{-2y\,x^3+1}{5y\,(y\,x^3-1)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2}{5x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{5R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{2\ln(R)}{5} + c_1 \tag{4}$$

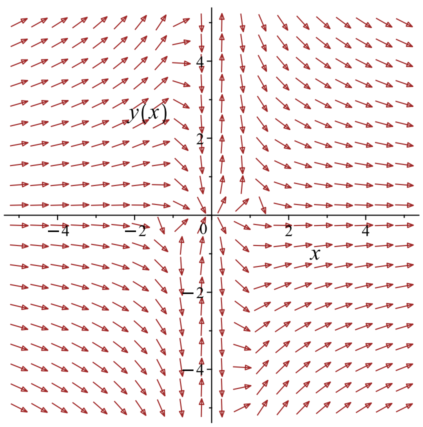
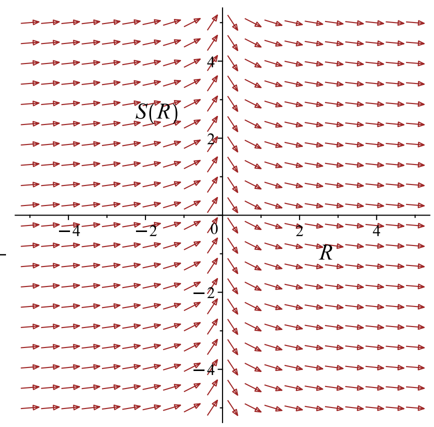
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{5} - \frac{\ln(yx^3-1)}{5} = -\frac{2\ln(x)}{5} + c_1$$

Which simplifies to

$$-\frac{\ln(y)}{5} - \frac{\ln(yx^3-1)}{5} = -\frac{2\ln(x)}{5} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y\,x^{3}+2)}{x(2y\,x^{3}-1)}$ 	$R = x$ $S = -\frac{\ln(y)}{5} - \frac{\ln\left(y\,x^{3} -\right.}{5}$	$\frac{dS}{dR} = -\frac{2}{5R}$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{5} - \frac{\ln(yx^{3}-1)}{5} = -\frac{2\ln(x)}{5} + c_{1} \tag{1}$$

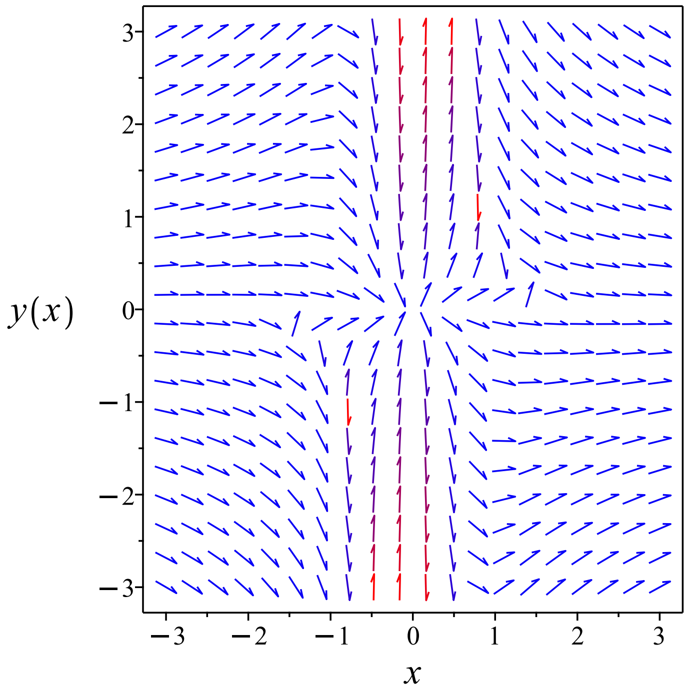


Figure 215: Slope field plot

Verification of solutions

$$-\frac{\ln(y)}{5} - \frac{\ln(y x^3 - 1)}{5} = -\frac{2\ln(x)}{5} + c_1$$

Verified OK.

4.14.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{-2y\,x^3 + 1}{x^2}\right) \mathrm{d}y = \left(-\frac{y(y\,x^3 + 2)}{x^3}\right) \mathrm{d}x$$
$$\left(\frac{y(y\,x^3 + 2)}{x^3}\right) \mathrm{d}x + \left(-\frac{-2y\,x^3 + 1}{x^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{y(y\,x^3 + 2)}{x^3}$$
$$N(x, y) = -\frac{-2y\,x^3 + 1}{x^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y(y\,x^3 + 2)}{x^3}\right)$$
$$= \frac{2y\,x^3 + 2}{x^3}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{-2y\,x^3+1}{x^2}\right)$$
$$= \frac{2y\,x^3+2}{x^3}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{y(y\,x^3+2)}{x^3}\,\mathrm{d}x$$
$$\phi = \frac{y(y\,x^3-1)}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y\,x^3-1}{x^2} + yx + f'(y) \tag{4}$$
$$= \frac{2y\,x^3-1}{x^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{-2y\,x^3+1}{x^2}$. Therefore equation (4) becomes

$$-\frac{-2y\,x^3+1}{x^2} = \frac{2y\,x^3-1}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y(y\,x^3 - 1)}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y(y\,x^3 - 1)}{x^2}$$

Summary
The solution(s) found are the following

$$\frac{y(yx^3 - 1)}{x^2} = c_1 \tag{1}$$

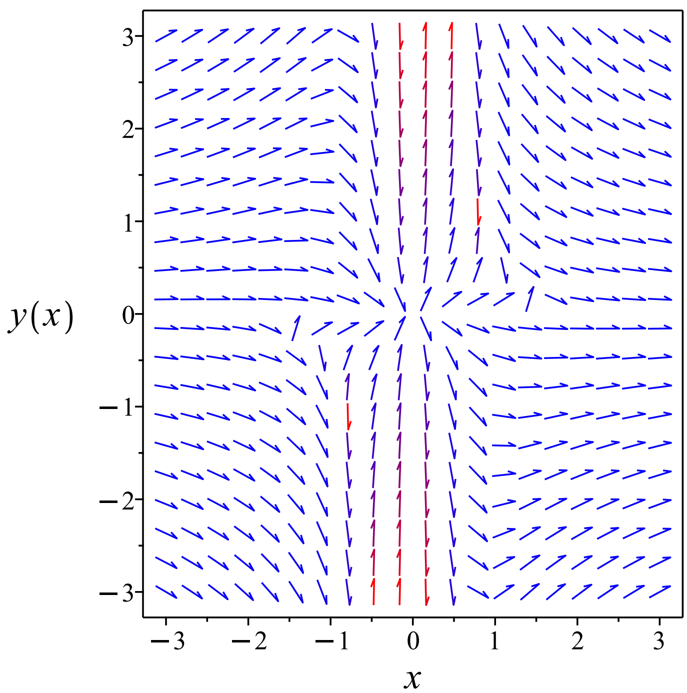


Figure 216: Slope field plot

Verification of solutions

$$\frac{y(yx^3-1)}{x^2}=c_1$$

Verified OK.

4.14.3 Maple step by step solution

Let's solve

$\frac{y(2+yx^3)}{x^3}-\frac{(1-2yx^3)y'}{x^2}=0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $\frac{y\,x^3+2}{x^3}+y=6y+\frac{2(-2y\,x^3+1)}{x^3}$

  - Simplify

    $\frac{2y\,x^3+2}{x^3}=\frac{2y\,x^3+2}{x^3}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\frac{y(y\,x^3+2)}{x^3}dx+f_1(y)$

- Evaluate integral

  $F(x,y)=y\left(yx-\frac{1}{x^2}\right)+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

$$-\frac{-2y\,x^3+1}{x^2} = 2yx - \frac{1}{x^2} + \frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = -\frac{-2y\,x^3+1}{x^2} - 2yx + \frac{1}{x^2}$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = y\left(yx - \frac{1}{x^2}\right)$$

- Substitute $F(x,y)$ into the solution of the ODE

$$y\left(yx - \frac{1}{x^2}\right) = c_1$$

- Solve for y

$$\left\{y = \frac{1+\sqrt{4c_1x^5+1}}{2x^3}, y = -\frac{-1+\sqrt{4c_1x^5+1}}{2x^3}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 57

```
dsolve(y(x)*(2+x^3*y(x))/x^3=(1-2*x^3*y(x))/x^2*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{c_1^3 - \sqrt{c_1^6 + 4c_1x^5}}{2x^3c_1^3}$$

$$y(x) = \frac{c_1^3 + \sqrt{c_1^6 + 4c_1x^5}}{2c_1^3x^3}$$

✓ Solution by Mathematica

Time used: 0.861 (sec). Leaf size: 80

```
DSolve[y[x]*(2+x^3*y[x])/x^3==(1-2*x^3*y[x])/x^2*y'[x],y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1 - \sqrt{\frac{1}{x^3}}x^2\sqrt{\frac{1}{x} + 4c_1x^4}}{2x^3}$$
$$y(x) \to \frac{1 + \sqrt{\frac{1}{x^3}}x^2\sqrt{\frac{1}{x} + 4c_1x^4}}{2x^3}$$

4.15 problem 16

Internal problem ID [1957]
Internal file name [`OUTPUT/1957_Sunday_February_25_2024_06_38_59_AM_63397742/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{y^2 \csc(x)^2 + 6yx - (2y\cot(x) - 3x^2)\, y' = 2}$$

4.15.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-2y\cot(x) + 3x^2\right)\mathrm{d}y &= \left(-y^2\csc(x)^2 - 6yx + 2\right)\mathrm{d}x\\ \left(y^2\csc(x)^2 + 6yx - 2\right)\mathrm{d}x + \left(-2y\cot(x) + 3x^2\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y^2\csc(x)^2 + 6yx - 2\\ N(x,y) &= -2y\cot(x) + 3x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\csc(x)^2 + 6yx - 2\right)\\ &= 2y\csc(x)^2 + 6x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-2y\cot(x) + 3x^2\right)\\ &= 2y\csc(x)^2 + 6x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y^2 \csc(x)^2 + 6yx - 2 \, \mathrm{d}x$$
$$\phi = -2x - \cot(x) \, y^2 + 3y \, x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -2y\cot(x) + 3x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -2y\cot(x) + 3x^2$. Therefore equation (4) becomes

$$-2y\cot(x) + 3x^2 = -2y\cot(x) + 3x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -2x - \cot(x) \, y^2 + 3y \, x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2x - \cot(x) \, y^2 + 3y \, x^2$$

Summary

The solution(s) found are the following

$$-2x - y^2\cot(x) + 3x^2y = c_1 \tag{1}$$

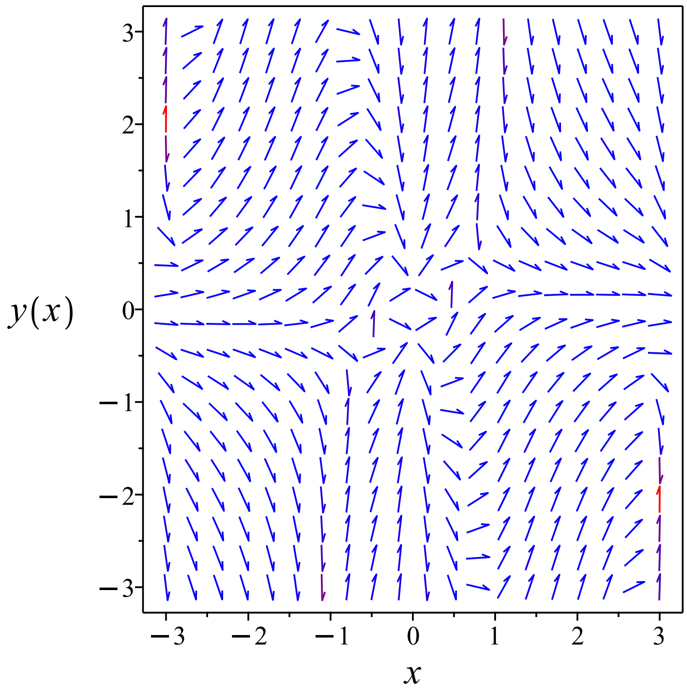


Figure 217: Slope field plot

Verification of solutions

$$-2x - y^2 \cot(x) + 3x^2 y = c_1$$

Verified OK.

4.15.2 Maple step by step solution

Let's solve

$y^2 \csc(x)^2 + 6yx - (2y \cot(x) - 3x^2) y' = 2$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - ∘ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $2y \csc(x)^2 + 6x = -2y\left(-1 - \cot(x)^2\right) + 6x$

- Simplify

  $2y \csc(x)^2 + 6x = 2y \csc(x)^2 + 6x$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int \left(y^2 \csc(x)^2 + 6yx - 2\right) dx + f_1(y)$

- Evaluate integral

  $F(x,y) = -2x - \cot(x)\, y^2 + 3y\, x^2 + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-2y \cot(x) + 3x^2 = -2y \cot(x) + 3x^2 + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = -2x - \cot(x)\, y^2 + 3y\, x^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $-2x - \cot(x)\, y^2 + 3y\, x^2 = c_1$

- Solve for y

  $\left\{y = \frac{3\tan(x)x^2}{2} - \frac{\sqrt{9\tan(x)^2x^4 - 4\tan(x)c_1 - 8\tan(x)x}}{2}, y = \frac{3\tan(x)x^2}{2} + \frac{\sqrt{9\tan(x)^2x^4 - 4\tan(x)c_1 - 8\tan(x)x}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
dsolve(y(x)^2*csc(x)^2+6*x*y(x)-2=(2*y(x)*cot(x)-3*x^2)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{3\tan(x)\,x^2}{2} - \frac{\sqrt{\tan(x)\,(9\tan(x)\,x^4 + 4c_1 - 8x)}}{2}$$
$$y(x) = \frac{3\tan(x)\,x^2}{2} + \frac{\sqrt{\tan(x)\,(9\tan(x)\,x^4 + 4c_1 - 8x)}}{2}$$

✓ Solution by Mathematica

Time used: 31.692 (sec). Leaf size: 201

```
DSolve[y[x]^2*Csc[x]^2+6*x*y[x]-2==(2*y[x]*Cot[x]-3*x^2)*y'[x],y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{3}{2}x^2\tan(x) - \frac{\csc(2x)\sqrt{-\left(\tan(x)\left(16\cos^2(x)\arcsin\left(\sqrt{\sin^2(x)}\right) - 9x^4e^{\text{arctanh}(\cos(2x))} + \cos(2x)\left(9x^4e^{\text{arctanh}(\cos(2x))}\right.\right.\right.}}{2\sqrt{\csc(2x)e^{\text{arctanh}(\cos(2x))}}}$$

$$y(x) \to \frac{3}{2}x^2\tan(x) + \frac{\csc(2x)\sqrt{-\left(\tan(x)\left(16\cos^2(x)\arcsin\left(\sqrt{\sin^2(x)}\right) - 9x^4e^{\text{arctanh}(\cos(2x))} + \cos(2x)\left(9x^4e^{\text{arctanh}(\cos(2x))}\right.\right.\right.}}{2\sqrt{\csc(2x)e^{\text{arctanh}(\cos(2x))}}}$$

4.16 problem 17

Internal problem ID [1958]
Internal file name [`OUTPUT/1958_Sunday_February_25_2024_06_40_26_AM_44326358/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational]
```

$$\boxed{\frac{2y}{x^3}+\frac{2x}{y^2}-\left(\frac{1}{x^2}+\frac{2x^2}{y^3}\right)y'=0}$$

4.16.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{2y(x^4+y^3)}{x\,(2x^4+y^3)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{2y(x^4+y^3)(b_3-a_2)}{x(2x^4+y^3)} - \frac{4y^2(x^4+y^3)^2 a_3}{x^2(2x^4+y^3)^2} \\
&- \left(-\frac{2y(x^4+y^3)}{x^2(2x^4+y^3)} + \frac{8y\,x^2}{2x^4+y^3} - \frac{16y\,x^2(x^4+y^3)}{(2x^4+y^3)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{2x^4+2y^3}{x(2x^4+y^3)} + \frac{6y^3}{x(2x^4+y^3)} - \frac{6y^3(x^4+y^3)}{x(2x^4+y^3)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4x^9b_1 - 4x^8ya_1 + 8x^6y^3b_2 - 8x^5y^4a_2 + 6x^5y^4b_3 - 6x^4y^5a_3 + 12x^5y^3b_1 - 14x^4y^4a_1 + x^2y^6b_2 + 2y^8a_3 + 2}{x^2(2x^4+y^3)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-4x^9b_1 + 4x^8ya_1 - 8x^6y^3b_2 + 8x^5y^4a_2 - 6x^5y^4b_3 + 6x^4y^5a_3 \\
&- 12x^5y^3b_1 + 14x^4y^4a_1 - x^2y^6b_2 - 2y^8a_3 - 2x\,y^6b_1 + 2y^7a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&4a_1v_1^8v_2 + 8a_2v_1^5v_2^4 + 6a_3v_1^4v_2^5 - 4b_1v_1^9 - 8b_2v_1^6v_2^3 - 6b_3v_1^5v_2^4 \\
&+ 14a_1v_1^4v_2^4 - 2a_3v_2^8 - 12b_1v_1^5v_2^3 - b_2v_1^2v_2^6 + 2a_1v_2^7 - 2b_1v_1v_2^6 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} -4b_1v_1^9 + 4a_1v_1^8v_2 - 8b_2v_1^6v_2^3 + (8a_2 - 6b_3)\, v_1^5v_2^4 - 12b_1v_1^5v_2^3 \\ + 6a_3v_1^4v_2^5 + 14a_1v_1^4v_2^4 - b_2v_1^2v_2^6 - 2b_1v_1v_2^6 - 2a_3v_2^8 + 2a_1v_2^7 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ 4a_1 &= 0 \\ 14a_1 &= 0 \\ -2a_3 &= 0 \\ 6a_3 &= 0 \\ -12b_1 &= 0 \\ -4b_1 &= 0 \\ -2b_1 &= 0 \\ -8b_2 &= 0 \\ -b_2 &= 0 \\ 8a_2 - 6b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= \frac{4a_2}{3} \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= \frac{4y}{3} \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= \frac{4y}{3} - \left(\frac{2y(x^4+y^3)}{x\,(2x^4+y^3)}\right)(x) \\
&= \frac{2x^4y - 2y^4}{6x^4+3y^3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2x^4y-2y^4}{6x^4+3y^3}} dy
\end{aligned}$$

Which results in

$$S = -\frac{3\ln\left(-x^4+y^3\right)}{2} + 3\ln\left(y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y(x^4+y^3)}{x\,(2x^4+y^3)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{6x^3}{x^4 - y^3} \\
S_y &= \frac{9y^2}{2x^4 - 2y^3} + \frac{3}{y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{3}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{3}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -3\ln(R) + c_1 \tag{4}
$$

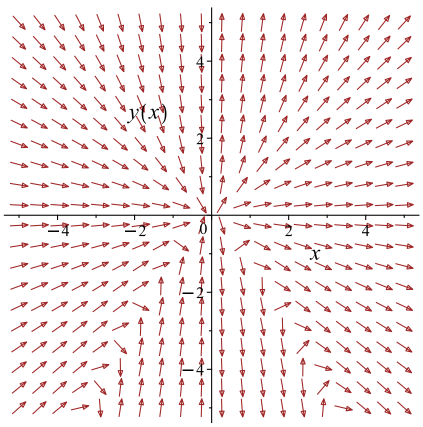
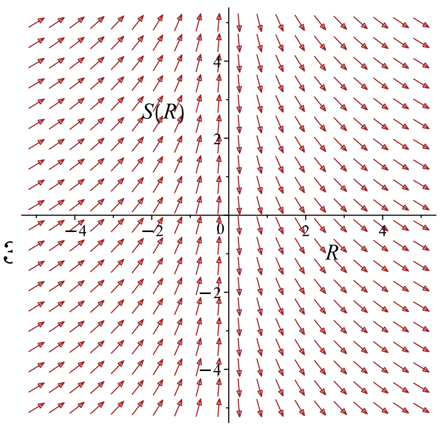
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{3\ln(-x^4 + y^3)}{2} + 3\ln(y) = -3\ln(x) + c_1
$$

Which simplifies to

$$
-\frac{3\ln(-x^4 + y^3)}{2} + 3\ln(y) = -3\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y(x^4+y^3)}{x(2x^4+y^3)}$ 	$R = x$ $S = -\frac{3\ln(-x^4+y^3)}{2} + 3$	$\frac{dS}{dR} = -\frac{3}{R}$ 

Summary
The solution(s) found are the following

$$-\frac{3\ln(-x^4+y^3)}{2} + 3\ln(y) = -3\ln(x) + c_1 \tag{1}$$

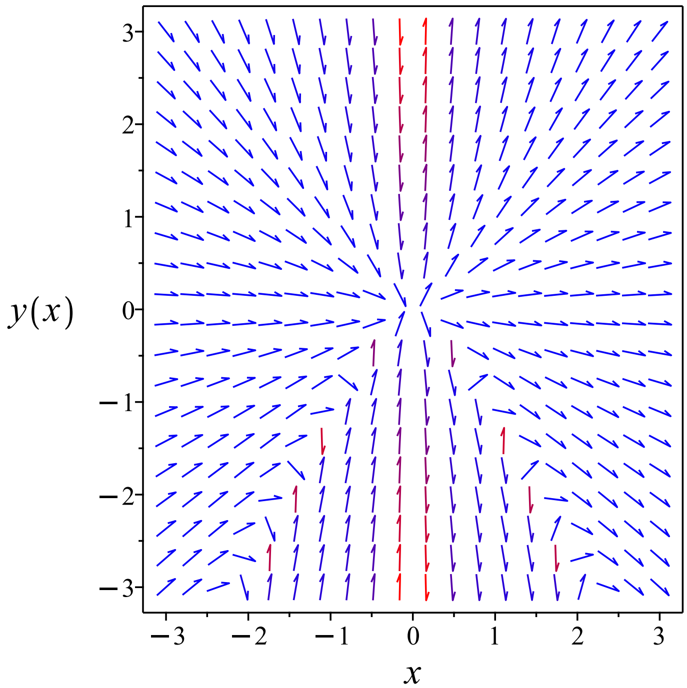


Figure 218: Slope field plot

Verification of solutions

$$-\frac{3\ln\left(-x^4+y^3\right)}{2}+3\ln\left(y\right)=-3\ln\left(x\right)+c_1$$

Verified OK.

4.16.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{x^2} - \frac{2x^2}{y^3}\right)\mathrm{d}y &= \left(-\frac{2y}{x^3} - \frac{2x}{y^2}\right)\mathrm{d}x \\ \left(\frac{2y}{x^3} + \frac{2x}{y^2}\right)\mathrm{d}x + \left(-\frac{1}{x^2} - \frac{2x^2}{y^3}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \frac{2y}{x^3} + \frac{2x}{y^2} \\ N(x,y) &= -\frac{1}{x^2} - \frac{2x^2}{y^3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2y}{x^3} + \frac{2x}{y^2}\right) \\ &= \frac{2}{x^3} - \frac{4x}{y^3}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{x^2} - \frac{2x^2}{y^3}\right) \\
&= \frac{2}{x^3} - \frac{4x}{y^3}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2y}{x^3} + \frac{2x}{y^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{y}{x^2} + \frac{x^2}{y^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{1}{x^2} - \frac{2x^2}{y^3} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{x^2} - \frac{2x^2}{y^3}$. Therefore equation (4) becomes

$$
-\frac{1}{x^2} - \frac{2x^2}{y^3} = -\frac{1}{x^2} - \frac{2x^2}{y^3} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y}{x^2} + \frac{x^2}{y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y}{x^2} + \frac{x^2}{y^2}$$

Summary
The solution(s) found are the following

$$-\frac{y}{x^2} + \frac{x^2}{y^2} = c_1 \tag{1}$$

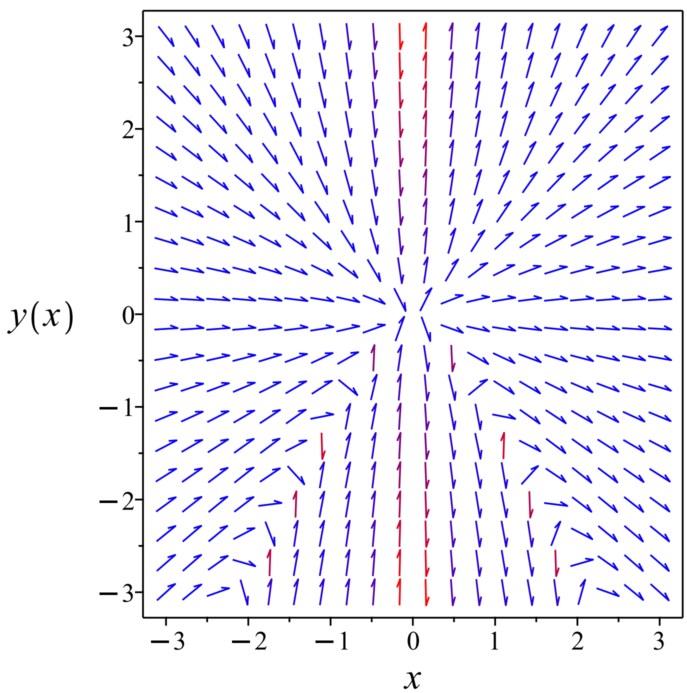


Figure 219: Slope field plot

Verification of solutions

$$-\frac{y}{x^2} + \frac{x^2}{y^2} = c_1$$

Verified OK.

4.16.3 Maple step by step solution

Let's solve

$\frac{2y}{x^3}+\frac{2x}{y^2}-\left(\frac{1}{x^2}+\frac{2x^2}{y^3}\right)y'=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $\frac{2}{x^3}-\frac{4x}{y^3}=\frac{2}{x^3}-\frac{4x}{y^3}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(\frac{2y}{x^3}+\frac{2x}{y^2}\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=-\frac{y}{x^2}+\frac{x^2}{y^2}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-\frac{1}{x^2}-\frac{2x^2}{y^3}=-\frac{1}{x^2}-\frac{2x^2}{y^3}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=0$

- Solve for $f_1(y)$

  $f_1(y)=0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = -\frac{y}{x^2} + \frac{x^2}{y^2}$
- Substitute $F(x, y)$ into the solution of the ODE

  $-\frac{y}{x^2} + \frac{x^2}{y^2} = c_1$
- Solve for y

  $$\left\{y = \left(\frac{\left(108x - 8c_1^3x^3 + 12\sqrt{-12c_1^3x^4 + 81x^2}\right)^{\frac{1}{3}}}{6} + \frac{2c_1^2x^2}{3\left(108x - 8c_1^3x^3 + 12\sqrt{-12c_1^3x^4 + 81x^2}\right)^{\frac{1}{3}}} - \frac{c_1x}{3}\right)x, y = \left(-\frac{\left(108x - \right.}{}\right.\right.$$

<u>Maple trace</u>

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ <u>Solution by Maple</u>

Time used: 0.625 (sec). Leaf size: 231

```
dsolve(2*(y(x)/x^3+x/y(x)^2)=(1/x^2+2*x^2/y(x)^3)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{x^{\frac{4}{3}} \operatorname{RootOf}\left(x^{14}\_Z^{18} + 6x^{\frac{38}{3}}\_Z^{16} + 15x^{\frac{34}{3}}\_Z^{14} + (20x^{10} - 27c_1x^8)\_Z^{12} + \left(15x^{\frac{26}{3}} - 81x^{\frac{20}{3}}c_1\right)\_Z^{10} + \left(\right.\right.}{\operatorname{RootOf}\left(x^{14}\_Z^{18} + 6x^{\frac{38}{3}}\_Z^{16} + 15x^{\frac{34}{3}}\_Z^{14} + (20x^{10} - 27c_1x^8)\_Z^{12} + \left(15x^{\frac{26}{3}} - 81x^{\frac{20}{3}}c_1\right)\_Z^{10} + \right.}$$

✓ Solution by Mathematica

Time used: 10.8 (sec). Leaf size: 414

```
DSolve[2*(y[x]/x^3+x/y[x]^2)==(1/x^2+2*x^2/y[x]^3)*y'[x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{3}\left(c_1x^2 + \frac{c_1{}^2x^4}{\sqrt[3]{c_1{}^3x^6 + \frac{27x^4}{2} + \frac{3}{2}\sqrt{3}\sqrt{x^8\left(27 + 4c_1{}^3x^2\right)}}} + \sqrt[3]{c_1{}^3x^6 + \frac{27x^4}{2} + \frac{3}{2}\sqrt{3}\sqrt{x^8\left(27 + 4c_1{}^3x^2\right)}}\right)$$

$$y(x) \to \frac{1}{12}\left(4c_1x^2 - \frac{2\left(1 + i\sqrt{3}\right)c_1{}^2x^4}{\sqrt[3]{c_1{}^3x^6 + \frac{27x^4}{2} + \frac{3}{2}\sqrt{3}\sqrt{x^8\left(27 + 4c_1{}^3x^2\right)}}} + i2^{2/3}\left(\sqrt{3} + i\right)\sqrt[3]{2c_1{}^3x^6 + 27x^4 + 3\sqrt{3}\sqrt{x^8\left(27 + 4c_1{}^3x^2\right)}}\right)$$

$$y(x) \to \frac{1}{12}\left(4c_1x^2 + \frac{2i\left(\sqrt{3} + i\right)c_1{}^2x^4}{\sqrt[3]{c_1{}^3x^6 + \frac{27x^4}{2} + \frac{3}{2}\sqrt{3}\sqrt{x^8\left(27 + 4c_1{}^3x^2\right)}}} - 2^{2/3}\left(1 + i\sqrt{3}\right)\sqrt[3]{2c_1{}^3x^6 + 27x^4 + 3\sqrt{3}\sqrt{x^8\left(27 + 4c_1{}^3x^2\right)}}\right)$$

4.17 problem 18

Internal problem ID [1959]
Internal file name [`OUTPUT/1959_Sunday_February_25_2024_06_40_28_AM_85370397/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{\cos(y) - (\sin(y)\, x - y^2)\, y' = 0}$$

4.17.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x\sin(y)+y^2)\,\mathrm{d}y&=(-\cos(y))\,\mathrm{d}x\\(\cos(y))\,\mathrm{d}x+(-x\sin(y)+y^2)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\cos(y)\\N(x,y)&=-x\sin(y)+y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(\cos(y))\\&=-\sin(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(-x\sin(y)+y^2)\\&=-\sin(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \cos(y) \, \mathrm{d}x$$
$$\phi = x\cos(y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x\sin(y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x\sin(y) + y^2$. Therefore equation (4) becomes

$$-x\sin(y) + y^2 = -x\sin(y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^2) \, \mathrm{d}y$$
$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\cos(y) + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\cos(y) + \frac{y^3}{3}$$

Summary
The solution(s) found are the following

$$x\cos(y)+\frac{y^3}{3}=c_1 \tag{1}$$

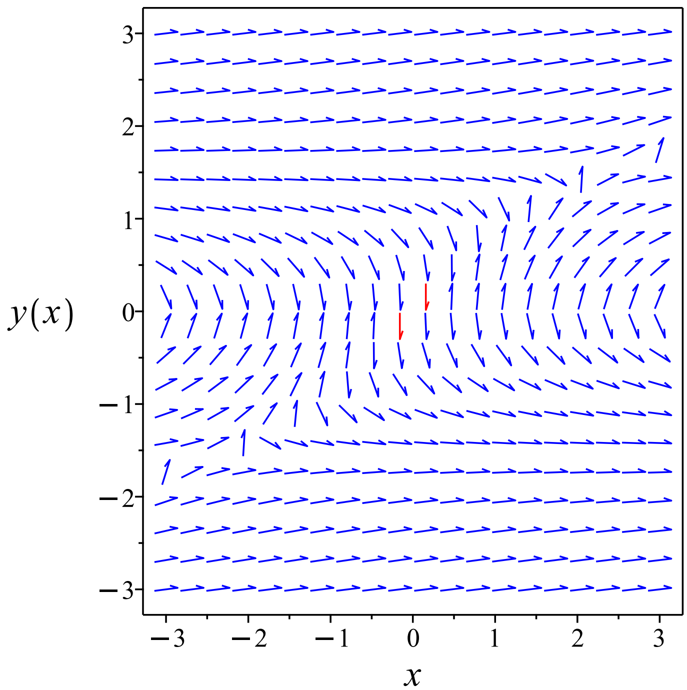


Figure 220: Slope field plot

Verification of solutions

$$x\cos(y)+\frac{y^3}{3}=c_1$$

Verified OK.

4.17.2 Maple step by step solution

Let's solve

$\cos(y)-(\sin(y)\,x-y^2)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $-\sin(y) = -\sin(y)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \cos(y) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x\cos(y) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $-x\sin(y) + y^2 = -x\sin(y) + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = y^2$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^3}{3}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = x\cos(y) + \frac{y^3}{3}$

- Substitute $F(x, y)$ into the solution of the ODE

  $x\cos(y) + \frac{y^3}{3} = c_1$

- Solve for y

  $y = \mathit{RootOf}\left(-\_Z^3 - 3x\cos(\_Z) + 3c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 18

```
dsolve(cos(y(x))-(x*sin(y(x))-y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$x+\frac{\sec\left(y(x)\right)\left(y(x)^{3}-3c_{1}\right)}{3}=0$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 23

```
DSolve[Cos[y[x]]-(x*Sin[y[x]]-y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x=-\frac{1}{3}y(x)^{3}\sec(y(x))+c_{1}\sec(y(x)),y(x)\right]$$

4.18 problem 19

Internal problem ID [1960]
Internal file name [`OUTPUT/1960_Sunday_February_25_2024_06_40_29_AM_76101662/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{2y\sin(yx) + \left(2x\sin(yx) + y^3\right)y' = 0}$$

4.18.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x\sin(yx)+y^3)\,\mathrm{d}y&=(-2y\sin(yx))\,\mathrm{d}x\\(2y\sin(yx))\,\mathrm{d}x+(2x\sin(yx)+y^3)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2y\sin(yx)\\N(x,y)&=2x\sin(yx)+y^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(2y\sin(yx))\\&=2\sin(yx)+2xy\cos(yx)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(2x\sin(yx)+y^3)\\&=2\sin(yx)+2xy\cos(yx)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2y \sin(yx) \, \mathrm{d}x$$

$$\phi = -2\cos(yx) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x \sin(yx) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x \sin(yx) + y^3$. Therefore equation (4) becomes

$$2x \sin(yx) + y^3 = 2x \sin(yx) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^3) \, \mathrm{d}y$$

$$f(y) = \frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\cos(yx) + \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\cos(yx) + \frac{y^4}{4}$$

Summary
The solution(s) found are the following

$$-2\cos(yx) + \frac{y^4}{4} = c_1 \tag{1}$$

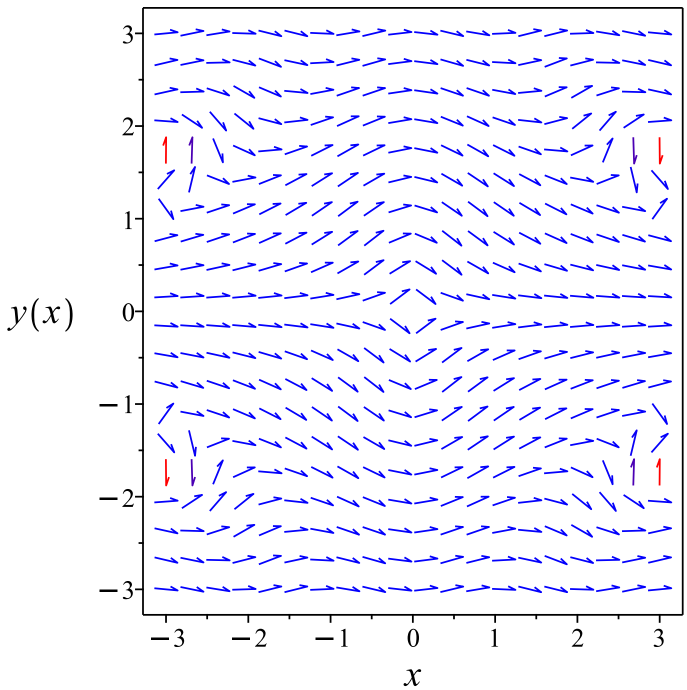


Figure 221: Slope field plot

Verification of solutions

$$-2\cos(yx) + \frac{y^4}{4} = c_1$$

Verified OK.

4.18.2 Maple step by step solution

Let's solve

$2y\sin(yx) + (2x\sin(yx) + y^3)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y) = 0$
- Compute derivative of lhs

  $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$
- Evaluate derivatives

  $2\sin(yx) + 2xy\cos(yx) = 2\sin(yx) + 2xy\cos(yx)$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int 2y\sin(yx)\,dx + f_1(y)$
- Evaluate integral

  $F(x,y) = -2\cos(yx) + f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $2x\sin(yx) + y^3 = 2x\sin(yx) + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = y^3$
- Solve for $f_1(y)$

  $f_1(y) = \frac{y^4}{4}$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = -2\cos(yx) + \frac{y^4}{4}$
- Substitute $F(x,y)$ into the solution of the ODE

  $-2\cos(yx) + \frac{y^4}{4} = c_1$
- Solve for y

  $y = \frac{\mathit{RootOf}\left(8\cos(\_Z)x^4 + 4c_1x^4 - \_Z^4\right)}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 26

```
dsolve(2*y(x)*sin(x*y(x))+(2*x*sin(x*y(x))+y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\text{RootOf}\left(-8\cos\left(\_Z\right)x^4 + 4c_1x^4 + \_Z^4\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.158 (sec). Leaf size: 22

```
DSolve[2*y[x]*Sin[x*y[x]]+(2*x*Sin[x*y[x]]+y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$\text{Solve}\left[\frac{y(x)^4}{4} - 2\cos(xy(x)) = c_1, y(x)\right]$$

4.19 problem 20

Internal problem ID [1961]
Internal file name [`OUTPUT/1961_Sunday_February_25_2024_06_43_22_AM_29961136/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\frac{x\cos\left(\frac{x}{y}\right)}{y}+\sin\left(\frac{x}{y}\right)-\frac{x^{2}\cos\left(\frac{x}{y}\right)y'}{y^{2}}=-\cos\left(x\right)}$$

4.19.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{x^2\cos\left(\frac{x}{y}\right)}{y^2}\right)\mathrm{d}y = \left(-\frac{x\cos\left(\frac{x}{y}\right)}{y} - \sin\left(\frac{x}{y}\right) - \cos(x)\right)\mathrm{d}x$$

$$\left(\frac{x\cos\left(\frac{x}{y}\right)}{y} + \sin\left(\frac{x}{y}\right) + \cos(x)\right)\mathrm{d}x + \left(-\frac{x^2\cos\left(\frac{x}{y}\right)}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{x\cos\left(\frac{x}{y}\right)}{y} + \sin\left(\frac{x}{y}\right) + \cos(x)$$
$$N(x, y) = -\frac{x^2\cos\left(\frac{x}{y}\right)}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x\cos\left(\frac{x}{y}\right)}{y} + \sin\left(\frac{x}{y}\right) + \cos(x)\right)$$
$$= \frac{x\left(x\sin\left(\frac{x}{y}\right) - 2\cos\left(\frac{x}{y}\right)y\right)}{y^3}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{x^2\cos\left(\frac{x}{y}\right)}{y^2}\right)$$
$$= \frac{x\left(x\sin\left(\frac{x}{y}\right) - 2\cos\left(\frac{x}{y}\right)y\right)}{y^3}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{x\cos\left(\frac{x}{y}\right)}{y} + \sin\left(\frac{x}{y}\right) + \cos(x)\,\mathrm{d}x$$
$$\phi = x\sin\left(\frac{x}{y}\right) + \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x^2\cos\left(\frac{x}{y}\right)}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x^2 \cos\left(\frac{x}{y}\right)}{y^2}$. Therefore equation (4) becomes

$$-\frac{x^2 \cos\left(\frac{x}{y}\right)}{y^2} = -\frac{x^2 \cos\left(\frac{x}{y}\right)}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x \sin\left(\frac{x}{y}\right) + \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x \sin\left(\frac{x}{y}\right) + \sin(x)$$

Summary

The solution(s) found are the following

$$x \sin\left(\frac{x}{y}\right) + \sin(x) = c_1 \tag{1}$$

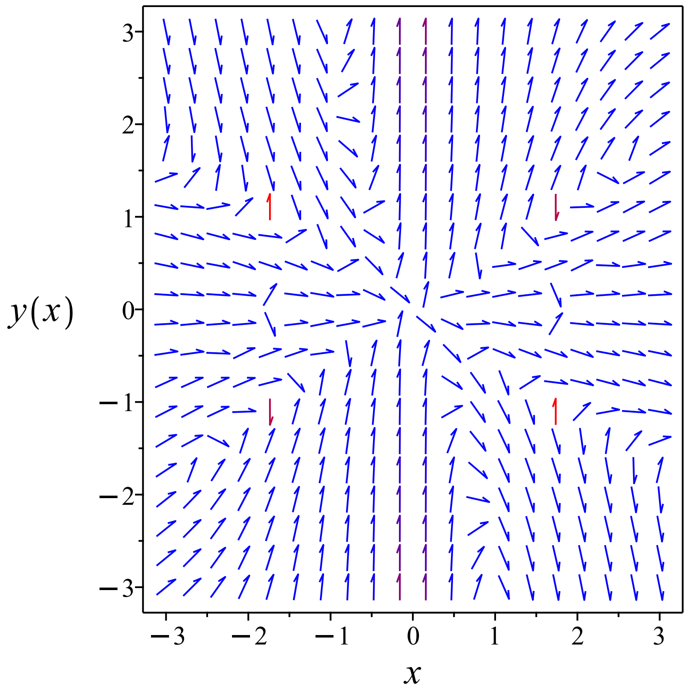


Figure 222: Slope field plot

Verification of solutions

$$x \sin\left(\frac{x}{y}\right)+\sin\left(x\right)=c_1$$

Verified OK.

4.19.2 Maple step by step solution

Let's solve

$\frac{x\cos\left(\frac{x}{y}\right)}{y}+\sin\left(\frac{x}{y}\right)-\frac{x^2\cos\left(\frac{x}{y}\right)y'}{y^2}=-\cos\left(x\right)$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$
  - Compute derivative of lhs

$F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

- Evaluate derivatives

  $-\frac{2x\cos\left(\frac{x}{y}\right)}{y^2} + \frac{x^2\sin\left(\frac{x}{y}\right)}{y^3} = -\frac{2x\cos\left(\frac{x}{y}\right)}{y^2} + \frac{x^2\sin\left(\frac{x}{y}\right)}{y^3}$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int \left(\frac{x\cos\left(\frac{x}{y}\right)}{y} + \sin\left(\frac{x}{y}\right) + \cos(x)\right) dx + f_1(y)$

- Evaluate integral

  $F(x,y) = y\left(\cos\left(\frac{x}{y}\right) + \frac{x\sin\left(\frac{x}{y}\right)}{y}\right) + \sin(x) - \cos\left(\frac{x}{y}\right) y + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-\frac{x^2\cos\left(\frac{x}{y}\right)}{y^2} = -\frac{x^2\cos\left(\frac{x}{y}\right)}{y^2} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = y\left(\cos\left(\frac{x}{y}\right) + \frac{x\sin\left(\frac{x}{y}\right)}{y}\right) + \sin(x) - \cos\left(\frac{x}{y}\right) y$

- Substitute $F(x,y)$ into the solution of the ODE

  $y\left(\cos\left(\frac{x}{y}\right) + \frac{x\sin\left(\frac{x}{y}\right)}{y}\right) + \sin(x) - \cos\left(\frac{x}{y}\right) y = c_1$

- Solve for y

  $y = \frac{x}{\arcsin\left(\frac{-\sin(x)+c_1}{x}\right)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve((x/y(x)*cos(x/y(x))+sin(x/y(x))+cos(x) )-x^2/y(x)^2*cos(x/y(x))*diff(y(x),x)=0,y(x),
```

$$y(x) = -\frac{x}{\arcsin\left(\frac{\sin(x)+c_1}{x}\right)}$$

✓ Solution by Mathematica

Time used: 25.647 (sec). Leaf size: 25

```
DSolve[(x/y[x]*Cos[x/y[x]]+Sin[x/y[x]]+Cos[x] )-x^2/y[x]^2*Cos[x/y[x]]*y'[x]==0,y[x],x,Inclu
```

$$y(x) \to -\frac{x}{\arcsin\left(\frac{\sin(x)+c_1}{x}\right)}$$
$$y(x) \to 0$$

4.20 problem 21

Internal problem ID [1962]
Internal file name [`OUTPUT/1962_Sunday_February_25_2024_06_43_27_AM_9783746/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{e^{yx} y + 2yx + \left(e^{yx} x + x^2\right) y' = 0}$$

4.20.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^{yx}x + x^2)\,\mathrm{d}y &= (-\mathrm{e}^{yx}y - 2yx)\,\mathrm{d}x\\ (\mathrm{e}^{yx}y + 2yx)\,\mathrm{d}x + (\mathrm{e}^{yx}x + x^2)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \mathrm{e}^{yx}y + 2yx\\ N(x, y) &= \mathrm{e}^{yx}x + x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\mathrm{e}^{yx}y + 2yx)\\ &= \mathrm{e}^{yx}yx + \mathrm{e}^{yx} + 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^{yx}x + x^2)\\ &= \mathrm{e}^{yx}yx + \mathrm{e}^{yx} + 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \mathrm{e}^{yx} y + 2yx \, \mathrm{d}x$$
$$\phi = \mathrm{e}^{yx} + y x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{yx} x + x^2 + f'(y) \tag{4}$$
$$= x(x + \mathrm{e}^{yx}) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{yx} x + x^2$. Therefore equation (4) becomes

$$\mathrm{e}^{yx} x + x^2 = x(x + \mathrm{e}^{yx}) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{yx} + y x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{yx} + y x^2$$

The solution becomes

$$y = \frac{-\operatorname{LambertW}\left(\frac{\mathrm{e}^{\frac{c_1}{x}}}{x}\right) + \frac{c_1}{x}}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{-\operatorname{LambertW}\left(\frac{e^{\frac{c_1}{x}}}{x}\right) + \frac{c_1}{x}}{x} \tag{1}$$

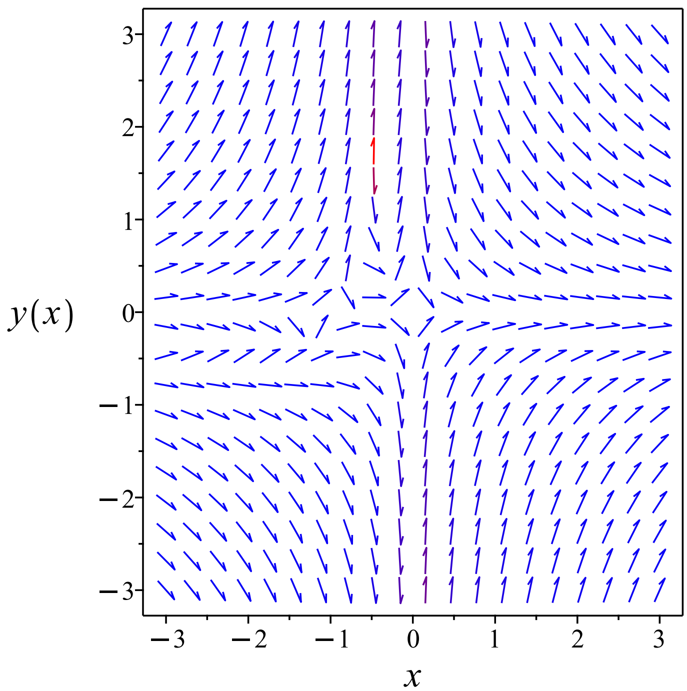


Figure 223: Slope field plot

Verification of solutions

$$y = \frac{-\operatorname{LambertW}\left(\frac{e^{\frac{c_1}{x}}}{x}\right) + \frac{c_1}{x}}{x}$$

Verified OK.

4.20.2 Maple step by step solution

Let's solve

$e^{yx}y + 2yx + (e^{yx}x + x^2)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $e^{yx} yx + e^{yx} + 2x = e^{yx} yx + e^{yx} + 2x$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (e^{yx} y + 2yx) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = e^{yx} + y x^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $e^{yx} x + x^2 = e^{yx} x + x^2 + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = e^{yx} + y x^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $e^{yx} + y x^2 = c_1$

- Solve for y

  $y = -\frac{x \mathit{LambertW}\left(\frac{e^{\frac{c_1}{x}}}{x}\right) - c_1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve((y(x)*exp(x*y(x))+2*x*y(x))+(x*exp(x*y(x))+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x \,\mathrm{LambertW}\left(\frac{\mathrm{e}^{-\frac{c_1}{x}}}{x}\right) - c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 3.117 (sec). Leaf size: 28

```
DSolve[(y[x]*Exp[x*y[x]]+2*x*y[x])+(x*Exp[x*y[x]]+x^2)*y'[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{c_1 - xW\left(\frac{e^{\frac{c_1}{x}}}{x}\right)}{x^2}$$

4.21 problem 22

Internal problem ID [1963]
Internal file name [`OUTPUT/1963_Sunday_February_25_2024_06_43_29_AM_49525481/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _exact, _rational, _dAlembert]`

$$\boxed{\frac{x^2+3y^2}{x(3x^2+4y^2)}+\frac{(y^2+2x^2)\,y'}{y(3x^2+4y^2)}=0}$$

4.21.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{x^2+3u(x)^2\,x^2}{x\left(3x^2+4u(x)^2\,x^2\right)}+\frac{\left(u(x)^2\,x^2+2x^2\right)\left(u'(x)\,x+u(x)\right)}{u(x)\,x\left(3x^2+4u(x)^2\,x^2\right)}=0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{4u^3+3u}{x(u^2+2)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{4u^3+3u}{u^2+2}$. Integrating both sides gives

$$\frac{1}{\frac{4u^3+3u}{u^2+2}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{4u^3+3u}{u^2+2}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{2\ln(u)}{3} - \frac{5\ln(4u^2+3)}{24} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\frac{2\ln(u)}{3} - \frac{5\ln\left(4u^2+3\right)}{24}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u^{\frac{2}{3}}}{(4u^2+3)^{\frac{5}{24}}} = \frac{c_3}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\left(\frac{c_3 \operatorname{RootOf}\left(x^3\_Z^{24} - 4\_Z^{15}c_3^3 - 3x^3\right)^5}{x}\right)^{\frac{3}{2}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{c_3 \operatorname{RootOf}\left(x^3\_Z^{24} - 4\_Z^{15}c_3^3 - 3x^3\right)^5}{x}\right)^{\frac{3}{2}} \tag{1}$$

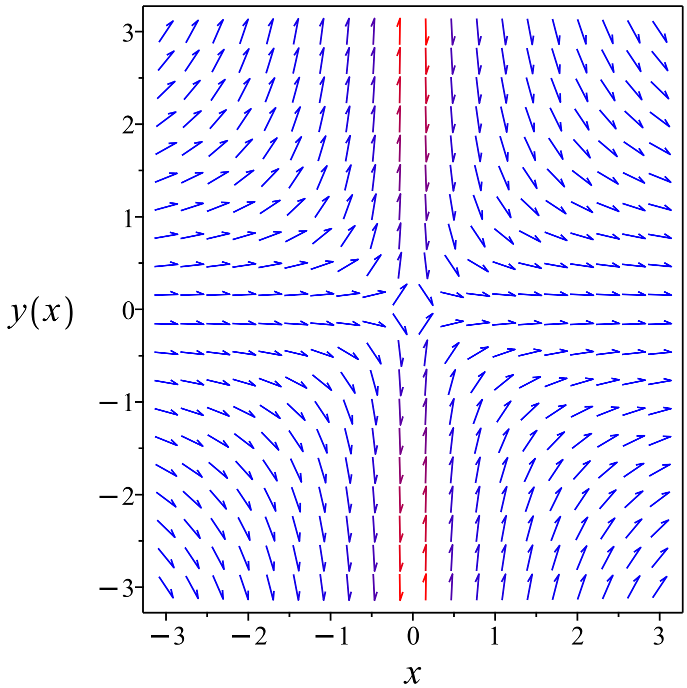


Figure 224: Slope field plot

Verification of solutions

$$y = x\left(\frac{c_3 \operatorname{RootOf}\left(x^3\_Z^{24} - 4\_Z^{15}c_3^3 - 3x^3\right)^5}{x}\right)^{\frac{3}{2}}$$

Verified OK.

4.21.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{(x^2 + 3y^2)\, y}{x\,(2x^2 + y^2)}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(x^2+3y^2)\, y(b_3-a_2)}{x\,(2x^2+y^2)} - \frac{(x^2+3y^2)^2\, y^2 a_3}{x^2\,(2x^2+y^2)^2} \\ & - \left(-\frac{2y}{2x^2+y^2} + \frac{(x^2+3y^2)\, y}{x^2\,(2x^2+y^2)} + \frac{4(x^2+3y^2)\, y}{(2x^2+y^2)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{6y^2}{x\,(2x^2+y^2)} + \frac{2(x^2+3y^2)\, y^2}{x\,(2x^2+y^2)^2} - \frac{x^2+3y^2}{(2x^2+y^2)\, x}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^6b_2 - 3x^4y^2a_3 + 21x^4y^2b_2 - 10x^3y^3a_2 + 10x^3y^3b_3 - 23x^2y^4a_3 + 4x^2y^4b_2 - 12y^6a_3 + 2x^5b_1 - 2x^4ya_1 + 1}{(2x^2+y^2)^2\, x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 6x^6b_2 - 3x^4y^2a_3 + 21x^4y^2b_2 - 10x^3y^3a_2 + 10x^3y^3b_3 - 23x^2y^4a_3 + 4x^2y^4b_2 \\ & - 12y^6a_3 + 2x^5b_1 - 2x^4ya_1 + 17x^3y^2b_1 - 17x^2y^3a_1 + 3x\,y^4b_1 - 3y^5a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -10a_2v_1^3v_2^3 - 3a_3v_1^4v_2^2 - 23a_3v_1^2v_2^4 - 12a_3v_2^6 + 6b_2v_1^6 + 21b_2v_1^4v_2^2 + 4b_2v_1^2v_2^4 \\ & + 10b_3v_1^3v_2^3 - 2a_1v_1^4v_2 - 17a_1v_1^2v_2^3 - 3a_1v_2^5 + 2b_1v_1^5 + 17b_1v_1^3v_2^2 + 3b_1v_1v_2^4 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$6b_2v_1^6 + 2b_1v_1^5 + (-3a_3 + 21b_2)\, v_1^4v_2^2 - 2a_1v_1^4v_2 + (-10a_2 + 10b_3)\, v_1^3v_2^3 + 17b_1v_1^3v_2^2 + (-23a_3 + 4b_2)\, v_1^2v_2^4 - 17a_1v_1^2v_2^3 + 3b_1v_1v_2^4 - 12a_3v_2^6 - 3a_1v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-17a_1 &= 0 \\
-3a_1 &= 0 \\
-2a_1 &= 0 \\
-12a_3 &= 0 \\
2b_1 &= 0 \\
3b_1 &= 0 \\
17b_1 &= 0 \\
6b_2 &= 0 \\
-10a_2 + 10b_3 &= 0 \\
-23a_3 + 4b_2 &= 0 \\
-3a_3 + 21b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{(x^2+3y^2)\,y}{x\,(2x^2+y^2)}\right)(x) \\
&= \frac{3y\,x^2+4y^3}{2x^2+y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{3y\,x^2+4y^3}{2x^2+y^2}}dy
\end{aligned}$$

Which results in

$$S = -\frac{5\ln(3x^2+4y^2)}{24} + \frac{2\ln(y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{(x^2+3y^2)\,y}{x\,(2x^2+y^2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{5x}{12x^2+16y^2}\\
S_y &= \frac{2x^2+y^2}{3y\,x^2+4y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{3}{4x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{3}{4R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{3\ln(R)}{4} + c_1 \tag{4}
$$

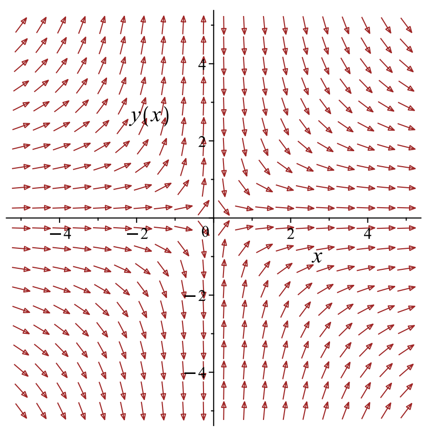
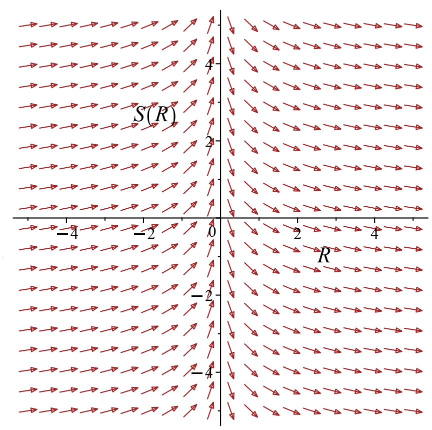
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{5\ln(3x^2+4y^2)}{24} + \frac{2\ln(y)}{3} = -\frac{3\ln(x)}{4} + c_1
$$

Which simplifies to

$$
-\frac{5\ln(3x^2+4y^2)}{24} + \frac{2\ln(y)}{3} = -\frac{3\ln(x)}{4} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(x^2+3y^2)y}{x(2x^2+y^2)}$ 	$R = x$ $S = -\frac{5\ln(3x^2+4y^2)}{24} +$	$\frac{dS}{dR} = -\frac{3}{4R}$ 

Summary

The solution(s) found are the following

$$-\frac{5\ln(3x^2+4y^2)}{24} + \frac{2\ln(y)}{3} = -\frac{3\ln(x)}{4} + c_1 \tag{1}$$

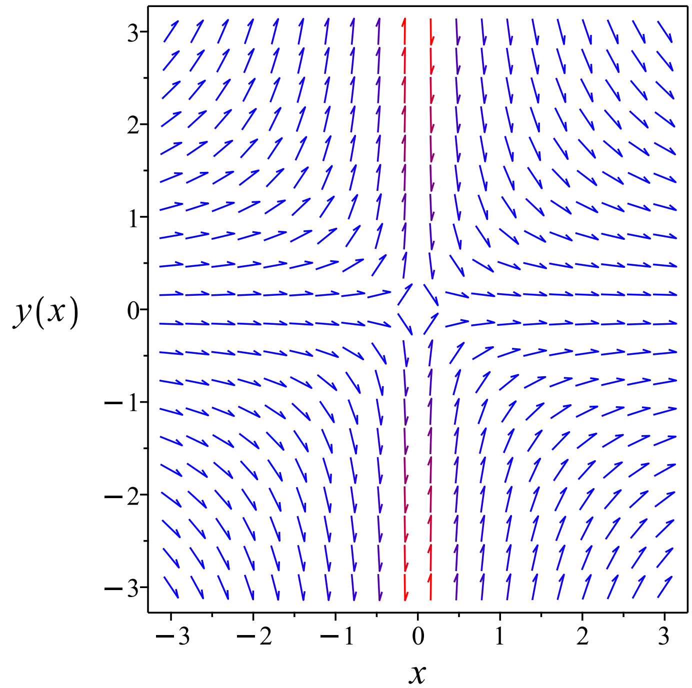


Figure 225: Slope field plot

Verification of solutions

$$-\frac{5\ln\left(3x^{2}+4y^{2}\right)}{24}+\frac{2\ln\left(y\right)}{3}=-\frac{3\ln\left(x\right)}{4}+c_{1}$$

Verified OK.

4.21.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{2x^2 + y^2}{y\,(3x^2 + 4y^2)}\right) \mathrm{d}y &= \left(-\frac{x^2 + 3y^2}{x\,(3x^2 + 4y^2)}\right) \mathrm{d}x \\
\left(\frac{x^2 + 3y^2}{x\,(3x^2 + 4y^2)}\right) \mathrm{d}x + \left(\frac{2x^2 + y^2}{y\,(3x^2 + 4y^2)}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \frac{x^2 + 3y^2}{x\,(3x^2 + 4y^2)} \\
N(x, y) &= \frac{2x^2 + y^2}{y\,(3x^2 + 4y^2)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^2 + 3y^2}{x\,(3x^2 + 4y^2)}\right) \\
&= \frac{10yx}{(3x^2 + 4y^2)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2x^2+y^2}{y\left(3x^2+4y^2\right)}\right) \\
&= \frac{10yx}{\left(3x^2+4y^2\right)^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{x^2+3y^2}{x\left(3x^2+4y^2\right)} \mathrm{d}x \\
\phi &= \frac{3\ln(x)}{4} - \frac{5\ln\left(3x^2+4y^2\right)}{24} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{5y}{3\left(3x^2+4y^2\right)} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2x^2+y^2}{y(3x^2+4y^2)}$. Therefore equation (4) becomes

$$
\frac{2x^2+y^2}{y\left(3x^2+4y^2\right)} = -\frac{5y}{9x^2+12y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{2}{3y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2}{3y}\right) \mathrm{d}y$$
$$f(y) = \frac{2\ln(y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3\ln(x)}{4} - \frac{5\ln(3x^2+4y^2)}{24} + \frac{2\ln(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3\ln(x)}{4} - \frac{5\ln(3x^2+4y^2)}{24} + \frac{2\ln(y)}{3}$$

Summary

The solution(s) found are the following

$$-\frac{5\ln(3x^2+4y^2)}{24} + \frac{2\ln(y)}{3} + \frac{3\ln(x)}{4} = c_1 \tag{1}$$

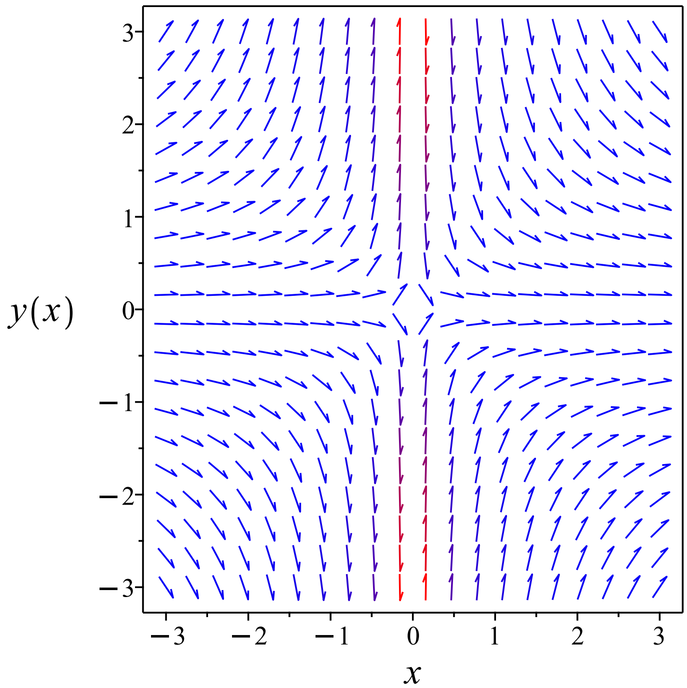


Figure 226: Slope field plot

Verification of solutions

$$-\frac{5\ln\left(3x^2+4y^2\right)}{24}+\frac{2\ln\left(y\right)}{3}+\frac{3\ln\left(x\right)}{4}=c_1$$

Verified OK.

4.21.4 Maple step by step solution

Let's solve

$\frac{x^2+3y^2}{x(3x^2+4y^2)}+\frac{(y^2+2x^2)y'}{y(3x^2+4y^2)}=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

- ◦ Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $$\frac{6y}{x(3x^2+4y^2)} - \frac{8(x^2+3y^2)y}{x(3x^2+4y^2)^2} = \frac{4x}{y(3x^2+4y^2)} - \frac{6(2x^2+y^2)x}{y(3x^2+4y^2)^2}$$

- Simplify

  $$\frac{10yx}{(3x^2+4y^2)^2} = \frac{10yx}{(3x^2+4y^2)^2}$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $$\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $$F(x,y) = \int \frac{x^2+3y^2}{x(3x^2+4y^2)}dx + f_1(y)$$

- Evaluate integral

  $$F(x,y) = \frac{3\ln(x)}{4} - \frac{5\ln(3x^2+4y^2)}{24} + f_1(y)$$

- Take derivative of $F(x,y)$ with respect to y

  $$N(x,y) = \frac{\partial}{\partial y}F(x,y)$$

- Compute derivative

  $$\frac{2x^2+y^2}{y(3x^2+4y^2)} = -\frac{5y}{3(3x^2+4y^2)} + \frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

  $$\frac{d}{dy}f_1(y) = \frac{2x^2+y^2}{y(3x^2+4y^2)} + \frac{5y}{3(3x^2+4y^2)}$$

- Solve for $f_1(y)$

  $$f_1(y) = \frac{2\ln(y)}{3}$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $$F(x,y) = \frac{3\ln(x)}{4} - \frac{5\ln(3x^2+4y^2)}{24} + \frac{2\ln(y)}{3}$$

- Substitute $F(x,y)$ into the solution of the ODE

  $$\frac{3\ln(x)}{4} - \frac{5\ln(3x^2+4y^2)}{24} + \frac{2\ln(y)}{3} = c_1$$

- Solve for y

  $$\left\{ y = -\frac{e^{-12c_1}\left(e^{\frac{4c_1}{3}}\right)^9 RootOf\left(x^9\_Z^{72} - 12x^{11}\_Z^{54} - 256\left(e^{\frac{4c_1}{3}}\right)^9\_Z^{45} + 54x^{13}\_Z^{36} - 108x^{15}\_Z^{18} + 81x^{17}\right)^9 \sqrt{-3x\left(-\right.}}{} \right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.437 (sec). Leaf size: 73

```
dsolve((x^2+3*y(x)^2)/(x*(3*x^2+4*y(x)^2))+(2*x^2+y(x)^2)/(y(x)*(3*x^2+4*y(x)^2))*diff(y(x),
```

$$y(x) = \operatorname{RootOf}\left(x^3 e^{3c_1}\_Z^{24} - 4\_Z^{15} - 3x^3 e^{3c_1}\right)^5 \sqrt{\frac{\operatorname{RootOf}\left(x^3 e^{3c_1}\_Z^{24} - 4\_Z^{15} - 3x^3 e^{3c_1}\right)^5}{x}}\, e^{-\frac{3c_1}{2}}$$

✓ Solution by Mathematica

Time used: 60.136 (sec). Leaf size: 1649

```
DSolve[(x^2+3*y[x]^2)/(x*(3*x^2+4*y[x]^2))+(2*x^2+y[x]^2)/(y[x]*(3*x^2+4*y[x]^2))*y'[x]==0,y
```

Too large to display

4.22 problem 23

Internal problem ID [1964]
Internal file name [`OUTPUT/1964_Sunday_February_25_2024_06_43_37_AM_26665600/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{\frac{x^2-y^2}{x\left(y^2+2x^2\right)}+\frac{\left(2y^2+x^2\right)y'}{y\left(y^2+2x^2\right)}=0}$$

4.22.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{x^2-u(x)^2\,x^2}{x\left(u\left(x\right)^2x^2+2x^2\right)}+\frac{\left(2u(x)^2\,x^2+x^2\right)\left(u'(x)\,x+u(x)\right)}{u\left(x\right)x\left(u\left(x\right)^2x^2+2x^2\right)}=0$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{u(u^2+2)}{x\left(2u^2+1\right)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2+2)}{2u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+2)}{2u^2+1}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u(u^2+2)}{2u^2+1}}\,du = \int -\frac{1}{x}\,dx$$
$$\frac{\ln(u)}{2} + \frac{3\ln(u^2+2)}{4} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln(u)}{2} + \frac{3\ln(u^2+2)}{4}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u}\,(u^2+2)^{\frac{3}{4}} = \frac{c_3}{x}$$

Therefore the solution y is

$$y = xu$$
$$= \frac{c_3^2}{x\,\mathrm{RootOf}\left(x^4\_Z^{16} - 2\_Z^{12}x^4 - c_3^4\right)^6}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3^2}{x\,\mathrm{RootOf}\left(x^4\_Z^{16} - 2\_Z^{12}x^4 - c_3^4\right)^6} \tag{1}$$

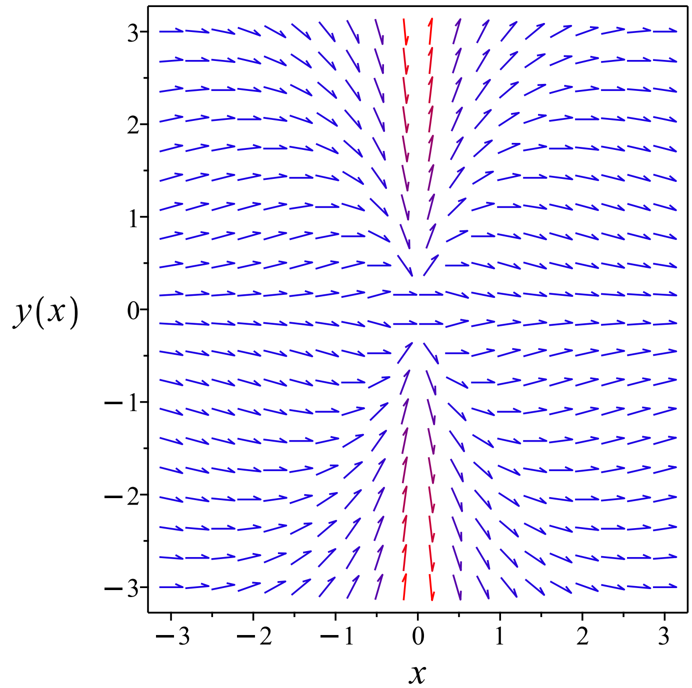


Figure 227: Slope field plot

Verification of solutions

$$y = \frac{c_3^2}{x \operatorname{RootOf}\left(x^4\_Z^{16} - 2\_Z^{12}x^4 - c_3^4\right)^6}$$

Verified OK.

4.22.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(-x^2 + y^2)}{x\,(x^2 + 2y^2)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{y(-x^2+y^2)\,(b_3-a_2)}{x\,(x^2+2y^2)} - \frac{y^2(-x^2+y^2)^2\,a_3}{x^2\,(x^2+2y^2)^2} \\
&- \left(-\frac{2y}{x^2+2y^2} - \frac{y(-x^2+y^2)}{x^2\,(x^2+2y^2)} - \frac{2y(-x^2+y^2)}{(x^2+2y^2)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{-x^2+y^2}{x\,(x^2+2y^2)} + \frac{2y^2}{x\,(x^2+2y^2)} - \frac{4y^2(-x^2+y^2)}{x\,(x^2+2y^2)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^6b_2 - 2x^4y^2a_3 - x^4y^2b_2 + 6x^3y^3a_2 - 6x^3y^3b_3 + 7x^2y^4a_3 + 2x^2y^4b_2 + y^6a_3 + x^5b_1 - x^4ya_1 - 5x^3y^2b_1 + 5}{(x^2+2y^2)^2\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&2x^6b_2 - 2x^4y^2a_3 - x^4y^2b_2 + 6x^3y^3a_2 - 6x^3y^3b_3 + 7x^2y^4a_3 + 2x^2y^4b_2 \\
&+ y^6a_3 + x^5b_1 - x^4ya_1 - 5x^3y^2b_1 + 5x^2y^3a_1 - 2x\,y^4b_1 + 2y^5a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&6a_2v_1^3v_2^3 - 2a_3v_1^4v_2^2 + 7a_3v_1^2v_2^4 + a_3v_2^6 + 2b_2v_1^6 - b_2v_1^4v_2^2 + 2b_2v_1^2v_2^4 \\
&- 6b_3v_1^3v_2^3 - a_1v_1^4v_2 + 5a_1v_1^2v_2^3 + 2a_1v_2^5 + b_1v_1^5 - 5b_1v_1^3v_2^2 - 2b_1v_1v_2^4 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^6 + b_1v_1^5 + (-2a_3 - b_2)\, v_1^4v_2^2 - a_1v_1^4v_2 + (6a_2 - 6b_3)\, v_1^3v_2^3 - 5b_1v_1^3v_2^2 + (7a_3 + 2b_2)\, v_1^2v_2^4 + 5a_1v_1^2v_2^3 - 2b_1v_1v_2^4 + a_3v_2^6 + 2a_1v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_3 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ 2a_1 &= 0 \\ 5a_1 &= 0 \\ -5b_1 &= 0 \\ -2b_1 &= 0 \\ 2b_2 &= 0 \\ 6a_2 - 6b_3 &= 0 \\ -2a_3 - b_2 &= 0 \\ 7a_3 + 2b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{y(-x^2+y^2)}{x\,(x^2+2y^2)}\right)(x) \\ &= \frac{2y\,x^2+y^3}{x^2+2y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2y\,x^2+y^3}{x^2+2y^2}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{2} + \frac{3\ln(2x^2+y^2)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(-x^2+y^2)}{x\,(x^2+2y^2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3x}{2x^2 + y^2} \\
S_y &= \frac{x^2 + 2y^2}{2y\,x^2 + y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

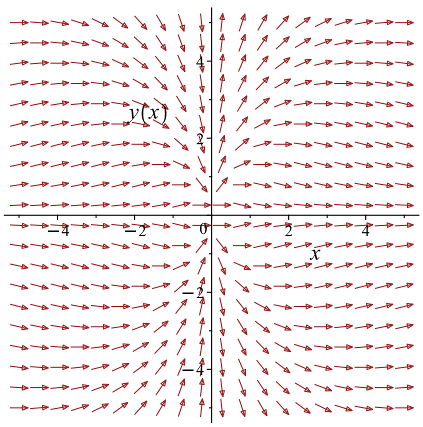
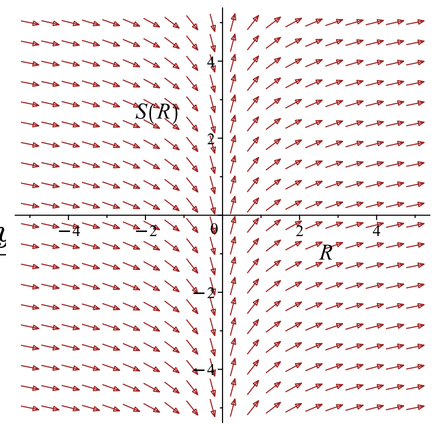
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y)}{2} + \frac{3\ln(y^2 + 2x^2)}{4} = \ln(x) + c_1
$$

Which simplifies to

$$
\frac{\ln(y)}{2} + \frac{3\ln(y^2 + 2x^2)}{4} = \ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(-x^2+y^2)}{x(x^2+2y^2)}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{3\ln(2x^2+y^2)}{4}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{2} + \frac{3\ln(y^2+2x^2)}{4} = \ln(x) + c_1 \tag{1}$$

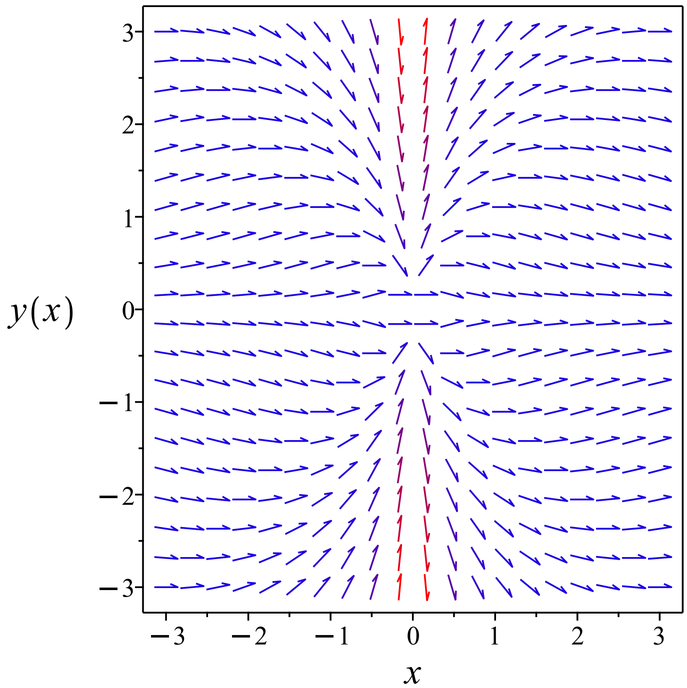


Figure 228: Slope field plot

<u>Verification of solutions</u>

$$\frac{\ln(y)}{2}+\frac{3\ln(y^2+2x^2)}{4}=\ln(x)+c_1$$

Verified OK.

4.22.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{x^2 + 2y^2}{y\,(2x^2 + y^2)}\right) \mathrm{d}y &= \left(-\frac{x^2 - y^2}{x\,(2x^2 + y^2)}\right) \mathrm{d}x \\
\left(\frac{x^2 - y^2}{x\,(2x^2 + y^2)}\right) \mathrm{d}x + \left(\frac{x^2 + 2y^2}{y\,(2x^2 + y^2)}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \frac{x^2 - y^2}{x\,(2x^2 + y^2)} \\
N(x, y) &= \frac{x^2 + 2y^2}{y\,(2x^2 + y^2)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^2 - y^2}{x\,(2x^2 + y^2)}\right) \\
&= -\frac{6yx}{(2x^2 + y^2)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x^2+2y^2}{y\left(2x^2+y^2\right)}\right)\\
&= -\frac{6yx}{\left(2x^2+y^2\right)^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2-y^2}{x\left(2x^2+y^2\right)}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + \frac{3\ln\left(2x^2+y^2\right)}{4} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{3y}{2\left(2x^2+y^2\right)} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2+2y^2}{y(2x^2+y^2)}$. Therefore equation (4) becomes

$$
\frac{x^2+2y^2}{y\left(2x^2+y^2\right)} = \frac{3y}{4x^2+2y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{3\ln(2x^2+y^2)}{4} + \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{3\ln(2x^2+y^2)}{4} + \frac{\ln(y)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{2} + \frac{3\ln(y^2+2x^2)}{4} - \ln(x) = c_1 \tag{1}$$

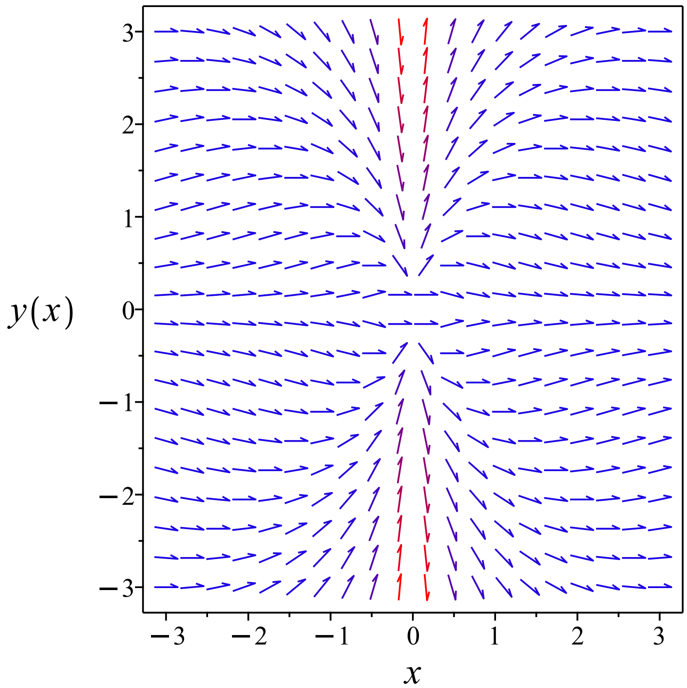


Figure 229: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{2} + \frac{3\ln(y^2 + 2x^2)}{4} - \ln(x) = c_1$$

Verified OK.

4.22.4 Maple step by step solution

Let's solve

$\frac{x^2-y^2}{x(y^2+2x^2)} + \frac{(2y^2+x^2)y'}{y(y^2+2x^2)} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives
  $-\frac{2y}{x(2x^2+y^2)}-\frac{2(x^2-y^2)y}{x(2x^2+y^2)^2}=\frac{2x}{y(2x^2+y^2)}-\frac{4(x^2+2y^2)x}{y(2x^2+y^2)^2}$
- Simplify
  $-\frac{6yx}{(2x^2+y^2)^2}=-\frac{6yx}{(2x^2+y^2)^2}$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form
  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x
  $F(x,y)=\int \frac{x^2-y^2}{x(2x^2+y^2)}dx+f_1(y)$
- Evaluate integral
  $F(x,y)=-\ln(x)+\frac{3\ln(2x^2+y^2)}{4}+f_1(y)$
- Take derivative of $F(x,y)$ with respect to y
  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative
  $\frac{x^2+2y^2}{y(2x^2+y^2)}=\frac{3y}{2(2x^2+y^2)}+\frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$
  $\frac{d}{dy}f_1(y)=\frac{x^2+2y^2}{y(2x^2+y^2)}-\frac{3y}{2(2x^2+y^2)}$
- Solve for $f_1(y)$
  $f_1(y)=\frac{\ln(y)}{2}$
- Substitute $f_1(y)$ into equation for $F(x,y)$
  $F(x,y)=-\ln(x)+\frac{3\ln(2x^2+y^2)}{4}+\frac{\ln(y)}{2}$
- Substitute $F(x,y)$ into the solution of the ODE
  $-\ln(x)+\frac{3\ln(2x^2+y^2)}{4}+\frac{\ln(y)}{2}=c_1$
- Solve for y
  $y=\frac{e^{2c_1}x^2}{RootOf\left(\_Z^{16}-2\_Z^{12}x^2-(e^{c_1})^4x^4\right)^6}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 6.063 (sec). Leaf size: 33

```
dsolve((x^2-y(x)^2)/(x*(2*x^2+y(x)^2))+(x^2+2*y(x)^2)/(y(x)*(2*x^2+y(x)^2))*diff(y(x),x)=0,y
```

$$y(x) = \frac{c_1 \operatorname{RootOf}\left(\_Z^{16} c_1^2 + 2x^4 \_Z^4 - x^4\right)^6}{x}$$

✓ Solution by Mathematica

Time used: 60.251 (sec). Leaf size: 3381

```
DSolve[(x^2-y[x]^2)/(x*(2*x^2+y[x]^2))+(x^2+2*y[x]^2)/(y[x]*(2*x^2+y[x]^2))*y'[x]==0,y[x],x,
```

Too large to display

4.23 problem 24

Internal problem ID [1965]
Internal file name [`OUTPUT/1965_Sunday_February_25_2024_06_43_48_AM_87658634/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 8, page 34
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\frac{2x^2}{x^2+y^2}+\ln\left(x^2+y^2\right)+\frac{2xyy'}{x^2+y^2}=0}$$

4.23.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{2yx}{x^2+y^2}\right)\mathrm{d}y &= \left(-\frac{2x^2}{x^2+y^2} - \ln\left(x^2+y^2\right)\right)\mathrm{d}x \\
\left(\frac{2x^2}{x^2+y^2} + \ln\left(x^2+y^2\right)\right)\mathrm{d}x + \left(\frac{2yx}{x^2+y^2}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \frac{2x^2}{x^2+y^2} + \ln\left(x^2+y^2\right) \\
N(x, y) &= \frac{2yx}{x^2+y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x^2}{x^2+y^2} + \ln\left(x^2+y^2\right)\right) \\
&= \frac{-2y\,x^2 + 2y^3}{\left(x^2+y^2\right)^2}
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2yx}{x^2+y^2}\right)\\ &= \frac{-2y\,x^2+2y^3}{\left(x^2+y^2\right)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2x^2}{x^2+y^2} + \ln\left(x^2+y^2\right)\mathrm{d}x\end{aligned}$$

$$\phi = \ln\left(x^2+y^2\right)x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2yx}{x^2+y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2yx}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{2yx}{x^2+y^2} = \frac{2yx}{x^2+y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln\left(x^2 + y^2\right) x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln\left(x^2 + y^2\right) x$$

Summary
The solution(s) found are the following

$$\ln\left(x^2 + y^2\right) x = c_1 \tag{1}$$

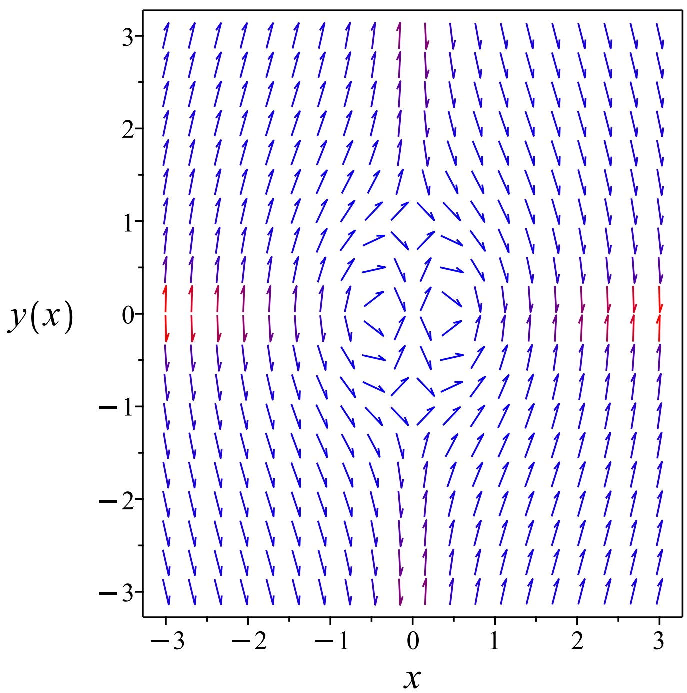


Figure 230: Slope field plot

Verification of solutions

$$\ln\left(x^2 + y^2\right) x = c_1$$

Verified OK.

4.23.2 Maple step by step solution

Let's solve

$\frac{2x^2}{x^2+y^2}+\ln(x^2+y^2)+\frac{2xyy'}{x^2+y^2}=0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $-\frac{4y\,x^2}{(x^2+y^2)^2}+\frac{2y}{x^2+y^2}=-\frac{4y\,x^2}{(x^2+y^2)^2}+\frac{2y}{x^2+y^2}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(\frac{2x^2}{x^2+y^2}+\ln(x^2+y^2)\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\ln(x^2+y^2)\,x+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\frac{2yx}{x^2+y^2}=\frac{2yx}{x^2+y^2}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=0$

- Solve for $f_1(y)$

  $f_1(y)=0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \ln(x^2+y^2)\,x$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\ln(x^2+y^2)\,x = c_1$$

- Solve for y

$$\left\{y = \sqrt{-x^2+\mathrm{e}^{\frac{c_1}{x}}}, y = -\sqrt{-x^2+\mathrm{e}^{\frac{c_1}{x}}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve((2*x^2/(x^2+y(x)^2)+ln(x^2+y(x)^2))+(2*x*y(x))/(x^2+y(x)^2)*diff(y(x),x)=0,y(x), sing
```

$$y(x) = \sqrt{-x^2+\mathrm{e}^{-\frac{c_1}{x}}}$$
$$y(x) = -\sqrt{-x^2+\mathrm{e}^{-\frac{c_1}{x}}}$$

✓ Solution by Mathematica

Time used: 0.823 (sec). Leaf size: 47

```
DSolve[(2*x^2/(x^2+y[x]^2)+Log[x^2+y[x]^2])+(2*x*y[x])/(x^2+y[x]^2)*y'[x]==0,y[x],x,IncludeS
```

$$y(x) \to -\sqrt{-x^2 + e^{\frac{c_1}{x}}}$$
$$y(x) \to \sqrt{-x^2 + e^{\frac{c_1}{x}}}$$

5 Exercise 9, page 38

5.1 problem 1

Internal problem ID [1966]
Internal file name [`OUTPUT/1966_Sunday_February_25_2024_06_43_49_AM_5528844/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y + y'x = -\ln(x)}$$

5.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = -\frac{\ln(x)}{x}$$

Hence the ode is

$$y' - \frac{y}{x} = -\frac{\ln(x)}{x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\frac{\ln(x)}{x}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x}\right)$$
$$d\left(\frac{y}{x}\right) = \left(-\frac{\ln(x)}{x^2}\right) dx$$

Integrating gives

$$\frac{y}{x} = \int -\frac{\ln(x)}{x^2} dx$$
$$\frac{y}{x} = \frac{\ln(x)}{x} + \frac{1}{x} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(\frac{\ln(x)}{x} + \frac{1}{x}\right) + c_1 x$$

which simplifies to

$$y = c_1 x + \ln(x) + 1$$

Summary

The solution(s) found are the following

$$y = c_1 x + \ln(x) + 1 \tag{1}$$

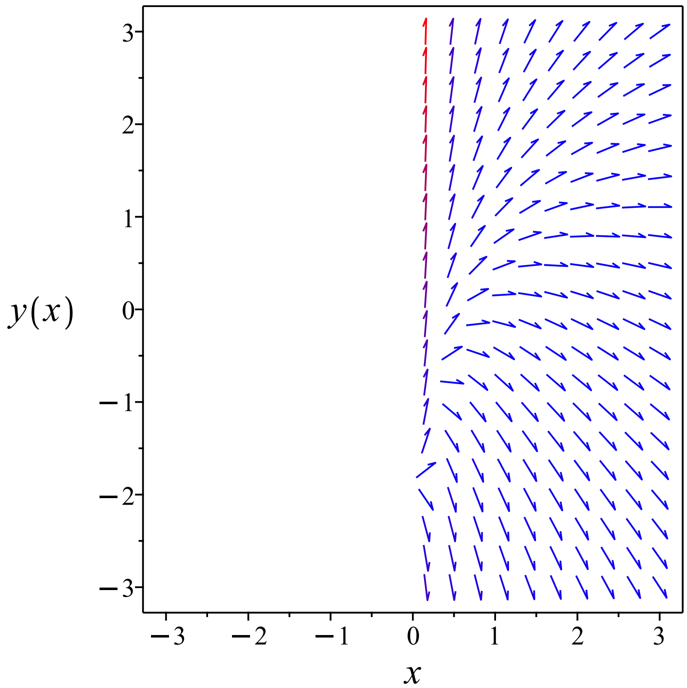


Figure 231: Slope field plot

Verification of solutions

$$y = c_1 x + \ln(x) + 1$$

Verified OK.

5.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-u(x)\, x + (u'(x)\, x + u(x))\, x = -\ln(x)$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int -\frac{\ln(x)}{x^2}\, dx \\ &= \frac{\ln(x)}{x} + \frac{1}{x} + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x\left(\frac{\ln(x)}{x} + \frac{1}{x} + c_2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{\ln(x)}{x} + \frac{1}{x} + c_2\right) \tag{1}$$

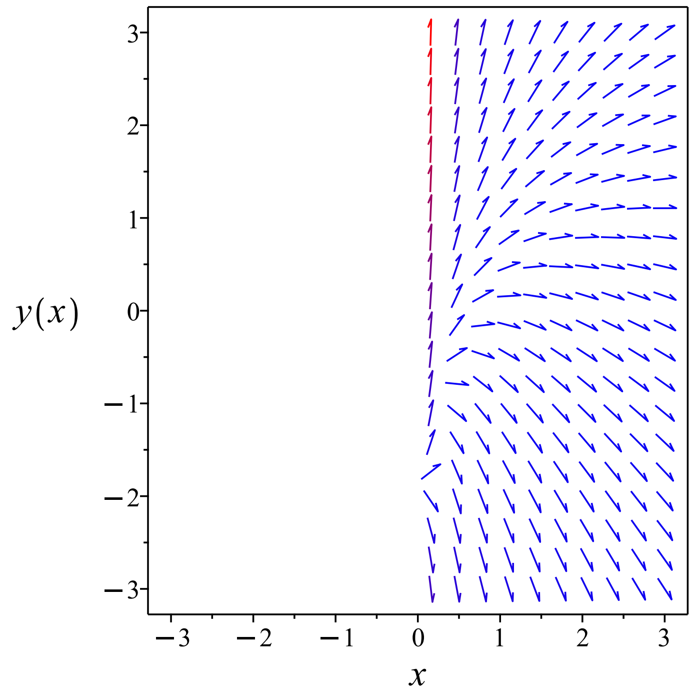


Figure 232: Slope field plot

Verification of solutions

$$y = x\left(\frac{\ln(x)}{x} + \frac{1}{x} + c_2\right)$$

Verified OK.

5.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\ln(x) - y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 125: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{x}dy\end{aligned}$$

Which results in

$$S=\frac{y}{x}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y}\tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=-\frac{\ln(x)-y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=-\frac{y}{x^2}\\S_y&=\frac{1}{x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=-\frac{\ln(x)}{x^2}\tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=-\frac{\ln(R)}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R)}{R} + \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \frac{\ln (x)}{x} + \frac{1}{x} + c_1$$

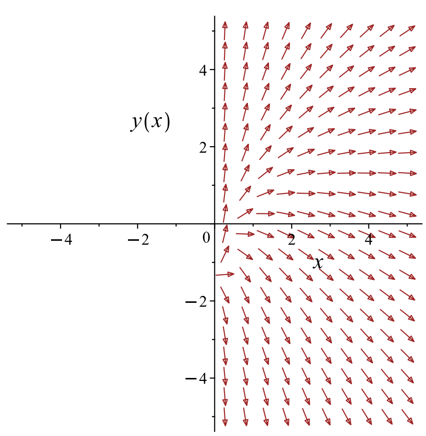
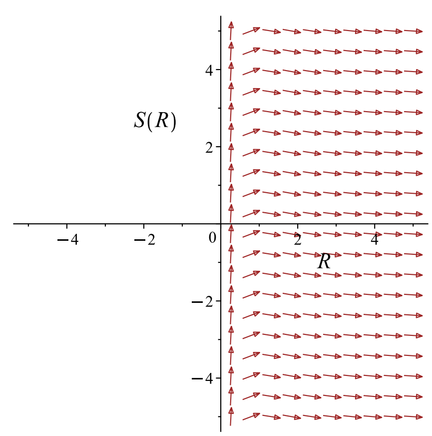
Which simplifies to

$$\frac{y}{x} = \frac{\ln (x)}{x} + \frac{1}{x} + c_1$$

Which gives

$$y = c_1 x + \ln (x) + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\ln(x)-y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = -\frac{\ln(R)}{R^2}$ 

Summary

The solution(s) found are the following

$$y = c_1 x + \ln (x) + 1 \tag{1}$$

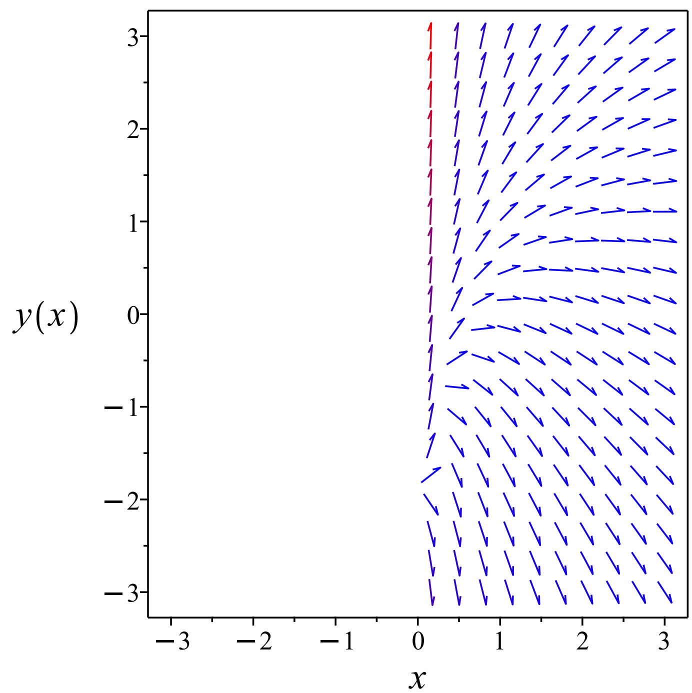


Figure 233: Slope field plot

Verification of solutions

$$y = c_1 x + \ln(x) + 1$$

Verified OK.

5.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x)\, \mathrm{d}y &= (-\ln(x) + y)\, \mathrm{d}x \\
(\ln(x) - y)\, \mathrm{d}x + (x)\, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \ln(x) - y \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\ln(x) - y) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-1) - (1)) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(\ln(x) - y) \\
&= \frac{\ln(x) - y}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(x) \\
&= \frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\ln(x) - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{\ln(x) - y}{x^2} \, \mathrm{d}x$$

$$\phi = \frac{-\ln(x) - 1 + y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-\ln(x) - 1 + y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-\ln(x) - 1 + y}{x}$$

The solution becomes

$$y = c_1 x + \ln(x) + 1$$

Summary

The solution(s) found are the following

$$y = c_1 x + \ln(x) + 1 \tag{1}$$

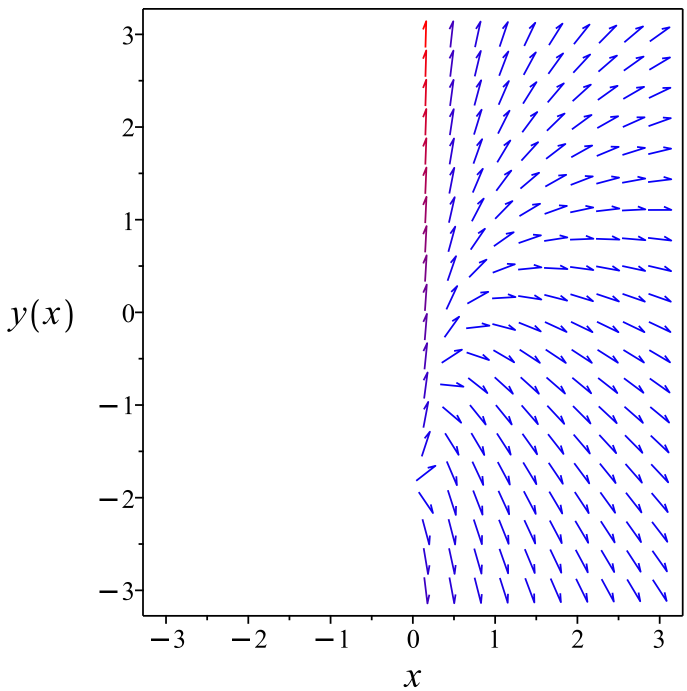


Figure 234: Slope field plot

Verification of solutions

$$y = c_1 x + \ln(x) + 1$$

Verified OK.

5.1.5 Maple step by step solution

Let's solve

$-y + y'x = -\ln(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} - \frac{\ln(x)}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = -\frac{\ln(x)}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = -\frac{\mu(x)\ln(x)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{\mu(x)\ln(x)}{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{\mu(x)\ln(x)}{x} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{\mu(x)\ln(x)}{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int -\frac{\ln(x)}{x^2} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x\left(\frac{\ln(x)}{x} + \frac{1}{x} + c_1\right)$

- Simplify

$$y = c_1x + \ln(x) + 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x*diff(y(x),x)+(ln(x)-y(x))=0,y(x), singsol=all)
```

$$y(x) = c_1x + \ln(x) + 1$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 13

```
DSolve[x*y'[x]+(Log[x]-y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x) + c_1x + 1$$

5.2 problem 2

Internal problem ID [1967]
Internal file name [`OUTPUT/1967_Sunday_February_25_2024_06_43_49_AM_61117948/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{yx + (x^2 + y)\, y' = 0}$$

5.2.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{yx}{x^2 + y} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{yx(b_3 - a_2)}{x^2 + y} - \frac{y^2x^2a_3}{(x^2+y)^2} - \left(-\frac{y}{x^2+y} + \frac{2y\,x^2}{(x^2+y)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{x}{x^2+y} + \frac{yx}{(x^2+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^4b_2 - 2y^2x^2a_3 + x^3b_1 - x^2ya_1 + 2x^2yb_2 + 2x\,y^2a_2 - x\,y^2b_3 + y^3a_3 + y^2a_1 + y^2b_2}{(x^2+y)^2} = 0$$

Setting the numerator to zero gives

$$2x^4b_2 - 2y^2x^2a_3 + x^3b_1 - x^2ya_1 + 2x^2yb_2 + 2x\,y^2a_2 - x\,y^2b_3 + y^3a_3 + y^2a_1 + y^2b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_3v_1^2v_2^2 + 2b_2v_1^4 - a_1v_1^2v_2 + 2a_2v_1v_2^2 + a_3v_2^3 + b_1v_1^3 + 2b_2v_1^2v_2 - b_3v_1v_2^2 + a_1v_2^2 + b_2v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^4 + b_1v_1^3 - 2a_3v_1^2v_2^2 + (-a_1 + 2b_2)\,v_1^2v_2 + (2a_2 - b_3)\,v_1v_2^2 + a_3v_2^3 + (a_1 + b_2)\,v_2^2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0 \\
b_1 &= 0 \\
-2a_3 &= 0 \\
2b_2 &= 0 \\
-a_1 + 2b_2 &= 0 \\
a_1 + b_2 &= 0 \\
2a_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 2y - \left(-\frac{yx}{x^2 + y}\right)(x) \\
&= \frac{3y\,x^2 + 2y^2}{x^2 + y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3y\,x^2+2y^2}{x^2+y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{3} + \frac{\ln(3x^2+2y)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{yx}{x^2+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{3x^2+2y} \\ S_y &= \frac{x^2+y}{3y\,x^2+2y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

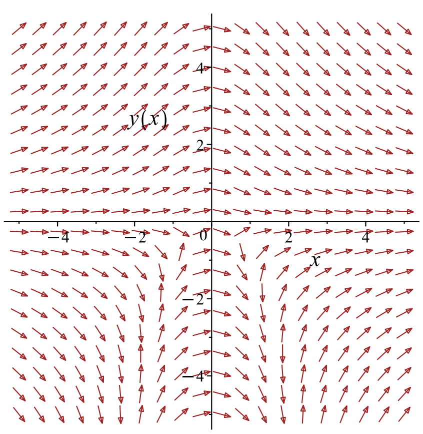
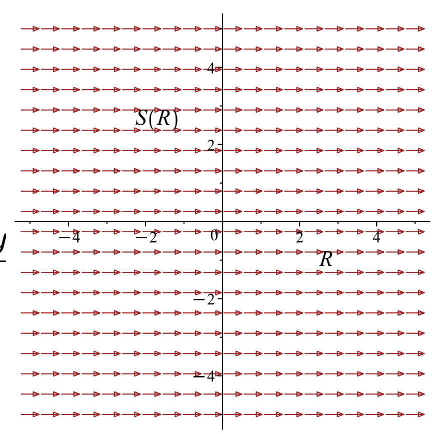
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{3} + \frac{\ln(3x^2 + 2y)}{6} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{3} + \frac{\ln(3x^2 + 2y)}{6} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{yx}{x^2+y}$ 	$R = x$ $S = \frac{\ln(y)}{3} + \frac{\ln(3x^2 + 2y)}{6}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{3} + \frac{\ln(3x^2 + 2y)}{6} = c_1 \tag{1}$$

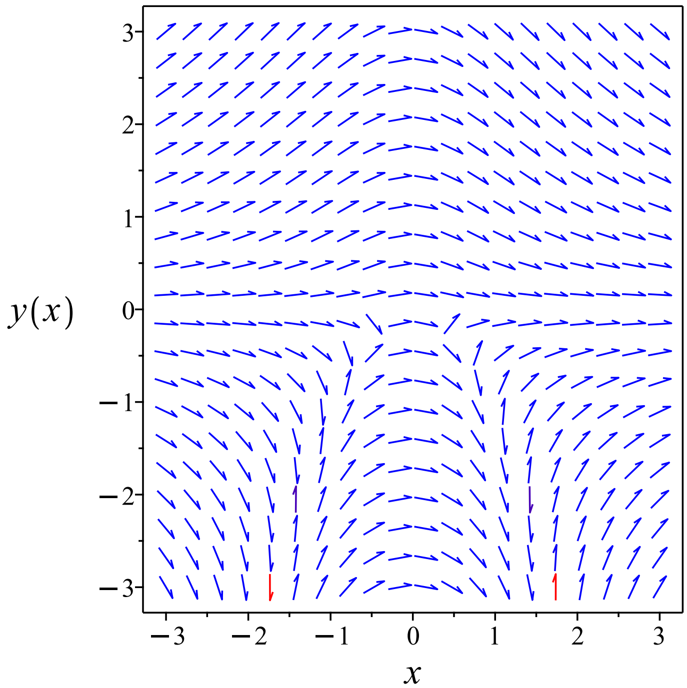


Figure 235: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{3} + \frac{\ln(3x^2 + 2y)}{6} = c_1$$

Verified OK.

5.2.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + y)\,\mathrm{d}y &= (-yx)\,\mathrm{d}x \\ (yx)\,\mathrm{d}x + (x^2 + y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = yx$$
$$N(x, y) = x^2 + y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(yx) \\ &= x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2+y}((x) - (2x)) \\
&= -\frac{x}{x^2+y}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{yx}((2x) - (x)) \\
&= \frac{1}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{1}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(y)} \\
&= y
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= y(yx) \\
&= x\,y^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y(x^2+y) \\
&= y(x^2+y)
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(x\,y^2\right) + \left(y\left(x^2+y\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int x\,y^2\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2y^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = y\,x^2 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y(x^2+y)$. Therefore equation (4) becomes

$$
y\left(x^2+y\right) = y\,x^2 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y^2
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(y^2\right)\mathrm{d}y \\
f(y) &= \frac{y^3}{3} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{2}x^2y^2 + \frac{1}{3}y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{2}x^2y^2 + \frac{1}{3}y^3$$

Summary

The solution(s) found are the following

$$\frac{x^2y^2}{2} + \frac{y^3}{3} = c_1 \tag{1}$$

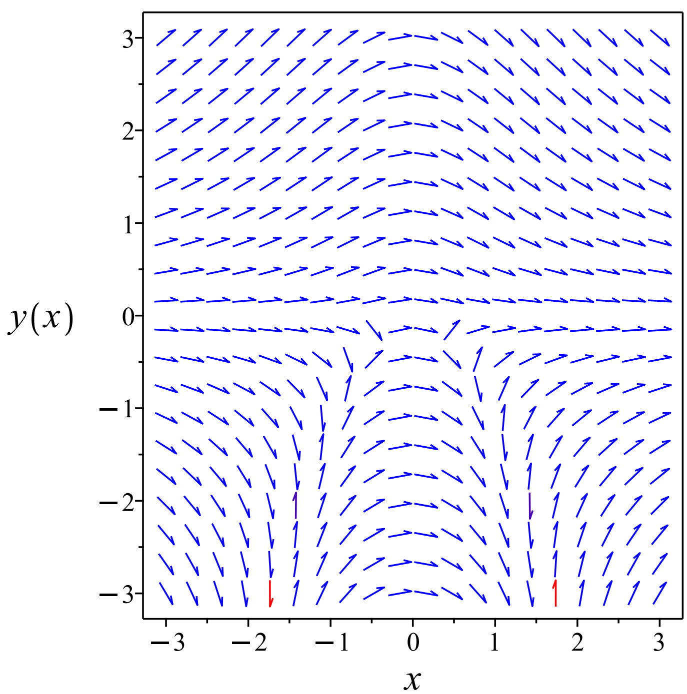


Figure 236: Slope field plot

Verification of solutions

$$\frac{x^2y^2}{2} + \frac{y^3}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 985

```
dsolve(x*y(x)+(x^2+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-1+\frac{\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}{x^2}+\frac{x^2}{\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}\right)x^2}{2}$$

$$y(x) = \frac{\left(-1+\frac{\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}{x^2}+\frac{x^2}{\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}\right)x^2}{2}$$

$$y(x) = \frac{\left(-1+\frac{\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}{x^2}+\frac{x^2}{\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}\right)x^2}{2}$$

$$y(x) = \frac{i\sqrt{3}\,x^4 - i\sqrt{3}\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{2}{3}} - x^4 - 2x^2\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}} - \left(2c_1^2-x^6+2c \cdots}{4\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}{4} - \frac{\left(i\sqrt{3}\,x^2+x^2+2\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}\right)x^2}{4\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{i\sqrt{3}\,x^4 - i\sqrt{3}\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{2}{3}} - x^4 - 2x^2\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}} - \left(2c_1^2-x^6+2c \cdots}{4\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}{4} - \frac{\left(i\sqrt{3}\,x^2+x^2+2\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}\right)x^2}{4\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{i\sqrt{3}\,x^4 - i\sqrt{3}\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{2}{3}} - x^4 - 2x^2\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}} - \left(2c_1^2-x^6+2c \cdots}{4\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}}$$

$$\left(i\sqrt{3}-1\right)\left(2c_1^2-x^6+2c_1\sqrt{-x^6+c_1^2}\right)^{\frac{1}{3}}$$

✓ Solution by Mathematica

Time used: 60.038 (sec). Leaf size: 397

```
DSolve[x*y[x]+(x^2+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 + \frac{1}{\text{Root}\left[\#1^6\left(x^{12}+e^{12c_1}\right)-6\#1^4x^8+4\#1^3x^6+9\#1^2x^4-12\#1x^2+4\&,1\right]}$$

$$y(x) \to -x^2 + \frac{1}{\text{Root}\left[\#1^6\left(x^{12}+e^{12c_1}\right)-6\#1^4x^8+4\#1^3x^6+9\#1^2x^4-12\#1x^2+4\&,2\right]}$$

$$y(x) \to -x^2 + \frac{1}{\text{Root}\left[\#1^6\left(x^{12}+e^{12c_1}\right)-6\#1^4x^8+4\#1^3x^6+9\#1^2x^4-12\#1x^2+4\&,3\right]}$$

$$y(x) \to -x^2 + \frac{1}{\text{Root}\left[\#1^6\left(x^{12}+e^{12c_1}\right)-6\#1^4x^8+4\#1^3x^6+9\#1^2x^4-12\#1x^2+4\&,4\right]}$$

$$y(x) \to -x^2 + \frac{1}{\text{Root}\left[\#1^6\left(x^{12}+e^{12c_1}\right)-6\#1^4x^8+4\#1^3x^6+9\#1^2x^4-12\#1x^2+4\&,5\right]}$$

$$y(x) \to -x^2 + \frac{1}{\text{Root}\left[\#1^6\left(x^{12}+e^{12c_1}\right)-6\#1^4x^8+4\#1^3x^6+9\#1^2x^4-12\#1x^2+4\&,6\right]}$$

5.3 problem 3

Internal problem ID [1968]
Internal file name [`OUTPUT/1968_Sunday_February_25_2024_06_43_50_AM_70868834/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(-2yx + x)\, y' + 2y = 0}$$

5.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{2y}{x\,(-1+2y)} \end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(y) = \frac{y}{-1+2y}$. Integrating both sides gives

$$\frac{1}{\frac{y}{-1+2y}}\, dy = \frac{2}{x}\, dx$$

$$\int \frac{1}{\frac{y}{-1+2y}}\, dy = \int \frac{2}{x}\, dx$$

$$2y - \ln(y) = 2\ln(x) + c_1$$

Which results in

$$y = -\frac{\text{LambertW}\left(-\frac{2\,\mathrm{e}^{-c_1}}{x^2}\right)}{2}$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$y = -\frac{\text{LambertW}\left(-\frac{2\,\mathrm{e}^{-c_1}}{x^2}\right)}{2}$$

gives

$$y = -\frac{\text{LambertW}\left(-\frac{2}{c_1 x^2}\right)}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\text{LambertW}\left(-\frac{2}{c_1 x^2}\right)}{2} \tag{1}$$

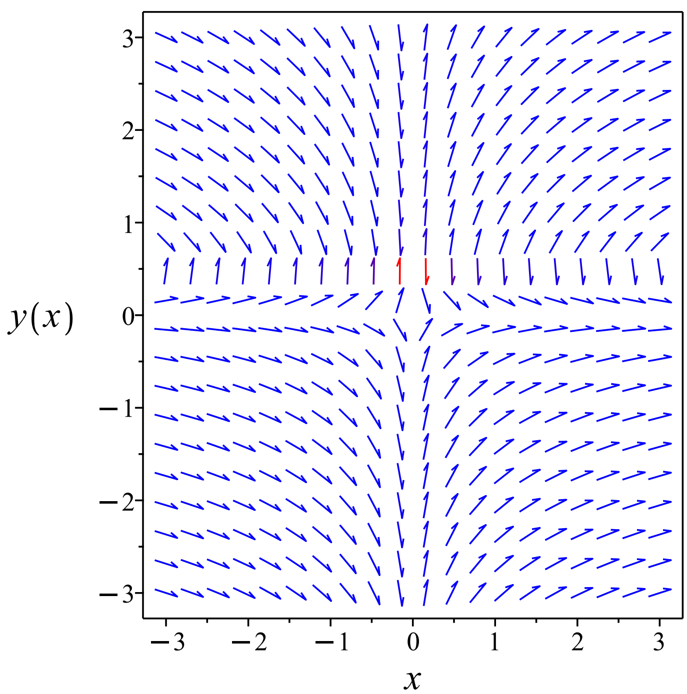


Figure 237: Slope field plot

Verification of solutions

$$y = -\frac{\text{LambertW}\left(-\frac{2}{c_1 x^2}\right)}{2}$$

Verified OK.

5.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{2y}{x(-1+2y)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 128: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x}{2} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x}{2}} dx \end{aligned}$$

Which results in

$$S = 2\ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y}{x(-1+2y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{2}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-1+2y}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{-1+2R}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2R - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\ln(x) = 2y - \ln(y) + c_1$$

Which simplifies to

$$2\ln(x) = 2y - \ln(y) + c_1$$

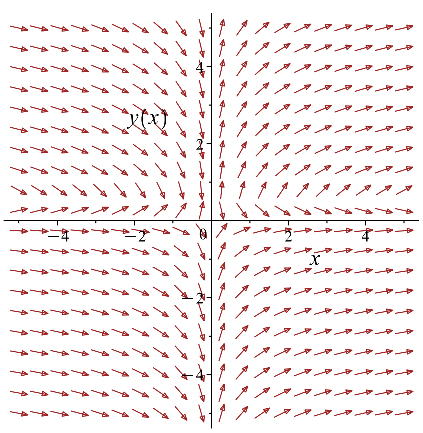
Which gives

$$y = -\frac{\operatorname{LambertW}\left(-\frac{2\,e^{c_1}}{x^2}\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{2y}{x(-1+2y)}$



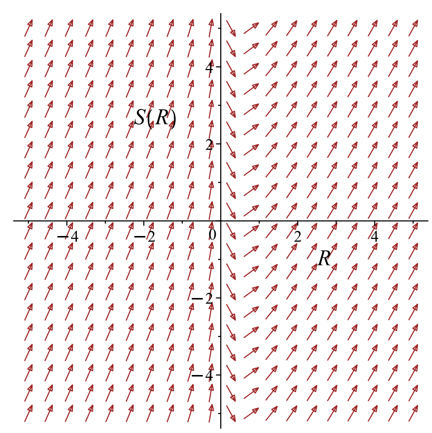
</td><td>

$R = y$

$S = 2\ln(x)$

</td><td>

$\frac{dS}{dR} = \frac{-1+2R}{R}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{\operatorname{LambertW}\left(-\frac{2\,e^{c_1}}{x^2}\right)}{2} \tag{1}$$

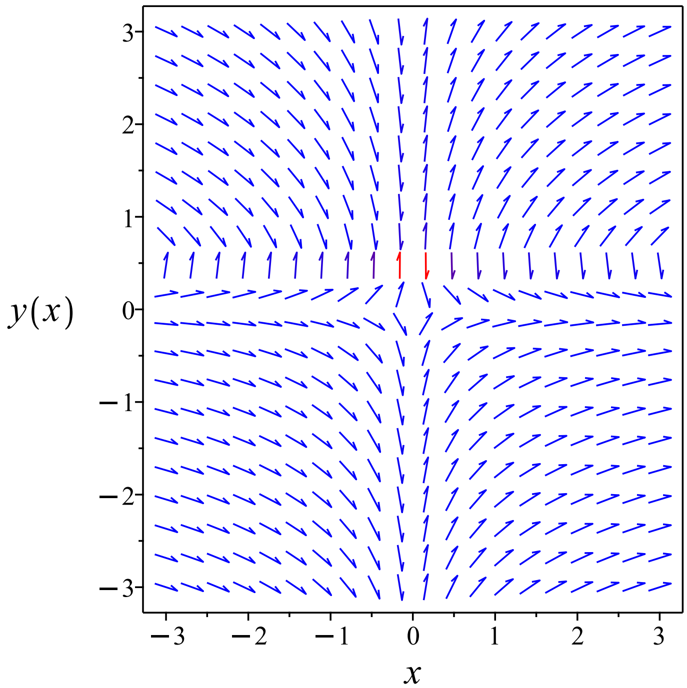


Figure 238: Slope field plot

Verification of solutions

$$y = -\frac{\operatorname{LambertW}\left(-\frac{2\,e^{c_1}}{x^2}\right)}{2}$$

Verified OK.

5.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-1 + 2y}{2y}\right) \mathrm{d}y = \left(\frac{1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{-1 + 2y}{2y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{-1 + 2y}{2y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-1+2y}{2y}\right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x \\ \phi &= -\ln(x) + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-1+2y}{2y}$. Therefore equation (4) becomes

$$\frac{-1+2y}{2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{-1+2y}{2y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{-1+2y}{2y}\right)\mathrm{d}y \\ f(y) &= y - \frac{\ln(y)}{2} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + y - \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + y - \frac{\ln(y)}{2}$$

The solution becomes

$$y = -\frac{\operatorname{LambertW}\left(-\frac{2\,\mathrm{e}^{-2c_1}}{x^2}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\operatorname{LambertW}\left(-\frac{2\,\mathrm{e}^{-2c_1}}{x^2}\right)}{2} \tag{1}$$

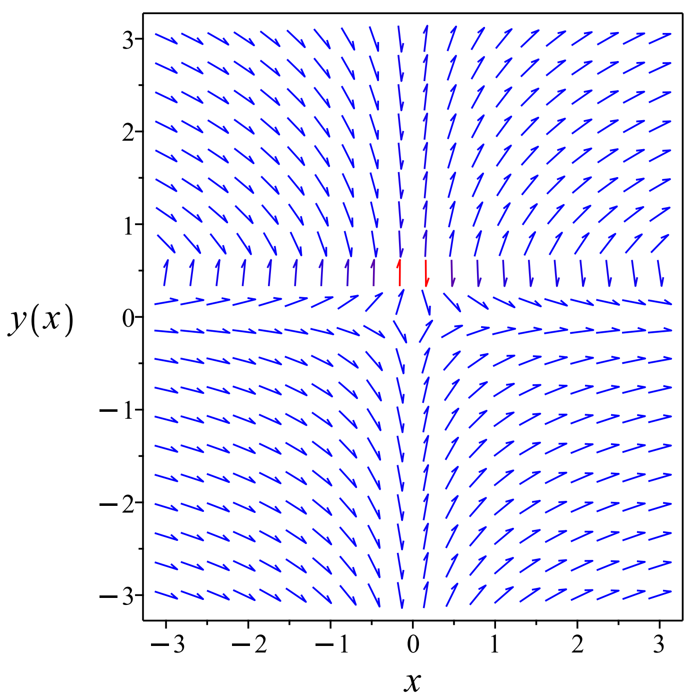


Figure 239: Slope field plot

Verification of solutions

$$y = -\frac{\text{LambertW}\left(-\frac{2\,e^{-2c_1}}{x^2}\right)}{2}$$

Verified OK.

5.3.4 Maple step by step solution

Let's solve

$(-2yx + x)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(2y-1)}{y} = \frac{2}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'(2y-1)}{y}dx = \int \frac{2}{x}dx + c_1$

- Evaluate integral

  $2y - \ln(y) = 2\ln(x) + c_1$

- Solve for y

  $y = \frac{e^{-\mathit{LambertW}\left(-\frac{2}{e^{c_1}x^2}\right)-c_1}}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve((x-2*x*y(x))*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\text{LambertW}\left(-\frac{2c_1}{x^2}\right)}{2}$$

✓ Solution by Mathematica

Time used: 2.94 (sec). Leaf size: 26

```
DSolve[(x-2*x*y[x])*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}W\left(-\frac{2e^{-c_1}}{x^2}\right)$$
$$y(x) \to 0$$

5.4 problem 4

Internal problem ID [1969]
Internal file name [`OUTPUT/1969_Sunday_February_25_2024_06_43_51_AM_7762154/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{x^2y + y^2 + x^3y' = 0}$$

5.4.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(x^2 + y)}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 131: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x\, y^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x\,y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(x^2+y)}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^2 y} \\ S_y &= \frac{1}{x\,y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{3R^3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{yx} = \frac{1}{3x^3} + c_1$$

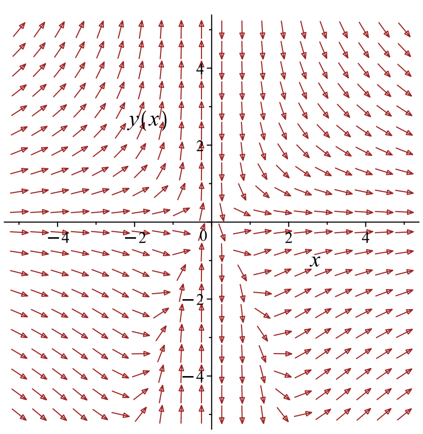
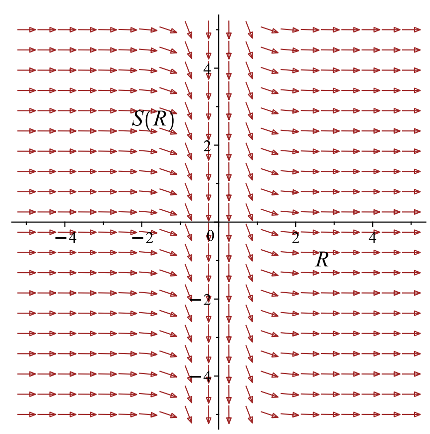
Which simplifies to

$$-\frac{1}{yx} = \frac{1}{3x^3} + c_1$$

Which gives

$$y = -\frac{3x^2}{3c_1 x^3 + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(x^2+y)}{x^3}$ 	$R = x$ $S = -\frac{1}{yx}$	$\frac{dS}{dR} = -\frac{1}{R^4}$ 

Summary

The solution(s) found are the following

$$y = -\frac{3x^2}{3c_1x^3 + 1} \tag{1}$$

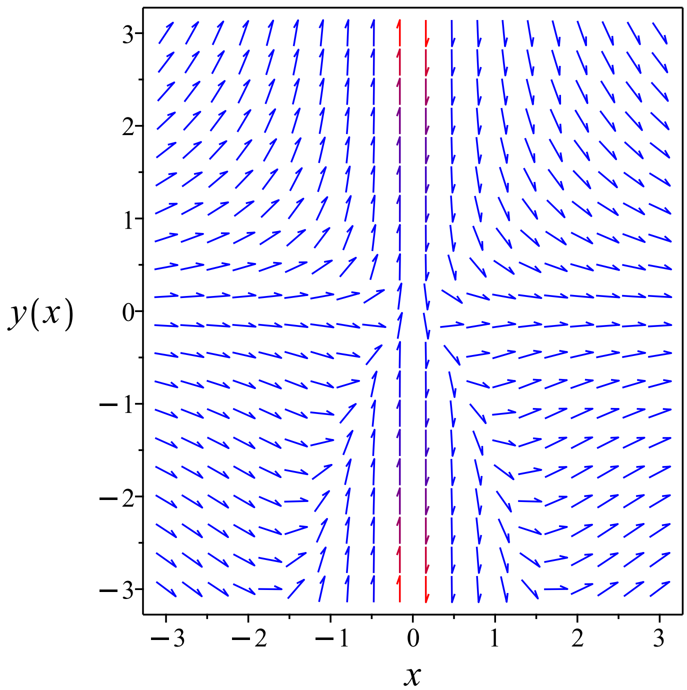


Figure 240: Slope field plot

Verification of solutions

$$y = -\frac{3x^2}{3c_1x^3 + 1}$$

Verified OK.

5.4.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(x^2 + y)}{x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - \frac{1}{x^3}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= -\frac{1}{x^3} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{yx} - \frac{1}{x^3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} - \frac{1}{x^3} \\ w' &= \frac{w}{x} + \frac{1}{x^3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{1}{x^3}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = \frac{1}{x^3}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(\frac{1}{x^3}\right)$$
$$\frac{d}{dx}\left(\frac{w}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{1}{x^3}\right)$$
$$d\left(\frac{w}{x}\right) = \frac{1}{x^4}\,dx$$

Integrating gives

$$\frac{w}{x} = \int \frac{1}{x^4}\,dx$$
$$\frac{w}{x} = -\frac{1}{3x^3} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = -\frac{1}{3x^2} + c_1 x$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = -\frac{1}{3x^2} + c_1 x$$

Or

$$y = \frac{1}{-\frac{1}{3x^2} + c_1 x}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-\frac{1}{3x^2} + c_1 x} \tag{1}$$

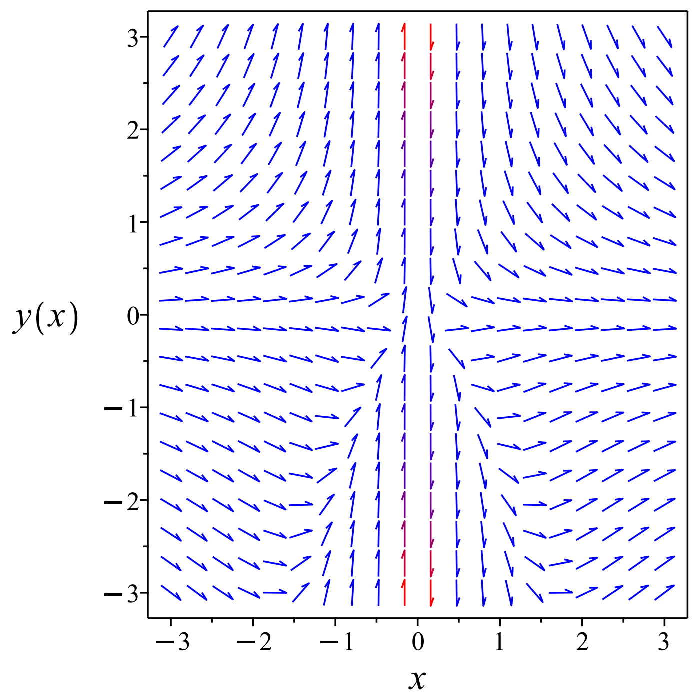


Figure 241: Slope field plot

Verification of solutions

$$y = \frac{1}{-\frac{1}{3x^2} + c_1 x}$$

Verified OK.

5.4.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(x^2 + y)}{x^3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y}{x} - \frac{y^2}{x^3}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = -\frac{1}{x^3}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^3}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{3}{x^4} \\ f_1 f_2 &= \frac{1}{x^4} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^3} - \frac{4u'(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x^3}$$

The above shows that

$$u'(x) = -\frac{3c_2}{x^4}$$

Using the above in (1) gives the solution

$$y = -\frac{3c_2}{x\left(c_1 + \frac{c_2}{x^3}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{3}{x\left(c_3 + \frac{1}{x^3}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{x\left(c_3 + \frac{1}{x^3}\right)} \tag{1}$$

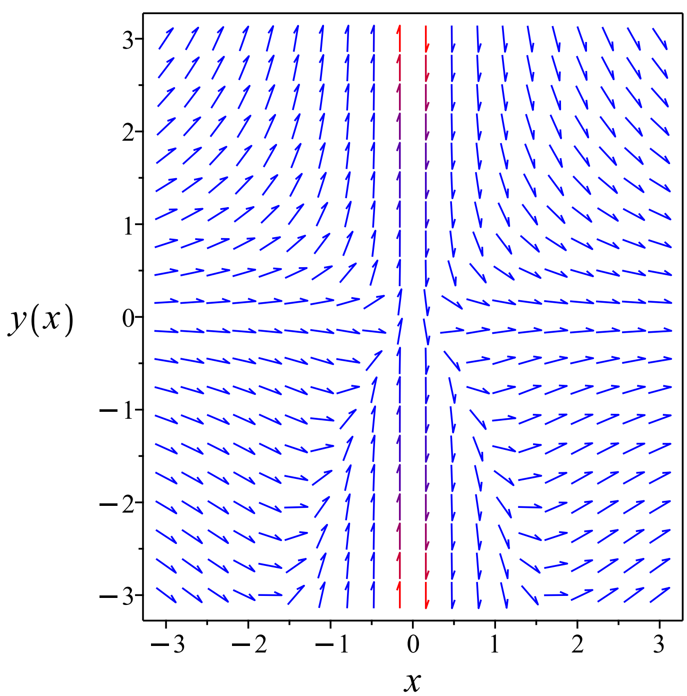


Figure 242: Slope field plot

Verification of solutions

$$y = -\frac{3}{x\left(c_3 + \frac{1}{x^3}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve((x^2*y(x)+y(x)^2)+x^3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3x^2}{3c_1x^3 - 1}$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 26

```
DSolve[(x^2*y[x]+y[x]^2)+x^3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3x^2}{-1 + 3c_1x^3}$$
$$y(x) \to 0$$

5.5 problem 5

Internal problem ID [1970]
Internal file name [`OUTPUT/1970_Sunday_February_25_2024_06_43_51_AM_8686326/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{xy^3 + x^2y^2y' = 1}$$

5.5.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x\,y^3 - 1}{x^2y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 133: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{y^2 x^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y^2 x^3}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3 x^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x\,y^3 - 1}{x^2 y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^3 x^2 \\ S_y &= x^3 y^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

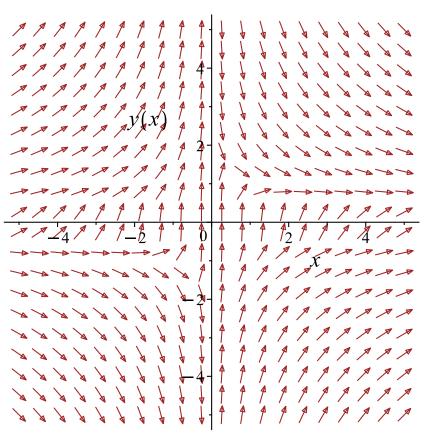
$$\frac{x^3y^3}{3} = \frac{x^2}{2} + c_1$$

Which simplifies to

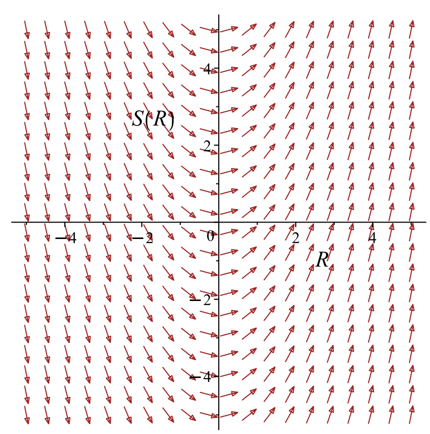
$$\frac{x^3y^3}{3} = \frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = -\frac{x\,y^3-1}{x^2y^2}$


</td><td>$R = x$

$S = \frac{y^3x^3}{3}$</td><td>$\frac{dS}{dR} = R$


</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{x^3y^3}{3} = \frac{x^2}{2} + c_1 \tag{1}$$

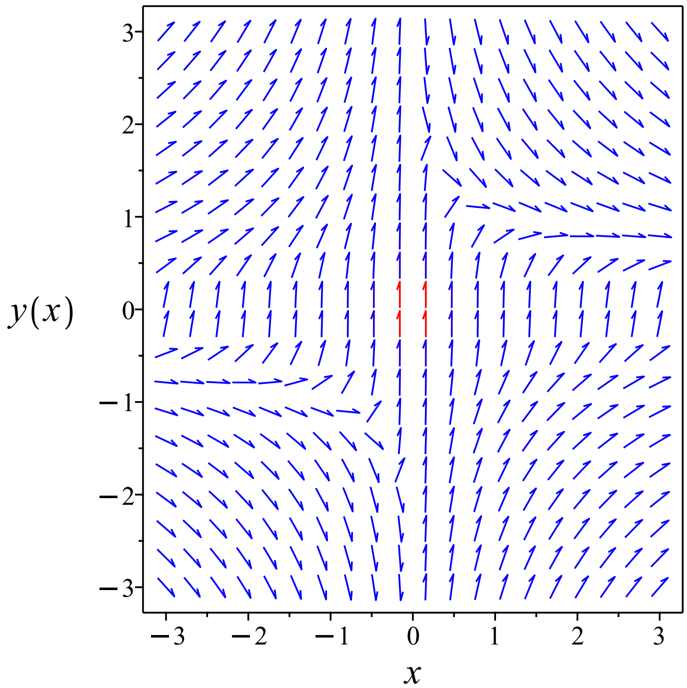


Figure 243: Slope field plot

<u>Verification of solutions</u>

$$\frac{x^3y^3}{3} = \frac{x^2}{2} + c_1$$

Verified OK.

5.5.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x\,y^3 - 1}{x^2y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{1}{x^2}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{1}{x^2} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = -\frac{y^3}{x} + \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= -\frac{w(x)}{x} + \frac{1}{x^2} \\ w' &= -\frac{3w}{x} + \frac{3}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{3w(x)}{x} = \frac{3}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{x} dx} \\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{3}{x^2}\right) \\ \frac{d}{dx}(x^3 w) &= (x^3)\left(\frac{3}{x^2}\right) \\ d(x^3 w) &= (3x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^3 w &= \int 3x\, dx \\ x^3 w &= \frac{3x^2}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$w(x) = \frac{3}{2x} + \frac{c_1}{x^3}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = \frac{3}{2x} + \frac{c_1}{x^3}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{(12x^2 + 8c_1)^{\frac{1}{3}}}{2x} \\ y(x) &= \frac{(12x^2 + 8c_1)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{4x} \\ y(x) &= -\frac{(12x^2 + 8c_1)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{4x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{(12x^2 + 8c_1)^{\frac{1}{3}}}{2x} \tag{1}$$

$$y = \frac{(12x^2 + 8c_1)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{4x} \tag{2}$$

$$y = -\frac{(12x^2 + 8c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{4x} \tag{3}$$

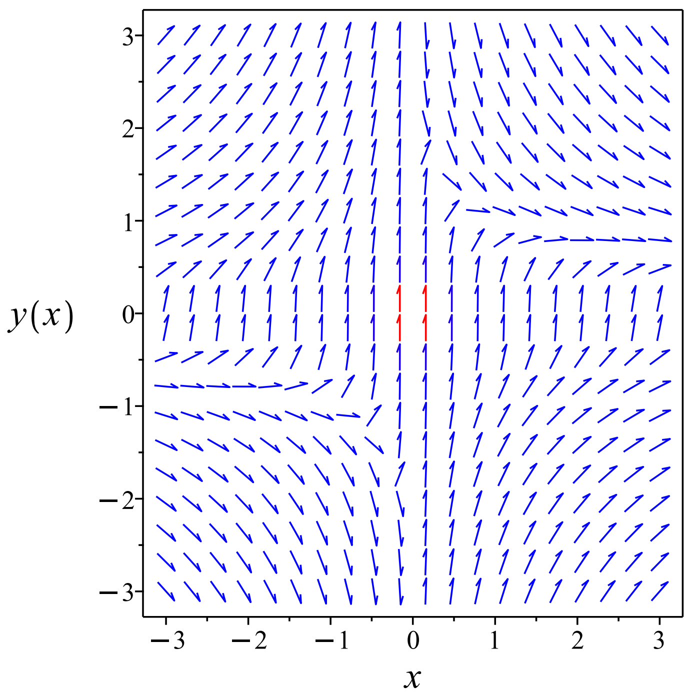


Figure 244: Slope field plot

Verification of solutions

$$
y=\frac{\left(12x^{2}+8c_{1}\right)^{\frac{1}{3}}}{2x}
$$

Verified OK.

$$
y=\frac{\left(12x^{2}+8c_{1}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4x}
$$

Verified OK.

$$
y=-\frac{\left(12x^{2}+8c_{1}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{4x}
$$

Verified OK.

5.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}
$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y)=0
$$

Hence

$$
\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial\phi}{\partial x}&=M\\
\frac{\partial\phi}{\partial y}&=N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(x^2y^2\right)\mathrm{d}y &= \left(-x\,y^3+1\right)\mathrm{d}x\\
\left(x\,y^3-1\right)\mathrm{d}x + \left(x^2y^2\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= x\,y^3-1\\
N(x,y) &= x^2y^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\,y^3-1\right)\\
&= 3x\,y^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2y^2\right)\\
&= 2x\,y^2
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\
&= \frac{1}{x^2y^2}\left(\left(3x\,y^2\right) - \left(2x\,y^2\right)\right)\\
&= \frac{1}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int \frac{1}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x(x\,y^3 - 1) \\ &= x(x\,y^3 - 1)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x(x^2y^2) \\ &= x^3y^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (x(x\,y^3 - 1)) + (x^3y^2)\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int x(x\,y^3 - 1)\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{1}{3}y^3x^3 - \frac{1}{2}x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 y^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 y^2$. Therefore equation (4) becomes

$$x^3 y^2 = x^3 y^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3} y^3 x^3 - \frac{1}{2} x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3} y^3 x^3 - \frac{1}{2} x^2$$

Summary

The solution(s) found are the following

$$\frac{x^3 y^3}{3} - \frac{x^2}{2} = c_1 \tag{1}$$

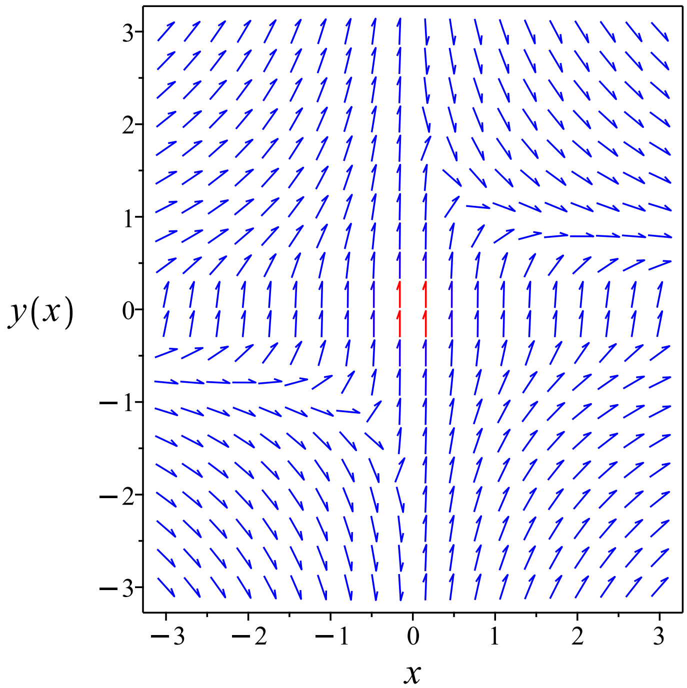


Figure 245: Slope field plot

Verification of solutions

$$\frac{x^3y^3}{3} - \frac{x^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 74

```
dsolve((x*y(x)^3-1)+x^2*y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(12x^2 + 8c_1)^{\frac{1}{3}}}{2x}$$
$$y(x) = -\frac{(12x^2 + 8c_1)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{4x}$$
$$y(x) = \frac{(12x^2 + 8c_1)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{4x}$$

✓ Solution by Mathematica

Time used: 0.208 (sec). Leaf size: 80

```
DSolve[(x*y[x]^3-1)+x^2*y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt[3]{-\frac{1}{2}}\sqrt[3]{3x^2 + 2c_1}}{x}$$
$$y(x) \to \frac{\sqrt[3]{\frac{3x^2}{2} + c_1}}{x}$$
$$y(x) \to \frac{(-1)^{2/3}\sqrt[3]{\frac{3x^2}{2} + c_1}}{x}$$

5.6 problem 6

Internal problem ID [1971]
Internal file name [`OUTPUT/1971_Sunday_February_25_2024_06_43_52_AM_47101479/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{(x^3y^3-1)\,y'+x^2y^4=0}$$

5.6.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{x^2y^4}{y^3x^3-1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{x^2 y^4 (b_3 - a_2)}{y^3 x^3 - 1} - \frac{x^4 y^8 a_3}{(y^3 x^3 - 1)^2} - \left(-\frac{2x\, y^4}{y^3 x^3 - 1} + \frac{3x^4 y^7}{(y^3 x^3 - 1)^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{4y^3 x^2}{y^3 x^3 - 1} + \frac{3x^5 y^6}{(y^3 x^3 - 1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^6 y^6 b_2 - 2x^4 y^8 a_3 + x^5 y^6 b_1 - x^4 y^7 a_1 - 6x^3 y^3 b_2 - 3x^2 y^4 a_2 - 3x^2 y^4 b_3 - 2x\, y^5 a_3 - 4x^2 y^3 b_1 - 2x\, y^4 a_1 + b_2}{(y^3 x^3 - 1)^2} = 0$$

Setting the numerator to zero gives

$$2x^6 y^6 b_2 - 2x^4 y^8 a_3 + x^5 y^6 b_1 - x^4 y^7 a_1 - 6x^3 y^3 b_2 - 3x^2 y^4 a_2 - 3x^2 y^4 b_3 - 2x\, y^5 a_3 - 4x^2 y^3 b_1 - 2x\, y^4 a_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_3 v_1^4 v_2^8 + 2b_2 v_1^6 v_2^6 - a_1 v_1^4 v_2^7 + b_1 v_1^5 v_2^6 - 3a_2 v_1^2 v_2^4 - 2a_3 v_1 v_2^5 - 6b_2 v_1^3 v_2^3 - 3b_3 v_1^2 v_2^4 - 2a_1 v_1 v_2^4 - 4b_1 v_1^2 v_2^3 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^6v_2^6 + b_1v_1^5v_2^6 - 2a_3v_1^4v_2^8 - a_1v_1^4v_2^7 - 6b_2v_1^3v_2^3 + (-3a_2 - 3b_3)\, v_1^2v_2^4 - 4b_1v_1^2v_2^3 - 2a_3v_1v_2^5 - 2a_1v_1v_2^4 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ b_2 &= 0 \\ -2a_1 &= 0 \\ -a_1 &= 0 \\ -2a_3 &= 0 \\ -4b_1 &= 0 \\ -6b_2 &= 0 \\ 2b_2 &= 0 \\ -3a_2 - 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{x^2 y^4}{y^3 x^3 - 1}\right)(-x) \\
&= -\frac{y}{y^3 x^3 - 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{y}{y^3 x^3 - 1}} dy
\end{aligned}$$

Which results in

$$S = -\frac{y^3 x^3}{3} + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^2 y^4}{y^3 x^3 - 1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -y^3x^2 \\
S_y &= -x^3y^2 + \frac{1}{y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{x^3y^3}{3} + \ln(y) = c_1
$$

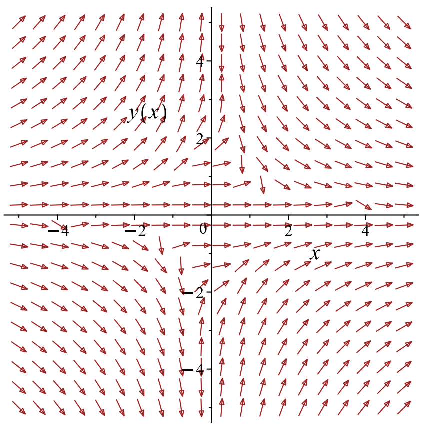
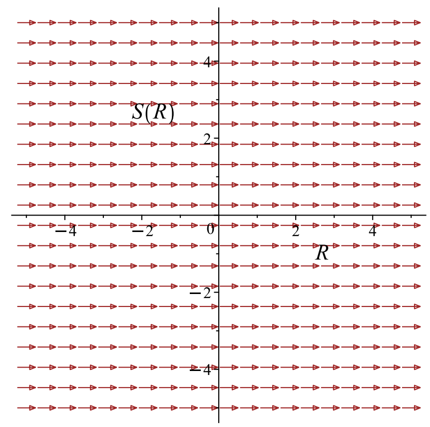
Which simplifies to

$$
-\frac{x^3y^3}{3} + \ln(y) = c_1
$$

Which gives

$$
y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\mathrm{e}^{3c_1}x^3\right)}{3}+c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2 y^4}{y^3 x^3 - 1}$ 	$R = x$ $S = -\frac{y^3 x^3}{3} + \ln(y)$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = e^{-\frac{\text{LambertW}\left(-e^{3c_1} x^3\right)}{3} + c_1} \tag{1}$$

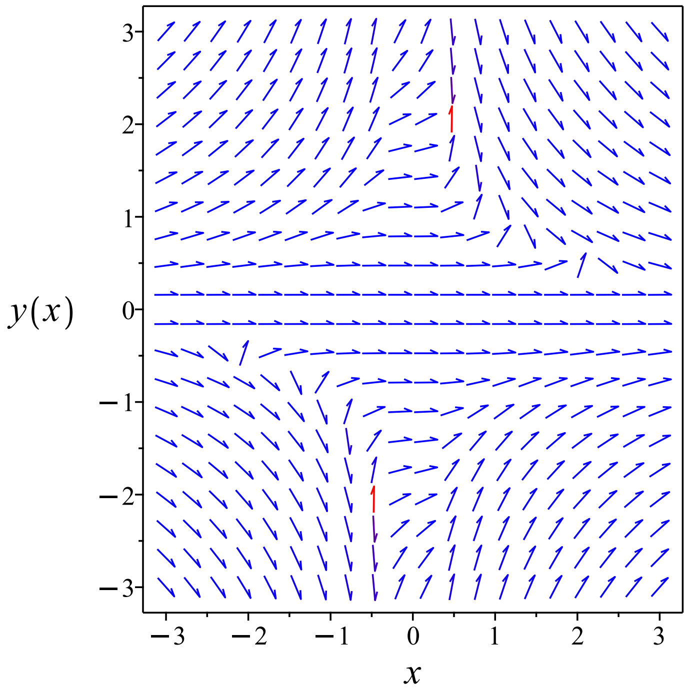


Figure 246: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\mathrm{e}^{3c_1}x^3\right)}{3}+c_1}$$

Verified OK.

5.6.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(y^3 x^3 - 1\right) \mathrm{d}y &= \left(-x^2 y^4\right) \mathrm{d}x \\
\left(x^2 y^4\right) \mathrm{d}x + \left(y^3 x^3 - 1\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x^2 y^4 \\
N(x, y) &= y^3 x^3 - 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 y^4\right) \\
&= 4y^3 x^2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^3 x^3 - 1\right) \\
&= 3y^3 x^2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{y^3x^3 - 1}\left(\left(4y^3x^2\right) - \left(3y^3x^2\right)\right) \\ &= \frac{y^3x^2}{y^3x^3 - 1} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{x^2y^4}\left(\left(3y^3x^2\right) - \left(4y^3x^2\right)\right) \\ &= -\frac{1}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{1}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(y)} \\ &= \frac{1}{y} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y}\left(x^2y^4\right) \\ &= y^3x^2 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y}\left(y^3x^3 - 1\right) \\ &= \frac{y^3x^3 - 1}{y} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$(y^3x^2) + \left(\frac{y^3x^3 - 1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int y^3x^2\,\mathrm{d}x$$
$$\phi = \frac{y^3x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3y^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^3x^3-1}{y}$. Therefore equation (4) becomes

$$\frac{y^3x^3 - 1}{y} = x^3y^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y^3x^3}{3} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^3x^3}{3} - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-x^3\mathrm{e}^{-3c_1}\right)}{3} - c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-x^3\mathrm{e}^{-3c_1}\right)}{3} - c_1} \tag{1}$$

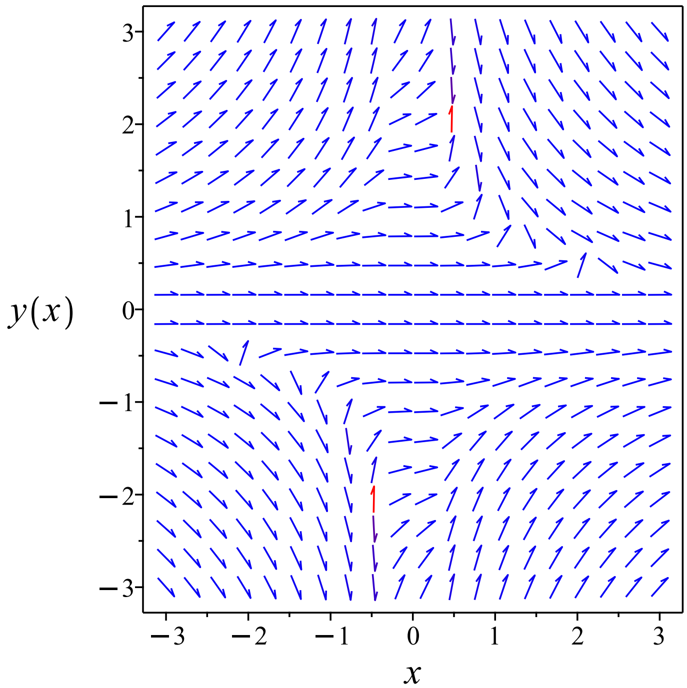


Figure 247: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-x^3 \mathrm{e}^{-3c_1}\right)}{3} - c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve((x^3*y(x)^3-1)*diff(y(x),x)+x^2*y(x)^4=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-c_1}}{\left(-\frac{x^3 e^{-3c_1}}{\text{LambertW}(-x^3 e^{-3c_1})}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 4.808 (sec). Leaf size: 90

```
DSolve[(x^3*y[x]^3-1)*y'[x]+x^2*y[x]^4==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt[3]{W\left(-e^{-3c_1}x^3\right)}}{x}$$
$$y(x) \to \frac{\sqrt[3]{-1}\sqrt[3]{W\left(-e^{-3c_1}x^3\right)}}{x}$$
$$y(x) \to -\frac{(-1)^{2/3}\sqrt[3]{W\left(-e^{-3c_1}x^3\right)}}{x}$$
$$y(x) \to 0$$

5.7 problem 7

Internal problem ID [1972]
Internal file name [OUTPUT/1972_Sunday_February_25_2024_06_43_54_AM_88563334/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{y(-x^2+y)+x^3y'=0}$$

5.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x(-x^2+u(x)\,x)+x^3(u'(x)\,x+u(x))=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2}{x^2} \end{aligned}$$

Where $f(x) = -\frac{1}{x^2}$ and $g(u) = u^2$. Integrating both sides gives

$$\frac{1}{u^2}\,du = -\frac{1}{x^2}\,dx$$
$$\int \frac{1}{u^2}\,du = \int -\frac{1}{x^2}\,dx$$
$$-\frac{1}{u} = \frac{1}{x} + c_2$$

The solution is

$$-\frac{1}{u(x)} - \frac{1}{x} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x}{y} - \frac{1}{x} - c_2 = 0$$
$$-\frac{x}{y} - \frac{1}{x} - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{x}{y} - \frac{1}{x} - c_2 = 0 \tag{1}$$

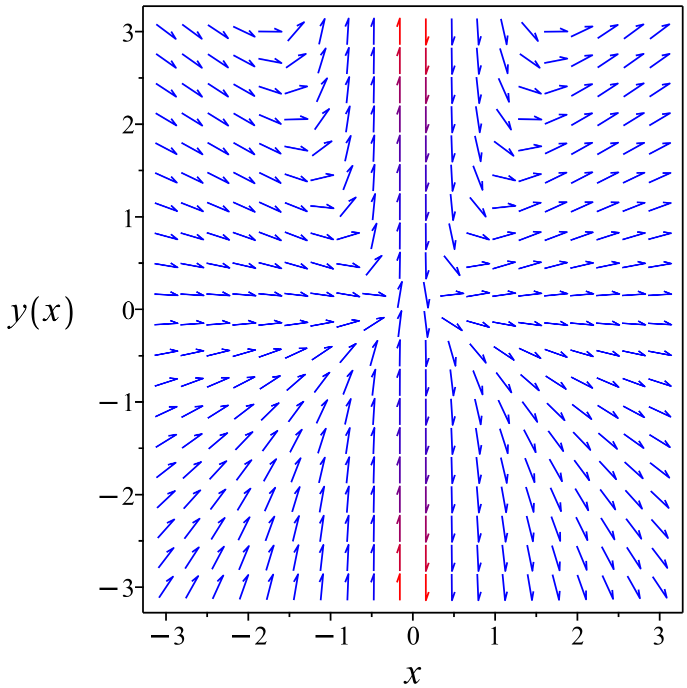


Figure 248: Slope field plot

Verification of solutions

$$-\frac{x}{y}-\frac{1}{x}-c_2=0$$

Verified OK.

5.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=-\frac{y(-x^2+y)}{x^3}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 135: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^2}{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(-x^2 + y)}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = \frac{1}{x} + c_1$$

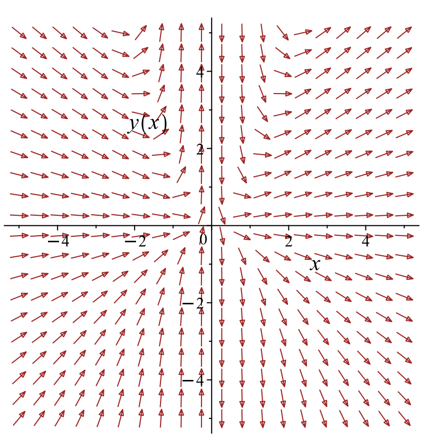
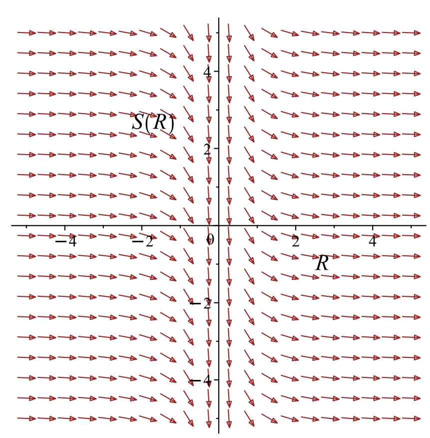
Which simplifies to

$$-\frac{x}{y} = \frac{1}{x} + c_1$$

Which gives

$$y = -\frac{x^2}{c_1 x + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-x^2+y)}{x^3}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = -\frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{c_1 x + 1} \tag{1}$$

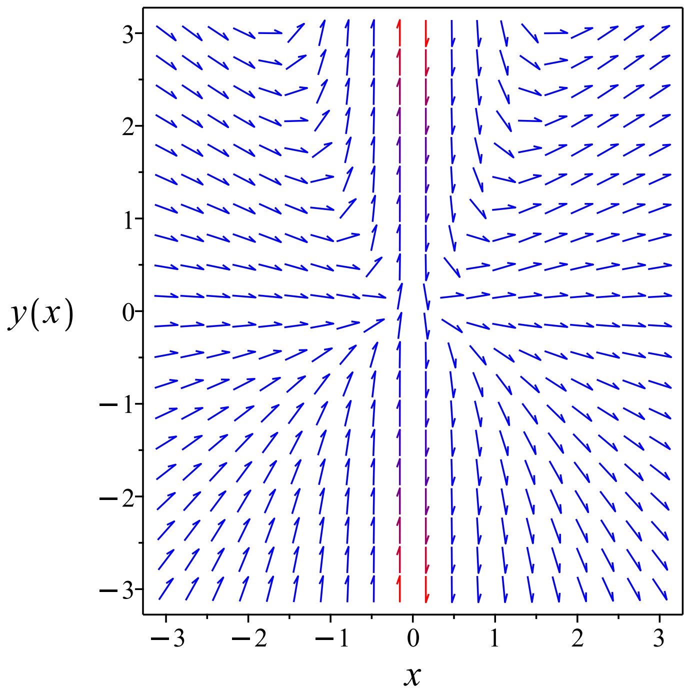


Figure 249: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{c_1 x + 1}$$

Verified OK.

5.7.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(-x^2 + y)}{x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x} y - \frac{1}{x^3} y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -\frac{1}{x^3} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{yx} - \frac{1}{x^3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} - \frac{1}{x^3} \\ w' &= -\frac{w}{x} + \frac{1}{x^3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^3} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = \frac{1}{x^3}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x^3}\right) \\ \frac{d}{dx}(xw) &= (x)\left(\frac{1}{x^3}\right) \\ d(xw) &= \frac{1}{x^2}\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} xw &= \int \frac{1}{x^2}\, dx \\ xw &= -\frac{1}{x} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{1}{x^2} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{c_1 x - 1}{x^2}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{c_1 x - 1}{x^2}$$

Or

$$y = \frac{x^2}{c_1 x - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{c_1 x - 1} \tag{1}$$

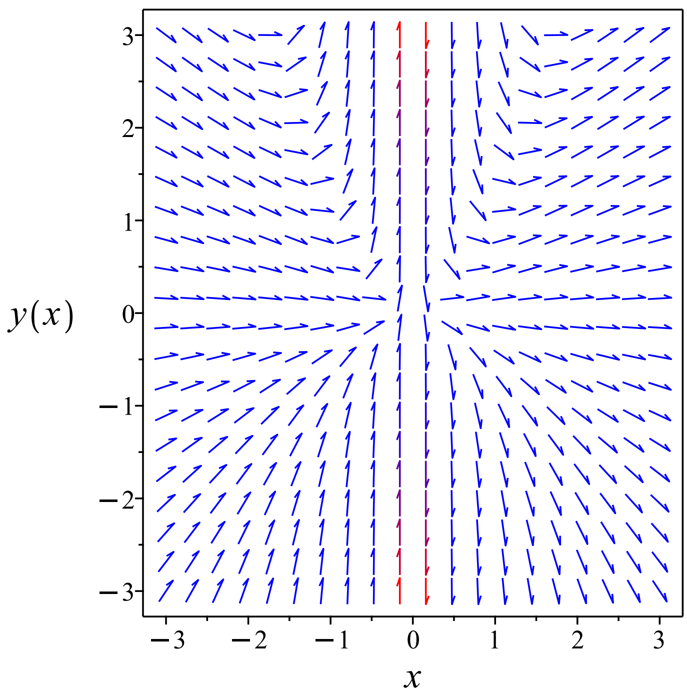


Figure 250: Slope field plot

Verification of solutions

$$y = \frac{x^2}{c_1 x - 1}$$

Verified OK.

5.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^3)\,\mathrm{d}y&=\left(-y\left(-x^2+y\right)\right)\mathrm{d}x\\ \left(y\left(-x^2+y\right)\right)\mathrm{d}x+(x^3)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y\left(-x^2+y\right)\\ N(x,y)&=x^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y(-x^2+y)\right) \\ &= -x^2+2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^3\right) \\ &= 3x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x^3}\left(\left(-x^2+2y\right) - \left(3x^2\right)\right) \\ &= \frac{-4x^2+2y}{x^3}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{y\left(x^2-y\right)}\left(\left(3x^2\right) - \left(-x^2+2y\right)\right) \\ &= \frac{-4x^2+2y}{y\left(x^2-y\right)}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(3x^2) - (-x^2 + 2y)}{x\,(y\,(-x^2 + y)) - y\,(x^3)} \\ &= -\frac{2}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(t)} \\ &= \frac{1}{t^2} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2 y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^2 y^2} \left(y\left(-x^2 + y\right)\right) \\ &= \frac{-x^2 + y}{y\,x^2} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^2 y^2} \left(x^3\right) \\ &= \frac{x}{y^2} \end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x^2 + y}{y\,x^2}\right) + \left(\frac{x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-x^2 + y}{y\,x^2}\,\mathrm{d}x$$
$$\phi = \frac{-x^2 - y}{yx} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{yx} - \frac{-x^2 - y}{y^2 x} + f'(y) \tag{4}$$
$$= \frac{x}{y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^2 - y}{yx} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^2 - y}{yx}$$

The solution becomes

$$y = -\frac{x^2}{c_1 x + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{c_1 x + 1} \tag{1}$$

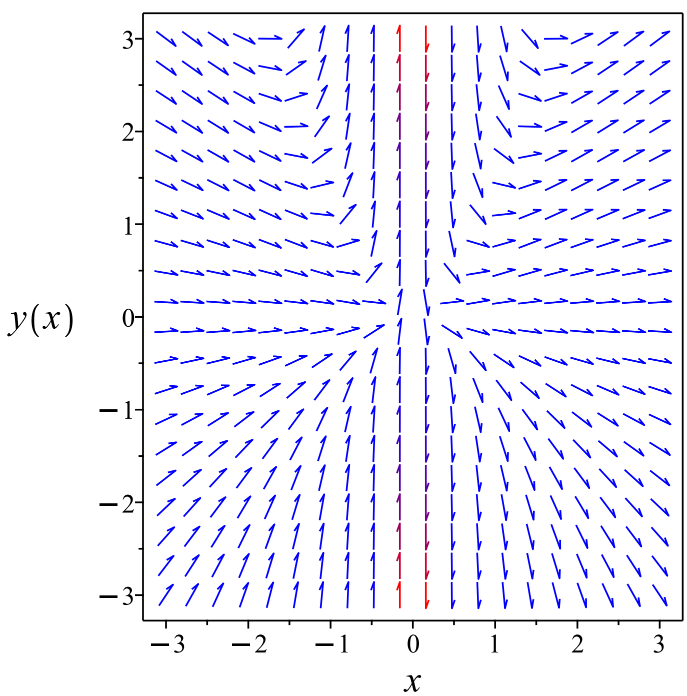


Figure 251: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{c_1 x + 1}$$

Verified OK.

5.7.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(-x^2 + y)}{x^3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} - \frac{y^2}{x^3}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x^3}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^3}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{3}{x^4} \\ f_1 f_2 &= -\frac{1}{x^4} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^3} - \frac{2u'(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 x}{c_1 + \frac{c_2}{x}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x}{c_3 + \frac{1}{x}}$$

Summary
The solution(s) found are the following

$$y = -\frac{x}{c_3 + \frac{1}{x}} \tag{1}$$

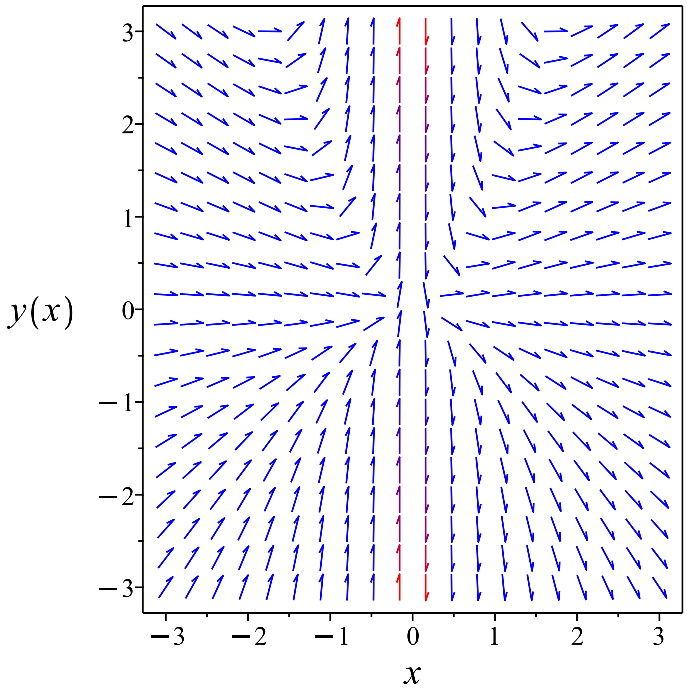


Figure 252: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_3 + \frac{1}{x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(y(x)*(y(x)-x^2)+x^3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{c_1 x - 1}$$

✓ Solution by Mathematica

Time used: 0.141 (sec). Leaf size: 22

```
DSolve[y[x]*(y[x]-x^2)+x^3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{-1 + c_1 x}$$
$$y(x) \to 0$$

5.8 problem 8

Internal problem ID [1973]
Internal file name [`OUTPUT/1973_Sunday_February_25_2024_06_43_55_AM_33078909/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y + xy^2 + (x - x^2 y)\, y' = 0}$$

5.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(yx+1)}{x\,(yx-1)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2+\frac{y(yx+1)\left(b_3-a_2\right)}{x\left(yx-1\right)}-\frac{y^2(yx+1)^2 a_3}{x^2\left(yx-1\right)^2} \\
& -\left(\frac{y^2}{x\left(yx-1\right)}-\frac{y(yx+1)}{x^2\left(yx-1\right)}-\frac{y^2(yx+1)}{x\left(yx-1\right)^2}\right)\left(x a_2+y a_3+a_1\right) \\
& -\left(\frac{yx+1}{x\left(yx-1\right)}+\frac{y}{yx-1}-\frac{y(yx+1)}{\left(yx-1\right)^2}\right)\left(x b_2+y b_3+b_1\right)=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^3 y^2 b_1-x^2 y^3 a_1-2 x^2 y^2 a_2-2 x^2 y^2 b_3-2 x^2 y b_1-2 x\, y^2 a_1-2 b_2 x^2+2 y^2 a_3-x b_1+y a_1}{x^2\left(yx-1\right)^2} = 0
$$

Setting the numerator to zero gives

$$
-x^3 y^2 b_1+x^2 y^3 a_1+2 x^2 y^2 a_2+2 x^2 y^2 b_3+2 x^2 y b_1+2 x\, y^2 a_1+2 b_2 x^2-2 y^2 a_3+x b_1-y a_1=0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$
a_1 v_1^2 v_2^3-b_1 v_1^3 v_2^2+2 a_2 v_1^2 v_2^2+2 b_3 v_1^2 v_2^2+2 a_1 v_1 v_2^2+2 b_1 v_1^2 v_2-2 a_3 v_2^2+2 b_2 v_1^2-a_1 v_2+b_1 v_1=0 \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_1v_1^3v_2^2 + a_1v_1^2v_2^3 + (2a_2 + 2b_3)\, v_1^2v_2^2 + 2b_1v_1^2v_2 + 2b_2v_1^2 + 2a_1v_1v_2^2 + b_1v_1 - 2a_3v_2^2 - a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ -b_1 &= 0 \\ 2b_1 &= 0 \\ 2b_2 &= 0 \\ 2a_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{y(yx + 1)}{x\,(yx - 1)}\right)(-x) \\ &= \frac{2y^2x}{yx - 1} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2y^2 x}{yx-1}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{2} + \frac{1}{2yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(yx + 1)}{x(yx - 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{2x^2 y} \\ S_y &= \frac{yx - 1}{2y^2 x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x \ln(y)\, y + 1}{2yx} = \frac{\ln(x)}{2} + c_1$$

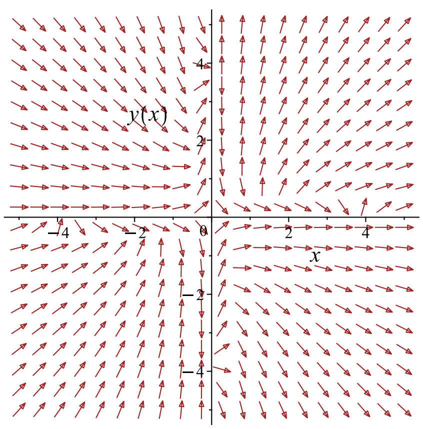
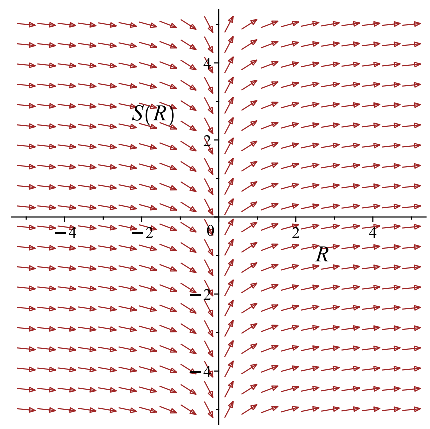
Which simplifies to

$$\frac{x \ln(y)\, y + 1}{2yx} = \frac{\ln(x)}{2} + c_1$$

Which gives

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{-2c_1}}{x^2}\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(yx+1)}{x(yx-1)}$ 	$R = x$ $S = \frac{\ln(y)\, yx + 1}{2yx}$	$\frac{dS}{dR} = \frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{1}{x \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-2c_1}}{x^2}\right)} \tag{1}$$

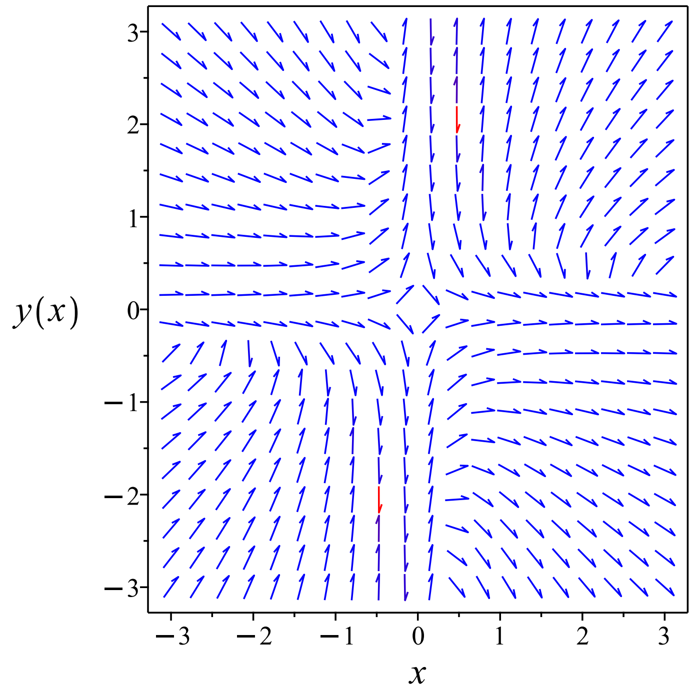


Figure 253: Slope field plot

Verification of solutions

$$y = -\frac{1}{x \,\text{LambertW}\left(-\frac{e^{-2c_1}}{x^2}\right)}$$

Verified OK.

5.8.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-y\,x^2 + x\right)\mathrm{d}y &= \left(-x\,y^2 - y\right)\mathrm{d}x \\ \left(x\,y^2 + y\right)\mathrm{d}x + \left(-y\,x^2 + x\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x\,y^2 + y \\ N(x, y) &= -y\,x^2 + x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\,y^2 + y\right) \\ &= 2yx + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-y\,x^2 + x\right) \\ &= -2yx + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{-y\,x^2 + x}((2yx + 1) - (-2yx + 1)) \\ &= -\frac{4y}{yx - 1} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{x\,y^2 + y}((-2yx + 1) - (2yx + 1)) \\ &= -\frac{4x}{yx + 1} \end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(-2yx + 1) - (2yx + 1)}{x\,(x\,y^2 + y) - y\,(-y\,x^2 + x)} \\ &= -\frac{2}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{x^2y^2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2y^2}(x\,y^2 + y) \\
&= \frac{yx + 1}{y\,x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2y^2}(-y\,x^2 + x) \\
&= \frac{-yx + 1}{x\,y^2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{yx + 1}{y\,x^2}\right) + \left(\frac{-yx + 1}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{yx+1}{y\,x^2} \, \mathrm{d}x$$
$$\phi = \ln(x) - \frac{1}{yx} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x\,y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-yx+1}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{-yx+1}{x\,y^2} = \frac{1}{x\,y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \frac{1}{yx} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{1}{yx} - \ln(y)$$

The solution becomes

$$y = -\frac{1}{x \,\mathrm{LambertW}\left(-\frac{e^{c_1}}{x^2}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x \,\mathrm{LambertW}\left(-\frac{e^{c_1}}{x^2}\right)} \tag{1}$$

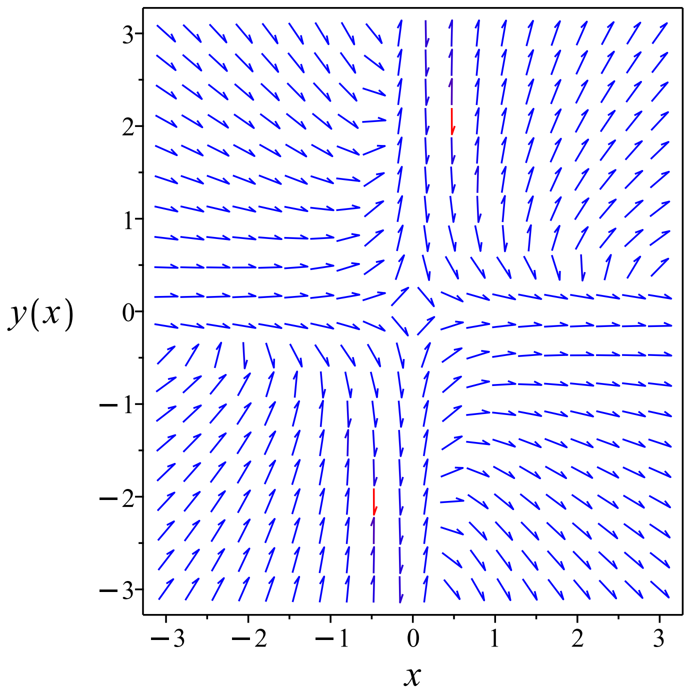


Figure 254: Slope field plot

Verification of solutions

$$y = -\frac{1}{x \,\mathrm{LambertW}\left(-\frac{e^{c_1}}{x^2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve((y(x)+x*y(x)^2)+(x-x^2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{\operatorname{LambertW}\left(-\frac{c_1}{x^2}\right) x}$$

✓ Solution by Mathematica

Time used: 5.848 (sec). Leaf size: 35

```
DSolve[(y[x]+x*y[x]^2)+(x-x^2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{xW\left(\frac{e^{-1+\frac{9c_1}{2^{2/3}}}}{x^2}\right)}$$
$$y(x) \to 0$$

5.9 problem 9

Internal problem ID [1974]
Internal file name [`OUTPUT/1974_Sunday_February_25_2024_06_43_56_AM_78755235/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

[`y=_G(x,y')`]

Unable to solve or complete the solution.

$$\left(x - x\sqrt{x^2 - y^2}\right) y' - y = 0$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`[0, (x^2-y^2)^(1/2)/((x^2-y^2)^(1/2)-1)]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 27

```
dsolve((x-x*sqrt(x^2-y(x)^2))*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x)-\arctan\left(\frac{y(x)}{\sqrt{x^{2}-y\left(x\right)^{2}}}\right)-c_{1}=0$$

✓ Solution by Mathematica

Time used: 0.529 (sec). Leaf size: 29

```
DSolve[(x-x*Sqrt[x^2-y[x]^2])*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\arctan\left(\frac{\sqrt{x^{2}-y(x)^{2}}}{y(x)}\right)+y(x)=c_{1},y(x)\right]$$

5.10 problem 10

Internal problem ID [1975]
Internal file name [`OUTPUT/1975_Sunday_February_25_2024_06_43_57_AM_63678344/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{2yx + \left(-x^2 + y\right) y' = 0}$$

5.10.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2yx}{-x^2 + y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{2yx(b_3 - a_2)}{-x^2 + y} - \frac{4y^2x^2a_3}{(-x^2 + y)^2} - \left(-\frac{2y}{-x^2 + y} - \frac{4y\,x^2}{(-x^2 + y)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{2x}{-x^2 + y} + \frac{2yx}{(-x^2 + y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4b_2 + 2y^2x^2a_3 + 2x^3b_1 - 2x^2ya_1 + 2x^2yb_2 - 4x\,y^2a_2 + 2x\,y^2b_3 - 2y^3a_3 - 2y^2a_1 - y^2b_2}{(x^2 - y)^2} = 0$$

Setting the numerator to zero gives

$$-x^4b_2 - 2y^2x^2a_3 - 2x^3b_1 + 2x^2ya_1 - 2x^2yb_2 + 4x\,y^2a_2 - 2x\,y^2b_3 + 2y^3a_3 + 2y^2a_1 + y^2b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_3v_1^2v_2^2 - b_2v_1^4 + 2a_1v_1^2v_2 + 4a_2v_1v_2^2 + 2a_3v_2^3 - 2b_1v_1^3 - 2b_2v_1^2v_2 - 2b_3v_1v_2^2 + 2a_1v_2^2 + b_2v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_2 v_1^4 - 2b_1 v_1^3 - 2a_3 v_1^2 v_2^2 + (2a_1 - 2b_2) v_1^2 v_2 + (4a_2 - 2b_3) v_1 v_2^2 + 2a_3 v_2^3 + (2a_1 + b_2) v_2^2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_3 &= 0 \\ 2a_3 &= 0 \\ -2b_1 &= 0 \\ -b_2 &= 0 \\ 2a_1 - 2b_2 &= 0 \\ 2a_1 + b_2 &= 0 \\ 4a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 2y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 2y - \left(-\frac{2yx}{-x^2 + y}\right)(x) \\ &= -\frac{2y^2}{x^2 - y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{2y^2}{x^2 - y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln (y)}{2} + \frac{x^2}{2y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2yx}{-x^2 + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{y} \\ S_y &= \frac{-x^2 + y}{2y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)\, y + x^2}{2y} = c_1$$

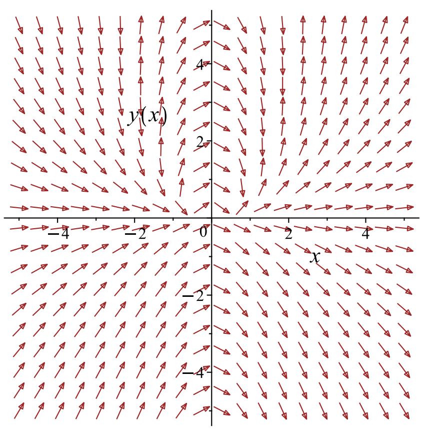
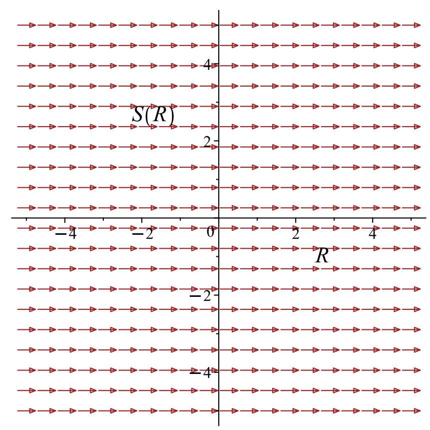
Which simplifies to

$$\frac{\ln(y)\, y + x^2}{2y} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x^2 \mathrm{e}^{-2c_1}\right)+2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2yx}{-x^2+y}$ 	$R = x$ $S = \frac{\ln(y)\, y + x^2}{2y}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x^2 \mathrm{e}^{-2c_1}\right)+2c_1} \tag{1}$$

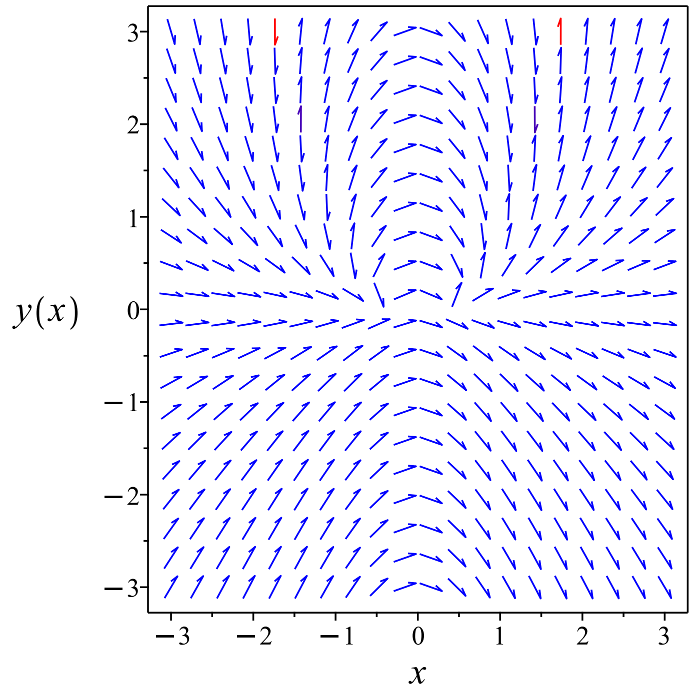


Figure 255: Slope field plot

Verification of solutions

$$y = e^{\text{LambertW}\left(-x^2 e^{-2c_1}\right)+2c_1}$$

Verified OK.

5.10.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(-x^2 + y) \,\mathrm{d}y &= (-2yx) \,\mathrm{d}x \\
(2yx) \,\mathrm{d}x + (-x^2 + y) \,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 2yx \\
N(x, y) &= -x^2 + y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2yx) \\
&= 2x
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2 + y) \\
&= -2x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{-x^2 + y}((2x) - (-2x)) \\ &= -\frac{4x}{x^2 - y} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{2yx}((-2x) - (2x)) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2}(2yx) \\ &= \frac{2x}{y} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(-x^2 + y) \\
&= \frac{-x^2 + y}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2x}{y}\right) + \left(\frac{-x^2 + y}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2x}{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2}{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{x^2}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^2+y}{y^2}$. Therefore equation (4) becomes

$$
\frac{-x^2 + y}{y^2} = -\frac{x^2}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(y) + \frac{x^2}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(y) + \frac{x^2}{y}$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x^2 \mathrm{e}^{-c_1}\right)+c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x^2 \mathrm{e}^{-c_1}\right)+c_1} \tag{1}$$

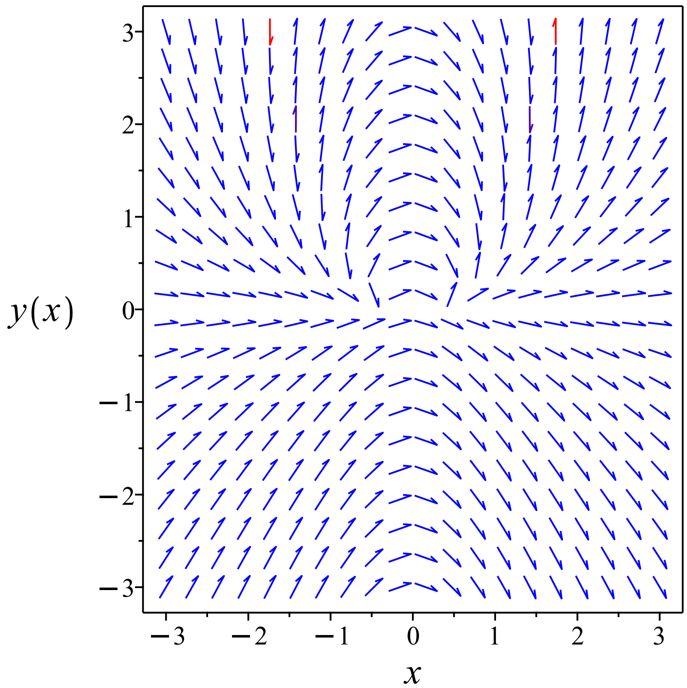


Figure 256: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}\left(-x^2 e^{-c_1}\right)+c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 18

```
dsolve(2*x*y(x)+(y(x)-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{\text{LambertW}\left(-c_1x^2\right)}$$

✓ Solution by Mathematica

Time used: 3.007 (sec). Leaf size: 285

```
DSolve[2*x*y[x]+(y[x]-x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\left(2-\frac{2\left(x^3+2xy(x)\right)}{\sqrt[3]{x^3}(x^2-y(x))}\right)\left(\frac{x^3+2xy(x)}{\sqrt[3]{x^3}(x^2-y(x))}+2\right)\left(\left(1-\frac{x\left(x^2+2y(x)\right)}{\sqrt[3]{x^3}(x^2-y(x))}\right)\log\left(\frac{2-\frac{2\left(x^3+2xy(x)\right)}{\sqrt[3]{x^3}(x^2-y(x))}}{\sqrt[3]{2}}\right)+\left(\frac{x^3+2x\cdots}{\sqrt[3]{x^3}(x^2\cdots}\right.\right.}{9\sqrt[3]{2}\left(-\frac{(x^2+2y(x))^3}{(x^2-y(x))^3}+\frac{3(x^3+2xy(x))}{\sqrt[3]{x^3}(x^2-y(x))}-2\right)}\right.$$

$$\left.+c_1, y(x)\right]$$

5.11 problem 11

Internal problem ID [1976]
Internal file name [`OUTPUT/1976_Sunday_February_25_2024_06_43_58_AM_77481508/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y - x(x^2 y - 1) y' = 0}$$

5.11.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y}{x(y x^2 - 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{y(b_3 - a_2)}{x\left(y\,x^2-1\right)} - \frac{y^2 a_3}{x^2\left(y\,x^2-1\right)^2} \\ & - \left(-\frac{y}{x^2\left(y\,x^2-1\right)} - \frac{2y^2}{\left(y\,x^2-1\right)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(\frac{1}{x\left(y\,x^2-1\right)} - \frac{yx}{\left(y\,x^2-1\right)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^6y^2b_2 - 2x^4yb_2 + 2x^3y^2a_2 + x^3y^2b_3 + 3x^2y^3a_3 + 3x^2y^2a_1 + 2b_2x^2 - 2y^2a_3 + xb_1 - ya_1}{x^2\left(y\,x^2-1\right)^2} = 0$$

Setting the numerator to zero gives

$$x^6y^2b_2 - 2x^4yb_2 + 2x^3y^2a_2 + x^3y^2b_3 + 3x^2y^3a_3 + 3x^2y^2a_1 + 2b_2x^2 - 2y^2a_3 + xb_1 - ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & b_2v_1^6v_2^2 + 2a_2v_1^3v_2^2 + 3a_3v_1^2v_2^3 - 2b_2v_1^4v_2 + b_3v_1^3v_2^2 \\ & + 3a_1v_1^2v_2^2 - 2a_3v_2^2 + 2b_2v_1^2 - a_1v_2 + b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2 v_1^6 v_2^2 - 2b_2 v_1^4 v_2 + (2a_2 + b_3) v_1^3 v_2^2 + 3a_3 v_1^2 v_2^3 + 3a_1 v_1^2 v_2^2 + 2b_2 v_1^2 + b_1 v_1 - 2a_3 v_2^2 - a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ b_2 &= 0 \\ -a_1 &= 0 \\ 3a_1 &= 0 \\ -2a_3 &= 0 \\ 3a_3 &= 0 \\ -2b_2 &= 0 \\ 2b_2 &= 0 \\ 2a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= -2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= -2y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -2y - \left(\frac{y}{x\,(y\,x^2 - 1)}\right)(x) \\
&= \frac{-2x^2y^2 + y}{y\,x^2 - 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x^2y^2+y}{y\,x^2-1}} dy
\end{aligned}$$

Which results in

$$S = -\ln(y) + \frac{\ln(2y\,x^2 - 1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{x\,(y\,x^2 - 1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2yx}{2y\,x^2-1} \\
S_y &= -\frac{1}{y}+\frac{x^2}{2y\,x^2-1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

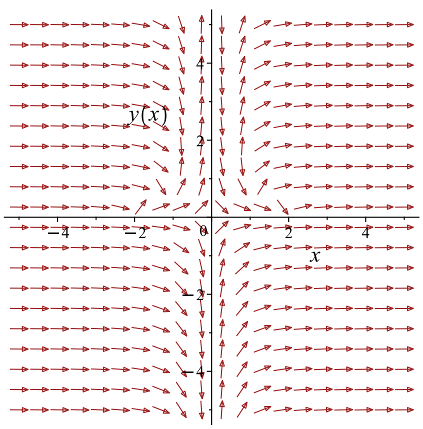
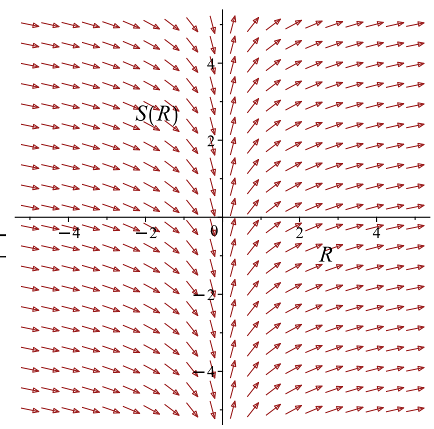
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(y) + \frac{\ln(2x^2y-1)}{2} = \ln(x) + c_1
$$

Which simplifies to

$$
-\ln(y) + \frac{\ln(2x^2y-1)}{2} = \ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x(y\,x^2-1)}$ 	$R = x$ $S = -\ln(y) + \frac{\ln(2y\,x^2 -}{2}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$-\ln(y) + \frac{\ln(2x^2y - 1)}{2} = \ln(x) + c_1 \tag{1}$$

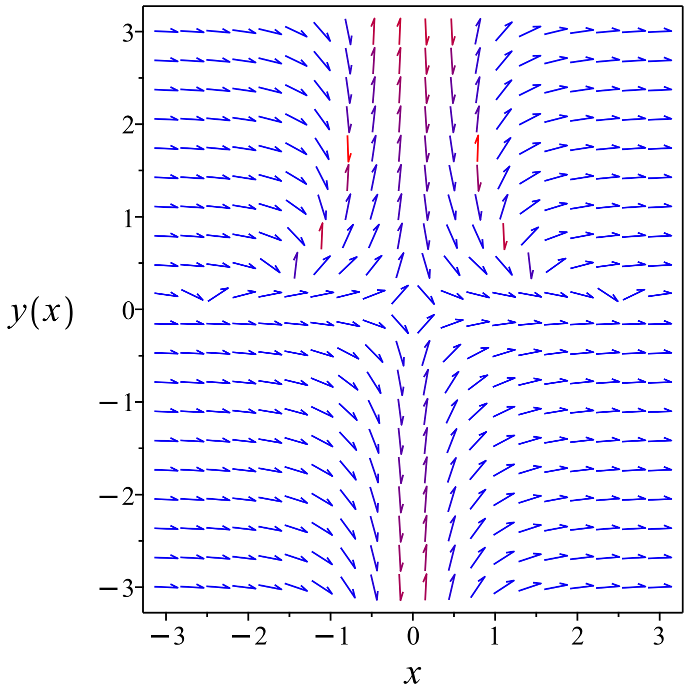


Figure 257: Slope field plot

Verification of solutions

$$-\ln(y)+\frac{\ln(2x^2y-1)}{2}=\ln(x)+c_1$$

Verified OK.

5.11.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x\left(y\,x^2-1\right)\right)\mathrm{d}y &= (-y)\,\mathrm{d}x\\ (y)\,\mathrm{d}x + \left(-x\left(y\,x^2-1\right)\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\\ N(x,y) &= -x\left(y\,x^2-1\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x\left(y\,x^2-1\right)\right)\\ &= -3y\,x^2+1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x\,(y\,x^2 - 1)}\left((1) - \left(-3y\,x^2 + 1\right)\right) \\
&= -\frac{3xy}{y\,x^2 - 1}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y}\left(\left(-3y\,x^2 + 1\right) - (1)\right) \\
&= -3x^2
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{\left(-3y\,x^2 + 1\right) - (1)}{x\,(y) - y\left(-x\,(y\,x^2 - 1)\right)} \\
&= -\frac{3}{yx}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{3}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{3}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(t)} \\
&= \frac{1}{t^3}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{y^3 x^3}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^3 x^3}(y) \\
&= \frac{1}{y^2 x^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^3 x^3}\left(-x\left(y\,x^2-1\right)\right) \\
&= \frac{-y\,x^2+1}{y^3 x^2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{1}{y^2 x^3}\right) + \left(\frac{-y\,x^2+1}{y^3 x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \frac{1}{y^2 x^3} \,\mathrm{d}x \\
\phi &= -\frac{1}{2x^2 y^2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x^2 y^3} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-y\,x^2+1}{y^3 x^2}$. Therefore equation (4) becomes

$$
\frac{-y\,x^2 + 1}{y^3 x^2} = \frac{1}{x^2 y^3} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{1}{y^2}\right) \mathrm{d}y \\
f(y) &= \frac{1}{y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{1}{2x^2 y^2} + \frac{1}{y} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{2x^2y^2} + \frac{1}{y}$$

Summary

The solution(s) found are the following

$$-\frac{1}{2x^2y^2} + \frac{1}{y} = c_1 \tag{1}$$

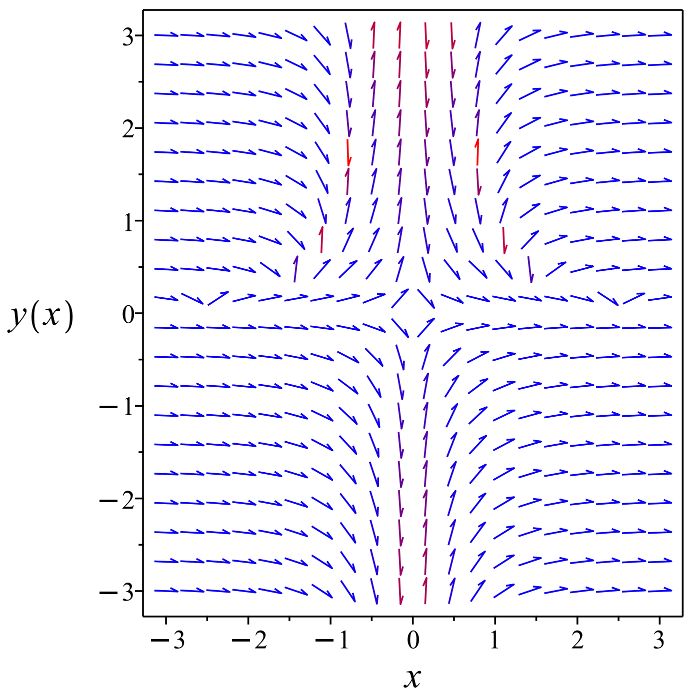


Figure 258: Slope field plot

Verification of solutions

$$-\frac{1}{2x^2y^2} + \frac{1}{y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 45

```
dsolve(y(x)=x*(x^2*y(x)-1)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{x + \sqrt{x^2 - c_1}}{c_1 x}$$
$$y(x) = \frac{x - \sqrt{x^2 - c_1}}{c_1 x}$$

✓ Solution by Mathematica

Time used: 1.509 (sec). Leaf size: 77

```
DSolve[y[x]==x*(x^2*y[x]-1)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{x^2 + \sqrt{-\frac{1}{x^3}} x^2 \sqrt{-x\left(x^2 + c_1\right)}}$$
$$y(x) \rightarrow \frac{x}{x^3 + \frac{\sqrt{-x(x^2+c_1)}}{\sqrt{-\frac{1}{x^3}}}}$$
$$y(x) \rightarrow 0$$

5.12 problem 12

Internal problem ID [1977]
Internal file name [`OUTPUT/1977_Sunday_February_25_2024_06_43_59_AM_17376835/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{e^{x} y' - 2 x y^{2} - e^{x} y = 0}$$

5.12.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= y(2yx + e^{x}) e^{-x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 137: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^2 \mathrm{e}^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 \mathrm{e}^{-x}} dy \end{aligned}$$

Which results in

$$S = -\frac{\mathrm{e}^x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y(2yx + \mathrm{e}^x)\, \mathrm{e}^{-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\mathrm{e}^x}{y} \\ S_y &= \frac{\mathrm{e}^x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{e^x}{y} = x^2 + c_1$$

Which simplifies to

$$-\frac{e^x}{y} = x^2 + c_1$$

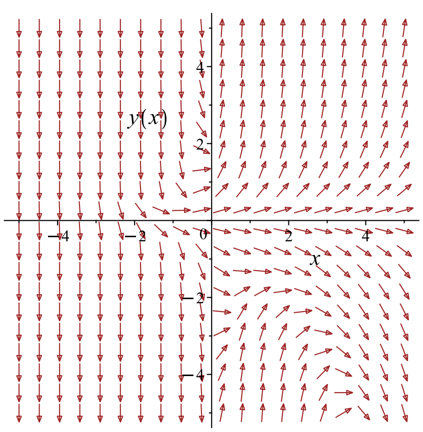
Which gives

$$y = -\frac{e^x}{x^2 + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = y(2yx + e^x)\,e^{-x}$



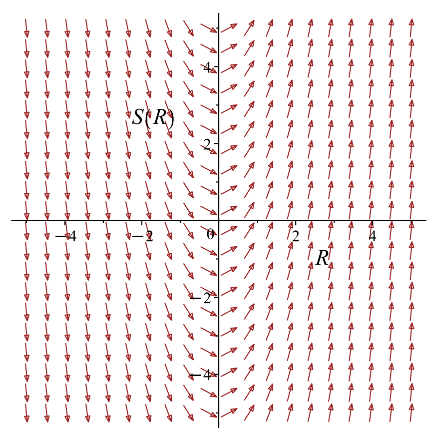
</td><td>

$R = x$

$S = -\frac{e^x}{y}$

</td><td>

$\frac{dS}{dR} = 2R$



</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = -\frac{e^x}{x^2 + c_1} \tag{1}$$

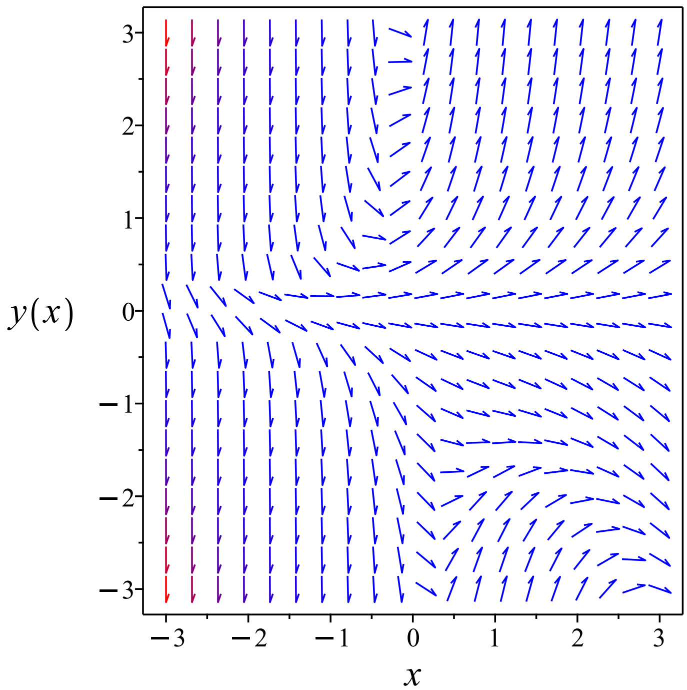


Figure 259: Slope field plot

Verification of solutions

$$y = -\frac{e^x}{x^2 + c_1}$$

Verified OK.

5.12.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y(2yx + e^x)\, e^{-x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = y + 2x\, e^{-x} y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= 1 \\ f_1(x) &= 2x\,\mathrm{e}^{-x} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{y} + 2x\,\mathrm{e}^{-x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= w(x) + 2x\,\mathrm{e}^{-x} \\ w' &= -w - 2x\,\mathrm{e}^{-x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= -2x\,\mathrm{e}^{-x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + w(x) = -2x\,\mathrm{e}^{-x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1 dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-2x\,\mathrm{e}^{-x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(w\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})\left(-2x\,\mathrm{e}^{-x}\right) \\
\mathrm{d}(w\,\mathrm{e}^{x}) &= (-2x)\ \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
w\,\mathrm{e}^{x} &= \int -2x\,\mathrm{d}x \\
w\,\mathrm{e}^{x} &= -x^2 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$
w(x) = -x^2\mathrm{e}^{-x} + c_1\mathrm{e}^{-x}
$$

which simplifies to

$$
w(x) = \mathrm{e}^{-x}\left(-x^2 + c_1\right)
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = \mathrm{e}^{-x}\left(-x^2 + c_1\right)
$$

Or

$$
y = \frac{\mathrm{e}^{x}}{-x^2 + c_1}
$$

Which is simplified to

$$y = \frac{e^x}{-x^2 + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{e^x}{-x^2 + c_1} \tag{1}$$

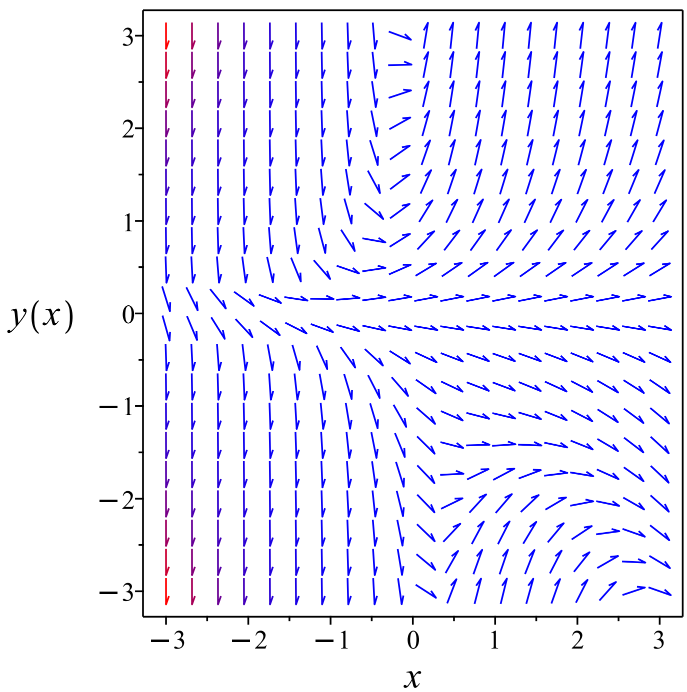


Figure 260: Slope field plot

Verification of solutions

$$y = \frac{e^x}{-x^2 + c_1}$$

Verified OK.

5.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^x)\,\mathrm{d}y&=\left(2x\,y^2+\mathrm{e}^x y\right)\mathrm{d}x\\ \left(-2x\,y^2-\mathrm{e}^x y\right)\mathrm{d}x+(\mathrm{e}^x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-2x\,y^2-\mathrm{e}^x y\\ N(x,y)&=\mathrm{e}^x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2x\,y^2 - \mathrm{e}^x y\right)\\ &= -4yx - \mathrm{e}^x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^x)\\ &= \mathrm{e}^x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \mathrm{e}^{-x}((-4yx - \mathrm{e}^x) - (\mathrm{e}^x))\\ &= -4\,\mathrm{e}^{-x}xy - 2\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{y\,(2yx + \mathrm{e}^x)}((\mathrm{e}^x) - (-4yx - \mathrm{e}^x))\\ &= -\frac{2}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y}\\ &= e^{\int -\frac{2}{y}\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)}\\ &= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2}\left(-2x\,y^2 - \mathrm{e}^x y\right) \\
&= \frac{-2yx - \mathrm{e}^x}{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(\mathrm{e}^x) \\
&= \frac{\mathrm{e}^x}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-2yx - \mathrm{e}^x}{y}\right) + \left(\frac{\mathrm{e}^x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-2yx - \mathrm{e}^x}{y}\,\mathrm{d}x \\
\phi &= \frac{-y\,x^2 - \mathrm{e}^x}{y} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -\frac{-y\,x^2-\mathrm{e}^x}{y^2}-\frac{x^2}{y}+f'(y) \\ &= \frac{\mathrm{e}^x}{y^2}+f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{\mathrm{e}^x}{y^2}$. Therefore equation (4) becomes

$$\frac{\mathrm{e}^x}{y^2}=\frac{\mathrm{e}^x}{y^2}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\frac{-y\,x^2-\mathrm{e}^x}{y}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=\frac{-y\,x^2-\mathrm{e}^x}{y}$$

The solution becomes

$$y=-\frac{\mathrm{e}^x}{x^2+c_1}$$

Summary

The solution(s) found are the following

$$y=-\frac{\mathrm{e}^x}{x^2+c_1} \tag{1}$$

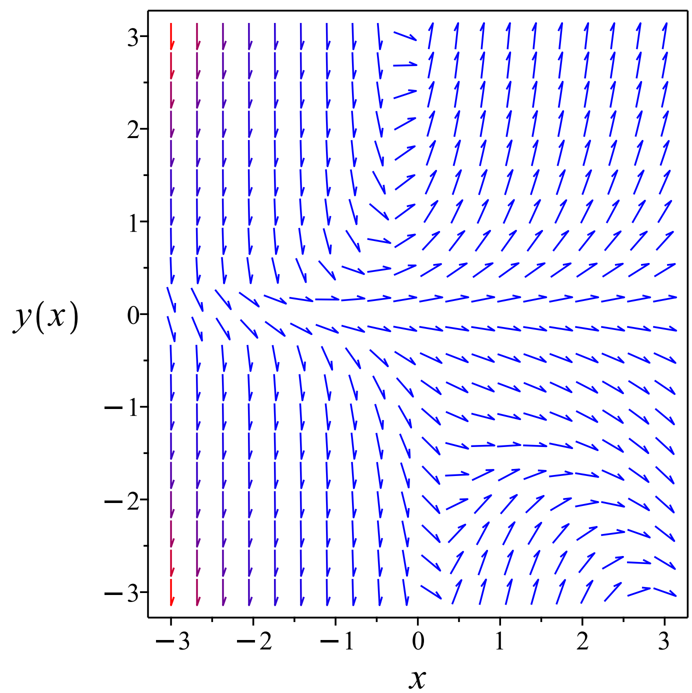


Figure 261: Slope field plot

Verification of solutions

$$y = -\frac{e^{x}}{x^{2}+c_{1}}$$

Verified OK.

5.12.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= y(2yx + e^{x})\, e^{-x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2\, e^{-x} x\, y^{2} + y$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 1$ and $f_2(x) = 2x\,\mathrm{e}^{-x}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{2x\,\mathrm{e}^{-x} u}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= 2\,\mathrm{e}^{-x} - 2x\,\mathrm{e}^{-x} \\
f_1 f_2 &= 2x\,\mathrm{e}^{-x} \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
2x\,\mathrm{e}^{-x} u''(x) - 2\,\mathrm{e}^{-x} u'(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_2 x^2 + c_1
$$

The above shows that

$$
u'(x) = 2c_2 x
$$

Using the above in (1) gives the solution

$$
y = -\frac{c_2 \mathrm{e}^{x}}{c_2 x^2 + c_1}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = -\frac{\mathrm{e}^{x}}{x^2 + c_3}
$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{x}}{x^{2}+c_{3}} \tag{1}$$

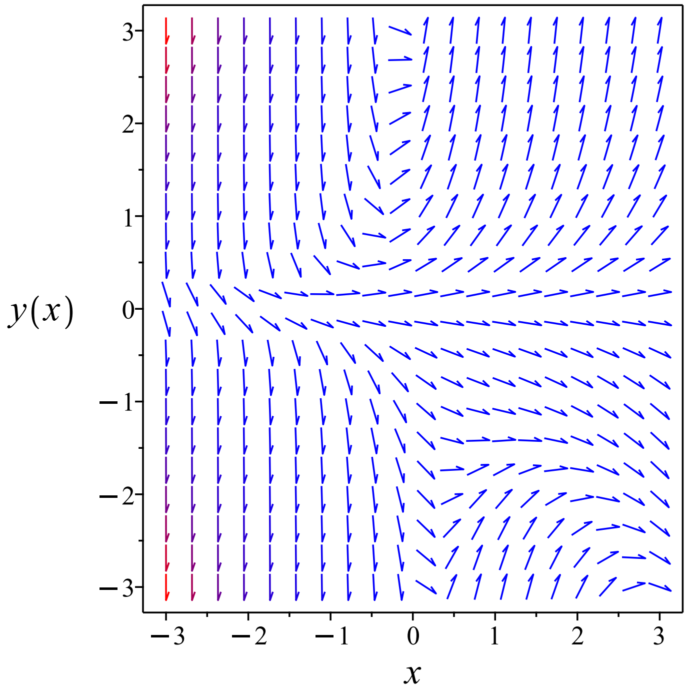


Figure 262: Slope field plot

Verification of solutions

$$y = -\frac{e^{x}}{x^{2}+c_{3}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(exp(x)*diff(y(x),x)=2*x*y(x)^2+exp(x)*y(x),y(x), singsol=all)
```

$$y(x) = \frac{e^x}{-x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.286 (sec). Leaf size: 25

```
DSolve[Exp[x]*y'[x]==2*x*y[x]^2+Exp[x]*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^x}{x^2 - c_1}$$
$$y(x) \to 0$$

5.13 problem 13

Internal problem ID [1978]
Internal file name [`OUTPUT/1978_Sunday_February_25_2024_06_43_59_AM_62142406/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{(x^2 + y^2 + x)\, y' - y = 0}$$

5.13.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + y^2 + x)\,\mathrm{d}y &= (y)\,\mathrm{d}x\\ (-y)\,\mathrm{d}x + (x^2 + y^2 + x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y\\ N(x, y) &= x^2 + y^2 + x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y^2 + x)\\ &= 1 + 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = -y$ and $N = x^2 + y^2 + x$ by this integrating factor the ode becomes exact. The new M, N are

$$M = -\frac{y}{x^2+y^2}$$
$$N = \frac{x^2+y^2+x}{x^2+y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{x^2+y^2+x}{x^2+y^2}\right)\mathrm{d}y = \left(\frac{y}{x^2+y^2}\right)\mathrm{d}x$$
$$\left(-\frac{y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{x^2+y^2+x}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{y}{x^2+y^2} \\
N(x, y) &= \frac{x^2+y^2+x}{x^2+y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) \\
&= \frac{-x^2+y^2}{\left(x^2+y^2\right)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x^2+y^2+x}{x^2+y^2}\right) \\
&= \frac{-x^2+y^2}{\left(x^2+y^2\right)^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{y}{x^2+y^2} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\arctan\left(\frac{x}{y}\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \\ &= \frac{x}{x^2+y^2} + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2+y^2+x}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{x^2+y^2+x}{x^2+y^2} = \frac{x}{x^2+y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\, \mathrm{d}y &= \int (1)\, \mathrm{d}y \\ f(y) &= y + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan\left(\frac{x}{y}\right) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan\left(\frac{x}{y}\right) + y$$

Summary

The solution(s) found are the following

$$-\arctan\left(\frac{x}{y}\right) + y = c_1 \tag{1}$$

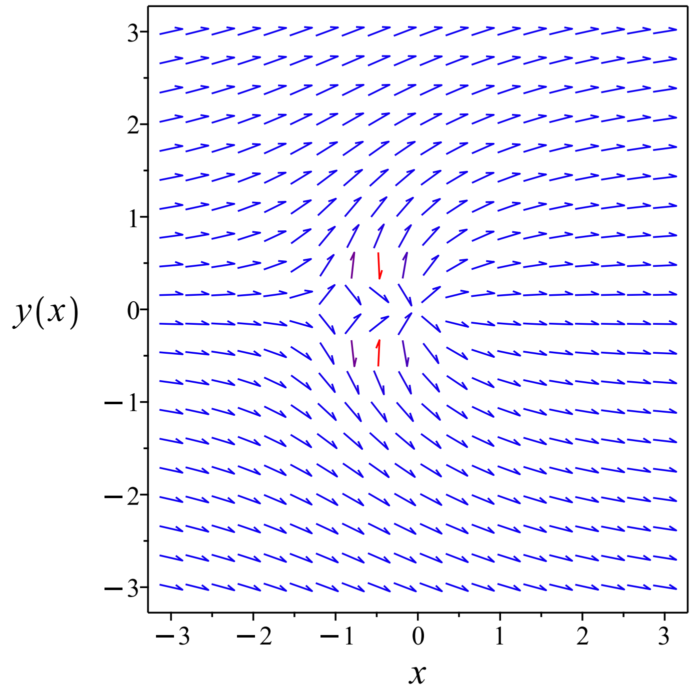


Figure 263: Slope field plot

Verification of solutions

$$-\arctan\left(\frac{x}{y}\right)+y=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying Riccati sub-methods:
   <- Riccati particular polynomial solution successful
<- inverse_Riccati successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 40

```
dsolve((x^2+y(x)^2+x)*diff(y(x),x)=y(x),y(x), singsol=all)
```

$$\frac{e^{-2iy(x)}(ix+y(x))+2c_1(x+iy(x))}{2iy\left(x\right)+2x}=0$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 18

```
DSolve[(x^2+y[x]^2+x)*y'[x]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[y(x)-\arctan\left(\frac{x}{y(x)}\right)=c_1,y(x)\right]$$

5.14 problem 14

Internal problem ID [1979]
Internal file name [`OUTPUT/1979_Sunday_February_25_2024_06_44_00_AM_59080884/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{(2x + 3x^2y)\, y' + y + 2xy^2 = 0}$$

5.14.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(2yx + 1)}{x\,(3yx + 2)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{y(2yx+1)\,(b_3-a_2)}{x\,(3yx+2)} - \frac{y^2(2yx+1)^2\,a_3}{x^2\,(3yx+2)^2} \\
&- \left(-\frac{2y^2}{x\,(3yx+2)} + \frac{y(2yx+1)}{x^2\,(3yx+2)} + \frac{3y^2(2yx+1)}{x\,(3yx+2)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{2yx+1}{x\,(3yx+2)} - \frac{2y}{3yx+2} + \frac{3y(2yx+1)}{(3yx+2)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$\frac{15x^4y^2b_2 - 10x^2y^4a_3 + 6x^3y^2b_1 - 6x^2y^3a_1 + 20x^3yb_2 + x^2y^2a_2 + x^2y^2b_3 - 10x\,y^3a_3 + 8x^2yb_1 - 6x\,y^2a_1 +}{x^2\,(3yx+2)^2}$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&15x^4y^2b_2 - 10x^2y^4a_3 + 6x^3y^2b_1 - 6x^2y^3a_1 + 20x^3yb_2 + x^2y^2a_2 + x^2y^2b_3 \\
&- 10x\,y^3a_3 + 8x^2yb_1 - 6x\,y^2a_1 + 6b_2x^2 - 3y^2a_3 + 2xb_1 - 2ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-10a_3v_1^2v_2^4 + 15b_2v_1^4v_2^2 - 6a_1v_1^2v_2^3 + 6b_1v_1^3v_2^2 + a_2v_1^2v_2^2 - 10a_3v_1v_2^3 + 20b_2v_1^3v_2 \\
&+ b_3v_1^2v_2^2 - 6a_1v_1v_2^2 + 8b_1v_1^2v_2 - 3a_3v_2^2 + 6b_2v_1^2 - 2a_1v_2 + 2b_1v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$15b_2v_1^4v_2^2 + 6b_1v_1^3v_2^2 + 20b_2v_1^3v_2 - 10a_3v_1^2v_2^4 - 6a_1v_1^2v_2^3 + (a_2 + b_3)\, v_1^2v_2^2 \\ + 8b_1v_1^2v_2 + 6b_2v_1^2 - 10a_3v_1v_2^3 - 6a_1v_1v_2^2 + 2b_1v_1 - 3a_3v_2^2 - 2a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -6a_1 &= 0 \\ -2a_1 &= 0 \\ -10a_3 &= 0 \\ -3a_3 &= 0 \\ 2b_1 &= 0 \\ 6b_1 &= 0 \\ 8b_1 &= 0 \\ 6b_2 &= 0 \\ 15b_2 &= 0 \\ 20b_2 &= 0 \\ a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(2yx+1)}{x\,(3yx+2)}\right)(-x) \\
&= \frac{x\,y^2+y}{3yx+2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x\,y^2+y}{3yx+2}} dy
\end{aligned}
$$

Which results in

$$
S = \ln(yx+1) + 2\ln(y)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y(2yx+1)}{x\,(3yx+2)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{y}{yx+1}\\
S_y &= \frac{x}{yx+1}+\frac{2}{y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

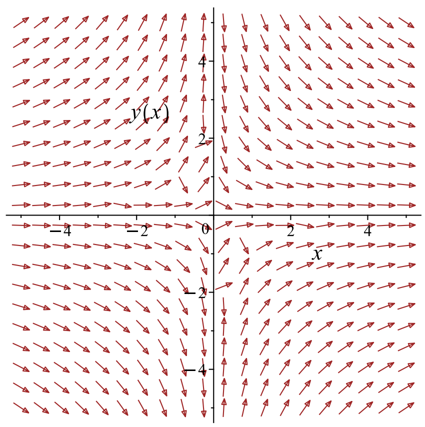
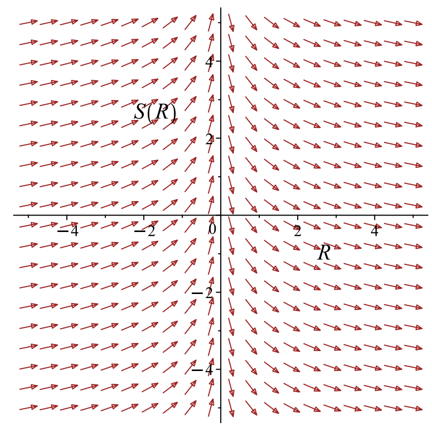
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(yx+1) + 2\ln(y) = -\ln(x) + c_1
$$

Which simplifies to

$$
\ln(yx+1) + 2\ln(y) = -\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2yx+1)}{x(3yx+2)}$ 	$R = x$ $S = \ln(yx+1) + 2\ln(y)$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\ln(yx+1) + 2\ln(y) = -\ln(x) + c_1 \tag{1}$$

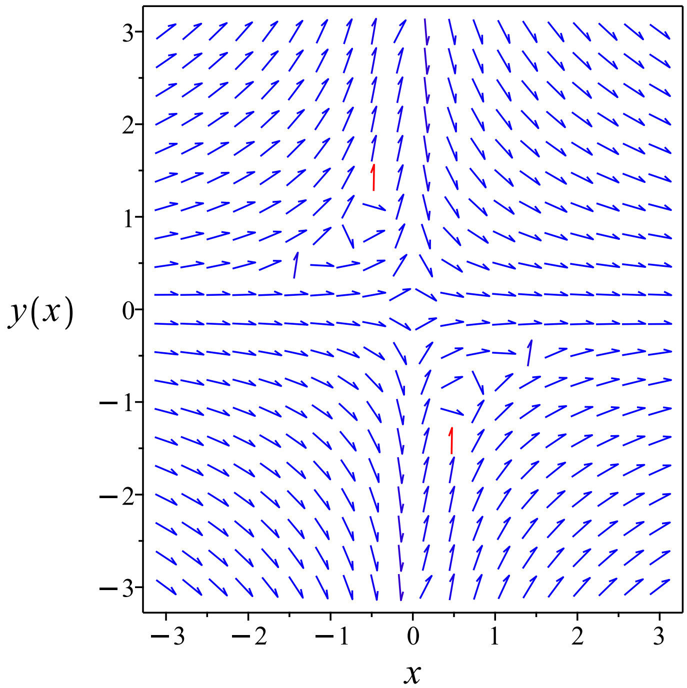


Figure 264: Slope field plot

Verification of solutions

$$\ln(yx+1)+2\ln(y)=-\ln(x)+c_1$$

Verified OK.

5.14.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3y\, x^2 + 2x)\, \mathrm{d}y &= (-2x\, y^2 - y)\, \mathrm{d}x \\ (2x\, y^2 + y)\, \mathrm{d}x + (3y\, x^2 + 2x)\, \mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2x\, y^2 + y \\ N(x, y) &= 3y\, x^2 + 2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x\, y^2 + y) \\ &= 4yx + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3y\, x^2 + 2x) \\ &= 6yx + 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{3y\,x^2 + 2x}((4yx + 1) - (6yx + 2)) \\
&= \frac{-2yx - 1}{3y\,x^2 + 2x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2x\,y^2 + y}((6yx + 2) - (4yx + 1)) \\
&= \frac{1}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{1}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(y)} \\
&= y
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= y\left(2x\,y^2 + y\right) \\
&= 2x\,y^3 + y^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y\left(3y\,x^2 + 2x\right) \\
&= 3x^2y^2 + 2yx
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(2x\,y^3 + y^2\right) + \left(3x^2y^2 + 2yx\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 2x\,y^3 + y^2\,\mathrm{d}x \\
\phi &= y^2x(yx + 1) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= 2yx(yx + 1) + x^2y^2 + f'(y) \qquad (4) \\
&= 3x^2y^2 + 2yx + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x^2y^2 + 2yx$. Therefore equation (4) becomes

$$
3x^2y^2 + 2yx = 3x^2y^2 + 2yx + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2x(yx + 1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^2x(yx + 1)$$

Summary
The solution(s) found are the following

$$y^2x(yx + 1) = c_1 \tag{1}$$

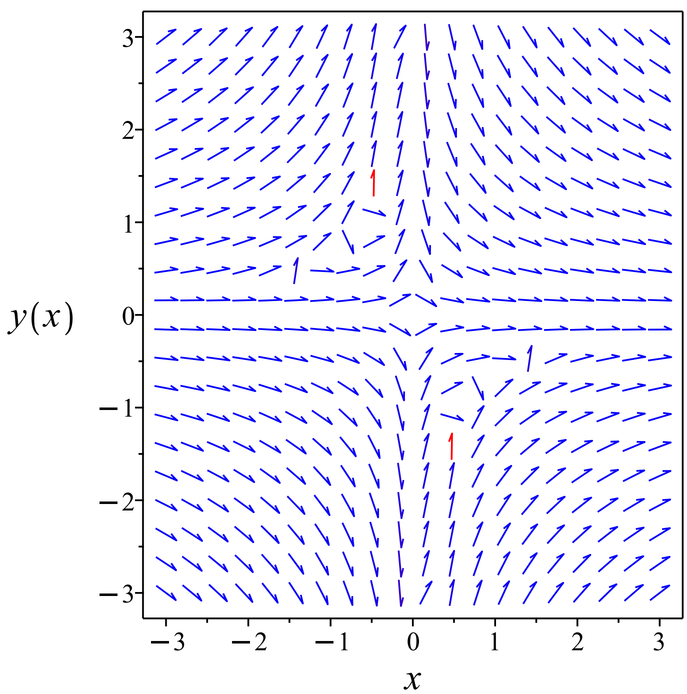


Figure 265: Slope field plot

Verification of solutions

$$y^2x(yx + 1) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 380

```
dsolve((2*x+3*x^2*y(x))*diff(y(x),x)+(y(x)+2*y(x)^2*x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2c_1^2 2^{\frac{1}{3}} - 2c_1\left(\left(3\sqrt{\frac{-12c_1+81x}{x}}\,x - 2c_1 + 27x\right)c_1^2\right)^{\frac{1}{3}} + 2^{\frac{2}{3}}\left(\left(3\sqrt{\frac{-12c_1+81x}{x}}\,x - 2c_1 + 27x\right)c_1^2\right)^{\frac{2}{3}}}{6c_1 x\left(\left(3\sqrt{\frac{-12c_1+81x}{x}}\,x - 2c_1 + 27x\right)c_1^2\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{2\left(i\sqrt{3}-1\right)c_1^2 2^{\frac{1}{3}} - \left(\left(3\sqrt{\frac{-12c_1+81x}{x}}\,x - 2c_1 + 27x\right)c_1^2\right)^{\frac{1}{3}}\left(\left(\left(3\sqrt{\frac{-12c_1+81x}{x}}\,x - 2c_1 + 27x\right)c_1^2\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right.\right.}{12\left(\left(3\sqrt{\frac{-12c_1+81x}{x}}\,x - 2c_1 + 27x\right)c_1^2\right)^{\frac{1}{3}} c_1 x}$$

$$y(x) = \frac{-2\left(1+i\sqrt{3}\right)c_1^2 2^{\frac{1}{3}} + \left(\left(3\sqrt{\frac{-12c_1+81x}{x}}\,x - 2c_1 + 27x\right)c_1^2\right)^{\frac{1}{3}}\left(\left(\left(3\sqrt{\frac{-12c_1+81x}{x}}\,x - 2c_1 + 27x\right)c_1^2\right)^{\frac{1}{3}}\left(i\sqrt{3}-\right.\right.}{12\left(\left(3\sqrt{\frac{-12c_1+81x}{x}}\,x - 2c_1 + 27x\right)c_1^2\right)^{\frac{1}{3}} c_1 x}$$

✓ Solution by Mathematica

Time used: 21.875 (sec). Leaf size: 380

```
DSolve[(2*x+3*x^2*y[x])*y'[x]+(y[x]+2*y[x]^2*x)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}\left(\frac{2}{\sqrt[3]{\frac{3}{2}\sqrt{3}\sqrt{c_1x^7(-4+27c_1x)}+\frac{27c_1x^4}{2}-x^3}} + \frac{2^{2/3}\sqrt[3]{3\sqrt{3}\sqrt{c_1x^7(-4+27c_1x)}+27c_1x^4-2x^3}}{x^2}-\frac{2}{x}\right)$$

$$y(x) \to \frac{1}{12}\left(-\frac{2(1+i\sqrt{3})}{\sqrt[3]{\frac{3}{2}\sqrt{3}\sqrt{c_1x^7(-4+27c_1x)}+\frac{27c_1x^4}{2}-x^3}} + \frac{i2^{2/3}(\sqrt{3}+i)\sqrt[3]{3\sqrt{3}\sqrt{c_1x^7(-4+27c_1x)}+27c_1x^4-2x^3}}{x^2}-\frac{4}{x}\right)$$

$$y(x) \to \frac{1}{12}\left(\frac{2i(\sqrt{3}+i)}{\sqrt[3]{\frac{3}{2}\sqrt{3}\sqrt{c_1x^7(-4+27c_1x)}+\frac{27c_1x^4}{2}-x^3}} - \frac{2^{2/3}(1+i\sqrt{3})\sqrt[3]{3\sqrt{3}\sqrt{c_1x^7(-4+27c_1x)}+27c_1x^4-2x^3}}{x^2}-\frac{4}{x}\right)$$

5.15 problem 15

Internal problem ID [1980]
Internal file name [`OUTPUT/1980_Sunday_February_25_2024_06_44_01_AM_91802935/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _Bernoulli]
```

$$\boxed{2y'x^2y-2xy^2=-\mathrm{e}^x x^4}$$

5.15.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2(u'(x)\,x+u(x))\,x^3u(x)-2x^3u(x)^2=-\mathrm{e}^x x^4$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{\mathrm{e}^x}{2u} \end{aligned}$$

Where $f(x) = -\frac{e^x}{2}$ and $g(u) = \frac{1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{1}{u}}\,du = -\frac{e^x}{2}\,dx$$

$$\int \frac{1}{\frac{1}{u}}\,du = \int -\frac{e^x}{2}\,dx$$

$$\frac{u^2}{2} = -\frac{e^x}{2} + c_2$$

The solution is

$$\frac{u(x)^2}{2} + \frac{e^x}{2} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{2x^2} + \frac{e^x}{2} - c_2 = 0$$

$$\frac{x^2 e^x - 2c_2 x^2 + y^2}{2x^2} = 0$$

Summary

The solution(s) found are the following

$$\frac{x^2 e^x - 2c_2 x^2 + y^2}{2x^2} = 0 \tag{1}$$

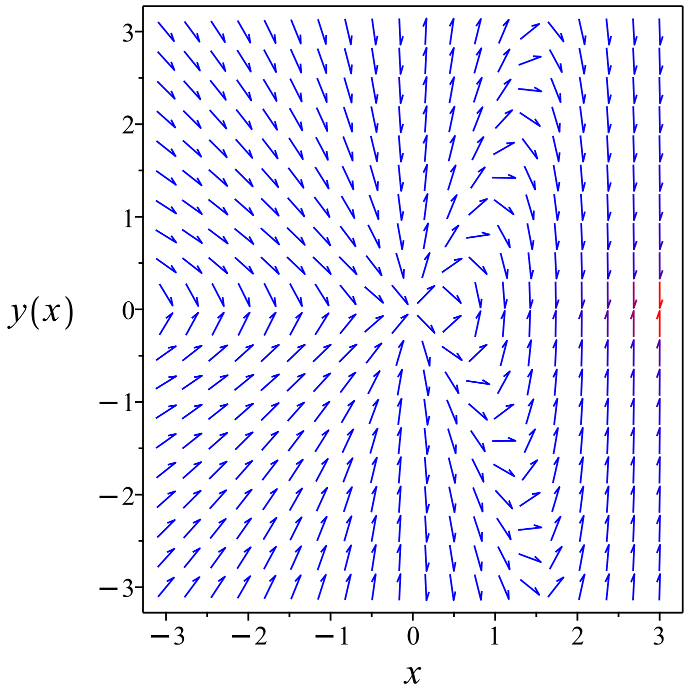


Figure 266: Slope field plot

Verification of solutions

$$\frac{x^2 e^x - 2c_2 x^2 + y^2}{2x^2} = 0$$

Verified OK.

5.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{e^x x^3 - 2y^2}{2yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 139: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x^2}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2}{y}} dy
\end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{e^x x^3 - 2y^2}{2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^2}{x^3} \\
S_y &= \frac{y}{x^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{e^x}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{e^R}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{e^{R}}{2} + c_1 \tag{4}$$

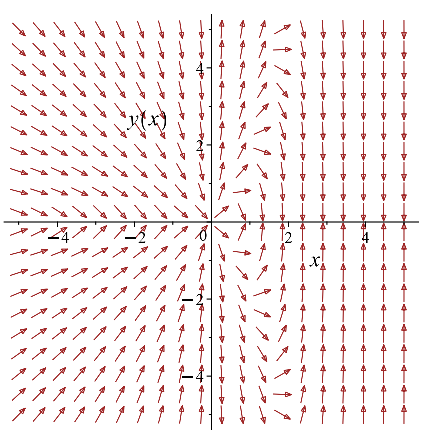
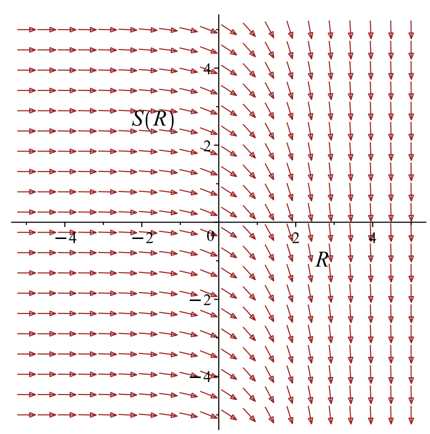
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^2} = -\frac{e^{x}}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x^2} = -\frac{e^{x}}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{e^{x}x^3 - 2y^2}{2yx}$ 	$R = x$ $S = \frac{y^2}{2x^2}$	$\frac{dS}{dR} = -\frac{e^{R}}{2}$ 

Summary
The solution(s) found are the following

$$\frac{y^2}{2x^2} = -\frac{e^{x}}{2} + c_1 \tag{1}$$

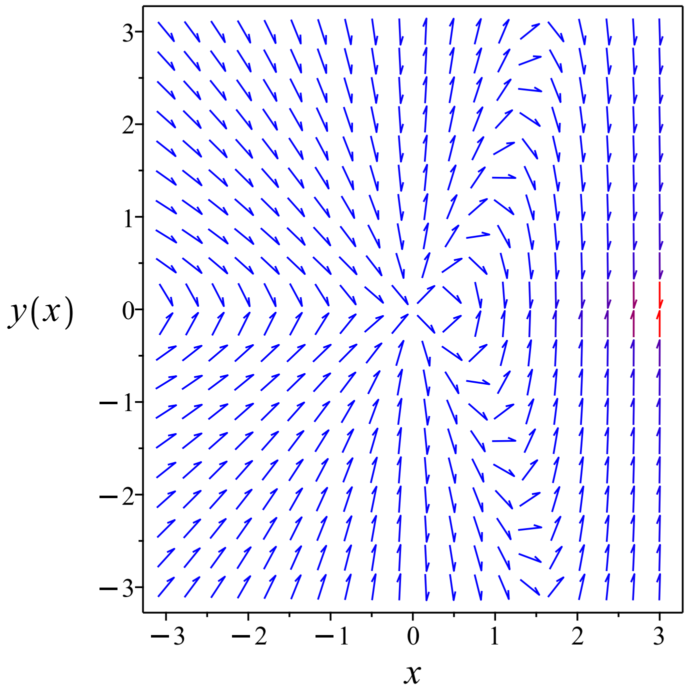


Figure 267: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} = -\frac{e^x}{2} + c_1$$

Verified OK.

5.15.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{e^x x^3 - 2y^2}{2yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{x^2 e^x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -\frac{x^2 e^x}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{x} - \frac{x^2 e^x}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{x} - \frac{x^2 e^x}{2} \\ w' &= \frac{2w}{x} - x^2 e^x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= -x^2 e^x \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = -x^2 e^x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-x^2 e^x\right) \\ \frac{d}{dx}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-x^2 e^x\right) \\ d\left(\frac{w}{x^2}\right) &= \left(-e^x\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int -e^x\, dx \\ \frac{w}{x^2} &= -e^x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = -x^2 e^x + c_1 x^2$$

which simplifies to

$$w(x) = x^2(-e^x + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x^2(-e^x + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{-e^x + c_1}\, x \\ y(x) &= -\sqrt{-e^x + c_1}\, x\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-e^{x} + c_1}\, x \tag{1}$$

$$y = -\sqrt{-e^{x} + c_1}\, x \tag{2}$$

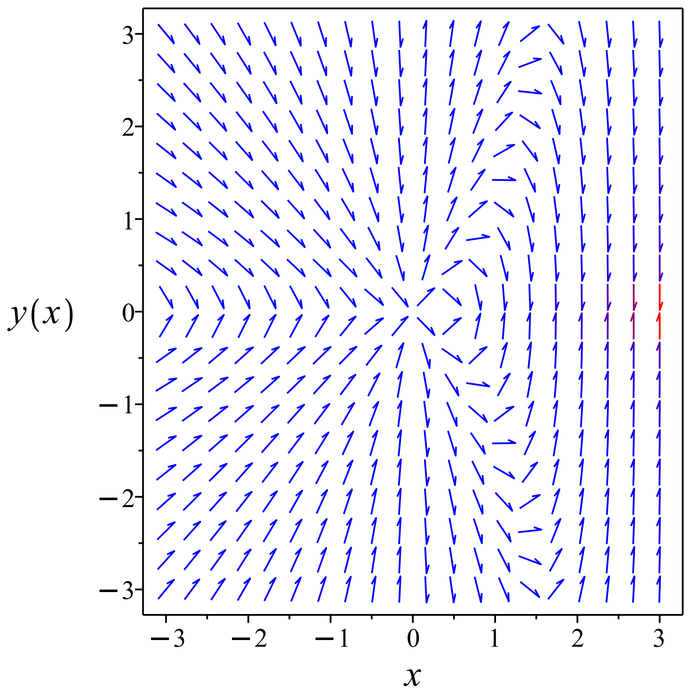


Figure 268: Slope field plot

Verification of solutions

$$y = \sqrt{-e^{x} + c_1}\, x$$

Verified OK.

$$y = -\sqrt{-e^{x} + c_1}\, x$$

Verified OK.

5.15.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(2y\,x^2\right)\mathrm{d}y&=\left(-\mathrm{e}^x x^4+2x\,y^2\right)\mathrm{d}x\\ \left(\mathrm{e}^x x^4-2x\,y^2\right)\mathrm{d}x+\left(2y\,x^2\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\mathrm{e}^x x^4-2x\,y^2\\ N(x,y)&=2y\,x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{x} x^{4} - 2x\, y^{2}\right)\\ &= -4yx\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2y\, x^{2}\right)\\ &= 4yx\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x^{2}y}((-4yx) - (4yx))\\ &= -\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{4}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)}\\ &= \frac{1}{x^{4}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^{4}}\left(\mathrm{e}^{x} x^{4} - 2x\, y^{2}\right)\\ &= \frac{\mathrm{e}^{x} x^{3} - 2y^{2}}{x^{3}}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^4}(2y\,x^2) \\
&= \frac{2y}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\mathrm{e}^x x^3 - 2y^2}{x^3}\right) + \left(\frac{2y}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{\mathrm{e}^x x^3 - 2y^2}{x^3}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2\mathrm{e}^x + y^2}{x^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{2y}{x^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y}{x^2}$. Therefore equation (4) becomes

$$
\frac{2y}{x^2} = \frac{2y}{x^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2 e^x + y^2}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2 e^x + y^2}{x^2}$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{x^2 e^x + y^2}{x^2} = c_1 \tag{1}$$

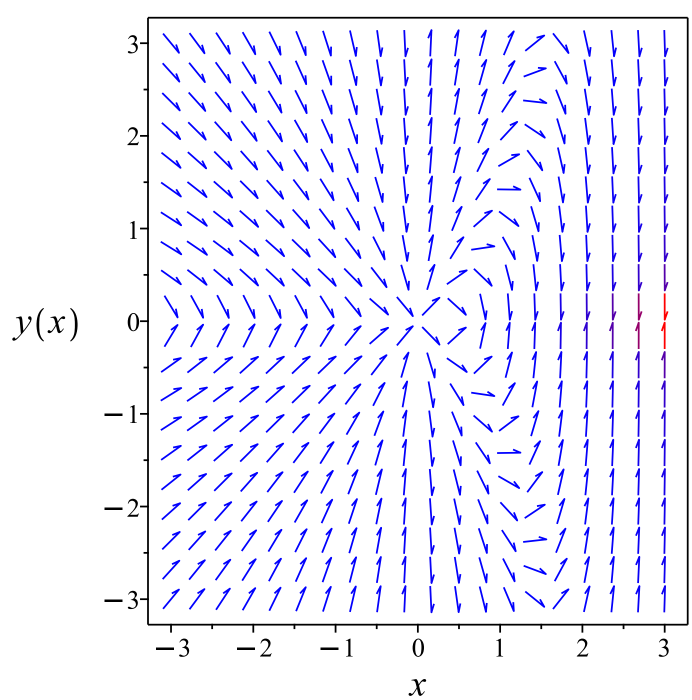


Figure 269: Slope field plot

Verification of solutions

$$\frac{x^2 e^x + y^2}{x^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(2*x^2*y(x)*diff(y(x),x)+(x^4*exp(x)-2*x*y(x)^2)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-e^x + c_1}\, x$$
$$y(x) = -\sqrt{-e^x + c_1}\, x$$

✓ Solution by Mathematica

Time used: 7.23 (sec). Leaf size: 45

```
DSolve[2*x^2*y[x]*y'[x]+(x^4*Exp[x]-2*x*y[x]^2)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2 (e^x - c_1)}$$
$$y(x) \to \sqrt{-x^2 (e^x - c_1)}$$

5.16 problem 20

Internal problem ID [1981]
Internal file name [`OUTPUT/1981_Sunday_February_25_2024_06_44_02_AM_16394885/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y(1-x^4y^2)+y'x=0}$$

With initial conditions

$$[y(1)=-1]$$

5.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y(x^4y^2-1)}{x} \end{aligned}$$

The x domain of $f(x,y)$ when $y=-1$ is

$$\{x<0 \lor 0<x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(x^4y^2-1)}{x}\right)\\ &= \frac{x^4y^2-1}{x} + 2x^3y^2\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

5.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \frac{y(x^4y^2-1)}{x}\\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 141: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^3 x^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^3 x^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{2x^2 y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(x^4 y^2 - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y^2 x^3} \\ S_y &= \frac{1}{x^2 y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

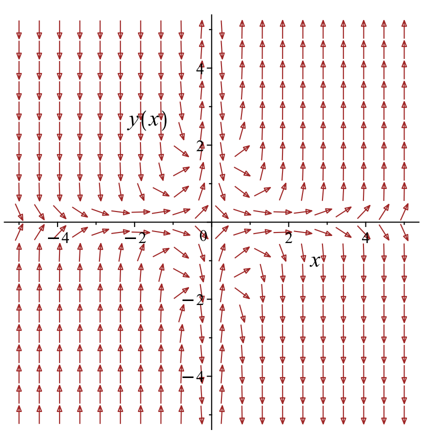
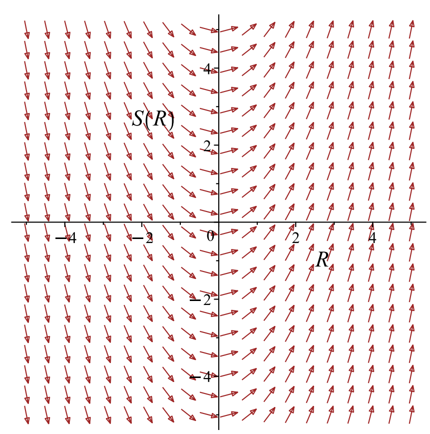
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{2x^2y^2} = \frac{x^2}{2} + c_1$$

Which simplifies to

$$-\frac{1}{2x^2y^2} = \frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x^4y^2-1)}{x}$ 	$R = x$ $S = -\frac{1}{2x^2y^2}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = \frac{1}{2} + c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$-\frac{1}{2x^2y^2} = \frac{x^2}{2} - 1$$

The above simplifies to

$$-x^4y^2 + 2x^2y^2 - 1 = 0$$

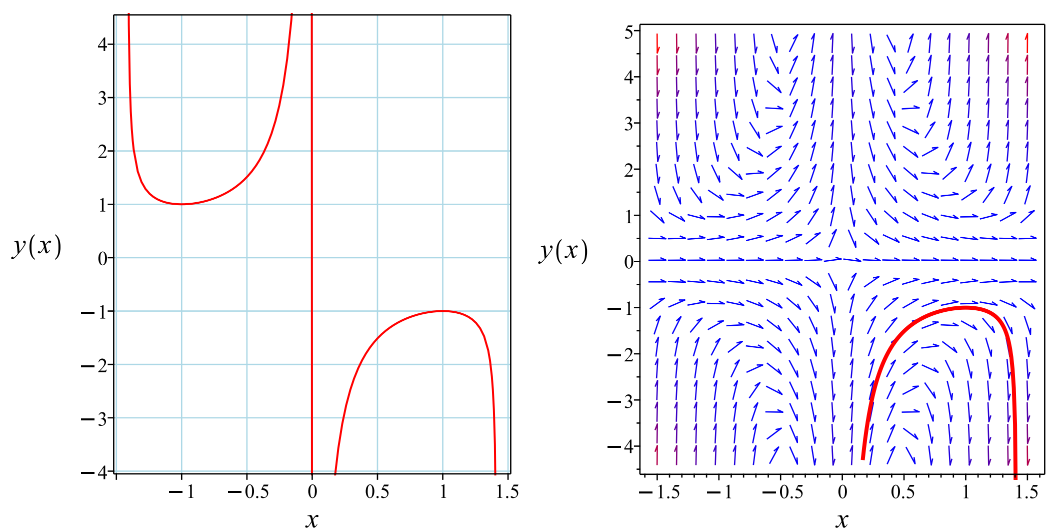
Solving for y from the above gives

$$y = -\frac{1}{\sqrt{-x^2 + 2x}}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{\sqrt{-x^2 + 2x}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{-x^2 + 2x}}$$

Verified OK.

5.16.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= \frac{y(x^4y^2 - 1)}{x}
\end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + x^3y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= x^3 \\
n &= 3
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y'\frac{1}{y^3} = -\frac{1}{x\,y^2} + x^3 \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2} \tag{5}
\end{aligned}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= -\frac{w(x)}{x} + x^3 \\
w' &= \frac{2w}{x} - 2x^3
\end{aligned}
\tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= -2x^3
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{2w(x)}{x} = -2x^3
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-2x^3\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-2x^3\right) \\
\mathrm{d}\left(\frac{w}{x^2}\right) &= (-2x)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x^2} &= \int -2x\,\mathrm{d}x \\
\frac{w}{x^2} &= -x^2 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$
w(x) = -x^4 + c_1 x^2
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = -x^4 + c_1 x^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - 1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{y^2} = -x^4 + 2x^2$$

The above simplifies to

$$x^4 y^2 - 2x^2 y^2 + 1 = 0$$

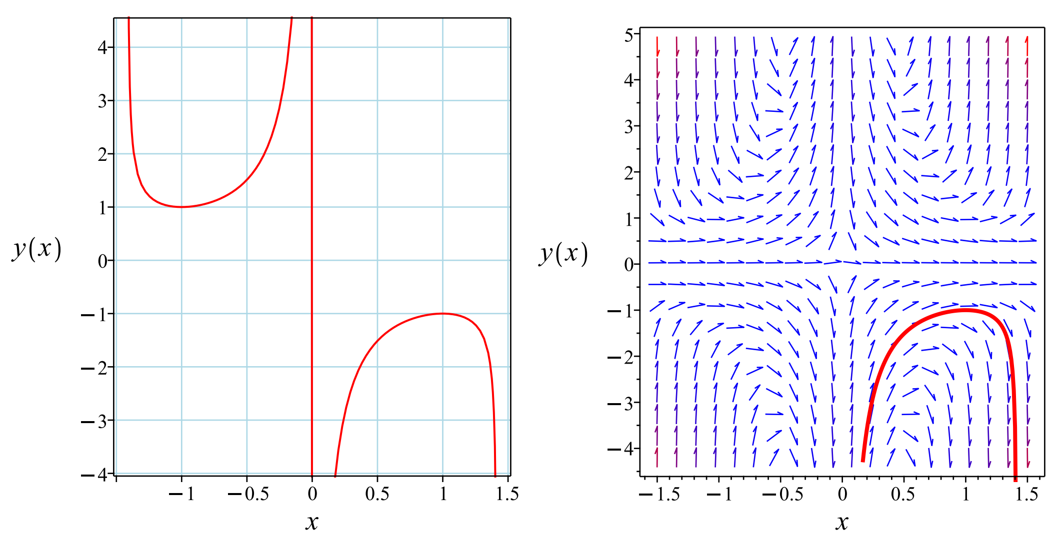
Solving for y from the above gives

$$y = -\frac{1}{\sqrt{-x^2 + 2x}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{1}{\sqrt{-x^2 + 2x}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{-x^2 + 2x}}$$

Verified OK.

5.16.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = \left(-y\left(-x^4y^2 + 1\right)\right)\mathrm{d}x$$
$$\left(y\left(-x^4y^2 + 1\right)\right)\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= y(-x^4y^2+1) \\
N(x,y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y(-x^4y^2+1)\right) \\
&= -3x^4y^2+1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}\left((-3x^4y^2+1) - (1)\right) \\
&= -3x^3y^2
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{y^3x^4 - y}\left((1) - (-3x^4y^2+1)\right) \\
&= -\frac{3x^4y}{x^4y^2-1}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(1) - (-3x^4y^2 + 1)}{x\,(y\,(-x^4y^2 + 1)) - y\,(x)} \\ &= -\frac{3}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{3}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(-\frac{3}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-3\ln(t)} \\ &= \frac{1}{t^3} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{y^3x^3}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^3x^3}\left(y\left(-x^4y^2 + 1\right)\right) \\ &= \frac{-x^4y^2 + 1}{y^2x^3} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^3x^3}(x) \\ &= \frac{1}{x^2y^3} \end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^4y^2+1}{y^2x^3}\right) + \left(\frac{1}{x^2y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-x^4y^2+1}{y^2x^3}\,\mathrm{d}x \\
\phi &= \frac{-x^4y^2-1}{2x^2y^2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\frac{x^2}{y} - \frac{-x^4y^2-1}{x^2y^3} + f'(y) \\
&= \frac{1}{x^2y^3} + f'(y)
\end{aligned} \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2y^3}$. Therefore equation (4) becomes

$$
\frac{1}{x^2y^3} = \frac{1}{x^2y^3} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^4y^2 - 1}{2x^2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^4y^2 - 1}{2x^2y^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$\frac{-x^4y^2 - 1}{2x^2y^2} = -1$$

The above simplifies to

$$-x^4y^2 + 2x^2y^2 - 1 = 0$$

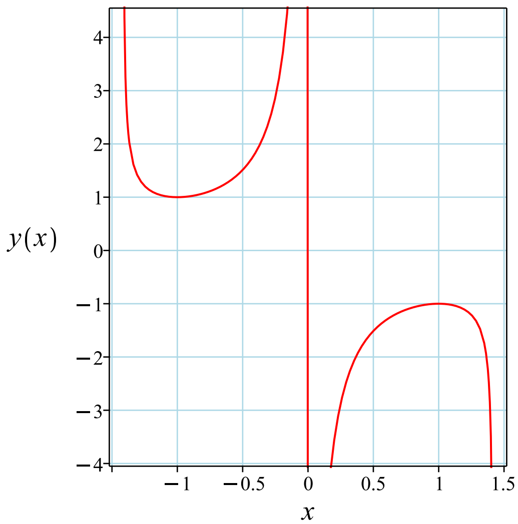
Solving for y from the above gives

$$y = -\frac{1}{\sqrt{-x^2 + 2x}}$$

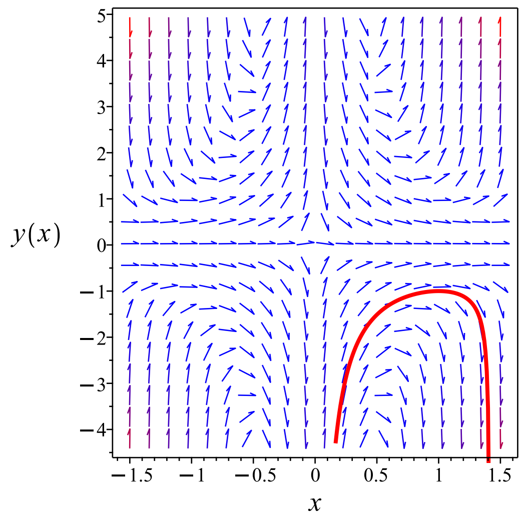
Summary

The solution(s) found are the following

$$y = -\frac{1}{\sqrt{-x^2 + 2x}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{-x^2 + 2x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 18

```
dsolve([y(x)*(1-x^4*y(x)^2)+x*diff(y(x),x)=0,y(1) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{1}{\sqrt{-x^2 + 2x}}$$

✓ Solution by Mathematica

Time used: 0.39 (sec). Leaf size: 21

```
DSolve[{y[x]*(1-x^4*y[x]^2)+x*y'[x]==0,{y[1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{-x^2 (x^2 - 2)}}$$

5.17 problem 21

Internal problem ID [1982]
Internal file name [`OUTPUT/1982_Sunday_February_25_2024_06_44_03_AM_97490773/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y(x^2-1)+x(x^2+1)\,y'=0}$$

With initial conditions

$$[y(1)=2]$$

5.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1-x^2}{x\,(x^2+1)} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{(1-x^2)\,y}{x\,(x^2+1)} = 0$$

The domain of $p(x) = -\frac{1-x^2}{x(x^2+1)}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

5.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(x^2-1)}{x\,(x^2+1)} \end{aligned}$$

Where $f(x) = -\frac{x^2-1}{x(x^2+1)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= -\frac{x^2-1}{x\,(x^2+1)}\,dx \\ \int \frac{1}{y}\,dy &= \int -\frac{x^2-1}{x\,(x^2+1)}\,dx \\ \ln(y) &= \ln(x) - \ln\left(x^2+1\right) + c_1 \\ y &= e^{\ln(x)-\ln(x^2+1)+c_1} \\ &= c_1 e^{\ln(x)-\ln(x^2+1)} \end{aligned}$$

Which can be simplified to become

$$y = \frac{c_1 x}{x^2 + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{c_1}{2}$$

$$c_1 = 4$$

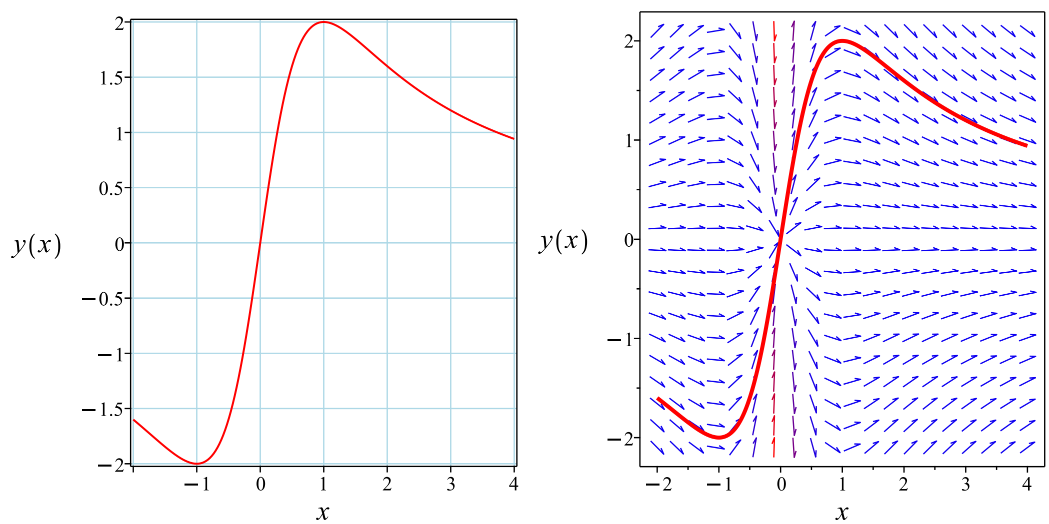
Substituting c_1 found above in the general solution gives

$$y = \frac{4x}{x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{4x}{x^2 + 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{4x}{x^2 + 1}$$

Verified OK.

5.17.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1-x^2}{x\left(x^2+1\right)}dx} \\
&= \mathrm{e}^{-\ln(x)+\ln\left(x^2+1\right)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{x^2+1}{x}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\left(x^2+1\right)y}{x}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\frac{\left(x^2+1\right)y}{x} = c_1
$$

Dividing both sides by the integrating factor $\mu = \frac{x^2+1}{x}$ results in

$$
y = \frac{c_1 x}{x^2+1}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = \frac{c_1}{2}
$$

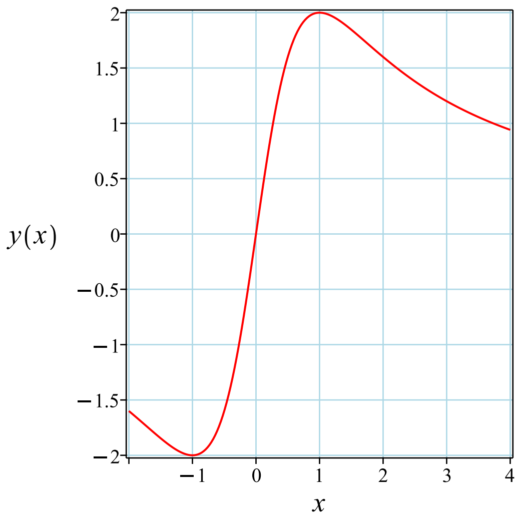
$$
c_1 = 4
$$

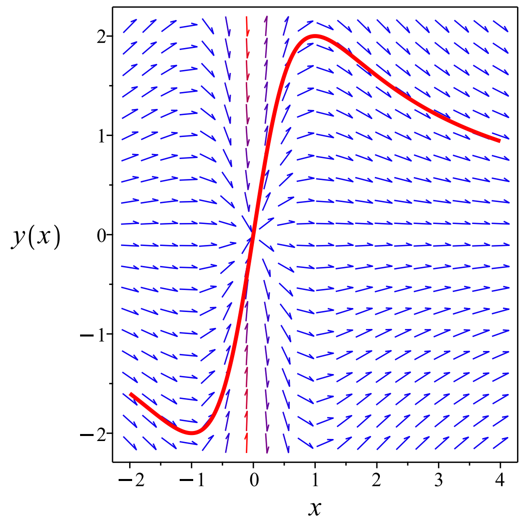
Substituting c_1 found above in the general solution gives

$$
y = \frac{4x}{x^2+1}
$$

Summary
The solution(s) found are the following

$$
y = \frac{4x}{x^2+1} \tag{1}
$$





(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{4x}{x^2 + 1}$$

Verified OK.

5.17.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x\left(x^2 - 1\right) + x\left(x^2 + 1\right)\left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2ux}{x^2 + 1} \end{aligned}$$

Where $f(x) = -\frac{2x}{x^2+1}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2x}{x^2+1}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2x}{x^2+1}\,dx \\
\ln(u) &= -\ln\left(x^2+1\right) + c_2 \\
u &= e^{-\ln(x^2+1)+c_2} \\
&= \frac{c_2}{x^2+1}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{x c_2}{x^2+1}
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{c_2}{2}$$

$$c_2 = 4$$

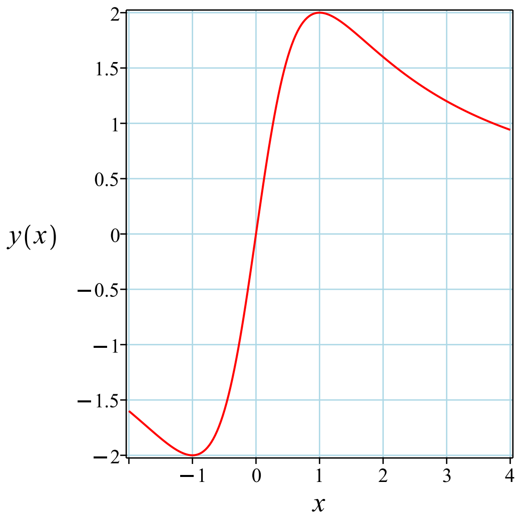
Substituting c_2 found above in the general solution gives

$$y = \frac{4x}{x^2+1}$$

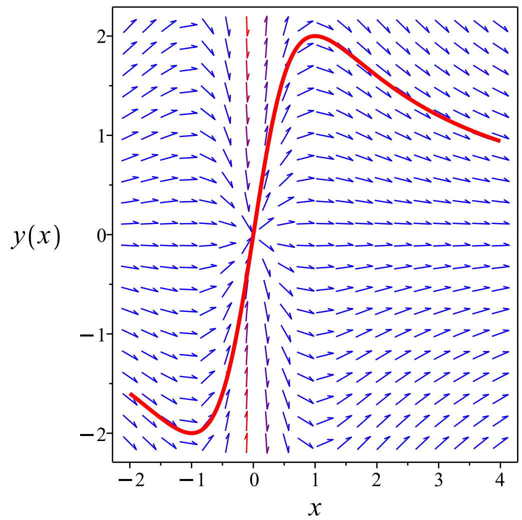
Summary

The solution(s) found are the following

$$y = \frac{4x}{x^2+1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{4x}{x^2+1}$$

Verified OK.

5.17.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(x^2-1)}{x\,(x^2+1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 143: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\ln(x) - \ln(x^2+1)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\ln(x)-\ln(x^2+1)}} dy \end{aligned}$$

Which results in

$$S = \frac{(x^2+1)\, y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(x^2-1)}{x\,(x^2+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y(x^2-1)}{x^2} \\ S_y &= \frac{x^2+1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x^2+1)\,y}{x} = c_1$$

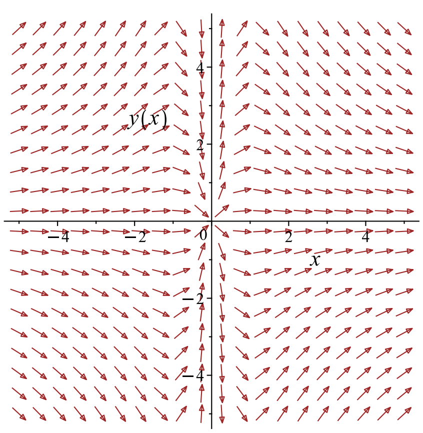
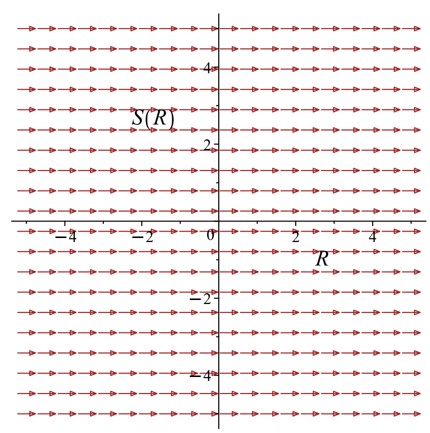
Which simplifies to

$$\frac{(x^2+1)\,y}{x} = c_1$$

Which gives

$$y = \frac{c_1 x}{x^2+1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y(x^2-1)}{x(x^2+1)}$

</td><td>$R = x$
$S = \frac{(x^2+1)\,y}{x}$</td><td>$\frac{dS}{dR} = 0$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{c_1}{2}$$

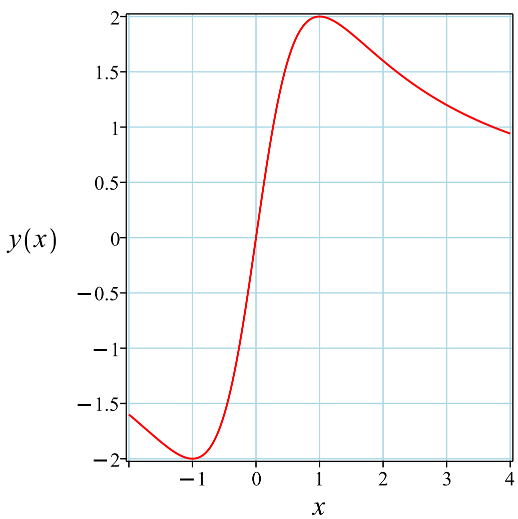
$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

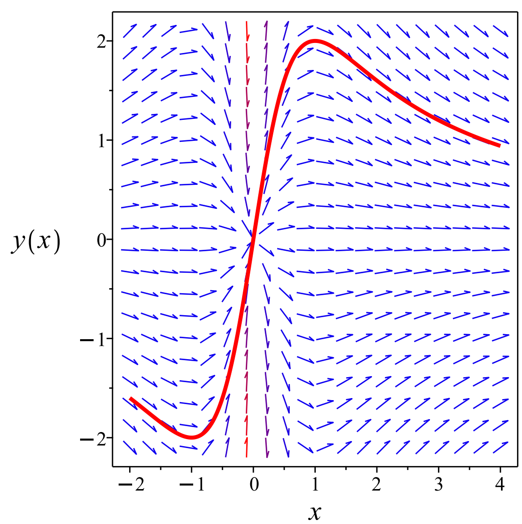
$$y = \frac{4x}{x^2+1}$$

Summary
The solution(s) found are the following

$$y = \frac{4x}{x^2+1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4x}{x^2+1}$$

Verified OK.

5.17.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y = \left(\frac{x^2 - 1}{x\left(x^2 + 1\right)}\right)\mathrm{d}x$$
$$\left(-\frac{x^2 - 1}{x\left(x^2 + 1\right)}\right)\mathrm{d}x + \left(-\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x^2 - 1}{x\left(x^2 + 1\right)}$$
$$N(x, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2-1}{x\,(x^2+1)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x^2-1}{x\,(x^2+1)}\,\mathrm{d}x\end{aligned}$$
$$\phi = \ln(x) - \ln\left(x^2+1\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \ln(x^2 + 1) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \ln(x^2 + 1) - \ln(y)$$

The solution becomes

$$y = \frac{x\,\mathrm{e}^{-c_1}}{x^2 + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{\mathrm{e}^{-c_1}}{2}$$

$$c_1 = -2\ln(2)$$

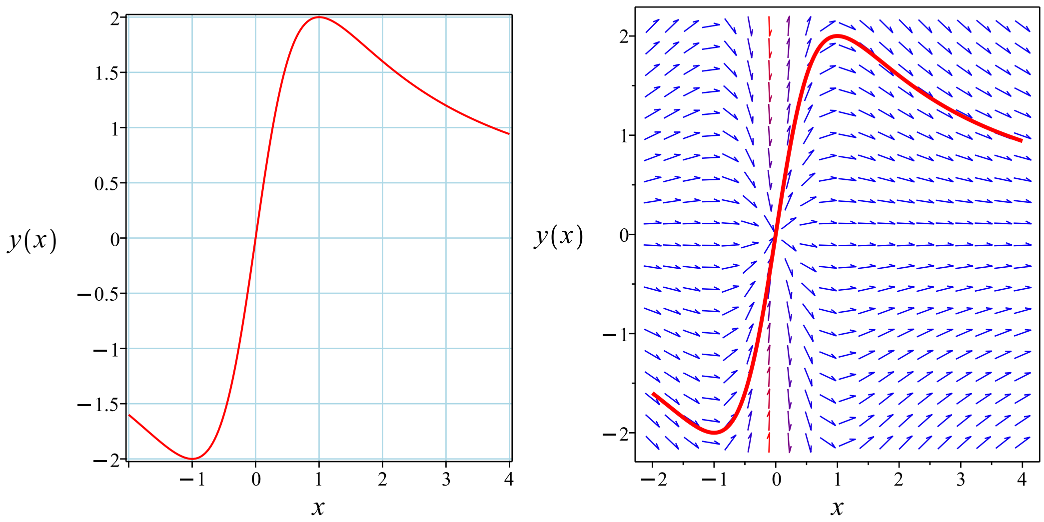
Substituting c_1 found above in the general solution gives

$$y = \frac{4x}{x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{4x}{x^2 + 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{4x}{x^2+1}$$

Verified OK.

5.17.7 Maple step by step solution

Let's solve

$[y(x^2-1)+x(x^2+1)\,y'=0, y(1)=2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{x^2-1}{x(x^2+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{x^2-1}{x(x^2+1)}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) - \ln(x^2+1) + c_1$

- Solve for y

  $y = \frac{x\,\mathrm{e}^{c_1}}{x^2+1}$

- Use initial condition $y(1) = 2$

  $2 = \frac{e^{c_1}}{2}$
- Solve for c_1

  $c_1 = 2\ln(2)$
- Substitute $c_1 = 2\ln(2)$ into general solution and simplify

  $y = \frac{4x}{x^2+1}$
- Solution to the IVP

  $y = \frac{4x}{x^2+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([y(x)*(x^2-1)+x*(x^2+1)*diff(y(x),x)=0,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{4x}{x^2+1}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 15

```
DSolve[{y[x]*(x^2-1)+x*(x^2+1)*y'[x]==0,{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4x}{x^2+1}$$

5.18 problem 22

Internal problem ID [1983]
Internal file name [`OUTPUT/1983_Sunday_February_25_2024_06_44_04_AM_36474790/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{x^2y^2 - y + \left(2yx^3 + x\right) y' = 0}$$

With initial conditions

$$[y(2) = -2]$$

5.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{y(y\,x^2 - 1)}{x\,(2y\,x^2 + 1)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = -2$ is

$$\left\{-\infty \leq x < 0, 0 < x < -\frac{1}{2}, \frac{1}{2} < x \leq \infty\right\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x, y)$ when $x = 2$ is

$$\left\{y < -\frac{1}{8} \vee -\frac{1}{8} < y\right\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y(y\,x^2-1)}{x\,(2y\,x^2+1)}\right)\\ &= -\frac{y\,x^2-1}{x\,(2y\,x^2+1)} - \frac{yx}{2y\,x^2+1} + \frac{2y(y\,x^2-1)\,x}{(2y\,x^2+1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -2$ is

$$\left\{-\infty \le x < 0, 0 < x < -\frac{1}{2}, \frac{1}{2} < x \le \infty\right\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\left\{y < -\frac{1}{8} \vee -\frac{1}{8} < y\right\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

5.18.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{y(y\,x^2-1)}{x\,(2y\,x^2+1)}\\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{y(y\,x^2-1)\,(b_3-a_2)}{x\,(2y\,x^2+1)} - \frac{y^2(y\,x^2-1)^2\,a_3}{x^2\,(2y\,x^2+1)^2} \\ & - \left(-\frac{2y^2}{2y\,x^2+1} + \frac{y(y\,x^2-1)}{x^2\,(2y\,x^2+1)} + \frac{4y^2(y\,x^2-1)}{(2y\,x^2+1)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{y\,x^2-1}{x\,(2y\,x^2+1)} - \frac{yx}{2y\,x^2+1} + \frac{2y(y\,x^2-1)\,x}{(2y\,x^2+1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^6y^2b_2 - 3x^4y^4a_3 + 2x^5y^2b_1 - 2x^4y^3a_1 + 6x^4yb_2 + 6x^3y^2a_2 + 3x^3y^2b_3 + 9x^2y^3a_3 + 2x^3yb_1 + 7x^2y^2a_1 - x}{(2y\,x^2+1)^2\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 6x^6y^2b_2 - 3x^4y^4a_3 + 2x^5y^2b_1 - 2x^4y^3a_1 + 6x^4yb_2 + 6x^3y^2a_2 \\ & + 3x^3y^2b_3 + 9x^2y^3a_3 + 2x^3yb_1 + 7x^2y^2a_1 - xb_1 + ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -3a_3v_1^4v_2^4 + 6b_2v_1^6v_2^2 - 2a_1v_1^4v_2^3 + 2b_1v_1^5v_2^2 + 6a_2v_1^3v_2^2 + 9a_3v_1^2v_2^3 \\ & + 6b_2v_1^4v_2 + 3b_3v_1^3v_2^2 + 7a_1v_1^2v_2^2 + 2b_1v_1^3v_2 + a_1v_2 - b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$6b_2v_1^6v_2^2 + 2b_1v_1^5v_2^2 - 3a_3v_1^4v_2^4 - 2a_1v_1^4v_2^3 + 6b_2v_1^4v_2 + (6a_2 + 3b_3)\, v_1^3v_2^2 \\ + 2b_1v_1^3v_2 + 9a_3v_1^2v_2^3 + 7a_1v_1^2v_2^2 - b_1v_1 + a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -2a_1 &= 0 \\ 7a_1 &= 0 \\ -3a_3 &= 0 \\ 9a_3 &= 0 \\ -b_1 &= 0 \\ 2b_1 &= 0 \\ 6b_2 &= 0 \\ 6a_2 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= -2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= -2y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -2y - \left(-\frac{y(y\,x^2 - 1)}{x\,(2y\,x^2 + 1)}\right)(x) \\
&= \frac{-3x^2y^2 - 3y}{2y\,x^2 + 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-3x^2y^2 - 3y}{2y\,x^2 + 1}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln\left(y(y\,x^2 + 1)\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(y\,x^2 - 1)}{x\,(2y\,x^2 + 1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{2yx}{3y\,x^2+3}\\
S_y &= \frac{-2y\,x^2-1}{3y\,(y\,x^2+1)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{3} + c_1 \tag{4}$$

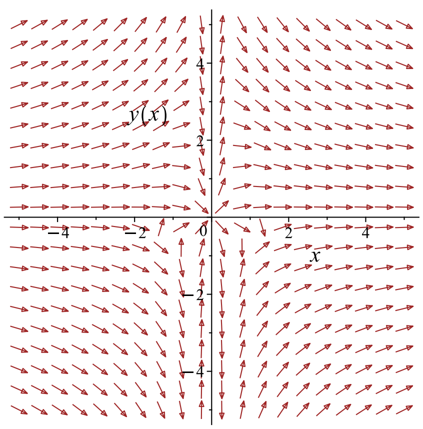
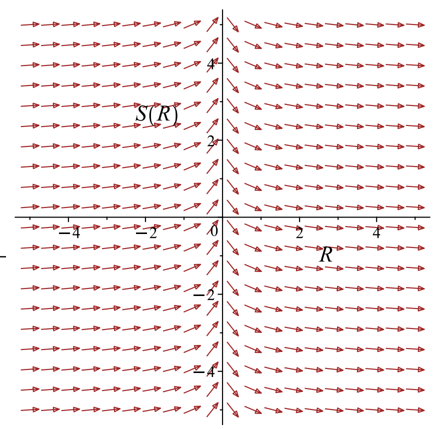
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{3} - \frac{\ln(x^2y+1)}{3} = -\frac{\ln(x)}{3} + c_1$$

Which simplifies to

$$-\frac{\ln(y)}{3} - \frac{\ln(x^2y+1)}{3} = -\frac{\ln(x)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y\,x^2-1)}{x(2y\,x^2+1)}$ 	$R = x$ $S = -\frac{\ln(y)}{3} - \frac{\ln(y\,x^2 +}{3}$	$\frac{dS}{dR} = -\frac{1}{3R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\ln(2)}{3} - \frac{2i\pi}{3} - \frac{\ln(7)}{3} = -\frac{\ln(2)}{3} + c_1$$

$$c_1 = -\frac{2i\pi}{3} - \frac{\ln(7)}{3}$$

Substituting c_1 found above in the general solution gives

$$-\frac{\ln(y)}{3} - \frac{\ln(y\,x^2+1)}{3} = -\frac{\ln(x)}{3} - \frac{2i\pi}{3} - \frac{\ln(7)}{3}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{3} - \frac{\ln(x^2y+1)}{3} = -\frac{\ln(x)}{3} - \frac{2i\pi}{3} - \frac{\ln(7)}{3} \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{3} - \frac{\ln(x^2y+1)}{3} = -\frac{\ln(x)}{3} - \frac{2i\pi}{3} - \frac{\ln(7)}{3}$$

Verified OK.

5.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(2y\,x^3+x\right)\mathrm{d}y&=\left(-x^2y^2+y\right)\mathrm{d}x\\ \left(x^2y^2-y\right)\mathrm{d}x+\left(2y\,x^3+x\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x^2y^2-y\\ N(x,y)&=2y\,x^3+x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 y^2 - y\right)\\ &= 2y\,x^2 - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2y\,x^3 + x\right)\\ &= 6y\,x^2 + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2y\,x^3 + x}\left(\left(2y\,x^2 - 1\right) - \left(6y\,x^2 + 1\right)\right)\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(x^2 y^2 - y\right)\\ &= \frac{y\left(y\,x^2 - 1\right)}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}\left(2y\,x^3+x\right) \\
&= \frac{2y\,x^2+1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y(y\,x^2-1)}{x^2}\right)+\left(\frac{2y\,x^2+1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\frac{y(y\,x^2-1)}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{y(y\,x^2+1)}{x}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = \frac{y\,x^2+1}{x}+yx+f'(y) \tag{4}
$$

$$
= \frac{2y\,x^2+1}{x}+f'(y)
$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{2y\,x^2+1}{x}$. Therefore equation (4) becomes

$$
\frac{2y\,x^2+1}{x} = \frac{2y\,x^2+1}{x}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y(y\,x^2+1)}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y(y\,x^2+1)}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$7 = c_1$$

$$c_1 = 7$$

Substituting c_1 found above in the general solution gives

$$\frac{y(y\,x^2+1)}{x} = 7$$

The above simplifies to

$$x^2y^2 - 7x + y = 0$$

Summary

The solution(s) found are the following

$$x^2y^2 - 7x + y = 0 \tag{1}$$

Verification of solutions

$$x^2y^2 - 7x + y = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.109 (sec). Leaf size: 20

```
dsolve([(x^2*y(x)^2-y(x))+(2*x^3*y(x)+x)*diff(y(x),x)=0,y(2) = -2],y(x), singsol=all)
```

$$y(x) = \frac{-1 - \sqrt{28x^3 + 1}}{2x^2}$$

✓ Solution by Mathematica
Time used: 0.648 (sec). Leaf size: 34

```
DSolve[{(x^2*y[x]^2-y[x])+(2*x^3*y[x]+x)*y'[x]==0,{y[2]==-2}},y[x],x,IncludeSingularSolution
```

$$y(x) \to -\frac{\sqrt{\frac{1}{x^2}}\sqrt{28x^3 + 1}x + 1}{2x^2}$$

5.19 problem 23

Internal problem ID [1984]
Internal file name [`OUTPUT/1984_Sunday_February_25_2024_06_44_05_AM_90096383/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{(x^2 + y^2 - 2y)\, y' = 2x}$$

With initial conditions

$$[y(1) = 0]$$

5.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{2x}{x^2 + y^2 - 2y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2 - 2y}\right) \\
&= -\frac{2x(2y-2)}{(x^2 + y^2 - 2y)^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

5.19.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+y^2-2y\right)\mathrm{d}y &= (2x)\,\mathrm{d}x\\ (-2x)\,\mathrm{d}x + \left(x^2+y^2-2y\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2x\\ N(x,y) &= x^2+y^2-2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2+y^2-2y\right)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2+y^2-2y}((0)-(2x))\\ &= -\frac{2x}{x^2+y^2-2y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{2x}((2x) - (0)) \\ &= -1 \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -1 \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-y} \\ &= \mathrm{e}^{-y} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-y}(-2x) \\ &= -2x\,\mathrm{e}^{-y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-y}\left(x^2 + y^2 - 2y\right) \\ &= \left(x^2 + y^2 - 2y\right)\mathrm{e}^{-y} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-2x\,\mathrm{e}^{-y}\right) + \left(\left(x^2 + y^2 - 2y\right)\mathrm{e}^{-y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2x \, \mathrm{e}^{-y} \, \mathrm{d}x$$
$$\phi = -x^2 \mathrm{e}^{-y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 \mathrm{e}^{-y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x^2 + y^2 - 2y) \, \mathrm{e}^{-y}$. Therefore equation (4) becomes

$$\left(x^2 + y^2 - 2y\right) \mathrm{e}^{-y} = x^2 \mathrm{e}^{-y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= \mathrm{e}^{-y} y^2 - 2y \, \mathrm{e}^{-y} \\ &= \mathrm{e}^{-y} y(y - 2) \end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y) \, \mathrm{d}y = \int \left(\mathrm{e}^{-y} y(y - 2)\right) \mathrm{d}y$$
$$f(y) = -\mathrm{e}^{-y} y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 \mathrm{e}^{-y} - \mathrm{e}^{-y} y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 e^{-y} - e^{-y} y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$-x^2 e^{-y} - e^{-y} y^2 = -1$$

Summary

The solution(s) found are the following

$$e^{-y}\left(-x^2 - y^2\right) = -1 \tag{1}$$

Verification of solutions

$$e^{-y}\left(-x^2 - y^2\right) = -1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✗ Solution by Maple

```
dsolve([(x^2+y(x)^2-2*y(x))*diff(y(x),x)=2*x,y(1) = 0],y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 29

```
DSolve[{(x^2+y[x]^2-2*y[x])*y'[x]==2*x,{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x^2\left(-e^{-y(x)}\right)-e^{-y(x)}y(x)^2=-1,y(x)\right]$$

5.20 problem 24

Internal problem ID [1985]
Internal file name [`OUTPUT/1985_Sunday_February_25_2024_06_44_07_AM_24494164/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

Unable to solve or complete the solution.

$$\boxed{y - x^2\sqrt{x^2 - y^2} - y'x = 0}$$

With initial conditions

$$[y(1) = 1]$$

Unable to determine ODE type.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(diff(y(x), x), x)+(y(x)*x^4-3*(diff(y(x), x))*x+3*y(x
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`[0, (x^2-y^2)^(1/2)]
```

✗ Solution by Maple

```
dsolve([(y(x)-x^2*sqrt(x^2-y(x)^2))-x*diff(y(x),x)=0,y(1) = 1],y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(y[x]-x^2*Sqrt[x^2-y[x]^2])-x*y'[x]==0,{y[1]==1}},y[x],x,IncludeSingularSolutions ->
```

{}

5.21 problem 25

Internal problem ID [1986]
Internal file name [`OUTPUT/1986_Sunday_February_25_2024_06_44_09_AM_84845897/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 9, page 38
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y(x+y^2)+x(-y^2+x)\,y'=0}$$

With initial conditions

$$[y(2)=2]$$

5.21.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(y^2+x)}{x\,(y^2-x)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{y(y^2+x)(b_3-a_2)}{x(y^2-x)} - \frac{y^2(y^2+x)^2 a_3}{x^2(y^2-x)^2} \\
&- \left(\frac{y}{x(y^2-x)} - \frac{y(y^2+x)}{x^2(y^2-x)} + \frac{y(y^2+x)}{x(y^2-x)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{y^2+x}{x(y^2-x)} + \frac{2y^2}{x(y^2-x)} - \frac{2y^2(y^2+x)}{x(y^2-x)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^3y^2b_2 - 2x^2y^3a_2 + 4x^2y^3b_3 - 4x\,y^4a_3 - x\,y^4b_1 + y^5a_1 + 2x^4b_2 - 2x^2y^2a_3 + 4x^2y^2b_1 - 2x\,y^3a_1 + x^3b_1 - }{x^2(-y^2+x)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&2x^3y^2b_2 - 2x^2y^3a_2 + 4x^2y^3b_3 - 4x\,y^4a_3 - x\,y^4b_1 + y^5a_1 \\
&+ 2x^4b_2 - 2x^2y^2a_3 + 4x^2y^2b_1 - 2x\,y^3a_1 + x^3b_1 - x^2ya_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&a_1v_2^5 - 2a_2v_1^2v_2^3 - 4a_3v_1v_2^4 - b_1v_1v_2^4 + 2b_2v_1^3v_2^2 + 4b_3v_1^2v_2^3 \\
&- 2a_1v_1v_2^3 - 2a_3v_1^2v_2^2 + 4b_1v_1^2v_2^2 + 2b_2v_1^4 - a_1v_1^2v_2 + b_1v_1^3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^4 + 2b_2v_1^3v_2^2 + b_1v_1^3 + (-2a_2 + 4b_3)\, v_1^2v_2^3 + (-2a_3 + 4b_1)\, v_1^2v_2^2 - a_1v_1^2v_2 + (-4a_3 - b_1)\, v_1v_2^4 - 2a_1v_1v_2^3 + a_1v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -2a_1 &= 0 \\ -a_1 &= 0 \\ 2b_2 &= 0 \\ -2a_2 + 4b_3 &= 0 \\ -4a_3 - b_1 &= 0 \\ -2a_3 + 4b_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 2x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{y(y^2+x)}{x\,(y^2-x)}\right)(2x) \\ &= \frac{y^3+3yx}{-y^2+x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^3+3yx}{-y^2+x}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{3} - \frac{2\ln(y^2+3x)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(y^2+x)}{x\,(y^2-x)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2}{y^2 + 3x} \\
S_y &= \frac{-y^2 + x}{y^3 + 3yx}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

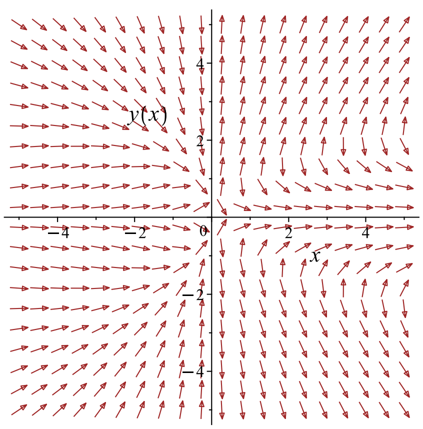
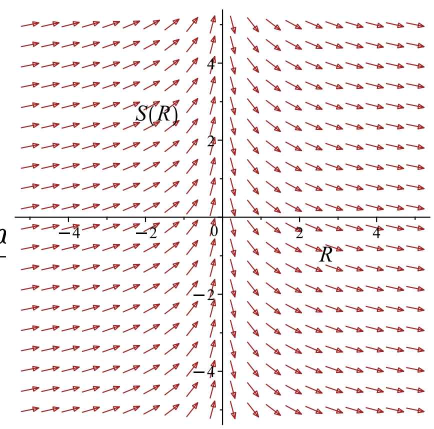
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{3} - \frac{2\ln(y^2 + 3x)}{3} = -\ln(x) + c_1$$

Which simplifies to

$$\frac{\ln(y)}{3} - \frac{2\ln(y^2 + 3x)}{3} = -\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(y^2+x)}{x(y^2-x)}$ 	$R = x$ $S = \frac{\ln(y)}{3} - \frac{2\ln(y^2+3x)}{3}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\ln(2)}{3} - \frac{2\ln(5)}{3} = -\ln(2) + c_1$$

$$c_1 = \frac{2\ln(2)}{3} - \frac{2\ln(5)}{3}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(y)}{3} - \frac{2\ln(y^2+3x)}{3} = -\ln(x) + \frac{2\ln(2)}{3} - \frac{2\ln(5)}{3}$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{3} - \frac{2\ln(y^2+3x)}{3} = -\ln(x) + \frac{2\ln(2)}{3} - \frac{2\ln(5)}{3} \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{3} - \frac{2\ln(y^2+3x)}{3} = -\ln(x) + \frac{2\ln(2)}{3} - \frac{2\ln(5)}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.984 (sec). Leaf size: 40

```
dsolve([y(x)*(x+y(x)^2)+x*(x-y(x)^2)*diff(y(x),x)=0,y(2) = 2],y(x), singsol=all)
```

$$y(x) = \mathrm{RootOf}\left(-3\ln(x) + 4\ln(2) - 4\ln(5) + 4\ln\left(\frac{\_Z^2 + 3x}{x}\right) - 2\ln\left(\frac{\_Z}{\sqrt{x}}\right)\right)$$

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y[x]*(x+y[x]^2)+x*(x-y[x]^2)*y'[x]==0,{y[2]==2}},y[x],x,IncludeSingularSolutions ->
```

$$\{\}$$

6 Exercise 10, page 41

6.1 problem 1

Internal problem ID [1987]
Internal file name [`OUTPUT/1987_Sunday_February_25_2024_06_44_13_AM_62589353/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{2y + y'x = x^2}$$

6.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = x$$

Hence the ode is

$$y' + \frac{2y}{x} = x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{2}{x} dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(x) \\
\frac{d}{dx}(y\,x^2) &= (x^2)(x) \\
d(y\,x^2) &= x^3\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,x^2 &= \int x^3\,dx \\
y\,x^2 &= \frac{x^4}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
y = \frac{x^2}{4} + \frac{c_1}{x^2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^2}{4} + \frac{c_1}{x^2} \tag{1}
$$

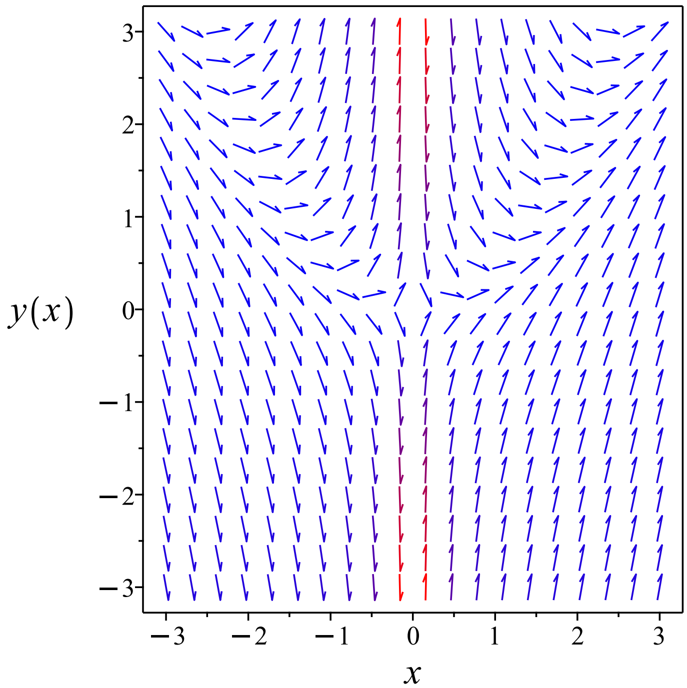


Figure 278: Slope field plot

Verification of solutions

$$y = \frac{x^2}{4} + \frac{c_1}{x^2}$$

Verified OK.

6.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^2 + 2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 146: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = y\, x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x^2 + 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2yx \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 y = \frac{x^4}{4} + c_1$$

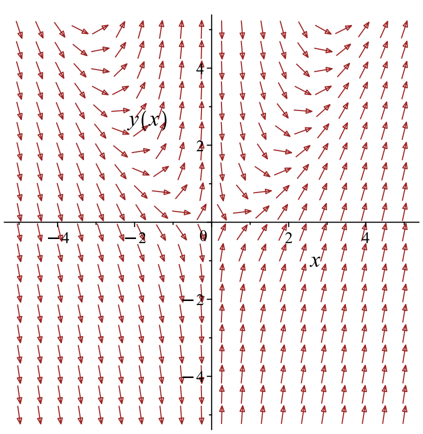
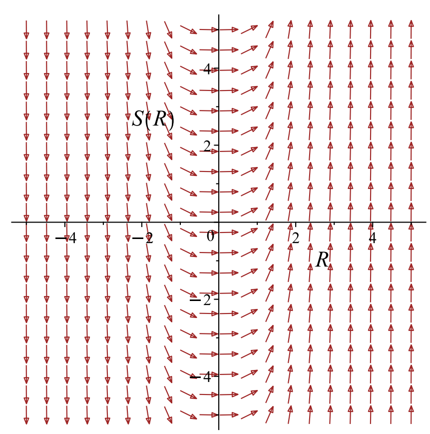
Which simplifies to

$$x^2 y = \frac{x^4}{4} + c_1$$

Which gives

$$y = \frac{x^4 + 4c_1}{4x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+2y}{x}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = R^3$ 

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 4c_1}{4x^2} \tag{1}$$

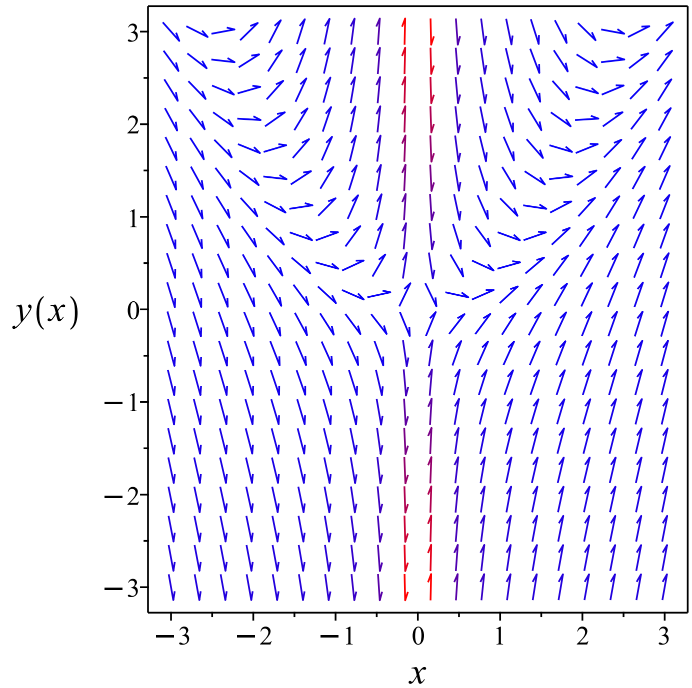


Figure 279: Slope field plot

Verification of solutions

$$y = \frac{x^4 + 4c_1}{4x^2}$$

Verified OK.

6.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x) \, \mathrm{d}y &= \left(x^2 - 2y\right) \mathrm{d}x \\
\left(-x^2 + 2y\right) \mathrm{d}x + (x) \, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^2 + 2y \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 + 2y\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((2) - (1)) \\
&= \frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x(-x^2 + 2y) \\
&= -x(x^2 - 2y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x(x) \\
&= x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-x(x^2 - 2y)) + (x^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x(x^2 - 2y) \, \mathrm{d}x$$
$$\phi = -\frac{(x^2 - 2y)^2}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 - 2y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 - 2y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (2y) \, \mathrm{d}y$$
$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(x^2 - 2y)^2}{4} + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(x^2 - 2y)^2}{4} + y^2$$

The solution becomes

$$y = \frac{x^4 + 4c_1}{4x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 4c_1}{4x^2} \tag{1}$$

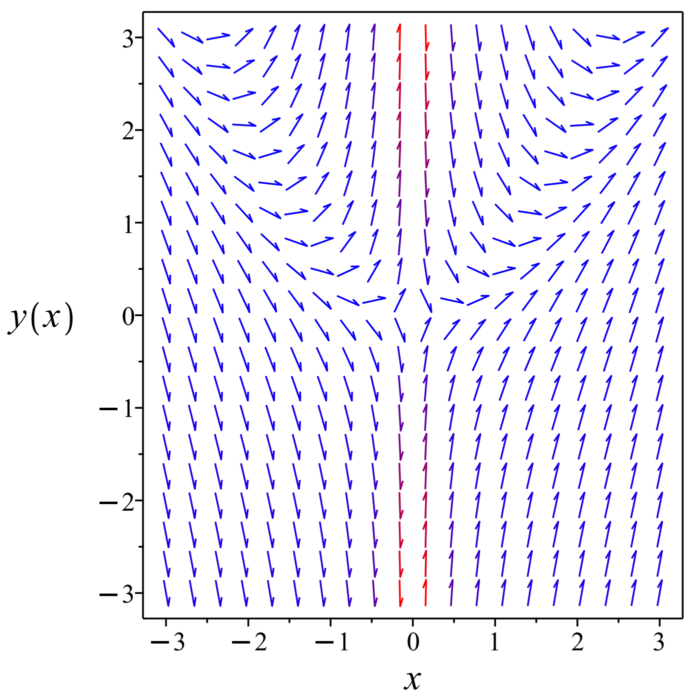


Figure 280: Slope field plot

Verification of solutions

$$y = \frac{x^4 + 4c_1}{4x^2}$$

Verified OK.

6.1.4 Maple step by step solution

Let's solve

$2y + y'x = x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int x^3 dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^4}{4} + c_1}{x^2}$

- Simplify

$$y = \frac{x^4+4c_1}{4x^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x)+2*y(x)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^4 + 4c_1}{4x^2}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 19

```
DSolve[x*y'[x]+2*y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{4} + \frac{c_1}{x^2}$$

6.2 problem 2

Internal problem ID [1988]
Internal file name [`OUTPUT/1988_Sunday_February_25_2024_06_44_13_AM_37754355/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - yx = e^{\frac{x^2}{2}} \cos(x)}$$

6.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -x$$
$$q(x) = e^{\frac{x^2}{2}} \cos(x)$$

Hence the ode is

$$y' - yx = e^{\frac{x^2}{2}} \cos(x)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -x dx} \\
&= e^{-\frac{x^2}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(e^{\frac{x^2}{2}} \cos(x)\right) \\
\frac{d}{dx}\left(e^{-\frac{x^2}{2}} y\right) &= \left(e^{-\frac{x^2}{2}}\right)\left(e^{\frac{x^2}{2}} \cos(x)\right) \\
d\left(e^{-\frac{x^2}{2}} y\right) &= \cos(x)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-\frac{x^2}{2}} y &= \int \cos(x)\, dx \\
e^{-\frac{x^2}{2}} y &= \sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x^2}{2}}$ results in

$$
y = e^{\frac{x^2}{2}} \sin(x) + c_1 e^{\frac{x^2}{2}}
$$

which simplifies to

$$
y = e^{\frac{x^2}{2}}(\sin(x) + c_1)
$$

Summary

The solution(s) found are the following

$$
y = e^{\frac{x^2}{2}}(\sin(x) + c_1) \tag{1}
$$

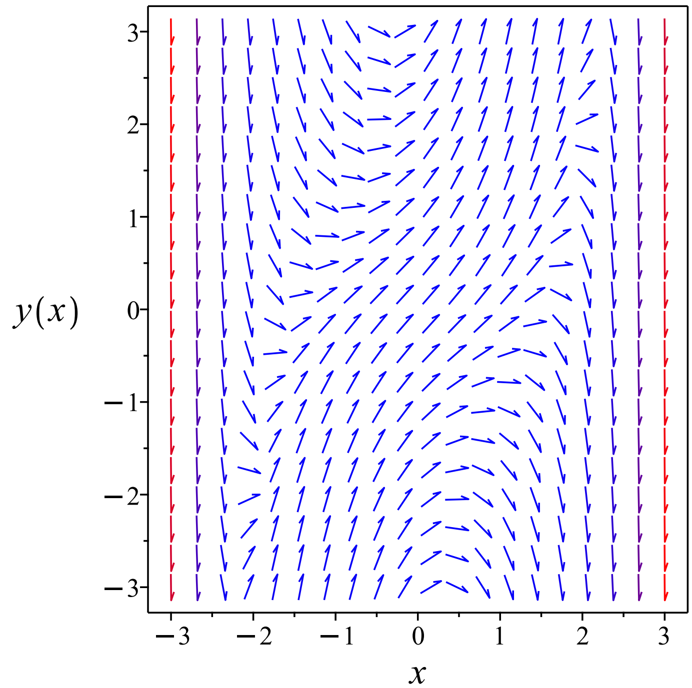


Figure 281: Slope field plot

Verification of solutions

$$y = e^{\frac{x^2}{2}}(\sin(x) + c_1)$$

Verified OK.

6.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = yx + e^{\frac{x^2}{2}} \cos(x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 149: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= e^{\frac{x^2}{2}}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{x^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{x^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = yx + e^{\frac{x^2}{2}} \cos(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -x\, e^{-\frac{x^2}{2}} y \\ S_y &= e^{-\frac{x^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-\frac{x^2}{2}} y = \sin(x) + c_1$$

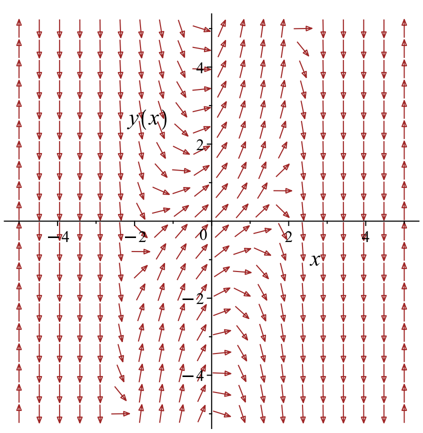
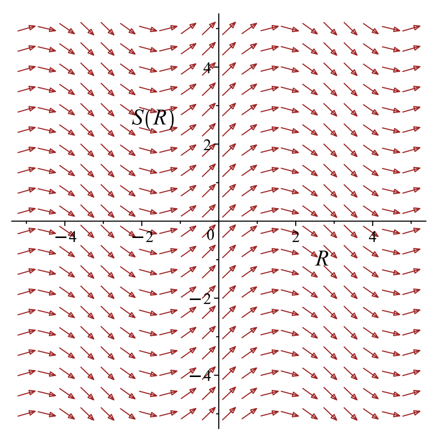
Which simplifies to

$$e^{-\frac{x^2}{2}} y = \sin(x) + c_1$$

Which gives

$$y = e^{\frac{x^2}{2}} (\sin(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = yx + e^{\frac{x^2}{2}} \cos(x)$ 	$R = x$ $S = e^{-\frac{x^2}{2}} y$	$\frac{dS}{dR} = \cos(R)$ 

Summary

The solution(s) found are the following

$$y = e^{\frac{x^2}{2}} (\sin(x) + c_1) \tag{1}$$

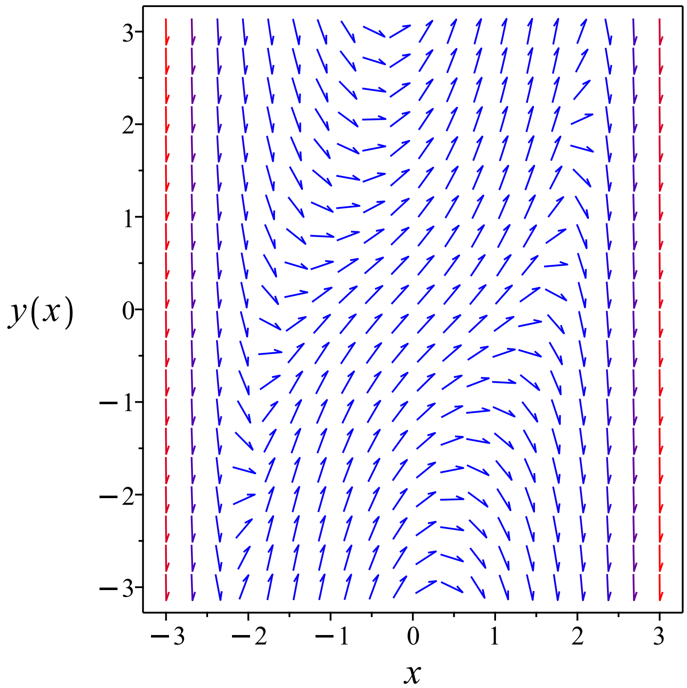


Figure 282: Slope field plot

Verification of solutions

$$y = e^{\frac{x^2}{2}}(\sin(x) + c_1)$$

Verified OK.

6.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1A}$$

Therefore

$$dy = \left(yx + e^{\frac{x^2}{2}} \cos(x) \right) dx$$
$$\left(-yx - e^{\frac{x^2}{2}} \cos(x) \right) dx + dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -yx - e^{\frac{x^2}{2}} \cos(x)$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(-yx - e^{\frac{x^2}{2}} \cos(x) \right)$$
$$= -x$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-x) - (0)) \\
&= -x
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -x \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{x^2}{2}} \\
&= \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{x^2}{2}}\left(-yx - \mathrm{e}^{\frac{x^2}{2}}\cos(x)\right) \\
&= -x\,\mathrm{e}^{-\frac{x^2}{2}} y - \cos(x)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{x^2}{2}}(1) \\
&= \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-x\,\mathrm{e}^{-\frac{x^2}{2}} y - \cos(x)\right) + \left(\mathrm{e}^{-\frac{x^2}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -x\,\mathrm{e}^{-\frac{x^2}{2}} y - \cos(x) \mathrm{d}x$$

$$\phi = \mathrm{e}^{-\frac{x^2}{2}} y - \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{x^2}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{x^2}{2}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{x^2}{2}} = \mathrm{e}^{-\frac{x^2}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-\frac{x^2}{2}} y - \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-\frac{x^2}{2}} y - \sin(x)$$

The solution becomes

$$y = e^{\frac{x^2}{2}} (\sin(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x^2}{2}} (\sin(x) + c_1) \tag{1}$$

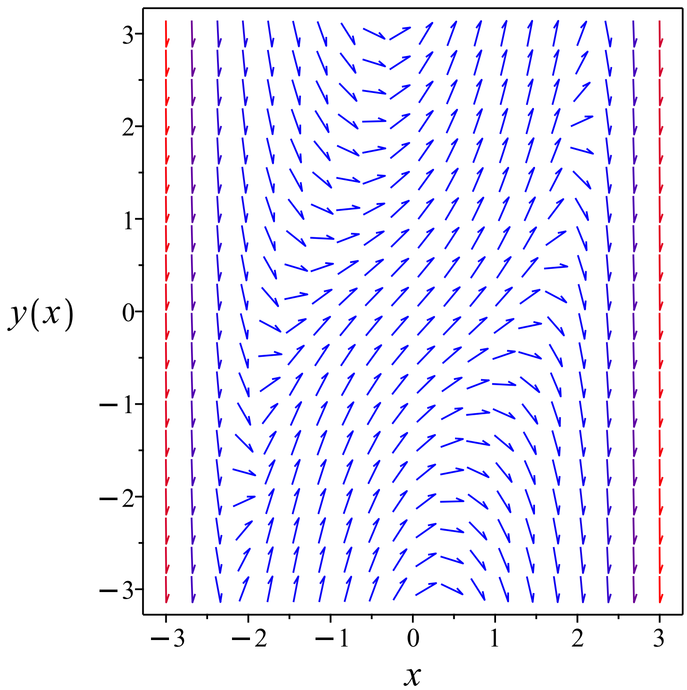


Figure 283: Slope field plot

Verification of solutions

$$y = e^{\frac{x^2}{2}} (\sin(x) + c_1)$$

Verified OK.

6.2.4 Maple step by step solution

Let's solve

$y' - yx = e^{\frac{x^2}{2}} \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = yx + e^{\frac{x^2}{2}} \cos(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - yx = e^{\frac{x^2}{2}} \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' - yx) = \mu(x) e^{\frac{x^2}{2}} \cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x)(y' - yx) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x) x$

- Solve to find the integrating factor

  $\mu(x) = e^{-\frac{x^2}{2}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) e^{\frac{x^2}{2}} \cos(x) dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \mu(x) e^{\frac{x^2}{2}} \cos(x) dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) e^{\frac{x^2}{2}} \cos(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{-\frac{x^2}{2}}$

  $y = \frac{\int e^{\frac{x^2}{2}} \cos(x) e^{-\frac{x^2}{2}} dx + c_1}{e^{-\frac{x^2}{2}}}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x) + c_1}{e^{-\frac{x^2}{2}}}$

- Simplify

  $$y = e^{\frac{x^2}{2}}(\sin(x) + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)-x*y(x)=exp(x^2/2)*cos(x),y(x), singsol=all)
```

$$y(x) = (\sin(x) + c_1)\, e^{\frac{x^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 20

```
DSolve[y'[x]-x*y[x]==Exp[x^2/2]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{x^2}{2}}(\sin(x) + c_1)$$

6.3 problem 3

Internal problem ID [1989]
Internal file name [`OUTPUT/1989_Sunday_February_25_2024_06_44_14_AM_63231234/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2yx + y' = 2x\,\mathrm{e}^{-x^2}}$$

6.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2x$$
$$q(x) = 2x\,\mathrm{e}^{-x^2}$$

Hence the ode is

$$2yx + y' = 2x\,\mathrm{e}^{-x^2}$$

The integrating factor μ is

$$\mu = e^{\int 2x dx}$$
$$= e^{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(2x\, e^{-x^2}\right)$$
$$\frac{d}{dx}\left(e^{x^2} y\right) = \left(e^{x^2}\right)\left(2x\, e^{-x^2}\right)$$
$$d\left(e^{x^2} y\right) = (2x)\, dx$$

Integrating gives

$$e^{x^2} y = \int 2x\, dx$$
$$e^{x^2} y = x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = x^2 e^{-x^2} + c_1 e^{-x^2}$$

which simplifies to

$$y = e^{-x^2}\left(x^2 + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-x^2}\left(x^2 + c_1\right) \tag{1}$$

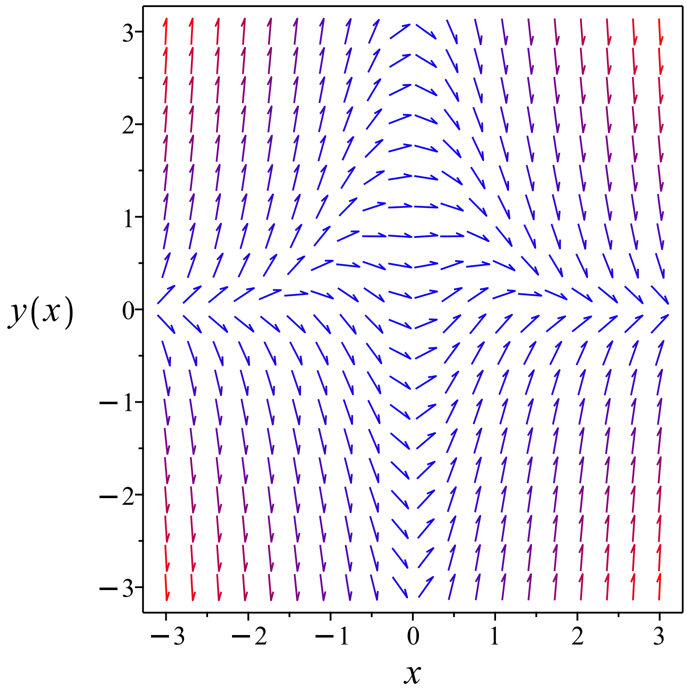


Figure 284: Slope field plot

Verification of solutions

$$y = e^{-x^2}(x^2 + c_1)$$

Verified OK.

6.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2yx + 2x\,e^{-x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 152: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-x^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x^2}} dy \end{aligned}$$

Which results in

$$S = e^{x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2yx + 2x\,e^{-x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2x\,e^{x^2} y \\ S_y &= e^{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x^2} y = x^2 + c_1$$

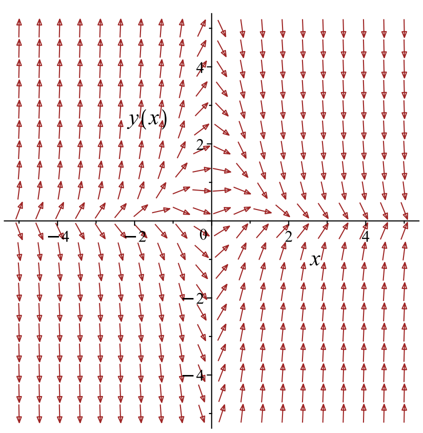
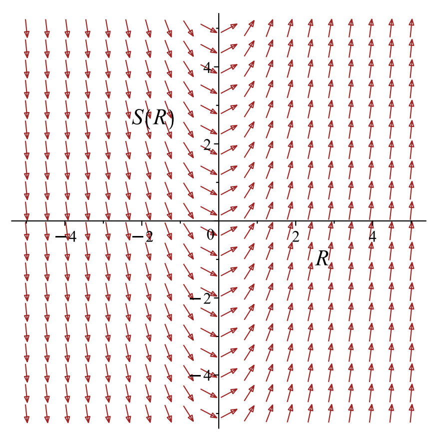
Which simplifies to

$$e^{x^2} y = x^2 + c_1$$

Which gives

$$y = e^{-x^2} \left(x^2 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2yx + 2x\,e^{-x^2}$ 	$R = x$ $S = e^{x^2} y$	$\frac{dS}{dR} = 2R$ 

Summary

The solution(s) found are the following

$$y = e^{-x^2} \left(x^2 + c_1\right) \tag{1}$$

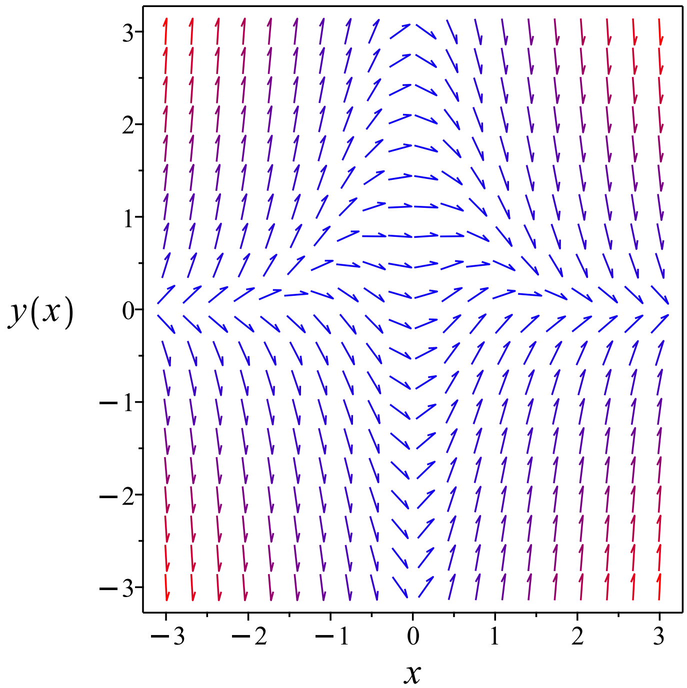


Figure 285: Slope field plot

Verification of solutions

$$y = e^{-x^2}(x^2 + c_1)$$

Verified OK.

6.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-2yx + 2x\, \mathrm{e}^{-x^2}\right) \mathrm{d}x$$
$$\left(2yx - 2x\, \mathrm{e}^{-x^2}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2yx - 2x\, \mathrm{e}^{-x^2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2yx - 2x\, \mathrm{e}^{-x^2}\right) \\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((2x) - (0)) \\ &= 2x \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 2x \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x^2} \\ &= \mathrm{e}^{x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x^2}\left(2yx - 2x\,\mathrm{e}^{-x^2}\right) \\ &= 2x\left(\mathrm{e}^{x^2} y - 1\right) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x^2}(1) \\ &= \mathrm{e}^{x^2} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(2x\left(\mathrm{e}^{x^2} y - 1\right)\right) + \left(\mathrm{e}^{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2x\left(\mathrm{e}^{x^2} y - 1\right) \mathrm{d}x$$
$$\phi = -x^2 + \mathrm{e}^{x^2} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{x^2} = \mathrm{e}^{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 + \mathrm{e}^{x^2} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 + \mathrm{e}^{x^2} y$$

The solution becomes

$$y = e^{-x^2}(x^2 + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{-x^2}(x^2 + c_1) \tag{1}$$

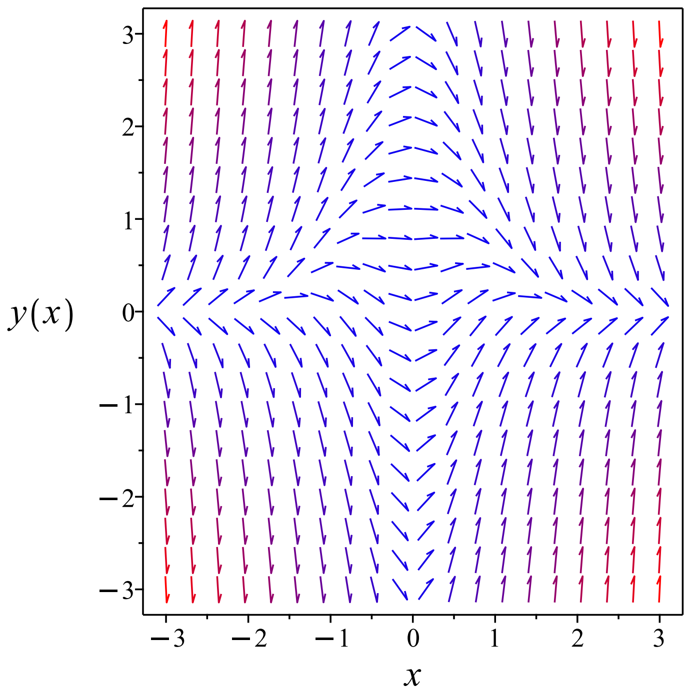


Figure 286: Slope field plot

Verification of solutions

$$y = e^{-x^2}(x^2 + c_1)$$

Verified OK.

6.3.4 Maple step by step solution

Let's solve

$2yx + y' = 2x\,\mathrm{e}^{-x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2yx + 2x\,\mathrm{e}^{-x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $2yx + y' = 2x\,\mathrm{e}^{-x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(2yx + y') = 2\mu(x)\,x\,\mathrm{e}^{-x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(2yx + y') = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)\,x$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,x\,\mathrm{e}^{-x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,x\,\mathrm{e}^{-x^2} dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)x\,\mathrm{e}^{-x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x^2}$

  $y = \frac{\int 2x\,\mathrm{e}^{-x^2}\mathrm{e}^{x^2} dx + c_1}{\mathrm{e}^{x^2}}$

- Evaluate the integrals on the rhs

  $y = \frac{x^2 + c_1}{\mathrm{e}^{x^2}}$

- Simplify

$$y = e^{-x^2}(x^2 + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+2*x*y(x)=2*x*exp(-x^2),y(x), singsol=all)
```

$$y(x) = (x^2 + c_1)\, e^{-x^2}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 19

```
DSolve[y'[x]+2*x*y[x]==2*x*Exp[-x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}(x^2 + c_1)$$

6.4 problem 4

Internal problem ID [1990]
Internal file name [`OUTPUT/1990_Sunday_February_25_2024_06_44_14_AM_85983421/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 3x^2 e^x}$$

6.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = 3x^2 e^x$$

Hence the ode is

$$y' - y = 3x^2 e^x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-1)dx} \\
&= e^{-x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(3x^2 e^x\right) \\
\frac{d}{dx}\left(e^{-x} y\right) &= \left(e^{-x}\right)\left(3x^2 e^x\right) \\
d\left(e^{-x} y\right) &= \left(3x^2\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-x} y &= \int 3x^2\, dx \\
e^{-x} y &= x^3 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$
y = e^x x^3 + c_1 e^x
$$

which simplifies to

$$
y = e^x\left(x^3 + c_1\right)
$$

Summary

The solution(s) found are the following

$$
y = e^x\left(x^3 + c_1\right) \tag{1}
$$

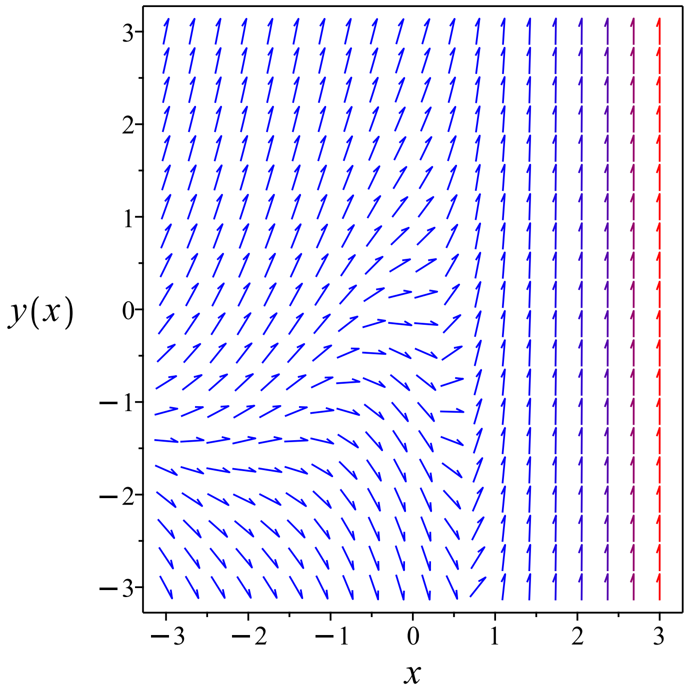


Figure 287: Slope field plot

<u>Verification of solutions</u>

$$y = e^{x}(x^3 + c_1)$$

Verified OK.

6.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y + 3x^2 e^{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 155: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x}} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y + 3x^2 e^{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{-x} = x^3 + c_1$$

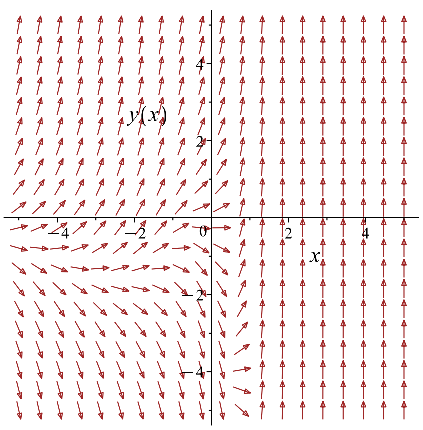
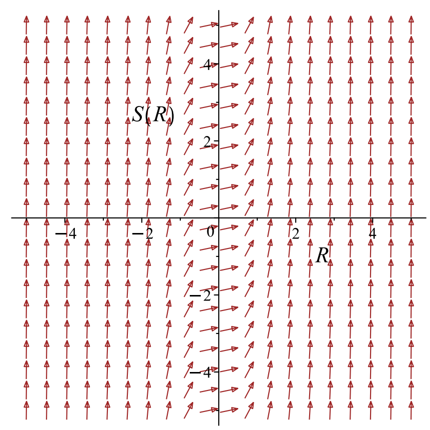
Which simplifies to

$$y\,\mathrm{e}^{-x} = x^3 + c_1$$

Which gives

$$y = \mathrm{e}^x \left(x^3 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y + 3x^2\mathrm{e}^x$ 	$R = x$ $S = \mathrm{e}^{-x} y$	$\frac{dS}{dR} = 3R^2$ 

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^x \left(x^3 + c_1\right) \tag{1}$$

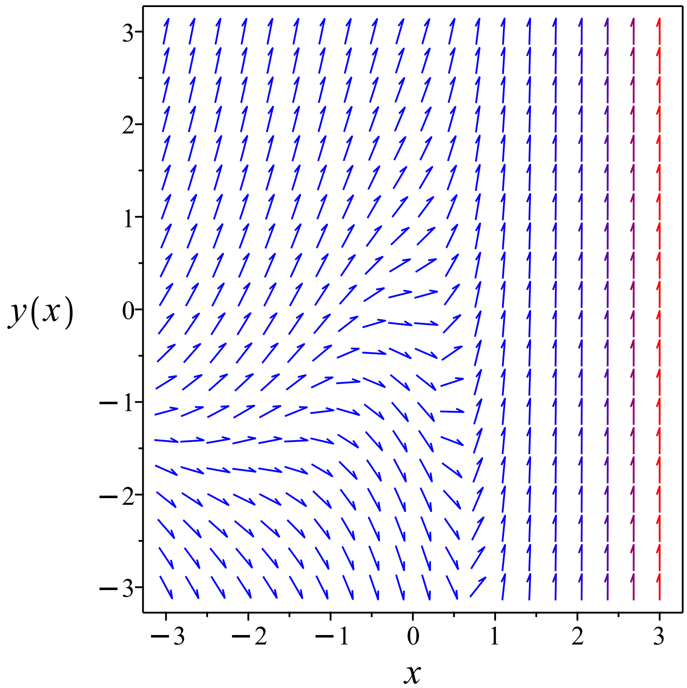


Figure 288: Slope field plot

Verification of solutions

$$y = e^{x}\left(x^{3}+c_1\right)$$

Verified OK.

6.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(y + 3x^2 \mathrm{e}^x\right) \mathrm{d}x \\
\left(-y - 3x^2 \mathrm{e}^x\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -y - 3x^2 \mathrm{e}^x \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - 3x^2 \mathrm{e}^x\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -1 \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-x} \\
&= \mathrm{e}^{-x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x}(-y - 3x^2\mathrm{e}^x) \\
&= -\mathrm{e}^{-x}y - 3x^2
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(1) \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-\mathrm{e}^{-x}y - 3x^2) + (\mathrm{e}^{-x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\mathrm{e}^{-x} y - 3x^2 \,\mathrm{d}x$$

$$\phi = -x^3 + \mathrm{e}^{-x} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 + \mathrm{e}^{-x} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 + \mathrm{e}^{-x} y$$

The solution becomes

$$y = \mathrm{e}^{x} (x^3 + c_1)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x} (x^3 + c_1) \tag{1}$$

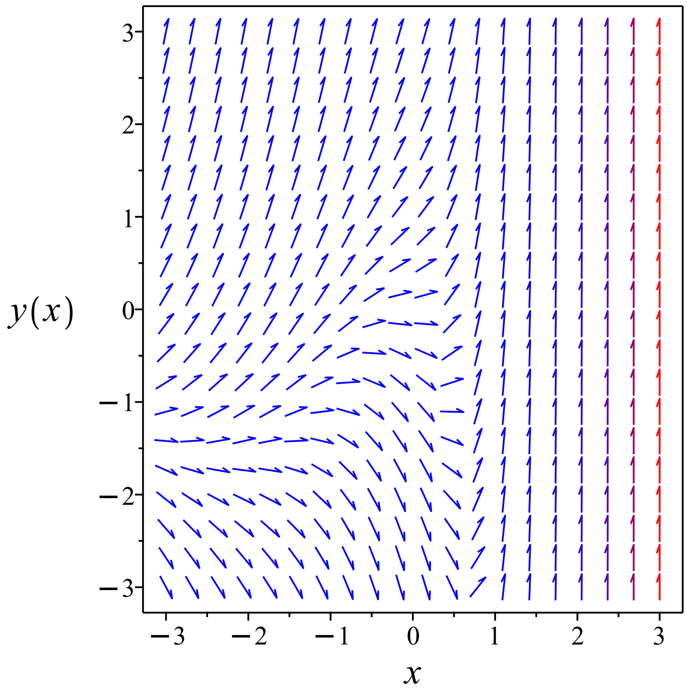


Figure 289: Slope field plot

Verification of solutions

$$y = e^x (x^3 + c_1)$$

Verified OK.

6.4.4 Maple step by step solution

Let's solve

$y' - y = 3x^2 e^x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + 3x^2 e^x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = 3x^2 e^x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) (y' - y) = 3\mu(x) x^2 e^x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\, (y' - y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 3\mu(x)\, x^2 \mathrm{e}^x dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 3\mu(x)\, x^2 \mathrm{e}^x dx + c_1$
- Solve for y

  $y = \frac{\int 3\mu(x)x^2\mathrm{e}^x dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int 3x^2\mathrm{e}^x\mathrm{e}^{-x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{x^3 + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = \mathrm{e}^x (x^3 + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=y(x)+3*x^2*exp(x),y(x), singsol=all)
```

$$y(x) = (x^3 + c_1) e^x$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 15

```
DSolve[y'[x]==y[x]+3*x^2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x (x^3 + c_1)$$

6.5 problem 5

Internal problem ID [1991]
Internal file name [`OUTPUT/1991_Sunday_February_25_2024_06_44_15_AM_15727751/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' + x = \mathrm{e}^{-y}}$$

6.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(y)x = q(y)$$

Where here

$$p(y) = 1$$
$$q(y) = \mathrm{e}^{-y}$$

Hence the ode is

$$x' + x = \mathrm{e}^{-y}$$

The integrating factor μ is

$$\mu = e^{\int 1dy}$$
$$= e^{y}$$

The ode becomes

$$\frac{d}{dy}(\mu x) = (\mu)\left(e^{-y}\right)$$
$$\frac{d}{dy}(e^{y} x) = (e^{y})\left(e^{-y}\right)$$
$$d(e^{y} x) = dy$$

Integrating gives

$$e^{y} x = \int dy$$
$$e^{y} x = y + c_1$$

Dividing both sides by the integrating factor $\mu = e^{y}$ results in

$$x = y\,e^{-y} + c_1 e^{-y}$$

which simplifies to

$$x = e^{-y}(y + c_1)$$

Summary

The solution(s) found are the following

$$x = e^{-y}(y + c_1) \tag{1}$$

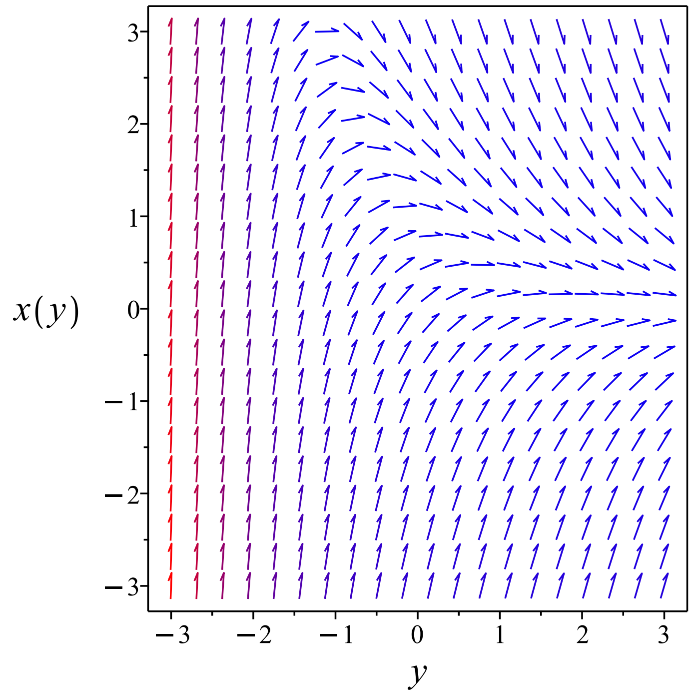


Figure 290: Slope field plot

Verification of solutions

$$x = \mathrm{e}^{-y}(y + c_1)$$

Verified OK.

6.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -x + \mathrm{e}^{-y}$$
$$x' = \omega(y, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_x - \xi_y) - \omega^2 \xi_x - \omega_y \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 158: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(y, x) &= 0 \\
\eta(y, x) &= \mathrm{e}^{-y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial x}\right) S(y, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-y}} dy \end{aligned}$$

Which results in

$$S = e^{y} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y, x) S_x}{R_y + \omega(y, x) R_x} \tag{2}$$

Where in the above R_y, R_x, S_y, S_x are all partial derivatives and $\omega(y, x)$ is the right hand side of the original ode given by

$$\omega(y, x) = -x + e^{-y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_y &= 1 \\ R_x &= 0 \\ S_y &= e^{y} x \\ S_x &= e^{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, x coordinates. This results in

$$e^y x = y + c_1$$

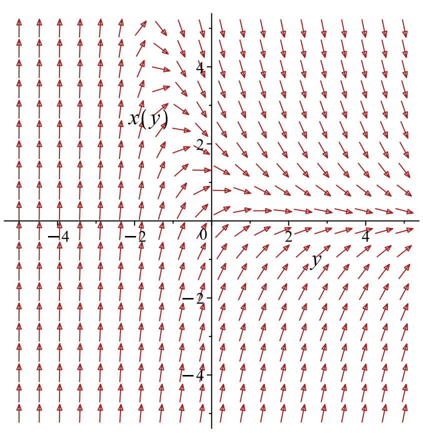
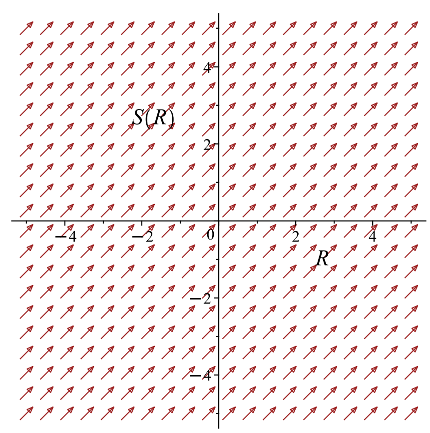
Which simplifies to

$$e^y x = y + c_1$$

Which gives

$$x = e^{-y}(y + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in y, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dy} = -x + e^{-y}$ 	$R = y$ $S = e^y x$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$x = e^{-y}(y + c_1) \tag{1}$$

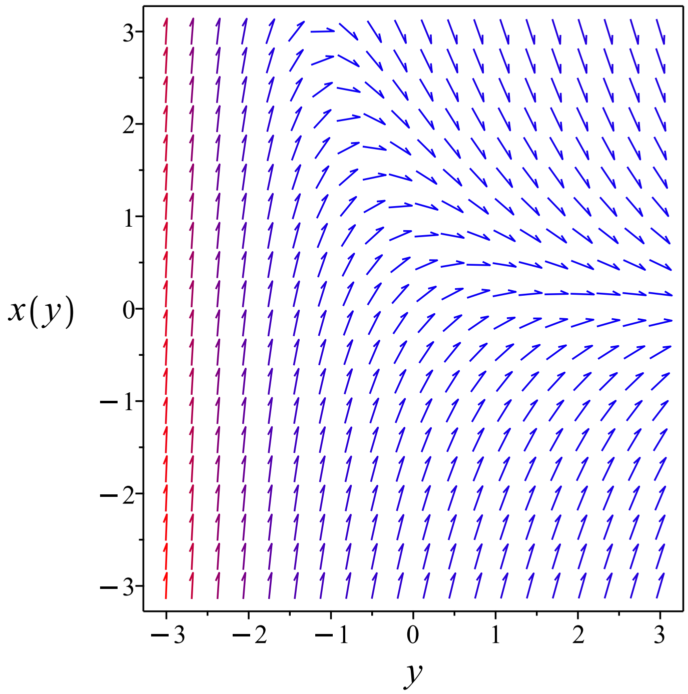


Figure 291: Slope field plot

Verification of solutions

$$x = e^{-y}(y + c_1)$$

Verified OK.

6.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y, x)\, \mathrm{d}y + N(y, x)\, \mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}x &= \left(-x + \mathrm{e}^{-y}\right) \mathrm{d}y \\ \left(x - \mathrm{e}^{-y}\right) \mathrm{d}y + \mathrm{d}x &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(y, x) &= x - \mathrm{e}^{-y} \\ N(y, x) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(x - \mathrm{e}^{-y}\right) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial y} &= \frac{\partial}{\partial y}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}y} \\ &= e^{\int 1 \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{y} \\ &= \mathrm{e}^{y} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{y}\left(x - \mathrm{e}^{-y}\right) \\ &= \mathrm{e}^{y} x - 1 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{y}(1) \\ &= \mathrm{e}^{y} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}y} &= 0 \\ \left(\mathrm{e}^{y} x - 1\right) + \left(\mathrm{e}^{y}\right)\frac{\mathrm{d}x}{\mathrm{d}y} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(y, x)$

$$\frac{\partial \phi}{\partial y} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \overline{M} \, \mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \mathrm{e}^{y} x - 1 \, \mathrm{d}y$$
$$\phi = -y + \mathrm{e}^{y} x + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both y and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \mathrm{e}^{y} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{y}$. Therefore equation (4) becomes

$$\mathrm{e}^{y} = \mathrm{e}^{y} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = -y + \mathrm{e}^{y} x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -y + \mathrm{e}^{y} x$$

The solution becomes

$$x = \mathrm{e}^{-y}(y + c_1)$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-y}(y + c_1) \tag{1}$$

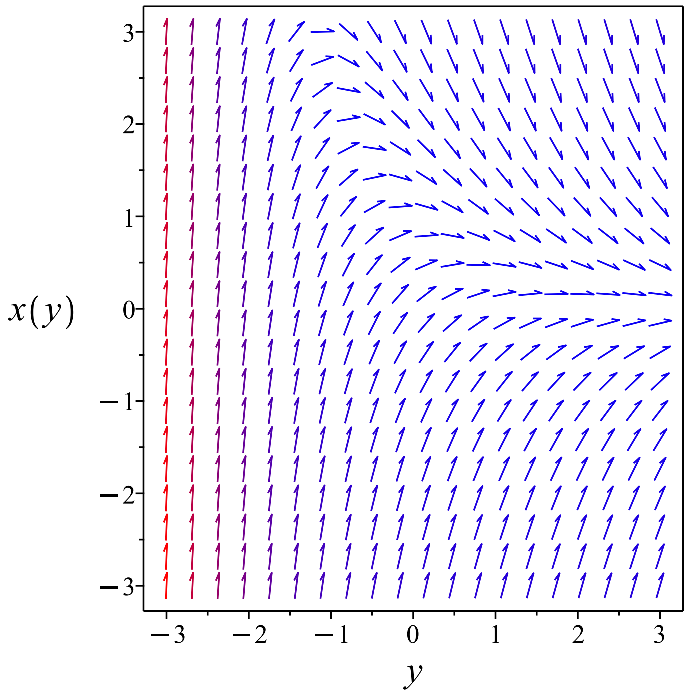


Figure 292: Slope field plot

Verification of solutions

$$x = e^{-y}(y + c_1)$$

Verified OK.

6.5.4 Maple step by step solution

Let's solve

$x' + x = e^{-y}$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -x + e^{-y}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + x = e^{-y}$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

$$\mu(y)\left(x'+x\right)=\mu(y)\,\mathrm{e}^{-y}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\,x)$

$$\mu(y)\left(x'+x\right)=\mu'(y)\,x+\mu(y)\,x'$$

- Isolate $\mu'(y)$

$$\mu'(y)=\mu(y)$$

- Solve to find the integrating factor

$$\mu(y)=\mathrm{e}^{y}$$

- Integrate both sides with respect to y

$$\int\left(\frac{d}{dy}(\mu(y)\,x)\right)dy=\int\mu(y)\,\mathrm{e}^{-y}dy+c_1$$

- Evaluate the integral on the lhs

$$\mu(y)\,x=\int\mu(y)\,\mathrm{e}^{-y}dy+c_1$$

- Solve for x

$$x=\frac{\int\mu(y)\mathrm{e}^{-y}dy+c_1}{\mu(y)}$$

- Substitute $\mu(y)=\mathrm{e}^{y}$

$$x=\frac{\int\mathrm{e}^{-y}\mathrm{e}^{y}dy+c_1}{\mathrm{e}^{y}}$$

- Evaluate the integrals on the rhs

$$x=\frac{y+c_1}{\mathrm{e}^{y}}$$

- Simplify

$$x=\mathrm{e}^{-y}(y+c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(x(y),y)+x(y)=exp(-y),x(y), singsol=all)
```

$$x(y) = (y + c_1)\,\mathrm{e}^{-y}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 15

```
DSolve[x'[y]+x[y]==Exp[-y],x[y],y,IncludeSingularSolutions -> True]
```

$$x(y) \to e^{-y}(y + c_1)$$

6.6 problem 6

Internal problem ID [1992]
Internal file name [`OUTPUT/1992_Sunday_February_25_2024_06_44_15_AM_82609578/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y x' + (y+1)\, x = \mathrm{e}^{y}}$$

6.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(y)x = q(y)$$

Where here

$$p(y) = -\frac{-1-y}{y}$$
$$q(y) = \frac{\mathrm{e}^{y}}{y}$$

Hence the ode is

$$x' - \frac{(-1-y)\, x}{y} = \frac{\mathrm{e}^{y}}{y}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{-1-y}{y} dy} \\
&= e^{y+\ln(y)}
\end{aligned}
$$

Which simplifies to

$$
\mu = e^{y} y
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dy}(\mu x) &= (\mu)\left(\frac{e^{y}}{y}\right) \\
\frac{d}{dy}(e^{y} x y) &= (e^{y} y)\left(\frac{e^{y}}{y}\right) \\
d(e^{y} x y) &= e^{2y}\, dy
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{y} x y &= \int e^{2y}\, dy \\
e^{y} x y &= \frac{e^{2y}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{y} y$ results in

$$
x = \frac{e^{-y} e^{2y}}{2y} + \frac{c_1 e^{-y}}{y}
$$

which simplifies to

$$
x = \frac{2c_1 e^{-y} + e^{y}}{2y}
$$

Summary
The solution(s) found are the following

$$
x = \frac{2c_1 e^{-y} + e^{y}}{2y} \tag{1}
$$

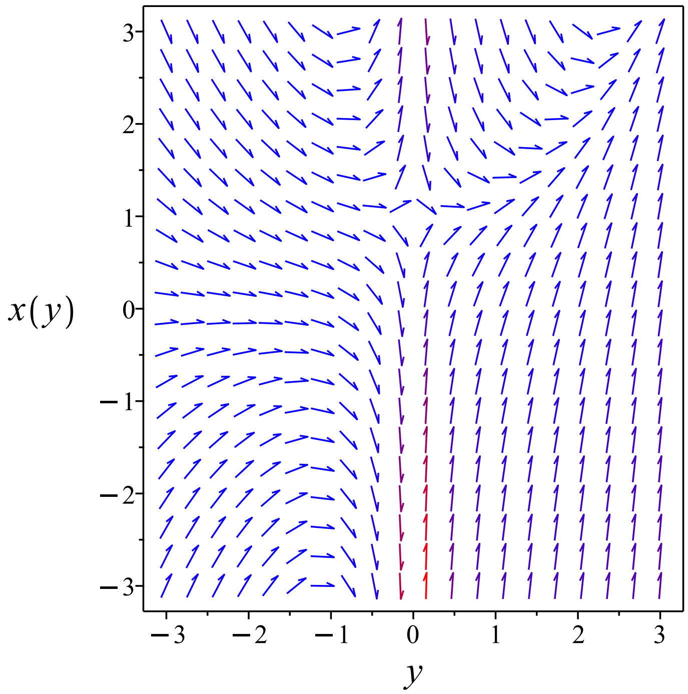


Figure 293: Slope field plot

Verification of solutions

$$x = \frac{2c_1 e^{-y} + e^y}{2y}$$

Verified OK.

6.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{-yx + e^y - x}{y}$$
$$x' = \omega(y, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_x - \xi_y) - \omega^2 \xi_x - \omega_y \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 161: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(y, x) &= 0 \\
\eta(y, x) &= \mathrm{e}^{-y-\ln(y)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial x}\right) S(y, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-y-\ln(y)}} dy \end{aligned}$$

Which results in

$$S = e^y x y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y,x) S_x}{R_y + \omega(y,x) R_x} \tag{2}$$

Where in the above R_y, R_x, S_y, S_x are all partial derivatives and $\omega(y,x)$ is the right hand side of the original ode given by

$$\omega(y,x) = \frac{-yx + e^y - x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_y &= 1 \\ R_x &= 0 \\ S_y &= e^y x(y+1) \\ S_x &= e^y y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{2y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, x coordinates. This results in

$$e^y xy = \frac{e^{2y}}{2} + c_1$$

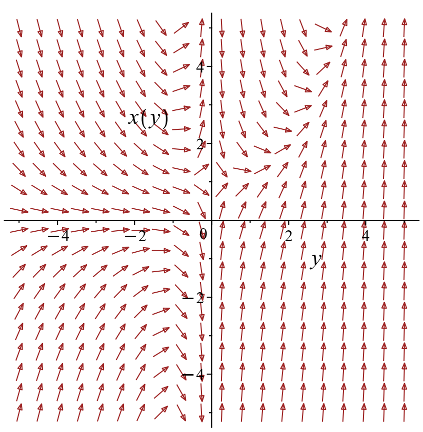
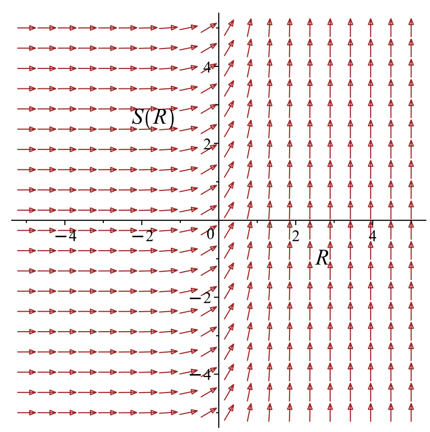
Which simplifies to

$$e^y xy = \frac{e^{2y}}{2} + c_1$$

Which gives

$$x = \frac{(e^{2y} + 2c_1) e^{-y}}{2y}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in y, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dy} = \frac{-yx+e^y-x}{y}$ 	$R = y$ $S = e^y xy$	$\frac{dS}{dR} = e^{2R}$ 

Summary
The solution(s) found are the following

$$x = \frac{(e^{2y} + 2c_1) e^{-y}}{2y} \tag{1}$$

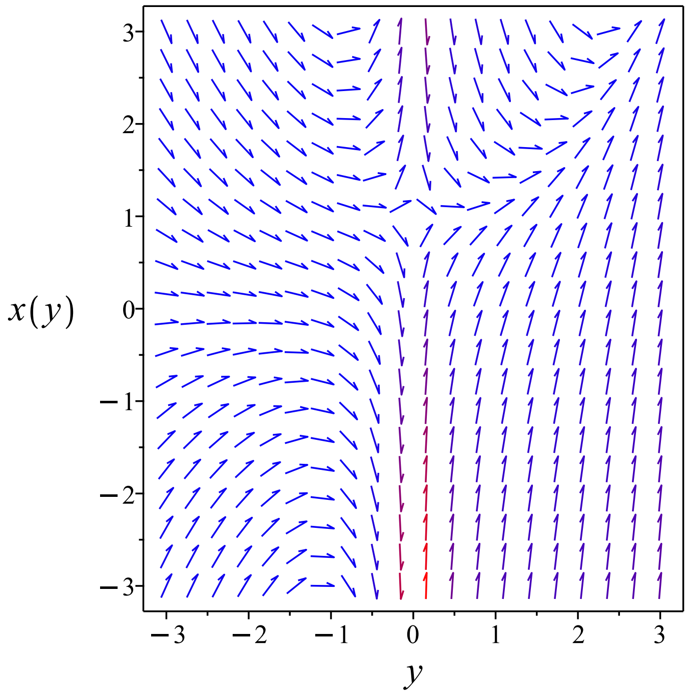


Figure 294: Slope field plot

Verification of solutions

$$x = \frac{(e^{2y} + 2c_1) e^{-y}}{2y}$$

Verified OK.

6.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y,x)\,\mathrm{d}y + N(y,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (y)\,\mathrm{d}x &= (-x(y+1) + \mathrm{e}^{y})\,\mathrm{d}y \\ (-\mathrm{e}^{y} + x(y+1))\,\mathrm{d}y + (y)\,\mathrm{d}x &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(y,x) &= -\mathrm{e}^{y} + x(y+1) \\ N(y,x) &= y \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-\mathrm{e}^{y} + x(y+1)) \\ &= y + 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial y} &= \frac{\partial}{\partial y}(y) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) \\
&= \frac{1}{y}((y+1) - (1)) \\
&= 1
\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}y} \\
&= e^{\int 1 \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{y} \\
&= \mathrm{e}^{y}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{y}(-\mathrm{e}^{y} + x(y+1)) \\
&= (-\mathrm{e}^{y} + x(y+1))\,\mathrm{e}^{y}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{y}(y) \\
&= \mathrm{e}^{y} y
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}y} &= 0 \\
((-\mathrm{e}^{y} + x(y+1))\,\mathrm{e}^{y}) + (\mathrm{e}^{y} y)\frac{\mathrm{d}x}{\mathrm{d}y} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(y, x)$

$$\frac{\partial \phi}{\partial y} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \overline{M} \,\mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int (-\mathrm{e}^y + x(y+1))\,\mathrm{e}^y \,\mathrm{d}y$$

$$\phi = \mathrm{e}^y xy - \frac{\mathrm{e}^{2y}}{2} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both y and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \mathrm{e}^y y + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^y y$. Therefore equation (4) becomes

$$\mathrm{e}^y y = \mathrm{e}^y y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^y xy - \frac{\mathrm{e}^{2y}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^y xy - \frac{\mathrm{e}^{2y}}{2}$$

The solution becomes

$$x = \frac{(e^{2y} + 2c_1)\, e^{-y}}{2y}$$

Summary

The solution(s) found are the following

$$x = \frac{(e^{2y} + 2c_1)\, e^{-y}}{2y} \tag{1}$$

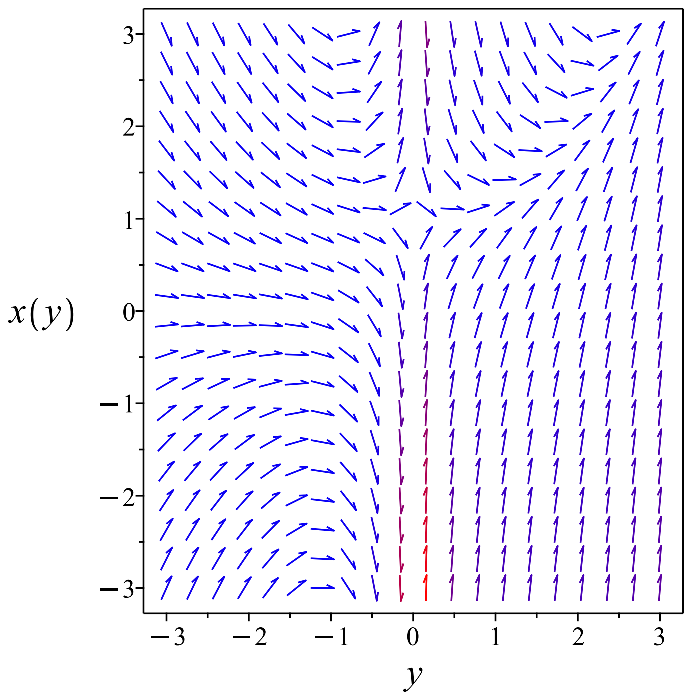


Figure 295: Slope field plot

Verification of solutions

$$x = \frac{(e^{2y} + 2c_1)\, e^{-y}}{2y}$$

Verified OK.

6.6.4 Maple step by step solution

Let's solve

$yx' + (y+1)\,x = \mathrm{e}^{y}$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -\frac{(y+1)x}{y} + \frac{\mathrm{e}^{y}}{y}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{(y+1)x}{y} = \frac{\mathrm{e}^{y}}{y}$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

  $\mu(y)\left(x' + \frac{(y+1)x}{y}\right) = \frac{\mu(y)\mathrm{e}^{y}}{y}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\,x)$

  $\mu(y)\left(x' + \frac{(y+1)x}{y}\right) = \mu'(y)\,x + \mu(y)\,x'$

- Isolate $\mu'(y)$

  $\mu'(y) = \frac{\mu(y)(y+1)}{y}$

- Solve to find the integrating factor

  $\mu(y) = \mathrm{e}^{y} y$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}(\mu(y)\,x)\right) dy = \int \frac{\mu(y)\mathrm{e}^{y}}{y} dy + c_1$

- Evaluate the integral on the lhs

  $\mu(y)\,x = \int \frac{\mu(y)\mathrm{e}^{y}}{y} dy + c_1$

- Solve for x

  $x = \frac{\int \frac{\mu(y)\mathrm{e}^{y}}{y} dy + c_1}{\mu(y)}$

- Substitute $\mu(y) = \mathrm{e}^{y} y$

  $x = \frac{\int (\mathrm{e}^{y})^2 dy + c_1}{\mathrm{e}^{y} y}$

- Evaluate the integrals on the rhs

  $x = \frac{\frac{(\mathrm{e}^{y})^2}{2} + c_1}{\mathrm{e}^{y} y}$

- Simplify

  $x = \frac{2c_1 e^{-y} + e^y}{2y}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(y*diff(x(y),y)+(1+y)*x(y)=exp(y),x(y), singsol=all)
```

$$x(y) = \frac{e^y + 2\,e^{-y} c_1}{2y}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 18

```
DSolve[y*x'[y]+(1+y)*x[y]==Exp[-y],x[y],y,IncludeSingularSolutions -> True]
```

$$x(y) \to \frac{e^{-y}(y + c_1)}{y}$$

6.7 problem 7

Internal problem ID [1993]
Internal file name [`OUTPUT/1993_Sunday_February_25_2024_06_44_15_AM_99315778/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (-3y + 2x)\, y' = 0}$$

6.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x + (-3u(x)\, x + 2x)\,(u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u(u-1)}{(3u-2)\, x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = \frac{u(u-1)}{3u-2}$. Integrating both sides gives

$$\frac{1}{\frac{u(u-1)}{3u-2}}\,du = -\frac{3}{x}\,dx$$
$$\int \frac{1}{\frac{u(u-1)}{3u-2}}\,du = \int -\frac{3}{x}\,dx$$
$$2\ln(u) + \ln(u-1) = -3\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u)+\ln(u-1)} = e^{-3\ln(x)+c_2}$$

Which simplifies to

$$u^2(u-1) = \frac{c_3}{x^3}$$

The solution is

$$u(x)^2\,(u(x)-1) = \frac{c_3}{x^3}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2\left(\frac{y}{x}-1\right)}{x^2} = \frac{c_3}{x^3}$$
$$\frac{y^2(-x+y)}{x^3} = \frac{c_3}{x^3}$$

Which simplifies to

$$-y^2(x-y) = c_3$$

Summary
The solution(s) found are the following

$$-y^2(x-y) = c_3 \tag{1}$$

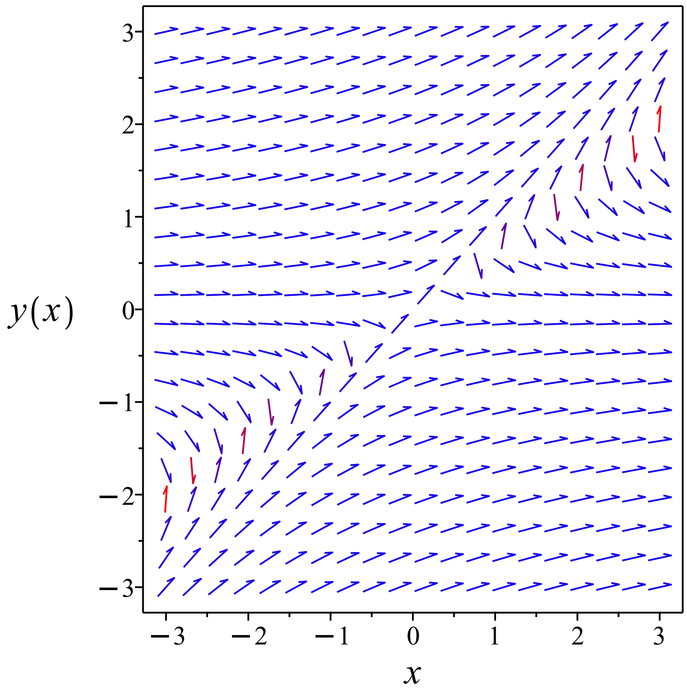


Figure 296: Slope field plot

Verification of solutions

$$-y^2(x-y) = c_3$$

Verified OK.

6.7.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y}{3y-2x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(b_3 - a_2)}{3y - 2x} - \frac{y^2 a_3}{(3y - 2x)^2} - \frac{2y(xa_2 + ya_3 + a_1)}{(3y - 2x)^2} - \left(\frac{1}{3y - 2x} - \frac{3y}{(3y - 2x)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^2b_2 - 12xyb_2 - 3y^2a_2 - 3y^2a_3 + 9y^2b_2 + 3y^2b_3 + 2xb_1 - 2ya_1}{(-3y + 2x)^2} = 0$$

Setting the numerator to zero gives

$$6x^2b_2 - 12xyb_2 - 3y^2a_2 - 3y^2a_3 + 9y^2b_2 + 3y^2b_3 + 2xb_1 - 2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-3a_2v_2^2 - 3a_3v_2^2 + 6b_2v_1^2 - 12b_2v_1v_2 + 9b_2v_2^2 + 3b_3v_2^2 - 2a_1v_2 + 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$6b_2v_1^2 - 12b_2v_1v_2 + 2b_1v_1 + (-3a_2 - 3a_3 + 9b_2 + 3b_3)\,v_2^2 - 2a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0 \\
2b_1 &= 0 \\
-12b_2 &= 0 \\
6b_2 &= 0 \\
-3a_2 - 3a_3 + 9b_2 + 3b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y}{3y - 2x}\right)(x) \\
&= \frac{3yx - 3y^2}{-3y + 2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3yx-3y^2}{-3y+2x}} dy \end{aligned}$$

Which results in

$$S = \frac{2\ln(y)}{3} + \frac{\ln(-x+y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{3y-2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{-3y+3x} \\ S_y &= \frac{-3y+2x}{3y(x-y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

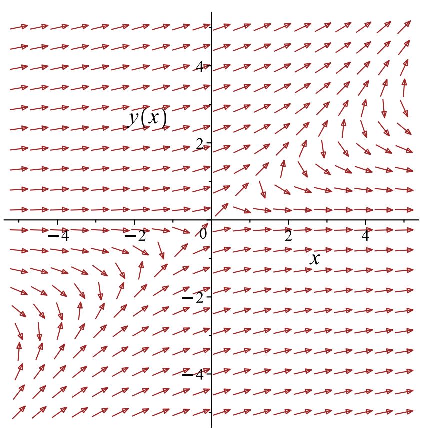
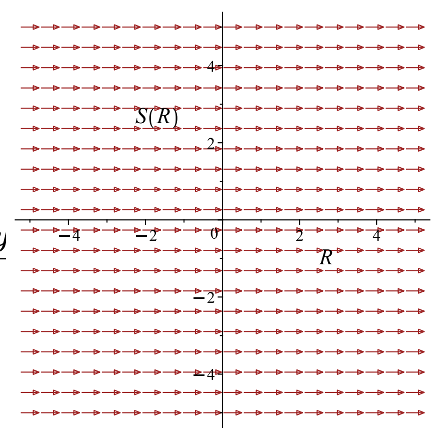
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(y)}{3} + \frac{\ln(-x+y)}{3} = c_1$$

Which simplifies to

$$\frac{2\ln(y)}{3} + \frac{\ln(-x+y)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{3y-2x}$ 	$R = x$ $S = \frac{2\ln(y)}{3} + \frac{\ln(-x+y)}{3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{2\ln(y)}{3} + \frac{\ln(-x+y)}{3} = c_1 \tag{1}$$

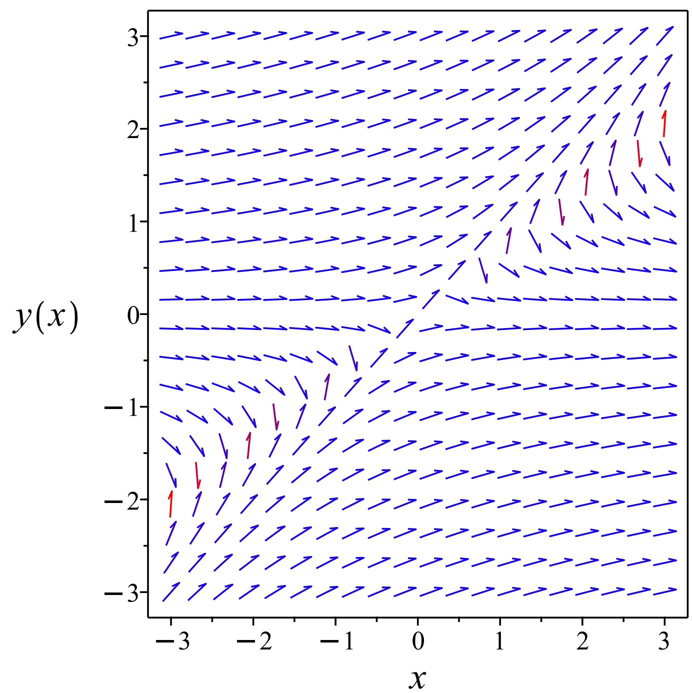


Figure 297: Slope field plot

Verification of solutions

$$\frac{2\ln(y)}{3}+\frac{\ln(-x+y)}{3}=c_1$$

Verified OK.

6.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-3y + 2x)\,\mathrm{d}y = (-y)\,\mathrm{d}x$$
$$(y)\,\mathrm{d}x + (-3y + 2x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y$$
$$N(x, y) = -3y + 2x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y)$$
$$= 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-3y + 2x)$$
$$= 2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-3y+2x}((1)-(2)) \\
&= -\frac{1}{-3y+2x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y}((2)-(1)) \\
&= \frac{1}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{1}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(y)} \\
&= y
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= y(y) \\
&= y^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y(-3y+2x) \\
&= y(-3y+2x)
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(y^2)+(y(-3y+2x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int y^2\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x\,y^2 + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2yx + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y(-3y+2x)$. Therefore equation (4) becomes

$$
y(-3y+2x) = 2yx + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -3y^2
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-3y^2\right)\mathrm{d}y \\
f(y) &= -y^3 + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,y^2 - y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,y^2 - y^3$$

Summary

The solution(s) found are the following

$$xy^2 - y^3 = c_1 \tag{1}$$

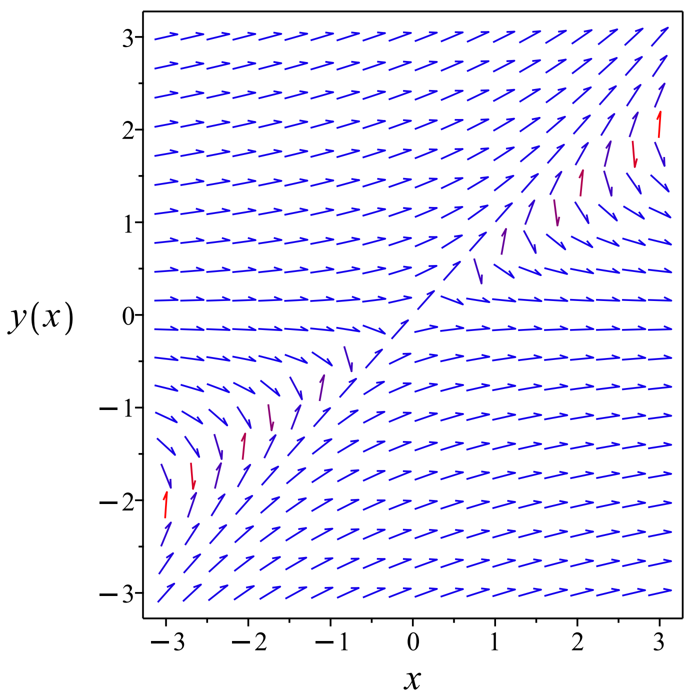


Figure 298: Slope field plot

Verification of solutions

$$xy^2 - y^3 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 309

```
dsolve(y(x)+(2*x-3*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-108c_1 + 8x^3 + 12\sqrt{3}\,\sqrt{c_1\left(-4x^3 + 27c_1\right)}\right)^{\frac{1}{3}}}{6} + \frac{2x^2}{3\left(-108c_1 + 8x^3 + 12\sqrt{3}\,\sqrt{c_1\left(-4x^3 + 27c_1\right)}\right)^{\frac{1}{3}}} + \frac{x}{3}$$

$$y(x) = \frac{\left(-1 - i\sqrt{3}\right)\left(-108c_1 + 8x^3 + 12\sqrt{3}\,\sqrt{-4c_1x^3 + 27c_1^2}\right)^{\frac{1}{3}}}{12} + \frac{\left(i\sqrt{3}\,x - x + \left(-108c_1 + 8x^3 + 12\sqrt{3}\,\sqrt{-4c_1x^3 + 27c_1^2}\right)^{\frac{1}{3}}\right)x}{3\left(-108c_1 + 8x^3 + 12\sqrt{3}\,\sqrt{-4c_1x^3 + 27c_1^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\left(i\sqrt{3} - 1\right)\left(-108c_1 + 8x^3 + 12\sqrt{3}\,\sqrt{-4c_1x^3 + 27c_1^2}\right)^{\frac{1}{3}}}{12} - \frac{\left(i\sqrt{3}\,x + x - \left(-108c_1 + 8x^3 + 12\sqrt{3}\,\sqrt{-4c_1x^3 + 27c_1^2}\right)^{\frac{1}{3}}\right)x}{3\left(-108c_1 + 8x^3 + 12\sqrt{3}\,\sqrt{-4c_1x^3 + 27c_1^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 60.066 (sec). Leaf size: 379

```
DSolve[y[x]+(2*x-3*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(\sqrt[3]{x^3+\frac{3}{2}\sqrt{3}\sqrt{e^{c_1}\left(-4x^3+27e^{c_1}\right)}-\frac{27e^{c_1}}{2}} + \frac{x^2}{\sqrt[3]{x^3+\frac{3}{2}\sqrt{3}\sqrt{e^{c_1}\left(-4x^3+27e^{c_1}\right)}-\frac{27e^{c_1}}{2}}}+x\right)$$

$$y(x) \to \frac{1}{12}\left(i2^{2/3}\left(\sqrt{3}+i\right)\sqrt[3]{2x^3+3\sqrt{3}\sqrt{e^{c_1}\left(-4x^3+27e^{c_1}\right)}-27e^{c_1}} - \frac{2\left(1+i\sqrt{3}\right)x^2}{\sqrt[3]{x^3+\frac{3}{2}\sqrt{3}\sqrt{e^{c_1}\left(-4x^3+27e^{c_1}\right)}-\frac{27e^{c_1}}{2}}}+4x\right)$$

$$y(x) \to \frac{1}{12}\left(-2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{2x^3+3\sqrt{3}\sqrt{e^{c_1}\left(-4x^3+27e^{c_1}\right)}-27e^{c_1}} + \frac{2i\left(\sqrt{3}+i\right)x^2}{\sqrt[3]{x^3+\frac{3}{2}\sqrt{3}\sqrt{e^{c_1}\left(-4x^3+27e^{c_1}\right)}-\frac{27e^{c_1}}{2}}}+4x\right)$$

6.8 problem 8

Internal problem ID [1994]
Internal file name [`OUTPUT/1994_Sunday_February_25_2024_06_44_17_AM_80735483/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x - 2y = 2x^4}$$

6.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = 2x^3$$

Hence the ode is

$$y' - \frac{2y}{x} = 2x^3$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x}dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(2x^3\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(2x^3\right)$$
$$d\left(\frac{y}{x^2}\right) = (2x)\ dx$$

Integrating gives

$$\frac{y}{x^2} = \int 2x\, dx$$
$$\frac{y}{x^2} = x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^4 + c_1 x^2$$

Summary
The solution(s) found are the following

$$y = x^4 + c_1 x^2 \tag{1}$$

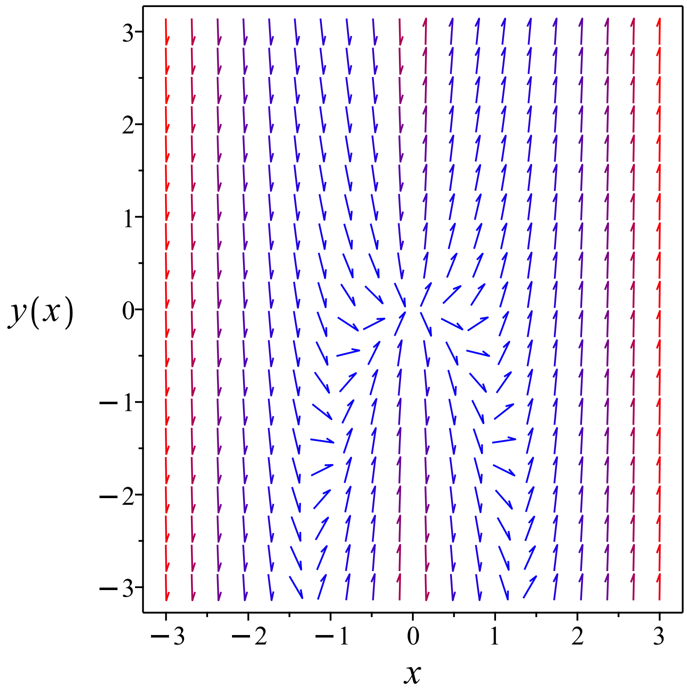


Figure 299: Slope field plot

Verification of solutions

$$y = x^4 + c_1x^2$$

Verified OK.

6.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{2x^4 + 2y}{x} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 164: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x^4 + 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = x^2 + c_1$$

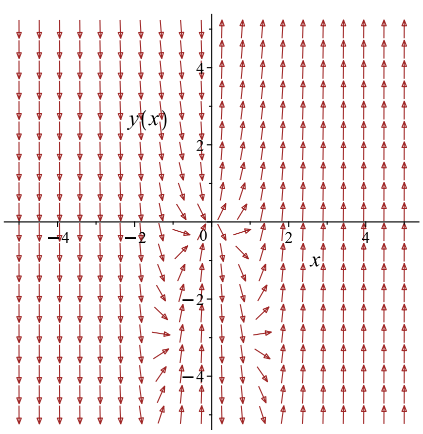
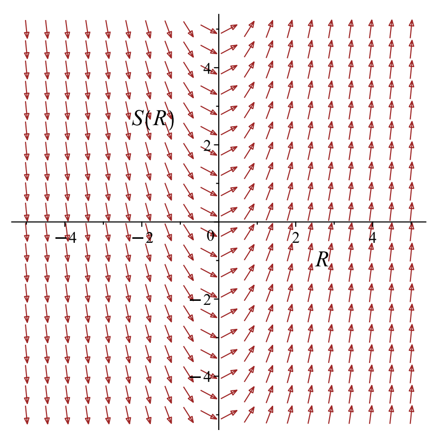
Which simplifies to

$$\frac{y}{x^2} = x^2 + c_1$$

Which gives

$$y = x^2(x^2 + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x^4+2y}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = 2R$ 

Summary

The solution(s) found are the following

$$y = x^2(x^2 + c_1) \tag{1}$$

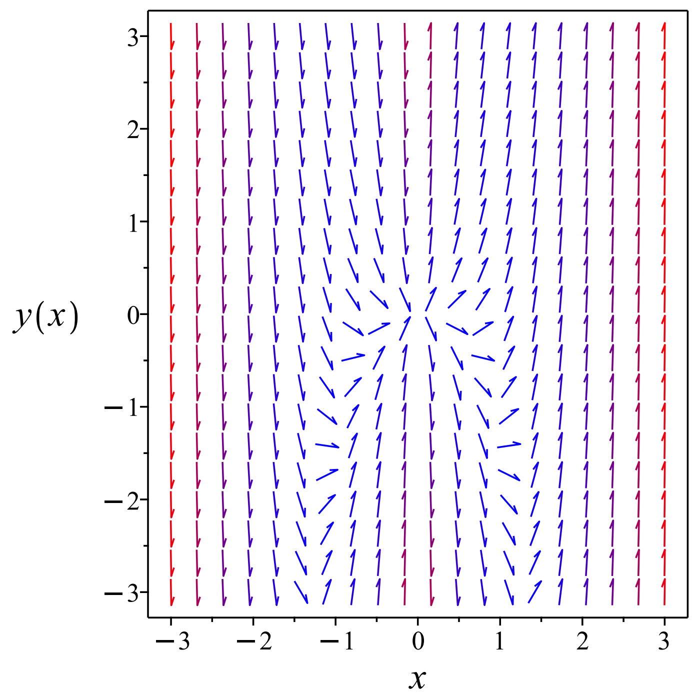


Figure 300: Slope field plot

Verification of solutions

$$y = x^2(x^2 + c_1)$$

Verified OK.

6.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x)\, \mathrm{d}y &= \left(2x^4 + 2y\right) \mathrm{d}x \\
\left(-2x^4 - 2y\right) \mathrm{d}x + (x)\, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -2x^4 - 2y \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2x^4 - 2y\right) \\
&= -2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-2) - (1)) \\ &= -\frac{3}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{3}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-3\ln(x)} \\ &= \frac{1}{x^3} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^3}(-2x^4 - 2y) \\ &= \frac{-2x^4 - 2y}{x^3} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^3}(x) \\ &= \frac{1}{x^2} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-2x^4 - 2y}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{-2x^4 - 2y}{x^3} \,\mathrm{d}x$$

$$\phi = \frac{-x^4 + y}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^4 + y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^4 + y}{x^2}$$

The solution becomes

$$y = x^2(x^2 + c_1)$$

Summary

The solution(s) found are the following

$$y = x^2(x^2 + c_1) \tag{1}$$

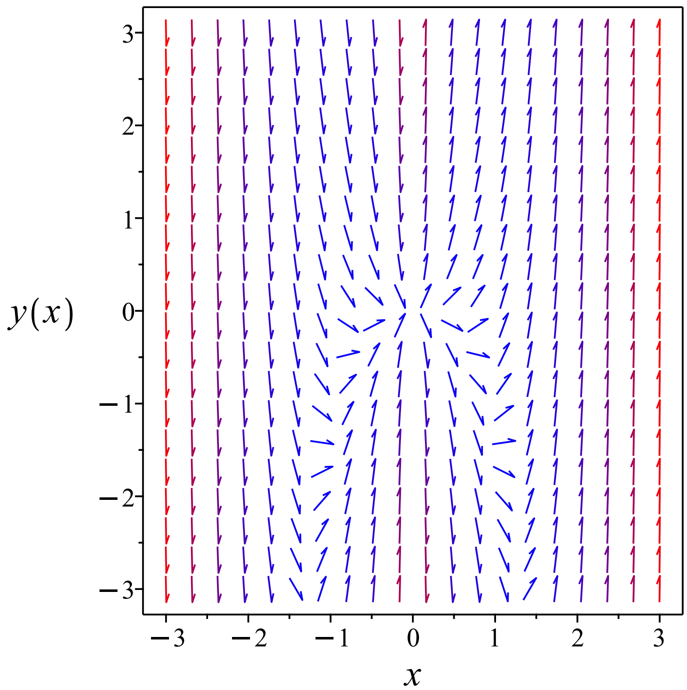


Figure 301: Slope field plot

Verification of solutions

$$y = x^2(x^2 + c_1)$$

Verified OK.

6.8.4 Maple step by step solution

Let's solve

$y'x - 2y = 2x^4$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} + 2x^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = 2x^3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = 2\mu(x)\, x^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 2\mu(x)\, x^3 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 2\mu(x)\, x^3 dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)x^3 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^2}$

  $y = x^2\left(\int 2x dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^2(x^2 + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x*diff(y(x),x)-2*(x^4+y(x))=0,y(x), singsol=all)
```

$$y(x) = (x^2 + c_1)\, x^2$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 15

```
DSolve[x*y'[x]-2*(x^4+y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(x^2 + c_1)$$

6.9 problem 9

Internal problem ID [1995]
Internal file name [`OUTPUT/1995_Sunday_February_25_2024_06_44_17_AM_818388/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_exponential_symmetries]]
```

$$\boxed{-(\mathrm{e}^{y}+x)\, y' = -1}$$

6.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{1}{\mathrm{e}^{y}+x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{b_3 - a_2}{e^y + x} - \frac{a_3}{(e^y + x)^2} + \frac{x a_2 + y a_3 + a_1}{(e^y + x)^2} + \frac{e^y (x b_2 + y b_3 + b_1)}{(e^y + x)^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{e^{2y} b_2 + 3\,e^y x b_2 + e^y y b_3 + x^2 b_2 - a_2 e^y + e^y b_1 + b_3 e^y + b_3 x + y a_3 + a_1 - a_3}{(e^y + x)^2} = 0$$

Setting the numerator to zero gives

$$e^{2y} b_2 + 3\,e^y x b_2 + e^y y b_3 + x^2 b_2 - a_2 e^y + e^y b_1 + b_3 e^y + b_3 x + y a_3 + a_1 - a_3 = 0 \tag{6E}$$

Simplifying the above gives

$$e^{2y} b_2 + 3\,e^y x b_2 + e^y y b_3 + x^2 b_2 - a_2 e^y + e^y b_1 + b_3 e^y + b_3 x + y a_3 + a_1 - a_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, e^y, e^{2y}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, e^y = v_3, e^{2y} = v_4\}$$

The above PDE (6E) now becomes

$$v_1^2 b_2 + 3 v_3 v_1 b_2 + v_3 v_2 b_3 - a_2 v_3 + v_2 a_3 + v_3 b_1 + v_4 b_2 + b_3 v_1 + b_3 v_3 + a_1 - a_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$v_1^2 b_2 + 3v_3 v_1 b_2 + b_3 v_1 + v_3 v_2 b_3 + v_2 a_3 + (-a_2 + b_1 + b_3) v_3 + v_4 b_2 + a_1 - a_3 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_3 &= 0 \\ b_2 &= 0 \\ b_3 &= 0 \\ 3b_2 &= 0 \\ a_1 - a_3 &= 0 \\ -a_2 + b_1 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_1 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 1 - \left(\frac{1}{e^y + x}\right)(x) \\ &= \frac{e^y}{e^y + x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^y}{e^y + x}} dy \end{aligned}$$

Which results in

$$S = \ln(e^y) - x\,e^{-y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{1}{e^y + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-y} \\ S_y &= 1 + x\,e^{-y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y - x\,\mathrm{e}^{-y} = c_1$$

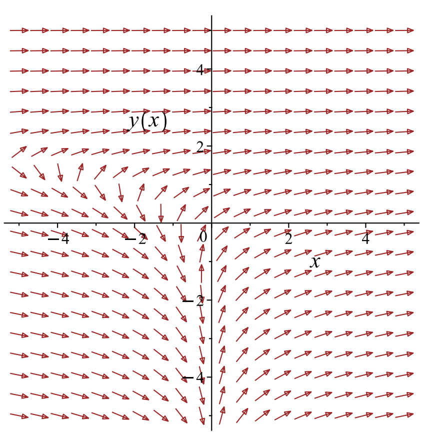
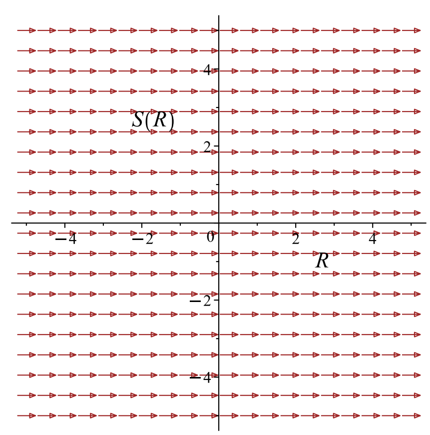
Which simplifies to

$$y - x\,\mathrm{e}^{-y} = c_1$$

Which gives

$$y = \mathrm{LambertW}\left(x\,\mathrm{e}^{-c_1}\right) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1}{\mathrm{e}^{y}+x}$ 	$R = x$ $S = y - x\,\mathrm{e}^{-y}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = \operatorname{LambertW}\left(x\,\mathrm{e}^{-c_1}\right) + c_1 \tag{1}$$

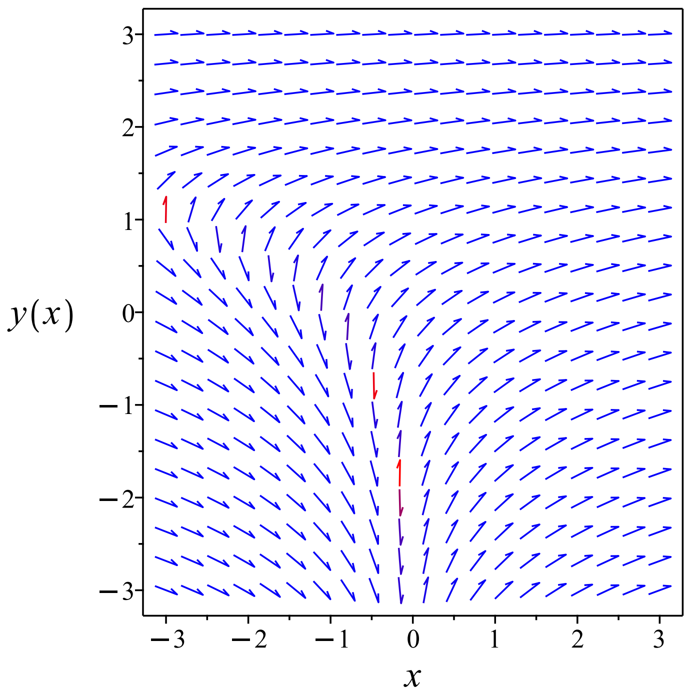


Figure 302: Slope field plot

Verification of solutions

$$y = \operatorname{LambertW}\left(x\,\mathrm{e}^{-c_1}\right) + c_1$$

Verified OK.

6.9.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-\mathrm{e}^y-x)\,\mathrm{d}y&=(-1)\,\mathrm{d}x\\ (1)\,\mathrm{d}x+(-\mathrm{e}^y-x)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=1\\ N(x,y)&=-\mathrm{e}^y-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(1)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\mathrm{e}^{y} - x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{\mathrm{e}^{y} + x}((0) - (-1)) \\
&= -\frac{1}{\mathrm{e}^{y} + x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -1 \,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-y} \\
&= \mathrm{e}^{-y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-y}(1) \\
&= \mathrm{e}^{-y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-y}(-\mathrm{e}^{y} - x) \\
&= -1 - x\,\mathrm{e}^{-y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\mathrm{e}^{-y}\right) + \left(-1 - x\,\mathrm{e}^{-y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{-y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x\,\mathrm{e}^{-y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -x\,\mathrm{e}^{-y} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -1 - x\,\mathrm{e}^{-y}$. Therefore equation (4) becomes

$$
-1 - x\,\mathrm{e}^{-y} = -x\,\mathrm{e}^{-y} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -1
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-1)\,\mathrm{d}y$$
$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,\mathrm{e}^{-y} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,\mathrm{e}^{-y} - y$$

The solution becomes

$$y = \mathrm{LambertW}\left(x\,\mathrm{e}^{c_1}\right) - c_1$$

Summary

The solution(s) found are the following

$$y = \mathrm{LambertW}\left(x\,\mathrm{e}^{c_1}\right) - c_1 \tag{1}$$

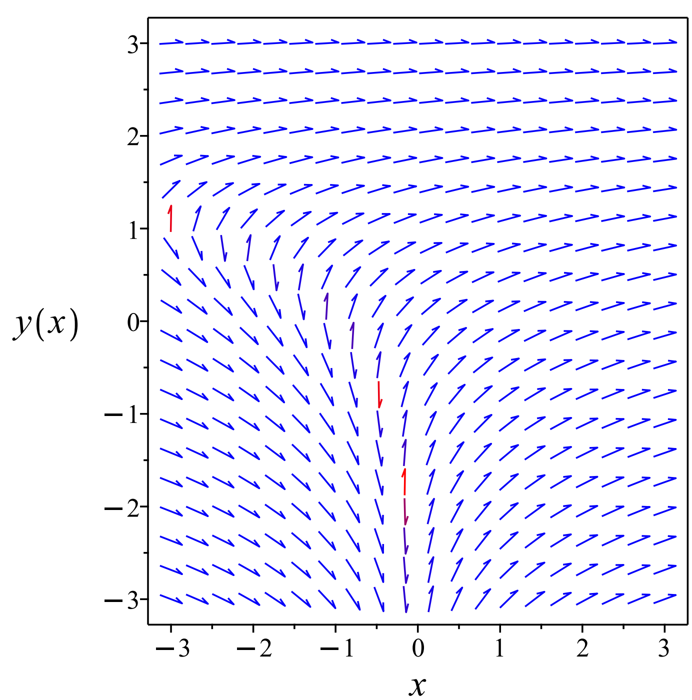


Figure 303: Slope field plot

Verification of solutions

$$y = \text{LambertW}\left(x\,\mathrm{e}^{c_1}\right) - c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve(1=(x+exp(y(x)))*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \text{LambertW}\left(x\,\mathrm{e}^{c_1}\right) - c_1$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 17

```
DSolve[1==(x+Exp[y[x]])*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to W(e^{c_1}x) - c_1$$

6.10 problem 10

Internal problem ID [1996]
Internal file name [`OUTPUT/1996_Sunday_February_25_2024_06_44_18_AM_13999829/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y^2x' + (y^2 + 2y)\, x = 1}$$

6.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(y)x = q(y)$$

Where here

$$p(y) = -\frac{-2-y}{y}$$
$$q(y) = \frac{1}{y^2}$$

Hence the ode is

$$x' - \frac{(-2-y)\, x}{y} = \frac{1}{y^2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{-2-y}{y}dy} \\
&= e^{y+2\ln(y)}
\end{aligned}$$

Which simplifies to

$$\mu = y^2 e^y$$

The ode becomes

$$\begin{aligned}
\frac{d}{dy}(\mu x) &= (\mu)\left(\frac{1}{y^2}\right) \\
\frac{d}{dy}\left(e^y x\, y^2\right) &= \left(y^2 e^y\right)\left(\frac{1}{y^2}\right) \\
d\left(e^y x\, y^2\right) &= e^y\, dy
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^y x\, y^2 &= \int e^y\, dy \\
e^y x\, y^2 &= e^y + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = y^2 e^y$ results in

$$x = \frac{e^{-y} e^y}{y^2} + \frac{c_1 e^{-y}}{y^2}$$

which simplifies to

$$x = \frac{c_1 e^{-y} + 1}{y^2}$$

Summary
The solution(s) found are the following

$$x = \frac{c_1 e^{-y} + 1}{y^2} \tag{1}$$

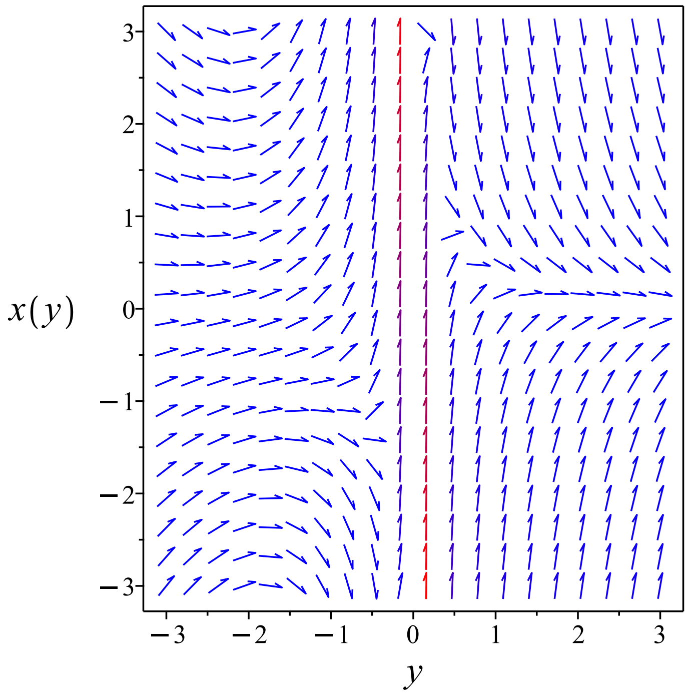


Figure 304: Slope field plot

Verification of solutions

$$x = \frac{c_1 \mathrm{e}^{-y} + 1}{y^2}$$

Verified OK.

6.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -\frac{x\,y^2 + 2yx - 1}{y^2}$$
$$x' = \omega(y, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_x - \xi_y) - \omega^2 \xi_x - \omega_y \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 167: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(y,x) &= 0 \\ \eta(y,x) &= \mathrm{e}^{-y-2\ln(y)}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dy}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial x}\right) S(y,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-y-2\ln(y)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{y} x\, y^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y, x) S_x}{R_y + \omega(y, x) R_x} \tag{2}$$

Where in the above R_y, R_x, S_y, S_x are all partial derivatives and $\omega(y, x)$ is the right hand side of the original ode given by

$$\omega(y, x) = -\frac{x\, y^2 + 2yx - 1}{y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_y &= 1 \\ R_x &= 0 \\ S_y &= \mathrm{e}^{y} xy(2 + y) \\ S_x &= y^2 \mathrm{e}^{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, x coordinates. This results in

$$e^{y} x y^{2} = e^{y} + c_1$$

Which simplifies to

$$e^{y} x y^{2} = e^{y} + c_1$$

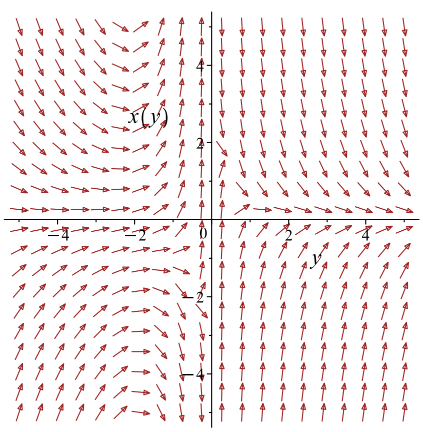
Which gives

$$x = \frac{(e^{y} + c_1)\, e^{-y}}{y^{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in y, x coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dx}{dy} = -\frac{x\,y^{2}+2yx-1}{y^{2}}$



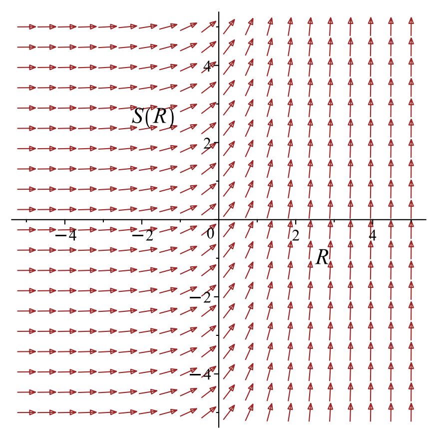
</td><td>

$R = y$

$S = e^{y} x\, y^{2}$

</td><td>

$\frac{dS}{dR} = e^{R}$



</td></tr>
</table>

Summary
The solution(s) found are the following

$$x = \frac{(e^{y} + c_1)\, e^{-y}}{y^{2}} \tag{1}$$

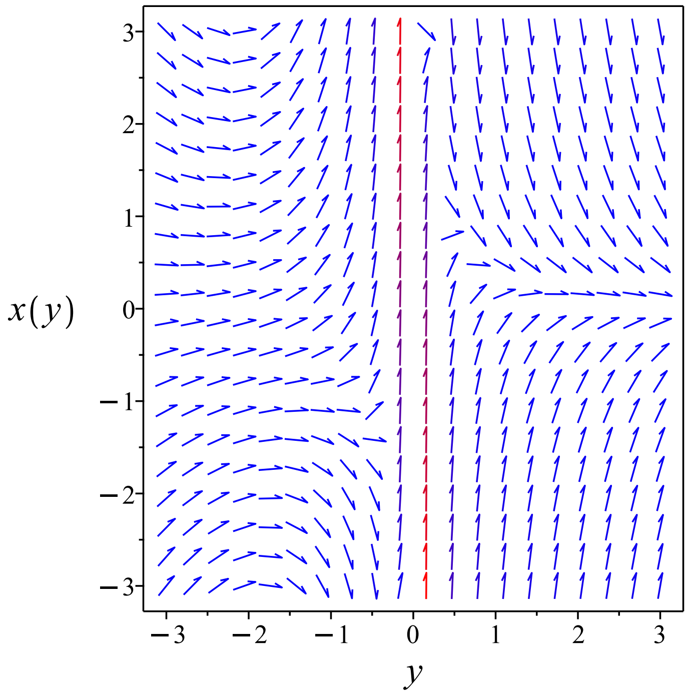


Figure 305: Slope field plot

Verification of solutions

$$x = \frac{(e^y + c_1)\, e^{-y}}{y^2}$$

Verified OK.

6.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y, x)\,\mathrm{d}y + N(y, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^2)\,\mathrm{d}x &= \left(-\left(y^2 + 2y\right)x + 1\right)\mathrm{d}y \\ \left(-1 + \left(y^2 + 2y\right)x\right)\mathrm{d}y + (y^2)\,\mathrm{d}x &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(y, x) &= -1 + \left(y^2 + 2y\right)x \\ N(y, x) &= y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-1 + \left(y^2 + 2y\right)x\right) \\ &= y^2 + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\right) \\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) \\
&= \frac{1}{y^2}\left(\left(y^2 + 2y\right) - (2y)\right) \\
&= 1
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}y} \\
&= e^{\int 1 \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{y} \\
&= \mathrm{e}^{y}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{y}\left(-1 + \left(y^2 + 2y\right) x\right) \\
&= \left(-1 + \left(y^2 + 2y\right) x\right) \mathrm{e}^{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{y}\left(y^2\right) \\
&= y^2 \mathrm{e}^{y}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}y} &= 0 \\
\left(\left(-1 + \left(y^2 + 2y\right) x\right) \mathrm{e}^{y}\right) + \left(y^2 \mathrm{e}^{y}\right) \frac{\mathrm{d}x}{\mathrm{d}y} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(y, x)$

$$\frac{\partial \phi}{\partial y} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \mathrm{d}y = \int \overline{M} \,\mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y} \mathrm{d}y = \int \left(-1 + \left(y^2 + 2y\right) x\right) \mathrm{e}^y \,\mathrm{d}y$$

$$\phi = \mathrm{e}^y \left(x\, y^2 - 1\right) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both y and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = y^2 \mathrm{e}^y + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = y^2 \mathrm{e}^y$. Therefore equation (4) becomes

$$y^2 \mathrm{e}^y = y^2 \mathrm{e}^y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^y \left(x\, y^2 - 1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^y \left(x\, y^2 - 1\right)$$

The solution becomes

$$x = \frac{(e^{y} + c_1) e^{-y}}{y^2}$$

Summary

The solution(s) found are the following

$$x = \frac{(e^{y} + c_1) e^{-y}}{y^2} \tag{1}$$

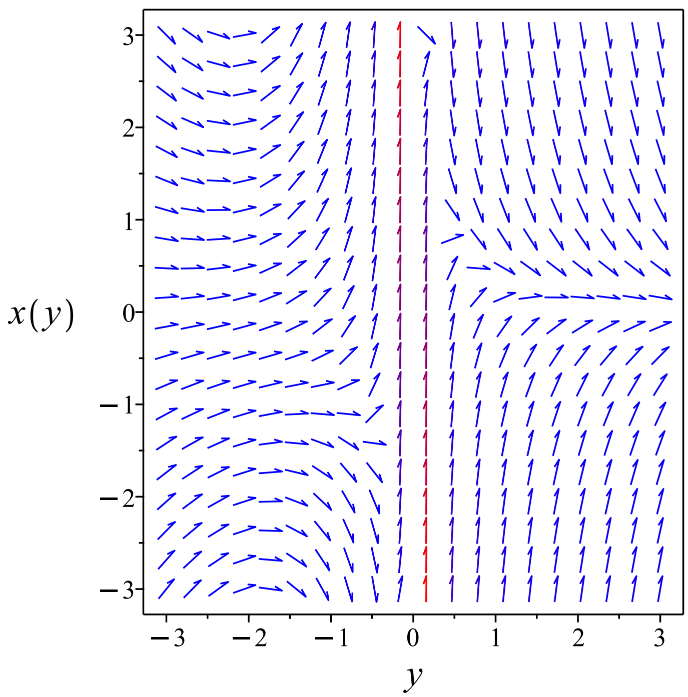


Figure 306: Slope field plot

Verification of solutions

$$x = \frac{(e^{y} + c_1) e^{-y}}{y^2}$$

Verified OK.

6.10.4 Maple step by step solution

Let's solve

$y^2x' + (y^2+2y)\,x = 1$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -\frac{(2+y)x}{y} + \frac{1}{y^2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{(2+y)x}{y} = \frac{1}{y^2}$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

  $\mu(y)\left(x' + \frac{(2+y)x}{y}\right) = \frac{\mu(y)}{y^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\,x)$

  $\mu(y)\left(x' + \frac{(2+y)x}{y}\right) = \mu'(y)\,x + \mu(y)\,x'$

- Isolate $\mu'(y)$

  $\mu'(y) = \frac{\mu(y)(2+y)}{y}$

- Solve to find the integrating factor

  $\mu(y) = y^2\mathrm{e}^y$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}(\mu(y)\,x)\right) dy = \int \frac{\mu(y)}{y^2} dy + c_1$

- Evaluate the integral on the lhs

  $\mu(y)\,x = \int \frac{\mu(y)}{y^2} dy + c_1$

- Solve for x

  $x = \frac{\int \frac{\mu(y)}{y^2} dy + c_1}{\mu(y)}$

- Substitute $\mu(y) = y^2\mathrm{e}^y$

  $x = \frac{\int \mathrm{e}^y dy + c_1}{y^2\mathrm{e}^y}$

- Evaluate the integrals on the rhs

  $x = \frac{\mathrm{e}^y + c_1}{y^2\mathrm{e}^y}$

- Simplify

  $x = \frac{c_1 e^{-y}+1}{y^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(y^2*diff(x(y),y)+(y^2+2*y)*x(y)=1,x(y), singsol=all)
```

$$x(y) = \frac{e^{-y} c_1 + 1}{y^2}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 19

```
DSolve[y^2*x'[y]+(y^2+2*y)*x[y]==1,x[y],y,IncludeSingularSolutions -> True]
```

$$x(y) \to \frac{1 + c_1 e^{-y}}{y^2}$$

6.11 problem 11

Internal problem ID [1997]
Internal file name [`OUTPUT/1997_Sunday_February_25_2024_06_44_18_AM_9875408/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x - 5y = x + 1}$$

6.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{5}{x}$$
$$q(x) = \frac{x+1}{x}$$

Hence the ode is

$$y' - \frac{5y}{x} = \frac{x+1}{x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{5}{x}dx}$$
$$= \frac{1}{x^5}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x+1}{x}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^5}\right) = \left(\frac{1}{x^5}\right)\left(\frac{x+1}{x}\right)$$
$$d\left(\frac{y}{x^5}\right) = \left(\frac{x+1}{x^6}\right)dx$$

Integrating gives

$$\frac{y}{x^5} = \int \frac{x+1}{x^6}\,dx$$
$$\frac{y}{x^5} = -\frac{1}{5x^5} - \frac{1}{4x^4} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^5}$ results in

$$y = x^5\left(-\frac{1}{5x^5} - \frac{1}{4x^4}\right) + c_1 x^5$$

which simplifies to

$$y = -\frac{1}{5} - \frac{1}{4}x + c_1 x^5$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{5} - \frac{1}{4}x + c_1 x^5 \tag{1}$$

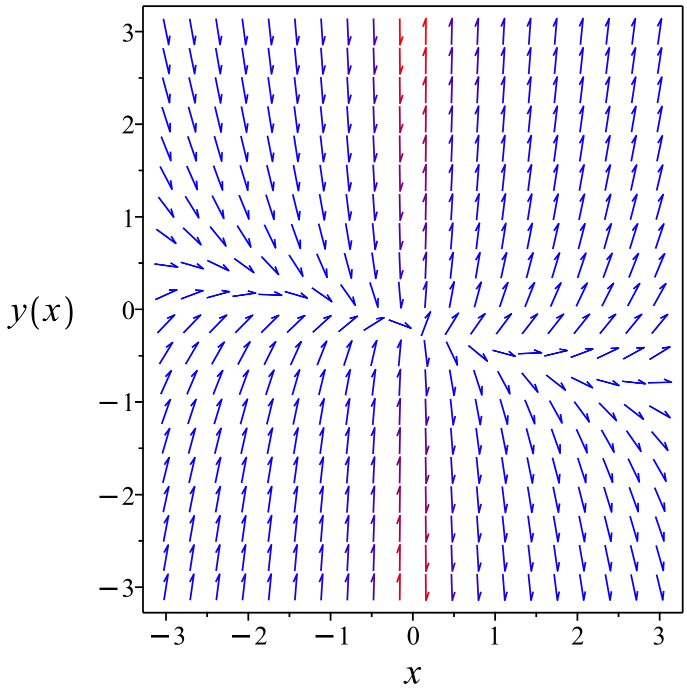


Figure 307: Slope field plot

Verification of solutions

$$y = -\frac{1}{5} - \frac{1}{4}x + c_1 x^5$$

Verified OK.

6.11.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{5Y(X) + 5y_0 + X + x_0 + 1}{X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = -\frac{1}{5}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{5Y(X) + X}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{5Y+X}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = 5Y + X$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= 5u + 1 \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{4u(X)+1}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{4u(X)+1}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X - 4u(X) - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= \frac{4u+1}{X} \end{aligned}$$

Where $f(X) = \frac{1}{X}$ and $g(u) = 4u + 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{4u+1}\,du &= \frac{1}{X}\,dX \\ \int \frac{1}{4u+1}\,du &= \int \frac{1}{X}\,dX \\ \frac{\ln(4u+1)}{4} &= \ln(X) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$(4u+1)^{\frac{1}{4}} = \mathrm{e}^{\ln(X)+c_2}$$

Which simplifies to

$$(4u+1)^{\frac{1}{4}} = c_3 X$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X\left(\frac{c_3^4 \mathrm{e}^{4c_2} X^4}{4} - \frac{1}{4}\right)$$

Using the solution for $Y(X)$

$$Y(X) = X\left(\frac{c_3^4 \mathrm{e}^{4c_2} X^4}{4} - \frac{1}{4}\right)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - \frac{1}{5}$$
$$X = x$$

Then the solution in y becomes

$$y + \frac{1}{5} = x\left(\frac{\mathrm{e}^{4c_2} c_3^4 x^4}{4} - \frac{1}{4}\right)$$

Summary
The solution(s) found are the following

$$y + \frac{1}{5} = x\left(\frac{\mathrm{e}^{4c_2} c_3^4 x^4}{4} - \frac{1}{4}\right) \tag{1}$$

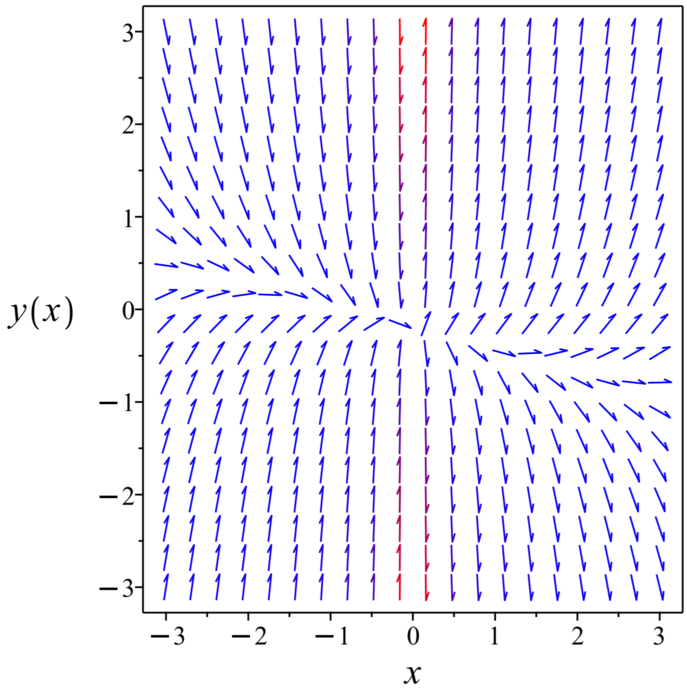


Figure 308: Slope field plot

Verification of solutions

$$y + \frac{1}{5} = x\left(\frac{e^{4c_2} c_3^4 x^4}{4} - \frac{1}{4}\right)$$

Verified OK.

6.11.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{5y + x + 1}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 170: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^5
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^5} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{5y + x + 1}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{5y}{x^6} \\ S_y &= \frac{1}{x^5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x+1}{x^6} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R+1}{R^6}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{5R^5} - \frac{1}{4R^4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^5} = -\frac{1}{5x^5} - \frac{1}{4x^4} + c_1$$

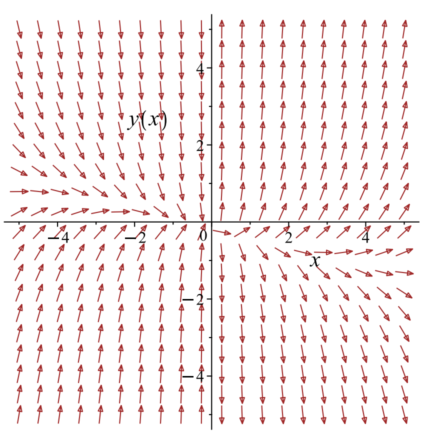
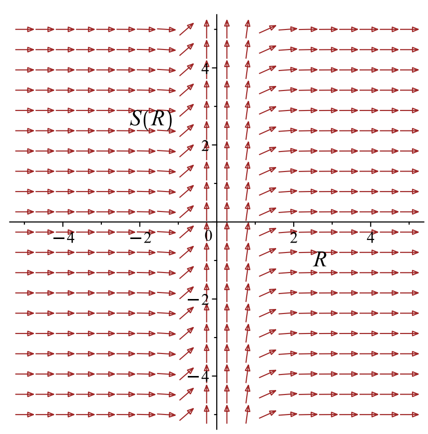
Which simplifies to

$$\frac{y}{x^5} = -\frac{1}{5x^5} - \frac{1}{4x^4} + c_1$$

Which gives

$$y = -\frac{1}{5} - \frac{1}{4}x + c_1 x^5$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{5y+x+1}{x}$ 	$R = x$ $S = \frac{y}{x^5}$	$\frac{dS}{dR} = \frac{R+1}{R^6}$ 

Summary
The solution(s) found are the following

$$y = -\frac{1}{5} - \frac{1}{4}x + c_1x^5 \tag{1}$$

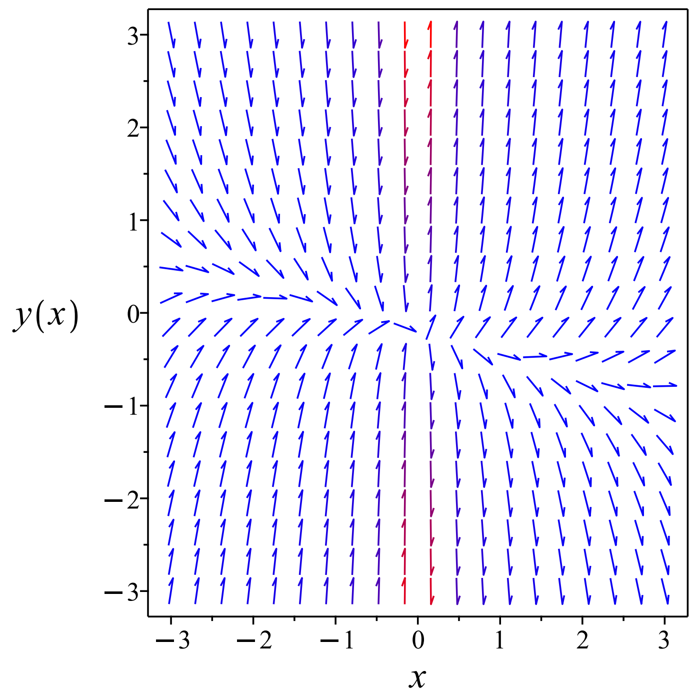


Figure 309: Slope field plot

Verification of solutions

$$y = -\frac{1}{5} - \frac{1}{4}x + c_1x^5$$

Verified OK.

6.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^{2} \phi}{\partial x \partial y}=\frac{\partial^{2} \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^{2} \phi}{\partial x \partial y}=\frac{\partial^{2} \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x) \mathrm{d} y &= (5 y+x+1) \mathrm{d} x \\
(-5 y-x-1) \mathrm{d} x+(x) \mathrm{d} y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -5 y-x-1 \\
N(x, y) &= x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-5 y-x-1) \\
&= -5
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-5) - (1))\\ &= -\frac{6}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{6}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-6\ln(x)}\\ &= \frac{1}{x^6}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^6}(-5y - x - 1)\\ &= \frac{-5y - x - 1}{x^6}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^6}(x)\\ &= \frac{1}{x^5}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-5y - x - 1}{x^6}\right) + \left(\frac{1}{x^5}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-5y - x - 1}{x^6} \mathrm{d}x$$
$$\phi = \frac{20y + 4 + 5x}{20x^5} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^5} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^5}$. Therefore equation (4) becomes

$$\frac{1}{x^5} = \frac{1}{x^5} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{20y + 4 + 5x}{20x^5} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{20y + 4 + 5x}{20x^5}$$

The solution becomes

$$y = -\frac{1}{5} - \frac{1}{4}x + c_1 x^5$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{5} - \frac{1}{4}x + c_1 x^5 \tag{1}$$

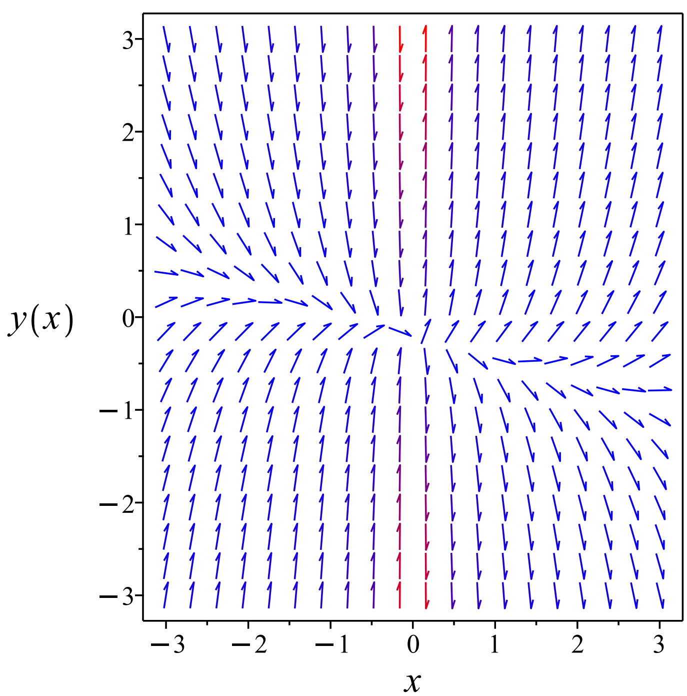


Figure 310: Slope field plot

Verification of solutions

$$y = -\frac{1}{5} - \frac{1}{4}x + c_1 x^5$$

Verified OK.

6.11.5 Maple step by step solution

Let's solve

$y'x - 5y = x + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{5y}{x} + \frac{x+1}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{5y}{x} = \frac{x+1}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{5y}{x}\right) = \frac{\mu(x)(x+1)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{5y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{5\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^5}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)(x+1)}{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)(x+1)}{x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)(x+1)}{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^5}$

$$y = x^5\left(\int \frac{x+1}{x^6}dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = x^5\left(-\frac{1}{5x^5} - \frac{1}{4x^4} + c_1\right)$$

- Simplify

$$y = -\frac{1}{5} - \frac{1}{4}x + c_1x^5$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x*diff(y(x),x)=5*y(x)+x+1,y(x), singsol=all)
```

$$y(x) = -\frac{1}{5} - \frac{1}{4}x + c_1x^5$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 20

```
DSolve[x*y'[x]==5*y[x]+x+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1x^5 - \frac{x}{4} - \frac{1}{5}$$

6.12 problem 12

Internal problem ID [1998]
Internal file name [`OUTPUT/1998_Sunday_February_25_2024_06_44_19_AM_95188671/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x^2 + y - 2yx = 2x^2}$$

6.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2x-1}{x^2}$$
$$q(x) = 2$$

Hence the ode is

$$y' - \frac{(2x-1)\,y}{x^2} = 2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2x-1}{x^2}dx} \\
&= e^{-2\ln(x)-\frac{1}{x}}
\end{aligned}
$$

Which simplifies to

$$\mu = \frac{e^{-\frac{1}{x}}}{x^2}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(2) \\
\frac{d}{dx}\left(\frac{e^{-\frac{1}{x}}y}{x^2}\right) &= \left(\frac{e^{-\frac{1}{x}}}{x^2}\right)(2) \\
d\left(\frac{e^{-\frac{1}{x}}y}{x^2}\right) &= \left(\frac{2\,e^{-\frac{1}{x}}}{x^2}\right)dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{e^{-\frac{1}{x}}y}{x^2} &= \int \frac{2\,e^{-\frac{1}{x}}}{x^2}\,dx \\
\frac{e^{-\frac{1}{x}}y}{x^2} &= 2\,e^{-\frac{1}{x}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{e^{-\frac{1}{x}}}{x^2}$ results in

$$y = 2x^2 e^{\frac{1}{x}} e^{-\frac{1}{x}} + c_1 x^2 e^{\frac{1}{x}}$$

which simplifies to

$$y = x^2\left(2 + c_1 e^{\frac{1}{x}}\right)$$

Summary

The solution(s) found are the following

$$y = x^2\left(2 + c_1 e^{\frac{1}{x}}\right) \tag{1}$$

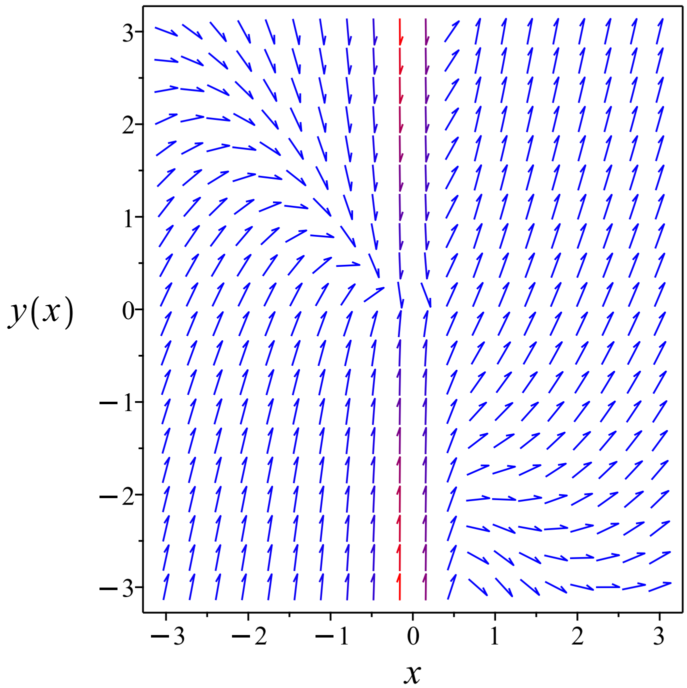


Figure 311: Slope field plot

Verification of solutions

$$y = x^2\left(2 + c_1 e^{\frac{1}{x}}\right)$$

Verified OK.

6.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x^2 + 2yx - y}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 173: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{2\ln(x)+\frac{1}{x}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{2\ln(x)+\frac{1}{x}}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{-\frac{1}{x}} y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2x^2 + 2yx - y}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y(1 - 2x) e^{-\frac{1}{x}}}{x^4} \\ S_y &= \frac{e^{-\frac{1}{x}}}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2 e^{-\frac{1}{x}}}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2 e^{-\frac{1}{R}}}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2\,\mathrm{e}^{-\frac{1}{R}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\mathrm{e}^{-\frac{1}{x}} y}{x^2} = 2\,\mathrm{e}^{-\frac{1}{x}} + c_1$$

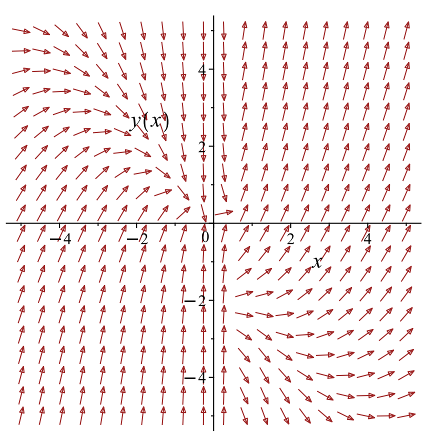
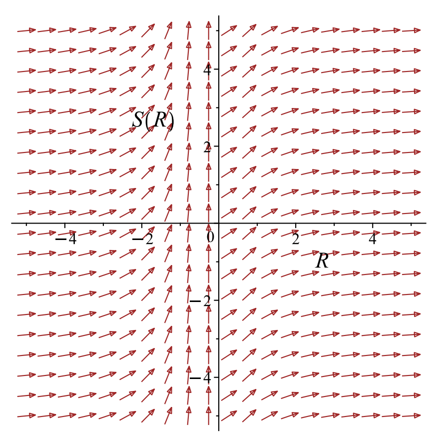
Which simplifies to

$$\frac{\mathrm{e}^{-\frac{1}{x}} y}{x^2} = 2\,\mathrm{e}^{-\frac{1}{x}} + c_1$$

Which gives

$$y = x^2 \left(2\,\mathrm{e}^{-\frac{1}{x}} + c_1\right) \mathrm{e}^{\frac{1}{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x^2+2yx-y}{x^2}$ 	$R = x$ $S = \frac{\mathrm{e}^{-\frac{1}{x}} y}{x^2}$	$\frac{dS}{dR} = \frac{2\,\mathrm{e}^{-\frac{1}{R}}}{R^2}$ 

Summary
The solution(s) found are the following

$$y = x^2 \left(2\,\mathrm{e}^{-\frac{1}{x}} + c_1\right) \mathrm{e}^{\frac{1}{x}} \tag{1}$$

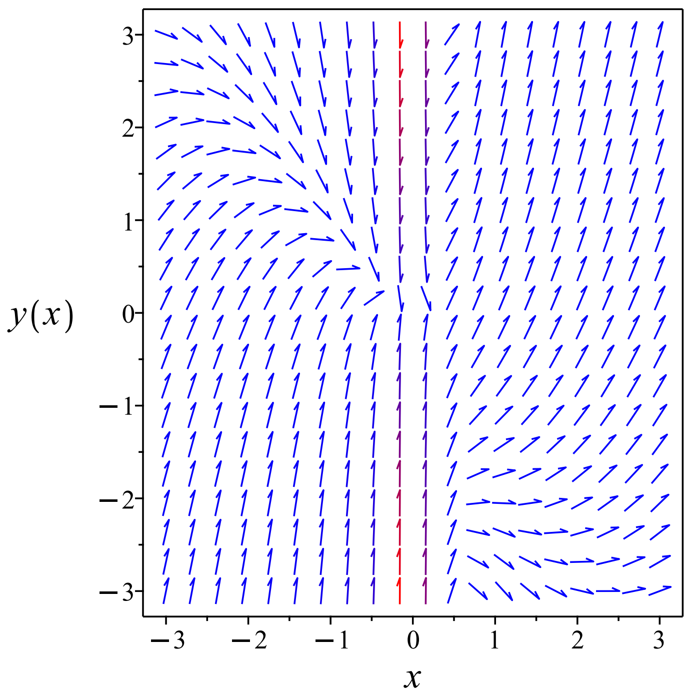


Figure 312: Slope field plot

Verification of solutions

$$y = x^2 \left(2\,\mathrm{e}^{-\frac{1}{x}} + c_1\right) \mathrm{e}^{\frac{1}{x}}$$

Verified OK.

6.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(x^2\right) \mathrm{d} y &= \left(2 x^2+2 y x-y\right) \mathrm{d} x \\
\left(-2 x^2-2 y x+y\right) \mathrm{d} x+\left(x^2\right) \mathrm{d} y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -2 x^2-2 y x+y \\
N(x, y) &= x^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2 x^2-2 y x+y\right) \\
&= 1-2 x
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2}((1-2x)-(2x))\\ &= \frac{-4x+1}{x^2}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{-4x+1}{x^2}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)-\frac{1}{x}}\\ &= \frac{\mathrm{e}^{-\frac{1}{x}}}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{\mathrm{e}^{-\frac{1}{x}}}{x^4}(-2x^2-2yx+y)\\ &= -\frac{2\left(x^2+yx-\frac{1}{2}y\right)\mathrm{e}^{-\frac{1}{x}}}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{\mathrm{e}^{-\frac{1}{x}}}{x^4}(x^2)\\ &= \frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-\frac{2\left(x^2 + yx - \frac{1}{2}y\right) \mathrm{e}^{-\frac{1}{x}}}{x^4}\right) + \left(\frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}\right) \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{2\left(x^2 + yx - \frac{1}{2}y\right) \mathrm{e}^{-\frac{1}{x}}}{x^4} \mathrm{d}x$$
$$\phi = \frac{(-2x^2 + y) \mathrm{e}^{-\frac{1}{x}}}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{-\frac{1}{x}}}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{-\frac{1}{x}}}{x^2}$. Therefore equation (4) becomes

$$\frac{\mathrm{e}^{-\frac{1}{x}}}{x^2} = \frac{\mathrm{e}^{-\frac{1}{x}}}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(-2x^2 + y)\,e^{-\frac{1}{x}}}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(-2x^2 + y)\,e^{-\frac{1}{x}}}{x^2}$$

The solution becomes

$$y = x^2\left(2\,e^{-\frac{1}{x}} + c_1\right)e^{\frac{1}{x}}$$

Summary

The solution(s) found are the following

$$y = x^2\left(2\,e^{-\frac{1}{x}} + c_1\right)e^{\frac{1}{x}} \tag{1}$$

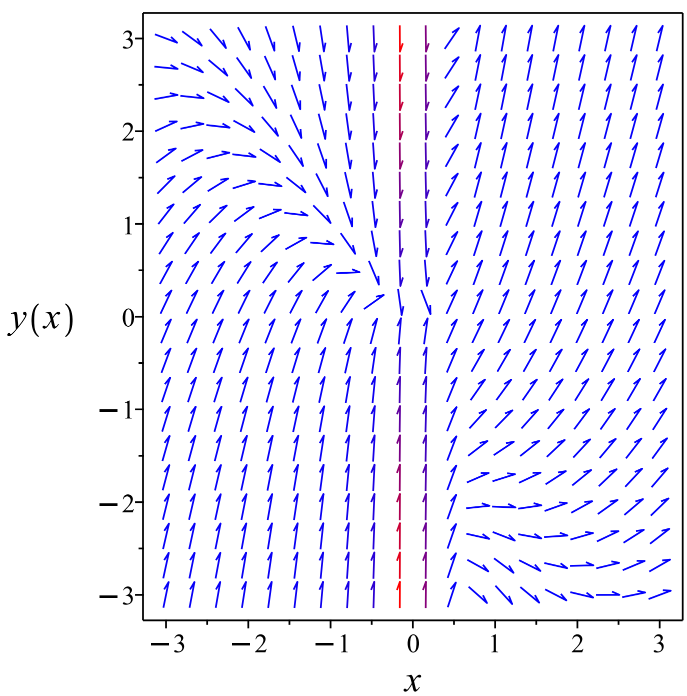


Figure 313: Slope field plot

Verification of solutions

$$y = x^2 \left(2\,e^{-\frac{1}{x}} + c_1\right) e^{\frac{1}{x}}$$

Verified OK.

6.12.4 Maple step by step solution

Let's solve

$y'x^2 + y - 2yx = 2x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2 + \frac{(2x-1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{(2x-1)y}{x^2} = 2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{(2x-1)y}{x^2}\right) = 2\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{(2x-1)y}{x^2}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)(2x-1)}{x^2}$

- Solve to find the integrating factor

  $\mu(x) = \frac{e^{-\frac{1}{x}}}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{e^{-\frac{1}{x}}}{x^2}$

$$y = \frac{x^2 \left(\int \frac{2\,\mathrm{e}^{-\frac{1}{x}}}{x^2} dx + c_1\right)}{\mathrm{e}^{-\frac{1}{x}}}$$

- Evaluate the integrals on the rhs

$$y = \frac{x^2 \left(2\,\mathrm{e}^{-\frac{1}{x}} + c_1\right)}{\mathrm{e}^{-\frac{1}{x}}}$$

- Simplify

$$y = x^2 \left(2 + c_1 \mathrm{e}^{\frac{1}{x}}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x^2*diff(y(x),x)+(y(x)-2*x*y(x)-2*x^2)=0,y(x), singsol=all)
```

$$y(x) = x^2 \left(2 + c_1 \mathrm{e}^{\frac{1}{x}}\right)$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 19

```
DSolve[x^2*y'[x]+(y[x]-2*x*y[x]-2*x^2)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 \left(2 + c_1 e^{\frac{1}{x}}\right)$$

6.13 problem 13

Internal problem ID [1999]
Internal file name [`OUTPUT/1999_Sunday_February_25_2024_06_44_21_AM_66280716/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x+1)\,y'+2y=\frac{e^{x}}{x+1}}$$

6.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{2}{x+1}$$
$$q(x)=\frac{e^{x}}{(x+1)^{2}}$$

Hence the ode is

$$y'+\frac{2y}{x+1}=\frac{e^{x}}{(x+1)^{2}}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{2}{x+1}dx} \\
&= (x+1)^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{e^x}{(x+1)^2}\right) \\
\frac{d}{dx}\left((x+1)^2 y\right) &= \left((x+1)^2\right)\left(\frac{e^x}{(x+1)^2}\right) \\
d\left((x+1)^2 y\right) &= e^x\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(x+1)^2 y &= \int e^x\, dx \\
(x+1)^2 y &= e^x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = (x+1)^2$ results in

$$
y = \frac{e^x}{(x+1)^2} + \frac{c_1}{(x+1)^2}
$$

which simplifies to

$$
y = \frac{e^x + c_1}{(x+1)^2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{e^x + c_1}{(x+1)^2} \tag{1}
$$

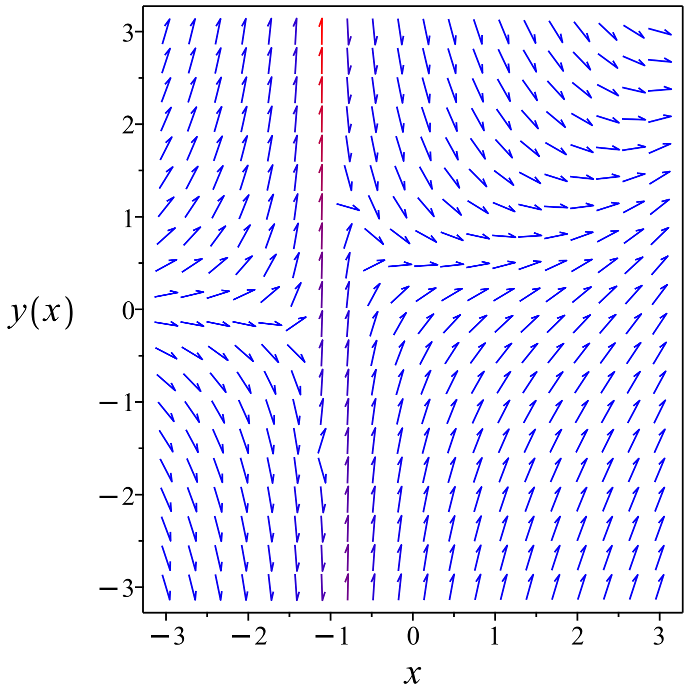


Figure 314: Slope field plot

Verification of solutions

$$y = \frac{e^{x} + c_1}{(x+1)^2}$$

Verified OK.

6.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-2yx + e^{x} - 2y}{(x+1)^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 176: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{(x+1)^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x+1)^2}} dy \end{aligned}$$

Which results in

$$S = (x+1)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-2yx + \mathrm{e}^x - 2y}{(x+1)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2(x+1)\, y \\ S_y &= (x+1)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x+1)^2 y = e^{x} + c_1$$

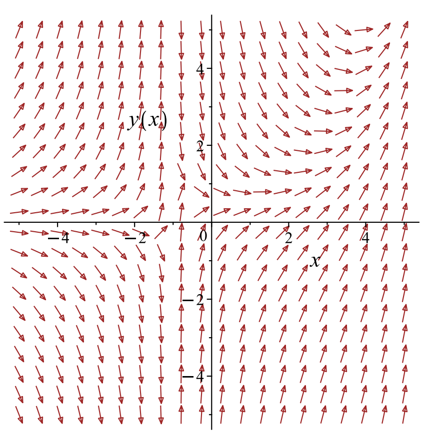
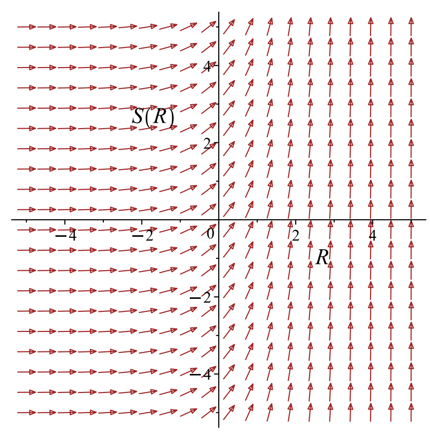
Which simplifies to

$$(x+1)^2 y = e^{x} + c_1$$

Which gives

$$y = \frac{e^{x} + c_1}{(x+1)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-2yx+e^{x}-2y}{(x+1)^2}$ 	$R = x$ $S = (x+1)^2 y$	$\frac{dS}{dR} = e^{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{x} + c_1}{(x+1)^2} \tag{1}$$

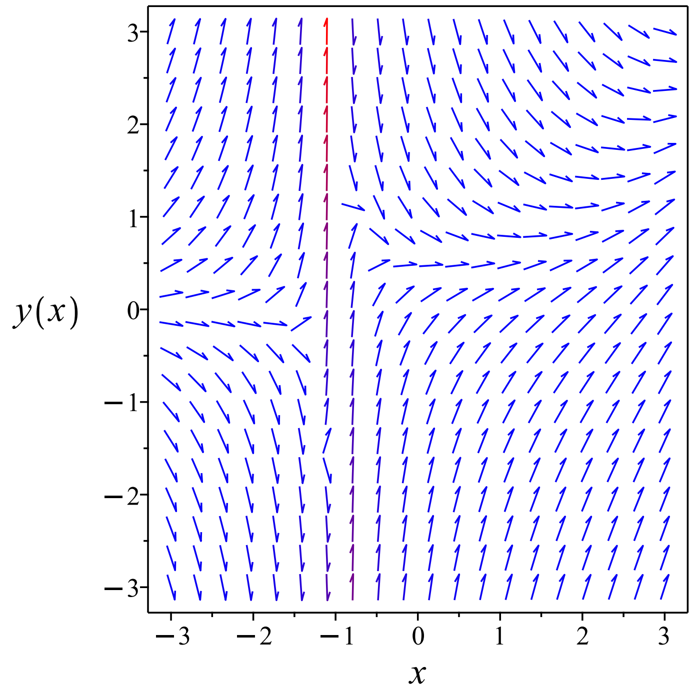


Figure 315: Slope field plot

Verification of solutions

$$y = \frac{e^x + c_1}{(x+1)^2}$$

Verified OK.

6.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \left((x+1)^2\right) \mathrm{d}y &= \left(-2yx + \mathrm{e}^x - 2y\right) \mathrm{d}x \\ \left(2yx - \mathrm{e}^x + 2y\right) \mathrm{d}x + \left((x+1)^2\right) \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 2yx - \mathrm{e}^x + 2y \\ N(x, y) &= (x+1)^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2yx - \mathrm{e}^x + 2y\right) \\ &= 2 + 2x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left((x+1)^2\right) \\ &= 2 + 2x \end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1)\\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 2yx - \mathrm{e}^{x} + 2y \,\mathrm{d}x\\
\phi &= y\,x^2 + 2yx - \mathrm{e}^{x} + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^2 + 2x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x+1)^2$. Therefore equation (4) becomes

$$
(x+1)^2 = x^2 + 2x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 1
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int (1) \,\mathrm{d}y\\
f(y) &= y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = y\,x^2 + 2yx - \mathrm{e}^{x} + y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 + 2yx - e^x + y$$

The solution becomes

$$y = \frac{e^x + c_1}{x^2 + 2x + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{e^x + c_1}{x^2 + 2x + 1} \tag{1}$$

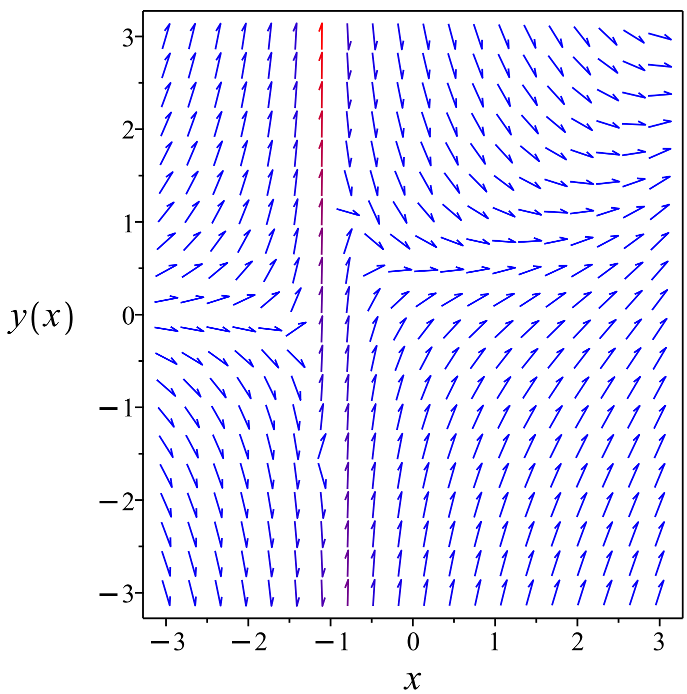


Figure 316: Slope field plot

Verification of solutions

$$y = \frac{e^x + c_1}{x^2 + 2x + 1}$$

Verified OK.

6.13.4 Maple step by step solution

Let's solve

$(x+1)\,y'+2y=\frac{{\mathrm e}^{x}}{x+1}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{2y}{x+1}+\frac{{\mathrm e}^{x}}{(x+1)^{2}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{2y}{x+1}=\frac{{\mathrm e}^{x}}{(x+1)^{2}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{2y}{x+1}\right)=\frac{\mu(x)\,{\mathrm e}^{x}}{(x+1)^{2}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{2y}{x+1}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{2\mu(x)}{x+1}$

- Solve to find the integrating factor

  $\mu(x)=(x+1)^{2}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)\,{\mathrm e}^{x}}{(x+1)^{2}}dx+c_{1}$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)\,{\mathrm e}^{x}}{(x+1)^{2}}dx+c_{1}$

- Solve for y

  $y=\frac{\int\frac{\mu(x)\,{\mathrm e}^{x}}{(x+1)^{2}}dx+c_{1}}{\mu(x)}$

- Substitute $\mu(x)=(x+1)^{2}$

  $y=\frac{\int{\mathrm e}^{x}dx+c_{1}}{(x+1)^{2}}$

- Evaluate the integrals on the rhs

  $y=\frac{{\mathrm e}^{x}+c_{1}}{(x+1)^{2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((x+1)*diff(y(x),x)+2*y(x)=exp(x)/(1+x),y(x), singsol=all)
```

$$y(x) = \frac{e^{x} + c_1}{(x+1)^2}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 17

```
DSolve[(x+1)*y'[x]+2*y[x]==Exp[x]/(1+x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x + c_1}{(x+1)^2}$$

6.14 problem 14

Internal problem ID [2000]
Internal file name [`OUTPUT/2000_Sunday_February_25_2024_06_44_21_AM_79110195/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]`

$$\boxed{\cos(y)^2 + (x - \tan(y))\, y' = 0}$$

6.14.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x - \tan(y))\,\mathrm{d}y &= \left(-\cos(y)^2\right)\mathrm{d}x\\ \left(\cos(y)^2\right)\mathrm{d}x + (x - \tan(y))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \cos(y)^2\\ N(x, y) &= x - \tan(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\cos(y)^2\right)\\ &= -\sin(2y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x - \tan(y))\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x - \tan(y)}\left((-2\cos(y)\sin(y)) - (1)\right) \\
&= \frac{-2\cos(y)^2\sin(y) - \cos(y)}{x\cos(y) - \sin(y)}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \sec(y)^2\left((1) - (-2\cos(y)\sin(y))\right) \\
&= 2\tan(y) + \sec(y)^2
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int 2\tan(y)+\sec(y)^2 \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\tan(y) - 2\ln(\cos(y))} \\
&= \frac{\mathrm{e}^{\tan(y)}}{\cos(y)^2}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{\tan(y)}}{\cos(y)^2}\left(\cos(y)^2\right) \\
&= \mathrm{e}^{\tan(y)}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{\tan(y)}}{\cos(y)^2}(x - \tan(y)) \\
&= \sec(y)^2(x - \tan(y))\,\mathrm{e}^{\tan(y)}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(\mathrm{e}^{\tan(y)}\right) + \left(\sec\left(y\right)^{2}\left(x - \tan\left(y\right)\right)\mathrm{e}^{\tan(y)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \mathrm{e}^{\tan(y)}\,\mathrm{d}x$$

$$\phi = \mathrm{e}^{\tan(y)}x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \left(1 + \tan\left(y\right)^{2}\right)\mathrm{e}^{\tan(y)}x + f'(y) \tag{4}$$

$$= \sec\left(y\right)^{2}\mathrm{e}^{\tan(y)}x + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec\left(y\right)^{2}\left(x - \tan\left(y\right)\right)\mathrm{e}^{\tan(y)}$. Therefore equation (4) becomes

$$\sec\left(y\right)^{2}\left(x - \tan\left(y\right)\right)\mathrm{e}^{\tan(y)} = \sec\left(y\right)^{2}\mathrm{e}^{\tan(y)}x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\sec\left(y\right)^{2}\mathrm{e}^{\tan(y)}\tan\left(y\right)$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\sec\left(y\right)^{2} \mathrm{e}^{\tan(y)} \tan\left(y\right)\right) \mathrm{d}y$$

$$f(y) = -\tan\left(y\right) \mathrm{e}^{\tan(y)} + \mathrm{e}^{\tan(y)} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{\tan(y)} x - \tan\left(y\right) \mathrm{e}^{\tan(y)} + \mathrm{e}^{\tan(y)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{\tan(y)} x - \tan\left(y\right) \mathrm{e}^{\tan(y)} + \mathrm{e}^{\tan(y)}$$

Summary

The solution(s) found are the following

$$\mathrm{e}^{\tan(y)} x - \tan\left(y\right) \mathrm{e}^{\tan(y)} + \mathrm{e}^{\tan(y)} = c_1 \tag{1}$$

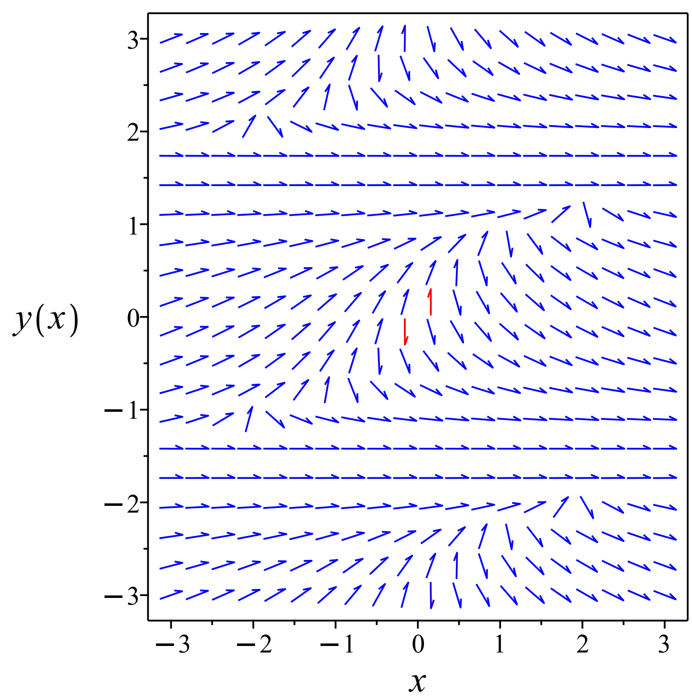


Figure 317: Slope field plot

Verification of solutions

$$e^{\tan(y)}x - \tan(y)\,e^{\tan(y)} + e^{\tan(y)} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(cos(y(x))^2+(x-tan(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arctan\left(\mathrm{LambertW}\left(-c_1 e^{-x-1}\right) + x + 1\right)$$

✓ Solution by Mathematica

Time used: 60.291 (sec). Leaf size: 21

```
DSolve[Cos[y[x]]^2+(x-Tan[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arctan\left(W\left(c_1\left(-e^{-x-1}\right)\right) + x + 1\right)$$

6.15 problem 15

Internal problem ID [2001]
Internal file name [`OUTPUT/2001_Sunday_February_25_2024_06_44_24_AM_82818777/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{2y - \left(y^4 + x\right) y' = 0}$$

6.15.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{2y}{y^4 + x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2+\frac{2y(b_3-a_2)}{y^4+x}-\frac{4y^2a_3}{(y^4+x)^2}+\frac{2y(xa_2+ya_3+a_1)}{(y^4+x)^2} \\ & -\left(\frac{2}{y^4+x}-\frac{8y^4}{(y^4+x)^2}\right)(xb_2+yb_3+b_1)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-y^8b_2-8x\,y^4b_2+2y^5a_2-8y^5b_3-6y^4b_1+x^2b_2+2y^2a_3+2xb_1-2ya_1}{(y^4+x)^2}=0$$

Setting the numerator to zero gives

$$y^8b_2+8x\,y^4b_2-2y^5a_2+8y^5b_3+6y^4b_1-x^2b_2-2y^2a_3-2xb_1+2ya_1=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$b_2v_2^8-2a_2v_2^5+8b_2v_1v_2^4+8b_3v_2^5+6b_1v_2^4-2a_3v_2^2-b_2v_1^2+2a_1v_2-2b_1v_1=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_2v_1^2+8b_2v_1v_2^4-2b_1v_1+b_2v_2^8+(-2a_2+8b_3)\,v_2^5+6b_1v_2^4-2a_3v_2^2+2a_1v_2=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_2 &= 0 \\
2a_1 &= 0 \\
-2a_3 &= 0 \\
-2b_1 &= 0 \\
6b_1 &= 0 \\
-b_2 &= 0 \\
8b_2 &= 0 \\
-2a_2 + 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= 4b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 4x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{2y}{y^4 + x}\right)(4x) \\
&= \frac{y^5 - 7yx}{y^4 + x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^5 - 7yx}{y^4 + x}} dy \end{aligned}$$

Which results in

$$S = \frac{2 \ln (y^4 - 7x)}{7} - \frac{\ln (y)}{7}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y}{y^4 + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{-y^4 + 7x} \\ S_y &= \frac{-y^4 - x}{y(-y^4 + 7x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

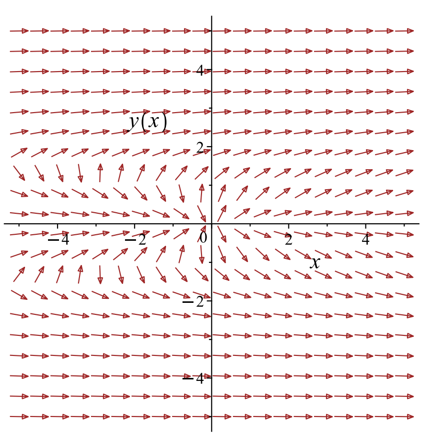
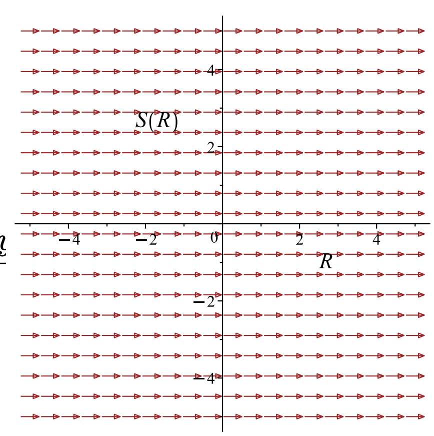
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(y^4 - 7x)}{7} - \frac{\ln(y)}{7} = c_1$$

Which simplifies to

$$\frac{2\ln(y^4 - 7x)}{7} - \frac{\ln(y)}{7} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y}{y^4+x}$ 	$R = x$ $S = \frac{2\ln(y^4 - 7x)}{7} - \frac{\ln(y)}{7}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{2\ln(y^4-7x)}{7}-\frac{\ln(y)}{7}=c_1 \tag{1}$$

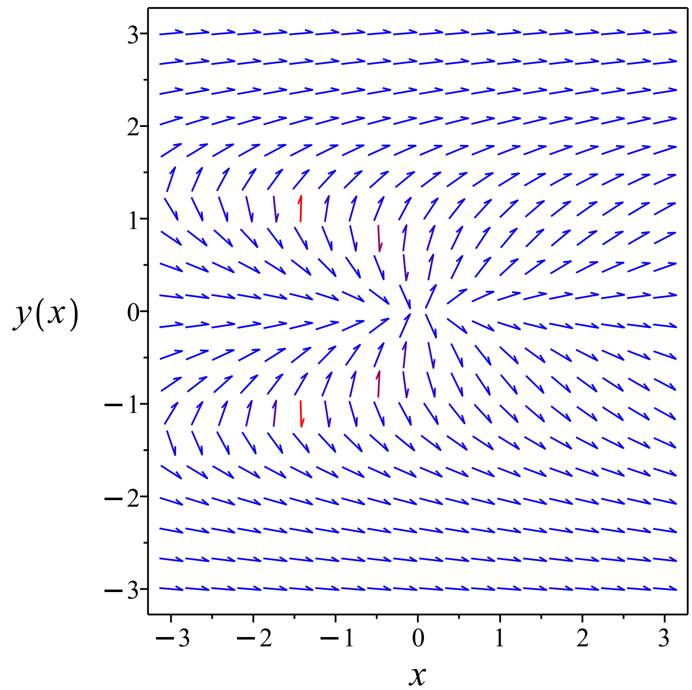


Figure 318: Slope field plot

Verification of solutions

$$\frac{2\ln(y^4-7x)}{7}-\frac{\ln(y)}{7}=c_1$$

Verified OK.

6.15.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y^4 - x)\, \mathrm{d}y &= (-2y)\, \mathrm{d}x \\ (2y)\, \mathrm{d}x + (-y^4 - x)\, \mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2y \\ N(x, y) &= -y^4 - x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y) \\ &= 2\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y^4 - x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{y^4 + x}((2) - (-1)) \\
&= -\frac{3}{y^4 + x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2y}((-1) - (2)) \\
&= -\frac{3}{2y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{3}{2y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{3\ln(y)}{2}} \\
&= \frac{1}{y^{\frac{3}{2}}}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^{\frac{3}{2}}}(2y) \\
&= \frac{2}{\sqrt{y}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^{\frac{3}{2}}}(-y^4 - x) \\
&= -\frac{y^4 + x}{y^{\frac{3}{2}}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2}{\sqrt{y}}\right) + \left(-\frac{y^4 + x}{y^{\frac{3}{2}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2}{\sqrt{y}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{2x}{\sqrt{y}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{x}{y^{\frac{3}{2}}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y^4+x}{y^{\frac{3}{2}}}$. Therefore equation (4) becomes

$$
-\frac{y^4 + x}{y^{\frac{3}{2}}} = -\frac{x}{y^{\frac{3}{2}}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^{\frac{5}{2}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(-y^{\frac{5}{2}}\right) \mathrm{d}y$$

$$f(y) = -\frac{2y^{\frac{7}{2}}}{7} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2x}{\sqrt{y}} - \frac{2y^{\frac{7}{2}}}{7} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{2x}{\sqrt{y}} - \frac{2y^{\frac{7}{2}}}{7}$$

Summary

The solution(s) found are the following

$$\frac{2x}{\sqrt{y}} - \frac{2y^{\frac{7}{2}}}{7} = c_1 \tag{1}$$

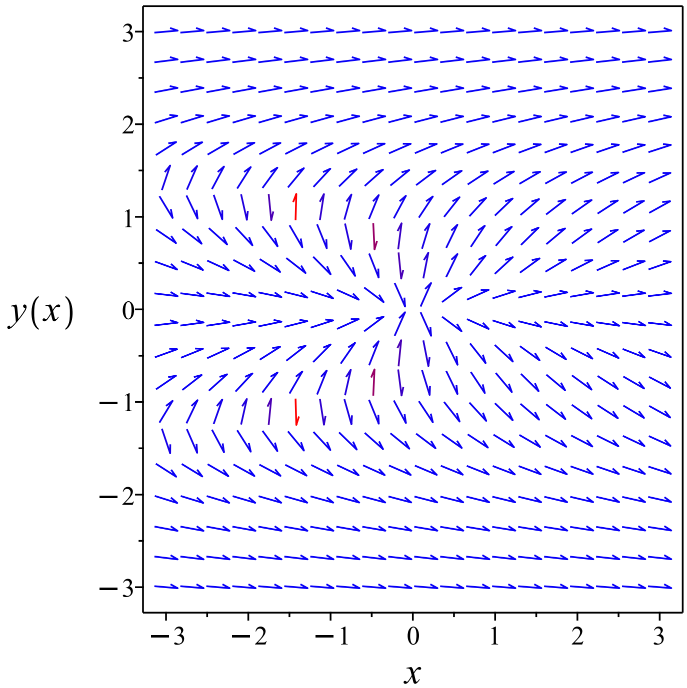


Figure 319: Slope field plot

Verification of solutions

$$\frac{2x}{\sqrt{y}} - \frac{2y^{\frac{7}{2}}}{7} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(2*y(x)=(y(x)^4+x)*diff(y(x),x),y(x), singsol=all)
```

$$x - \frac{y(x)^4}{7} - \sqrt{y(x)}\, c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.102 (sec). Leaf size: 257

```
DSolve[2*y[x]==(y[x]^4+x)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned}
y(x) &\to \text{Root}\left[\#1^8 - 14\#1^4 x - 49\#1 c_1{}^2 + 49x^2\&, 1\right] \\
y(x) &\to \text{Root}\left[\#1^8 - 14\#1^4 x - 49\#1 c_1{}^2 + 49x^2\&, 2\right] \\
y(x) &\to \text{Root}\left[\#1^8 - 14\#1^4 x - 49\#1 c_1{}^2 + 49x^2\&, 3\right] \\
y(x) &\to \text{Root}\left[\#1^8 - 14\#1^4 x - 49\#1 c_1{}^2 + 49x^2\&, 4\right] \\
y(x) &\to \text{Root}\left[\#1^8 - 14\#1^4 x - 49\#1 c_1{}^2 + 49x^2\&, 5\right] \\
y(x) &\to \text{Root}\left[\#1^8 - 14\#1^4 x - 49\#1 c_1{}^2 + 49x^2\&, 6\right] \\
y(x) &\to \text{Root}\left[\#1^8 - 14\#1^4 x - 49\#1 c_1{}^2 + 49x^2\&, 7\right] \\
y(x) &\to \text{Root}\left[\#1^8 - 14\#1^4 x - 49\#1 c_1{}^2 + 49x^2\&, 8\right]
\end{aligned}$$

6.16 problem 16

Internal problem ID [2002]
Internal file name [`OUTPUT/2002_Sunday_February_25_2024_06_44_25_AM_31396177/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(\theta) r' - 2r\sin(\theta) = 2}$$

6.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(\theta)r = q(\theta)$$

Where here

$$p(\theta) = -2\tan(\theta)$$
$$q(\theta) = 2\sec(\theta)$$

Hence the ode is

$$r' - 2\tan(\theta) r = 2\sec(\theta)$$

The integrating factor μ is

$$\mu = e^{\int -2\tan(\theta)d\theta}$$
$$= \cos(\theta)^2$$

The ode becomes

$$\frac{d}{d\theta}(\mu r) = (\mu)(2\sec(\theta))$$
$$\frac{d}{d\theta}\left(r\cos(\theta)^2\right) = \left(\cos(\theta)^2\right)(2\sec(\theta))$$
$$d\left(r\cos(\theta)^2\right) = (2\cos(\theta))\, d\theta$$

Integrating gives

$$r\cos(\theta)^2 = \int 2\cos(\theta)\, d\theta$$
$$r\cos(\theta)^2 = 2\sin(\theta) + c_1$$

Dividing both sides by the integrating factor $\mu = \cos(\theta)^2$ results in

$$r = 2\sec(\theta)^2\sin(\theta) + c_1\sec(\theta)^2$$

which simplifies to

$$r = \sec(\theta)^2(2\sin(\theta) + c_1)$$

Summary

The solution(s) found are the following

$$r = \sec(\theta)^2(2\sin(\theta) + c_1) \tag{1}$$

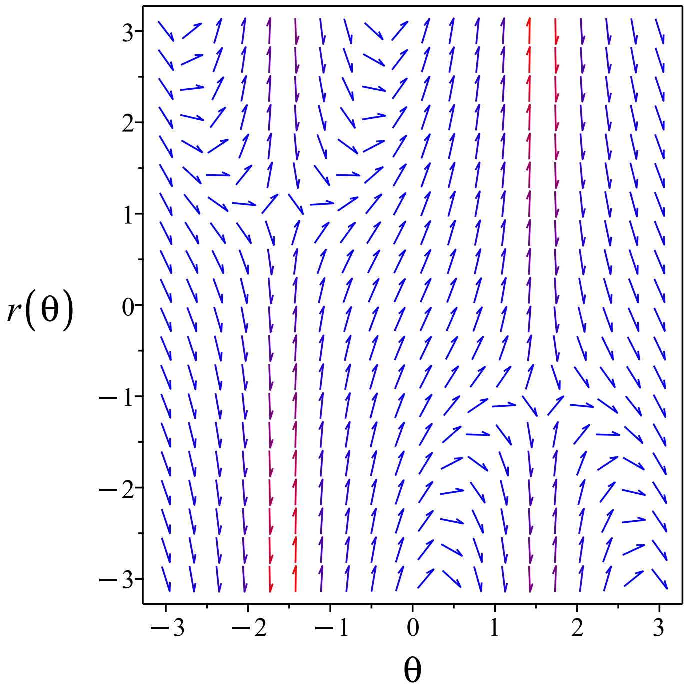


Figure 320: Slope field plot

Verification of solutions

$$r = \sec(\theta)^2 (2\sin(\theta) + c_1)$$

Verified OK.

6.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = \frac{2 + 2r\sin(\theta)}{\cos(\theta)}$$
$$r' = \omega(\theta, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2 \xi_r - \omega_\theta \xi - \omega_r \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 179: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, r) &= 0 \\
\eta(\theta, r) &= \frac{1}{\cos(\theta)^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cos(\theta)^2}} dy \end{aligned}$$

Which results in

$$S = r\cos(\theta)^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r)S_r}{R_\theta + \omega(\theta, r)R_r} \tag{2}$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = \frac{2 + 2r\sin(\theta)}{\cos(\theta)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 1 \\ R_r &= 0 \\ S_\theta &= -\sin(2\theta)\, r \\ S_r &= \cos(\theta)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\cos(\theta) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2\sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$\cos(\theta)^2 r = 2\sin(\theta) + c_1$$

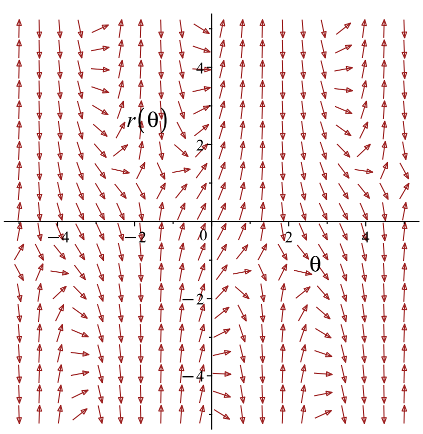
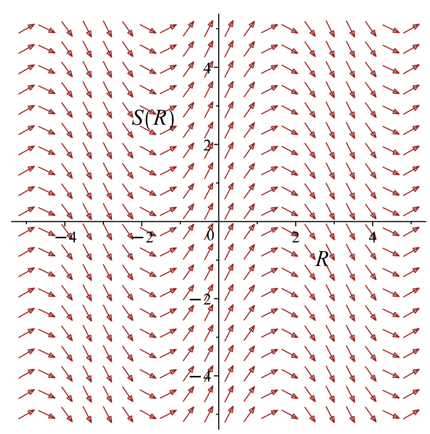
Which simplifies to

$$\cos(\theta)^2 r = 2\sin(\theta) + c_1$$

Which gives

$$r = \frac{2\sin(\theta) + c_1}{\cos(\theta)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{d\theta} = \frac{2+2r\sin(\theta)}{\cos(\theta)}$ 	$R = \theta$ $S = r\cos(\theta)^2$	$\frac{dS}{dR} = 2\cos(R)$ 

Summary

The solution(s) found are the following

$$r = \frac{2\sin(\theta) + c_1}{\cos(\theta)^2} \tag{1}$$

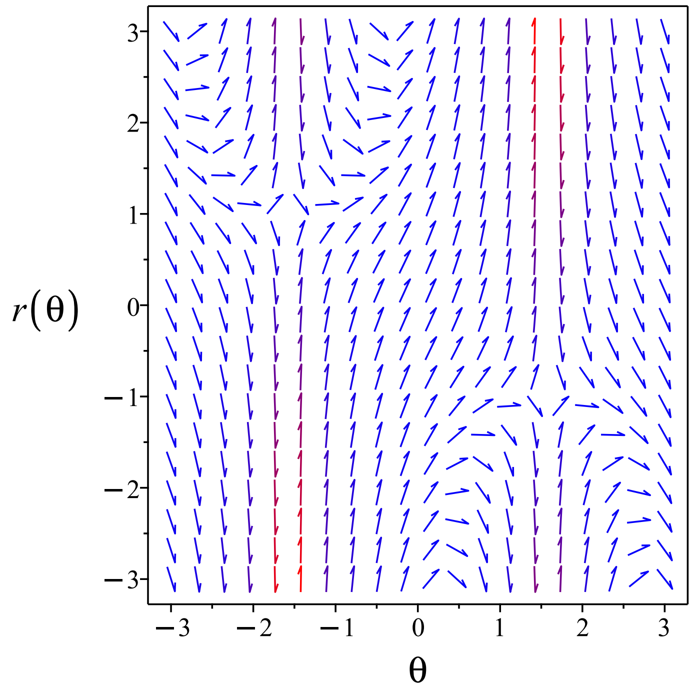


Figure 321: Slope field plot

Verification of solutions

$$r = \frac{2\sin(\theta) + c_1}{\cos(\theta)^2}$$

Verified OK.

6.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta, r)\, \mathrm{d}\theta + N(\theta, r)\, \mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(\theta))\, \mathrm{d}r &= (2 + 2r \sin(\theta))\, \mathrm{d}\theta\\ (-2r \sin(\theta) - 2)\, \mathrm{d}\theta + (\cos(\theta))\, \mathrm{d}r &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(\theta, r) &= -2r \sin(\theta) - 2\\ N(\theta, r) &= \cos(\theta)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}(-2r \sin(\theta) - 2)\\ &= -2 \sin(\theta)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(\cos(\theta))\\ &= -\sin(\theta)\end{aligned}$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial r} - \frac{\partial N}{\partial \theta}\right) \\
&= \sec(\theta)\left((-2\sin(\theta)) - (-\sin(\theta))\right) \\
&= -\tan(\theta)
\end{aligned}
$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}\theta} \\
&= e^{\int -\tan(\theta) \, \mathrm{d}\theta}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\cos(\theta))} \\
&= \cos(\theta)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \cos(\theta)(-2r\sin(\theta) - 2) \\
&= (-2r\sin(\theta) - 2)\cos(\theta)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \cos(\theta)(\cos(\theta)) \\
&= \cos(\theta)^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0 \\
((-2r\sin(\theta) - 2)\cos(\theta)) + \left(\cos(\theta)^2\right)\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(\theta, r)$

$$
\frac{\partial \phi}{\partial \theta} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial r} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta = \int \overline{M} \, \mathrm{d}\theta$$
$$\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta = \int (-2r\sin(\theta) - 2)\cos(\theta) \, \mathrm{d}\theta$$
$$\phi = -\sin(\theta)(r\sin(\theta) + 2) + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = -\sin(\theta)^2 + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \cos(\theta)^2$. Therefore equation (4) becomes

$$\cos(\theta)^2 = -\sin(\theta)^2 + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$\begin{aligned} f'(r) &= \sin(\theta)^2 + \cos(\theta)^2 \\ &= 1 \end{aligned}$$

Integrating the above w.r.t r results in

$$\int f'(r) \, \mathrm{d}r = \int (1) \, \mathrm{d}r$$
$$f(r) = r + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(r)$ into equation (3) gives ϕ

$$\phi = -\sin(\theta)(r\sin(\theta) + 2) + r + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(\theta)(r\sin(\theta) + 2) + r$$

The solution becomes

$$r = -\frac{2\sin\left(\theta\right)+c_1}{\sin\left(\theta\right)^2-1}$$

<u>Summary</u>
The solution(s) found are the following

$$r = -\frac{2\sin\left(\theta\right)+c_1}{\sin\left(\theta\right)^2-1} \tag{1}$$

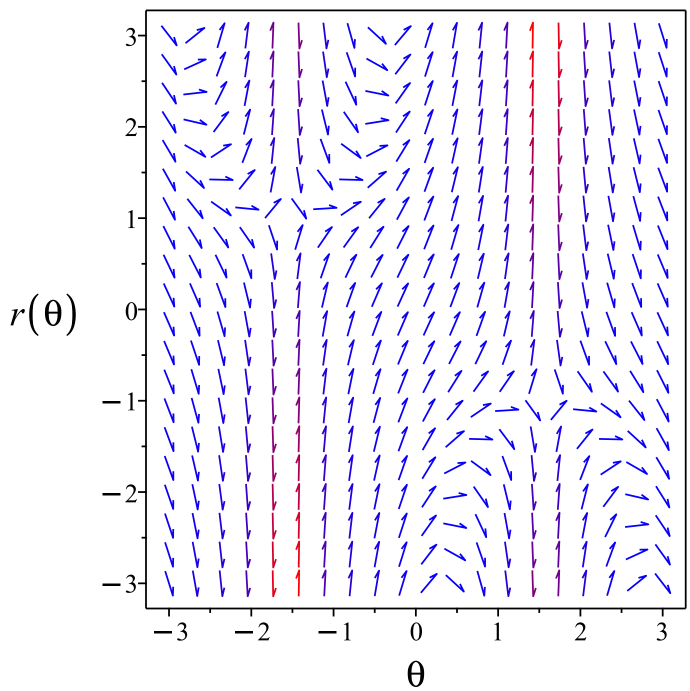


Figure 322: Slope field plot

<u>Verification of solutions</u>

$$r = -\frac{2\sin\left(\theta\right)+c_1}{\sin\left(\theta\right)^2-1}$$

Verified OK.

6.16.4 Maple step by step solution

Let's solve

$\cos(\theta) r' - 2r\sin(\theta) = 2$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

  $r' = \frac{2\sin(\theta) r}{\cos(\theta)} + \frac{2}{\cos(\theta)}$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

  $r' - \frac{2\sin(\theta) r}{\cos(\theta)} = \frac{2}{\cos(\theta)}$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

  $\mu(\theta)\left(r' - \frac{2\sin(\theta) r}{\cos(\theta)}\right) = \frac{2\mu(\theta)}{\cos(\theta)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta) r)$

  $\mu(\theta)\left(r' - \frac{2\sin(\theta) r}{\cos(\theta)}\right) = \mu'(\theta) r + \mu(\theta) r'$

- Isolate $\mu'(\theta)$

  $\mu'(\theta) = -\frac{2\mu(\theta)\sin(\theta)}{\cos(\theta)}$

- Solve to find the integrating factor

  $\mu(\theta) = \cos(\theta)^2$

- Integrate both sides with respect to θ

  $\int \left(\frac{d}{d\theta}(\mu(\theta) r)\right) d\theta = \int \frac{2\mu(\theta)}{\cos(\theta)} d\theta + c_1$

- Evaluate the integral on the lhs

  $\mu(\theta) r = \int \frac{2\mu(\theta)}{\cos(\theta)} d\theta + c_1$

- Solve for r

  $r = \frac{\int \frac{2\mu(\theta)}{\cos(\theta)} d\theta + c_1}{\mu(\theta)}$

- Substitute $\mu(\theta) = \cos(\theta)^2$

  $r = \frac{\int 2\cos(\theta) d\theta + c_1}{\cos(\theta)^2}$

- Evaluate the integrals on the rhs

  $r = \frac{2\sin(\theta) + c_1}{\cos(\theta)^2}$

- Simplify

  $$r = \sec(\theta)^2 (2\sin(\theta) + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(cos(theta)*diff(r(theta),theta)=2+2*r(theta)*sin(theta),r(theta), singsol=all)
```

$$r(\theta) = \sec(\theta)^2 (2\sin(\theta) + c_1)$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 17

```
DSolve[Cos[\[Theta]]*r'[\[Theta]]==2+2*r[\[Theta]]*Sin[\[Theta]],r[\[Theta]],\[Theta],Includ
```

$$r(\theta) \to \sec^2(\theta)(2\sin(\theta) + c_1)$$

6.17 problem 17

Internal problem ID [2003]
Internal file name [`OUTPUT/2003_Sunday_February_25_2024_06_44_26_AM_37625534/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\sin(\theta) r' + \tan(\theta) r = \cos(\theta) - 1}$$

6.17.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(\theta)r = q(\theta)$$

Where here

$$p(\theta) = \sec(\theta)$$
$$q(\theta) = \cot(\theta) - \csc(\theta)$$

Hence the ode is

$$r' + \sec(\theta) r = \cot(\theta) - \csc(\theta)$$

The integrating factor μ is

$$\mu = e^{\int \sec(\theta)d\theta}$$
$$= \sec(\theta) + \tan(\theta)$$

The ode becomes

$$\frac{d}{d\theta}(\mu r) = (\mu)(\cot(\theta) - \csc(\theta))$$
$$\frac{d}{d\theta}((\sec(\theta) + \tan(\theta))r) = (\sec(\theta) + \tan(\theta))(\cot(\theta) - \csc(\theta))$$
$$d((\sec(\theta) + \tan(\theta))r) = \left(\frac{-\cos(\theta) + \sin(\theta) + 1}{-\cos(\theta) + \sin(\theta) - 1}\right) d\theta$$

Integrating gives

$$(\sec(\theta) + \tan(\theta))r = \int \frac{-\cos(\theta) + \sin(\theta) + 1}{-\cos(\theta) + \sin(\theta) - 1} d\theta$$
$$(\sec(\theta) + \tan(\theta))r = 2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) + \theta + c_1$$

Dividing both sides by the integrating factor $\mu = \sec(\theta) + \tan(\theta)$ results in

$$r = \frac{2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) + \theta}{\sec(\theta) + \tan(\theta)} + \frac{c_1}{\sec(\theta) + \tan(\theta)}$$

which simplifies to

$$r = \frac{2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) + \theta + c_1}{\sec(\theta) + \tan(\theta)}$$

Summary

The solution(s) found are the following

$$r = \frac{2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) + \theta + c_1}{\sec(\theta) + \tan(\theta)} \tag{1}$$

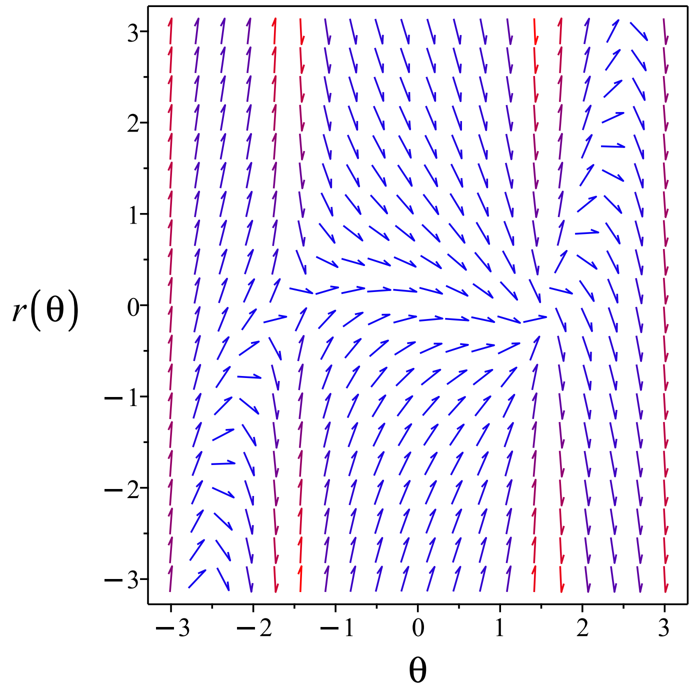


Figure 323: Slope field plot

Verification of solutions

$$r = \frac{2\ln\left(\tan\left(\frac{\theta}{2}\right)-1\right)+\theta+c_1}{\sec\left(\theta\right)+\tan\left(\theta\right)}$$

Verified OK.

6.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = \frac{-1-\tan\left(\theta\right)r+\cos\left(\theta\right)}{\sin\left(\theta\right)}$$
$$r' = \omega(\theta, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2\xi_r - \omega_\theta\xi - \omega_r\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 182: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, r) &= 0 \\
\eta(\theta, r) &= \frac{1}{\sec(\theta) + \tan(\theta)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sec(\theta)+\tan(\theta)}} dy \end{aligned}$$

Which results in

$$S = (\sec(\theta) + \tan(\theta))\, r$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r) S_r}{R_\theta + \omega(\theta, r) R_r} \tag{2}$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = \frac{-1 - \tan(\theta)\, r + \cos(\theta)}{\sin(\theta)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 1 \\ R_r &= 0 \\ S_\theta &= -\frac{r}{\sin(\theta) - 1} \\ S_r &= \sec(\theta) + \tan(\theta) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-\cos(\theta) + \sin(\theta) + 1}{-\cos(\theta) + \sin(\theta) - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{-\cos(R) + \sin(R) + 1}{-\cos(R) + \sin(R) - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2\ln\left(\tan\left(\frac{R}{2}\right) - 1\right) + R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$(\sec(\theta) + \tan(\theta))\, r = 2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) + \theta + c_1$$

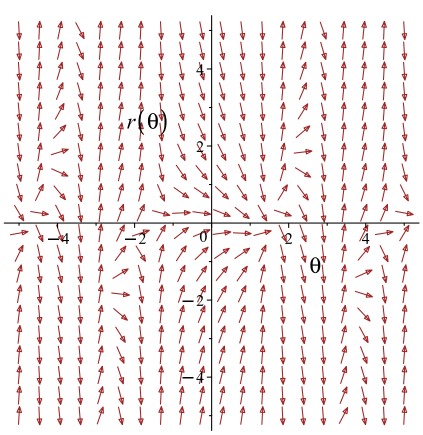
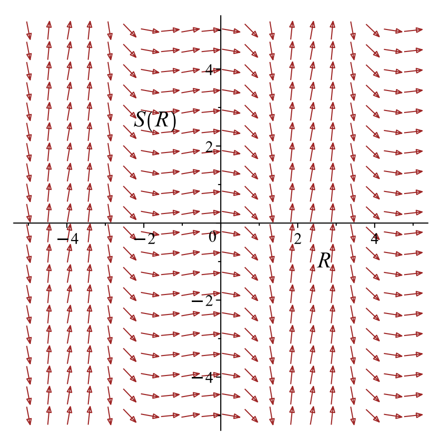
Which simplifies to

$$(\sec(\theta) + \tan(\theta))\, r = 2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) + \theta + c_1$$

Which gives

$$r = \frac{2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) + \theta + c_1}{\sec(\theta) + \tan(\theta)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{d\theta} = \frac{-1-\tan(\theta)r+\cos(\theta)}{\sin(\theta)}$ 	$R = \theta$ $S = (\sec(\theta) + \tan(\theta))\, r$	$\frac{dS}{dR} = \frac{-\cos(R)+\sin(R)+1}{-\cos(R)+\sin(R)-1}$ 

Summary

The solution(s) found are the following

$$r = \frac{2\ln\left(\tan\left(\frac{\theta}{2}\right)-1\right)+\theta+c_1}{\sec\left(\theta\right)+\tan\left(\theta\right)} \tag{1}$$

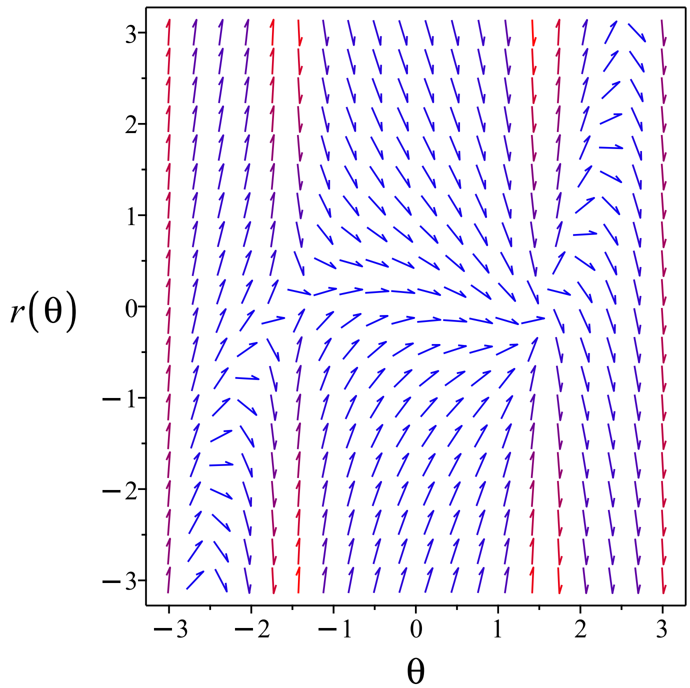


Figure 324: Slope field plot

Verification of solutions

$$r = \frac{2\ln\left(\tan\left(\frac{\theta}{2}\right)-1\right)+\theta+c_1}{\sec\left(\theta\right)+\tan\left(\theta\right)}$$

Verified OK.

6.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta,r)\,\mathrm{d}\theta+N(\theta,r)\,\mathrm{d}r=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\sin(\theta))\,\mathrm{d}r&=(-1-\tan(\theta)\,r+\cos(\theta))\,\mathrm{d}\theta\\ (1+\tan(\theta)\,r-\cos(\theta))\,\mathrm{d}\theta+(\sin(\theta))\,\mathrm{d}r&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(\theta,r)&=1+\tan(\theta)\,r-\cos(\theta)\\ N(\theta,r)&=\sin(\theta)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r}=\frac{\partial N}{\partial\theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}(1+\tan(\theta)\, r-\cos(\theta)) \\ &= \tan(\theta)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(\sin(\theta)) \\ &= \cos(\theta)\end{aligned}$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial r}-\frac{\partial N}{\partial \theta}\right) \\ &= \csc(\theta)\left((\tan(\theta))-(\cos(\theta))\right) \\ &= -\cot(\theta)+\sec(\theta)\end{aligned}$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}\theta} \\ &= e^{\int -\cot(\theta)+\sec(\theta) \, \mathrm{d}\theta}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(\sin(\theta))+\ln(\sec(\theta)+\tan(\theta))} \\ &= \frac{\sec(\theta)+\tan(\theta)}{\sin(\theta)}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{\sec(\theta)+\tan(\theta)}{\sin(\theta)}(1+\tan(\theta)\, r-\cos(\theta)) \\ &= \frac{(-r-\sin(\theta))\cos(\theta)-r}{(\cos(\theta)+1)(\sin(\theta)-1)}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{\sec(\theta)+\tan(\theta)}{\sin(\theta)}(\sin(\theta)) \\ &= \sec(\theta)+\tan(\theta)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$$
$$\left(\frac{(-r-\sin(\theta))\cos(\theta)-r}{(\cos(\theta)+1)(\sin(\theta)-1)}\right) + (\sec(\theta)+\tan(\theta))\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$$

The following equations are now set up to solve for the function $\phi(\theta, r)$

$$\frac{\partial \phi}{\partial \theta} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial r} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta}\,\mathrm{d}\theta = \int \overline{M}\,\mathrm{d}\theta$$
$$\int \frac{\partial \phi}{\partial \theta}\,\mathrm{d}\theta = \int \frac{(-r-\sin(\theta))\cos(\theta)-r}{(\cos(\theta)+1)(\sin(\theta)-1)}\,\mathrm{d}\theta$$
$$\phi = -\frac{2r}{\tan\left(\frac{\theta}{2}\right)-1} - 2\ln\left(\tan\left(\frac{\theta}{2}\right)-1\right) - \theta + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = -\frac{2}{\tan\left(\frac{\theta}{2}\right)-1} + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \sec(\theta) + \tan(\theta)$. Therefore equation (4) becomes

$$\sec(\theta) + \tan(\theta) = -\frac{2}{\tan\left(\frac{\theta}{2}\right)-1} + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = \frac{\tan\left(\frac{\theta}{2}\right)\tan(\theta) + \tan\left(\frac{\theta}{2}\right)\sec(\theta) - \tan(\theta) - \sec(\theta) + 2}{\tan\left(\frac{\theta}{2}\right)-1}$$
$$= -1$$

Integrating the above w.r.t r results in

$$\int f'(r)\,\mathrm{d}r = \int (-1)\,\mathrm{d}r$$
$$f(r) = -r + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(r)$ into equation (3) gives ϕ

$$\phi = -\frac{2r}{\tan\left(\frac{\theta}{2}\right) - 1} - 2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) - \theta - r + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{2r}{\tan\left(\frac{\theta}{2}\right) - 1} - 2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) - \theta - r$$

The solution becomes

$$r = -\frac{2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right)\tan\left(\frac{\theta}{2}\right) + c_1\tan\left(\frac{\theta}{2}\right) + \theta\tan\left(\frac{\theta}{2}\right) - 2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) - c_1 - \theta}{1 + \tan\left(\frac{\theta}{2}\right)}$$

Summary

The solution(s) found are the following

$$r = -\frac{2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right)\tan\left(\frac{\theta}{2}\right) + c_1\tan\left(\frac{\theta}{2}\right) + \theta\tan\left(\frac{\theta}{2}\right) - 2\ln\left(\tan\left(\frac{\theta}{2}\right) - 1\right) - c_1 - \theta}{1 + \tan\left(\frac{\theta}{2}\right)} \tag{1}$$

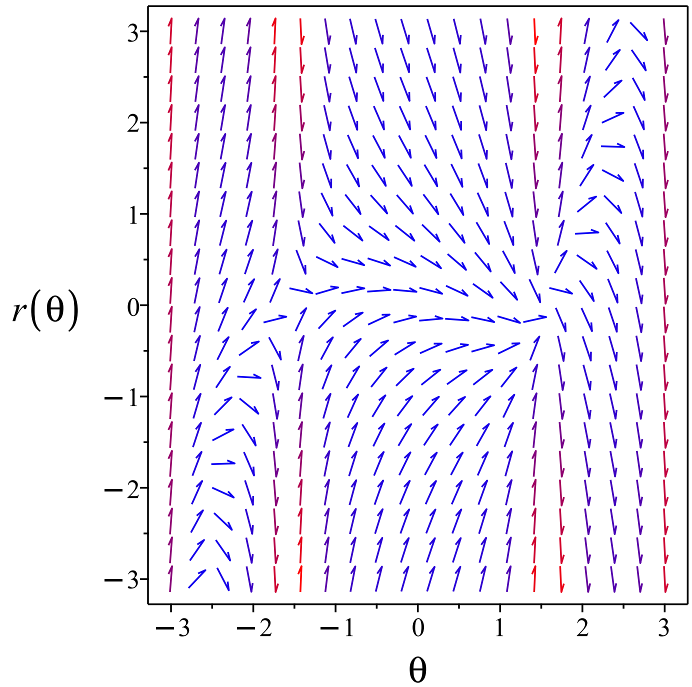


Figure 325: Slope field plot

Verification of solutions

$$r = -\frac{2\ln\left(\tan\left(\frac{\theta}{2}\right)-1\right)\tan\left(\frac{\theta}{2}\right)+c_1\tan\left(\frac{\theta}{2}\right)+\theta\tan\left(\frac{\theta}{2}\right)-2\ln\left(\tan\left(\frac{\theta}{2}\right)-1\right)-c_1-\theta}{1+\tan\left(\frac{\theta}{2}\right)}$$

Verified OK.

6.17.4 Maple step by step solution

Let's solve

$\sin(\theta)\, r' + \tan(\theta)\, r = \cos(\theta) - 1$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

  $r' = -\frac{\tan(\theta) r}{\sin(\theta)} + \frac{\cos(\theta)-1}{\sin(\theta)}$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

  $r' + \frac{\tan(\theta) r}{\sin(\theta)} = \frac{\cos(\theta)-1}{\sin(\theta)}$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

$$\mu(\theta) \left(r' + \frac{\tan(\theta) r}{\sin(\theta)}\right) = \frac{\mu(\theta) (\cos(\theta) - 1)}{\sin(\theta)}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta)\, r)$

$$\mu(\theta) \left(r' + \frac{\tan(\theta) r}{\sin(\theta)}\right) = \mu'(\theta)\, r + \mu(\theta)\, r'$$

- Isolate $\mu'(\theta)$

$$\mu'(\theta) = \frac{\mu(\theta) \tan(\theta)}{\sin(\theta)}$$

- Solve to find the integrating factor

$$\mu(\theta) = \sec(\theta) + \tan(\theta)$$

- Integrate both sides with respect to θ

$$\int \left(\frac{d}{d\theta}(\mu(\theta)\, r)\right) d\theta = \int \frac{\mu(\theta) (\cos(\theta) - 1)}{\sin(\theta)} d\theta + c_1$$

- Evaluate the integral on the lhs

$$\mu(\theta)\, r = \int \frac{\mu(\theta) (\cos(\theta) - 1)}{\sin(\theta)} d\theta + c_1$$

- Solve for r

$$r = \frac{\int \frac{\mu(\theta)(\cos(\theta)-1)}{\sin(\theta)} d\theta + c_1}{\mu(\theta)}$$

- Substitute $\mu(\theta) = \sec(\theta) + \tan(\theta)$

$$r = \frac{\int \frac{(\sec(\theta)+\tan(\theta))(\cos(\theta)-1)}{\sin(\theta)} d\theta + c_1}{\sec(\theta)+\tan(\theta)}$$

- Evaluate the integrals on the rhs

$$r = \frac{\theta - \ln(\sec(\theta)+\tan(\theta)) + \ln(-\cot(\theta)+\csc(\theta)) - \ln(\tan(\theta)) + c_1}{\sec(\theta)+\tan(\theta)}$$

- Simplify

$$r = (\theta - \ln(\sec(\theta) + \tan(\theta)) + \ln(-\cot(\theta) + \csc(\theta)) - \ln(\tan(\theta)) + c_1)(\sec(\theta) - \tan(\theta))$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(sin(theta)*diff(r(theta),theta)+1+r(theta)*tan(theta)=cos(theta),r(theta), singsol=al
```

$$r(\theta) = \frac{2\ln\left(\tan\left(\frac{\theta}{2}\right)-1\right)+\theta+c_1}{\sec\left(\theta\right)+\tan\left(\theta\right)}$$

✓ Solution by Mathematica

Time used: 6.359 (sec). Leaf size: 171

```
DSolve[Sin[\[Theta]]*r'[\[Theta]]+1+r[\[Theta]]*Tan[\[Theta]]==Cos[\[Theta]],r[\[Theta]],\[T
```

$r(\theta)$

$$\to \frac{1}{4}e^{-\coth^{-1}(\sin(\theta))}\left(-\frac{\sqrt{2}\sqrt{-\cot^2(\theta)}\left(\frac{2\left(\sqrt{\sin^2(\theta)}-1\right)\left(\sqrt{\cos(2\theta)-1}\operatorname{arctanh}\left(\sqrt{\cos^2(\theta)}\right)+\sqrt{2}\sqrt{\sin^2(\theta)}\log\left(\sqrt{\cos(2\theta)+1}-\sqrt{co\cdots}\right.\right.}{\sqrt{-\sin^2(\theta)}(\csc(\theta)-1)}\cdots}{\sqrt{\cos^2(\theta)}\cdots}\cdots + 4c_1\right)$$

6.18 problem 18

Internal problem ID [2004]
Internal file name [`OUTPUT/2004_Sunday_February_25_2024_06_44_29_AM_74054515/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{yx' - x(2+3y) = 2y\,e^{3y}}$$

6.18.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(y)x = q(y)$$

Where here

$$p(y) = -\frac{2+3y}{y}$$
$$q(y) = 2\,e^{3y}$$

Hence the ode is

$$x' - \frac{(2+3y)\,x}{y} = 2\,e^{3y}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2+3y}{y}dy} \\
&= \mathrm{e}^{-3y-2\ln(y)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\mathrm{e}^{-3y}}{y^2}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y}(\mu x) &= (\mu)\left(2\,\mathrm{e}^{3y}\right) \\
\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{e}^{-3y}x}{y^2}\right) &= \left(\frac{\mathrm{e}^{-3y}}{y^2}\right)\left(2\,\mathrm{e}^{3y}\right) \\
\mathrm{d}\left(\frac{\mathrm{e}^{-3y}x}{y^2}\right) &= \left(\frac{2}{y^2}\right)\,\mathrm{d}y
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\mathrm{e}^{-3y}x}{y^2} &= \int \frac{2}{y^2}\,\mathrm{d}y \\
\frac{\mathrm{e}^{-3y}x}{y^2} &= -\frac{2}{y} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{-3y}}{y^2}$ results in

$$
x = -2y\,\mathrm{e}^{3y} + c_1\mathrm{e}^{3y}y^2
$$

which simplifies to

$$
x = \mathrm{e}^{3y}y(c_1 y - 2)
$$

Summary
The solution(s) found are the following

$$
x = \mathrm{e}^{3y}y(c_1 y - 2) \tag{1}
$$

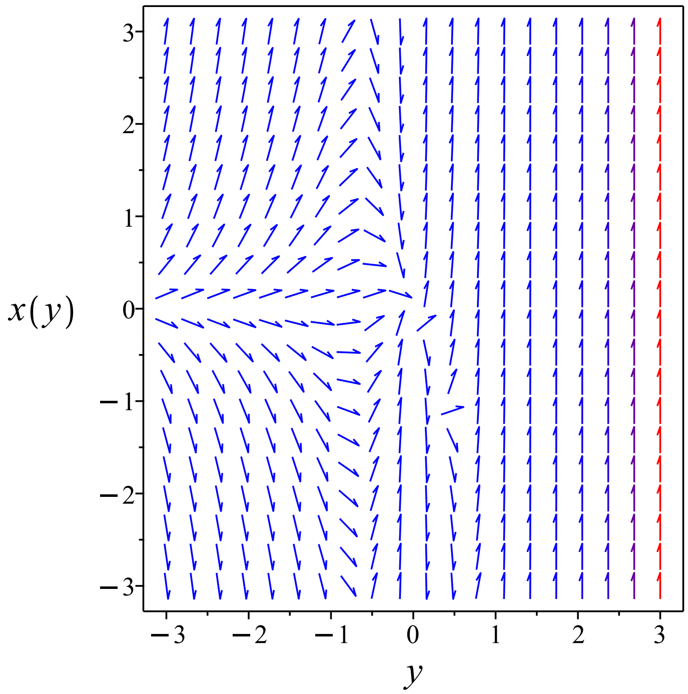


Figure 326: Slope field plot

Verification of solutions

$$x = e^{3y} y(c_1 y - 2)$$

Verified OK.

6.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = \frac{2y\,e^{3y} + 3yx + 2x}{y}$$
$$x' = \omega(y, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_x - \xi_y) - \omega^2 \xi_x - \omega_y \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 185: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(y, x) &= 0 \\ \eta(y, x) &= \mathrm{e}^{3y+2\ln(y)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dy}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial x}\right) S(y, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{3y+2\ln(y)}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{-3y} x}{y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y, x) S_x}{R_y + \omega(y, x) R_x} \tag{2}$$

Where in the above R_y, R_x, S_y, S_x are all partial derivatives and $\omega(y, x)$ is the right hand side of the original ode given by

$$\omega(y, x) = \frac{2y\, e^{3y} + 3yx + 2x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_y &= 1 \\ R_x &= 0 \\ S_y &= \frac{x(-3y-2)\, e^{-3y}}{y^3} \\ S_x &= \frac{e^{-3y}}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{2}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to y, x coordinates. This results in

$$\frac{e^{-3y}x}{y^2} = -\frac{2}{y} + c_1$$

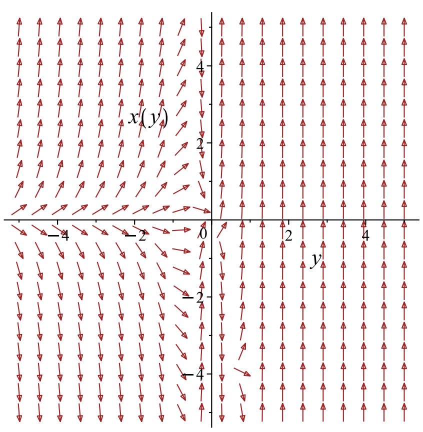
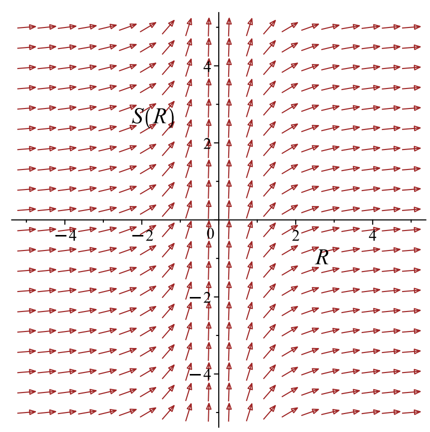
Which simplifies to

$$\frac{e^{-3y}x}{y^2} = -\frac{2}{y} + c_1$$

Which gives

$$x = e^{3y}y(c_1 y - 2)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in y, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dy} = \frac{2y\,e^{3y}+3yx+2x}{y}$ 	$R = y$ $S = \frac{e^{-3y}x}{y^2}$	$\frac{dS}{dR} = \frac{2}{R^2}$ 

Summary
The solution(s) found are the following

$$x = e^{3y} y(c_1 y - 2) \tag{1}$$

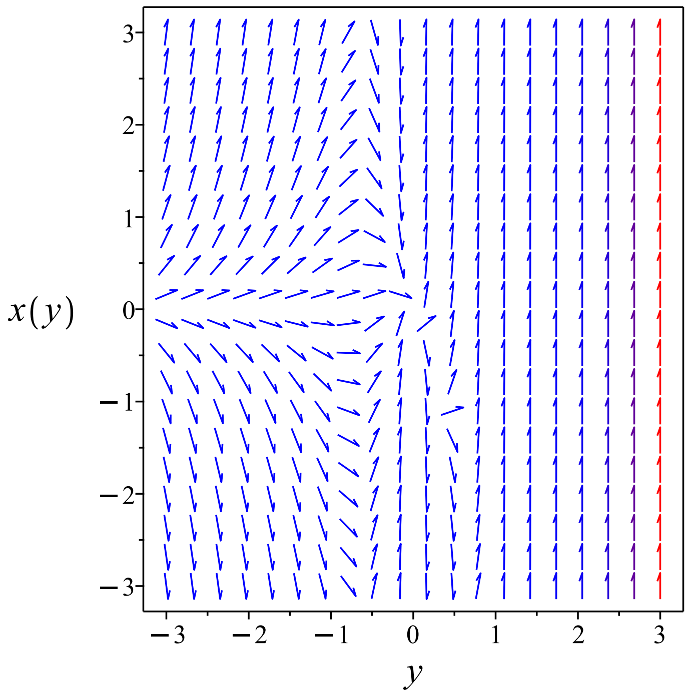


Figure 327: Slope field plot

Verification of solutions

$$x = e^{3y} y(c_1 y - 2)$$

Verified OK.

6.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y, x)\,\mathrm{d}y + N(y, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y)\,\mathrm{d}x &= \left(2y\,\mathrm{e}^{3y} + x(2+3y)\right)\mathrm{d}y \\ \left(-2y\,\mathrm{e}^{3y} - x(2+3y)\right)\mathrm{d}y + (y)\,\mathrm{d}x &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(y, x) &= -2y\,\mathrm{e}^{3y} - x(2+3y) \\ N(y, x) &= y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-2y\,\mathrm{e}^{3y} - x(2+3y)\right) \\ &= -3y - 2\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial y} &= \frac{\partial}{\partial y}(y) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) \\
&= \frac{1}{y}((-3y-2)-(1)) \\
&= \frac{-3y-3}{y}
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}y} \\
&= e^{\int \frac{-3y-3}{y} \,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3y-3\ln(y)} \\
&= \frac{\mathrm{e}^{-3y}}{y^3}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{-3y}}{y^3}\left(-2y\,\mathrm{e}^{3y} - x(2+3y)\right) \\
&= \frac{x(-3y-2)\,\mathrm{e}^{-3y} - 2y}{y^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{-3y}}{y^3}(y) \\
&= \frac{\mathrm{e}^{-3y}}{y^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}y} = 0$$
$$\left(\frac{x(-3y-2)\,\mathrm{e}^{-3y}-2y}{y^3}\right) + \left(\frac{\mathrm{e}^{-3y}}{y^2}\right)\frac{\mathrm{d}x}{\mathrm{d}y} = 0$$

The following equations are now set up to solve for the function $\phi(y, x)$

$$\frac{\partial \phi}{\partial y} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \overline{M}\,\mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \frac{x(-3y-2)\,\mathrm{e}^{-3y}-2y}{y^3}\,\mathrm{d}y$$
$$\phi = \frac{\mathrm{e}^{-3y}x+2y}{y^2} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both y and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \frac{\mathrm{e}^{-3y}}{y^2} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{\mathrm{e}^{-3y}}{y^2}$. Therefore equation (4) becomes

$$\frac{\mathrm{e}^{-3y}}{y^2} = \frac{\mathrm{e}^{-3y}}{y^2} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{e^{-3y}x + 2y}{y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{e^{-3y}x + 2y}{y^2}$$

The solution becomes

$$x = e^{3y}y(c_1y - 2)$$

Summary

The solution(s) found are the following

$$x = e^{3y}y(c_1y - 2) \tag{1}$$

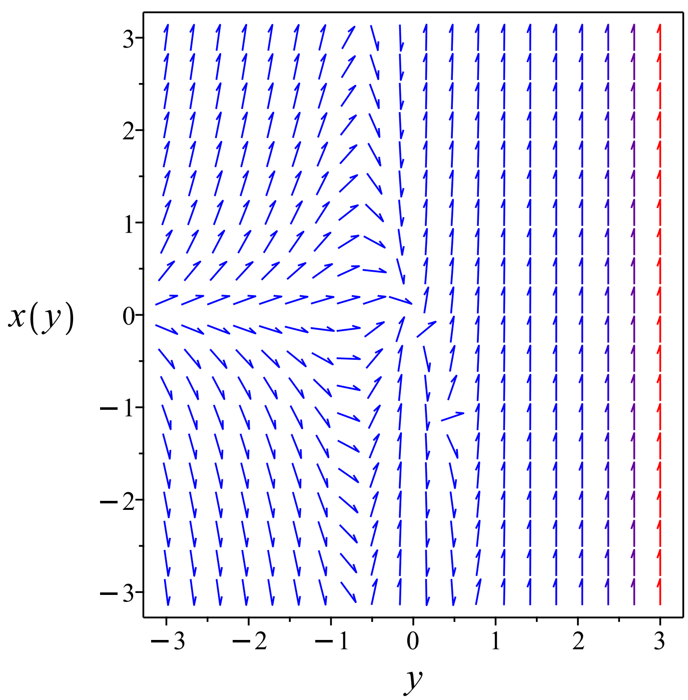


Figure 328: Slope field plot

Verification of solutions

$$x = e^{3y} y(c_1 y - 2)$$

Verified OK.

6.18.4 Maple step by step solution

Let's solve

$yx' - x(2 + 3y) = 2y\, e^{3y}$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = \frac{(2+3y)x}{y} + 2\, e^{3y}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' - \frac{(2+3y)x}{y} = 2\, e^{3y}$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

  $\mu(y) \left(x' - \frac{(2+3y)x}{y}\right) = 2\mu(y)\, e^{3y}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\, x)$

  $\mu(y) \left(x' - \frac{(2+3y)x}{y}\right) = \mu'(y)\, x + \mu(y)\, x'$

- Isolate $\mu'(y)$

  $\mu'(y) = -\frac{\mu(y)(2+3y)}{y}$

- Solve to find the integrating factor

  $\mu(y) = \frac{e^{-3y}}{y^2}$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}(\mu(y)\, x)\right) dy = \int 2\mu(y)\, e^{3y} dy + c_1$

- Evaluate the integral on the lhs

  $\mu(y)\, x = \int 2\mu(y)\, e^{3y} dy + c_1$

- Solve for x

  $x = \frac{\int 2\mu(y) e^{3y} dy + c_1}{\mu(y)}$

- Substitute $\mu(y) = \frac{e^{-3y}}{y^2}$

$$x = \frac{y^2 \left(\int \frac{2\,e^{3y} e^{-3y}}{y^2} dy + c_1\right)}{e^{-3y}}$$

- Evaluate the integrals on the rhs

$$x = \frac{y^2 \left(-\frac{2}{y} + c_1\right)}{e^{-3y}}$$

- Simplify

$$x = e^{3y} y (c_1 y - 2)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(y*diff(x(y),y)=2*y*exp(3*y)+x(y)*(3*y+2),x(y), singsol=all)
```

$$x(y) = (c_1 y - 2)\, y\, e^{3y}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 18

```
DSolve[y*x'[y]==2*y*Exp[3*y]+x[y]*(3*y+2),x[y],y,IncludeSingularSolutions -> True]
```

$$x(y) \to e^{3y} y(-2 + c_1 y)$$

6.19 problem 19

Internal problem ID [2005]
Internal file name [`OUTPUT/2005_Sunday_February_25_2024_06_44_29_AM_23231500/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x)*G(y)
   ,0]`]]
```

$$\boxed{y^2 + (2yx - y^2)\, y' = -1}$$

With initial conditions

$$[y(0) = -1]$$

6.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y^2+1}{y(-2x+y)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=-1$ is

$$\left\{x < -\frac{1}{2} \lor -\frac{1}{2} < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^2+1}{y\,(-2x+y)}\right)\\ &= \frac{2}{-2x+y} - \frac{y^2+1}{y^2\,(-2x+y)} - \frac{y^2+1}{y\,(-2x+y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\left\{x < -\frac{1}{2} \lor -\frac{1}{2} < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

6.19.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(2yx - y^2\right)\mathrm{d}y &= \left(-y^2 - 1\right)\mathrm{d}x \\
\left(y^2 + 1\right)\mathrm{d}x + \left(2yx - y^2\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= y^2 + 1 \\
N(x, y) &= 2yx - y^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 + 1\right) \\
&= 2y
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2yx - y^2\right) \\
&= 2y
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int y^2 + 1 \mathrm{d}x \\ \phi &= \left(y^2 + 1\right) x + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2yx + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2yx - y^2$. Therefore equation (4) becomes

$$2yx - y^2 = 2yx + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^2$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) \mathrm{d}y &= \int \left(-y^2\right) \mathrm{d}y \\ f(y) &= -\frac{y^3}{3} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(y^2 + 1\right) x - \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(y^2 + 1\right) x - \frac{y^3}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{3} = c_1$$

$$c_1 = \frac{1}{3}$$

Substituting c_1 found above in the general solution gives

$$(y^2 + 1) x - \frac{y^3}{3} = \frac{1}{3}$$

Summary

The solution(s) found are the following

$$xy^2 + x - \frac{y^3}{3} = \frac{1}{3} \tag{1}$$

Verification of solutions

$$xy^2 + x - \frac{y^3}{3} = \frac{1}{3}$$

Verified OK.

6.19.3 Maple step by step solution

Let's solve

$[y^2 + (2yx - y^2) y' = -1, y(0) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2y = 2y$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (y^2+1)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=(y^2+1)\,x+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $2yx-y^2=2yx+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-y^2$

- Solve for $f_1(y)$

  $f_1(y)=-\frac{y^3}{3}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=(y^2+1)\,x-\frac{y^3}{3}$

- Substitute $F(x,y)$ into the solution of the ODE

  $(y^2+1)\,x-\frac{y^3}{3}=c_1$

- Solve for y

  $$\left\{y=\frac{\left(-12c_1+12x+8x^3+4\sqrt{-12c_1x^3+12x^4+9c_1^2-18c_1x+9x^2}\right)^{\frac{1}{3}}}{2}+\frac{2x^2}{\left(-12c_1+12x+8x^3+4\sqrt{-12c_1x^3+12x^4+9c_1^2-18c_1x+9x^2}\right.}\right.$$

- Use initial condition $y(0)=-1$

  $$-1=\frac{\left(-12c_1+4\sqrt{9}\,\sqrt{c_1^2}\right)^{\frac{1}{3}}}{2}$$

- Solve for c_1

  $c_1=\mathit{RootOf}\left(2+12^{\frac{1}{3}}\left(\mathrm{csgn}(\_Z)\_Z-\_Z\right)^{\frac{1}{3}}\right)$

- Substitute $c_1 = RootOf\left(2+12^{\frac{1}{3}}(\mathrm{csgn}(\_Z)\_Z-\_Z)^{\frac{1}{3}}\right)$ into general solution and simplify

$$y = \frac{4x^2+2x\left(-12RootOf\left(2+12^{\frac{1}{3}}\left(\_Z\left(\mathrm{csgn}\left(\_Z\right)-1\right)\right)^{\frac{1}{3}}\right)+12x+8x^3+4\sqrt{3}\sqrt{\left(x-RootOf\left(2+12^{\frac{1}{3}}\left(\_Z\left(\mathrm{csgn}\left(\_Z\right)-1\right)\right)^{\frac{1}{3}}\right.\right.}}{2\left(-12RootOf\left(2+12^{\frac{1}{3}}\left(\_Z\right.\right.\right.}$$

- Use initial condition $y(0)=-1$

$$-1 = -\frac{\left(-12c_1+4\sqrt{9}\sqrt{c_1^2}\right)^{\frac{1}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\left(-12c_1+4\sqrt{9}\sqrt{c_1^2}\right)^{\frac{1}{3}}}{4}$$

- Solve for c_1

$$c_1 = RootOf\left(\mathrm{I}\sqrt{3}\,12^{\frac{1}{3}}(\mathrm{csgn}(\_Z)\_Z-\_Z)^{\frac{1}{3}}+12^{\frac{1}{3}}(\mathrm{csgn}(\_Z)\_Z-\_Z)^{\frac{1}{3}}-4\right)$$

- Substitute $c_1 = RootOf\left(\mathrm{I}\sqrt{3}\,12^{\frac{1}{3}}(\mathrm{csgn}(\_Z)\_Z-\_Z)^{\frac{1}{3}}+12^{\frac{1}{3}}(\mathrm{csgn}(\_Z)\_Z-\_Z)^{\frac{1}{3}}-4\right)$ into g

$$y = \frac{\left(-\mathrm{I}\sqrt{3}-1\right)\left(-12RootOf\left(\mathrm{I}\,2^{\frac{2}{3}}3^{\frac{5}{6}}\left(\_Z\left(\mathrm{csgn}\left(\_Z\right)-1\right)\right)^{\frac{1}{3}}+2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\_Z\left(\mathrm{csgn}\left(\_Z\right)-1\right)\right)^{\frac{1}{3}}-4\right)+12x+8x^3+4\sqrt{3}\sqrt{\left(x-R\right.}}{}$$

- Use initial condition $y(0)=-1$

$$-1 = -\frac{\left(-12c_1+4\sqrt{9}\sqrt{c_1^2}\right)^{\frac{1}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\left(-12c_1+4\sqrt{9}\sqrt{c_1^2}\right)^{\frac{1}{3}}}{4}$$

- Solve for c_1

$$c_1 = RootOf\left(\mathrm{I}\sqrt{3}\,12^{\frac{1}{3}}(\mathrm{csgn}(\_Z)\_Z-\_Z)^{\frac{1}{3}}-12^{\frac{1}{3}}(\mathrm{csgn}(\_Z)\_Z-\_Z)^{\frac{1}{3}}+4\right)$$

- Substitute $c_1 = RootOf\left(\mathrm{I}\sqrt{3}\,12^{\frac{1}{3}}(\mathrm{csgn}(\_Z)\_Z-\_Z)^{\frac{1}{3}}-12^{\frac{1}{3}}(\mathrm{csgn}(\_Z)\_Z-\_Z)^{\frac{1}{3}}+4\right)$ into g

$$y = \frac{\left(-12RootOf\left(-\mathrm{I}\,2^{\frac{2}{3}}3^{\frac{5}{6}}\left(\_Z\left(\mathrm{csgn}\left(\_Z\right)-1\right)\right)^{\frac{1}{3}}+2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\_Z\left(\mathrm{csgn}\left(\_Z\right)-1\right)\right)^{\frac{1}{3}}-4\right)+12x+8x^3+4\sqrt{3}\sqrt{\left(x-RootOf\left(-\right.\right.}}{}$$

- Solutions to the IVP

$$\left\{ y = \frac{4x^2+2x\left(-12 RootOf\left(2+12^{\frac{1}{3}}\left(\_Z\left(\operatorname{csgn}\left(\_Z\right)-1\right)\right)^{\frac{1}{3}}\right)+12x+8x^3+4\sqrt{3}\sqrt{\left(x-RootOf\left(2+12^{\frac{1}{3}}\left(\_Z\left(\operatorname{csgn}\left(\_Z\right)-1\right)\right.\right.\right.}}}{2\left(-12 RootOf\left(2+12^{\frac{1}{3}}\left(\right.\right.\right.}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.313 (sec). Leaf size: 137

```
dsolve([(y(x)^2+1)+(2*x*y(x)-y(x)^2)*diff(y(x),x)=0,y(0) = -1],y(x), singsol=all)
```

$$y(x) = \frac{\left(-4+12x+8x^3+4\sqrt{12x^4-4x^3+9x^2-6x+1}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4} - \frac{\left(i\sqrt{3}\,x+x-\left(-4+12x+8x^3+4\sqrt{12x^4-4x^3+9x^2-6x+1}\right)^{\frac{1}{3}}\right)x}{\left(-4+12x+8x^3+4\sqrt{12x^4-4x^3+9x^2-6x+1}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 4.803 (sec). Leaf size: 100

```
DSolve[{(y[x]^2+1)+(2*x*y[x]-y[x]^2)*y'[x]==0,{y[0]==-1}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{\sqrt[3]{2}x^2}{\sqrt[3]{-2x^3+\sqrt{12x^4-4x^3+9x^2-6x+1}-3x+1}} - \frac{\sqrt[3]{-2x^3+\sqrt{12x^4-4x^3+9x^2-6x+1}-3x+1}}{\sqrt[3]{2}} + x$$

6.20 problem 20

Internal problem ID [2006]
Internal file name [`OUTPUT/2006_Sunday_February_25_2024_06_44_30_AM_53134904/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cot(x) = \sec(x)}$$

6.20.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot(x)$$
$$q(x) = \sec(x)$$

Hence the ode is

$$y' + y\cot(x) = \sec(x)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \cot(x)dx} \\
&= \sin(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\sec(x)) \\
\frac{d}{dx}(\sin(x)\, y) &= (\sin(x))(\sec(x)) \\
d(\sin(x)\, y) &= \tan(x)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sin(x)\, y &= \int \tan(x)\, dx \\
\sin(x)\, y &= -\ln(\cos(x)) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$
y = -\csc(x)\ln(\cos(x)) + \csc(x)\, c_1
$$

which simplifies to

$$
y = \csc(x)(-\ln(\cos(x)) + c_1)
$$

Summary

The solution(s) found are the following

$$
y = \csc(x)(-\ln(\cos(x)) + c_1) \tag{1}
$$

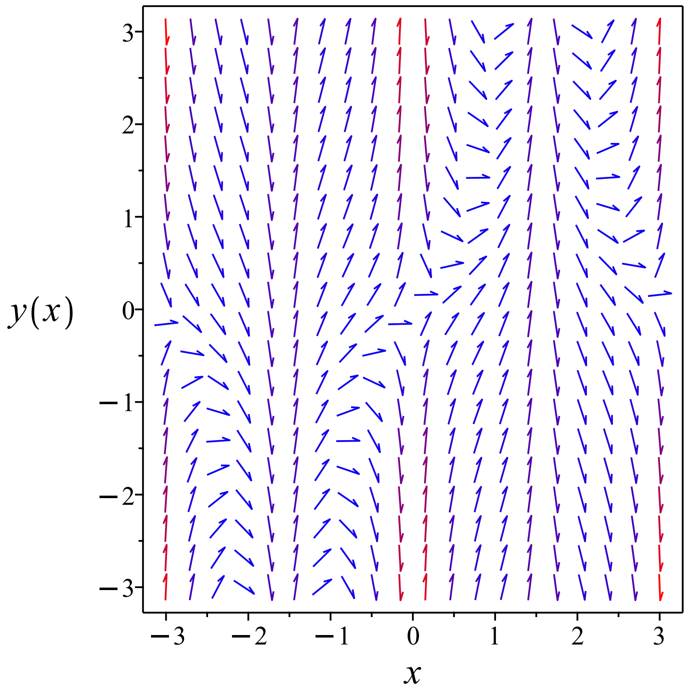


Figure 329: Slope field plot

Verification of solutions

$$y = \csc(x)\left(-\ln(\cos(x)) + c_1\right)$$

Verified OK.

6.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y\cot(x) + \sec(x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 189: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sin(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y\cot(x) + \sec(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\, y \\ S_y &= \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(\cos(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(x) y = -\ln(\cos(x)) + c_1$$

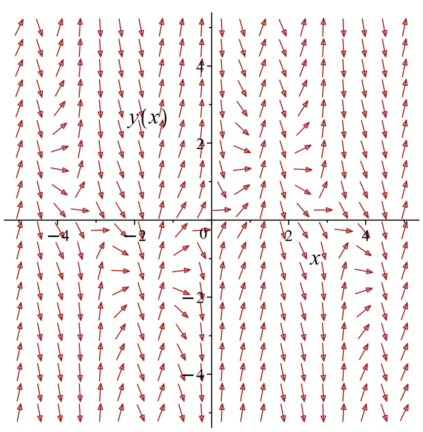
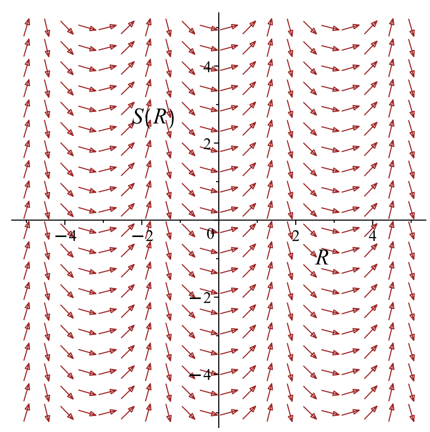
Which simplifies to

$$\sin(x) y = -\ln(\cos(x)) + c_1$$

Which gives

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\cot(x) + \sec(x)$ 	$R = x$ $S = \sin(x) y$	$\frac{dS}{dR} = \tan(R)$ 

Summary

The solution(s) found are the following

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)} \tag{1}$$

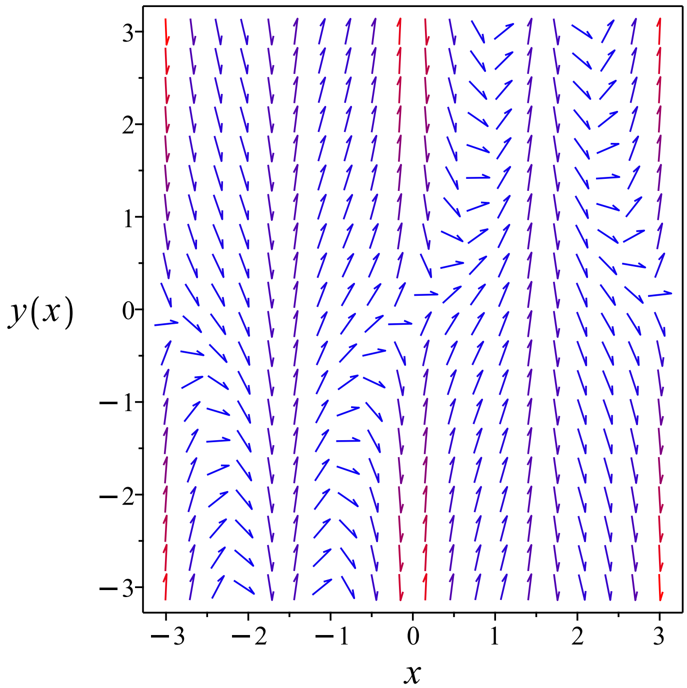


Figure 330: Slope field plot

Verification of solutions

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)}$$

Verified OK.

6.20.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y \cot(x) + \sec(x))\,\mathrm{d}x$$
$$(y \cot(x) - \sec(x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y \cot(x) - \sec(x)$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y \cot(x) - \sec(x)) \\ &= \cot(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cot(x)) - (0)) \\ &= \cot(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cot(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\sin(x))} \\ &= \sin(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sin(x)(y \cot(x) - \sec(x)) \\ &= \cos(x) y - \tan(x) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sin(x)(1) \\ &= \sin(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\cos(x) y - \tan(x)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \cos(x) y - \tan(x) \mathrm{d}x$$
$$\phi = \sin(x) y + \ln(\cos(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$\sin(x) = \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x) y + \ln(\cos(x)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x) y + \ln(\cos(x))$$

The solution becomes

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)} \tag{1}$$

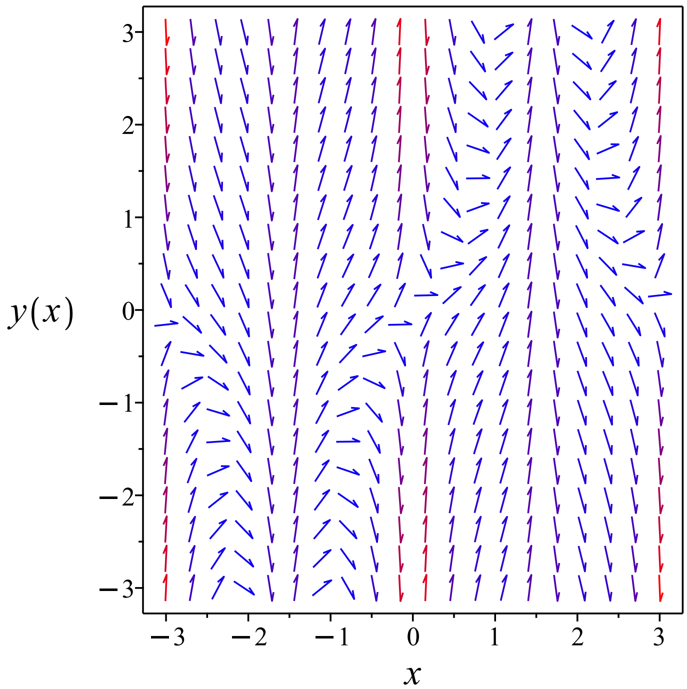


Figure 331: Slope field plot

Verification of solutions

$$y = -\frac{\ln(\cos(x)) - c_1}{\sin(x)}$$

Verified OK.

6.20.4 Maple step by step solution

Let's solve

$y' + y\cot(x) = \sec(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -y \cot(x) + \sec(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + y \cot(x) = \sec(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x) (y' + y \cot(x)) = \mu(x) \sec(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

$\mu(x) (y' + y \cot(x)) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

$\mu'(x) = \mu(x) \cot(x)$

- Solve to find the integrating factor

$\mu(x) = \sin(x)$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) \sec(x) dx + c_1$

- Evaluate the integral on the lhs

$\mu(x) y = \int \mu(x) \sec(x) dx + c_1$

- Solve for y

$y = \frac{\int \mu(x) \sec(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sin(x)$

$y = \frac{\int \sec(x) \sin(x) dx + c_1}{\sin(x)}$

- Evaluate the integrals on the rhs

$y = \frac{\ln(\sec(x)) + c_1}{\sin(x)}$

- Simplify

$y = \csc(x) (\ln(\sec(x)) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+(y(x)*cot(x)-sec(x))=0,y(x), singsol=all)
```

$$y(x)=\csc(x)(-\ln(\cos(x))+c_1)$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 16

```
DSolve[y'[x]+(y[x]*Cot[x]-Sec[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\csc(x)(-\log(\cos(x))+c_1)$$

6.21 problem 21

Internal problem ID [2007]
Internal file name [`OUTPUT/2007_Sunday_February_25_2024_06_44_31_AM_68684515/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y + y^3 + 4(x y^2 - 1) y' = 0}$$

With initial conditions

$$[y(0) = 1]$$

6.21.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{y(y^2 + 1)}{4 (x y^2 - 1)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y(y^2+1)}{4\,(x\,y^2-1)}\right)\\ &= -\frac{y^2+1}{4\,(x\,y^2-1)} - \frac{y^2}{2\,(x\,y^2-1)} + \frac{y^2(y^2+1)\,x}{2\,(x\,y^2-1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 1 \vee 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

6.21.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(4x\,y^2-4\right)\mathrm{d}y &= \left(-y^3-y\right)\mathrm{d}x\\ \left(y^3+y\right)\mathrm{d}x + \left(4x\,y^2-4\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y^3+y\\ N(x,y) &= 4x\,y^2-4\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^3+y\right)\\ &= 3y^2+1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(4x\,y^2-4\right)\\ &= 4y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{4x\,y^2-4}\left(\left(3y^2+1\right)-\left(4y^2\right)\right)\\ &= \frac{-y^2+1}{4x\,y^2-4}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^3+y}\left(\left(4y^2\right) - \left(3y^2+1\right)\right) \\
&= \frac{y^2-1}{y^3+y}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{y^2-1}{y^3+y} \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\ln(y)+\ln(y^2+1)} \\
&= \frac{y^2+1}{y}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{y^2+1}{y}\left(y^3+y\right) \\
&= \left(y^2+1\right)^2
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{y^2+1}{y}\left(4x\,y^2-4\right) \\
&= \frac{4(y^2+1)\,(x\,y^2-1)}{y}
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(y^2+1\right)^2\right) + \left(\frac{4(y^2+1)\,(x\,y^2-1)}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\overline{M}\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\left(y^{2}+1\right)^{2}\mathrm{d}x$$
$$\phi=\left(y^{2}+1\right)^{2}x+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=4\left(y^{2}+1\right)xy+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{4(y^{2}+1)\,(x\,y^{2}-1)}{y}$. Therefore equation (4) becomes

$$\frac{4(y^{2}+1)\,(x\,y^{2}-1)}{y}=4\left(y^{2}+1\right)xy+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-\frac{4(y^{2}+1)}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int\left(\frac{-4y^{2}-4}{y}\right)\mathrm{d}y$$
$$f(y)=-2y^{2}-4\ln(y)+c_{1}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=\left(y^{2}+1\right)^{2}x-2y^{2}-4\ln(y)+c_{1}$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y^2+1)^2 x - 2y^2 - 4\ln(y)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$(y^2+1)^2 x - 2y^2 - 4\ln(y) = -2$$

Summary

The solution(s) found are the following

$$(1+y^2)^2 x - 2y^2 - 4\ln(y) = -2 \tag{1}$$

Verification of solutions

$$(1+y^2)^2 x - 2y^2 - 4\ln(y) = -2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 34

```
dsolve([(y(x)+y(x)^3)+4*(x*y(x)^2-1)*diff(y(x),x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = e^{\operatorname{RootOf}\left(-x\,e^{4\_Z}-2x\,e^{2\_Z}+2\,e^{2\_Z}+4\_Z-x-2\right)}$$

✓ Solution by Mathematica

Time used: 0.207 (sec). Leaf size: 37

```
DSolve[{(y[x]+y[x]^3)+4*(x*y[x]^2-1)*y'[x]==0,{y[0]==1}},y[x],x,IncludeSingularSolutions ->
```

$$\text{Solve}\left[x=\frac{2y(x)^2+4\log(y(x))}{(y(x)^2+1)^2}-\frac{2}{(y(x)^2+1)^2},y(x)\right]$$

6.22 problem 22

Internal problem ID [2008]
Internal file name [`OUTPUT/2008_Sunday_February_25_2024_06_44_32_AM_70263782/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2y - yx + y'x = 3}$$

With initial conditions

$$[y(1) = 1]$$

6.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-2+x}{x}$$
$$q(x) = \frac{3}{x}$$

Hence the ode is

$$y' - \frac{(-2+x)\,y}{x} = \frac{3}{x}$$

The domain of $p(x) = -\frac{-2+x}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{3}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

6.22.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-2+x}{x}dx} \\ &= \mathrm{e}^{-x+2\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = x^2 \mathrm{e}^{-x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{3}{x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x}x^2 y\right) &= \left(x^2\mathrm{e}^{-x}\right)\left(\frac{3}{x}\right) \\ \mathrm{d}\left(\mathrm{e}^{-x}x^2 y\right) &= \left(3x\,\mathrm{e}^{-x}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-x}x^2 y &= \int 3x\,\mathrm{e}^{-x}\,\mathrm{d}x \\ \mathrm{e}^{-x}x^2 y &= -3(x+1)\,\mathrm{e}^{-x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2\mathrm{e}^{-x}$ results in

$$y = -\frac{3\,\mathrm{e}^{x}(x+1)\,\mathrm{e}^{-x}}{x^2} + \frac{\mathrm{e}^{x}c_1}{x^2}$$

which simplifies to

$$y = \frac{c_1 e^x - 3x - 3}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e c_1 - 6$$

$$c_1 = 7\,e^{-1}$$

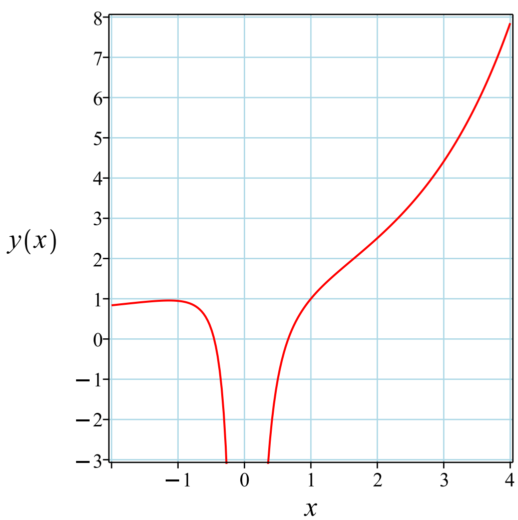
Substituting c_1 found above in the general solution gives

$$y = \frac{7\,e^{x-1} - 3x - 3}{x^2}$$

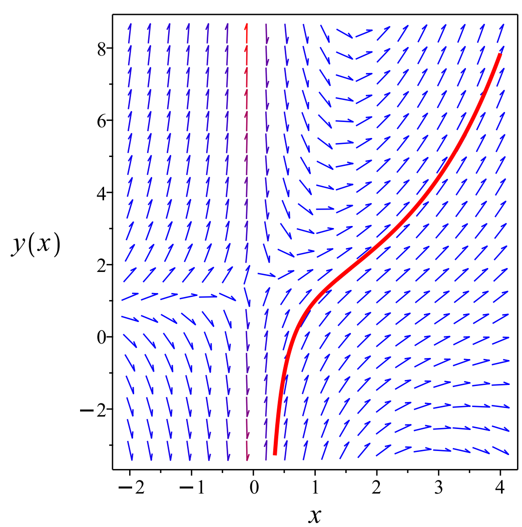
Summary

The solution(s) found are the following

$$y = \frac{7\,e^{x-1} - 3x - 3}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{7\,e^{x-1} - 3x - 3}{x^2}$$

Verified OK.

6.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{yx - 2y + 3}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 192: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{x-2\ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{x-2\ln(x)}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{-x} x^2 y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{yx - 2y + 3}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\mathrm{e}^{-x} y x (-2 + x) \\
S_y &= x^2 \mathrm{e}^{-x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3x\,\mathrm{e}^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R\,\mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -3(R+1)\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x}x^2y = -3(x+1)\,\mathrm{e}^{-x} + c_1$$

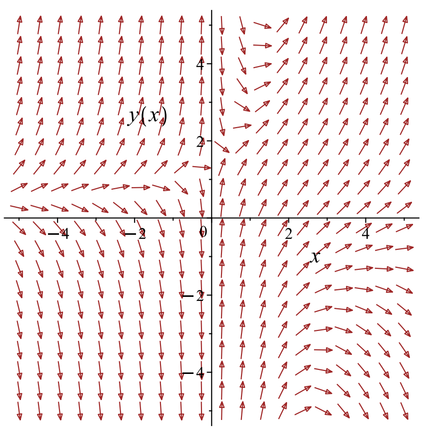
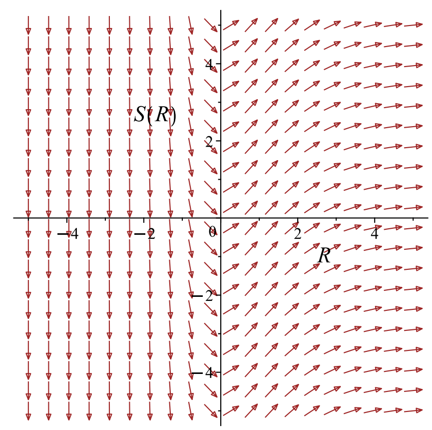
Which simplifies to

$$\mathrm{e}^{-x}x^2y = -3(x+1)\,\mathrm{e}^{-x} + c_1$$

Which gives

$$y = -\frac{(3x\,\mathrm{e}^{-x} + 3\,\mathrm{e}^{-x} - c_1)\,\mathrm{e}^{x}}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{yx-2y+3}{x}$ 	$R = x$ $S = \mathrm{e}^{-x} x^2 y$	$\frac{dS}{dR} = 3R\,\mathrm{e}^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e} c_1 - 6$$

$$c_1 = 7\,\mathrm{e}^{-1}$$

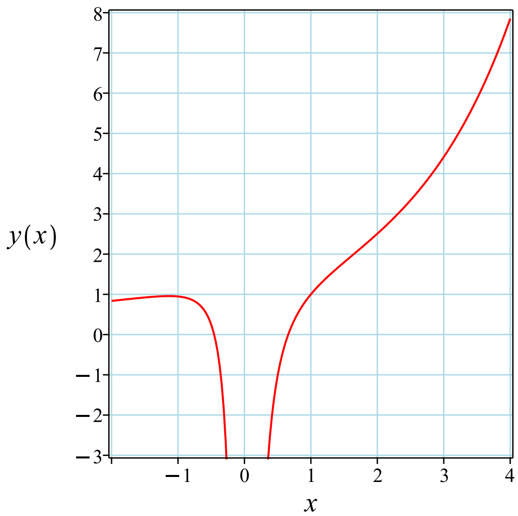
Substituting c_1 found above in the general solution gives

$$y = \frac{-3x\,\mathrm{e}^{-x}\mathrm{e}^{x} - 3\,\mathrm{e}^{-x}\mathrm{e}^{x} + 7\,\mathrm{e}^{x}\mathrm{e}^{-1}}{x^2}$$

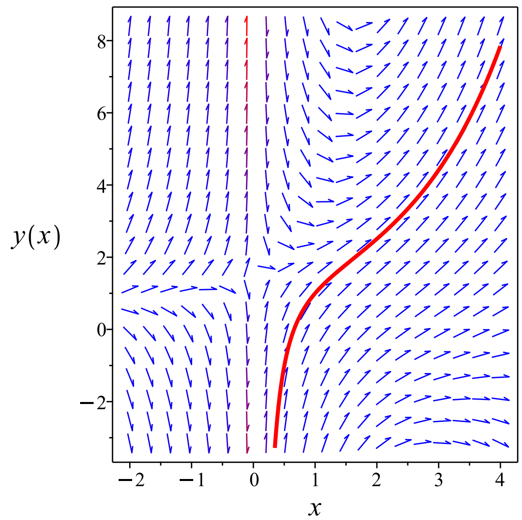
Summary

The solution(s) found are the following

$$y = \frac{-3x\,\mathrm{e}^{-x}\mathrm{e}^{x} - 3\,\mathrm{e}^{-x}\mathrm{e}^{x} + 7\,\mathrm{e}^{x}\mathrm{e}^{-1}}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-3x\,\mathrm{e}^{-x}\mathrm{e}^{x} - 3\,\mathrm{e}^{-x}\mathrm{e}^{x} + 7\,\mathrm{e}^{x}\mathrm{e}^{-1}}{x^2}$$

Verified OK.

6.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= (yx-2y+3)\,\mathrm{d}x\\ (-yx+2y-3)\,\mathrm{d}x+(x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -yx+2y-3\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-yx+2y-3)\\ &= 2-x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((2-x)-(1))\\ &= \frac{1-x}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1-x}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)-x} \\
&= x \, \mathrm{e}^{-x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x \, \mathrm{e}^{-x}(-yx + 2y - 3) \\
&= -x \, \mathrm{e}^{-x}(yx - 2y + 3)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x \, \mathrm{e}^{-x}(x) \\
&= x^2 \mathrm{e}^{-x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-x \, \mathrm{e}^{-x}(yx - 2y + 3)\right) + \left(x^2 \mathrm{e}^{-x}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{e}^{-x}(yx - 2y + 3) \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \left(y \, x^2 + 3x + 3\right) \mathrm{e}^{-x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$x^2 \mathrm{e}^{-x} = x^2 \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(y\, x^2 + 3x + 3\right) \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(y\, x^2 + 3x + 3\right) \mathrm{e}^{-x}$$

The solution becomes

$$y = -\frac{\left(3x\, \mathrm{e}^{-x} + 3\, \mathrm{e}^{-x} - c_1\right) \mathrm{e}^{x}}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e} c_1 - 6$$

$$c_1 = 7\, \mathrm{e}^{-1}$$

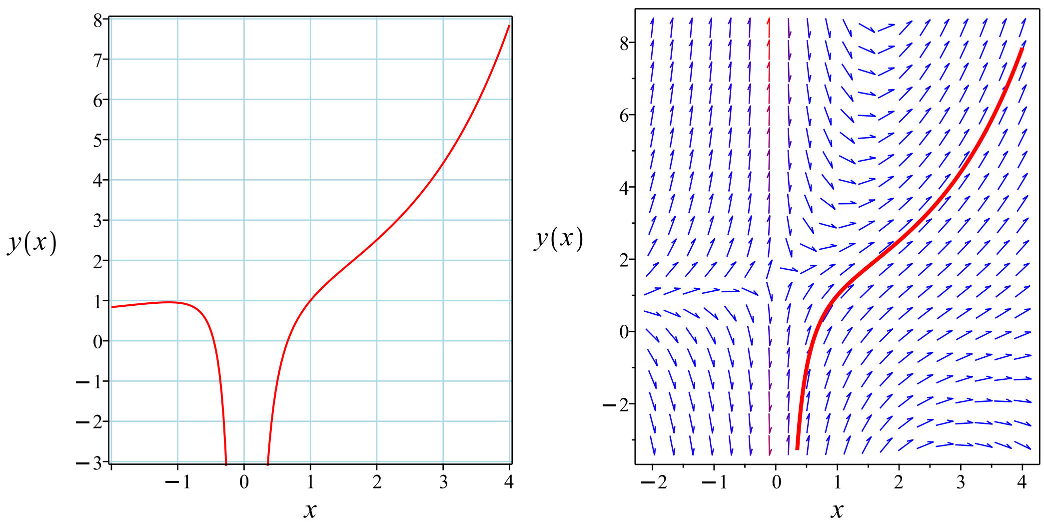
Substituting c_1 found above in the general solution gives

$$y = \frac{-3x\, \mathrm{e}^{-x} \mathrm{e}^{x} - 3\, \mathrm{e}^{-x} \mathrm{e}^{x} + 7\, \mathrm{e}^{x} \mathrm{e}^{-1}}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-3x\,\mathrm{e}^{-x}\mathrm{e}^{x} - 3\,\mathrm{e}^{-x}\mathrm{e}^{x} + 7\,\mathrm{e}^{x}\mathrm{e}^{-1}}{x^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{-3x\,\mathrm{e}^{-x}\mathrm{e}^{x} - 3\,\mathrm{e}^{-x}\mathrm{e}^{x} + 7\,\mathrm{e}^{x}\mathrm{e}^{-1}}{x^2}$$

Verified OK.

6.22.5 Maple step by step solution

Let's solve

$[2y - yx + y'x = 3, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{(-2+x)y}{x} + \frac{3}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{(-2+x)y}{x} = \frac{3}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'-\frac{(-2+x)y}{x}\right)=\frac{3\mu(x)}{x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-\frac{(-2+x)y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)(-2+x)}{x}$
- Solve to find the integrating factor

  $\mu(x)=x^2\mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{3\mu(x)}{x}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{3\mu(x)}{x}dx+c_1$
- Solve for y

  $y=\frac{\int\frac{3\mu(x)}{x}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=x^2\mathrm{e}^{-x}$

  $y=\frac{\int 3x\,\mathrm{e}^{-x}dx+c_1}{x^2\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y=\frac{-3(x+1)\mathrm{e}^{-x}+c_1}{x^2\mathrm{e}^{-x}}$
- Simplify

  $y=\frac{c_1\mathrm{e}^{x}-3x-3}{x^2}$
- Use initial condition $y(1)=1$

  $1=\mathrm{e}c_1-6$
- Solve for c_1

  $c_1=\frac{7}{\mathrm{e}}$
- Substitute $c_1=\frac{7}{\mathrm{e}}$ into general solution and simplify

  $y=\frac{7\,\mathrm{e}^{x-1}-3x-3}{x^2}$
- Solution to the IVP

  $y=\frac{7\,\mathrm{e}^{x-1}-3x-3}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve([(2*y(x)-x*y(x)-3)+x*diff(y(x),x)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{-3x - 3 + 7\,\mathrm{e}^{x-1}}{x^2}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 24

```
DSolve[{(2*y[x]-x*y[x]-3)+x*y'[x]==0,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{7e^x - 3e(x+1)}{ex^2}$$

6.23 problem 23

Internal problem ID [2009]
Internal file name [`OUTPUT/2009_Sunday_February_25_2024_06_44_35_AM_6972371/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y + 2(x - 2y^2)\, y' = 0}$$

With initial conditions

$$[y(2) = -1]$$

6.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y}{4y^2 - 2x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -1$ is

$$\{x < 2 \lor 2 < x\}$$

But the point $x_0 = 2$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

6.23.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{y}{4y^2 - 2x} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
b_2 + \frac{y(b_3 - a_2)}{4y^2 - 2x} - \frac{y^2 a_3}{4\left(2y^2 - x\right)^2} - \frac{y(x a_2 + y a_3 + a_1)}{2\left(2y^2 - x\right)^2} \\
- \left(\frac{1}{4y^2 - 2x} - \frac{2y^2}{\left(2y^2 - x\right)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{16y^4 b_2 - 12x\, y^2 b_2 - 4y^3 a_2 + 8y^3 b_3 + 6x^2 b_2 - 3y^2 a_3 + 4y^2 b_1 + 2x b_1 - 2y a_1}{4\left(-2y^2 + x\right)^2} = 0$$

Setting the numerator to zero gives

$$16y^4 b_2 - 12x\, y^2 b_2 - 4y^3 a_2 + 8y^3 b_3 + 6x^2 b_2 - 3y^2 a_3 + 4y^2 b_1 + 2x b_1 - 2y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$16b_2v_2^4 - 4a_2v_2^3 - 12b_2v_1v_2^2 + 8b_3v_2^3 - 3a_3v_2^2 + 4b_1v_2^2 + 6b_2v_1^2 - 2a_1v_2 + 2b_1v_1 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$6b_2v_1^2 - 12b_2v_1v_2^2 + 2b_1v_1 + 16b_2v_2^4 + (-4a_2 + 8b_3)\,v_2^3 + (-3a_3 + 4b_1)\,v_2^2 - 2a_1v_2 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ 2b_1 &= 0 \\ -12b_2 &= 0 \\ 6b_2 &= 0 \\ 16b_2 &= 0 \\ -4a_2 + 8b_3 &= 0 \\ -3a_3 + 4b_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 2x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y}{4y^2 - 2x}\right)(2x) \\
&= \frac{-2y^3 + 2yx}{-2y^2 + x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2y^3 + 2yx}{-2y^2 + x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(y)}{2} + \frac{\ln(y^2 - x)}{4}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{y}{4y^2 - 2x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{-4y^2 + 4x} \\
S_y &= \frac{1}{2y} - \frac{y}{-2y^2 + 2x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

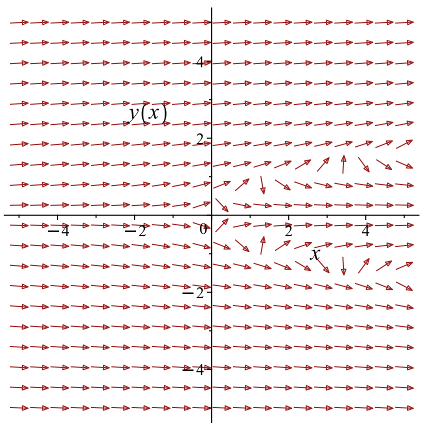
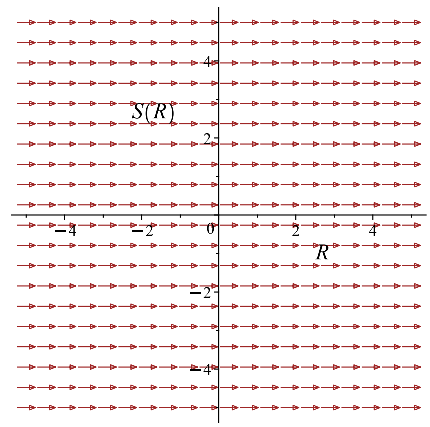
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y)}{2} + \frac{\ln(y^2 - x)}{4} = c_1
$$

Which simplifies to

$$
\frac{\ln(y)}{2} + \frac{\ln(y^2 - x)}{4} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{4y^2-2x}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(y^2-x)}{4}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3i\pi}{4} = c_1$$

$$c_1 = \frac{3i\pi}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(y)}{2} + \frac{\ln(y^2-x)}{4} = \frac{3i\pi}{4}$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{2} + \frac{\ln(y^2-x)}{4} = \frac{3i\pi}{4} \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{2} + \frac{\ln(y^2-x)}{4} = \frac{3i\pi}{4}$$

Verified OK.

6.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-4y^2 + 2x\right)\mathrm{d}y &= (-y)\,\mathrm{d}x\\ (y)\,\mathrm{d}x + \left(-4y^2 + 2x\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\\ N(x,y) &= -4y^2 + 2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-4y^2 + 2x) \\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{-4y^2 + 2x}((1) - (2)) \\ &= -\frac{1}{-4y^2 + 2x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y}((2) - (1)) \\ &= \frac{1}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int \frac{1}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(y)} \\
&= y
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= y(y) \\
&= y^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y\left(-4y^2 + 2x\right) \\
&= 2y\left(-2y^2 + x\right)
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(y^2\right) + \left(2y\left(-2y^2 + x\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int y^2\,\mathrm{d}x \\
\phi &= x\,y^2 + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2yx + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y(-2y^2 + x)$. Therefore equation (4) becomes

$$2y(-2y^2 + x) = 2yx + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -4y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-4y^3)\,\mathrm{d}y$$

$$f(y) = -y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -y^4 + x\,y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -y^4 + x\,y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$-y^4 + x\,y^2 = 1$$

Summary

The solution(s) found are the following

$$y^2(-y^2+x)=1 \tag{1}$$

Verification of solutions

$$y^2(-y^2+x)=1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 41

```
dsolve([y(x)+2*(x-2*y(x)^2)*diff(y(x),x)=0,y(2) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2\sqrt{x^2-4}+2x}}{2}$$
$$y(x) = -\frac{\sqrt{2\sqrt{x^2-4}+2x}}{2}$$

✓ Solution by Mathematica

Time used: 2.237 (sec). Leaf size: 55

```
DSolve[{y[x]+2*(x-2*y[x]^2)*y'[x]==0,{y[2]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{x-\sqrt{x^2-4}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{\sqrt{x^2-4}+x}}{\sqrt{2}}$$

6.24 problem 24

Internal problem ID [2010]
Internal file name [`OUTPUT/2010_Sunday_February_25_2024_06_44_36_AM_85647746/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 10, page 41
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\left(x^2-1\right) y'+4y=-\left(x^2-1\right)^2}$$

With initial conditions

$$[y(0)=-6]$$

6.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{4}{x^2-1}$$
$$q(x)=1-x^2$$

Hence the ode is

$$y' + \frac{4y}{x^2 - 1} = 1 - x^2$$

The domain of $p(x) = \frac{4}{x^2-1}$ is

$$\{-\infty \leq x < -1, -1 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1 - x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.24.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{4}{x^2-1}dx} \\ &= \frac{(1-x^2)^2}{(x+1)^4}\end{aligned}$$

Which simplifies to

$$\mu = \frac{(x-1)^2}{(x+1)^2}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(1-x^2\right) \\ \frac{d}{dx}\left(\frac{(x-1)^2 y}{(x+1)^2}\right) &= \left(\frac{(x-1)^2}{(x+1)^2}\right)\left(1-x^2\right) \\ d\left(\frac{(x-1)^2 y}{(x+1)^2}\right) &= \left(-\frac{(x-1)^3}{x+1}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{(x-1)^2 y}{(x+1)^2} &= \int -\frac{(x-1)^3}{x+1}\, dx \\ \frac{(x-1)^2 y}{(x+1)^2} &= -\frac{x^3}{3} + 2x^2 - 7x + 8\ln(x+1) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{(x-1)^2}{(x+1)^2}$ results in

$$y = \frac{(x+1)^2 \left(-\frac{x^3}{3} + 2x^2 - 7x + 8\ln(x+1)\right)}{(x-1)^2} + \frac{c_1 (x+1)^2}{(x-1)^2}$$

which simplifies to

$$y = -\frac{(x+1)^2 \left(x^3 - 6x^2 - 24\ln(x+1) - 3c_1 + 21x\right)}{3(x-1)^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -6$ in the above solution gives an equation to solve for the constant of integration.

$$-6 = c_1$$

$$c_1 = -6$$

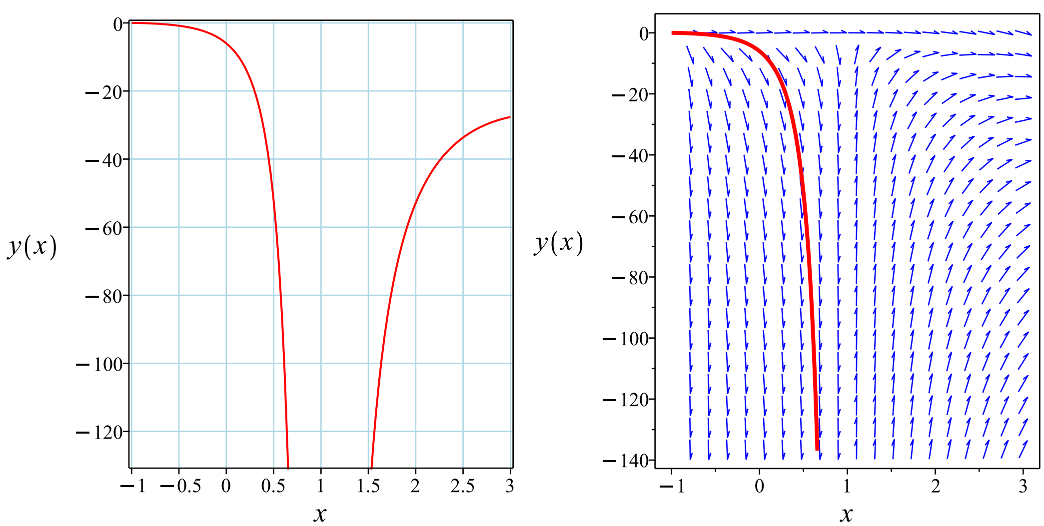
Substituting c_1 found above in the general solution gives

$$y = \frac{-x^5 + 4x^4 + 24\ln(x+1)x^2 - 10x^3 + 48\ln(x+1)x - 54x^2 + 24\ln(x+1) - 57x - 18}{3x^2 - 6x + 3}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^5 + 4x^4 + 24\ln(x+1)x^2 - 10x^3 + 48\ln(x+1)x - 54x^2 + 24\ln(x+1) - 57x - 18}{3x^2 - 6x + 3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-x^5 + 4x^4 + 24\ln(x+1)x^2 - 10x^3 + 48\ln(x+1)x - 54x^2 + 24\ln(x+1) - 57x - 18}{3x^2 - 6x + 3}$$

Verified OK.

6.24.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^4 - 2x^2 + 4y + 1}{x^2 - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 195: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{(x+1)^4}{(1-x^2)^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{(x+1)^4}{(1-x^2)^2}} dy \end{aligned}$$

Which results in

$$S = \frac{(1 - x^2)^2 y}{(x + 1)^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^4 - 2x^2 + 4y + 1}{x^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{4(x - 1) y}{(x + 1)^3} \\ S_y &= \frac{(x - 1)^2}{(x + 1)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{(x - 1)^3}{x + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{(R - 1)^3}{R + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R^3}{3} + 2R^2 - 7R + 8\ln(R+1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x-1)^2 y}{(x+1)^2} = -\frac{x^3}{3} + 2x^2 - 7x + 8\ln(x+1) + c_1$$

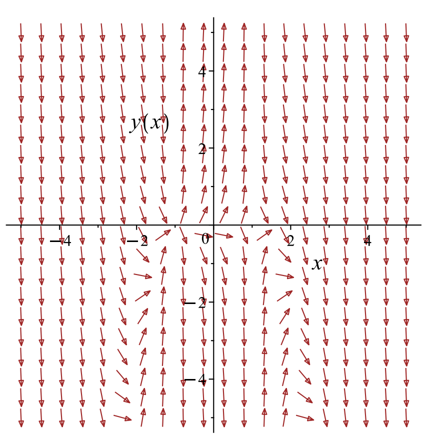
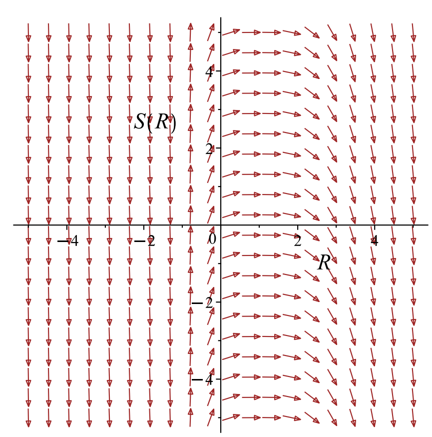
Which simplifies to

$$\frac{(x-1)^2 y}{(x+1)^2} = -\frac{x^3}{3} + 2x^2 - 7x + 8\ln(x+1) + c_1$$

Which gives

$$y = \frac{(x+1)^2 \left(-x^3 + 6x^2 + 24\ln(x+1) + 3c_1 - 21x\right)}{3(x-1)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^4-2x^2+4y+1}{x^2-1}$ 	$R = x$ $S = \frac{(x-1)^2 y}{(x+1)^2}$	$\frac{dS}{dR} = -\frac{(R-1)^3}{R+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -6$ in the above solution gives an equation to solve for the constant of integration.

$$-6 = c_1$$

$$c_1 = -6$$

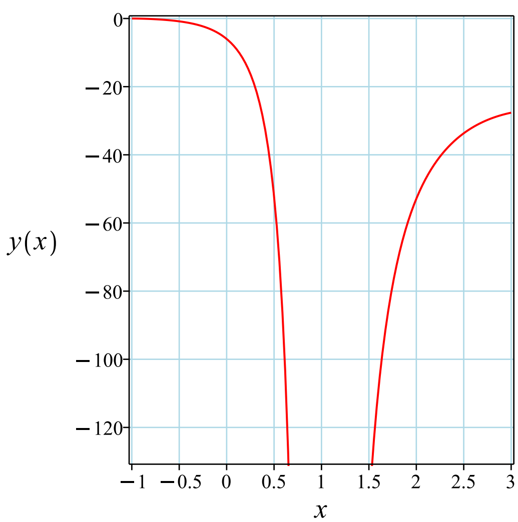
Substituting c_1 found above in the general solution gives

$$y = \frac{-x^5 + 4x^4 + 24\ln(x+1)x^2 - 10x^3 + 48\ln(x+1)x - 54x^2 + 24\ln(x+1) - 57x - 18}{3x^2 - 6x + 3}$$

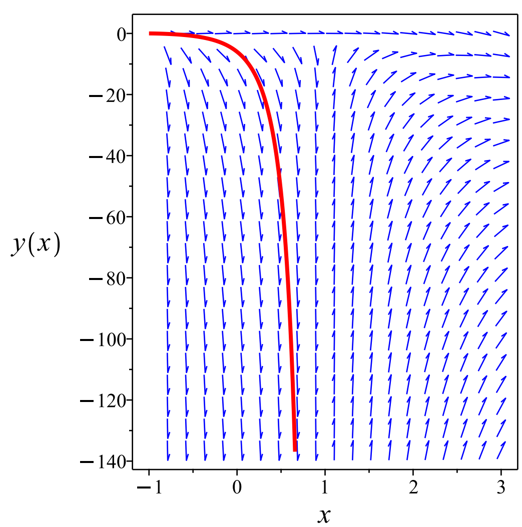
Summary

The solution(s) found are the following

$$y = \frac{-x^5 + 4x^4 + 24\ln(x+1)x^2 - 10x^3 + 48\ln(x+1)x - 54x^2 + 24\ln(x+1) - 57x - 18}{3x^2 - 6x + 3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-x^5 + 4x^4 + 24\ln(x+1)x^2 - 10x^3 + 48\ln(x+1)x - 54x^2 + 24\ln(x+1) - 57x - 18}{3x^2 - 6x + 3}$$

Verified OK.

6.24.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2-1)\,\mathrm{d}y &= \left(-(x^2-1)^2 - 4y\right)\mathrm{d}x\\ \left((x^2-1)^2 + 4y\right)\mathrm{d}x + (x^2-1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= (x^2-1)^2 + 4y\\ N(x,y) &= x^2 - 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\left(x^2-1\right)^2 + 4y\right)\\ &= 4\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2-1\right)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2-1}((4) - (2x))\\ &= \frac{4-2x}{x^2-1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int \frac{4-2x}{x^2-1} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x-1)-3\ln(x+1)}\\ &= \frac{x-1}{(x+1)^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{x-1}{(x+1)^3}\left(\left(x^2-1\right)^2 + 4y\right)\\ &= \frac{(x^4-2x^2+4y+1)(x-1)}{(x+1)^3}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{x-1}{(x+1)^3}\left(x^2-1\right) \\
&= \frac{(x-1)^2}{(x+1)^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{(x^4-2x^2+4y+1)(x-1)}{(x+1)^3}\right) + \left(\frac{(x-1)^2}{(x+1)^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{(x^4-2x^2+4y+1)(x-1)}{(x+1)^3}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^3}{3} - 2x^2 + 7x - 8\ln(x+1) + \frac{4y}{(x+1)^2} - \frac{4y}{x+1} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{4}{(x+1)^2} - \frac{4}{x+1} + f'(y) \tag{4}
$$

$$
= -\frac{4x}{(x+1)^2} + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(x-1)^2}{(x+1)^2}$. Therefore equation (4) becomes

$$\frac{(x-1)^2}{(x+1)^2} = -\frac{4x}{(x+1)^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (1)\,\mathrm{d}y$$

$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^3}{3} - 2x^2 + 7x - 8\ln(x+1) + \frac{4y}{(x+1)^2} - \frac{4y}{x+1} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^3}{3} - 2x^2 + 7x - 8\ln(x+1) + \frac{4y}{(x+1)^2} - \frac{4y}{x+1} + y$$

The solution becomes

$$y = \frac{(x+1)^2(-x^3 + 6x^2 + 24\ln(x+1) + 3c_1 - 21x)}{3x^2 - 6x + 3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -6$ in the above solution gives an equation to solve for the constant of integration.

$$-6 = c_1$$

$$c_1 = -6$$

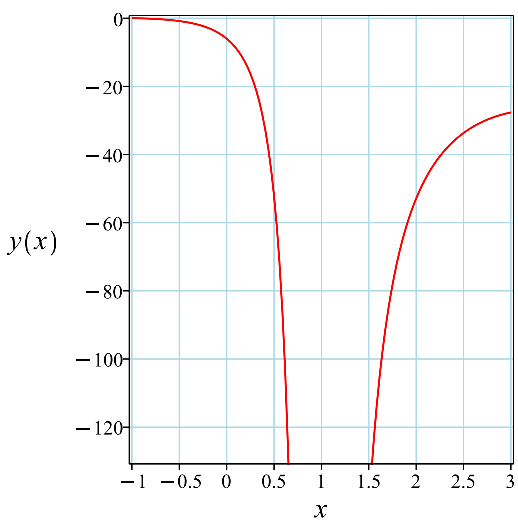
Substituting c_1 found above in the general solution gives

$$y = \frac{-x^5 + 4x^4 + 24\ln(x+1)\,x^2 - 10x^3 + 48\ln(x+1)\,x - 54x^2 + 24\ln(x+1) - 57x - 18}{3x^2 - 6x + 3}$$

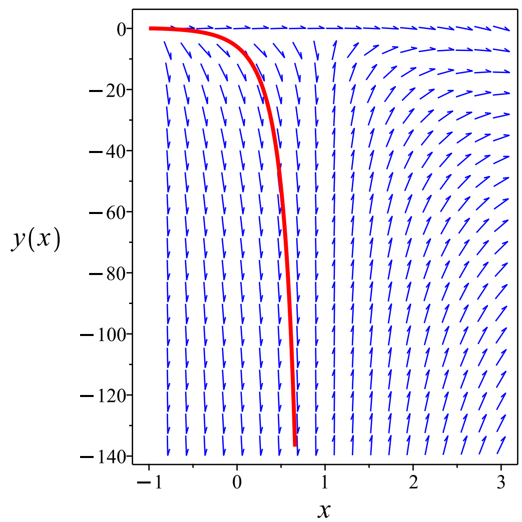
Summary

The solution(s) found are the following

$$y = \frac{-x^5 + 4x^4 + 24\ln(x+1)\,x^2 - 10x^3 + 48\ln(x+1)\,x - 54x^2 + 24\ln(x+1) - 57x - 18}{3x^2 - 6x + 3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-x^5 + 4x^4 + 24\ln(x+1)\,x^2 - 10x^3 + 48\ln(x+1)\,x - 54x^2 + 24\ln(x+1) - 57x - 18}{3x^2 - 6x + 3}$$

Verified OK.

6.24.5 Maple step by step solution

Let's solve

$$\left[(x^2-1)\,y' + 4y = -(x^2-1)^2, y(0) = -6\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{4y}{x^2-1} + 1 - x^2$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{4y}{x^2-1} = 1 - x^2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{4y}{x^2-1}\right) = \mu(x)\left(1 - x^2\right)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{4y}{x^2-1}\right) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \frac{4\mu(x)}{x^2-1}$
- Solve to find the integrating factor

  $\mu(x) = \frac{(x-1)^2}{(x+1)^2}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\left(1 - x^2\right) dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\left(1 - x^2\right) dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(1-x^2)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{(x-1)^2}{(x+1)^2}$

  $y = \frac{(x+1)^2\left(\int \frac{(1-x^2)(x-1)^2}{(x+1)^2}dx + c_1\right)}{(x-1)^2}$
- Evaluate the integrals on the rhs

  $y = \frac{(x+1)^2\left(-\frac{x^3}{3} + 2x^2 - 7x + 8\ln(x+1) + c_1\right)}{(x-1)^2}$
- Simplify

  $y = -\frac{(x+1)^2(x^3 - 6x^2 - 24\ln(x+1) - 3c_1 + 21x)}{3(x-1)^2}$
- Use initial condition $y(0) = -6$

  $-6 = c_1$
- Solve for c_1

$c_1 = -6$

- Substitute $c_1 = -6$ into general solution and simplify

  $y = -\frac{(x^3-6x^2-24\ln(x+1)+18+21x)(x+1)^2}{3(x-1)^2}$

- Solution to the IVP

  $y = -\frac{(x^3-6x^2-24\ln(x+1)+18+21x)(x+1)^2}{3(x-1)^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 38

```
dsolve([(x^2-1)*diff(y(x),x)+(x^2-1)^2+4*y(x)=0,y(0) = -6],y(x), singsol=all)
```

$$y(x) = \frac{\left(-\frac{x^3}{3} + 2x^2 - 7x + 8\ln(x+1) - 6\right)(x+1)^4}{(x^2-1)^2}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 38

```
DSolve[{(x^2-1)*y'[x]+(x^2-1)^2+4*y[x]==0,{y[0]==-6}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -\frac{(x+1)^2(x^3-6x^2+21x-24\log(x+1)+18)}{3(x-1)^2}$$

7 Exercise 11, page 45

7.1 problem 1

Internal problem ID [2011]
Internal file name [`OUTPUT/2011_Sunday_February_25_2024_06_44_37_AM_63844631/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{3y'y^2 - xy^3 = \mathrm{e}^{\frac{x^2}{2}} \cos(x)}$$

7.1.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x\,y^3 + \mathrm{e}^{\frac{x^2}{2}} \cos(x)}{3y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 198: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{e^{\frac{x^2}{2}}}{y^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{\frac{x^2}{2}}}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3 e^{-\frac{x^2}{2}}}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x\, y^3 + e^{\frac{x^2}{2}} \cos(x)}{3y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^3 x\, e^{-\frac{x^2}{2}}}{3} \\ S_y &= y^2 e^{-\frac{x^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cos(x)}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\cos(R)}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\sin(R)}{3} + c_1 \tag{4}$$

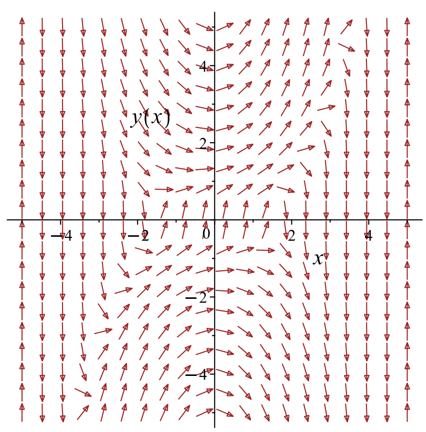
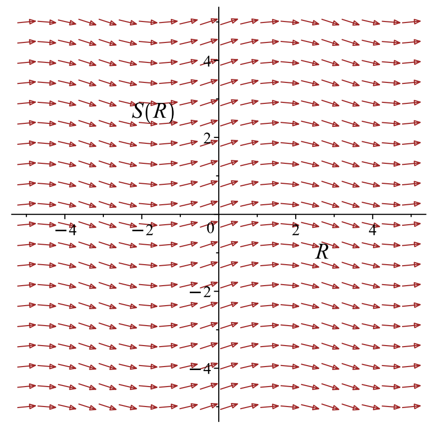
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3 e^{-\frac{x^2}{2}}}{3} = \frac{\sin(x)}{3} + c_1$$

Which simplifies to

$$\frac{y^3 e^{-\frac{x^2}{2}}}{3} = \frac{\sin(x)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\,y^3 + e^{\frac{x^2}{2}} \cos(x)}{3y^2}$  	$R = x$ $S = \frac{y^3 e^{-\frac{x^2}{2}}}{3}$	$\frac{dS}{dR} = \frac{\cos(R)}{3}$ 

Summary
The solution(s) found are the following

$$\frac{y^3 e^{-\frac{x^2}{2}}}{3} = \frac{\sin(x)}{3} + c_1 \tag{1}$$

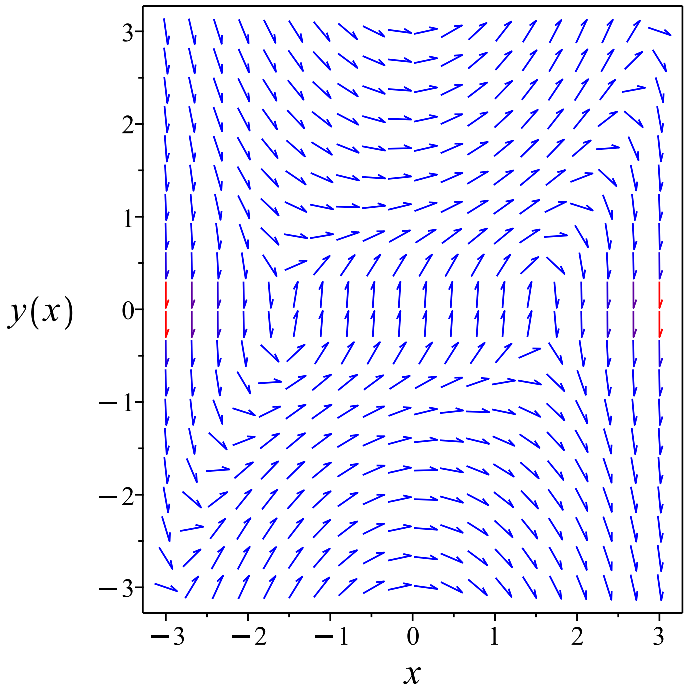


Figure 338: Slope field plot

Verification of solutions

$$\frac{y^3 e^{-\frac{x^2}{2}}}{3} = \frac{\sin(x)}{3} + c_1$$

Verified OK.

7.1.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x\,y^3 + e^{\frac{x^2}{2}}\cos(x)}{3y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{x}{3}y + \frac{e^{\frac{x^2}{2}}\cos(x)}{3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= \frac{x}{3} \\
f_1(x) &= \frac{e^{\frac{x^2}{2}} \cos(x)}{3} \\
n &= -2
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = \frac{x\,y^3}{3} + \frac{e^{\frac{x^2}{2}} \cos(x)}{3} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= y^3
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
\frac{w'(x)}{3} &= \frac{w(x)\,x}{3} + \frac{e^{\frac{x^2}{2}} \cos(x)}{3} \\
w' &= xw + e^{\frac{x^2}{2}} \cos(x)
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -x \\
q(x) &= \mathrm{e}^{\frac{x^2}{2}} \cos(x)
\end{aligned}
$$

Hence the ode is

$$
w'(x) - w(x)\, x = \mathrm{e}^{\frac{x^2}{2}} \cos(x)
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -x dx} \\
&= \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\mathrm{e}^{\frac{x^2}{2}} \cos(x)\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{x^2}{2}} w\right) &= \left(\mathrm{e}^{-\frac{x^2}{2}}\right)\left(\mathrm{e}^{\frac{x^2}{2}} \cos(x)\right) \\
\mathrm{d}\left(\mathrm{e}^{-\frac{x^2}{2}} w\right) &= \cos(x)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-\frac{x^2}{2}} w &= \int \cos(x)\, \mathrm{d}x \\
\mathrm{e}^{-\frac{x^2}{2}} w &= \sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{x^2}{2}}$ results in

$$
w(x) = \mathrm{e}^{\frac{x^2}{2}} \sin(x) + c_1 \mathrm{e}^{\frac{x^2}{2}}
$$

which simplifies to

$$
w(x) = \mathrm{e}^{\frac{x^2}{2}} (\sin(x) + c_1)
$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$
y^3 = \mathrm{e}^{\frac{x^2}{2}} (\sin(x) + c_1)
$$

Solving for y gives

$$y(x) = \left(e^{\frac{x^2}{2}}\left(\sin(x) + c_1\right)\right)^{\frac{1}{3}}$$
$$y(x) = \frac{\left(e^{\frac{x^2}{2}}\left(\sin(x) + c_1\right)\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2}$$
$$y(x) = -\frac{\left(e^{\frac{x^2}{2}}\left(\sin(x) + c_1\right)\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \left(e^{\frac{x^2}{2}}\left(\sin(x) + c_1\right)\right)^{\frac{1}{3}} \tag{1}$$
$$y = \frac{\left(e^{\frac{x^2}{2}}\left(\sin(x) + c_1\right)\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2} \tag{2}$$
$$y = -\frac{\left(e^{\frac{x^2}{2}}\left(\sin(x) + c_1\right)\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2} \tag{3}$$

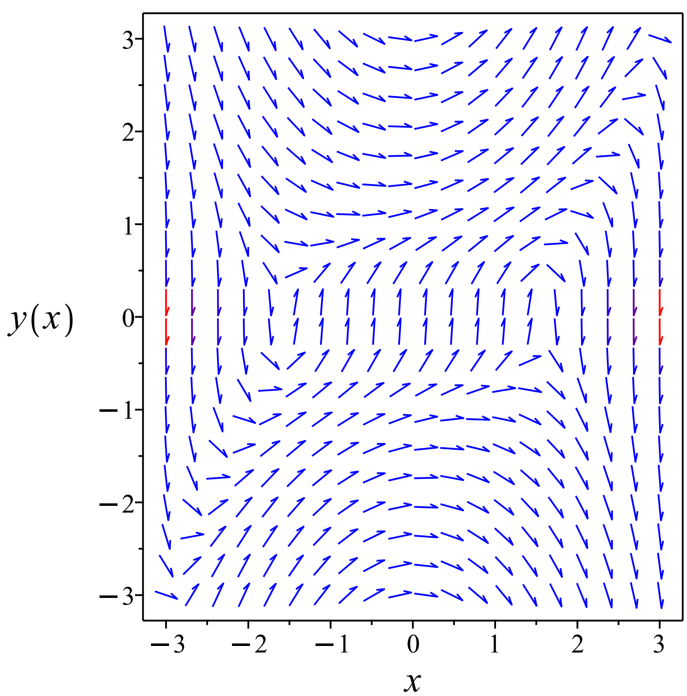


Figure 339: Slope field plot

Verification of solutions

$$y = \left(e^{\frac{x^2}{2}} (\sin (x) + c_1)\right)^{\frac{1}{3}}$$

Verified OK.

$$y = \frac{\left(e^{\frac{x^2}{2}} (\sin (x) + c_1)\right)^{\frac{1}{3}} (i\sqrt{3} - 1)}{2}$$

Verified OK.

$$y = -\frac{\left(e^{\frac{x^2}{2}} (\sin (x) + c_1)\right)^{\frac{1}{3}} (1 + i\sqrt{3})}{2}$$

Verified OK.

7.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3y^2)\,\mathrm{d}y &= \left(x\,y^3 + \mathrm{e}^{\frac{x^2}{2}}\cos(x)\right)\mathrm{d}x\\ \left(-x\,y^3 - \mathrm{e}^{\frac{x^2}{2}}\cos(x)\right)\mathrm{d}x + (3y^2)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\,y^3 - \mathrm{e}^{\frac{x^2}{2}}\cos(x)\\ N(x,y) &= 3y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x\,y^3 - \mathrm{e}^{\frac{x^2}{2}}\cos(x)\right)\\ &= -3x\,y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3y^2)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{3y^2}\left((-3x\,y^2) - (0)\right)\\ &= -x\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, dx} \\ &= e^{\int -x \, dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{x^2}{2}} \\ &= \mathrm{e}^{-\frac{x^2}{2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-\frac{x^2}{2}} \left(-x\, y^3 - \mathrm{e}^{\frac{x^2}{2}} \cos(x)\right) \\ &= -y^3 x\, \mathrm{e}^{-\frac{x^2}{2}} - \cos(x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-\frac{x^2}{2}} (3y^2) \\ &= 3y^2 \mathrm{e}^{-\frac{x^2}{2}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-y^3 x\, \mathrm{e}^{-\frac{x^2}{2}} - \cos(x)\right) + \left(3y^2 \mathrm{e}^{-\frac{x^2}{2}}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -y^3 x \,\mathrm{e}^{-\frac{x^2}{2}} - \cos(x) \mathrm{d}x$$

$$\phi = y^3 \mathrm{e}^{-\frac{x^2}{2}} - \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3y^2 \mathrm{e}^{-\frac{x^2}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^2 \mathrm{e}^{-\frac{x^2}{2}}$. Therefore equation (4) becomes

$$3y^2 \mathrm{e}^{-\frac{x^2}{2}} = 3y^2 \mathrm{e}^{-\frac{x^2}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y^3 \mathrm{e}^{-\frac{x^2}{2}} - \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^3 \mathrm{e}^{-\frac{x^2}{2}} - \sin(x)$$

Summary

The solution(s) found are the following

$$y^3 \mathrm{e}^{-\frac{x^2}{2}} - \sin(x) = c_1 \tag{1}$$

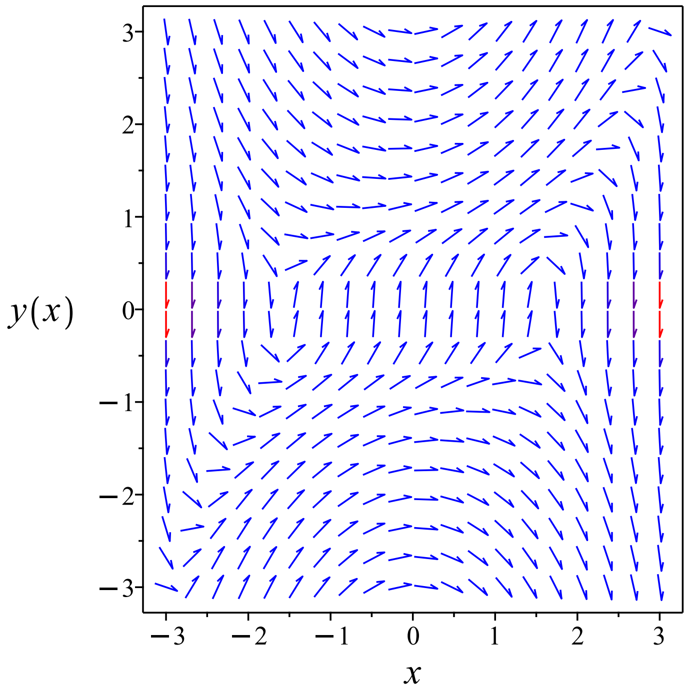


Figure 340: Slope field plot

Verification of solutions

$$y^3 e^{-\frac{x^2}{2}} - \sin(x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 88

```
dsolve(3*y(x)^2*diff(y(x),x)-x*y(x)^3=exp(x^2/2)*cos(x),y(x), singsol=all)
```

$$y(x) = \left(\left(\sin\left(x\right)+c_1\right)\mathrm{e}^{-x^2}\right)^{\frac{1}{3}}\mathrm{e}^{\frac{x^2}{2}}$$

$$y(x) = -\frac{\left(\left(\sin\left(x\right)+c_1\right)\mathrm{e}^{-x^2}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)\mathrm{e}^{\frac{x^2}{2}}}{2}$$

$$y(x) = \frac{\left(\left(\sin\left(x\right)+c_1\right)\mathrm{e}^{-x^2}\right)^{\frac{1}{3}}\mathrm{e}^{\frac{x^2}{2}}\left(i\sqrt{3}-1\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.426 (sec). Leaf size: 81

```
DSolve[3*y[x]^2*y'[x]-x*y[x]^3==Exp[x^2/2]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{x^2}{6}}\sqrt[3]{\sin(x)+c_1}$$

$$y(x) \to -\sqrt[3]{-1}e^{\frac{x^2}{6}}\sqrt[3]{\sin(x)+c_1}$$

$$y(x) \to (-1)^{2/3}e^{\frac{x^2}{6}}\sqrt[3]{\sin(x)+c_1}$$

7.2 problem 2

Internal problem ID [2012]
Internal file name [`OUTPUT/2012_Sunday_February_25_2024_06_44_38_AM_90715307/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y^3 y' + x y^4 = x\,\mathrm{e}^{-x^2}}$$

7.2.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x\left(-y^4 + \mathrm{e}^{-x^2}\right)}{y^3} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 200: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{\mathrm{e}^{-2x^2}}{y^3}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{-2x^2}}{y^3}} dy \end{aligned}$$

Which results in

$$S = \frac{y^4 e^{2x^2}}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x\left(-y^4 + e^{-x^2}\right)}{y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^4 x \, e^{2x^2} \\ S_y &= e^{2x^2} y^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{x^2} x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R^2} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{R^2}}{2} + c_1 \tag{4}$$

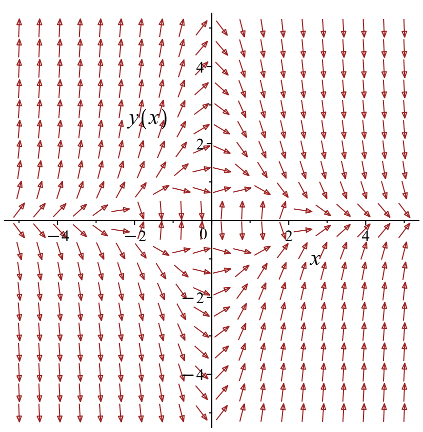
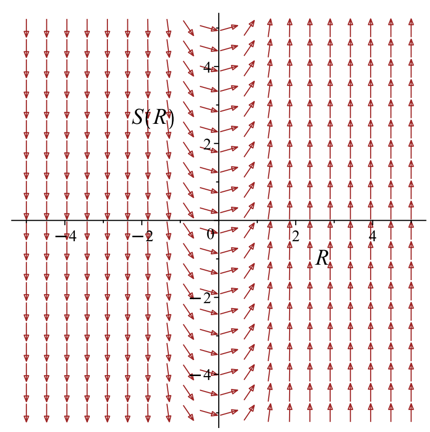
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^4 e^{2x^2}}{4} = \frac{e^{x^2}}{2} + c_1$$

Which simplifies to

$$\frac{y^4 e^{2x^2}}{4} = \frac{e^{x^2}}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\left(-y^4+e^{-x^2}\right)}{y^3}$ 	$R = x$ $S = \frac{y^4 e^{2x^2}}{4}$	$\frac{dS}{dR} = e^{R^2} R$ 

Summary

The solution(s) found are the following

$$\frac{y^4 e^{2x^2}}{4} = \frac{e^{x^2}}{2} + c_1 \tag{1}$$

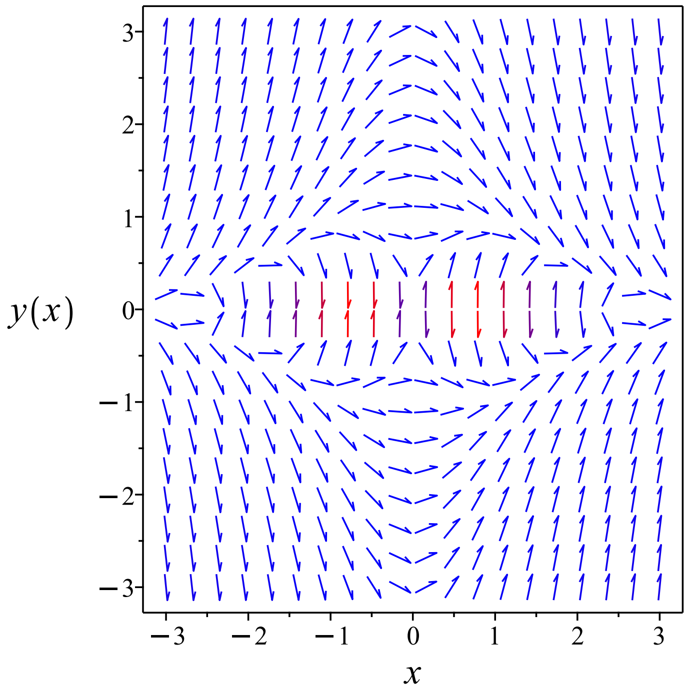


Figure 341: Slope field plot

Verification of solutions

$$\frac{y^4 e^{2x^2}}{4} = \frac{e^{x^2}}{2} + c_1$$

Verified OK.

7.2.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x\left(-y^4 + e^{-x^2}\right)}{y^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -xy + x\,e^{-x^2}\frac{1}{y^3} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -x \\ f_1(x) &= x\,\mathrm{e}^{-x^2} \\ n &= -3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^3}$ gives

$$y'y^3 = -x\,y^4 + x\,\mathrm{e}^{-x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^4 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 4y^3y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{4} &= -w(x)\,x + x\,\mathrm{e}^{-x^2} \\ w' &= -4xw + 4x\,\mathrm{e}^{-x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 4x \\ q(x) &= 4x\,\mathrm{e}^{-x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + 4w(x)\,x = 4x\,\mathrm{e}^{-x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 4x dx}\\ &= \mathrm{e}^{2x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(4x\,\mathrm{e}^{-x^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x^2} w\right) &= \left(\mathrm{e}^{2x^2}\right)\left(4x\,\mathrm{e}^{-x^2}\right)\\ \mathrm{d}\left(\mathrm{e}^{2x^2} w\right) &= \left(4\,\mathrm{e}^{x^2} x\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{2x^2} w &= \int 4\,\mathrm{e}^{x^2} x\,\mathrm{d}x\\ \mathrm{e}^{2x^2} w &= 2\,\mathrm{e}^{x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x^2}$ results in

$$w(x) = 2\,\mathrm{e}^{-2x^2}\mathrm{e}^{x^2} + c_1\mathrm{e}^{-2x^2}$$

which simplifies to

$$w(x) = 2\,\mathrm{e}^{-x^2} + c_1\mathrm{e}^{-2x^2}$$

Replacing w in the above by y^4 using equation (5) gives the final solution.

$$y^4 = 2\,\mathrm{e}^{-x^2} + c_1\mathrm{e}^{-2x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \mathrm{e}^{-x^2}\left(\left(2\,\mathrm{e}^{x^2} + c_1\right)\mathrm{e}^{2x^2}\right)^{\frac{1}{4}}\\ y(x) &= i\mathrm{e}^{-x^2}\left(\left(2\,\mathrm{e}^{x^2} + c_1\right)\mathrm{e}^{2x^2}\right)^{\frac{1}{4}}\\ y(x) &= -\mathrm{e}^{-x^2}\left(\left(2\,\mathrm{e}^{x^2} + c_1\right)\mathrm{e}^{2x^2}\right)^{\frac{1}{4}}\\ y(x) &= -i\mathrm{e}^{-x^2}\left(\left(2\,\mathrm{e}^{x^2} + c_1\right)\mathrm{e}^{2x^2}\right)^{\frac{1}{4}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x^2}\left(\left(2\,\mathrm{e}^{x^2}+c_1\right)\mathrm{e}^{2x^2}\right)^{\frac{1}{4}} \tag{1}$$

$$y = i\mathrm{e}^{-x^2}\left(\left(2\,\mathrm{e}^{x^2}+c_1\right)\mathrm{e}^{2x^2}\right)^{\frac{1}{4}} \tag{2}$$

$$y = -\mathrm{e}^{-x^2}\left(\left(2\,\mathrm{e}^{x^2}+c_1\right)\mathrm{e}^{2x^2}\right)^{\frac{1}{4}} \tag{3}$$

$$y = -i\mathrm{e}^{-x^2}\left(\left(2\,\mathrm{e}^{x^2}+c_1\right)\mathrm{e}^{2x^2}\right)^{\frac{1}{4}} \tag{4}$$

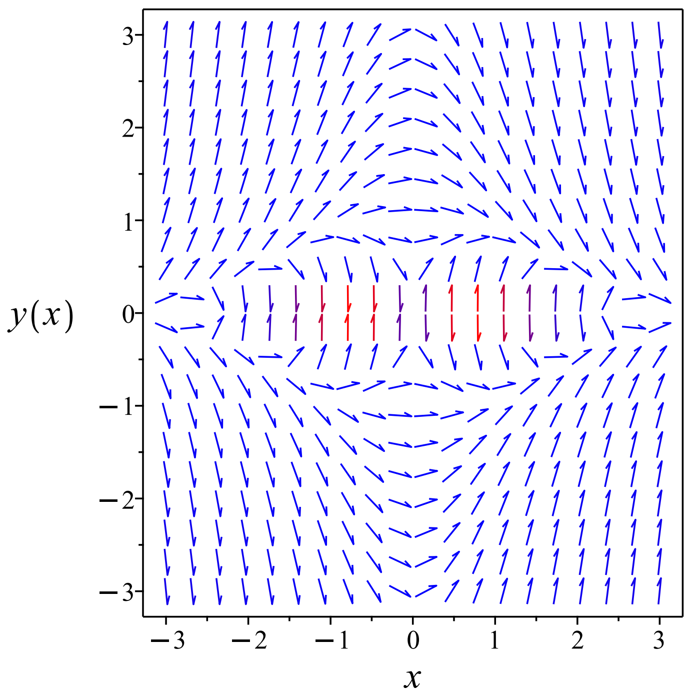


Figure 342: Slope field plot

Verification of solutions

$$y = e^{-x^2}\left(\left(2\,e^{x^2}+c_1\right)e^{2x^2}\right)^{\frac{1}{4}}$$

Verified OK.

$$y = i e^{-x^2}\left(\left(2\,e^{x^2}+c_1\right)e^{2x^2}\right)^{\frac{1}{4}}$$

Verified OK.

$$y = -e^{-x^2}\left(\left(2\,e^{x^2}+c_1\right)e^{2x^2}\right)^{\frac{1}{4}}$$

Verified OK.

$$y = -i e^{-x^2}\left(\left(2\,e^{x^2}+c_1\right)e^{2x^2}\right)^{\frac{1}{4}}$$

Verified OK.

7.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^3)\,\mathrm{d}y &= \left(-x\,y^4 + x\,\mathrm{e}^{-x^2}\right)\mathrm{d}x\\ \left(x\,y^4 - x\,\mathrm{e}^{-x^2}\right)\mathrm{d}x + (y^3)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x\,y^4 - x\,\mathrm{e}^{-x^2}\\ N(x,y) &= y^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\,y^4 - x\,\mathrm{e}^{-x^2}\right)\\ &= 4x\,y^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^3)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{y^3}\left((4x\,y^3) - (0)\right)\\ &= 4x\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 4x \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2x^2} \\ &= \mathrm{e}^{2x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{2x^2} \left(x\, y^4 - x\, \mathrm{e}^{-x^2} \right) \\ &= x\, \mathrm{e}^{x^2} \left(\mathrm{e}^{x^2} y^4 - 1 \right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{2x^2} \left(y^3 \right) \\ &= \mathrm{e}^{2x^2} y^3\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(x\, \mathrm{e}^{x^2} \left(\mathrm{e}^{x^2} y^4 - 1 \right) \right) + \left(\mathrm{e}^{2x^2} y^3 \right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x \, \mathrm{e}^{x^2} \left(\mathrm{e}^{x^2} y^4 - 1\right) \mathrm{d}x$$

$$\phi = \frac{y^4 \mathrm{e}^{2x^2}}{4} - \frac{\mathrm{e}^{x^2}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x^2} y^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x^2} y^3$. Therefore equation (4) becomes

$$\mathrm{e}^{2x^2} y^3 = \mathrm{e}^{2x^2} y^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y^4 \mathrm{e}^{2x^2}}{4} - \frac{\mathrm{e}^{x^2}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^4 \mathrm{e}^{2x^2}}{4} - \frac{\mathrm{e}^{x^2}}{2}$$

Summary

The solution(s) found are the following

$$\frac{y^4 \mathrm{e}^{2x^2}}{4} - \frac{\mathrm{e}^{x^2}}{2} = c_1 \tag{1}$$

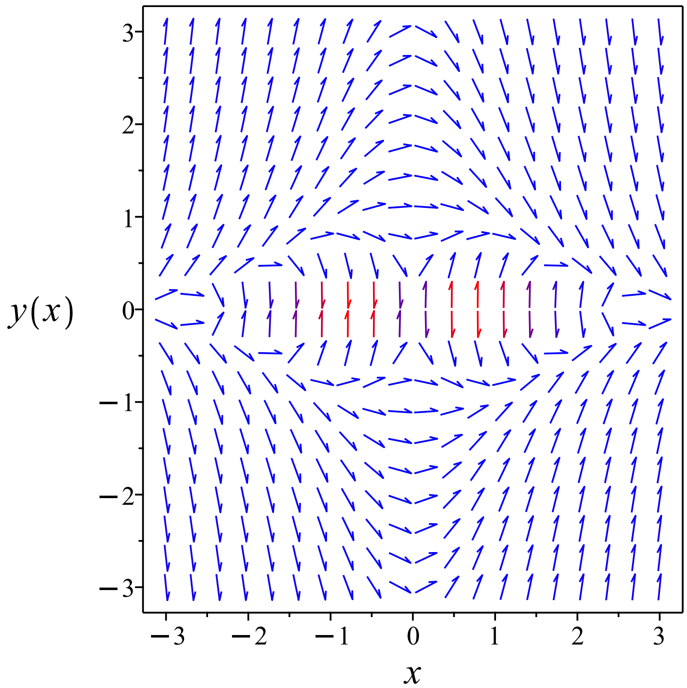


Figure 343: Slope field plot

Verification of solutions

$$\frac{y^4 e^{2x^2}}{4} - \frac{e^{x^2}}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 114

```
dsolve(y(x)^3*diff(y(x),x)+x*y(x)^4=x*exp(-x^2),y(x), singsol=all)
```

$$y(x) = e^{-x^2}\left(\left(2\,e^{x^2}+c_1\right)e^{2x^2}\right)^{\frac{1}{4}}$$
$$y(x) = -e^{-x^2}\left(\left(2\,e^{x^2}+c_1\right)e^{2x^2}\right)^{\frac{1}{4}}$$
$$y(x) = -ie^{-x^2}\left(\left(2\,e^{x^2}+c_1\right)e^{2x^2}\right)^{\frac{1}{4}}$$
$$y(x) = ie^{-x^2}\left(\left(2\,e^{x^2}+c_1\right)e^{2x^2}\right)^{\frac{1}{4}}$$

✓ Solution by Mathematica

Time used: 0.388 (sec). Leaf size: 200

```
DSolve[y[x]^3*y'[x]+x*y[x]^4==x*exp[-x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{-\frac{x^2}{2}}\sqrt[4]{4\int_1^x e^{2K[1]^2}\exp\left(-K[1]^2\right)K[1]dK[1]+c_1}$$
$$y(x) \to -ie^{-\frac{x^2}{2}}\sqrt[4]{4\int_1^x e^{2K[1]^2}\exp\left(-K[1]^2\right)K[1]dK[1]+c_1}$$
$$y(x) \to ie^{-\frac{x^2}{2}}\sqrt[4]{4\int_1^x e^{2K[1]^2}\exp\left(-K[1]^2\right)K[1]dK[1]+c_1}$$
$$y(x) \to e^{-\frac{x^2}{2}}\sqrt[4]{4\int_1^x e^{2K[1]^2}\exp\left(-K[1]^2\right)K[1]dK[1]+c_1}$$

7.3 problem 3

Internal problem ID [2013]
Internal file name [`OUTPUT/2013_Sunday_February_25_2024_06_44_39_AM_87492246/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{\cosh(y)\, y' + \sinh(y) = \mathrm{e}^{-x}}$$

7.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cosh(y))\,\mathrm{d}y &= \left(-\sinh(y) + \mathrm{e}^{-x}\right)\mathrm{d}x\\ \left(\sinh(y) - \mathrm{e}^{-x}\right)\mathrm{d}x + (\cosh(y))\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \sinh(y) - \mathrm{e}^{-x}\\ N(x, y) &= \cosh(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\sinh(y) - \mathrm{e}^{-x}\right)\\ &= \cosh(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cosh(y))\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \operatorname{sech}(y)\left((\cosh(y)) - (0)\right)\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int 1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x}\\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{x}\left(\sinh\left(y\right)-\mathrm{e}^{-x}\right)\\ &= \sinh\left(y\right)\mathrm{e}^{x}-1\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{x}\left(\cosh\left(y\right)\right)\\ &= \cosh\left(y\right)\mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\sinh\left(y\right)\mathrm{e}^{x}-1\right)+\left(\cosh\left(y\right)\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\sinh\left(y\right)\mathrm{e}^{x}-1\,\mathrm{d}x\end{aligned}$$

$$\phi = -x+\sinh\left(y\right)\mathrm{e}^{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cosh(y)\,e^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cosh(y)\,e^{x}$. Therefore equation (4) becomes

$$\cosh(y)\,e^{x} = \cosh(y)\,e^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \sinh(y)\,e^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \sinh(y)\,e^{x}$$

The solution becomes

$$y = \operatorname{arcsinh}\left(e^{-x}(x + c_1)\right)$$

Summary

The solution(s) found are the following

$$y = \operatorname{arcsinh}\left(e^{-x}(x + c_1)\right) \tag{1}$$

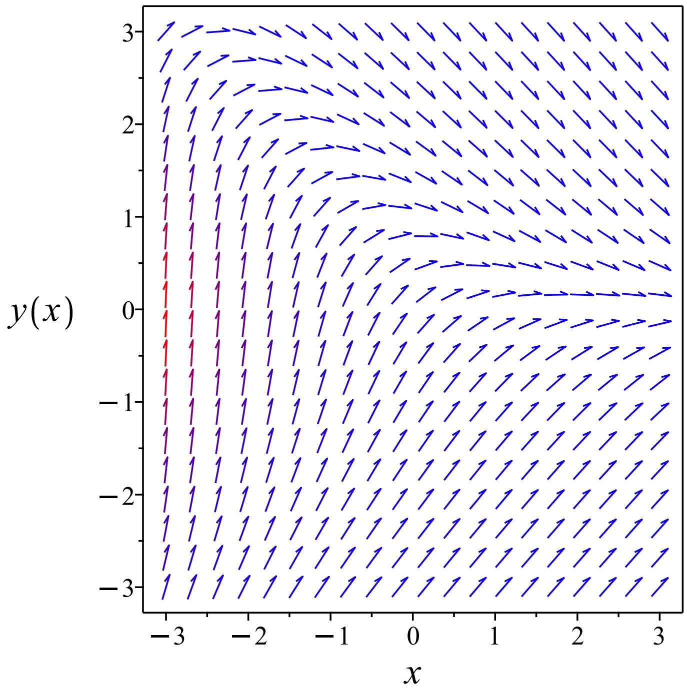


Figure 344: Slope field plot

Verification of solutions

$$y = \operatorname{arcsinh}\left(\mathrm{e}^{-x}(x + c_1)\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(cosh(y(x))*diff(y(x),x)+(sinh(y(x))-exp(-x) )=0,y(x), singsol=all)
```

$$y(x) = -\operatorname{arcsinh}\left((c_1 - x)\,\mathrm{e}^{-x}\right)$$

✓ Solution by Mathematica

Time used: 14.919 (sec). Leaf size: 16

```
DSolve[Cosh[y[x]]*y'[x]+(Sinh[y[x]]-Exp[-x] )==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \operatorname{arcsinh}\left(e^{-x}(x + c_1)\right)$$

7.4 problem 4

Internal problem ID [2014]
Internal file name [`OUTPUT/2014_Sunday_February_25_2024_06_44_40_AM_87973462/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{\sin(\theta)\,\theta' + \cos(\theta) = t\,\mathrm{e}^{-t}}$$

7.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,\theta)\,\mathrm{d}t + N(t,\theta)\,\mathrm{d}\theta = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\sin(\theta))\,\mathrm{d}\theta &= \left(-\cos(\theta) + t\,\mathrm{e}^{-t}\right)\mathrm{d}t\\ \left(\cos(\theta) - t\,\mathrm{e}^{-t}\right)\mathrm{d}t + (\sin(\theta))\,\mathrm{d}\theta &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,\theta) &= \cos(\theta) - t\,\mathrm{e}^{-t}\\ N(t,\theta) &= \sin(\theta)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(\cos(\theta) - t\,\mathrm{e}^{-t}\right)\\ &= -\sin(\theta)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(\sin(\theta))\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial \theta} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial \theta} - \frac{\partial N}{\partial t}\right)\\ &= \csc(\theta)\left((-\sin(\theta)) - (0)\right)\\ &= -1\end{aligned}$$

Since A does not depend on θ, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int -1\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-t}\\ &= \mathrm{e}^{-t}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-t}\left(\cos\left(\theta\right) - t\,\mathrm{e}^{-t}\right)\\ &= -\left(-\cos\left(\theta\right) + t\,\mathrm{e}^{-t}\right)\mathrm{e}^{-t}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-t}(\sin\left(\theta\right))\\ &= \mathrm{e}^{-t}\sin\left(\theta\right)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= 0\\ \left(-\left(-\cos\left(\theta\right) + t\,\mathrm{e}^{-t}\right)\mathrm{e}^{-t}\right) + \left(\mathrm{e}^{-t}\sin\left(\theta\right)\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,\theta)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial\theta} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int\overline{M}\,\mathrm{d}t\\ \int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int -\left(-\cos\left(\theta\right) + t\,\mathrm{e}^{-t}\right)\mathrm{e}^{-t}\,\mathrm{d}t\end{aligned}$$

$$\phi = \frac{(1+2t)\,\mathrm{e}^{-2t}}{4} - \mathrm{e}^{-t}\cos\left(\theta\right) + f(\theta) \tag{3}$$

Where $f(\theta)$ is used for the constant of integration since ϕ is a function of both t and θ. Taking derivative of equation (3) w.r.t θ gives

$$\frac{\partial \phi}{\partial \theta} = \mathrm{e}^{-t} \sin(\theta) + f'(\theta) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial \theta} = \mathrm{e}^{-t} \sin(\theta)$. Therefore equation (4) becomes

$$\mathrm{e}^{-t} \sin(\theta) = \mathrm{e}^{-t} \sin(\theta) + f'(\theta) \tag{5}$$

Solving equation (5) for $f'(\theta)$ gives

$$f'(\theta) = 0$$

Therefore

$$f(\theta) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(\theta)$ into equation (3) gives ϕ

$$\phi = \frac{(1+2t)\,\mathrm{e}^{-2t}}{4} - \mathrm{e}^{-t} \cos(\theta) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(1+2t)\,\mathrm{e}^{-2t}}{4} - \mathrm{e}^{-t} \cos(\theta)$$

Summary

The solution(s) found are the following

$$\frac{(1+2t)\,\mathrm{e}^{-2t}}{4} - \mathrm{e}^{-t} \cos(\theta) = c_1 \tag{1}$$

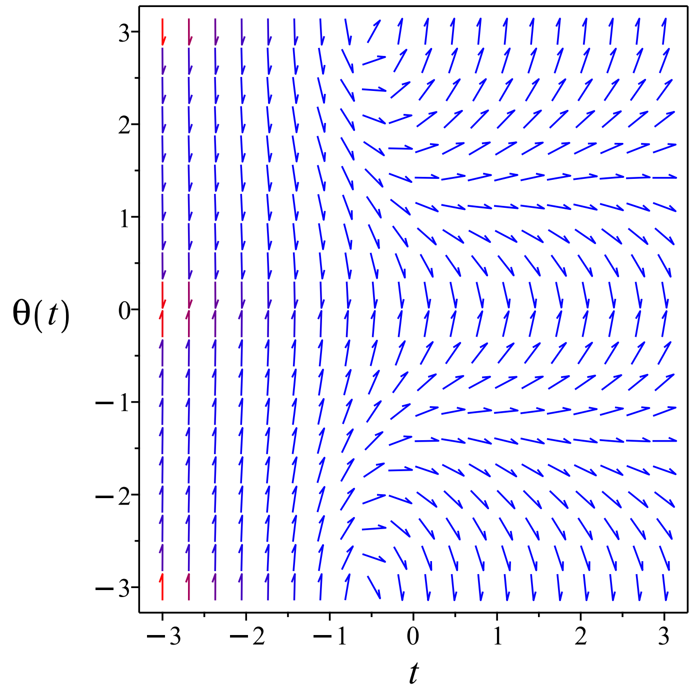


Figure 345: Slope field plot

Verification of solutions

$$\frac{(1+2t)\,e^{-2t}}{4} - e^{-t}\cos(\theta) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(sin(theta(t))*diff(theta(t),t)+(cos(theta(t))-t*exp(-t) )=0,theta(t), singsol=all)
```

$$\theta(t) = \arccos\left(\frac{(1+2t)\,\mathrm{e}^{-t}}{4} + \mathrm{e}^{t} c_1\right)$$

✓ Solution by Mathematica

Time used: 21.418 (sec). Leaf size: 59

```
DSolve[Sin[\[Theta][t]]*\[Theta]'[t]+(Cos[\[Theta][t]]-t*Exp[-t] )==0,\[Theta][t],t,IncludeS
```

$$\theta(t) \to -\arccos\left(\frac{1}{4}e^{-t}(2t+4c_1e^{2t}+1)\right)$$
$$\theta(t) \to \arccos\left(\frac{1}{4}e^{-t}(2t+4c_1e^{2t}+1)\right)$$

7.5 problem 5

Internal problem ID [2015]
Internal file name [`OUTPUT/2015_Sunday_February_25_2024_06_44_42_AM_91240055/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y'xy + y^2 = x^2}$$

7.5.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2u(x) + u(x)^2\,x^2 = x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^2-1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2-1}{u}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{2u^2-1}{u}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(2u^2-1)}{4} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$(2u^2-1)^{\frac{1}{4}} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$(2u^2-1)^{\frac{1}{4}} = \frac{c_3}{x}$$

Which simplifies to

$$(2u(x)^2-1)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

The solution is

$$(2u(x)^2-1)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{2y^2}{x^2}-1\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

$$\left(\frac{2y^2-x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Which simplifies to

$$\left(-\frac{-2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\left(-\frac{-2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x} \tag{1}$$

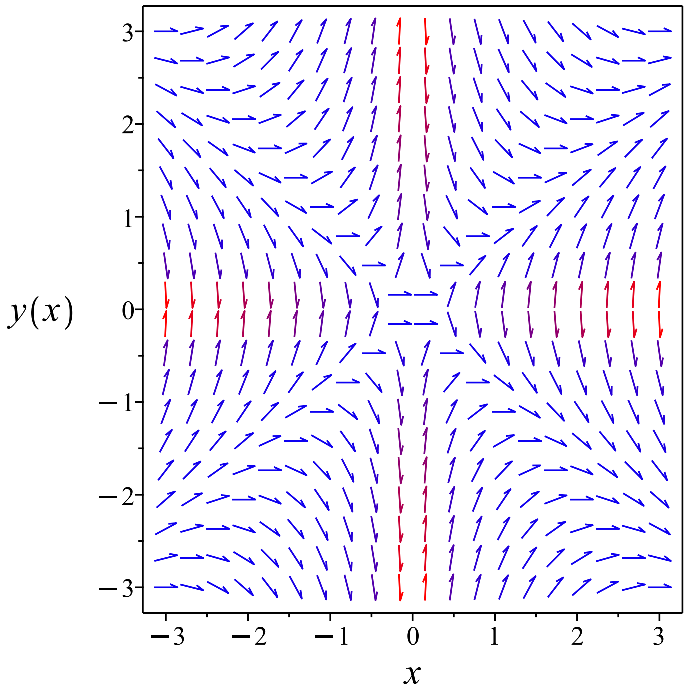


Figure 346: Slope field plot

Verification of solutions

$$\left(-\frac{-2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

7.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^2+y^2}{yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 202: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^2y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2 y}} dy \end{aligned}$$

Which results in

$$S = \frac{x^2 y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x^2 + y^2}{yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= x\, y^2 \\ S_y &= y\, x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \tag{4}$$

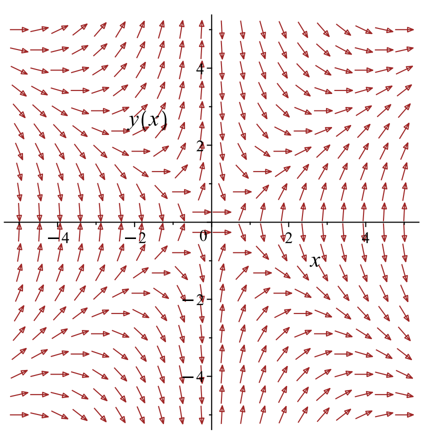
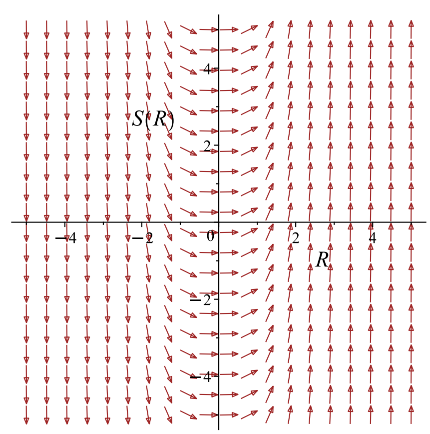
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2y^2}{2} = \frac{x^4}{4} + c_1$$

Which simplifies to

$$\frac{x^2y^2}{2} = \frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+y^2}{yx}$ 	$R = x$ $S = \frac{x^2y^2}{2}$	$\frac{dS}{dR} = R^3$ 

Summary
The solution(s) found are the following

$$\frac{x^2y^2}{2} = \frac{x^4}{4} + c_1 \tag{1}$$

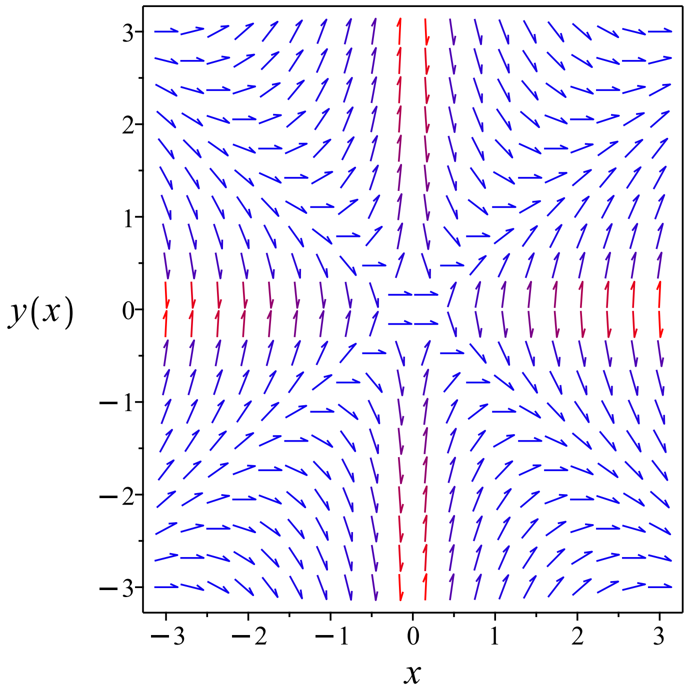


Figure 347: Slope field plot

Verification of solutions

$$\frac{x^2y^2}{2} = \frac{x^4}{4} + c_1$$

Verified OK.

7.5.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-x^2+y^2}{yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= x \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{y^2}{x} + x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{x} + x \\
w' &= -\frac{2w}{x} + 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= 2x
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = 2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(2x) \\ \frac{d}{dx}(x^2 w) &= (x^2)(2x) \\ d(x^2 w) &= (2x^3)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 w &= \int 2x^3\, dx \\ x^2 w &= \frac{x^4}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = \frac{x^2}{2} + \frac{c_1}{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{x^2}{2} + \frac{c_1}{x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{2x^4 + 4c_1}}{2x} \\ y(x) &= -\frac{\sqrt{2x^4 + 4c_1}}{2x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2x^4 + 4c_1}}{2x} \tag{1}$$

$$y = -\frac{\sqrt{2x^4 + 4c_1}}{2x} \tag{2}$$

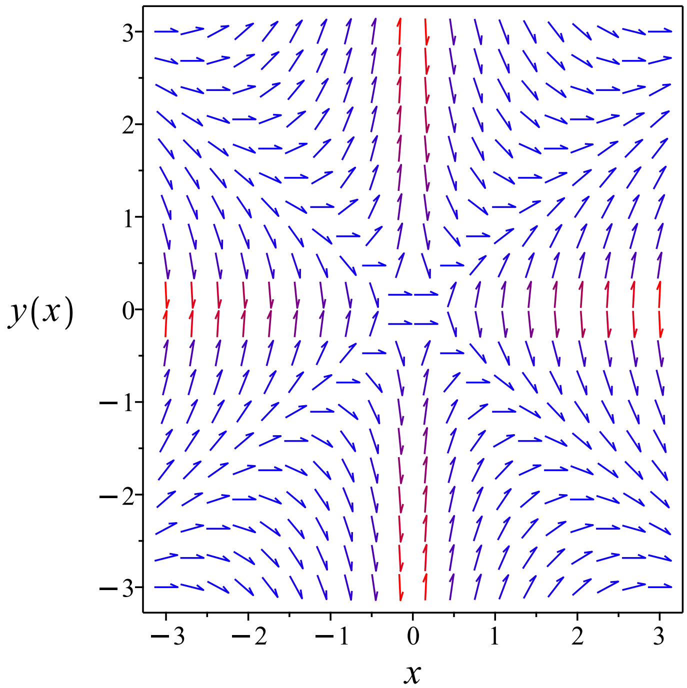


Figure 348: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2x^4 + 4c_1}}{2x}$$

Verified OK.

$$y = -\frac{\sqrt{2x^4 + 4c_1}}{2x}$$

Verified OK.

7.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(yx)\,\mathrm{d}y &= \left(x^2 - y^2\right)\mathrm{d}x \\
\left(-x^2 + y^2\right)\mathrm{d}x + (yx)\,\mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -x^2 + y^2 \\
N(x, y) &= yx
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 + y^2\right) \\
&= 2y
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(yx) \\
&= y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{yx}((2y) - (y)) \\
&= \frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x(-x^2 + y^2) \\
&= -x^3 + x\,y^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x(yx) \\
&= y\,x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-x^3 + x\,y^2) + (y\,x^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x^3 + x\,y^2 \, \mathrm{d}x$$

$$\phi = -\frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = y\,x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y\,x^2$. Therefore equation (4) becomes

$$y\,x^2 = y\,x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{4}x^4 + \frac{1}{2}x^2y^2$$

Summary

The solution(s) found are the following

$$\frac{x^2 y^2}{2} - \frac{x^4}{4} = c_1 \tag{1}$$

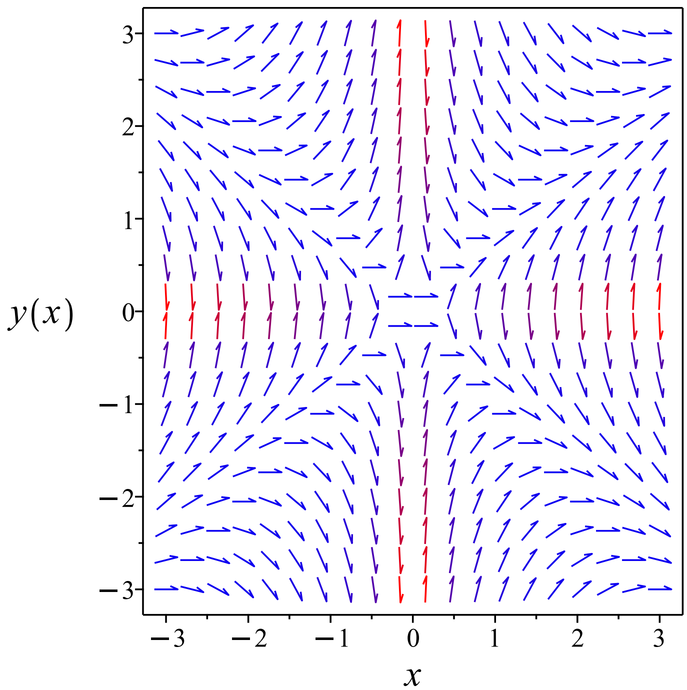


Figure 349: Slope field plot

Verification of solutions

$$\frac{x^2 y^2}{2} - \frac{x^4}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(x*y(x)*diff(y(x),x)=(x^2-y(x)^2),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2x^4 + 4c_1}}{2x}$$
$$y(x) = \frac{\sqrt{2x^4 + 4c_1}}{2x}$$

✓ Solution by Mathematica

Time used: 0.21 (sec). Leaf size: 46

```
DSolve[x*y[x]*y'[x]==(x^2-y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{x^4}{2} + c_1}}{x}$$
$$y(x) \to \frac{\sqrt{\frac{x^4}{2} + c_1}}{x}$$

7.6 problem 6

Internal problem ID [2016]
Internal file name [`OUTPUT/2016_Sunday_February_25_2024_06_44_44_AM_84748663/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y' - yx - \sqrt{y}\, x\, e^{x^2} = 0}$$

7.6.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= yx + \sqrt{y}\, x\, e^{x^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 204: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \sqrt{y}\, \mathrm{e}^{\frac{x^2}{4}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{y}\, \mathrm{e}^{\frac{x^2}{4}}} dy \end{aligned}$$

Which results in

$$S = 2\sqrt{y}\, \mathrm{e}^{-\frac{x^2}{4}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = yx + \sqrt{y}\, x\, \mathrm{e}^{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\sqrt{y}\, x\, \mathrm{e}^{-\frac{x^2}{4}} \\ S_y &= \frac{\mathrm{e}^{-\frac{x^2}{4}}}{\sqrt{y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\, \mathrm{e}^{\frac{3x^2}{4}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\, \mathrm{e}^{\frac{3R^2}{4}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{2\,\mathrm{e}^{\frac{3R^2}{4}}}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\sqrt{y}\,\mathrm{e}^{-\frac{x^2}{4}} = \frac{2\,\mathrm{e}^{\frac{3x^2}{4}}}{3} + c_1$$

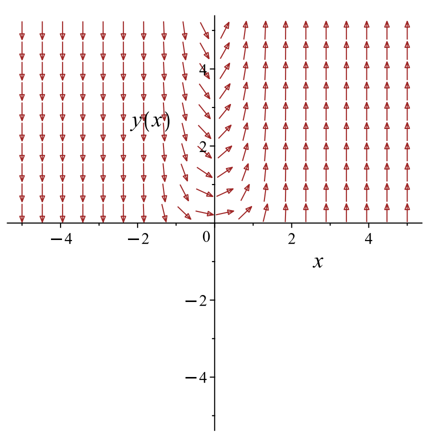
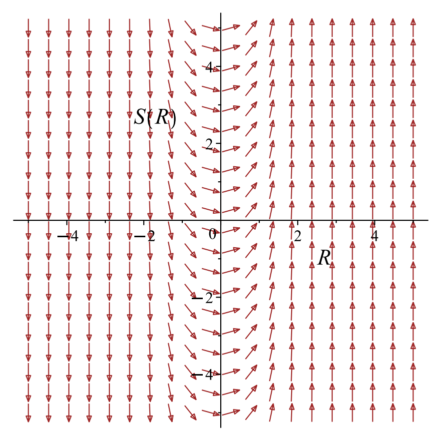
Which simplifies to

$$2\sqrt{y}\,\mathrm{e}^{-\frac{x^2}{4}} = \frac{2\,\mathrm{e}^{\frac{3x^2}{4}}}{3} + c_1$$

Which gives

$$y = \frac{\left(2\,\mathrm{e}^{\frac{3x^2}{4}} + 3c_1\right)^2 \mathrm{e}^{\frac{x^2}{2}}}{36}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = yx + \sqrt{y}\,x\,\mathrm{e}^{x^2}$ 	$R = x$ $S = 2\sqrt{y}\,\mathrm{e}^{-\frac{x^2}{4}}$	$\frac{dS}{dR} = R\,\mathrm{e}^{\frac{3R^2}{4}}$ 

Summary
The solution(s) found are the following

$$y = \frac{\left(2\,\mathrm{e}^{\frac{3x^2}{4}} + 3c_1\right)^2 \mathrm{e}^{\frac{x^2}{2}}}{36} \tag{1}$$

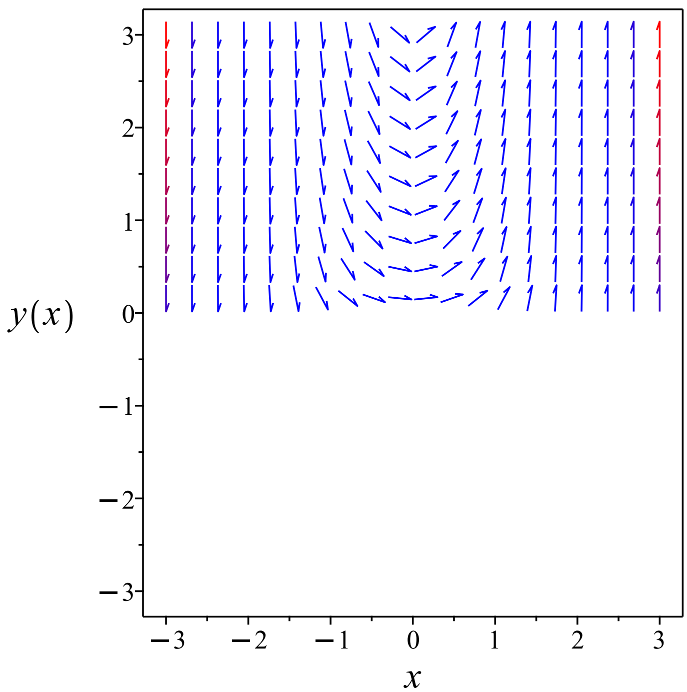


Figure 350: Slope field plot

Verification of solutions

$$y = \frac{\left(2\,\mathrm{e}^{\frac{3x^2}{4}} + 3c_1\right)^2 \mathrm{e}^{\frac{x^2}{2}}}{36}$$

Verified OK.

7.6.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= yx + \sqrt{y}\, x\, \mathrm{e}^{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = xy + \mathrm{e}^{x^2} x\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= x \\ f_1(x) &= \mathrm{e}^{x^2}x \\ n &= \frac{1}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y'\frac{1}{\sqrt{y}} = x\sqrt{y} + \mathrm{e}^{x^2}x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \sqrt{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{2\sqrt{y}}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 2w'(x) &= w(x)\,x + \mathrm{e}^{x^2}x \\ w' &= \frac{xw}{2} + \frac{\mathrm{e}^{x^2}x}{2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{x}{2} \\
q(x) &= \frac{e^{x^2} x}{2}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{w(x)\, x}{2} = \frac{e^{x^2} x}{2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{x}{2} dx} \\
&= e^{-\frac{x^2}{4}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu) \left(\frac{e^{x^2} x}{2}\right) \\
\frac{d}{dx}\left(e^{-\frac{x^2}{4}} w\right) &= \left(e^{-\frac{x^2}{4}}\right) \left(\frac{e^{x^2} x}{2}\right) \\
d\left(e^{-\frac{x^2}{4}} w\right) &= \left(\frac{x\, e^{\frac{3x^2}{4}}}{2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-\frac{x^2}{4}} w &= \int \frac{x\, e^{\frac{3x^2}{4}}}{2}\, dx \\
e^{-\frac{x^2}{4}} w &= \frac{e^{\frac{3x^2}{4}}}{3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x^2}{4}}$ results in

$$
w(x) = \frac{e^{\frac{x^2}{4}} e^{\frac{3x^2}{4}}}{3} + c_1 e^{\frac{x^2}{4}}
$$

which simplifies to

$$
w(x) = \frac{e^{x^2}}{3} + c_1 e^{\frac{x^2}{4}}
$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$\sqrt{y} = \frac{e^{x^2}}{3} + c_1 e^{\frac{x^2}{4}}$$

Summary

The solution(s) found are the following

$$\sqrt{y} = \frac{e^{x^2}}{3} + c_1 e^{\frac{x^2}{4}} \tag{1}$$

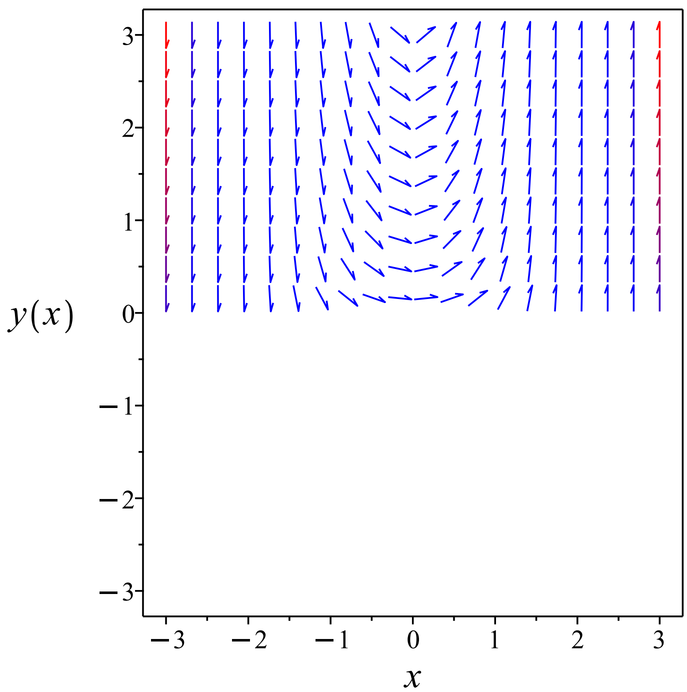


Figure 351: Slope field plot

Verification of solutions

$$\sqrt{y} = \frac{e^{x^2}}{3} + c_1 e^{\frac{x^2}{4}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)-x*y(x)=sqrt(y(x))*x*exp(x^2),y(x), singsol=all)
```

$$\sqrt{y(x)}-\frac{e^{x^2}}{3}-e^{\frac{x^2}{4}}c_1=0$$

✓ Solution by Mathematica

Time used: 0.307 (sec). Leaf size: 34

```
DSolve[y'[x]-x*y[x]==Sqrt[y[x]]*x*Exp[x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{9}e^{\frac{x^2}{2}}\left(e^{\frac{3x^2}{4}}+3c_1\right)^2$$

7.7 problem 7

Internal problem ID [2017]
Internal file name [`OUTPUT/2017_Sunday_February_25_2024_06_44_44_AM_8156806/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{tx' + x(1 - x^2t^4) = 0}$$

7.7.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} x' &= \frac{x(x^2t^4 - 1)}{t} \\ x' &= \omega(t, x) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 206: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, x) &= 0 \\
\eta(t, x) &= x^3 t^2
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^3 t^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{2t^2 x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = \frac{x(x^2 t^4 - 1)}{t}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= \frac{1}{t^3 x^2} \\ S_x &= \frac{1}{x^3 t^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = t \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

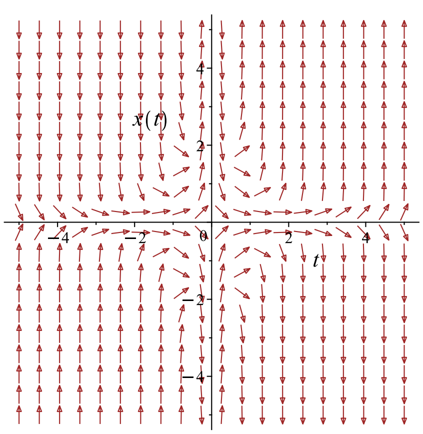
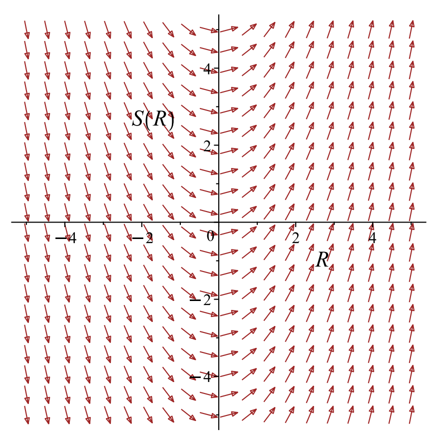
To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$-\frac{1}{2t^2x^2} = \frac{t^2}{2} + c_1$$

Which simplifies to

$$-\frac{1}{2t^2x^2} = \frac{t^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = \frac{x(x^2t^4-1)}{t}$ 	$R = t$ $S = -\frac{1}{2t^2x^2}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$-\frac{1}{2t^2x^2} = \frac{t^2}{2} + c_1 \tag{1}$$

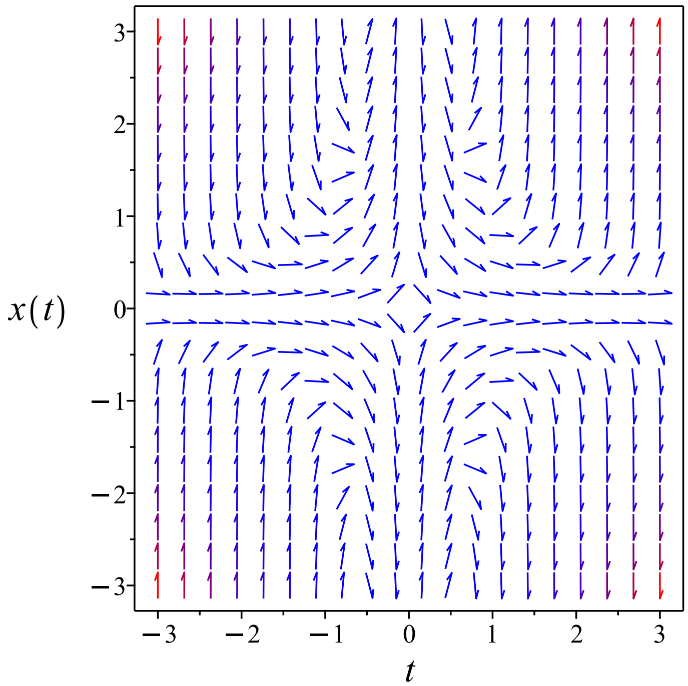


Figure 352: Slope field plot

Verification of solutions

$$-\frac{1}{2t^2x^2} = \frac{t^2}{2} + c_1$$

Verified OK.

7.7.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= \frac{x(x^2t^4 - 1)}{t} \end{aligned}$$

This is a Bernoulli ODE.

$$x' = -\frac{1}{t}x + t^3x^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$x' = f_0(t)x + f_1(t)x^n \tag{2}$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_0(t)x^{1-n} + f_1(t) \tag{3}$$

The next step is use the substitution $w = x^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(t) &= -\frac{1}{t} \\
f_1(t) &= t^3 \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $x^n = x^3$ gives

$$
x' \frac{1}{x^3} = -\frac{1}{t\,x^2} + t^3 \tag{4}
$$

Let

$$
\begin{aligned}
w &= x^{1-n} \\
&= \frac{1}{x^2} \tag{5}
\end{aligned}
$$

Taking derivative of equation (5) w.r.t t gives

$$
w' = -\frac{2}{x^3} x' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(t)}{2} &= -\frac{w(t)}{t} + t^3 \\
w' &= \frac{2w}{t} - 2t^3 \tag{7}
\end{aligned}
$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(t) + p(t)w(t) = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= -\frac{2}{t} \\
q(t) &= -2t^3
\end{aligned}
$$

Hence the ode is

$$w'(t) - \frac{2w(t)}{t} = -2t^3$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{2}{t}dt} \\
&= \frac{1}{t^2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu w) &= (\mu)\left(-2t^3\right) \\
\frac{d}{dt}\left(\frac{w}{t^2}\right) &= \left(\frac{1}{t^2}\right)\left(-2t^3\right) \\
d\left(\frac{w}{t^2}\right) &= (-2t)\ dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{w}{t^2} &= \int -2t\,dt \\
\frac{w}{t^2} &= -t^2 + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{t^2}$ results in

$$w(t) = -t^4 + c_1 t^2$$

Replacing w in the above by $\frac{1}{x^2}$ using equation (5) gives the final solution.

$$\frac{1}{x^2} = -t^4 + c_1 t^2$$

Solving for x gives

$$\begin{aligned}
x(t) &= \frac{1}{\sqrt{-t^2 + c_1}\,t} \\
x(t) &= -\frac{1}{\sqrt{-t^2 + c_1}\,t}
\end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{1}{\sqrt{-t^2 + c_1}\,t} \tag{1}$$

$$x = -\frac{1}{\sqrt{-t^2 + c_1}\,t} \tag{2}$$

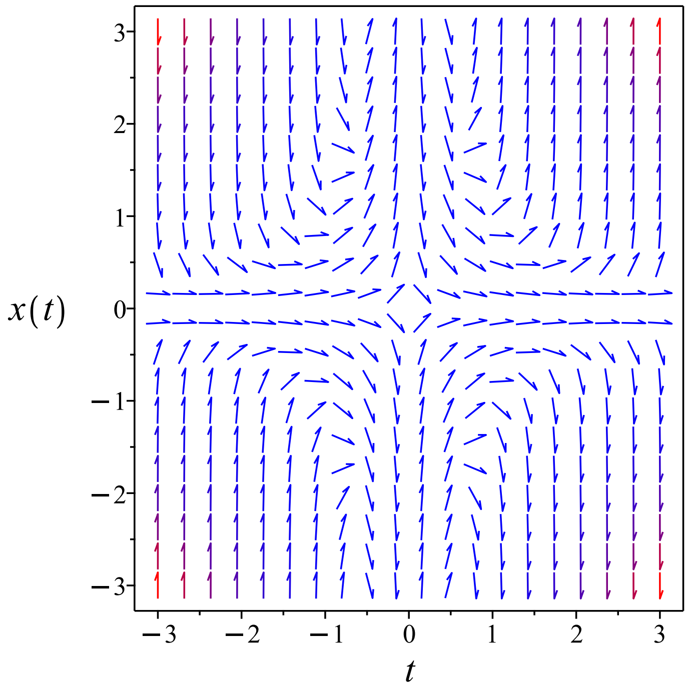


Figure 353: Slope field plot

Verification of solutions

$$x = \frac{1}{\sqrt{-t^2 + c_1 t}}$$

Verified OK.

$$x = -\frac{1}{\sqrt{-t^2 + c_1 t}}$$

Verified OK.

7.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}(t)\,\mathrm{d}x&=\left(-x\left(-x^2t^4+1\right)\right)\mathrm{d}t\\ \left(x\left(-x^2t^4+1\right)\right)\mathrm{d}t+(t)\,\mathrm{d}x&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,x)&=x\left(-x^2t^4+1\right)\\ N(t,x)&=t\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x}&=\frac{\partial}{\partial x}\left(x\left(-x^2t^4+1\right)\right)\\ &=-3x^2t^4+1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(t) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= \frac{1}{t}\left(\left(-3x^2t^4+1\right) - (1)\right) \\
&= -3t^3x^2
\end{aligned}
$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}\right) \\
&= -\frac{1}{x^3t^4 - x}\left((1) - \left(-3x^2t^4+1\right)\right) \\
&= -\frac{3t^4x}{x^2t^4 - 1}
\end{aligned}
$$

Since B depends on t, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = tx$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}}{xM - yN} \\
&= \frac{(1) - \left(-3x^2t^4+1\right)}{t\left(x\left(-x^2t^4+1\right)\right) - x\left(t\right)} \\
&= -\frac{3}{tx}
\end{aligned}
$$

Replacing all powers of terms tx by t gives

$$
R = -\frac{3}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}\mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(-\frac{3}{t}\right) \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(t)} \\ &= \frac{1}{t^3}\end{aligned}$$

Now t is replaced back with tx giving

$$\mu = \frac{1}{t^3 x^3}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{t^3 x^3}\left(x\left(-x^2 t^4 + 1\right)\right) \\ &= \frac{-x^2 t^4 + 1}{x^2 t^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{t^3 x^3}(t) \\ &= \frac{1}{x^3 t^2}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ \left(\frac{-x^2 t^4 + 1}{x^2 t^3}\right) + \left(\frac{1}{x^3 t^2}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \frac{-x^2t^4+1}{x^2t^3} \, \mathrm{d}t$$
$$\phi = \frac{-x^2t^4-1}{2t^2x^2} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -\frac{t^2}{x} - \frac{-x^2t^4-1}{t^2x^3} + f'(x) \tag{4}$$
$$= \frac{1}{x^3t^2} + f'(x)$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{x^3t^2}$. Therefore equation (4) becomes

$$\frac{1}{x^3t^2} = \frac{1}{x^3t^2} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{-x^2t^4-1}{2t^2x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^2t^4-1}{2t^2x^2}$$

Summary

The solution(s) found are the following

$$\frac{-x^2t^4-1}{2t^2x^2}=c_1 \tag{1}$$

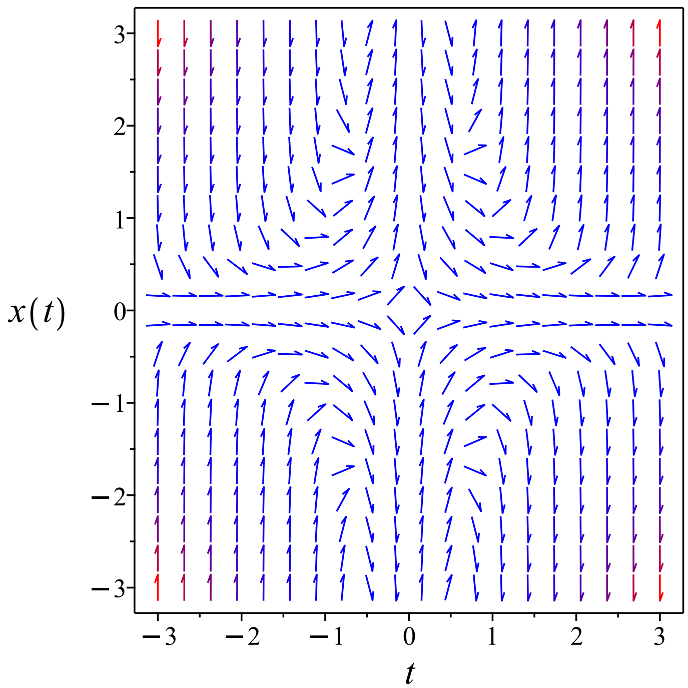


Figure 354: Slope field plot

Verification of solutions

$$\frac{-x^2t^4-1}{2t^2x^2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(t*diff(x(t),t)+x(t)*(1-x(t)^2*t^4)=0,x(t), singsol=all)
```

$$x(t) = \frac{1}{\sqrt{-t^2 + c_1}\, t}$$
$$x(t) = -\frac{1}{\sqrt{-t^2 + c_1}\, t}$$

✓ Solution by Mathematica

Time used: 0.37 (sec). Leaf size: 48

```
DSolve[t*x'[t]+x[t]*(1-x[t]^2*t^4)==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{1}{\sqrt{-t^4 + c_1 t^2}}$$
$$x(t) \to \frac{1}{\sqrt{-t^4 + c_1 t^2}}$$
$$x(t) \to 0$$

7.8 problem 8

Internal problem ID [2018]
Internal file name [OUTPUT/2018_Sunday_February_25_2024_06_44_45_AM_47149512/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y'x^2 + y^2 - yx = 0}$$

7.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 + u(x)^2\,x^2 - u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u^2}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{u^2}\, du &= \int -\frac{1}{x}\, dx \\
-\frac{1}{u} &= -\ln(x) + c_2
\end{aligned}
$$

The solution is

$$
-\frac{1}{u(x)} + \ln(x) - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
-\frac{x}{y} + \ln(x) - c_2 &= 0 \\
-\frac{x}{y} + \ln(x) - c_2 &= 0
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
-\frac{x}{y} + \ln(x) - c_2 = 0 \tag{1}
$$

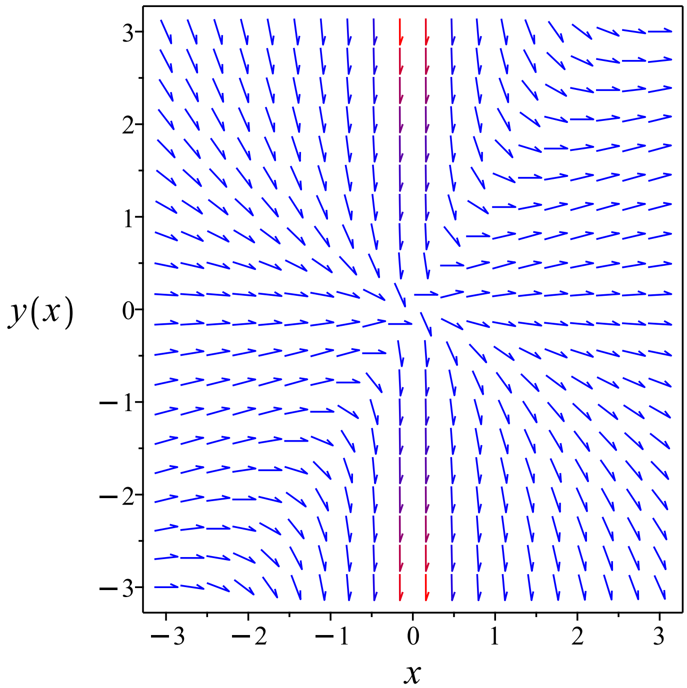


Figure 355: Slope field plot

Verification of solutions

$$-\frac{x}{y}+\ln(x)-c_2=0$$

Verified OK.

7.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=-\frac{y(-x+y)}{x^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 208: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^2}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-x + y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = -\ln(x) + c_1$$

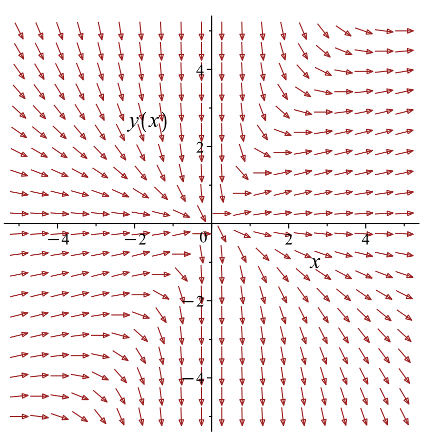
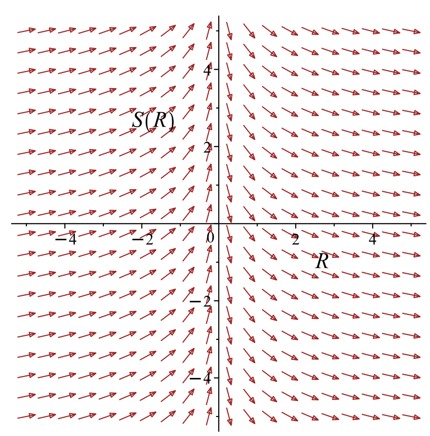
Which simplifies to

$$-\frac{x}{y} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y(-x+y)}{x^2}$
</td><td>$R = x$
$S = -\frac{x}{y}$</td><td>$\frac{dS}{dR} = -\frac{1}{R}$
</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = \frac{x}{\ln(x) - c_1} \tag{1}$$

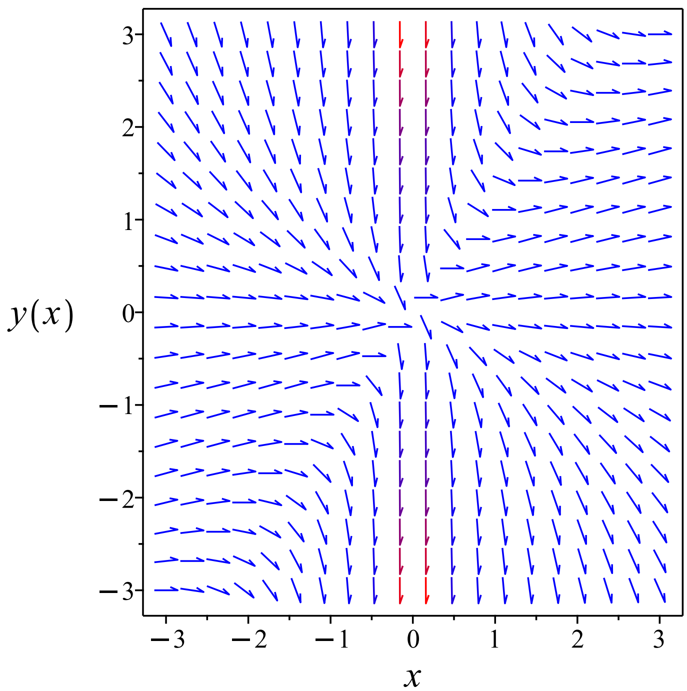


Figure 356: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) - c_1}$$

Verified OK.

7.8.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(-x+y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -\frac{1}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{yx} - \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} - \frac{1}{x^2} \\ w' &= -\frac{w}{x} + \frac{1}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = \frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x^2}\right) \\ \frac{d}{dx}(xw) &= (x)\left(\frac{1}{x^2}\right) \\ d(xw) &= \frac{1}{x}\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} xw &= \int \frac{1}{x}\, dx \\ xw &= \ln(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{\ln(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) + c_1} \tag{1}$$

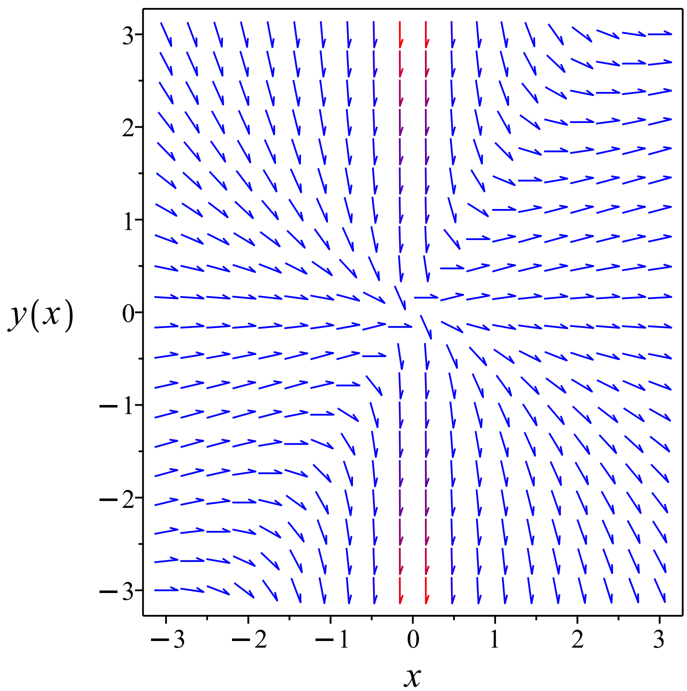


Figure 357: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) + c_1}$$

Verified OK.

7.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y&=(yx-y^2)\,\mathrm{d}x\\ (-yx+y^2)\,\mathrm{d}x+(x^2)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-yx+y^2\\ N(x,y)&=x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-yx + y^2\right)\\ &= -x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2\right)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y^2 - yx$ and $N = x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{y^2 - yx}{xy^2}\\ N &= \frac{x}{y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{x}{y^2}\right)\mathrm{d}y &= \left(-\frac{-yx + y^2}{y^2x}\right)\mathrm{d}x\\ \left(\frac{-yx + y^2}{y^2x}\right)\mathrm{d}x + \left(\frac{x}{y^2}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \frac{-yx + y^2}{y^2x}\\ N(x, y) &= \frac{x}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-yx + y^2}{y^2x}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{-yx + y^2}{y^2 x} \,\mathrm{d}x$$
$$\phi = \ln (x) - \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln (x) - \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{x}{y}$$

The solution becomes

$$y = \frac{x}{\ln(x) - c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) - c_1} \tag{1}$$

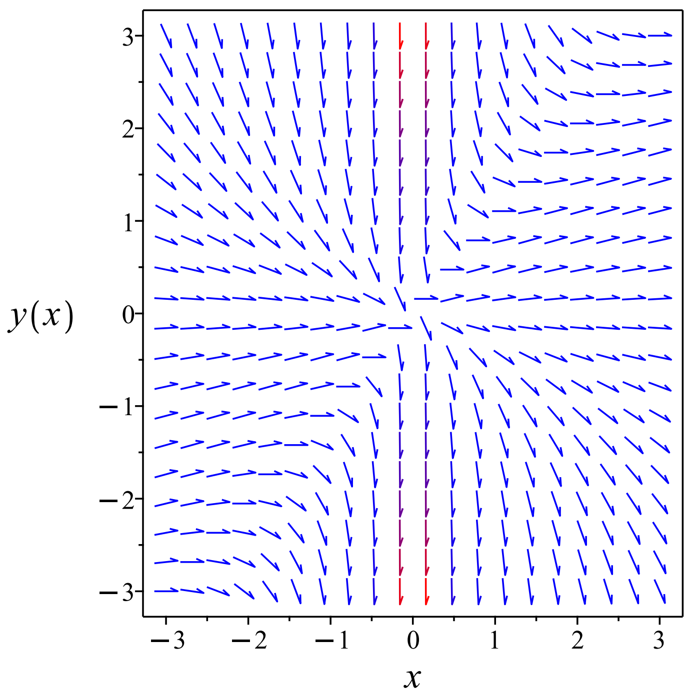


Figure 358: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) - c_1}$$

Verified OK.

7.8.5 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= -\frac{y(-x+y)}{x^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = \frac{y}{x} - \frac{y^2}{x^2}
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{u}{x^2}} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= \frac{2}{x^3} \\
f_1 f_2 &= -\frac{1}{x^3} \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{u''(x)}{x^2} - \frac{u'(x)}{x^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 + c_2 \ln(x)
$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x}{c_1 + c_2 \ln(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{c_3 + \ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{c_3 + \ln(x)} \tag{1}$$

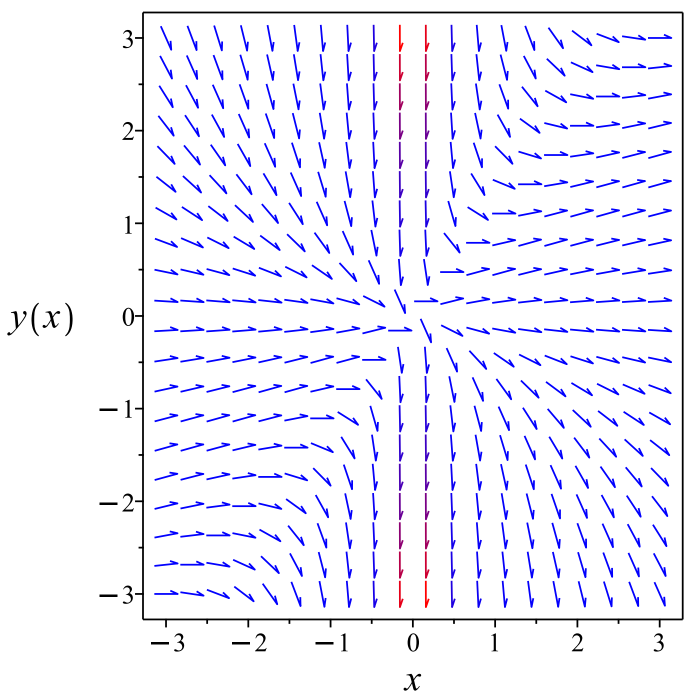


Figure 359: Slope field plot

Verification of solutions

$$y = \frac{x}{c_3 + \ln(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 12

```
dsolve(x^2*diff(y(x),x)+y(x)^2=x*y(x),y(x), singsol=all)
```

$$y(x) = \frac{x}{\ln(x) + c_1}$$

✓ Solution by Mathematica
Time used: 0.134 (sec). Leaf size: 19

```
DSolve[x^2*y'[x]+y[x]^2==x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{\log(x) + c_1}$$
$$y(x) \to 0$$

7.9 problem 9

Internal problem ID [2019]
Internal file name [`OUTPUT/2019_Sunday_February_25_2024_06_44_46_AM_33833873/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{\csc(y)\cot(y)\,y' - \csc(y) = e^{x}}$$

7.9.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\csc(y)\cot(y))\,\mathrm{d}y&=(\csc(y)+\mathrm{e}^x)\,\mathrm{d}x\\(-\csc(y)-\mathrm{e}^x)\,\mathrm{d}x+(\csc(y)\cot(y))\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\csc(y)-\mathrm{e}^x\\N(x,y)&=\csc(y)\cot(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-\csc(y)-\mathrm{e}^x)\\&=\csc(y)\cot(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(\csc(y)\cot(y))\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=\sin(y)\tan(y)\left((\csc(y)\cot(y))-(0)\right)\\&=1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}(-\csc(y) - \mathrm{e}^{x}) \\
&= -(\csc(y) + \mathrm{e}^{x})\,\mathrm{e}^{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(\csc(y)\cot(y)) \\
&= \csc(y)\,\mathrm{e}^{x}\cot(y)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-(\csc(y) + \mathrm{e}^{x})\,\mathrm{e}^{x}) + (\csc(y)\,\mathrm{e}^{x}\cot(y))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -(\csc(y) + \mathrm{e}^{x})\,\mathrm{e}^{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\csc(y)\,\mathrm{e}^{x} - \frac{\mathrm{e}^{2x}}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \csc(y)\,\mathrm{e}^{x}\cot(y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \csc(y)\,\mathrm{e}^{x}\cot(y)$. Therefore equation (4) becomes

$$\csc(y)\,\mathrm{e}^{x}\cot(y) = \csc(y)\,\mathrm{e}^{x}\cot(y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\csc(y)\,\mathrm{e}^{x} - \frac{\mathrm{e}^{2x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\csc(y)\,\mathrm{e}^{x} - \frac{\mathrm{e}^{2x}}{2}$$

The solution becomes

$$y = -\operatorname{arccsc}\left(\frac{(\mathrm{e}^{2x} + 2c_1)\,\mathrm{e}^{-x}}{2}\right)$$

Summary

The solution(s) found are the following

$$y = -\operatorname{arccsc}\left(\frac{(\mathrm{e}^{2x} + 2c_1)\,\mathrm{e}^{-x}}{2}\right) \tag{1}$$

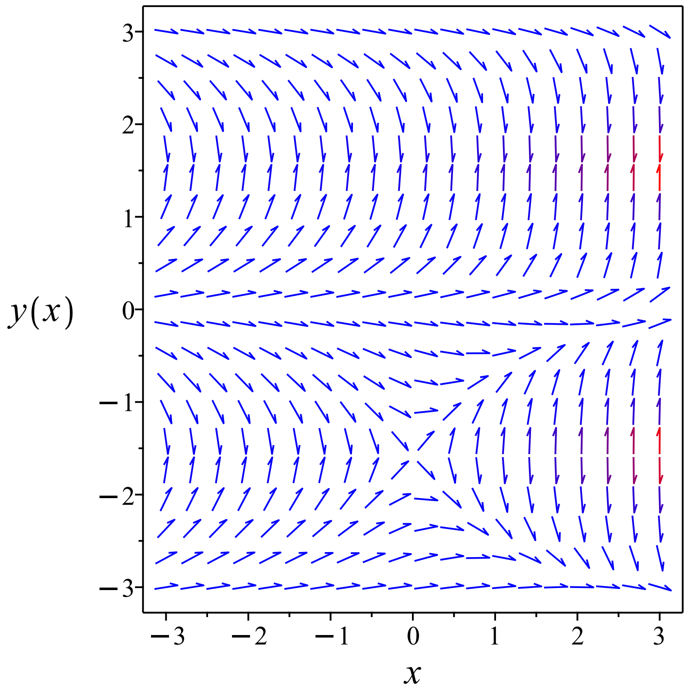


Figure 360: Slope field plot

Verification of solutions

$$y = -\operatorname{arccsc}\left(\frac{(e^{2x} + 2c_1)\, e^{-x}}{2}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(csc(y(x))*cot(y(x))*diff(y(x),x)=(csc(y(x))+exp(x)),y(x), singsol=all)
```

$$y(x) = \operatorname{arccsc}\left(-\frac{e^{x}}{2} + e^{-x} c_1\right)$$

✓ Solution by Mathematica

Time used: 1.115 (sec). Leaf size: 30

```
DSolve[Csc[y[x]]*Cot[y[x]]*y'[x]==(Csc[y[x]]+Exp[x]),y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\csc^{-1}\left(\frac{e^x}{2} - c_1 e^{-x}\right)$$
$$y(x) \to 0$$

7.10 problem 10

Internal problem ID [2020]
Internal file name [`OUTPUT/2020_Sunday_February_25_2024_06_44_49_AM_36336011/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - yx - \frac{x}{y} = 0}$$

7.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(y^2+1)\,x}{y} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{y^2+1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{y}}\,dy = x\,dx$$

$$\int \frac{1}{\frac{y^2+1}{y}}\, dy = \int x\, dx$$

$$\frac{\ln(y^2+1)}{2} = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = e^{\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = c_2 e^{\frac{x^2}{2}}$$

The solution is

$$\sqrt{1+y^2} = c_2 e^{\frac{x^2}{2}+c_1}$$

Summary
The solution(s) found are the following

$$\sqrt{1+y^2} = c_2 e^{\frac{x^2}{2}+c_1} \tag{1}$$

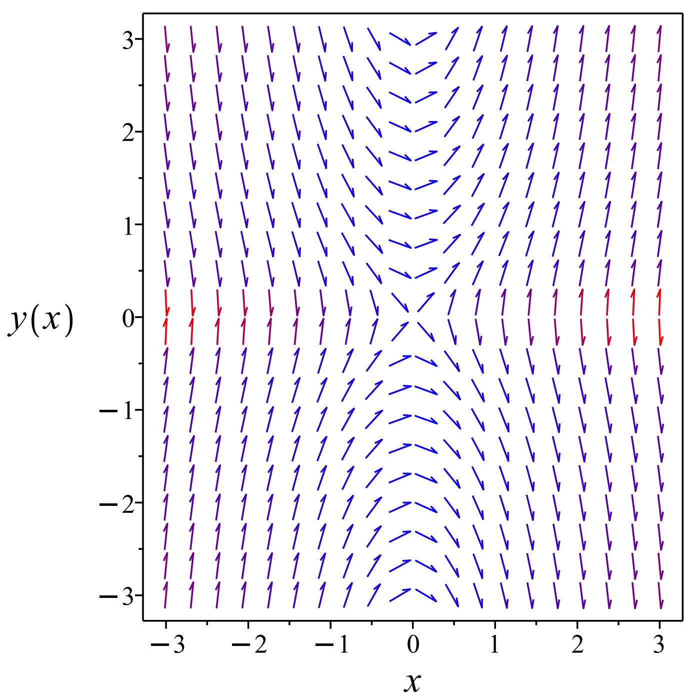


Figure 361: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} = c_2 \mathrm{e}^{\frac{x^2}{2}+c_1}$$

Verified OK.

7.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{(y^2+1)\,x}{y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 210: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{(y^2 + 1)\, x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

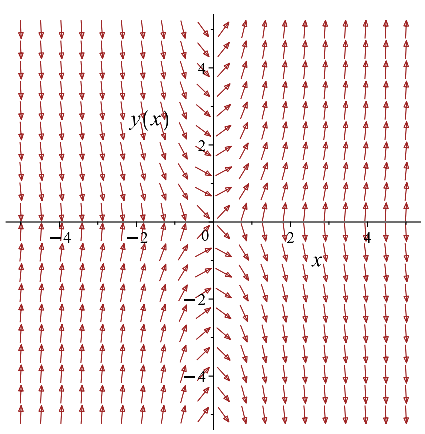
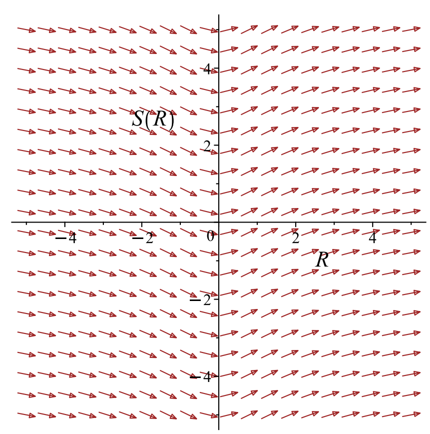
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(y^2+1)x}{y}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{R}{R^2+1}$ 

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = \frac{\ln(1+y^2)}{2} + c_1 \tag{1}$$

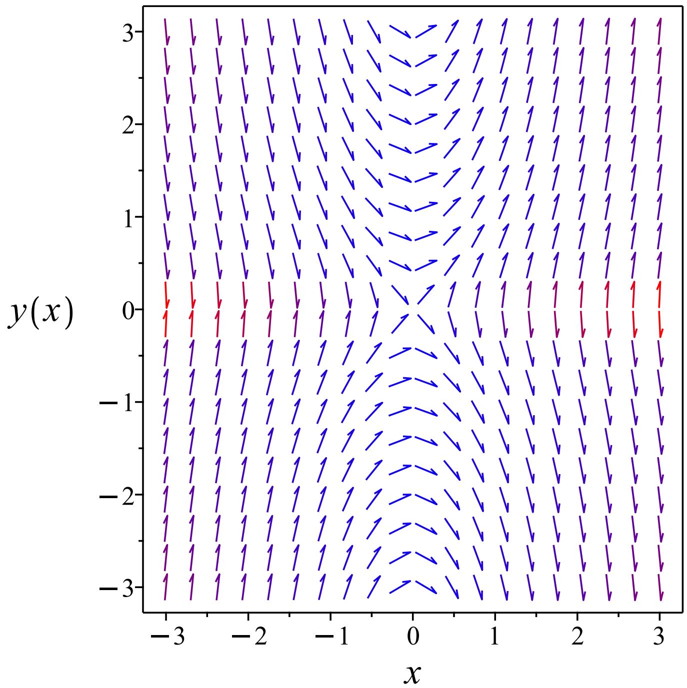


Figure 362: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

Verified OK.

7.10.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{(y^2+1)\,x}{y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = xy + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= x \\
f_1(x) &= x \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = x\,y^2 + x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= w(x)\,x + x \\
w' &= 2xw + 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -2x \\
q(x) &= 2x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - 2w(x)\,x = 2x
$$

The integrating factor μ is

$$\mu = e^{\int -2x dx}$$
$$= e^{-x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(2x)$$
$$\frac{d}{dx}\left(e^{-x^2} w\right) = \left(e^{-x^2}\right)(2x)$$
$$d\left(e^{-x^2} w\right) = \left(2x\, e^{-x^2}\right) dx$$

Integrating gives

$$e^{-x^2} w = \int 2x\, e^{-x^2} dx$$
$$e^{-x^2} w = -e^{-x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$w(x) = -e^{-x^2} e^{x^2} + c_1 e^{x^2}$$

which simplifies to

$$w(x) = -1 + c_1 e^{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -1 + c_1 e^{x^2}$$

Solving for y gives

$$y(x) = \sqrt{-1 + c_1 e^{x^2}}$$
$$y(x) = -\sqrt{-1 + c_1 e^{x^2}}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-1 + c_1 e^{x^2}} \tag{1}$$
$$y = -\sqrt{-1 + c_1 e^{x^2}} \tag{2}$$

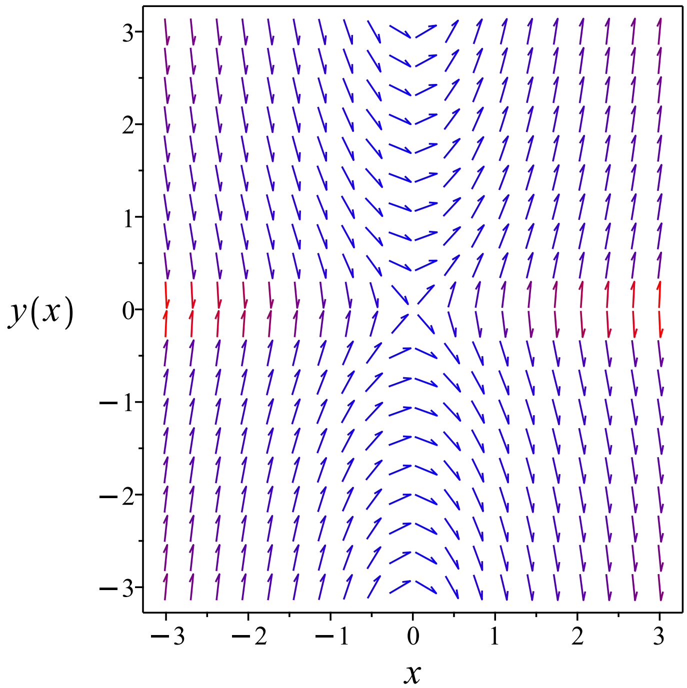


Figure 363: Slope field plot

Verification of solutions

$$y = \sqrt{-1 + c_1 e^{x^2}}$$

Verified OK.

$$y = -\sqrt{-1 + c_1 e^{x^2}}$$

Verified OK.

7.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^{2} \phi}{\partial x \partial y}=\frac{\partial^{2} \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^{2} \phi}{\partial x \partial y}=\frac{\partial^{2} \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{y}{y^{2}+1}\right) \mathrm{d} y &= (x) \mathrm{d} x \\
(-x) \mathrm{d} x+\left(\frac{y}{y^{2}+1}\right) \mathrm{d} y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{y}{y^{2}+1}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{y^2+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{y^2+1}$. Therefore equation (4) becomes

$$\frac{y}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{y^2+1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{y}{y^2+1}\right)\mathrm{d}y\\ f(y) &= \frac{\ln(y^2+1)}{2} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(y^2+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(y^2+1)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

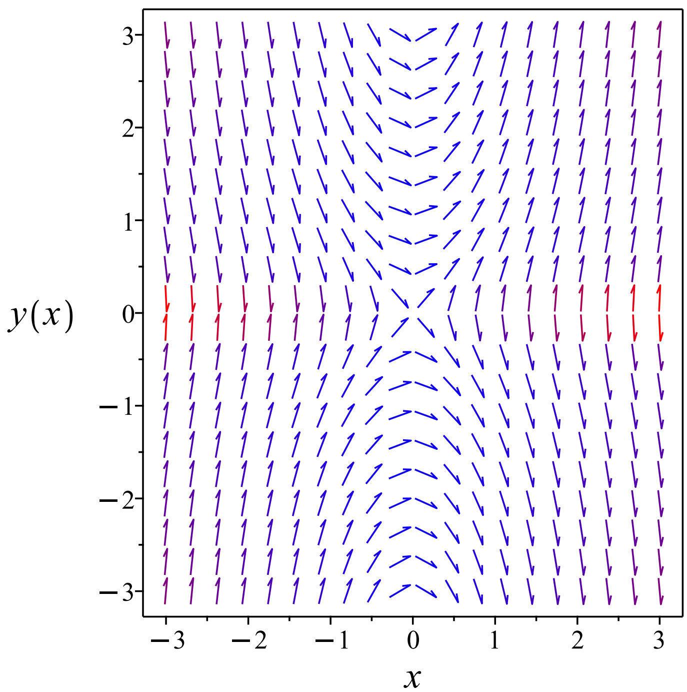


Figure 364: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{\ln(1+y^2)}{2} = c_1$$

Verified OK.

7.10.5 Maple step by step solution

Let's solve

$y' - yx - \frac{x}{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{yy'}{1+y^2} = x$

- Integrate both sides with respect to x

  $\int \frac{yy'}{1+y^2}dx = \int x dx + c_1$

- Evaluate integral

  $\frac{\ln(1+y^2)}{2} = \frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{e^{x^2+2c_1} - 1}, y = -\sqrt{e^{x^2+2c_1} - 1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)-x*y(x)=x/y(x),y(x), singsol=all)
```

$$y(x) = \sqrt{e^{x^2} c_1 - 1}$$
$$y(x) = -\sqrt{e^{x^2} c_1 - 1}$$

✓ Solution by Mathematica

Time used: 6.967 (sec). Leaf size: 57

```
DSolve[y'[x]-x*y[x]==x/y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-1 + e^{x^2+2c_1}}$$
$$y(x) \to \sqrt{-1 + e^{x^2+2c_1}}$$
$$y(x) \to -i$$
$$y(x) \to i$$

7.11 problem 11

Internal problem ID [2021]
Internal file name [`OUTPUT/2021_Sunday_February_25_2024_06_44_49_AM_29075328/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y'x + y - y^2x^2\cos(x) = 0}$$

7.11.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(\cos(x)\,x^2y - 1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 213: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x\, y^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x\,y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(\cos(x)\, x^2 y - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^2 y} \\ S_y &= \frac{1}{x\,y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{yx} = \sin(x) + c_1$$

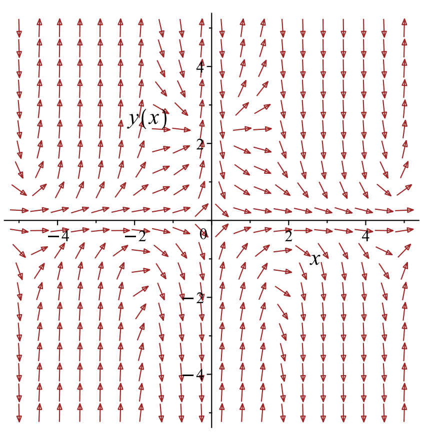
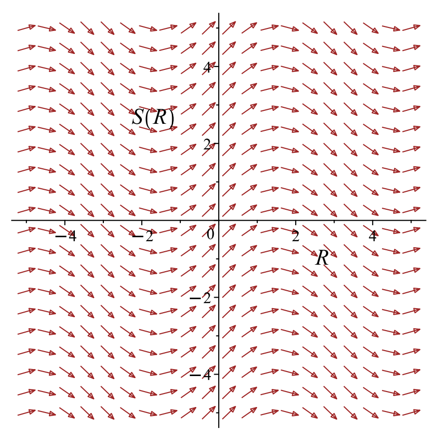
Which simplifies to

$$-\frac{1}{yx} = \sin(x) + c_1$$

Which gives

$$y = -\frac{1}{x(\sin(x) + c_1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(\cos(x)x^2y-1)}{x}$ 	$R = x$ $S = -\frac{1}{yx}$	$\frac{dS}{dR} = \cos(R)$ 

Summary
The solution(s) found are the following

$$y = -\frac{1}{x(\sin(x) + c_1)} \tag{1}$$

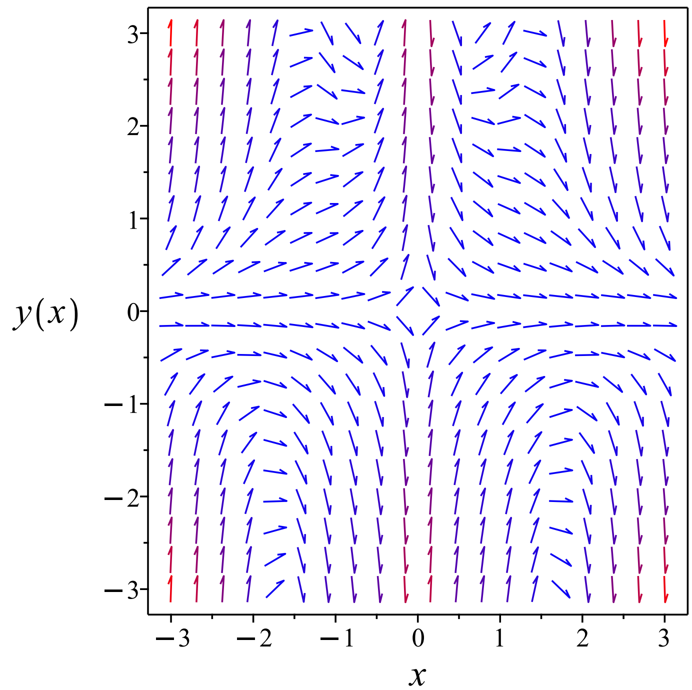


Figure 365: Slope field plot

Verification of solutions

$$y = -\frac{1}{x(\sin(x) + c_1)}$$

Verified OK.

7.11.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(\cos(x)\, x^2 y - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x} y + x\cos(x)\, y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= x\cos(x) \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{yx} + x\cos(x) \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} + x\cos(x) \\ w' &= \frac{w}{x} - x\cos(x) \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -x\cos(x)
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{w(x)}{x} = -x\cos(x)
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-x\cos(x)) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(-x\cos(x)) \\
\mathrm{d}\left(\frac{w}{x}\right) &= (-\cos(x))\ \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x} &= \int -\cos(x)\ \mathrm{d}x \\
\frac{w}{x} &= -\sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
w(x) = -\sin(x)\,x + c_1 x
$$

which simplifies to

$$
w(x) = x(-\sin(x) + c_1)
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = x(-\sin(x) + c_1)
$$

Or

$$y = \frac{1}{x \left(-\sin \left(x \right)+c_{1}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x \left(-\sin \left(x \right)+c_{1}\right)} \tag{1}$$

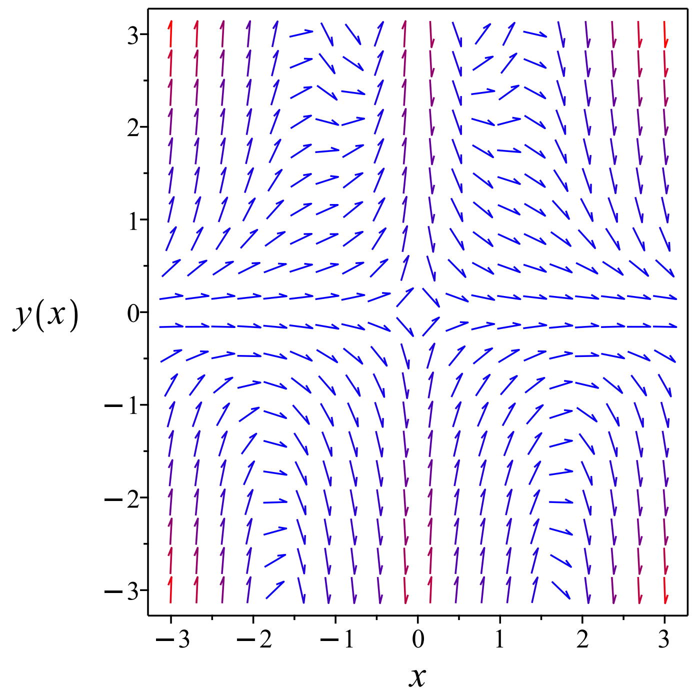


Figure 366: Slope field plot

Verification of solutions

$$y = \frac{1}{x \left(-\sin \left(x \right)+c_{1}\right)}$$

Verified OK.

7.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(-y + y^2x^2\cos(x)\right)\mathrm{d}x\\ \left(-y^2x^2\cos(x) + y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y^2x^2\cos(x) + y\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y^2 x^2 \cos(x) + y\right) \\
&= -2\cos(x)\, x^2 y + 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}\left(\left(-2\cos(x)\, x^2 y + 1\right) - (1)\right) \\
&= -2yx\cos(x)
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{y\left(\cos(x)\, x^2 y - 1\right)}\left((1) - \left(-2\cos(x)\, x^2 y + 1\right)\right) \\
&= -\frac{2\cos(x)\, x^2}{\cos(x)\, x^2 y - 1}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (-2\cos(x)\, x^2 y + 1)}{x\,(-y^2 x^2 \cos(x) + y) - y\,(x)} \\
&= -\frac{2}{yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \, \mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2 y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2 y^2}\left(-y^2 x^2 \cos(x) + y\right) \\
&= \frac{-\cos(x)\, x^2 y + 1}{y\, x^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2 y^2}(x) \\
&= \frac{1}{x\, y^2}
\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-\cos\left(x\right)x^{2}y+1}{y\,x^{2}}\right) + \left(\frac{1}{x\,y^{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int\overline{M}\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int\frac{-\cos\left(x\right)x^{2}y+1}{y\,x^{2}}\,\mathrm{d}x$$
$$\phi = \frac{-\sin\left(x\right)y-\frac{1}{x}}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= -\frac{-\sin\left(x\right)y-\frac{1}{x}}{y^{2}} - \frac{\sin\left(x\right)}{y} + f'(y) \\ &= \frac{1}{x\,y^{2}} + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{x\,y^{2}}$. Therefore equation (4) becomes

$$\frac{1}{x\,y^{2}} = \frac{1}{x\,y^{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-\sin(x)\, y - \frac{1}{x}}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-\sin(x)\, y - \frac{1}{x}}{y}$$

The solution becomes

$$y = -\frac{1}{x(\sin(x) + c_1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x(\sin(x) + c_1)} \tag{1}$$

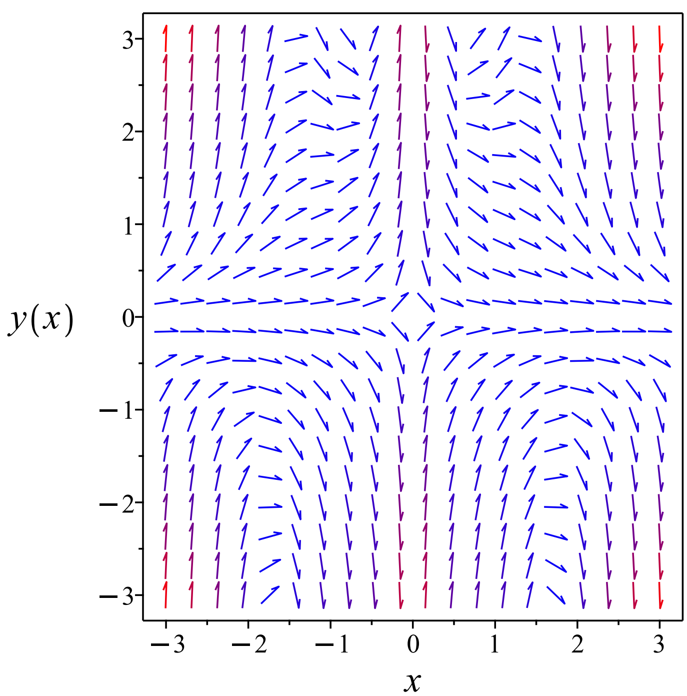


Figure 367: Slope field plot

Verification of solutions

$$y = -\frac{1}{x(\sin(x) + c_1)}$$

Verified OK.

7.11.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(\cos(x)\, x^2 y - 1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x\, y^2 \cos(x) - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = x \cos(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{x \cos(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \cos(x) - \sin(x)\, x \\ f_1 f_2 &= -\cos(x) \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$x \cos(x)\, u''(x) + \sin(x)\, x u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + c_2 \sin(x)$$

The above shows that

$$u'(x) = c_2 \cos(x)$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{x (c_1 + c_2 \sin(x))}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{x (c_3 + \sin(x))}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x (c_3 + \sin(x))} \tag{1}$$

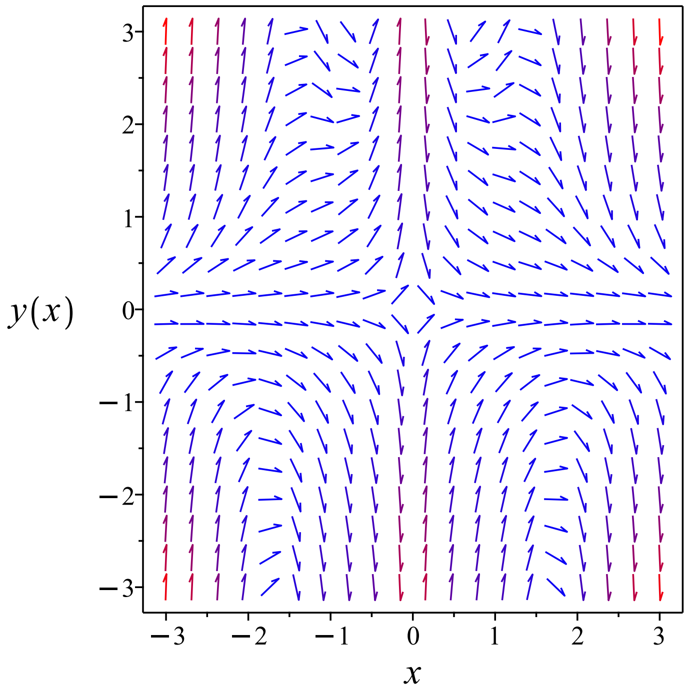


Figure 368: Slope field plot

Verification of solutions

$$y = -\frac{1}{x\left(c_3 + \sin\left(x\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x)+y(x)=y(x)^2*x^2*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{1}{(-\sin(x) + c_1)\, x}$$

✓ Solution by Mathematica

Time used: 0.177 (sec). Leaf size: 22

```
DSolve[x*y'[x]+y[x]==y[x]^2*x^2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{-x\sin(x) + c_1 x}$$
$$y(x) \to 0$$

7.12 problem 12

Internal problem ID [2022]
Internal file name [`OUTPUT/2022_Sunday_February_25_2024_06_44_51_AM_77530360/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{r' + \left(r - \frac{1}{r}\right)\theta = 0}$$

7.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} r' &= F(\theta, r) \\ &= f(\theta)g(r) \\ &= -\frac{\theta(r^2-1)}{r} \end{aligned}$$

Where $f(\theta) = -\theta$ and $g(r) = \frac{r^2-1}{r}$. Integrating both sides gives

$$\frac{1}{\frac{r^2-1}{r}}\,dr = -\theta\,d\theta$$

$$\int \frac{1}{\frac{r^2-1}{r}}\,dr = \int -\theta\,d\theta$$

$$\frac{\ln(r-1)}{2} + \frac{\ln(r+1)}{2} = -\frac{\theta^2}{2} + c_1$$

The above can be written as

$$\begin{aligned}\left(\frac{1}{2}\right)(\ln(r-1) + \ln(r+1)) &= -\frac{\theta^2}{2} + 2c_1\\ \ln(r-1) + \ln(r+1) &= (2)\left(-\frac{\theta^2}{2} + 2c_1\right)\\ &= -\theta^2 + 4c_1\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(r-1)+\ln(r+1)} = e^{-\theta^2+2c_1}$$

Which simplifies to

$$\begin{aligned}r^2 - 1 &= 2c_1 e^{-\theta^2}\\ &= c_2 e^{-\theta^2}\end{aligned}$$

The solution is

$$r^2 - 1 = c_2 e^{-\theta^2}$$

Summary

The solution(s) found are the following

$$r^2 - 1 = c_2 e^{-\theta^2} \tag{1}$$

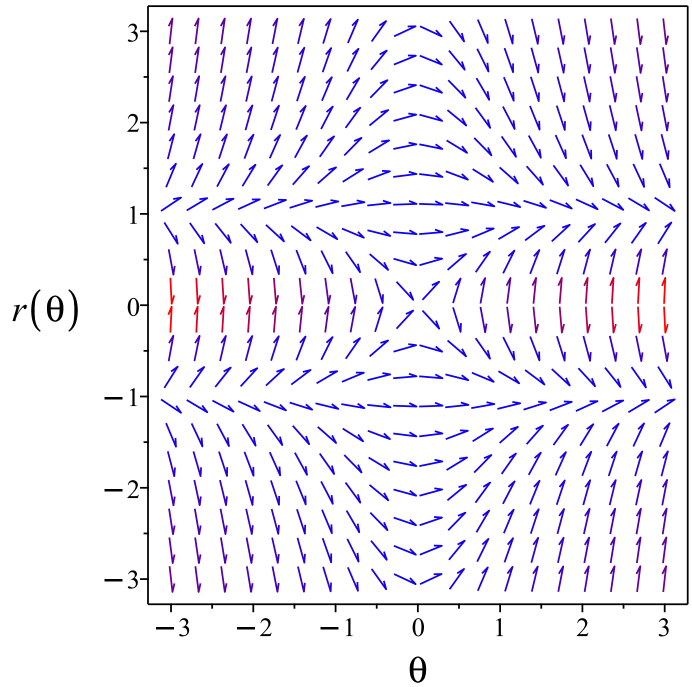


Figure 369: Slope field plot

Verification of solutions

$$r^2 - 1 = c_2 \mathrm{e}^{-\theta^2}$$

Verified OK.

7.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = -\frac{\theta(r^2 - 1)}{r}$$
$$r' = \omega(\theta, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2 \xi_r - \omega_\theta \xi - \omega_r \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 215: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, r) &= -\frac{1}{\theta} \\
\eta(\theta, r) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = r$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} d\theta \\ &= \int \frac{1}{-\frac{1}{\theta}} d\theta \end{aligned}$$

Which results in

$$S = -\frac{\theta^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r) S_r}{R_\theta + \omega(\theta, r) R_r} \tag{2}$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = -\frac{\theta(r^2 - 1)}{r}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 0 \\ R_r &= 1 \\ S_\theta &= -\theta \\ S_r &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{r}{r^2 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R-1)}{2} + \frac{\ln(R+1)}{2} + c_1 \tag{4}$$

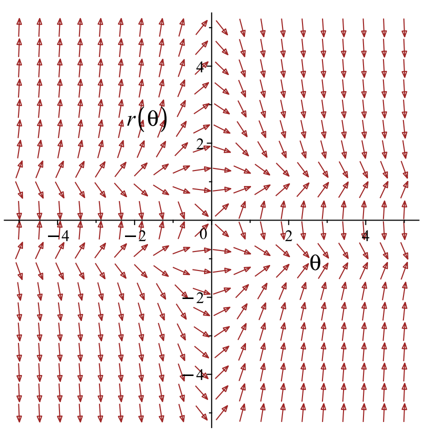
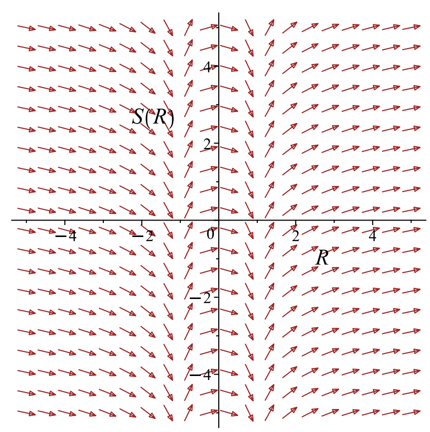
To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$-\frac{\theta^2}{2} = \frac{\ln(r-1)}{2} + \frac{\ln(r+1)}{2} + c_1$$

Which simplifies to

$$-\frac{\theta^2}{2} = \frac{\ln(r-1)}{2} + \frac{\ln(r+1)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in θ, r coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dr}{d\theta} = -\frac{\theta(r^2-1)}{r}$
</td><td>$R = r$
$S = -\frac{\theta^2}{2}$</td><td>$\frac{dS}{dR} = \frac{R}{R^2-1}$
</td></tr>
</table>

Summary
The solution(s) found are the following

$$-\frac{\theta^2}{2} = \frac{\ln(r-1)}{2} + \frac{\ln(r+1)}{2} + c_1 \tag{1}$$

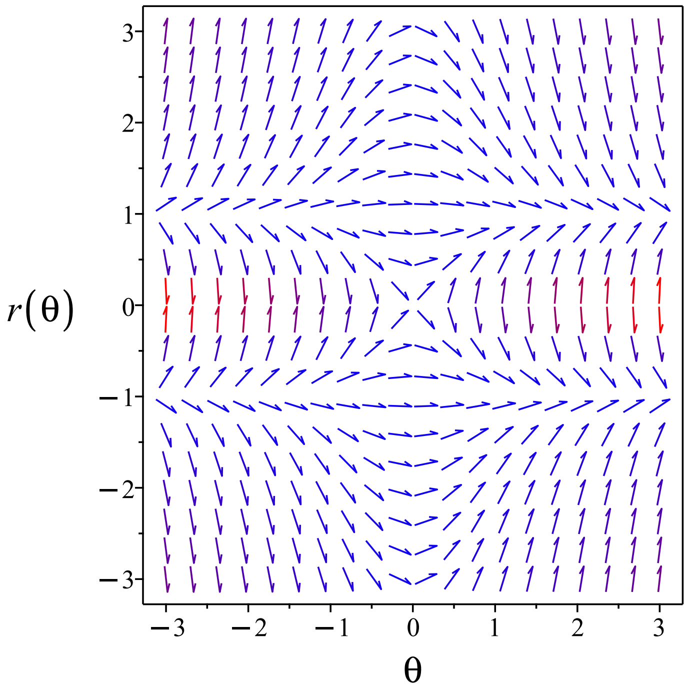


Figure 370: Slope field plot

Verification of solutions

$$-\frac{\theta^2}{2} = \frac{\ln(r-1)}{2} + \frac{\ln(r+1)}{2} + c_1$$

Verified OK.

7.12.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} r' &= F(\theta, r) \\ &= -\frac{\theta(r^2-1)}{r} \end{aligned}$$

This is a Bernoulli ODE.

$$r' = -\theta r + \theta \frac{1}{r} \tag{1}$$

The standard Bernoulli ODE has the form

$$r' = f_0(\theta) r + f_1(\theta) r^n \tag{2}$$

The first step is to divide the above equation by r^n which gives

$$\frac{r'}{r^n} = f_0(\theta) r^{1-n} + f_1(\theta) \tag{3}$$

The next step is use the substitution $w = r^{1-n}$ in equation (3) which generates a new ODE in $w(\theta)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $r(\theta)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(\theta) &= -\theta \\ f_1(\theta) &= \theta \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $r^n = \frac{1}{r}$ gives

$$r'r = -\theta r^2 + \theta \tag{4}$$

Let

$$\begin{aligned} w &= r^{1-n} \\ &= r^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t θ gives

$$w' = 2rr' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(\theta)}{2} &= -w(\theta)\,\theta + \theta \\ w' &= -2\theta w + 2\theta \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(\theta)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(\theta) + p(\theta)w(\theta) = q(\theta)$$

Where here

$$\begin{aligned} p(\theta) &= 2\theta \\ q(\theta) &= 2\theta \end{aligned}$$

Hence the ode is

$$w'(\theta) + 2w(\theta)\,\theta = 2\theta$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 2\theta d\theta} \\
&= e^{\theta^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{d\theta}(\mu w) &= (\mu)(2\theta) \\
\frac{d}{d\theta}\left(e^{\theta^2} w\right) &= \left(e^{\theta^2}\right)(2\theta) \\
d\left(e^{\theta^2} w\right) &= \left(2\theta\, e^{\theta^2}\right)\, d\theta
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{\theta^2} w &= \int 2\theta\, e^{\theta^2}\, d\theta \\
e^{\theta^2} w &= e^{\theta^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{\theta^2}$ results in

$$
w(\theta) = e^{-\theta^2} e^{\theta^2} + c_1 e^{-\theta^2}
$$

which simplifies to

$$
w(\theta) = 1 + c_1 e^{-\theta^2}
$$

Replacing w in the above by r^2 using equation (5) gives the final solution.

$$
r^2 = 1 + c_1 e^{-\theta^2}
$$

Solving for r gives

$$
\begin{aligned}
r(\theta) &= \sqrt{1 + c_1 e^{-\theta^2}} \\
r(\theta) &= -\sqrt{1 + c_1 e^{-\theta^2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
r = \sqrt{1 + c_1 e^{-\theta^2}} \tag{1}
$$

$$
r = -\sqrt{1 + c_1 e^{-\theta^2}} \tag{2}
$$

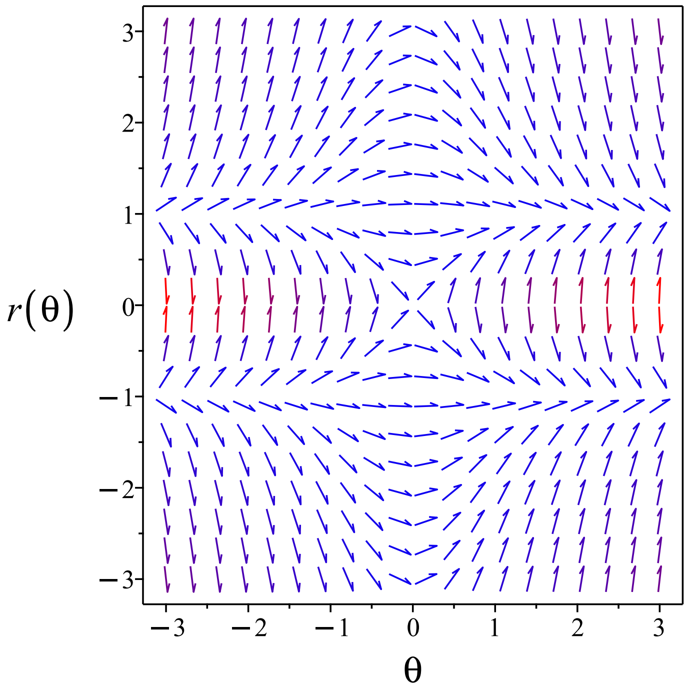


Figure 371: Slope field plot

<u>Verification of solutions</u>

$$r = \sqrt{1 + c_1 e^{-\theta^2}}$$

Verified OK.

$$r = -\sqrt{1 + c_1 e^{-\theta^2}}$$

Verified OK.

7.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta,r)\,\mathrm{d}\theta+N(\theta,r)\,\mathrm{d}r=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{r}{r^2-1}\right)\mathrm{d}r&=(\theta)\,\mathrm{d}\theta\\ (-\theta)\,\mathrm{d}\theta+\left(-\frac{r}{r^2-1}\right)\mathrm{d}r&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(\theta,r)&=-\theta\\ N(\theta,r)&=-\frac{r}{r^2-1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r}=\frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r}&=\frac{\partial}{\partial r}(-\theta)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(-\frac{r}{r^2-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(\theta, r)$

$$
\frac{\partial \phi}{\partial \theta} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial r} = N \tag{2}
$$

Integrating (1) w.r.t. θ gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial \theta} \,\mathrm{d}\theta &= \int M \,\mathrm{d}\theta \\
\int \frac{\partial \phi}{\partial \theta} \,\mathrm{d}\theta &= \int -\theta \,\mathrm{d}\theta
\end{aligned}
$$

$$
\phi = -\frac{\theta^2}{2} + f(r) \tag{3}
$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$
\frac{\partial \phi}{\partial r} = 0 + f'(r) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = -\frac{r}{r^2-1}$. Therefore equation (4) becomes

$$
-\frac{r}{r^2-1} = 0 + f'(r) \tag{5}
$$

Solving equation (5) for $f'(r)$ gives

$$
f'(r) = -\frac{r}{r^2-1}
$$

Integrating the above w.r.t r gives

$$
\begin{aligned}
\int f'(r) \,\mathrm{d}r &= \int \left(-\frac{r}{r^2-1}\right) \mathrm{d}r \\
f(r) &= -\frac{\ln(r-1)}{2} - \frac{\ln(r+1)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(r)$ into equation (3) gives ϕ

$$\phi = -\frac{\theta^2}{2} - \frac{\ln(r-1)}{2} - \frac{\ln(r+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\theta^2}{2} - \frac{\ln(r-1)}{2} - \frac{\ln(r+1)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\theta^2}{2} - \frac{\ln(r-1)}{2} - \frac{\ln(r+1)}{2} = c_1 \tag{1}$$

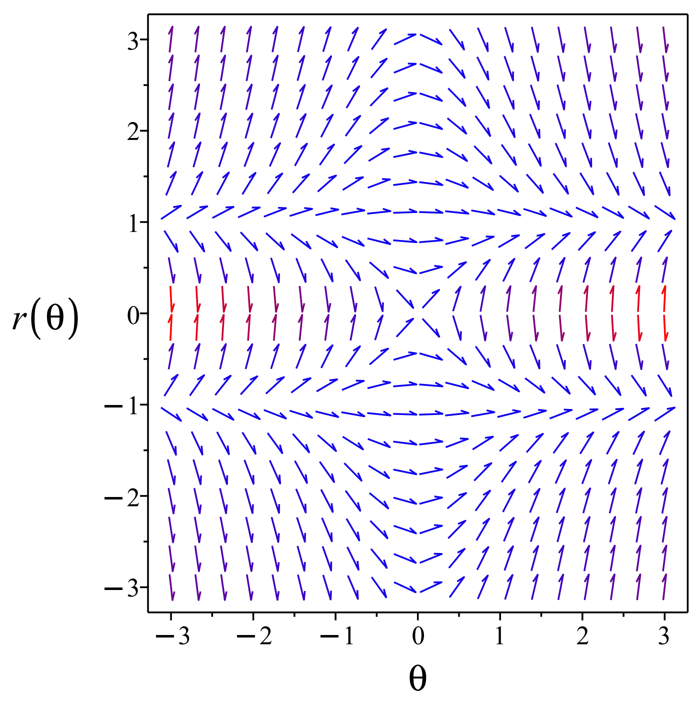


Figure 372: Slope field plot

Verification of solutions

$$-\frac{\theta^2}{2} - \frac{\ln(r-1)}{2} - \frac{\ln(r+1)}{2} = c_1$$

Verified OK.

7.12.5 Maple step by step solution

Let's solve

$r' + \left(r - \frac{1}{r}\right)\theta = 0$

- Highest derivative means the order of the ODE is 1

  r'

- Separate variables

  $\frac{r'}{r-\frac{1}{r}} = -\theta$

- Integrate both sides with respect to θ

  $\int \frac{r'}{r-\frac{1}{r}}d\theta = \int -\theta d\theta + c_1$

- Evaluate integral

  $\frac{\ln(r-1)}{2} + \frac{\ln(r+1)}{2} = -\frac{\theta^2}{2} + c_1$

- Solve for r

  $\left\{r = \frac{\left(1-\sqrt{1+e^{-\theta^2+2c_1}}\right)e^{\theta^2}+(e^{c_1})^2}{\left(1-\sqrt{1+e^{-\theta^2+2c_1}}\right)e^{\theta^2}}, r = \frac{\left(1+\sqrt{1+e^{-\theta^2+2c_1}}\right)e^{\theta^2}+(e^{c_1})^2}{\left(1+\sqrt{1+e^{-\theta^2+2c_1}}\right)e^{\theta^2}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(r(theta),theta)+(r(theta)-1/r(theta))*theta=0,r(theta), singsol=all)
```

$$r(\theta) = \sqrt{e^{-\theta^2}c_1 + 1}$$
$$r(\theta) = -\sqrt{e^{-\theta^2}c_1 + 1}$$

✓ Solution by Mathematica

Time used: 1.969 (sec). Leaf size: 57

```
DSolve[r'[\[Theta]]+(r[\[Theta]]-1/r[\[Theta]])*\[Theta]==0,r[\[Theta]],\[Theta],IncludeSing
```

$$
\begin{aligned}
r(\theta) &\to -\sqrt{1+e^{-\theta^2+2c_1}} \\
r(\theta) &\to \sqrt{1+e^{-\theta^2+2c_1}} \\
r(\theta) &\to -1 \\
r(\theta) &\to 1
\end{aligned}
$$

7.13 problem 13

Internal problem ID [2023]
Internal file name [`OUTPUT/2023_Sunday_February_25_2024_06_44_51_AM_93884988/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{2y + y'x - 3x^3 y^{\frac{4}{3}} = 0}$$

7.13.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y - 3x^3 y^{\frac{4}{3}}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 218: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^{\frac{2}{3}} y^{\frac{4}{3}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x^{\frac{2}{3}} y^{\frac{4}{3}}} dy
\end{aligned}$$

Which results in

$$S = -\frac{3}{y^{\frac{1}{3}} x^{\frac{2}{3}}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2y - 3x^3 y^{\frac{4}{3}}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2}{y^{\frac{1}{3}} x^{\frac{5}{3}}} \\
S_y &= \frac{1}{x^{\frac{2}{3}} y^{\frac{4}{3}}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3x^{\frac{4}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^{\frac{4}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{9R^{\frac{7}{3}}}{7} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{3}{y^{\frac{1}{3}}x^{\frac{2}{3}}} = \frac{9x^{\frac{7}{3}}}{7} + c_1$$

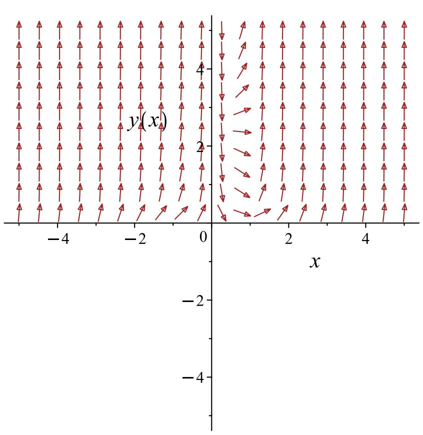
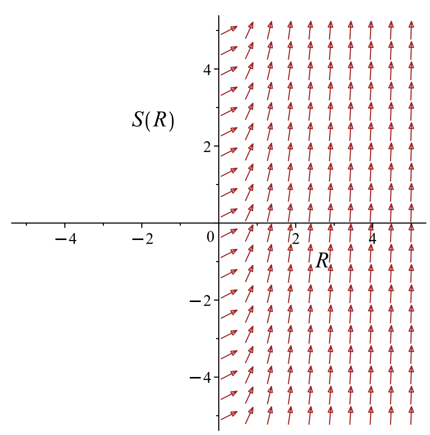
Which simplifies to

$$-\frac{3}{y^{\frac{1}{3}}x^{\frac{2}{3}}} = \frac{9x^{\frac{7}{3}}}{7} + c_1$$

Which gives

$$y = -\frac{9261}{\left(7c_1x^{\frac{2}{3}} + 9x^3\right)^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y-3x^3y^{\frac{4}{3}}}{x}$ 	$R = x$ $S = -\frac{3}{y^{\frac{1}{3}}x^{\frac{2}{3}}}$	$\frac{dS}{dR} = 3R^{\frac{4}{3}}$ 

Summary
The solution(s) found are the following

$$y = -\frac{9261}{\left(7c_1 x^{\frac{2}{3}} + 9x^3\right)^3} \tag{1}$$

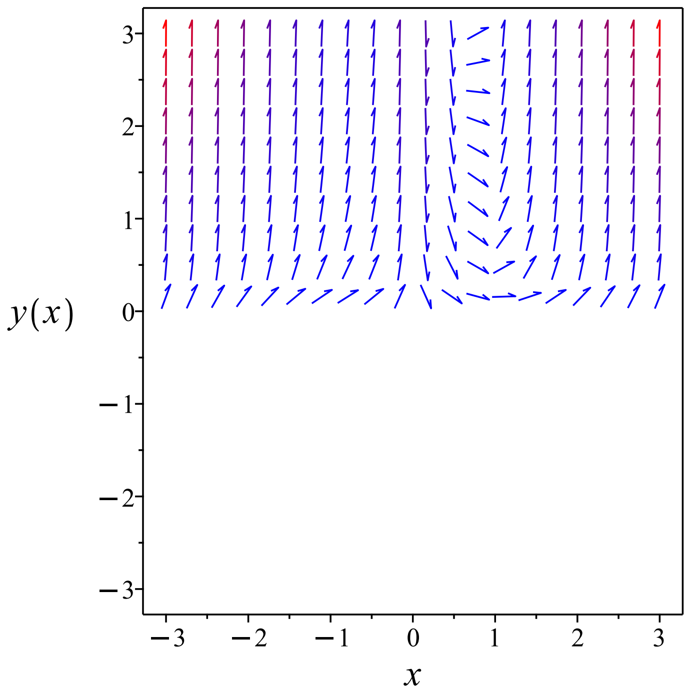


Figure 373: Slope field plot

Verification of solutions

$$y = -\frac{9261}{\left(7c_1 x^{\frac{2}{3}} + 9x^3\right)^3}$$

Verified OK.

7.13.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{2y - 3x^3 y^{\frac{4}{3}}}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{x} y + 3x^2 y^{\frac{4}{3}} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2}{x} \\ f_1(x) &= 3x^2 \\ n &= \frac{4}{3} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^{\frac{4}{3}}$ gives

$$y' \frac{1}{y^{\frac{4}{3}}} = -\frac{2}{x\, y^{\frac{1}{3}}} + 3x^2 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^{\frac{1}{3}}} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{3y^{\frac{4}{3}}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-3w'(x) &= -\frac{2w(x)}{x} + 3x^2 \\
w' &= \frac{2w}{3x} - x^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{3x} \\
q(x) &= -x^2
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{2w(x)}{3x} = -x^2
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2}{3x}dx} \\
&= \frac{1}{x^{\frac{2}{3}}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(-x^2\right) \\
\frac{d}{dx}\left(\frac{w}{x^{\frac{2}{3}}}\right) &= \left(\frac{1}{x^{\frac{2}{3}}}\right)\left(-x^2\right) \\
d\left(\frac{w}{x^{\frac{2}{3}}}\right) &= \left(-x^{\frac{4}{3}}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x^{\frac{2}{3}}} &= \int -x^{\frac{4}{3}} dx \\
\frac{w}{x^{\frac{2}{3}}} &= -\frac{3x^{\frac{7}{3}}}{7} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^{\frac{2}{3}}}$ results in

$$w(x) = -\frac{3x^3}{7} + c_1 x^{\frac{2}{3}}$$

Replacing w in the above by $\frac{1}{y^{\frac{1}{3}}}$ using equation (5) gives the final solution.

$$\frac{1}{y^{\frac{1}{3}}} = -\frac{3x^3}{7} + c_1 x^{\frac{2}{3}}$$

Summary

The solution(s) found are the following

$$\frac{1}{y^{\frac{1}{3}}} = -\frac{3x^3}{7} + c_1 x^{\frac{2}{3}} \tag{1}$$

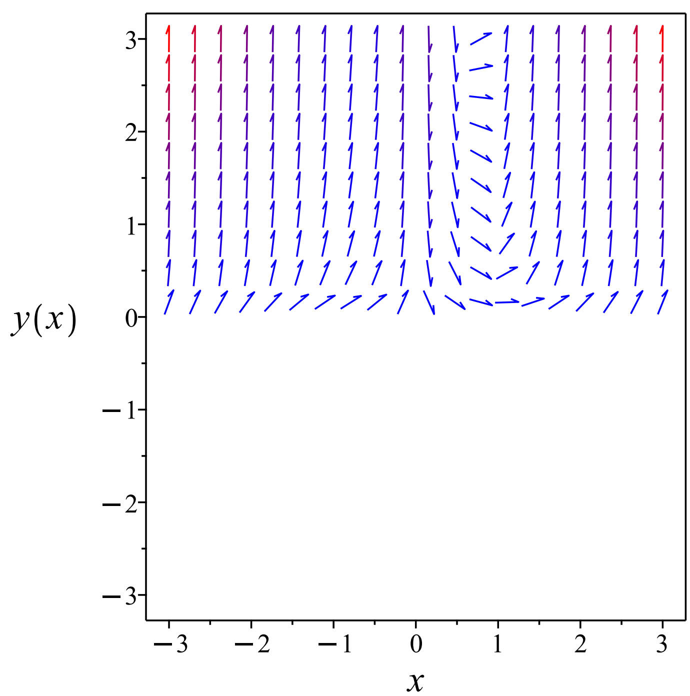


Figure 374: Slope field plot

Verification of solutions

$$\frac{1}{y^{\frac{1}{3}}} = -\frac{3x^3}{7} + c_1 x^{\frac{2}{3}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x)+2*y(x)=3*x^3*y(x)^(4/3),y(x), singsol=all)
```

$$\frac{1}{y(x)^{\frac{1}{3}}}+\frac{3x^3}{7}-x^{\frac{2}{3}}c_1=0$$

✓ Solution by Mathematica
Time used: 0.159 (sec). Leaf size: 29

```
DSolve[x*y'[x]+2*y[x]==3*x^3*y[x]^(4/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{343}{x^2\left(3x^{7/3}-7c_1\right)^3}$$
$$y(x)\to 0$$

7.14 problem 14

Internal problem ID [2024]
Internal file name [`OUTPUT/2024_Sunday_February_25_2024_06_44_52_AM_69882385/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{3y' + \frac{2y}{x+1} - \frac{x}{y^2} = 0}$$

7.14.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{2y^3 - x^2 - x}{3\,(x+1)\,y^2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 220: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{y^2 (x + 1)^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y^2(x+1)^2}} dy \end{aligned}$$

Which results in

$$S = \frac{(x+1)^2 y^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y^3 - x^2 - x}{3(x+1)y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2(x+1)y^3}{3} \\ S_y &= (x+1)^2 y^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x(x+1)^2}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R(R+1)^2}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{12}R^4 + \frac{2}{9}R^3 + \frac{1}{6}R^2 + c_1 \tag{4}$$

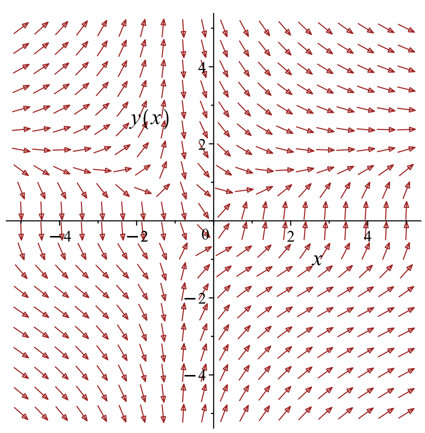
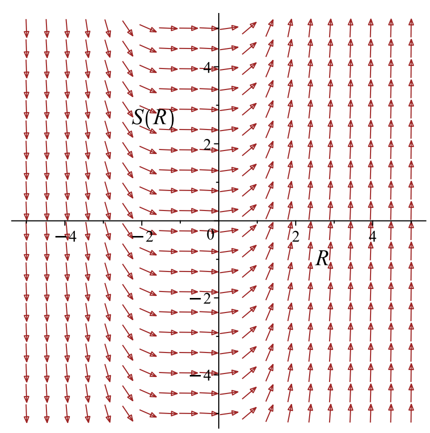
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x+1)^2 y^3}{3} = \frac{1}{12}x^4 + \frac{2}{9}x^3 + \frac{1}{6}x^2 + c_1$$

Which simplifies to

$$\frac{(x+1)^2 y^3}{3} = \frac{1}{12}x^4 + \frac{2}{9}x^3 + \frac{1}{6}x^2 + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y^3 - x^2 - x}{3(x+1)y^2}$ 	$R = x$ $S = \frac{(x+1)^2 y^3}{3}$	$\frac{dS}{dR} = \frac{R(R+1)^2}{3}$ 

Summary
The solution(s) found are the following

$$\frac{(x+1)^2 y^3}{3} = \frac{1}{12}x^4 + \frac{2}{9}x^3 + \frac{1}{6}x^2 + c_1 \tag{1}$$

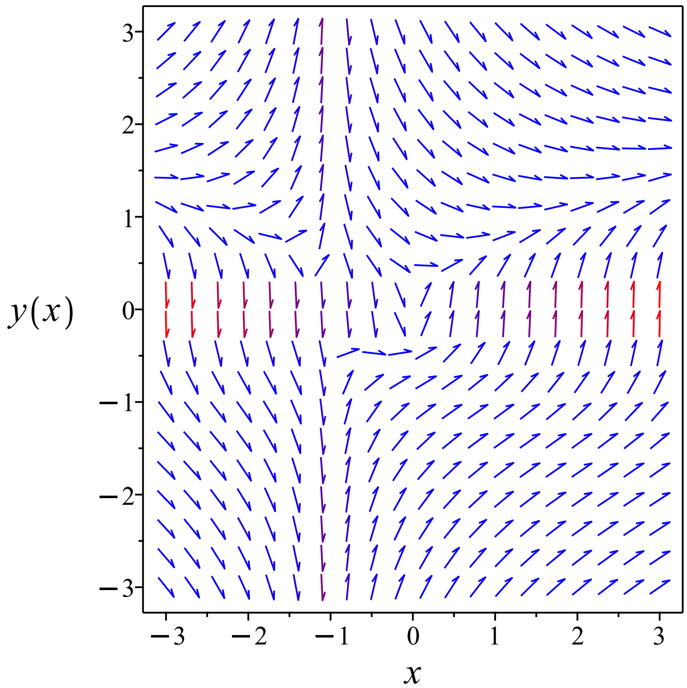


Figure 375: Slope field plot

Verification of solutions

$$\frac{(x+1)^2 y^3}{3} = \frac{1}{12}x^4 + \frac{2}{9}x^3 + \frac{1}{6}x^2 + c_1$$

Verified OK.

7.14.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{2y^3 - x^2 - x}{3(x+1)y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{3(x+1)}y - \frac{-x^2 - x}{3(x+1)}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2}{3(x+1)} \\ f_1(x) &= -\frac{-x^2 - x}{3(x+1)} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = -\frac{2y^3}{3(x+1)} - \frac{-x^2 - x}{3(x+1)} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= -\frac{2w(x)}{3(x+1)} - \frac{-x^2 - x}{3(x+1)} \\ w' &= -\frac{2w}{x+1} - \frac{-x^2 - x}{x+1} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x+1} \\
q(x) &= x
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{2w(x)}{x+1} = x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{2}{x+1} dx} \\
&= (x+1)^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(x) \\
\frac{d}{dx}\left((x+1)^2 w\right) &= \left((x+1)^2\right)(x) \\
d\left((x+1)^2 w\right) &= \left(x(x+1)^2\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(x+1)^2 w &= \int x(x+1)^2 \, dx \\
(x+1)^2 w &= \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = (x+1)^2$ results in

$$
w(x) = \frac{\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2}{(x+1)^2} + \frac{c_1}{(x+1)^2}
$$

which simplifies to

$$
w(x) = \frac{3x^4 + 8x^3 + 6x^2 + 12c_1}{12(x+1)^2}
$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$
y^3 = \frac{3x^4 + 8x^3 + 6x^2 + 12c_1}{12(x+1)^2}
$$

Solving for y gives

$$y(x) = \frac{\left(\left(3x^4+8x^3+6x^2+12c_1\right)(x+1)^4\right)^{\frac{1}{3}} 18^{\frac{1}{3}}}{6(x+1)^2}$$

$$y(x) = \frac{18^{\frac{1}{3}} \left(\left(3x^4+8x^3+6x^2+12c_1\right)(x+1)^4\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{12(x+1)^2}$$

$$y(x) = -\frac{18^{\frac{1}{3}} \left(\left(3x^4+8x^3+6x^2+12c_1\right)(x+1)^4\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{12(x+1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\left(3x^4+8x^3+6x^2+12c_1\right)(x+1)^4\right)^{\frac{1}{3}} 18^{\frac{1}{3}}}{6(x+1)^2} \tag{1}$$

$$y = \frac{18^{\frac{1}{3}} \left(\left(3x^4+8x^3+6x^2+12c_1\right)(x+1)^4\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{12(x+1)^2} \tag{2}$$

$$y = -\frac{18^{\frac{1}{3}} \left(\left(3x^4+8x^3+6x^2+12c_1\right)(x+1)^4\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{12(x+1)^2} \tag{3}$$

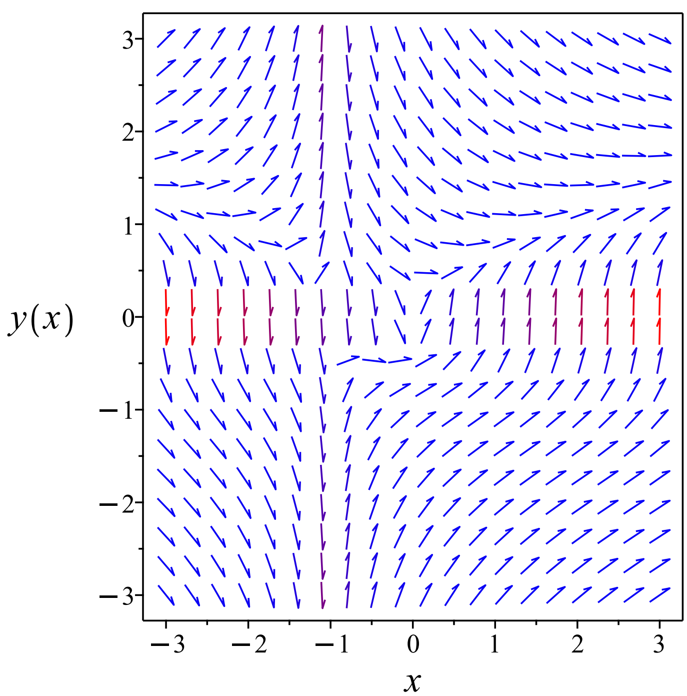


Figure 376: Slope field plot

Verification of solutions

$$y = \frac{\left(\left(3x^4+8x^3+6x^2+12c_1\right)\left(x+1\right)^4\right)^{\frac{1}{3}} 18^{\frac{1}{3}}}{6\left(x+1\right)^2}$$

Verified OK.

$$y = \frac{18^{\frac{1}{3}} \left(\left(3x^4+8x^3+6x^2+12c_1\right)\left(x+1\right)^4\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{12\left(x+1\right)^2}$$

Verified OK.

$$y = -\frac{18^{\frac{1}{3}} \left(\left(3x^4+8x^3+6x^2+12c_1\right)\left(x+1\right)^4\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{12\left(x+1\right)^2}$$

Verified OK.

7.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(3(x+1)\, y^2\right) \mathrm{d}y &= \left(-2y^3 + x^2 + x\right) \mathrm{d}x \\ \left(2y^3 - x^2 - x\right) \mathrm{d}x + \left(3(x+1)\, y^2\right) \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 2y^3 - x^2 - x \\ N(x, y) &= 3(x+1)\, y^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y^3 - x^2 - x\right) \\ &= 6y^2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3(x+1)\, y^2\right) \\ &= 3y^2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{3\,(x+1)\, y^2}\left(\left(6y^2\right) - \left(3y^2\right)\right) \\ &= \frac{1}{x+1} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x+1} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x+1)} \\ &= x+1\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x+1\left(2y^3 - x^2 - x\right) \\ &= -\left(-2y^3 + x^2 + x\right)(x+1)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x+1\left(3(x+1)\, y^2\right) \\ &= 3(x+1)^2\, y^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\left(-2y^3 + x^2 + x\right)(x+1)\right) + \left(3(x+1)^2\, y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\left(-2y^3 + x^2 + x\right)(x+1) \, \mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^4}{4} - \frac{2x^3}{3} + \frac{(2y^3 - 1)\, x^2}{2} + 2x\, y^3 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= 3x^2y^2 + 6x\,y^2 + f'(y) \\ &= 3x\,y^2(2+x) + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3(x+1)^2\,y^2$. Therefore equation (4) becomes

$$3(x+1)^2\,y^2 = 3x\,y^2(2+x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y^2$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int (3y^2)\,\mathrm{d}y \\ f(y) &= y^3 + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^4}{4} - \frac{2x^3}{3} + \frac{(2y^3-1)\,x^2}{2} + 2x\,y^3 + y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^4}{4} - \frac{2x^3}{3} + \frac{(2y^3-1)\,x^2}{2} + 2x\,y^3 + y^3$$

Summary

The solution(s) found are the following

$$-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{(2y^3-1)\,x^2}{2} + 2xy^3 + y^3 = c_1 \tag{1}$$

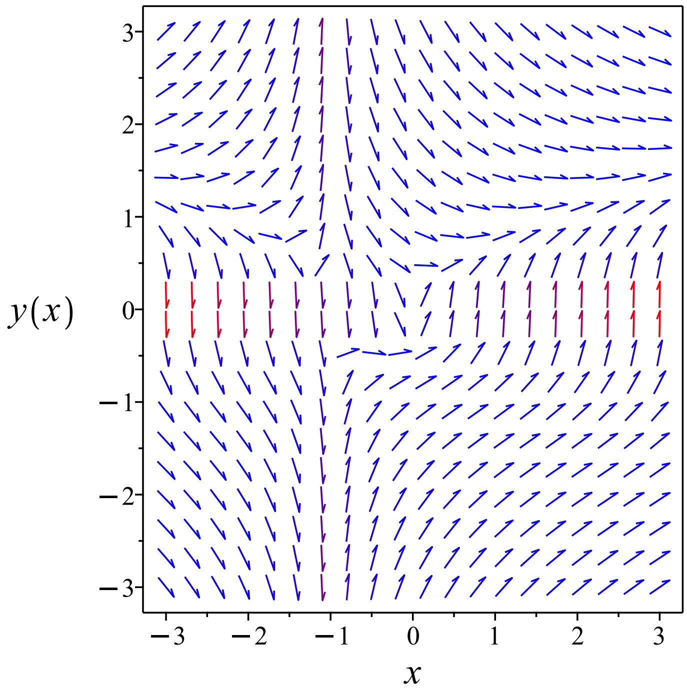


Figure 377: Slope field plot

Verification of solutions

$$-\frac{x^4}{4}-\frac{2x^3}{3}+\frac{(2y^3-1)\,x^2}{2}+2xy^3+y^3=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 137

```
dsolve(3*diff(y(x),x)+2*y(x)/(x+1)=x/y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(3x^4+8x^3+6x^2+12c_1\right)\left(x+1\right)^4\right)^{\frac{1}{3}} 18^{\frac{1}{3}}}{6\left(x+1\right)^2}$$

$$y(x) = -\frac{18^{\frac{1}{3}} \left(\left(3x^4+8x^3+6x^2+12c_1\right)\left(x+1\right)^4\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{12\left(x+1\right)^2}$$

$$y(x) = \frac{18^{\frac{1}{3}} \left(\left(3x^4+8x^3+6x^2+12c_1\right)\left(x+1\right)^4\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{12\left(x+1\right)^2}$$

✓ Solution by Mathematica

Time used: 3.9 (sec). Leaf size: 144

```
DSolve[3*y'[x]+2*y[x]/(x+1)==x/y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{\frac{3x^4+8x^3+6x^2+12c_1}{(x+1)^2}}}{2^{2/3}\sqrt[3]{3}}$$

$$y(x) \to -\frac{\sqrt[3]{-\frac{1}{3}}\sqrt[3]{\frac{3x^4+8x^3+6x^2+12c_1}{(x+1)^2}}}{2^{2/3}}$$

$$y(x) \to \frac{(-1)^{2/3}\sqrt[3]{\frac{3x^4+8x^3+6x^2+12c_1}{(x+1)^2}}}{2^{2/3}\sqrt[3]{3}}$$

7.15 problem 15

Internal problem ID [2025]
Internal file name [`OUTPUT/2025_Sunday_February_25_2024_06_44_54_AM_22703019/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\cos(y)\, y' + (\sin(y) - 1)\cos(x) = 0}$$

7.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\cos(x)\left(\tan(y) - \sec(y)\right) \end{aligned}$$

Where $f(x) = -\cos(x)$ and $g(y) = \tan(y) - \sec(y)$. Integrating both sides gives

$$\frac{1}{\tan(y) - \sec(y)}\, dy = -\cos(x)\, dx$$

$$\int \frac{1}{\tan(y) - \sec(y)}\, dy = \int -\cos(x)\, dx$$

$$\ln(\sin(y) - 1) = -\sin(x) + c_1$$

Raising both side to exponential gives

$$\sin(y) - 1 = e^{-\sin(x)+c_1}$$

Which simplifies to

$$\sin(y) - 1 = c_2 e^{-\sin(x)}$$

Summary
The solution(s) found are the following

$$y = \arcsin\left(1 + c_2 e^{-\sin(x)+c_1}\right) \tag{1}$$

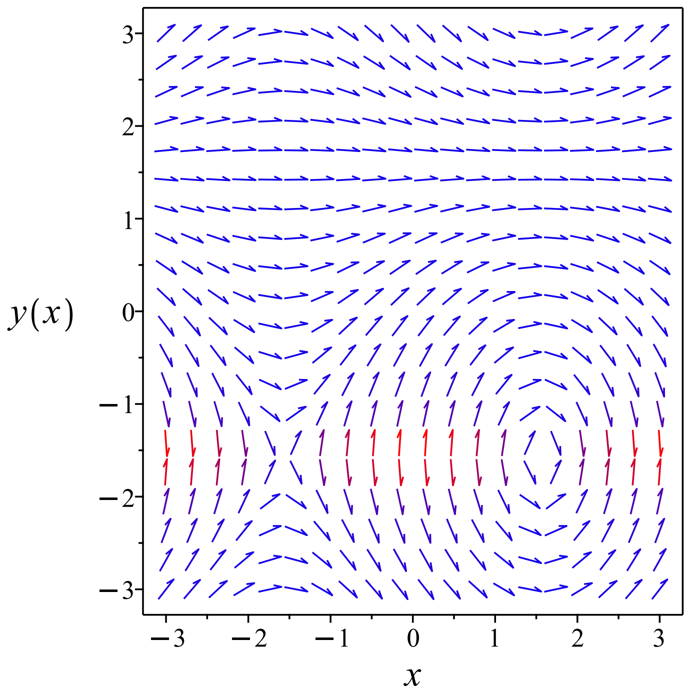


Figure 378: Slope field plot

Verification of solutions

$$y = \arcsin\left(1 + c_2 e^{-\sin(x)+c_1}\right)$$

Verified OK.

7.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{(\sin(y) - 1)\cos(x)}{\cos(y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 222: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{1}{\cos\left(x\right)} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{1}{\cos(x)}} dx
\end{aligned}
$$

Which results in

$$
S = -\sin\left(x\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{\left(\sin\left(y\right)-1\right)\cos\left(x\right)}{\cos\left(y\right)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\cos(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\cos(y)}{\sin(y) - 1} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{\cos(R)}{\sin(R) - 1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(\sin(R) - 1) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\sin(x) = \ln(\sin(y) - 1) + c_1
$$

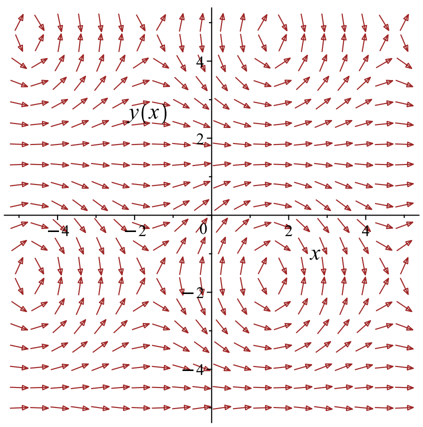
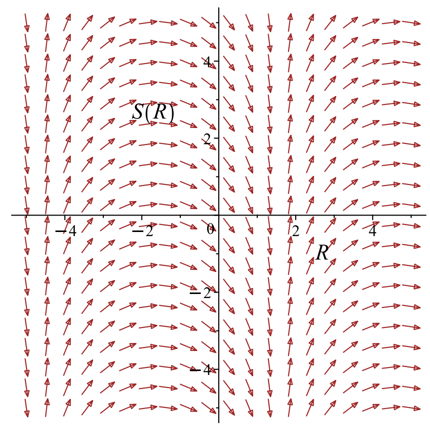
Which simplifies to

$$
-\sin(x) = \ln(\sin(y) - 1) + c_1
$$

Which gives

$$
y = \arcsin\left(e^{-\sin(x) - c_1} + 1\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(\sin(y)-1)\cos(x)}{\cos(y)}$ 	$R = y$ $S = -\sin(x)$	$\frac{dS}{dR} = \frac{\cos(R)}{\sin(R)-1}$ 

Summary

The solution(s) found are the following

$$y = \arcsin\left(e^{-\sin(x)-c_1} + 1\right) \tag{1}$$

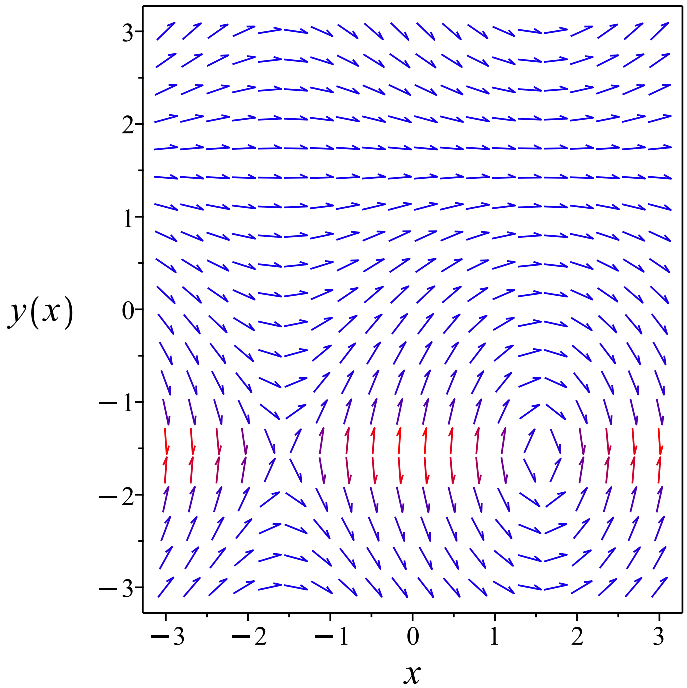


Figure 379: Slope field plot

Verification of solutions

$$y = \arcsin\left(\mathrm{e}^{-\sin(x)-c_1} + 1\right)$$

Verified OK.

7.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{\cos(y)}{\sin(y) - 1}\right) \mathrm{d}y &= (\cos(x))\, \mathrm{d}x \\
(-\cos(x))\, \mathrm{d}x + \left(-\frac{\cos(y)}{\sin(y) - 1}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\cos(x) \\
N(x, y) &= -\frac{\cos(y)}{\sin(y) - 1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x)) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\cos(y)}{\sin(y)-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, dx &= \int M\, dx \\
\int \frac{\partial \phi}{\partial x}\, dx &= \int -\cos(x)\, dx
\end{aligned}
$$

$$
\phi = -\sin(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\cos(y)}{\sin(y)-1}$. Therefore equation (4) becomes

$$
-\frac{\cos(y)}{\sin(y)-1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{\cos(y)}{\sin(y)-1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\, dy &= \int \left(-\frac{\cos(y)}{\sin(y)-1}\right) dy \\
f(y) &= -\ln(\sin(y)-1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) - \ln(\sin(y) - 1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) - \ln(\sin(y) - 1)$$

Summary

The solution(s) found are the following

$$-\sin(x) - \ln(\sin(y) - 1) = c_1 \tag{1}$$

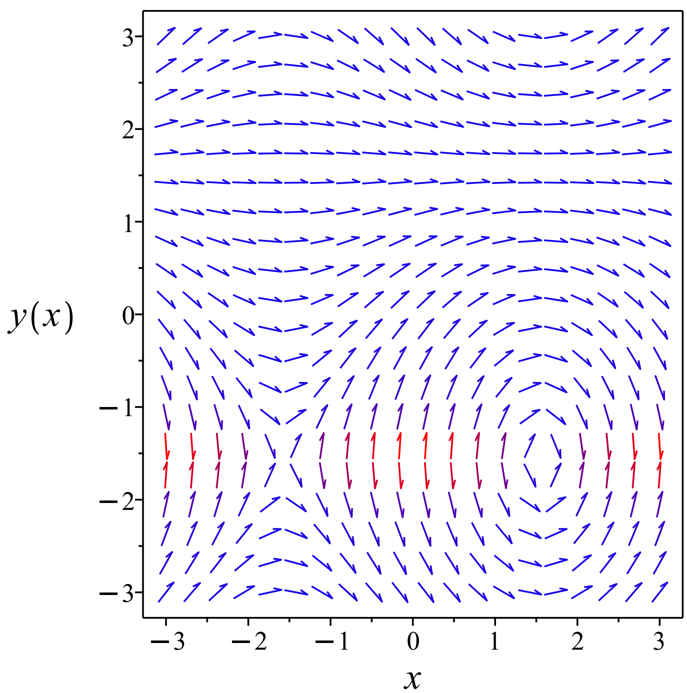


Figure 380: Slope field plot

Verification of solutions

$$-\sin(x) - \ln(\sin(y) - 1) = c_1$$

Verified OK.

7.15.4 Maple step by step solution

Let's solve

$\cos(y) y' + (\sin(y) - 1)\cos(x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{\cos(y)y'}{\sin(y)-1} = -\cos(x)$

- Integrate both sides with respect to x

  $\int \frac{\cos(y)y'}{\sin(y)-1}dx = \int -\cos(x)\, dx + c_1$

- Evaluate integral

  $\ln(\sin(y) - 1) = -\sin(x) + c_1$

- Solve for y

  $y = \arcsin\left(e^{-\sin(x)+c_1} + 1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 16

```
dsolve(cos(y(x))*diff(y(x),x)+(sin(y(x))-1)*cos(x)=0,y(x), singsol=all)
```

$$y(x) = \arcsin\left(\frac{e^{-\sin(x)} + c_1}{c_1}\right)$$

✓ Solution by Mathematica

Time used: 60.309 (sec). Leaf size: 225

```
DSolve[Cos[y[x]]*y'[x]+(Sin[y[x]]-1)*Cos[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$
$$y(x) \to -2\arccos\left(-\frac{1}{8}e^{-\sin(x)}\left(c_1e^{\frac{\sin(x)}{2}} + \sqrt{e^{\sin(x)}\left(32e^{\sin(x)} - c_1{}^2\right)}\right)\right)$$
$$y(x) \to 2\arccos\left(-\frac{1}{8}e^{-\sin(x)}\left(c_1e^{\frac{\sin(x)}{2}} + \sqrt{e^{\sin(x)}\left(32e^{\sin(x)} - c_1{}^2\right)}\right)\right)$$
$$y(x) \to -2\arccos\left(\frac{1}{8}e^{-\sin(x)}\left(\sqrt{e^{\sin(x)}\left(32e^{\sin(x)} - c_1{}^2\right)} - c_1e^{\frac{\sin(x)}{2}}\right)\right)$$
$$y(x) \to 2\arccos\left(\frac{1}{8}e^{-\sin(x)}\left(\sqrt{e^{\sin(x)}\left(32e^{\sin(x)} - c_1{}^2\right)} - c_1e^{\frac{\sin(x)}{2}}\right)\right)$$

7.16 problem 16

Internal problem ID [2026]
Internal file name [`OUTPUT/2026_Sunday_February_25_2024_06_45_28_AM_51073074/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{\left(x \tan \left(y\right)^{2}-x \right) y^{\prime}-\tan \left(y\right)=2 x^{2}}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-y(x)/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/2)*(2*y(x)*x+1)/x^2, y(x)`      *** Su
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(1/2)*(2*y(x)*x-1)/x^2, y(x)`      *** Su
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
```

✗ Solution by Maple

```
dsolve((x*tan(y(x))^2-x)*diff(y(x),x)=(2*x^2+tan(y(x))),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x*Tan[y[x]]^2-x)*y'[x]==(2*x^2+Tan[y[x]]),y[x],x,IncludeSingularSolutions -> True]
```

Not solved

7.17 problem 17

Internal problem ID [2027]
Internal file name [`OUTPUT/2027_Sunday_February_25_2024_06_45_30_AM_63951085/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y' + \cos(x)\, y - y^3 \sin(x) = 0}$$

7.17.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\cos(x)\, y + y^3 \sin(x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 225: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= y^3 e^{2\sin(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^3 e^{2\sin(x)}} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{-2\sin(x)}}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\cos(x)\, y + y^3 \sin(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\cos(x)\, e^{-2\sin(x)}}{y^2} \\ S_y &= \frac{e^{-2\sin(x)}}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-2\sin(x)} \sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-2\sin(R)} \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \int e^{-2\sin(R)} \sin(R)\, dR + c_1 \tag{4}$$

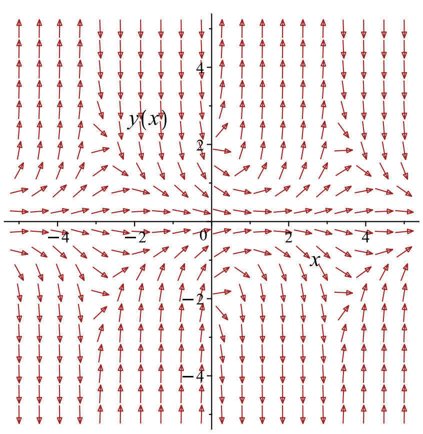
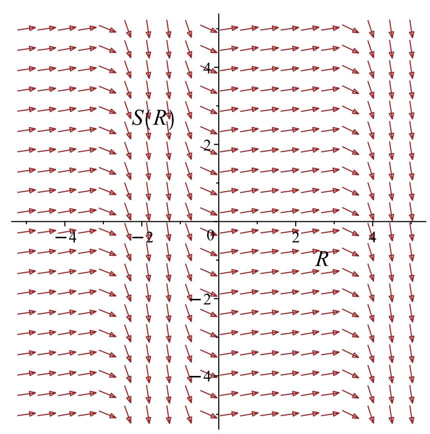
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{e^{-2\sin(x)}}{2y^2} = \int e^{-2\sin(x)} \sin(x)\, dx + c_1$$

Which simplifies to

$$-\frac{e^{-2\sin(x)}}{2y^2} = \int e^{-2\sin(x)} \sin(x)\, dx + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\cos(x)\, y + y^3 \sin(x)$ 	$R = x$ $S = -\frac{e^{-2\sin(x)}}{2y^2}$	$\frac{dS}{dR} = e^{-2\sin(R)} \sin(R)$ 

Summary

The solution(s) found are the following

$$-\frac{e^{-2\sin(x)}}{2y^2} = \int e^{-2\sin(x)} \sin(x)\, dx + c_1 \tag{1}$$

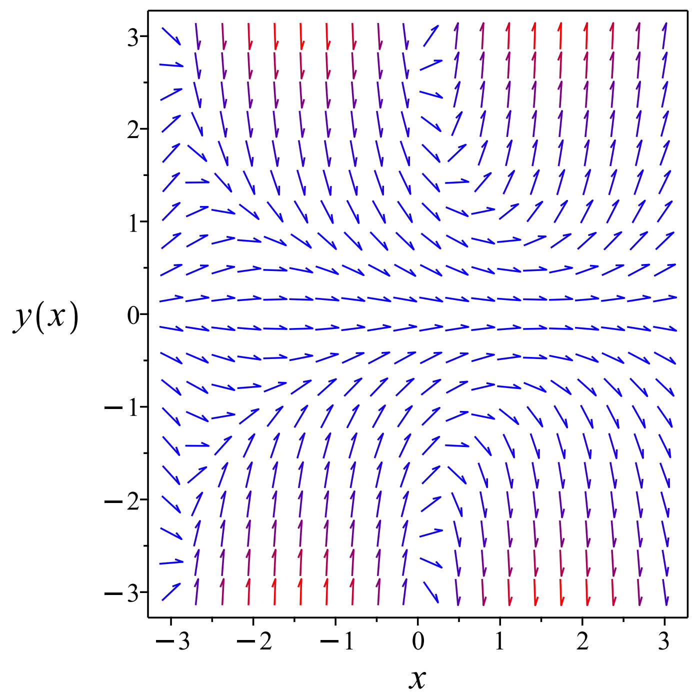


Figure 381: Slope field plot

Verification of solutions

$$-\frac{e^{-2\sin(x)}}{2y^2} = \int e^{-2\sin(x)} \sin(x)\, dx + c_1$$

Verified OK.

7.17.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\cos(x)\, y + y^3 \sin(x) \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\cos(x)\, y + \sin(x)\, y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\cos(x) \\ f_1(x) &= \sin(x) \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = -\frac{\cos(x)}{y^2} + \sin(x) \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= -\cos(x)\, w(x) + \sin(x) \\ w' &= 2\cos(x)\, w - 2\sin(x) \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2\cos(x) \\ q(x) &= -2\sin(x) \end{aligned}$$

Hence the ode is

$$w'(x) - 2\cos(x)\, w(x) = -2\sin(x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -2\cos(x)dx}\\ &= e^{-2\sin(x)}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-2\sin(x))\\ \frac{d}{dx}\left(e^{-2\sin(x)} w\right) &= \left(e^{-2\sin(x)}\right)(-2\sin(x))\\ d\left(e^{-2\sin(x)} w\right) &= \left(-2\,e^{-2\sin(x)}\sin(x)\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-2\sin(x)} w &= \int -2\,e^{-2\sin(x)}\sin(x)\, dx\\ e^{-2\sin(x)} w &= \int -2\,e^{-2\sin(x)}\sin(x)\, dx + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2\sin(x)}$ results in

$$w(x) = e^{2\sin(x)}\left(\int -2\,e^{-2\sin(x)}\sin(x)\, dx\right) + c_1 e^{2\sin(x)}$$

which simplifies to

$$w(x) = e^{2\sin(x)}\left(-2\left(\int e^{-2\sin(x)}\sin(x)\, dx\right) + c_1\right)$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = e^{2\sin(x)}\left(-2\left(\int e^{-2\sin(x)}\sin(x)\, dx\right) + c_1\right)$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{\sqrt{-e^{2\sin(x)}\left(2\left(\int e^{-2\sin(x)}\sin(x)\, dx\right) - c_1\right)}}\\ y(x) &= -\frac{1}{\sqrt{e^{2\sin(x)}\left(-2\left(\int e^{-2\sin(x)}\sin(x)\, dx\right) + c_1\right)}}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{\sqrt{-e^{2\sin(x)}\left(2\left(\int e^{-2\sin(x)}\sin(x)\,dx\right)-c_1\right)}} \tag{1}$$

$$y = -\frac{1}{\sqrt{e^{2\sin(x)}\left(-2\left(\int e^{-2\sin(x)}\sin(x)\,dx\right)+c_1\right)}} \tag{2}$$

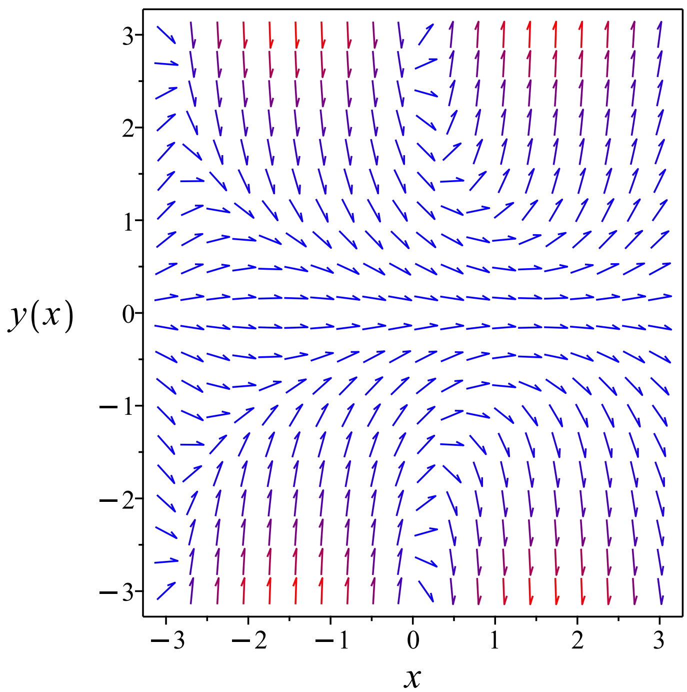


Figure 382: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{-e^{2\sin(x)}\left(2\left(\int e^{-2\sin(x)}\sin(x)\,dx\right)-c_1\right)}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{e^{2\sin(x)}\left(-2\left(\int e^{-2\sin(x)}\sin(x)\,dx\right)+c_1\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 86

```
dsolve(diff(y(x),x)+y(x)*cos(x)=y(x)^3*sin(x),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{\left(c_1 - 2\left(\int e^{-2\sin(x)}\sin(x)\,dx\right)\right)e^{-2\sin(x)}}}{c_1 - 2\left(\int e^{-2\sin(x)}\sin(x)\,dx\right)}$$

$$y(x) = \frac{\sqrt{\left(c_1 - 2\left(\int e^{-2\sin(x)}\sin(x)\,dx\right)\right)e^{-2\sin(x)}}}{c_1 - 2\left(\int e^{-2\sin(x)}\sin(x)\,dx\right)}$$

✓ Solution by Mathematica

Time used: 10.83 (sec). Leaf size: 84

```
DSolve[y'[x]+y[x]*Cos[x]==y[x]^3*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{e^{2\sin(x)}\left(-2\int_1^x e^{-2\sin(K[1])}\sin(K[1])dK[1] + c_1\right)}}$$

$$y(x) \to \frac{1}{\sqrt{e^{2\sin(x)}\left(-2\int_1^x e^{-2\sin(K[1])}\sin(K[1])dK[1] + c_1\right)}}$$

$$y(x) \to 0$$

7.18 problem 18

Internal problem ID [2028]
Internal file name [`OUTPUT/2028_Sunday_February_25_2024_06_45_32_AM_48582685/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Bernoulli]
```

$$\boxed{y' + y - y^2 e^{-t} = 0}$$

With initial conditions

$$[y(0) = 2]$$

7.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t, y) \\ &= -y + e^{-t} y^2 \end{aligned}$$

The t domain of $f(t, y)$ when $y = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $f(t, y)$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-y + \mathrm{e}^{-t} y^2\right)\\ &= 2\,\mathrm{e}^{-t} y - 1\end{aligned}$$

The t domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

7.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= -y + \mathrm{e}^{-t} y^2\\ y' &= \omega(t, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 227: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t, y) &= 0 \\ \eta(t, y) &= \mathrm{e}^t y^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}\right) S(t, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^t y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{-t}}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y) S_y}{R_t + \omega(t,y) R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -y + e^{-t} y^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_y &= 0 \\ S_t &= \frac{e^{-t}}{y} \\ S_y &= \frac{e^{-t}}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{e^{-2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, y coordinates. This results in

$$-\frac{e^{-t}}{y} = -\frac{e^{-2t}}{2} + c_1$$

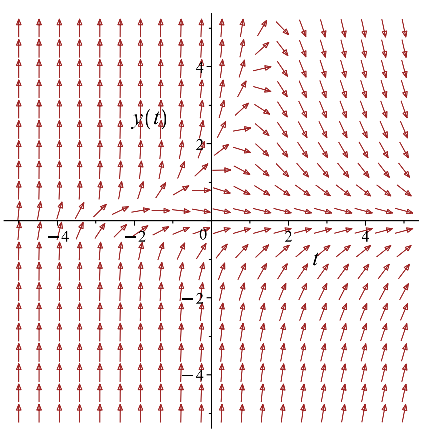
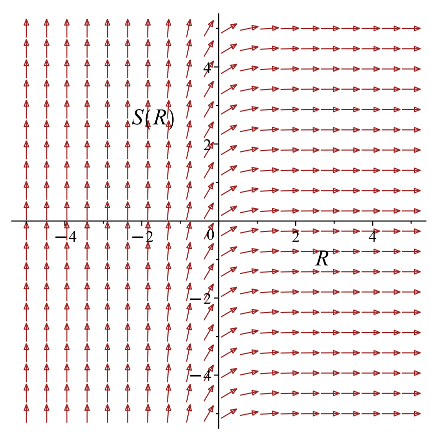
Which simplifies to

$$-\frac{e^{-t}}{y} = -\frac{e^{-2t}}{2} + c_1$$

Which gives

$$y = \frac{2\,e^{-t}}{e^{-2t} - 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -y + e^{-t}y^2$ 	$R = t$ $S = -\frac{e^{-t}}{y}$	$\frac{dS}{dR} = e^{-2R}$ 

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{2}{-1 + 2c_1}$$

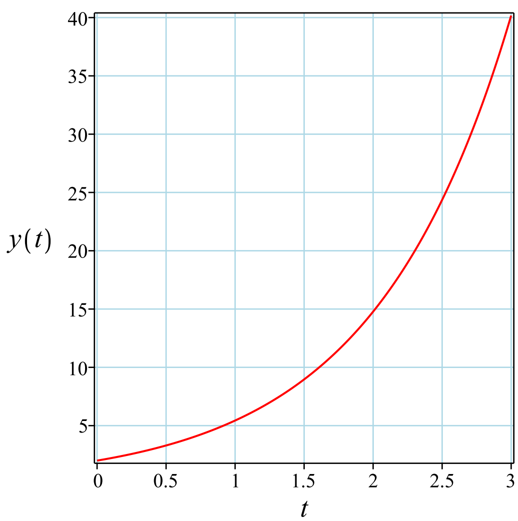
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

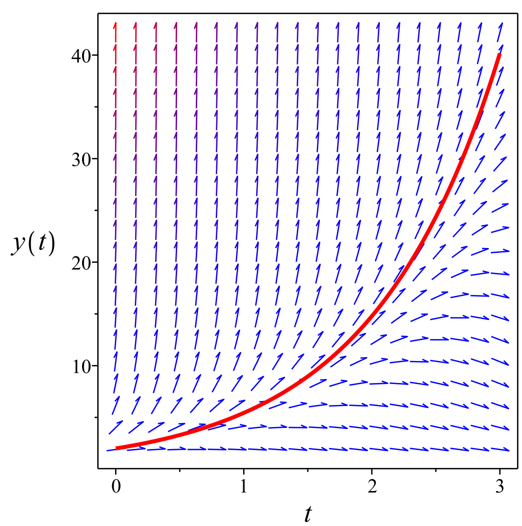
$$y = 2\,\mathrm{e}^t$$

Summary
The solution(s) found are the following

$$y = 2\,\mathrm{e}^t \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^t$$

Verified OK.

7.18.3 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(t,y) \\
&= -y + e^{-t}y^2
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = -y + e^{-t}y^2 \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(t)y + f_1(t)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(t)y^{1-n} + f_1(t) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(t) &= -1 \\
f_1(t) &= e^{-t} \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y'\frac{1}{y^2} = -\frac{1}{y} + e^{-t} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t t gives

$$
w' = -\frac{1}{y^2}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(t) &= -w(t) + \mathrm{e}^{-t} \\
w' &= w - \mathrm{e}^{-t}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(t) + p(t)w(t) = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= -1 \\
q(t) &= -\mathrm{e}^{-t}
\end{aligned}
$$

Hence the ode is

$$
w'(t) - w(t) = -\mathrm{e}^{-t}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-1)dt} \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu w) &= (\mu)\left(-\mathrm{e}^{-t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-t} w\right) &= \left(\mathrm{e}^{-t}\right)\left(-\mathrm{e}^{-t}\right) \\
\mathrm{d}\left(\mathrm{e}^{-t} w\right) &= \left(-\mathrm{e}^{-2t}\right) \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-t} w &= \int -\mathrm{e}^{-2t} \, \mathrm{d}t \\
\mathrm{e}^{-t} w &= \frac{\mathrm{e}^{-2t}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-t}$ results in

$$
w(t) = \frac{\mathrm{e}^{t} \mathrm{e}^{-2t}}{2} + c_1 \mathrm{e}^{t}
$$

which simplifies to

$$w(t) = \frac{e^{-t}}{2} + c_1 e^t$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{e^{-t}}{2} + c_1 e^t$$

Or

$$y = \frac{1}{\frac{e^{-t}}{2} + c_1 e^t}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{2}{2c_1 + 1}$$

$$c_1 = 0$$

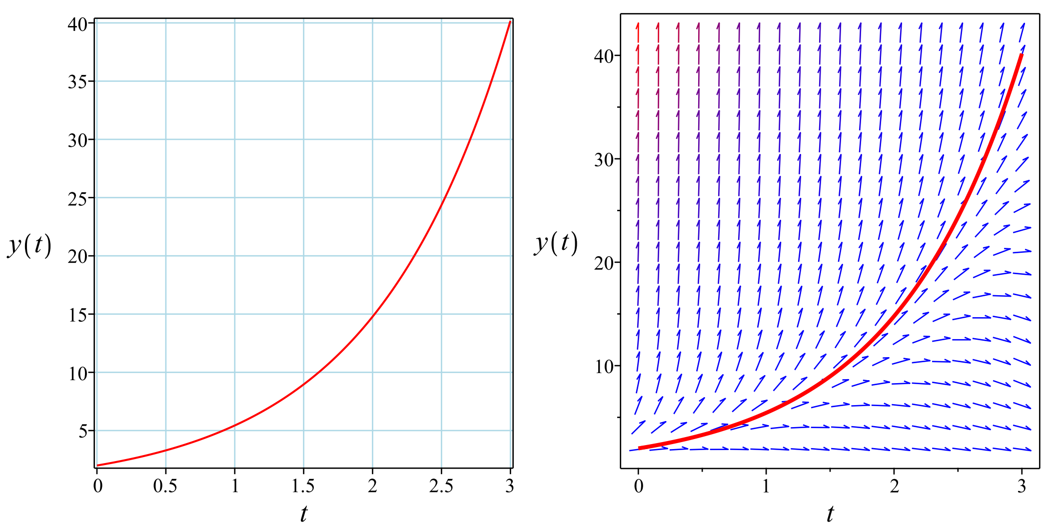
Substituting c_1 found above in the general solution gives

$$y = 2\, e^t$$

Summary

The solution(s) found are the following

$$y = 2\, e^t \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{t}$$

Verified OK.

7.18.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(t, y) \\ &= -y + \mathrm{e}^{-t} y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y + \mathrm{e}^{-t} y^2$$

With Riccati ODE standard form

$$y' = f_0(t) + f_1(t)y + f_2(t)y^2$$

Shows that $f_0(t) = 0$, $f_1(t) = -1$ and $f_2(t) = \mathrm{e}^{-t}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\mathrm{e}^{-t} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(t) - (f_2' + f_1 f_2)\, u'(t) + f_2^2 f_0 u(t) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\mathrm{e}^{-t} \\ f_1 f_2 &= -\mathrm{e}^{-t} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\mathrm{e}^{-t} u''(t) + 2\,\mathrm{e}^{-t} u'(t) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(t) = c_1 + \mathrm{e}^{-2t} c_2
$$

The above shows that

$$
u'(t) = -2\,\mathrm{e}^{-2t} c_2
$$

Using the above in (1) gives the solution

$$
y = \frac{2\,\mathrm{e}^{-2t} c_2 \mathrm{e}^{t}}{c_1 + \mathrm{e}^{-2t} c_2}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{2\,\mathrm{e}^{-t}}{c_3 + \mathrm{e}^{-2t}}
$$

Initial conditions are used to solve for c_3. Substituting $t = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = \frac{2}{c_3 + 1}
$$

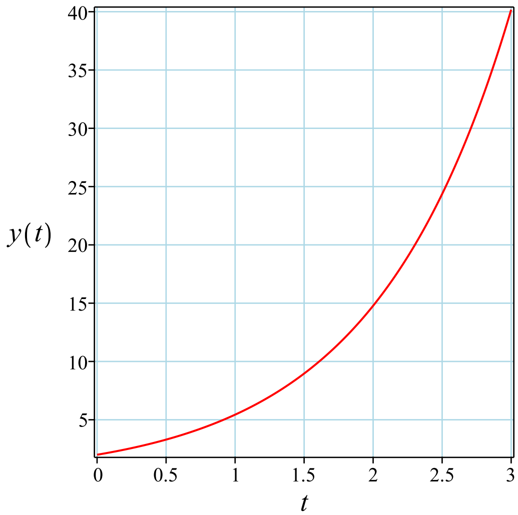
$$
c_3 = 0
$$

Substituting c_3 found above in the general solution gives

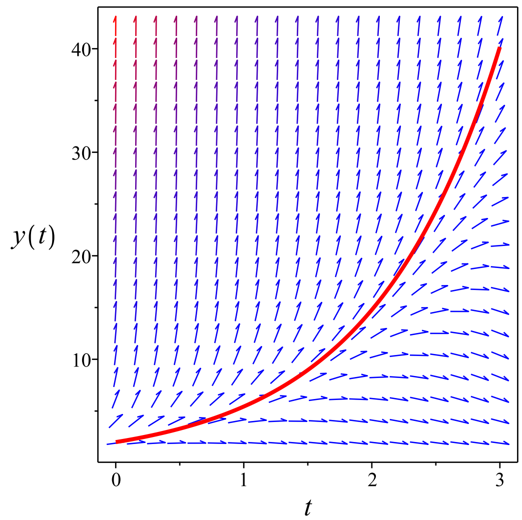
$$
y = 2\,\mathrm{e}^{t}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = 2\,\mathrm{e}^{t} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{t}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 8

```
dsolve([diff(y(t),t)+y(t)=y(t)^2*exp(-t),y(0) = 2],y(t), singsol=all)
```

$$y(t) = 2\,\mathrm{e}^{t}$$

✓ Solution by Mathematica

Time used: 0.281 (sec). Leaf size: 10

```
DSolve[{y'[t]+y[t]==y[t]^2*Exp[-t],{y[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2e^t$$

7.19 problem 19

Internal problem ID [2029]
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Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

$$\boxed{y' - x\left(1 - e^{2y - x^2}\right) = 0}$$

With initial conditions

$$[y(0) = 0]$$

7.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -x\left(e^{-x^2+2y} - 1\right) \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-x\left(\mathrm{e}^{-x^2+2y}-1\right)\right) \\
&= -2x\,\mathrm{e}^{-x^2+2y}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

7.19.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -x\left(\mathrm{e}^{-x^2+2y}-1\right) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$
\xi = x^2 a_4 + yx a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x^2 b_4 + yx b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&2xb_4 + yb_5 + b_2 - x\left(\mathrm{e}^{-x^2+2y}-1\right)(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3) \\
&- x^2\left(\mathrm{e}^{-x^2+2y}-1\right)^2 (xa_5 + 2ya_6 + a_3) \\
&- \left(-\mathrm{e}^{-x^2+2y} + 1 + 2x^2\mathrm{e}^{-x^2+2y}\right)(x^2 a_4 + yx a_5 + y^2 a_6 + x a_2 + y a_3 + a_1) \\
&+ 2x\,\mathrm{e}^{-x^2+2y}\left(x^2 b_4 + yx b_5 + y^2 b_6 + x b_2 + y b_3 + b_1\right) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$-2\,\mathrm{e}^{-2x^2+4y}x^2ya_6-2\,\mathrm{e}^{-x^2+2y}x^3ya_5-2\,\mathrm{e}^{-x^2+2y}x^2y^2a_6+4\,\mathrm{e}^{-x^2+2y}x^2ya_6+2\,\mathrm{e}^{-x^2+2y}x^2yb_5+2\,\mathrm{e}^{-x^2+2y}x\,y^2b_6+2\,\mathrm{e}^{-x^2+2y}xya_5-2\,\mathrm{e}^{-x^2+2y}xyb_6-a_1+b_2-2\,\mathrm{e}^{-x^2+2y}x^2ya_3+2\,\mathrm{e}^{-x^2+2y}xyb_3-2\,\mathrm{e}^{-x^2+2y}x^3a_2-2xa_2-ya_3-2\,\mathrm{e}^{-x^2+2y}x^2a_1+2\,\mathrm{e}^{-x^2+2y}x^2a_3+2\,\mathrm{e}^{-x^2+2y}x^2b_2+2\,\mathrm{e}^{-x^2+2y}xa_2+2\,\mathrm{e}^{-x^2+2y}xb_1-\mathrm{e}^{-x^2+2y}xb_3+\mathrm{e}^{-x^2+2y}ya_3-x^2a_3+2xb_4+yb_5+x^2b_5-x^3a_5+xb_3+\mathrm{e}^{-x^2+2y}a_1-3x^2a_4-y^2a_6-2x^2ya_6-2\,\mathrm{e}^{-x^2+2y}x^4a_4+2\,\mathrm{e}^{-x^2+2y}x^3a_5+2\,\mathrm{e}^{-x^2+2y}x^3b_4+3\,\mathrm{e}^{-x^2+2y}x^2a_4-\mathrm{e}^{-x^2+2y}x^2b_5+\mathrm{e}^{-x^2+2y}y^2a_6+2xyb_6-\mathrm{e}^{-2x^2+4y}x^2a_3-\mathrm{e}^{-2x^2+4y}x^3a_5-2yxa_5=0$$

Setting the numerator to zero gives

$$-2\,\mathrm{e}^{-2x^2+4y}x^2ya_6-2\,\mathrm{e}^{-x^2+2y}x^3ya_5-2\,\mathrm{e}^{-x^2+2y}x^2y^2a_6+4\,\mathrm{e}^{-x^2+2y}x^2ya_6+2\,\mathrm{e}^{-x^2+2y}x^2yb_5+2\,\mathrm{e}^{-x^2+2y}x\,y^2b_6+2\,\mathrm{e}^{-x^2+2y}xya_5-2\,\mathrm{e}^{-x^2+2y}xyb_6-a_1+b_2-2\,\mathrm{e}^{-x^2+2y}x^2ya_3+2\,\mathrm{e}^{-x^2+2y}xyb_3-2\,\mathrm{e}^{-x^2+2y}x^3a_2-2xa_2-ya_3-2\,\mathrm{e}^{-x^2+2y}x^2a_1+2\,\mathrm{e}^{-x^2+2y}x^2a_3+2\,\mathrm{e}^{-x^2+2y}x^2b_2+2\,\mathrm{e}^{-x^2+2y}xa_2+2\,\mathrm{e}^{-x^2+2y}xb_1-\mathrm{e}^{-x^2+2y}xb_3+\mathrm{e}^{-x^2+2y}ya_3-x^2a_3+2xb_4+yb_5+x^2b_5-x^3a_5+xb_3+\mathrm{e}^{-x^2+2y}a_1-3x^2a_4-y^2a_6-2x^2ya_6-2\,\mathrm{e}^{-x^2+2y}x^4a_4+2\,\mathrm{e}^{-x^2+2y}x^3a_5+2\,\mathrm{e}^{-x^2+2y}x^3b_4+3\,\mathrm{e}^{-x^2+2y}x^2a_4-\mathrm{e}^{-x^2+2y}x^2b_5+\mathrm{e}^{-x^2+2y}y^2a_6+2xyb_6-\mathrm{e}^{-2x^2+4y}x^2a_3-\mathrm{e}^{-2x^2+4y}x^3a_5-2yxa_5=0 \tag{6E}$$

Simplifying the above gives

$$-2\,\mathrm{e}^{-2x^2+4y}x^2ya_6-2\,\mathrm{e}^{-x^2+2y}x^3ya_5-2\,\mathrm{e}^{-x^2+2y}x^2y^2a_6+4\,\mathrm{e}^{-x^2+2y}x^2ya_6+2\,\mathrm{e}^{-x^2+2y}x^2yb_5+2\,\mathrm{e}^{-x^2+2y}x\,y^2b_6+2\,\mathrm{e}^{-x^2+2y}xya_5-2\,\mathrm{e}^{-x^2+2y}xyb_6-a_1+b_2-2\,\mathrm{e}^{-x^2+2y}x^2ya_3+2\,\mathrm{e}^{-x^2+2y}xyb_3-2\,\mathrm{e}^{-x^2+2y}x^3a_2-2xa_2-ya_3-2\,\mathrm{e}^{-x^2+2y}x^2a_1+2\,\mathrm{e}^{-x^2+2y}x^2a_3+2\,\mathrm{e}^{-x^2+2y}x^2b_2+2\,\mathrm{e}^{-x^2+2y}xa_2+2\,\mathrm{e}^{-x^2+2y}xb_1-\mathrm{e}^{-x^2+2y}xb_3+\mathrm{e}^{-x^2+2y}ya_3-x^2a_3+2xb_4+yb_5+x^2b_5-x^3a_5+xb_3+\mathrm{e}^{-x^2+2y}a_1-3x^2a_4-y^2a_6-2x^2ya_6-2\,\mathrm{e}^{-x^2+2y}x^4a_4+2\,\mathrm{e}^{-x^2+2y}x^3a_5+2\,\mathrm{e}^{-x^2+2y}x^3b_4+3\,\mathrm{e}^{-x^2+2y}x^2a_4-\mathrm{e}^{-x^2+2y}x^2b_5+\mathrm{e}^{-x^2+2y}y^2a_6+2xyb_6-\mathrm{e}^{-2x^2+4y}x^2a_3-\mathrm{e}^{-2x^2+4y}x^3a_5-2yxa_5=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\mathrm{e}^{-2x^2+4y},\mathrm{e}^{-x^2+2y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^{-2x^2+4y} = v_3, \mathrm{e}^{-x^2+2y} = v_4\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-2v_4v_1^4a_4 - 2v_4v_1^3v_2a_5 - 2v_4v_1^2v_2^2a_6 - 2v_4v_1^3a_2 - 2v_4v_1^2v_2a_3 - v_3v_1^3a_5 \\
&+ 2v_4v_1^3a_5 - 2v_3v_1^2v_2a_6 + 4v_4v_1^2v_2a_6 + 2v_4v_1^3b_4 + 2v_4v_1^2v_2b_5 + 2v_4v_1v_2^2b_6 \\
&- 2v_4v_1^2a_1 - v_3v_1^2a_3 + 2v_4v_1^2a_3 + 3v_4v_1^2a_4 - v_1^3a_5 + 2v_4v_1v_2a_5 - 2v_1^2v_2a_6 \\
&+ v_4v_2^2a_6 + 2v_4v_1^2b_2 + 2v_4v_1v_2b_3 - v_4v_1^2b_5 - 2v_4v_1v_2b_6 + 2v_4v_1a_2 \\
&- v_1^2a_3 + v_4v_2a_3 - 3v_1^2a_4 - 2v_2v_1a_5 - v_2^2a_6 + 2v_4v_1b_1 - v_4v_1b_3 + v_1^2b_5 \\
&+ 2v_1v_2b_6 + v_4a_1 - 2v_1a_2 - v_2a_3 + v_1b_3 + 2v_1b_4 + v_2b_5 - a_1 + b_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-2v_4v_1^4a_4 - 2v_4v_1^3v_2a_5 - v_3v_1^3a_5 + (-2a_2 + 2a_5 + 2b_4)\, v_1^3v_4 \\
&- v_1^3a_5 - 2v_4v_1^2v_2^2a_6 - 2v_3v_1^2v_2a_6 + (-2a_3 + 4a_6 + 2b_5)\, v_1^2v_2v_4 \\
&- 2v_1^2v_2a_6 - v_3v_1^2a_3 + (-2a_1 + 2a_3 + 3a_4 + 2b_2 - b_5)\, v_1^2v_4 \\
&+ (-a_3 - 3a_4 + b_5)\, v_1^2 + 2v_4v_1v_2^2b_6 + (2a_5 + 2b_3 - 2b_6)\, v_1v_2v_4 \\
&+ (-2a_5 + 2b_6)\, v_1v_2 + (2a_2 + 2b_1 - b_3)\, v_1v_4 + (-2a_2 + b_3 + 2b_4)\, v_1 \\
&+ v_4v_2^2a_6 - v_2^2a_6 + v_4v_2a_3 + (-a_3 + b_5)\, v_2 + v_4a_1 - a_1 + b_2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
a_3 &= 0 \\
a_6 &= 0 \\
-a_3 &= 0 \\
-2a_4 &= 0 \\
-2a_5 &= 0 \\
-a_5 &= 0 \\
-2a_6 &= 0 \\
-a_6 &= 0 \\
2b_6 &= 0 \\
-a_1 + b_2 &= 0 \\
-a_3 + b_5 &= 0 \\
-2a_5 + 2b_6 &= 0 \\
-2a_2 + 2a_5 + 2b_4 &= 0 \\
-2a_2 + b_3 + 2b_4 &= 0 \\
2a_2 + 2b_1 - b_3 &= 0 \\
-2a_3 + 4a_6 + 2b_5 &= 0 \\
-a_3 - 3a_4 + b_5 &= 0 \\
2a_5 + 2b_3 - 2b_6 &= 0 \\
-2a_1 + 2a_3 + 3a_4 + 2b_2 - b_5 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= b_4\\
a_3 &= 0\\
a_4 &= 0\\
a_5 &= 0\\
a_6 &= 0\\
b_1 &= -b_4\\
b_2 &= 0\\
b_3 &= 0\\
b_4 &= b_4\\
b_5 &= 0\\
b_6 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= x^2 - 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= x^2 - 1 - \left(-x\left(\mathrm{e}^{-x^2+2y} - 1\right)\right)(x)\\
&= x^2\mathrm{e}^{-x^2}\mathrm{e}^{2y} - 1\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2 e^{-x^2} e^{2y} - 1} dy \end{aligned}$$

Which results in

$$S = -\ln\left(e^y\right) + \frac{e^{x^2} e^{-x^2} \ln\left(-x^2 e^{2y} + e^{x^2}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -x\left(e^{-x^2+2y} - 1\right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x\left(e^{x^2} - e^{2y}\right)}{-x^2 e^{2y} + e^{x^2}} \\ S_y &= \frac{e^{x^2}}{x^2 e^{2y} - e^{x^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-y + \frac{\ln\left(-x^2 e^{2y} + e^{x^2}\right)}{2} = c_1$$

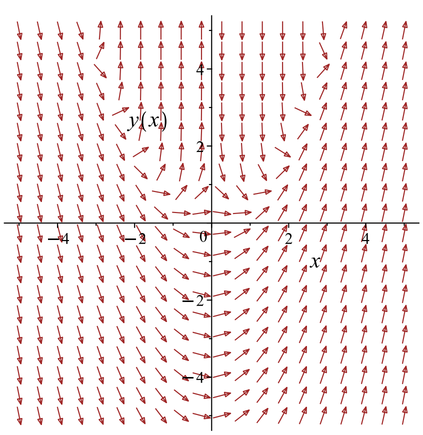
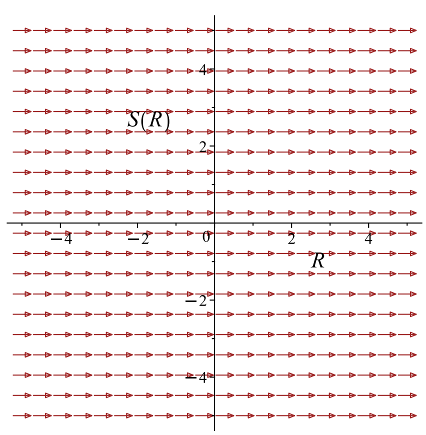
Which simplifies to

$$-y + \frac{\ln\left(-x^2 e^{2y} + e^{x^2}\right)}{2} = c_1$$

Which gives

$$y = \frac{\ln\left(\frac{1}{e^{2c_1} + x^2}\right)}{2} + \frac{x^2}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x\left(e^{-x^2+2y} - 1\right)$  	$R = x$ $S = -y + \frac{\ln\left(-x^2 e^{2y} + e\right.}{2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln\left(e^{-2c_1}\right)}{2}$$

$$c_1 = 0$$

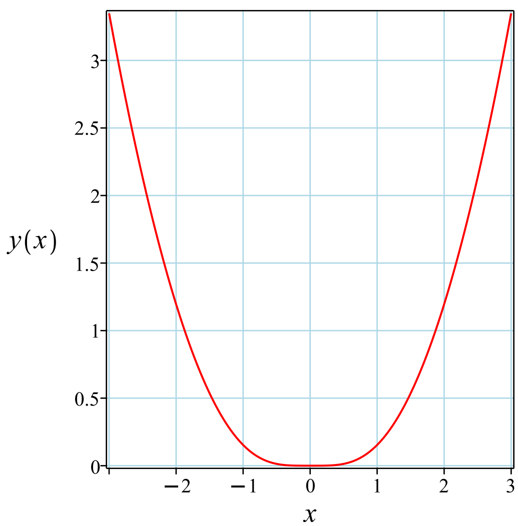
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln\left(\frac{1}{x^2+1}\right)}{2} + \frac{x^2}{2}$$

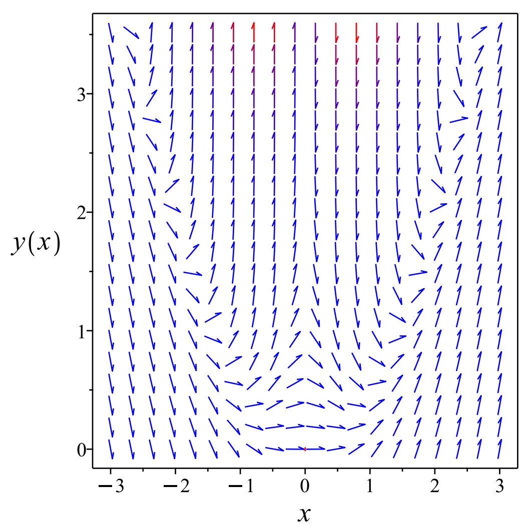
Summary

The solution(s) found are the following

$$y = \frac{\ln\left(\frac{1}{x^2+1}\right)}{2} + \frac{x^2}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\ln\left(\frac{1}{x^2+1}\right)}{2} + \frac{x^2}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3`[x, x^2-1]
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 18

```
dsolve([diff(y(x),x)=x*(1-exp(2*y(x)-x^2)),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} - \frac{\ln(x^2+1)}{2}$$

✓ Solution by Mathematica

Time used: 0.5 (sec). Leaf size: 21

```
DSolve[{y'[x]==x*(1-Exp[2*y[x]-x^2]),{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x^2 - \log\left(x^2+1\right)\right)$$

7.20 problem 20

Internal problem ID [2030]
Internal file name [`OUTPUT/2030_Sunday_February_25_2024_06_45_35_AM_21923956/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{2y - (x^2y^4 + x)\, y' = 0}$$

With initial conditions

$$[y(1) = 1]$$

7.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{2y}{x\,(x\,y^4+1)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty \le x < -1, -1 < x < 0, 0 < x \le \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2y}{x\,(x\,y^4+1)}\right) \\ = \frac{2}{(x\,y^4+1)\,x} - \frac{8y^4}{(x\,y^4+1)^2}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty \leq x < -1, -1 < x < 0, 0 < x \leq \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

7.20.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{2y}{x\,(x\,y^4+1)} \\ y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{2y(b_3 - a_2)}{x\,(x\,y^4+1)} - \frac{4y^2 a_3}{x^2\,(x\,y^4+1)^2} \\
&- \left(-\frac{2y}{x^2\,(x\,y^4+1)} - \frac{2y^5}{x\,(x\,y^4+1)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(\frac{2}{(x\,y^4+1)\,x} - \frac{8y^4}{(x\,y^4+1)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{x^4y^8b_2 + 8x^3y^4b_2 + 2x^2y^5a_2 + 8x^2y^5b_3 + 4x\,y^6a_3 + 6x^2y^4b_1 + 4x\,y^5a_1 - b_2x^2 - 2y^2a_3 - 2xb_1 + 2ya_1}{x^2\,(x\,y^4+1)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&x^4y^8b_2 + 8x^3y^4b_2 + 2x^2y^5a_2 + 8x^2y^5b_3 + 4x\,y^6a_3 \\
&+ 6x^2y^4b_1 + 4x\,y^5a_1 - b_2x^2 - 2y^2a_3 - 2xb_1 + 2ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&b_2v_1^4v_2^8 + 2a_2v_1^2v_2^5 + 4a_3v_1v_2^6 + 8b_2v_1^3v_2^4 + 8b_3v_1^2v_2^5 \\
&+ 4a_1v_1v_2^5 + 6b_1v_1^2v_2^4 - 2a_3v_2^2 - b_2v_1^2 + 2a_1v_2 - 2b_1v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
& b_2 v_1^4 v_2^8 + 8 b_2 v_1^3 v_2^4 + (2 a_2 + 8 b_3) v_1^2 v_2^5 + 6 b_1 v_1^2 v_2^4 - b_2 v_1^2 \\
& + 4 a_3 v_1 v_2^6 + 4 a_1 v_1 v_2^5 - 2 b_1 v_1 - 2 a_3 v_2^2 + 2 a_1 v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
2a_1 &= 0 \\
4a_1 &= 0 \\
-2a_3 &= 0 \\
4a_3 &= 0 \\
-2b_1 &= 0 \\
6b_1 &= 0 \\
-b_2 &= 0 \\
8b_2 &= 0 \\
2a_2 + 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -4b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -4x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{2y}{x\,(x\,y^4+1)}\right)(-4x) \\
&= \frac{y^5x+9y}{x\,y^4+1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^5x+9y}{x\,y^4+1}} dy
\end{aligned}$$

Which results in

$$S = \frac{2\ln(x\,y^4+9)}{9} + \frac{\ln(y)}{9}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y}{x\,(x\,y^4+1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2y^4}{9x\,y^4+81} \\
S_y &= \frac{x\,y^4+1}{y\,(x\,y^4+9)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{2}{9x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2}{9R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{2\ln(R)}{9} + c_1 \tag{4}
$$

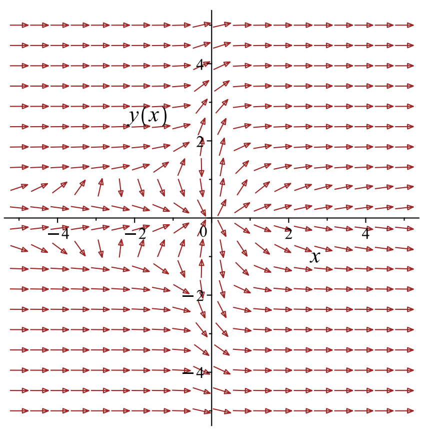
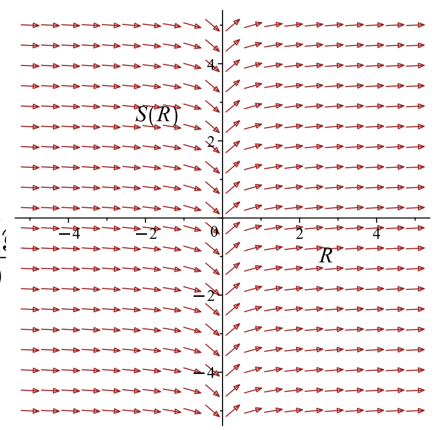
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2\ln(xy^4+9)}{9} + \frac{\ln(y)}{9} = \frac{2\ln(x)}{9} + c_1
$$

Which simplifies to

$$
\frac{2\ln(xy^4+9)}{9} + \frac{\ln(y)}{9} = \frac{2\ln(x)}{9} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y}{x(x\,y^4+1)}$ 	$R = x$ $S = \frac{2\ln(x\,y^4+9)}{9} + \frac{\ln(}{9}$	$\frac{dS}{dR} = \frac{2}{9R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{2\ln(2)}{9} + \frac{2\ln(5)}{9} = c_1$$

$$c_1 = \frac{2\ln(2)}{9} + \frac{2\ln(5)}{9}$$

Substituting c_1 found above in the general solution gives

$$\frac{2\ln(x\,y^4+9)}{9} + \frac{\ln(y)}{9} = \frac{2\ln(x)}{9} + \frac{2\ln(2)}{9} + \frac{2\ln(5)}{9}$$

Summary

The solution(s) found are the following

$$\frac{2\ln(xy^4+9)}{9} + \frac{\ln(y)}{9} = \frac{2\ln(x)}{9} + \frac{2\ln(2)}{9} + \frac{2\ln(5)}{9} \tag{1}$$

Verification of solutions

$$\frac{2\ln(xy^4+9)}{9} + \frac{\ln(y)}{9} = \frac{2\ln(x)}{9} + \frac{2\ln(2)}{9} + \frac{2\ln(5)}{9}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.797 (sec). Leaf size: 26

```
dsolve([2*y(x)=(x^2*y(x)^4+x)*diff(y(x),x),y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{100x^2}{\operatorname{RootOf}\left(\_Z^9 - 100000000x^9 - 9\_Z^8\right)^2}$$

✓ Solution by Mathematica

Time used: 4.207 (sec). Leaf size: 33

```
DSolve[{2*y[x]==(x^2*y[x]^4+x)*y'[x],{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \operatorname{Root}\left[\#1^9x^2 + 18\#1^5x + 81\#1 - 100x^2\&, 1\right]$$

7.21 problem 21

Internal problem ID [2031]
Internal file name [`OUTPUT/2031_Sunday_February_25_2024_06_45_37_AM_90454972/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]`

Unable to solve or complete the solution.

$$\boxed{xy(1+xy^2)\,y' = -1}$$

With initial conditions

$$[y(1) = 0]$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
<- Bernoulli successful
<- inverse_Riccati successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 66

```
dsolve([1+x*y(x)*(1+x*y(x)^2)*diff(y(x),x)=0,y(1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-2\left(\operatorname{LambertW}\left(-1, -\frac{3\,\mathrm{e}^{-\frac{2x+1}{2x}}}{2}\right)x + x + \frac{1}{2}\right)x}}{x}$$

$$y(x) = -\frac{\sqrt{-2\left(\operatorname{LambertW}\left(-1, -\frac{3\,\mathrm{e}^{-\frac{2x+1}{2x}}}{2}\right)x + x + \frac{1}{2}\right)x}}{x}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{1+x*y[x]*(1+x*y[x]^2)*y'[x]==0,{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

{}

7.22 problem 22

Internal problem ID [2032]
Internal file name [`OUTPUT/2032_Sunday_February_25_2024_06_45_37_AM_37374244/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 11, page 45
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{(1-x^2)\,y'+yx-x(1-x^2)\sqrt{y}=0}$$

With initial conditions

$$[y(0)=1]$$

7.22.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y)\\ &= \frac{x\left(x^2\sqrt{y}-\sqrt{y}+y\right)}{x^2-1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x(x^2\sqrt{y}-\sqrt{y}+y)}{x^2-1}\right)\\ &= \frac{x\left(\frac{x^2}{2\sqrt{y}}-\frac{1}{2\sqrt{y}}+1\right)}{x^2-1}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

7.22.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \frac{x(x^2\sqrt{y}-\sqrt{y}+y)}{x^2-1}\\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 229: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \sqrt{y}\, \mathrm{e}^{\frac{\ln(x-1)}{4}+\frac{\ln(x+1)}{4}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{y}\, \mathrm{e}^{\frac{\ln(x-1)}{4}+\frac{\ln(x+1)}{4}}} dy \end{aligned}$$

Which results in

$$S = 2\sqrt{y}\, \mathrm{e}^{\ln\left(\frac{1}{(x-1)^{\frac{1}{4}}}\right)+\ln\left(\frac{1}{(x+1)^{\frac{1}{4}}}\right)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x\left(x^2\sqrt{y} - \sqrt{y} + y\right)}{x^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\sqrt{y}\, x}{(x-1)^{\frac{5}{4}} (x+1)^{\frac{5}{4}}} \\ S_y &= \frac{1}{\sqrt{y}\, (x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x}{(x+1)^{\frac{1}{4}} (x-1)^{\frac{1}{4}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{(R+1)^{\frac{1}{4}} (R-1)^{\frac{1}{4}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{2(R-1)^{\frac{3}{4}} (R+1)^{\frac{3}{4}}}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\sqrt{y}}{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}} = \frac{2(x-1)^{\frac{3}{4}} (x+1)^{\frac{3}{4}}}{3} + c_1$$

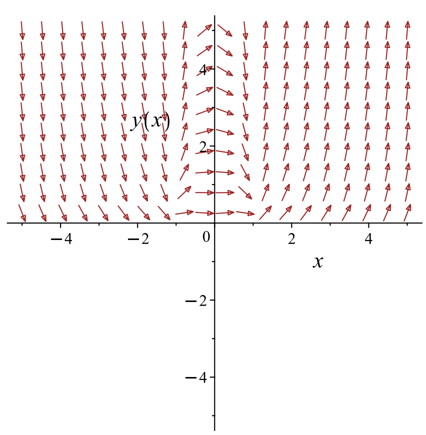
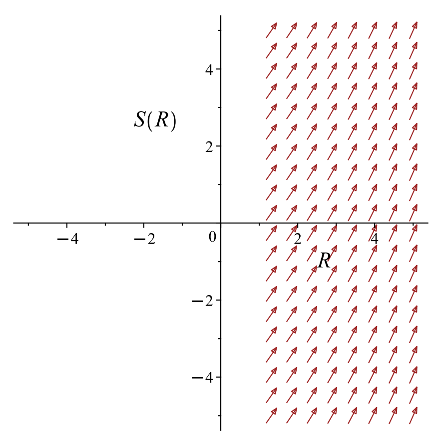
Which simplifies to

$$\frac{2\sqrt{y}}{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}} = \frac{2(x-1)^{\frac{3}{4}} (x+1)^{\frac{3}{4}}}{3} + c_1$$

Which gives

$$y = \frac{c_1(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}} x^2}{3} + \frac{c_1^2\sqrt{x-1}\,\sqrt{x+1}}{4} + \frac{x^4}{9} - \frac{c_1(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}}{3} - \frac{2x^2}{9} + \frac{1}{9}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(x^2\sqrt{y}-\sqrt{y}+y)}{x^2-1}$ 	$R = x$ $S = \frac{2\sqrt{y}}{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}}$	$\frac{dS}{dR} = \frac{R}{(R+1)^{\frac{1}{4}} (R-1)^{\frac{1}{4}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{i c_1^2}{4} - \frac{\sqrt{2}\, c_1}{6} - \frac{i c_1 \sqrt{2}}{6} + \frac{1}{9}$$

$$c_1 = \left(-\frac{2}{3} + \frac{2i}{3}\right)\sqrt{2}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{2\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x+1)^{\frac{1}{4}}\,x^2}{9} + \frac{2i\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x+1)^{\frac{1}{4}}\,x^2}{9} - \frac{4i\sqrt{x-1}\,\sqrt{x+1}}{9} + \frac{x^4}{9} + \frac{2\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x}{9}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &-\frac{2\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x+1)^{\frac{1}{4}}\,x^2}{9} + \frac{2i\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x+1)^{\frac{1}{4}}\,x^2}{9} - \frac{4i\sqrt{x-1}\,\sqrt{x+1}}{9} \\ &+ \frac{x^4}{9} + \frac{2\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x+1)^{\frac{1}{4}}}{9} - \frac{2i\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x+1)^{\frac{1}{4}}}{9} - \frac{2x^2}{9} + \frac{1}{9} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = &-\frac{2\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x+1)^{\frac{1}{4}}\,x^2}{9} + \frac{2i\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x+1)^{\frac{1}{4}}\,x^2}{9} - \frac{4i\sqrt{x-1}\,\sqrt{x+1}}{9} \\ &+ \frac{x^4}{9} + \frac{2\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x+1)^{\frac{1}{4}}}{9} - \frac{2i\sqrt{2}\,(x-1)^{\frac{1}{4}}\,(x+1)^{\frac{1}{4}}}{9} - \frac{2x^2}{9} + \frac{1}{9} \end{aligned}$$

Verified OK.

7.22.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x\left(x^2\sqrt{y} - \sqrt{y} + y\right)}{x^2 - 1} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{x}{x^2 - 1} y + x\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{x}{x^2-1} \\ f_1(x) &= x \\ n &= \frac{1}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y'\frac{1}{\sqrt{y}} = \frac{x\sqrt{y}}{x^2-1} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \sqrt{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{2\sqrt{y}}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 2w'(x) &= \frac{xw(x)}{x^2-1} + x \\ w' &= \frac{xw}{2x^2-2} + \frac{x}{2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{x}{2x^2-2} \\
q(x) &= \frac{x}{2}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{x w(x)}{2x^2-2} = \frac{x}{2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{x}{2x^2-2}dx} \\
&= e^{-\frac{\ln(x-1)}{4}-\frac{\ln(x+1)}{4}}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{1}{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{x}{2}\right) \\
\frac{d}{dx}\left(\frac{w}{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}}\right) &= \left(\frac{1}{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}}\right)\left(\frac{x}{2}\right) \\
d\left(\frac{w}{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}}\right) &= \left(\frac{x}{2(x+1)^{\frac{1}{4}}(x-1)^{\frac{1}{4}}}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}} &= \int \frac{x}{2(x+1)^{\frac{1}{4}}(x-1)^{\frac{1}{4}}} dx \\
\frac{w}{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}} &= \frac{(x-1)^{\frac{3}{4}}(x+1)^{\frac{3}{4}}}{3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}}$ results in

$$
w(x) = \frac{(x-1)(x+1)}{3} + c_1(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}
$$

which simplifies to

$$
w(x) = \frac{x^2}{3} - \frac{1}{3} + c_1(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}
$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$\sqrt{y} = \frac{x^2}{3} - \frac{1}{3} + c_1(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{3} + \frac{\sqrt{2}\, c_1}{2} + \frac{i c_1 \sqrt{2}}{2}$$

$$c_1 = \left(\frac{2}{3} - \frac{2i}{3}\right)\sqrt{2}$$

Substituting c_1 found above in the general solution gives

$$\sqrt{y} = \frac{x^2}{3} - \frac{1}{3} + \frac{2\sqrt{2}\,(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}}{3} - \frac{2i\sqrt{2}\,(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}}{3}$$

Solving for y from the above gives

$$y = \left(\frac{4}{9} - \frac{4i}{9}\right)(x+1)^{\frac{5}{4}} (x-1)^{\frac{5}{4}} \sqrt{2} + \frac{x^4}{9} - \frac{16i\sqrt{x-1}\,\sqrt{x+1}}{9} - \frac{2x^2}{9} + \frac{1}{9}$$

Summary

The solution(s) found are the following

$$y = \left(\frac{4}{9} - \frac{4i}{9}\right)(x+1)^{\frac{5}{4}} (x-1)^{\frac{5}{4}} \sqrt{2} + \frac{x^4}{9} - \frac{16i\sqrt{x-1}\,\sqrt{x+1}}{9} - \frac{2x^2}{9} + \frac{1}{9} \tag{1}$$

Verification of solutions

$$y = \left(\frac{4}{9} - \frac{4i}{9}\right)(x+1)^{\frac{5}{4}} (x-1)^{\frac{5}{4}} \sqrt{2} + \frac{x^4}{9} - \frac{16i\sqrt{x-1}\,\sqrt{x+1}}{9} - \frac{2x^2}{9} + \frac{1}{9}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.547 (sec). Leaf size: 46

```
dsolve([(1-x^2)*diff(y(x),x)+x*y(x)=x*(1-x^2)*sqrt(y(x)),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \left(\frac{4}{9} - \frac{4i}{9}\right)(x+1)^{\frac{5}{4}}\sqrt{2}\,(x-1)^{\frac{5}{4}} + \frac{x^4}{9} - \frac{16i\sqrt{x-1}\,\sqrt{x+1}}{9} - \frac{2x^2}{9} + \frac{1}{9}$$

✓ Solution by Mathematica

Time used: 0.228 (sec). Leaf size: 130

```
DSolve[{(1-x^2)*y'[x]+x*y[x]==x*(1-x^2)*Sqrt[y[x]],{y[0]==1}},y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{1}{9}\left(x^4 + \left(4(-1)^{3/4}\sqrt[4]{x^2-1} - 2\right)x^2 - 4i\sqrt{x^2-1} - 4(-1)^{3/4}\sqrt[4]{x^2-1} + 1\right)$$
$$y(x) \to \frac{1}{9}\left(x^4 - 2\left(4(-1)^{3/4}\sqrt[4]{x^2-1} + 1\right)x^2 - 16i\sqrt{x^2-1} + 8(-1)^{3/4}\sqrt[4]{x^2-1} + 1\right)$$

8 Exercise 12, page 46

8.1 problem 1

Internal problem ID [2033]
Internal file name [`OUTPUT/2033_Sunday_February_25_2024_06_45_39_AM_40975539/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(1-x)\,y'-y=1}$$

8.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-y-1}{x-1} \end{aligned}$$

Where $f(x) = \frac{1}{x-1}$ and $g(y) = -y - 1$. Integrating both sides gives

$$\frac{1}{-y-1}\, dy = \frac{1}{x-1}\, dx$$
$$\int \frac{1}{-y-1}\, dy = \int \frac{1}{x-1}\, dx$$
$$-\ln(y+1) = \ln(x-1) + c_1$$

Raising both side to exponential gives

$$\frac{1}{y+1} = \mathrm{e}^{\ln(x-1)+c_1}$$

Which simplifies to

$$\frac{1}{y+1} = (x-1)\, c_2$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(c_2 \mathrm{e}^{\ln(x-1)+c_1} - 1\right) \mathrm{e}^{-c_1}}{c_2\,(x-1)} \tag{1}$$

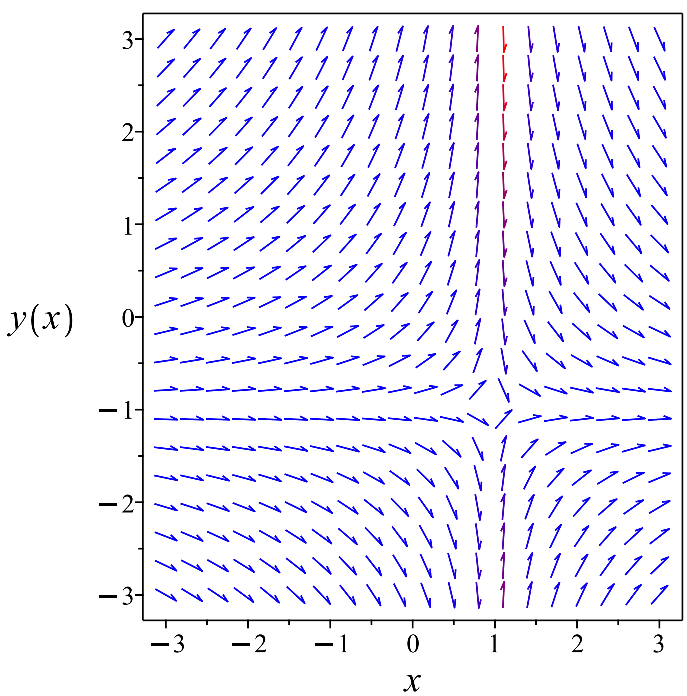


Figure 387: Slope field plot

Verification of solutions

$$y = -\frac{\left(c_2 e^{\ln(x-1)+c_1} - 1\right) e^{-c_1}}{c_2 (x-1)}$$

Verified OK.

8.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x-1}$$
$$q(x) = -\frac{1}{x-1}$$

Hence the ode is

$$y' + \frac{y}{x-1} = -\frac{1}{x-1}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x-1}dx}$$
$$= x - 1$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\frac{1}{x-1}\right)$$
$$\frac{d}{dx}((x-1)\,y) = (x-1)\left(-\frac{1}{x-1}\right)$$
$$d((x-1)\,y) = -1\,dx$$

Integrating gives

$$(x-1)\,y = \int -1\,dx$$
$$(x-1)\,y = -x + c_1$$

Dividing both sides by the integrating factor $\mu = x - 1$ results in

$$y = -\frac{x}{x-1} + \frac{c_1}{x-1}$$

which simplifies to

$$y = \frac{-x + c_1}{x - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-x + c_1}{x - 1} \tag{1}$$

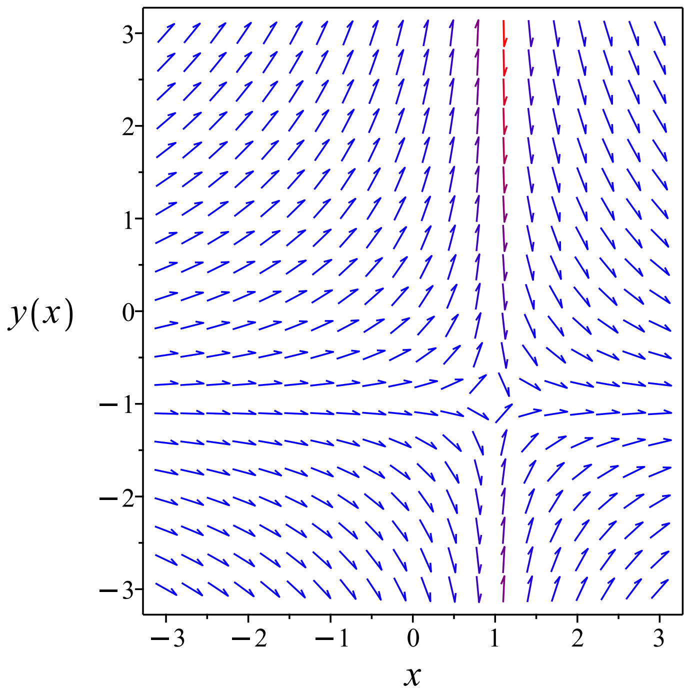


Figure 388: Slope field plot

Verification of solutions

$$y = \frac{-x + c_1}{x - 1}$$

Verified OK.

8.1.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{1+y}{1-x} \tag{1}$$

Which becomes

$$0 = (1-x)\,dy + (-y-1)\,dx \tag{2}$$

But the RHS is complete differential because

$$(1-x)\,dy + (-y-1)\,dx = d(-x(y+1)+y)$$

Hence (2) becomes

$$0 = d(-x(y+1)+y)$$

Integrating both sides gives gives these solutions

$$y = \frac{-x+c_1}{x-1} + c_1$$

Summary

The solution(s) found are the following

$$y = \frac{-x+c_1}{x-1} + c_1 \tag{1}$$

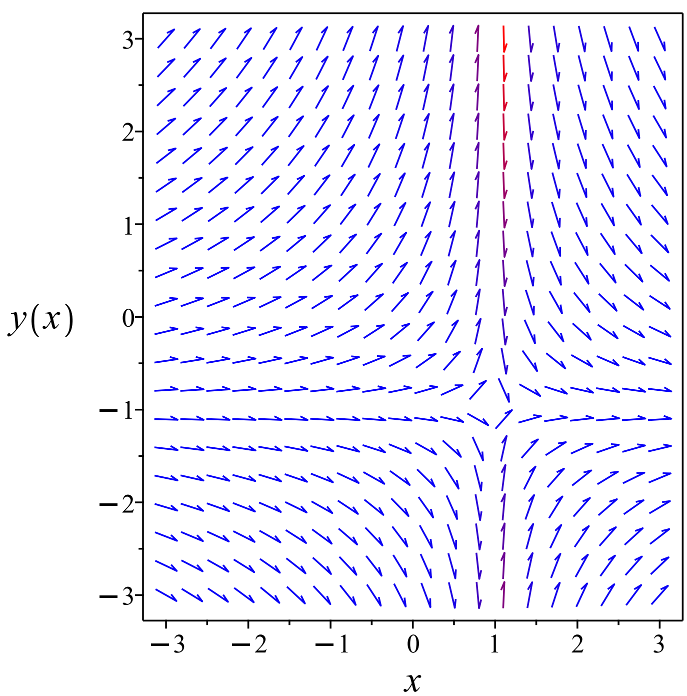


Figure 389: Slope field plot

Verification of solutions

$$y = \frac{-x + c_1}{x - 1} + c_1$$

Verified OK.

8.1.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{1 + Y(X) + y_0}{X + x_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{du}{dX}X + u = -u$$
$$\frac{du}{dX} = -\frac{2u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) + \frac{2u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X + 2u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{2u}{X} \end{aligned}$$

Where $f(X) = -\frac{2}{X}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{X}\,dX \\ \int \frac{1}{u}\,du &= \int -\frac{2}{X}\,dX \\ \ln(u) &= -2\ln(X) + c_2 \\ u &= \mathrm{e}^{-2\ln(X)+c_2} \\ &= \frac{c_2}{X^2} \end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \frac{c_2}{X}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{c_2}{X}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x + x_0 \end{aligned}$$

Or

$$Y = y - 1$$
$$X = x + 1$$

Then the solution in y becomes

$$1 + y = \frac{c_2}{x - 1}$$

Summary
The solution(s) found are the following

$$1 + y = \frac{c_2}{x - 1} \tag{1}$$

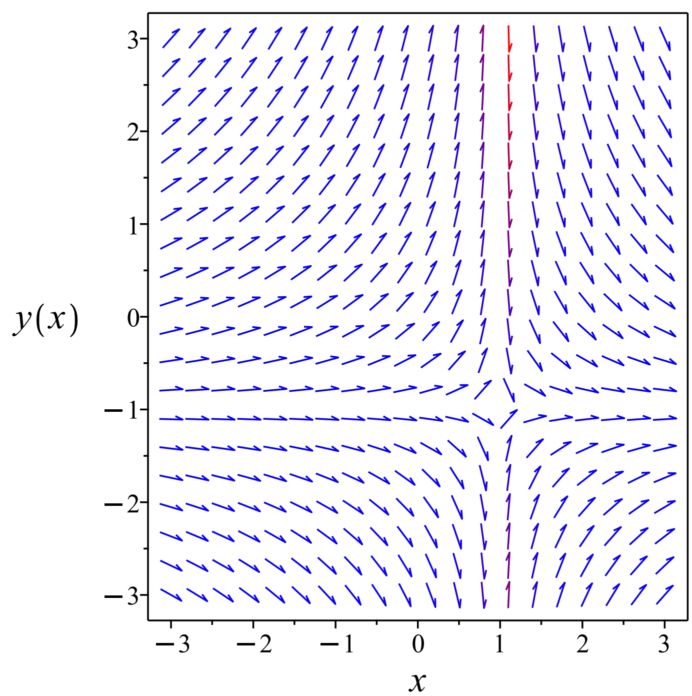


Figure 390: Slope field plot

Verification of solutions

$$1 + y = \frac{c_2}{x - 1}$$

Verified OK.

8.1.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y+1}{x-1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 231: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = \frac{1}{x - 1} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$S = \int \frac{1}{\eta} dy$$
$$= \int \frac{1}{\frac{1}{x-1}} dy$$

Which results in

$$S = (x - 1) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y + 1}{x - 1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \\
S_y &= x - 1
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y(x-1) = -x + c_1$$

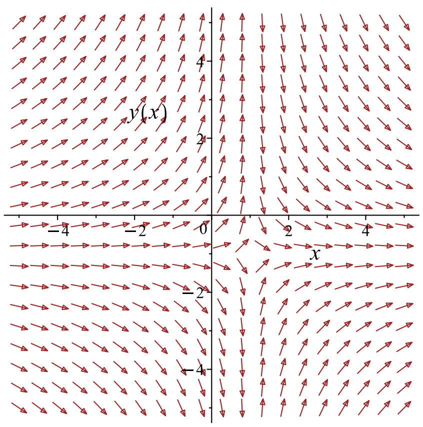
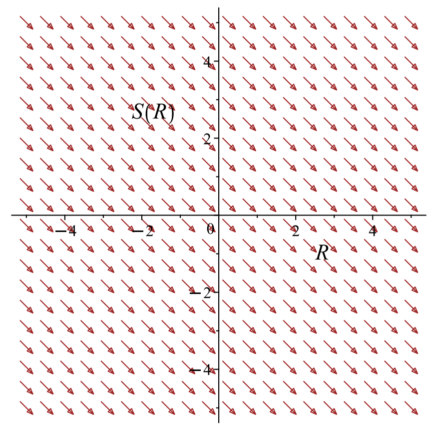
Which simplifies to

$$y(x-1) = -x + c_1$$

Which gives

$$y = \frac{-x + c_1}{x - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y+1}{x-1}$ 	$R = x$ $S = (x-1)\, y$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \frac{-x + c_1}{x - 1} \tag{1}$$

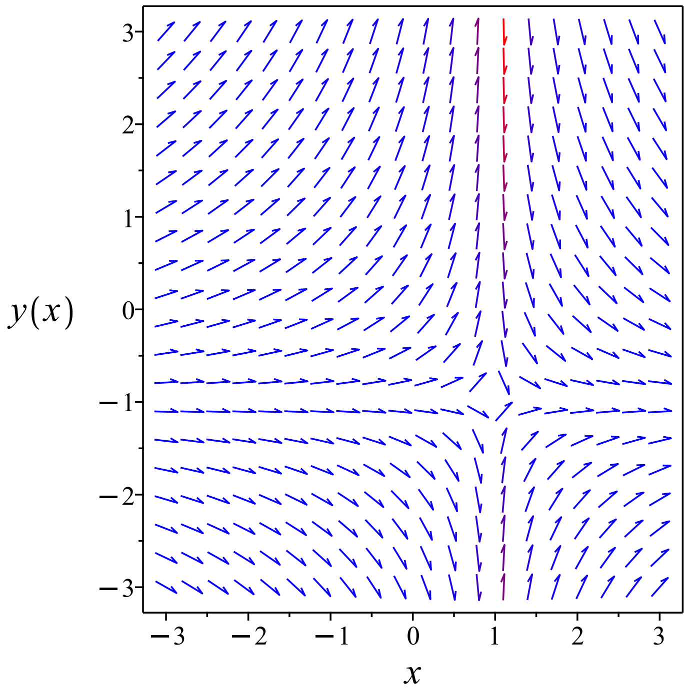


Figure 391: Slope field plot

Verification of solutions

$$y = \frac{-x + c_1}{x - 1}$$

Verified OK.

8.1.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{-y-1}\right) \mathrm{d}y &= \left(\frac{1}{x-1}\right) \mathrm{d}x \\
\left(-\frac{1}{x-1}\right) \mathrm{d}x + \left(\frac{1}{-y-1}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x-1} \\
N(x, y) &= \frac{1}{-y-1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x-1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-y-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x-1} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x-1) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y-1}$. Therefore equation (4) becomes

$$
\frac{1}{-y-1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{1}{y+1}\right) \mathrm{d}y \\
f(y) &= -\ln(y+1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x-1) - \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x-1) - \ln(y+1)$$

The solution becomes

$$y = -\frac{(x\,\mathrm{e}^{c_1} - \mathrm{e}^{c_1} - 1)\,\mathrm{e}^{-c_1}}{x-1}$$

Summary

The solution(s) found are the following

$$y = -\frac{(x\,\mathrm{e}^{c_1} - \mathrm{e}^{c_1} - 1)\,\mathrm{e}^{-c_1}}{x-1} \tag{1}$$

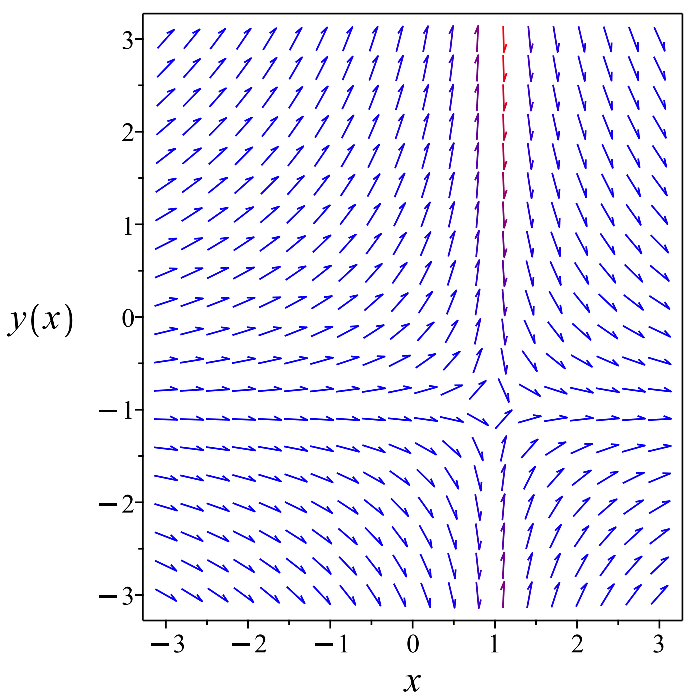


Figure 392: Slope field plot

Verification of solutions

$$y = -\frac{(x\,e^{c_1} - e^{c_1} - 1)\,e^{-c_1}}{x-1}$$

Verified OK.

8.1.7 Maple step by step solution

Let's solve

$(1-x)\,y' - y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int ((1-x)\,y' - y)\,dx = \int 1dx + c_1$

- Evaluate integral

  $-y(x-1) = x + c_1$

- Solve for y

  $y = -\frac{x+c_1}{x-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1-x)*diff(y(x),x)-(1+y(x))=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 - x}{x-1}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 22

```
DSolve[(1-x)*y'[x]-(1+y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x + c_1}{1 - x}$$
$$y(x) \to -1$$

8.2 problem 2

Internal problem ID [2034]
Internal file name [`OUTPUT/2034_Sunday_February_25_2024_06_45_40_AM_5992474/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2 + (x^2 + yx)\, y' = 0}$$

8.2.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\, x^2 + (x^2 + u(x)\, x^2)\, (u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^2 + u}{x\,(u+1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2+u}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+u}{u+1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{2u^2+u}{u+1}}\, du = \int -\frac{1}{x}\, dx$$

$$\ln(u) - \frac{\ln(2u+1)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\ln(u) - \frac{\ln(2u+1)}{2}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u}{\sqrt{2u+1}} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)}{\sqrt{2u(x)+1}} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x\sqrt{\frac{2y}{x}+1}} = \frac{c_3}{x}$$

$$\frac{y}{\sqrt{\frac{x+2y}{x}}\, x} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{y}{\sqrt{\frac{x+2y}{x}}} = c_3$$

Summary

The solution(s) found are the following

$$\frac{y}{\sqrt{\frac{x+2y}{x}}} = c_3 \tag{1}$$

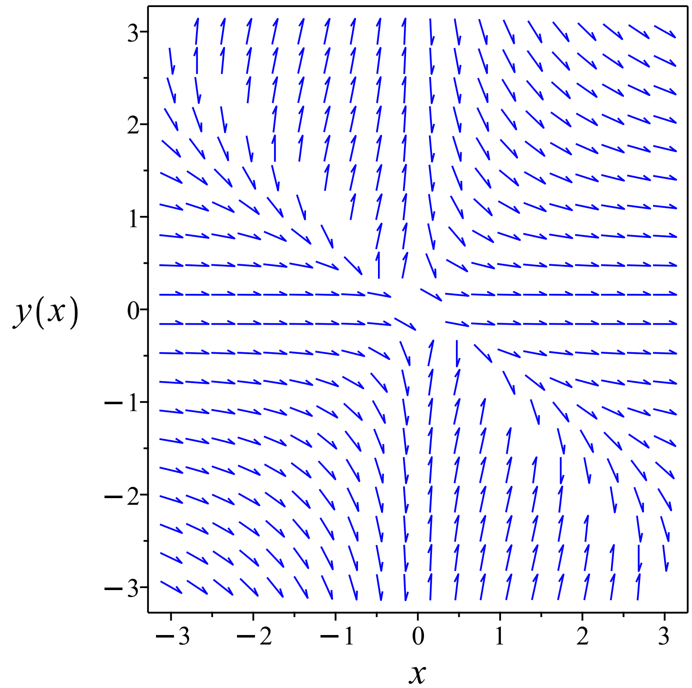


Figure 393: Slope field plot

Verification of solutions

$$\frac{y}{\sqrt{\frac{x+2y}{x}}} = c_3$$

Verified OK.

8.2.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y^2}{x(x+y)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y^2(b_3 - a_2)}{x(x+y)} - \frac{y^4 a_3}{x^2(x+y)^2} - \left(\frac{y^2}{x^2(x+y)} + \frac{y^2}{x(x+y)^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{2y}{x(x+y)} + \frac{y^2}{x(x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^4 b_2 + 4x^3 y b_2 - x^2 y^2 a_2 + 2x^2 y^2 b_2 + x^2 y^2 b_3 - 2x\,y^3 a_3 - 2y^4 a_3 + 2x^2 y b_1 - 2x\,y^2 a_1 + x\,y^2 b_1 - y^3 a_1}{x^2(x+y)^2} = 0$$

Setting the numerator to zero gives

$$x^4 b_2 + 4x^3 y b_2 - x^2 y^2 a_2 + 2x^2 y^2 b_2 + x^2 y^2 b_3 - 2x\,y^3 a_3 - 2y^4 a_3 + 2x^2 y b_1 - 2x\,y^2 a_1 + x\,y^2 b_1 - y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_1^2 v_2^2 - 2a_3 v_1 v_2^3 - 2a_3 v_2^4 + b_2 v_1^4 + 4b_2 v_1^3 v_2 + 2b_2 v_1^2 v_2^2 + b_3 v_1^2 v_2^2 - 2a_1 v_1 v_2^2 - a_1 v_2^3 + 2b_1 v_1^2 v_2 + b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
& b_2 v_1^4 + 4 b_2 v_1^3 v_2 + (-a_2 + 2b_2 + b_3) v_1^2 v_2^2 + 2 b_1 v_1^2 v_2 \\
& - 2 a_3 v_1 v_2^3 + (-2a_1 + b_1) v_1 v_2^2 - 2 a_3 v_2^4 - a_1 v_2^3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-a_1 &= 0 \\
-2a_3 &= 0 \\
2b_1 &= 0 \\
4b_2 &= 0 \\
-2a_1 + b_1 &= 0 \\
-a_2 + 2b_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y^2}{x\,(x+y)}\right)(x) \\
&= \frac{yx + 2y^2}{x+y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{yx+2y^2}{x+y}} dy \end{aligned}$$

Which results in

$$S = \ln(y) - \frac{\ln(x + 2y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2}{x(x + y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{2x + 4y} \\ S_y &= \frac{x + y}{y(x + 2y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

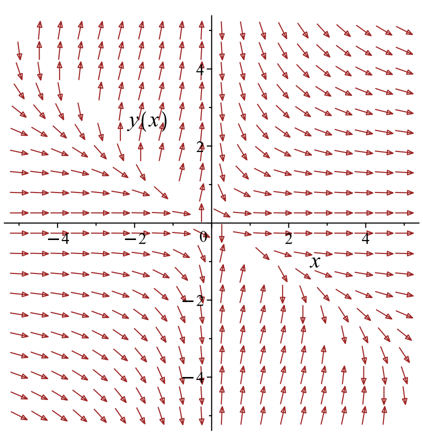
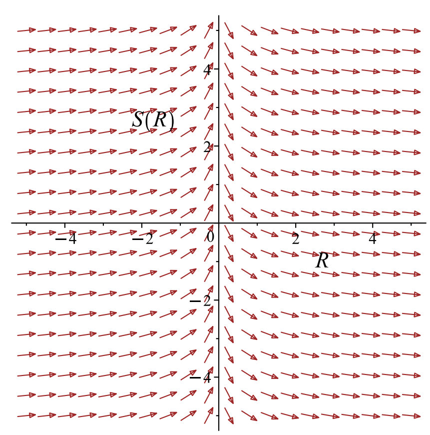
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - \frac{\ln(x+2y)}{2} = -\frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$\ln(y) - \frac{\ln(x+2y)}{2} = -\frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2}{x(x+y)}$ 	$R = x$ $S = \ln(y) - \frac{\ln(x+2y)}{2}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary
The solution(s) found are the following

$$\ln(y) - \frac{\ln(x+2y)}{2} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

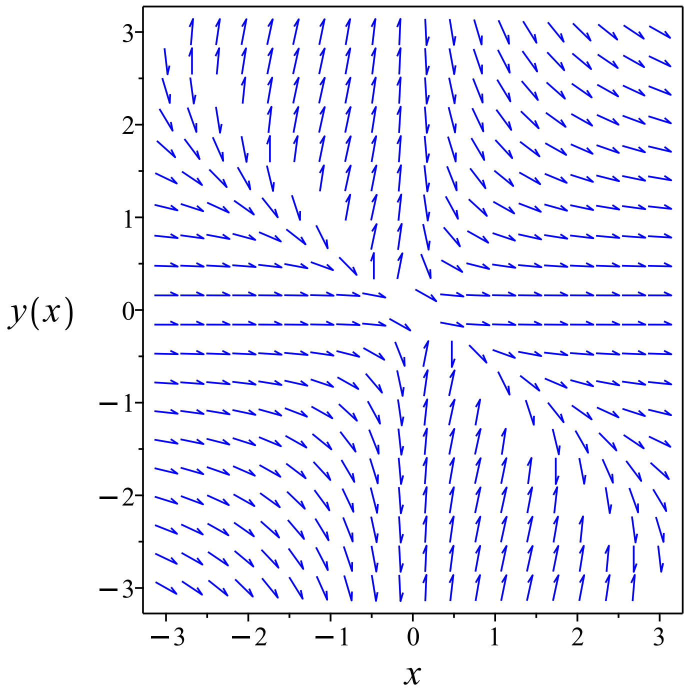


Figure 394: Slope field plot

Verification of solutions

$$\ln(y) - \frac{\ln(x+2y)}{2} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 45

```
dsolve(y(x)^2+(x*y(x)+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1+\sqrt{c_1x^2+1}}{c_1x}$$
$$y(x) = \frac{1-\sqrt{c_1x^2+1}}{c_1x}$$

✓ Solution by Mathematica

Time used: 2.48 (sec). Leaf size: 80

```
DSolve[y[x]^2+(x*y[x]+x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2c_1} - \sqrt{e^{2c_1}\left(x^2+e^{2c_1}\right)}}{x}$$
$$y(x) \to \frac{\sqrt{e^{2c_1}\left(x^2+e^{2c_1}\right)}+e^{2c_1}}{x}$$
$$y(x) \to 0$$

8.3 problem 3

Internal problem ID [2035]
Internal file name [`OUTPUT/2035_Sunday_February_25_2024_06_45_42_AM_46144366/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(x-2y)\,y'=-2x}$$

8.3.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x-(x-2u(x)\,x)\,(u'(x)\,x+u(x))=-2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{2(u^2+1)}{x\,(2u-1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^2+1}{2u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{2u-1}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{u^2+1}{2u-1}}\,du = \int -\frac{2}{x}\,dx$$
$$\ln\left(u^2+1\right) - \arctan\left(u\right) = -2\ln\left(x\right) + c_2$$

The solution is

$$\ln\left(u(x)^2+1\right) - \arctan\left(u(x)\right) + 2\ln\left(x\right) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{y^2}{x^2}+1\right) - \arctan\left(\frac{y}{x}\right) + 2\ln\left(x\right) - c_2 = 0$$
$$\ln\left(\frac{y^2}{x^2}+1\right) - \arctan\left(\frac{y}{x}\right) + 2\ln\left(x\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{y^2}{x^2}+1\right) - \arctan\left(\frac{y}{x}\right) + 2\ln\left(x\right) - c_2 = 0 \tag{1}$$

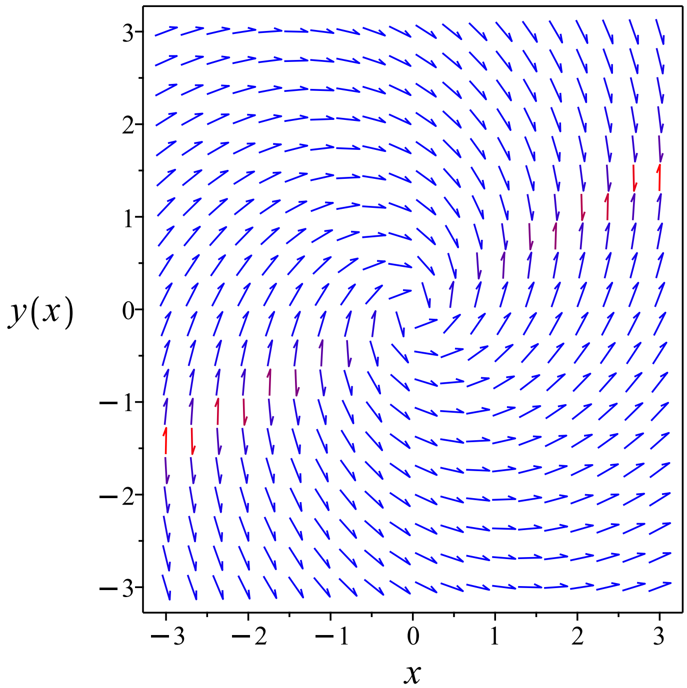


Figure 395: Slope field plot

Verification of solutions

$$\ln\left(\frac{y^2}{x^2}+1\right)-\arctan\left(\frac{y}{x}\right)+2\ln(x)-c_2=0$$

Verified OK.

8.3.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2x+y}{-x+2y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(2x+y)(b_3-a_2)}{-x+2y} - \frac{(2x+y)^2 a_3}{(-x+2y)^2} \\ & - \left(-\frac{2}{-x+2y} - \frac{2x+y}{(-x+2y)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(-\frac{1}{-x+2y} + \frac{4x+2y}{(-x+2y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + 4x^2a_3 + 4x^2b_2 - 2x^2b_3 - 8xya_2 + 4xya_3 + 4xyb_2 + 8xyb_3 - 2y^2a_2 - 4y^2a_3 - 4y^2b_2 + 2y^2b_3 + 5}{(x-2y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^2a_2 - 4x^2a_3 - 4x^2b_2 + 2x^2b_3 + 8xya_2 - 4xya_3 - 4xyb_2 \\ & - 8xyb_3 + 2y^2a_2 + 4y^2a_3 + 4y^2b_2 - 2y^2b_3 - 5xb_1 + 5ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -2a_2v_1^2 + 8a_2v_1v_2 + 2a_2v_2^2 - 4a_3v_1^2 - 4a_3v_1v_2 + 4a_3v_2^2 - 4b_2v_1^2 \\ & - 4b_2v_1v_2 + 4b_2v_2^2 + 2b_3v_1^2 - 8b_3v_1v_2 - 2b_3v_2^2 + 5a_1v_2 - 5b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-2a_2 - 4a_3 - 4b_2 + 2b_3)\, v_1^2 + (8a_2 - 4a_3 - 4b_2 - 8b_3)\, v_1 v_2 \\
&- 5b_1 v_1 + (2a_2 + 4a_3 + 4b_2 - 2b_3)\, v_2^2 + 5a_1 v_2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
5a_1 &= 0 \\
-5b_1 &= 0 \\
-2a_2 - 4a_3 - 4b_2 + 2b_3 &= 0 \\
2a_2 + 4a_3 + 4b_2 - 2b_3 &= 0 \\
8a_2 - 4a_3 - 4b_2 - 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{2x + y}{-x + 2y}\right)(x) \\
&= \frac{-2x^2 - 2y^2}{x - 2y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x^2 - 2y^2}{x - 2y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2)}{2} - \frac{\arctan\left(\frac{y}{x}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x + y}{-x + 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x + y}{2x^2 + 2y^2} \\ S_y &= \frac{-x + 2y}{2x^2 + 2y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

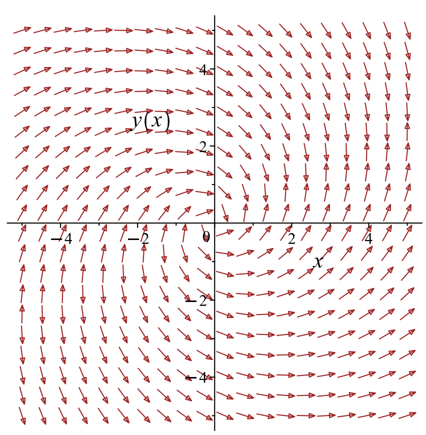
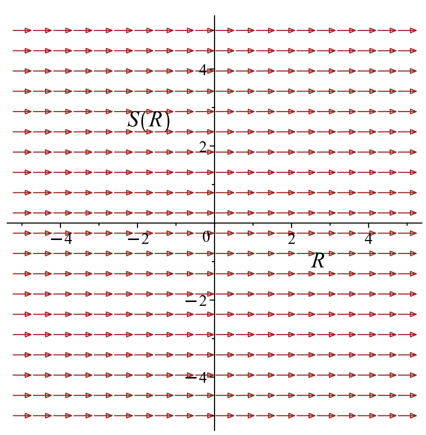
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+y^2)}{2} - \frac{\arctan\left(\frac{y}{x}\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(x^2+y^2)}{2} - \frac{\arctan\left(\frac{y}{x}\right)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+y}{-x+2y}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \frac{\text{arcta}}{2}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2} - \frac{\arctan\left(\frac{y}{x}\right)}{2} = c_1 \tag{1}$$

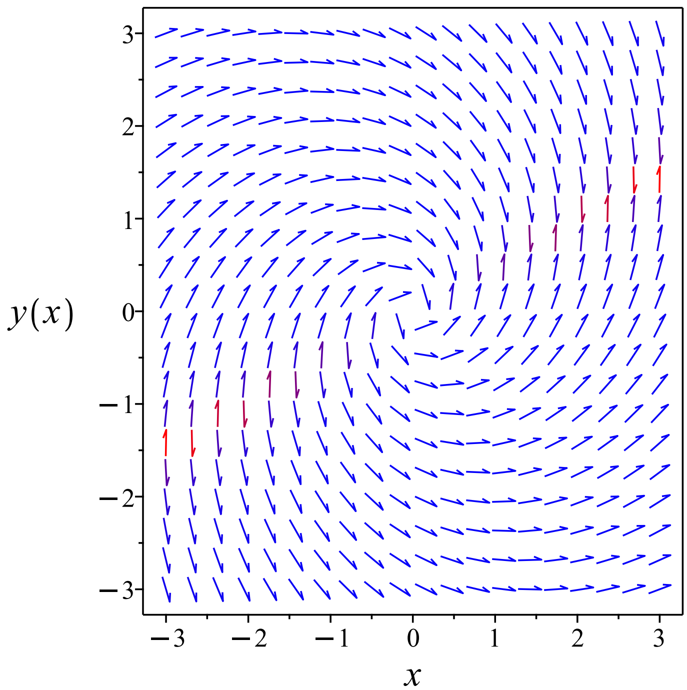


Figure 396: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2} - \frac{\arctan\left(\frac{y}{x}\right)}{2} = c_1$$

Verified OK.

8.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x+2y)\,\mathrm{d}y &= (-2x-y)\,\mathrm{d}x\\ (2x+y)\,\mathrm{d}x + (-x+2y)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x+y\\ N(x,y) &= -x+2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x+y)\\ &= 1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x+2y) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = 2x + y$ and $N = -x + 2y$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{2x+y}{x^2+y^2} \\
N &= \frac{-x+2y}{x^2+y^2}
\end{aligned}
$$

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x+2y}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{2x+y}{x^2+y^2}\right)\mathrm{d}x$$

$$\left(\frac{2x+y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{-x+2y}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{2x+y}{x^2+y^2}$$
$$N(x,y) = \frac{-x+2y}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x+y}{x^2+y^2}\right)\\ &= \frac{x^2-4yx-y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+2y}{x^2+y^2}\right)\\ &= \frac{x^2-4yx-y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{2x + y}{x^2 + y^2} \,\mathrm{d}x$$
$$\phi = \ln\left(x^2 + y^2\right) + \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2} + 1\right)} + f'(y) \tag{4}$$
$$= \frac{-x + 2y}{x^2 + y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+2y}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{-x + 2y}{x^2 + y^2} = \frac{-x + 2y}{x^2 + y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln\left(x^2 + y^2\right) + \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln\left(x^2 + y^2\right) + \arctan\left(\frac{x}{y}\right)$$

Summary
The solution(s) found are the following

$$\ln\left(x^2+y^2\right)+\arctan\left(\frac{x}{y}\right)=c_1 \tag{1}$$

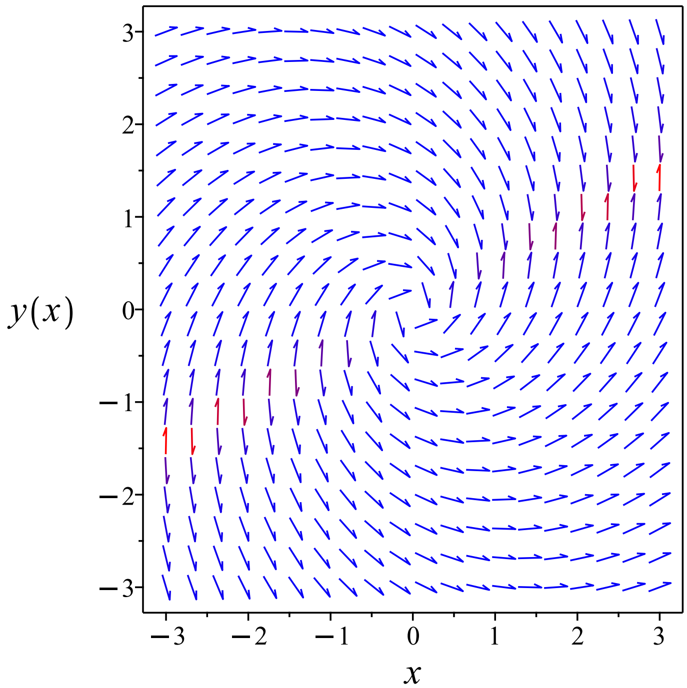


Figure 397: Slope field plot

Verification of solutions

$$\ln\left(x^2+y^2\right)+\arctan\left(\frac{x}{y}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve((2*x+y(x))-(x-2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\operatorname{RootOf}\left(\ln\left(\sec\left(\_Z\right)^2\right) - \_Z + 2\ln\left(x\right) + 2c_1\right)\right)x$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 32

```
DSolve[(2*x+y[x])-(x-2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = -2\log(x) + c_1, y(x)\right]$$

8.4 problem 4

Internal problem ID [2036]
Internal file name [`OUTPUT/2036_Sunday_February_25_2024_06_45_43_AM_36833978/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{x \ln(x) y' + y = x}$$

8.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x \ln(x)}$$
$$q(x) = \frac{1}{\ln(x)}$$

Hence the ode is

$$y' + \frac{y}{x \ln(x)} = \frac{1}{\ln(x)}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{x \ln(x)} dx} \\
&= \ln(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{\ln(x)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)\, y) &= (\ln(x))\left(\frac{1}{\ln(x)}\right) \\
\mathrm{d}(\ln(x)\, y) &= \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\ln(x)\, y &= \int \mathrm{d}x \\
\ln(x)\, y &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \ln(x)$ results in

$$
y = \frac{x}{\ln(x)} + \frac{c_1}{\ln(x)}
$$

which simplifies to

$$
y = \frac{x + c_1}{\ln(x)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x + c_1}{\ln(x)} \tag{1}
$$

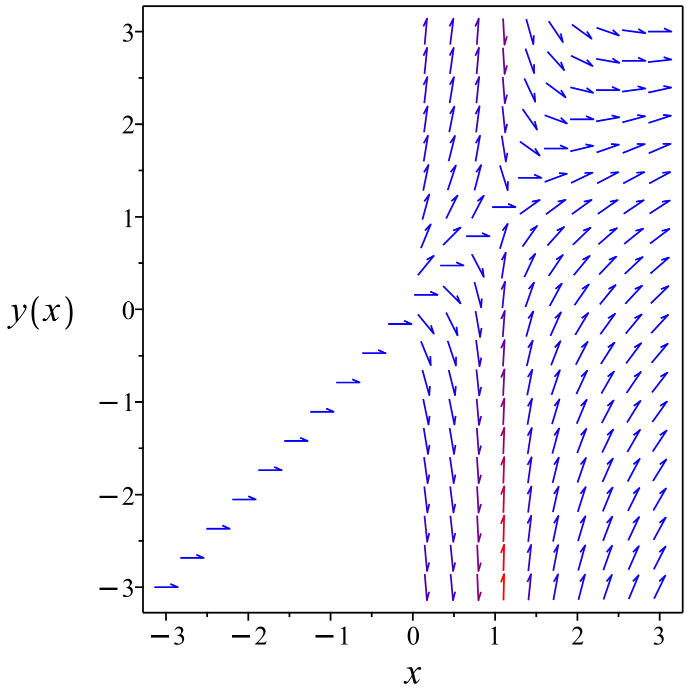


Figure 398: Slope field plot

Verification of solutions

$$y = \frac{x + c_1}{\ln(x)}$$

Verified OK.

8.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x + y}{x \ln(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 234: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \ln(x) y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x + y}{x \ln(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x} \\ S_y &= \ln(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x)\, y = x + c_1$$

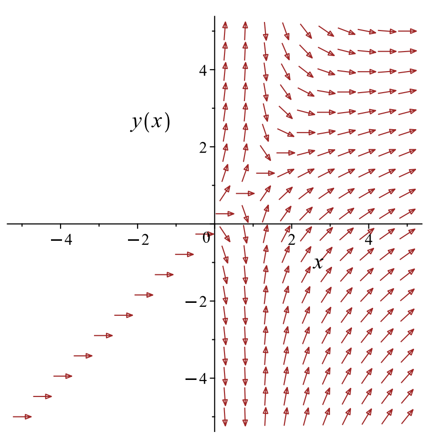
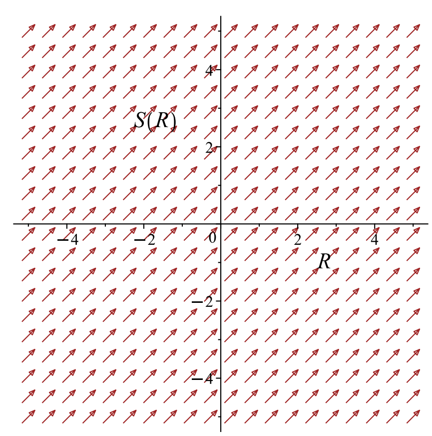
Which simplifies to

$$\ln(x)\, y = x + c_1$$

Which gives

$$y = \frac{x + c_1}{\ln(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x+y}{x\ln(x)}$ 	$R = x$ $S = \ln(x)\, y$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = \frac{x + c_1}{\ln(x)} \tag{1}$$

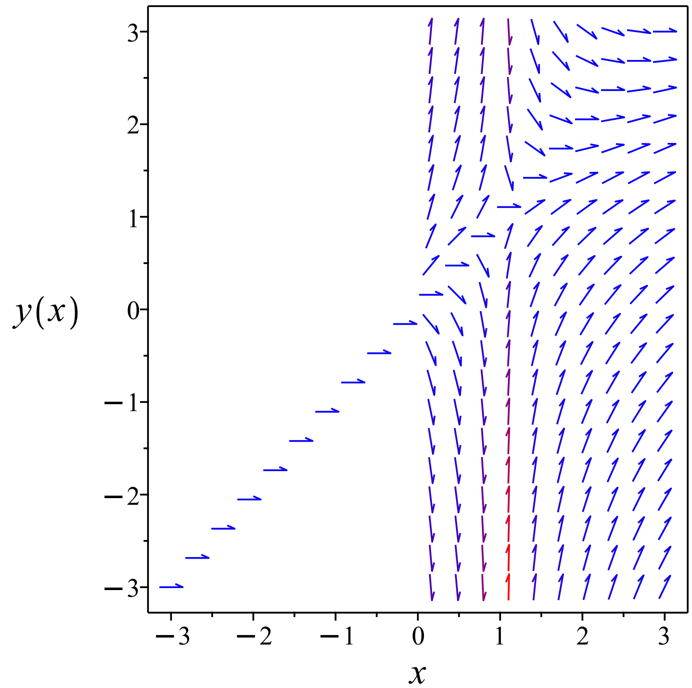


Figure 399: Slope field plot

Verification of solutions

$$y = \frac{x + c_1}{\ln(x)}$$

Verified OK.

8.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x \ln (x))\, \mathrm{d}y &= (x - y)\, \mathrm{d}x \\
(-x + y)\, \mathrm{d}x + (x \ln (x))\, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x + y \\
N(x, y) &= x \ln (x)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x + y) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x \ln (x)) \\
&= \ln (x) + 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x\ln(x)}((1) - (\ln(x) + 1)) \\ &= -\frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(x)} \\ &= \frac{1}{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x}(-x + y) \\ &= \frac{-x + y}{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x}(x\ln(x)) \\ &= \ln(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-x + y}{x}\right) + (\ln(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{-x+y}{x}\,\mathrm{d}x$$

$$\phi = \ln(x)\,y - x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \ln(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \ln(x)$. Therefore equation (4) becomes

$$\ln(x) = \ln(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x)\,y - x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x)\,y - x$$

The solution becomes

$$y = \frac{x + c_1}{\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x + c_1}{\ln(x)} \tag{1}$$

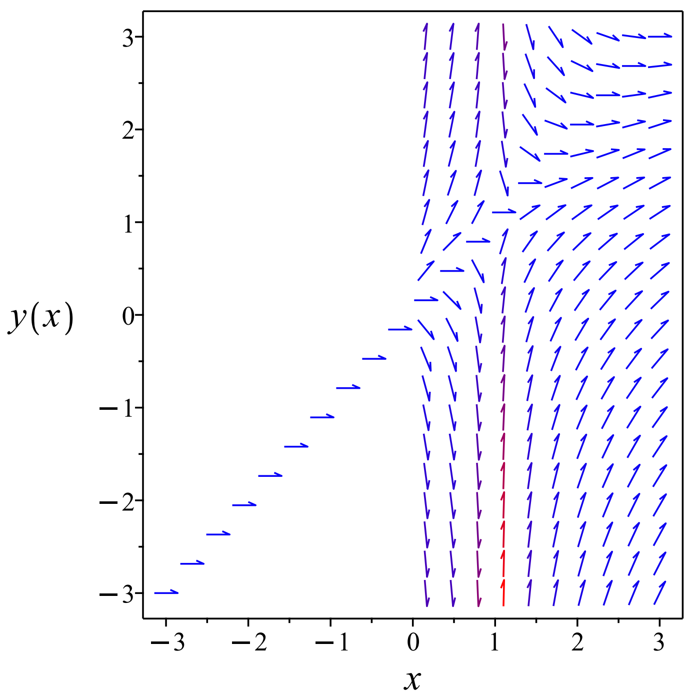


Figure 400: Slope field plot

Verification of solutions

$$y = \frac{x + c_1}{\ln(x)}$$

Verified OK.

8.4.4 Maple step by step solution

Let's solve

$x \ln(x) y' + y = x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x \ln(x)} + \frac{1}{\ln(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x \ln(x)} = \frac{1}{\ln(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{y}{x \ln(x)}\right) = \frac{\mu(x)}{\ln(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{y}{x \ln(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x \ln(x)}$

- Solve to find the integrating factor

  $\mu(x) = \ln(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{\ln(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{\ln(x)} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{\ln(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \ln(x)$

  $y = \frac{\int 1 dx + c_1}{\ln(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{x + c_1}{\ln(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*ln(x)*diff(y(x),x)+(y(x)-x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 + x}{\ln(x)}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 14

```
DSolve[x*Log[x]*y'[x]+(y[x]-x)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x + c_1}{\log(x)}$$

8.5 problem 5

Internal problem ID [2037]
Internal file name [`OUTPUT/2037_Sunday_February_25_2024_06_45_43_AM_90859865/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-2y+(y-2)\,y'=-x-1}$$

8.5.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=\frac{-X-x_0+2Y(X)+2y_0-1}{Y(X)+y_0-2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=3$$
$$y_0=2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=\frac{-X+2Y(X)}{Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{-X + 2Y}{Y} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -X + 2Y$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= -\frac{1}{u} + 2 \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-\frac{1}{u(X)} + 2 - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{-\frac{1}{u(X)} + 2 - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) u(X) X + u(X)^2 - 2u(X) + 1 = 0
$$

Or

$$
\left(\frac{d}{dX} u(X)\right) u(X) X + (u(X) - 1)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{(u - 1)^2}{uX}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{(u-1)^2}{u}$. Integrating both sides gives

$$\frac{1}{\frac{(u-1)^2}{u}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{(u-1)^2}{u}}\,du = \int -\frac{1}{X}\,dX$$

$$\ln(u-1) - \frac{1}{u-1} = -\ln(X) + c_2$$

The solution is

$$\ln(u(X)-1) - \frac{1}{u(X)-1} + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\ln\left(\frac{Y(X)}{X} - 1\right) - \frac{1}{\frac{Y(X)}{X} - 1} + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\ln\left(\frac{Y(X) - X}{X}\right) + \frac{X}{-Y(X) + X} + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = 2 + y$$
$$X = x + 3$$

Then the solution in y becomes

$$\ln\left(\frac{1 - x + y}{x - 3}\right) + \frac{x - 3}{x - 1 - y} + \ln(x-3) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{1 - x + y}{x - 3}\right) + \frac{x - 3}{x - 1 - y} + \ln(x-3) - c_2 = 0 \tag{1}$$

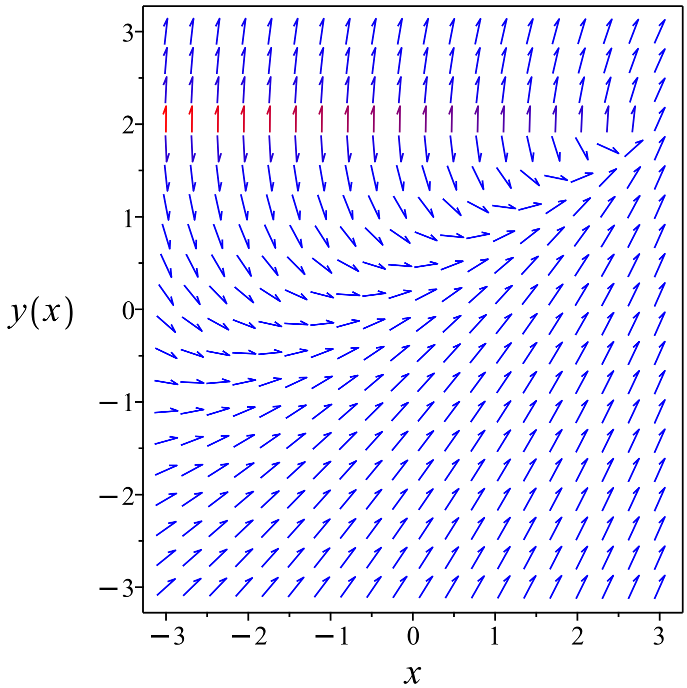


Figure 401: Slope field plot

Verification of solutions

$$\ln\left(\frac{1-x+y}{x-3}\right)+\frac{x-3}{x-1-y}+\ln(x-3)-c_2=0$$

Verified OK.

8.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-x+2y-1}{y-2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-x+2y-1)(b_3-a_2)}{y-2} - \frac{(-x+2y-1)^2 a_3}{(y-2)^2} + \frac{xa_2+ya_3+a_1}{y-2} - \left(\frac{2}{y-2} - \frac{-x+2y-1}{(y-2)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_3 + x^2b_2 - 2xya_2 - 4xya_3 + 2xyb_3 + 2y^2a_2 + 3y^2a_3 - y^2b_2 - 2y^2b_3 + 4xa_2 + 2xa_3 + xb_1 - 3xb_2 - 2x}{(y-2)^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_3 - x^2b_2 + 2xya_2 + 4xya_3 - 2xyb_3 - 2y^2a_2 - 3y^2a_3 + y^2b_2 + 2y^2b_3 - 4xa_2 - 2xa_3 - xb_1 + 3xb_2 + 2xb_3 + ya_1 + 5ya_2 + 2ya_3 - 4yb_2 - 2yb_3 - 2a_1 - 2a_2 - a_3 + 3b_1 + 4b_2 + 2b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 - 3a_3v_2^2 - b_2v_1^2 + b_2v_2^2 - 2b_3v_1v_2 + 2b_3v_2^2 + a_1v_2 - 4a_2v_1 + 5a_2v_2 - 2a_3v_1 + 2a_3v_2 - b_1v_1 + 3b_2v_1 - 4b_2v_2 + 2b_3v_1 - 2b_3v_2 - 2a_1 - 2a_2 - a_3 + 3b_1 + 4b_2 + 2b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_3 - b_2) v_1^2 + (2a_2 + 4a_3 - 2b_3) v_1 v_2 + (-4a_2 - 2a_3 - b_1 + 3b_2 + 2b_3) v_1 + (-2a_2 - 3a_3 + b_2 + 2b_3) v_2^2 + (a_1 + 5a_2 + 2a_3 - 4b_2 - 2b_3) v_2 - 2a_1 - 2a_2 - a_3 + 3b_1 + 4b_2 + 2b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_3 - b_2 &= 0 \\
2a_2 + 4a_3 - 2b_3 &= 0 \\
-2a_2 - 3a_3 + b_2 + 2b_3 &= 0 \\
a_1 + 5a_2 + 2a_3 - 4b_2 - 2b_3 &= 0 \\
-4a_2 - 2a_3 - b_1 + 3b_2 + 2b_3 &= 0 \\
-2a_1 - 2a_2 - a_3 + 3b_1 + 4b_2 + 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -4b_2 - 3b_3 \\
a_2 &= 2b_2 + b_3 \\
a_3 &= -b_2 \\
b_1 &= -3b_2 - 2b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x - 3 \\
\eta &= y - 2
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - 2 - \left(\frac{-x + 2y - 1}{y - 2}\right)(x - 3) \\
&= \frac{x^2 - 2yx + y^2 - 2x + 2y + 1}{y - 2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2 - 2yx + y^2 - 2x + 2y + 1}{y - 2}} dy \end{aligned}$$

Which results in

$$S = \ln(-x + y + 1) - \frac{x - 3}{-x + y + 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x + 2y - 1}{y - 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x - 2y + 1}{(x - y - 1)^2} \\ S_y &= \frac{y - 2}{(x - y - 1)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x-1-y)\ln(1-x+y)+x-3}{x-1-y} = c_1$$

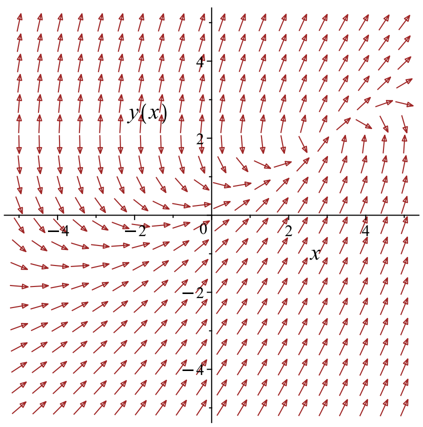
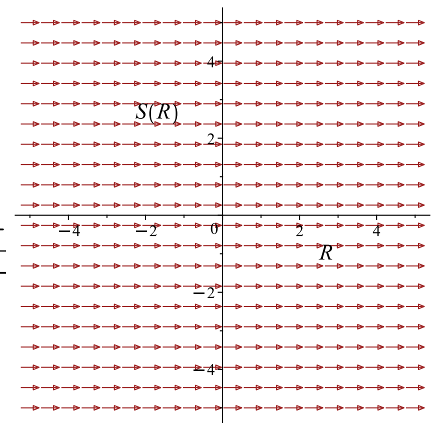
Which simplifies to

$$\frac{(x-1-y)\ln(1-x+y)+x-3}{x-1-y} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left((x-3)\mathrm{e}^{-c_1}\right)+c_1} + x - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+2y-1}{y-2}$ 	$R = x$ $S = \frac{(x-y-1)\ln(-x+}{x-y-}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left((x-3)\mathrm{e}^{-c_1}\right)+c_1} + x - 1 \tag{1}$$

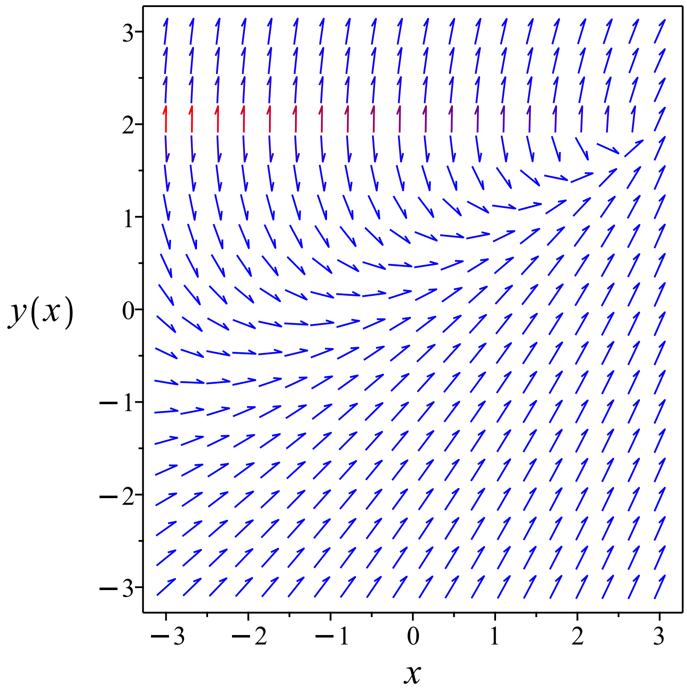


Figure 402: Slope field plot

Verification of solutions

$$y = e^{\text{LambertW}\left((x-3)e^{-c_1}\right)+c_1} + x - 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 28

```
dsolve((x-2*y(x)+1)+(y(x)-2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(x-1)\,\mathrm{LambertW}\,(-c_1(x-3)) + x - 3}{\mathrm{LambertW}\,(-c_1\,(x-3))}$$

✓ Solution by Mathematica

Time used: 0.773 (sec). Leaf size: 135

```
DSolve[(x-2*y[x]+1)+(y[x]-2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\frac{2^{2/3}\left(x\log\left(\frac{x-3}{y(x)-2}\right) - \log\left(\frac{3(x-3)}{y(x)-2}\right) - x\log\left(\frac{y(x)-x+1}{y(x)-2}\right) + \log\left(\frac{y(x)-x+1}{y(x)-2}\right) + y(x)\left(-\log\left(\frac{x-3}{y(x)-2}\right) + \mathrm{lo}\right.\right.}{9(-y(x)+x-1)}\right.$$

8.6 problem 6

Internal problem ID [2038]
Internal file name [`OUTPUT/2038_Sunday_February_25_2024_06_45_45_AM_99613365/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{2yx - 2xy^3 + \left(x^2 + y^2 - 3x^2y^2\right) y' = -x^3}$$

8.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(-3x^2y^2 + x^2 + y^2\right)\mathrm{d}y &= \left(2x\,y^3 - x^3 - 2yx\right)\mathrm{d}x \\
\left(-2x\,y^3 + x^3 + 2yx\right)\mathrm{d}x + \left(-3x^2y^2 + x^2 + y^2\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -2x\,y^3 + x^3 + 2yx \\
N(x, y) &= -3x^2y^2 + x^2 + y^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2x\,y^3 + x^3 + 2yx\right) \\
&= -6x\,y^2 + 2x
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-3x^2y^2 + x^2 + y^2\right) \\
&= -6x\,y^2 + 2x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int -2x\,y^3 + x^3 + 2yx\,\mathrm{d}x$$

$$\phi = \frac{\left(-2y^3 + x^2 + 2y\right)^2}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= \frac{(-2y^3 + x^2 + 2y)\,(-6y^2 + 2)}{2} + f'(y) \qquad (4)\\ &= -3\big(-2y^3 + x^2 + 2y\big)\left(y^2 - \frac{1}{3}\right) + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = -3x^2y^2 + x^2 + y^2$. Therefore equation (4) becomes

$$-3x^2y^2 + x^2 + y^2 = -3\big(-2y^3 + x^2 + 2y\big)\left(y^2 - \frac{1}{3}\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -6y^5 + 8y^3 + y^2 - 2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-6y^5 + 8y^3 + y^2 - 2y\right)\mathrm{d}y$$

$$f(y) = -y^6 + 2y^4 + \frac{1}{3}y^3 - y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\left(-2y^3 + x^2 + 2y\right)^2}{4} - y^6 + 2y^4 + \frac{y^3}{3} - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\left(-2y^3 + x^2 + 2y\right)^2}{4} - y^6 + 2y^4 + \frac{y^3}{3} - y^2$$

Summary

The solution(s) found are the following

$$\frac{\left(-2y^3 + x^2 + 2y\right)^2}{4} - y^6 + 2y^4 + \frac{y^3}{3} - y^2 = c_1 \tag{1}$$

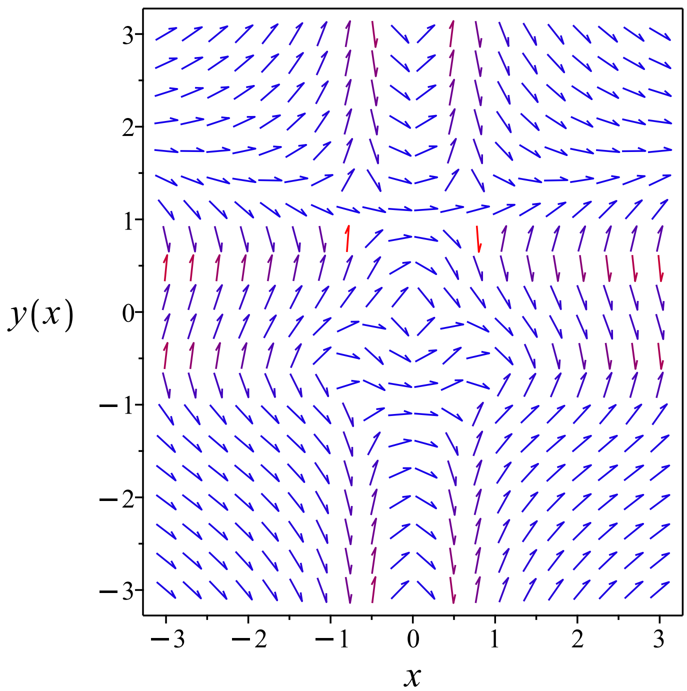


Figure 403: Slope field plot

Verification of solutions

$$\frac{\left(-2y^3 + x^2 + 2y\right)^2}{4} - y^6 + 2y^4 + \frac{y^3}{3} - y^2 = c_1$$

Verified OK.

8.6.2 Maple step by step solution

Let's solve

$2yx - 2xy^3 + (x^2 + y^2 - 3x^2y^2)\,y' = -x^3$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right)y' = 0$

  - Evaluate derivatives

    $-6x\,y^2 + 2x = -6x\,y^2 + 2x$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (-2x\,y^3 + x^3 + 2yx)\,dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{(-2y^3+x^2+2y)^2}{4} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $-3x^2y^2 + x^2 + y^2 = \frac{(-2y^3+x^2+2y)(-6y^2+2)}{2} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -3x^2y^2 + x^2 + y^2 - \frac{(-2y^3+x^2+2y)(-6y^2+2)}{2}$

- Solve for $f_1(y)$

  $f_1(y) = -y^6 + 2y^4 + \frac{1}{3}y^3 - y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = \frac{\left(-2y^3+x^2+2y\right)^2}{4} - y^6 + 2y^4 + \frac{y^3}{3} - y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{\left(-2y^3+x^2+2y\right)^2}{4} - y^6 + 2y^4 + \frac{y^3}{3} - y^2 = c_1$$

- Solve for y

$$\left\{ y = \frac{\left(\left(3x^4+\sqrt{\frac{27x^{10}-9x^8-216c_1x^6-64x^6+72c_1x^4+432c_1^2x^2-144c_1^2}{3x^2-1}}-12c_1\right)\left(3x^2-1\right)^2\right)^{\frac{1}{3}}}{2(3x^2-1)} + \frac{}{\left(\left(3x^4+\sqrt{\frac{27x^{10}-9x^8-216c_1x^6-64x^6}{3x^2}}\right.\right.} \right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 540

```
dsolve((2*x*y(x)-2*x*y(x)^3+x^3)+(x^2+y(x)^2-3*x^2*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{12x^4 - 4x^2 + \left(\left(3x^4 + \sqrt{\frac{27x^{10}-9x^8+(216c_1-64)x^6-72c_1x^4+432c_1^2x^2-144c_1^2}{3x^2-1}} + 12c_1\right)(3x^2-1)^2\right)^{\frac{2}{3}}}{\left(\left(3x^4 + \sqrt{\frac{27x^{10}-9x^8+(216c_1-64)x^6-72c_1x^4+432c_1^2x^2-144c_1^2}{3x^2-1}} + 12c_1\right)(3x^2-1)^2\right)^{\frac{1}{3}}(6x^2-2)}$$

$$y(x) = \frac{(-1-i\sqrt{3})\left(\left(3x^4 + \sqrt{\frac{27x^{10}-9x^8+(216c_1-64)x^6-72c_1x^4+432c_1^2x^2-144c_1^2}{3x^2-1}} + 12c_1\right)(3x^2-1)^2\right)^{\frac{1}{3}} + \frac{}{\left(\left(3x^4+\sqrt{\frac{27x^{10}-}{}}\right.\right.}}{12x^2-4}$$

$$y(x) = \frac{\frac{(i\sqrt{3}-1)\left(\left(3x^4+\sqrt{\frac{27x^{10}-9x^8+(216c_1-64)x^6-72c_1x^4+432c_1^2x^2-144c_1^2}{3x^2-1}}+12c_1\right)(3x^2-1)^2\right)^{\frac{2}{3}}}{4} + 3(-1-i\sqrt{3})\left(x^2-\frac{1}{3}\right)x^2}{\left(\left(3x^4 + \sqrt{\frac{27x^{10}-9x^8+(216c_1-64)x^6-72c_1x^4+432c_1^2x^2-144c_1^2}{3x^2-1}} + 12c_1\right)(3x^2-1)^2\right)^{\frac{1}{3}}(3x^2-1)}$$

✓ Solution by Mathematica

Time used: 36.799 (sec). Leaf size: 723

```
DSolve[(2*x*y[x]-2*x*y[x]^3+x^3)+(x^2+y[x]^2-3*x^2*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{12x^4 - 4x^2 + \left(27x^8 - 18x^6 + 3(1+36c_1)x^4 - 72c_1x^2 + \sqrt{(3x^2-1)^3 (27x^{10} - 9x^8 + 8(-8+27c_1)x^6 -}\right.}{2(3x^2-1)\sqrt[3]{27x^8 - 18x^6 + 3(1+36c_1)x^4 - 72c_1x^2 + \sqrt{(3x^2-1)^3 (27x^{10} - 9x^8 + 8(-8+27c_1)x^6}}}$$

$$y(x) \to \frac{-12i(\sqrt{3}-i)x^4 + (4+4i\sqrt{3})x^2 + i(\sqrt{3}+i)\left(27x^8 - 18x^6 + 3(1+36c_1)x^4 - 72c_1x^2 + \sqrt{(3x^2-1)^3}\right.}{4(3x^2-1)\sqrt[3]{27x^8 - 18x^6 + 3(1+36c_1)x^4 - 72c_1x^2 + \sqrt{(3x^2-1)^3 (27x^{10} - 9x^8 +}}}$$

$$y(x) \to \frac{12i(\sqrt{3}+i)x^4 + (4-4i\sqrt{3})x^2 - i(\sqrt{3}-i)\left(27x^8 - 18x^6 + 3(1+36c_1)x^4 - 72c_1x^2 + \sqrt{(3x^2-1)^3 (}\right.}{4(3x^2-1)\sqrt[3]{27x^8 - 18x^6 + 3(1+36c_1)x^4 - 72c_1x^2 + \sqrt{(3x^2-1)^3 (27x^{10} - 9x^8 +}}}$$

8.7 problem 7

Internal problem ID [2039]
Internal file name [`OUTPUT/2039_Sunday_February_25_2024_06_45_46_AM_57034262/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{2\,e^{x}+t\,e^{x}x' = t^{2}}$$

8.7.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$x' = -\frac{(-t^{2}+2\,e^{x})\,e^{-x}}{t}$$
$$x' = \omega(t,x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t+\omega(\eta_x-\xi_t)-\omega^2\xi_x-\omega_t\xi-\omega_x\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2+xa_3+a_1 \tag{1E}$$
$$\eta = tb_2+xb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(-t^2 + 2\,\mathrm{e}^x)\,\mathrm{e}^{-x}(b_3 - a_2)}{t} - \frac{(-t^2 + 2\,\mathrm{e}^x)^2\,\mathrm{e}^{-2x} a_3}{t^2} \\ &- \left(2\,\mathrm{e}^{-x} + \frac{(-t^2 + 2\,\mathrm{e}^x)\,\mathrm{e}^{-x}}{t^2}\right)(t a_2 + x a_3 + a_1) \\ &- \left(-\frac{2}{t} + \frac{(-t^2 + 2\,\mathrm{e}^x)\,\mathrm{e}^{-x}}{t}\right)(t b_2 + x b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{(\mathrm{e}^x t^4 b_2 + \mathrm{e}^x t^3 x b_3 + b_2 \mathrm{e}^{2x} t^2 - 2\,\mathrm{e}^x t^3 a_2 + \mathrm{e}^x t^3 b_1 + \mathrm{e}^x t^3 b_3 - \mathrm{e}^x t^2 x a_3 - t^4 a_3 - 2\,\mathrm{e}^{2x} t b_3 - 2\,\mathrm{e}^{2x} x a_3 - \mathrm{e}^x t^2 a_1 +}{t^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &\mathrm{e}^x t^4 b_2 + \mathrm{e}^x t^3 x b_3 + b_2 \mathrm{e}^{2x} t^2 - 2\,\mathrm{e}^x t^3 a_2 + \mathrm{e}^x t^3 b_1 + \mathrm{e}^x t^3 b_3 - \mathrm{e}^x t^2 x a_3 - t^4 a_3 \\ &- 2\,\mathrm{e}^{2x} t b_3 - 2\,\mathrm{e}^{2x} x a_3 - \mathrm{e}^x t^2 a_1 + 4\,\mathrm{e}^x t^2 a_3 - 2\,\mathrm{e}^{2x} a_1 - 4\,\mathrm{e}^{2x} a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &\mathrm{e}^x t^4 b_2 + \mathrm{e}^x t^3 x b_3 + b_2 \mathrm{e}^{2x} t^2 - 2\,\mathrm{e}^x t^3 a_2 + \mathrm{e}^x t^3 b_1 + \mathrm{e}^x t^3 b_3 - \mathrm{e}^x t^2 x a_3 - t^4 a_3 \\ &- 2\,\mathrm{e}^{2x} t b_3 - 2\,\mathrm{e}^{2x} x a_3 - \mathrm{e}^x t^2 a_1 + 4\,\mathrm{e}^x t^2 a_3 - 2\,\mathrm{e}^{2x} a_1 - 4\,\mathrm{e}^{2x} a_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x, \mathrm{e}^x, \mathrm{e}^{2x}\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2, \mathrm{e}^x = v_3, \mathrm{e}^{2x} = v_4\}$$

The above PDE (6E) now becomes

$$v_3v_1^4b_2 + v_3v_1^3v_2b_3 - 2v_3v_1^3a_2 - v_1^4a_3 - v_3v_1^2v_2a_3 + v_3v_1^3b_1 + v_3v_1^3b_3 - v_3v_1^2a_1 + 4v_3v_1^2a_3 + b_2v_4v_1^2 - 2v_4v_2a_3 - 2v_4v_1b_3 - 2v_4a_1 - 4v_4a_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$v_3v_1^4b_2 - v_1^4a_3 + v_3v_1^3v_2b_3 + (-2a_2 + b_1 + b_3)v_1^3v_3 - v_3v_1^2v_2a_3 + (-a_1 + 4a_3)v_1^2v_3 + b_2v_4v_1^2 - 2v_4v_1b_3 - 2v_4v_2a_3 + (-2a_1 - 4a_3)v_4 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ b_3 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ -2b_3 &= 0 \\ -2a_1 - 4a_3 &= 0 \\ -a_1 + 4a_3 &= 0 \\ -2a_2 + b_1 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 2a_2 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= t \\ \eta &= 2 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t,x)\,\xi \\
&= 2 - \left(-\frac{(-t^2 + 2\,\mathrm{e}^x)\,\mathrm{e}^{-x}}{t}\right)(t) \\
&= \left(-t^2 + 4\,\mathrm{e}^x\right)\mathrm{e}^{-x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{(-t^2 + 4\,\mathrm{e}^x)\,\mathrm{e}^{-x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(-t^2 + 4\,\mathrm{e}^x\right)}{4}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}
$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$
\omega(t,x) = -\frac{(-t^2 + 2\,\mathrm{e}^x)\,\mathrm{e}^{-x}}{t}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1 \\
R_x &= 0 \\
S_t &= \frac{t}{2t^2 - 8\,\mathrm{e}^x} \\
S_x &= -\frac{\mathrm{e}^x}{t^2 - 4\,\mathrm{e}^x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2t} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\frac{\ln(-t^2 + 4\,\mathrm{e}^x)}{4} = -\frac{\ln(t)}{2} + c_1$$

Which simplifies to

$$\frac{\ln(-t^2 + 4\,\mathrm{e}^x)}{4} = -\frac{\ln(t)}{2} + c_1$$

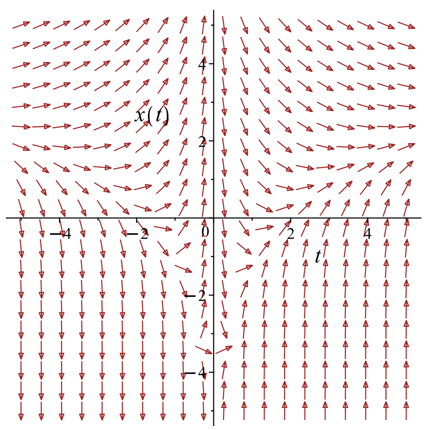
Which gives

$$x = \ln\left(\frac{t^4 + \mathrm{e}^{4c_1}}{4t^2}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, x coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dx}{dt} = -\frac{\left(-t^2+2\,\mathrm{e}^{x}\right)\mathrm{e}^{-x}}{t}$



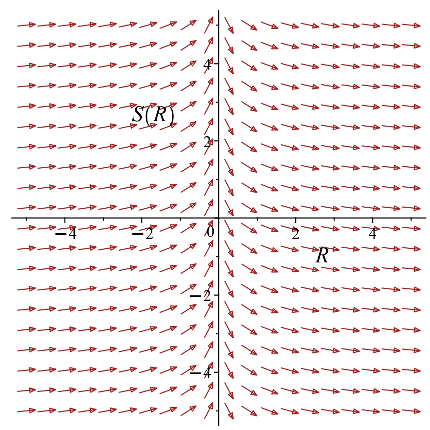
</td><td>

$R = t$

$S = \frac{\ln\left(-t^2+4\,\mathrm{e}^{x}\right)}{4}$

</td><td>

$\frac{dS}{dR} = -\frac{1}{2R}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$x = \ln\left(\frac{t^4 + \mathrm{e}^{4c_1}}{4t^2}\right) \tag{1}$$

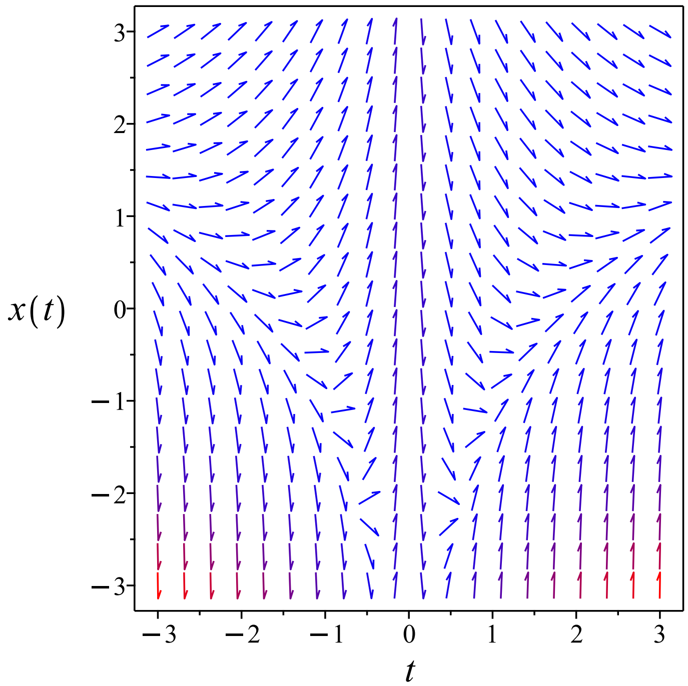


Figure 404: Slope field plot

Verification of solutions

$$x = \ln\left(\frac{t^4 + e^{4c_1}}{4t^2}\right)$$

Verified OK.

8.7.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$(\mathrm{e}^x t)\,\mathrm{d}x = \left(-2\,\mathrm{e}^x + t^2\right)\mathrm{d}t$$
$$\left(-t^2 + 2\,\mathrm{e}^x\right)\mathrm{d}t + (\mathrm{e}^x t)\,\mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, x) = -t^2 + 2\,\mathrm{e}^x$$
$$N(t, x) = \mathrm{e}^x t$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-t^2 + 2\,\mathrm{e}^x\right)\\ &= 2\,\mathrm{e}^x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(\mathrm{e}^x t)\\ &= \mathrm{e}^x\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= \frac{\mathrm{e}^{-x}}{t}((2\,\mathrm{e}^{x}) - (\mathrm{e}^{x})) \\
&= \frac{1}{t}
\end{aligned}
$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}t} \\
&= e^{\int \frac{1}{t} \,\mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(t)} \\
&= t
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= t\left(-t^{2} + 2\,\mathrm{e}^{x}\right) \\
&= -t\left(-2\,\mathrm{e}^{x} + t^{2}\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= t(\mathrm{e}^{x} t) \\
&= \mathrm{e}^{x} t^{2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\left(-t\left(-2\,\mathrm{e}^{x} + t^{2}\right)\right) + \left(\mathrm{e}^{x} t^{2}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int \overline{M} \,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \,\mathrm{d}t = \int -t(-2\,\mathrm{e}^{x} + t^2) \,\mathrm{d}t$$
$$\phi = -\frac{(-2\,\mathrm{e}^{x} + t^2)^2}{4} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = (-2\,\mathrm{e}^{x} + t^2)\,\mathrm{e}^{x} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{x} t^2$. Therefore equation (4) becomes

$$\mathrm{e}^{x} t^2 = (-2\,\mathrm{e}^{x} + t^2)\,\mathrm{e}^{x} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 2\,\mathrm{e}^{2x}$$

Integrating the above w.r.t x gives

$$\int f'(x) \,\mathrm{d}x = \int (2\,\mathrm{e}^{2x}) \,\mathrm{d}x$$
$$f(x) = \mathrm{e}^{2x} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{(-2\,\mathrm{e}^{x} + t^2)^2}{4} + \mathrm{e}^{2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\left(-2\,e^{x} + t^2\right)^2}{4} + e^{2x}$$

The solution becomes

$$x = \ln\left(\frac{t^4 + 4c_1}{4t^2}\right)$$

Summary

The solution(s) found are the following

$$x = \ln\left(\frac{t^4 + 4c_1}{4t^2}\right) \tag{1}$$

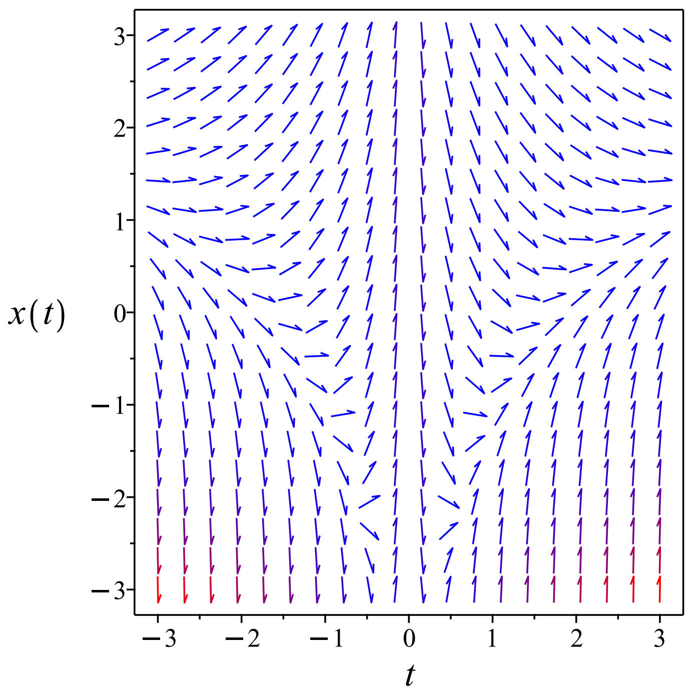


Figure 405: Slope field plot

Verification of solutions

$$x = \ln\left(\frac{t^4 + 4c_1}{4t^2}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 2/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      <- quadrature successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve((2*exp(x(t))-t^2)+(t*exp(x(t)))*diff(x(t),t)=0,x(t), singsol=all)
```

$$x(t) = -2\ln(2) + \ln\left(\frac{t^4 + c_1}{t^2}\right)$$

✓ Solution by Mathematica

Time used: 1.926 (sec). Leaf size: 20

```
DSolve[(2*Exp[x[t]]-t^2)+(t*Exp[x[t]])*x'[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \log\left(\frac{t^2}{4} + \frac{c_1}{t^2}\right)$$

8.8 problem 8

Internal problem ID [2040]
Internal file name [`OUTPUT/2040_Sunday_February_25_2024_06_45_47_AM_54025058/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2y - y'xy = -6}$$

8.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{6+2y}{yx} \end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(y) = \frac{3+y}{y}$. Integrating both sides gives

$$\frac{1}{\frac{3+y}{y}}\,dy = \frac{2}{x}\,dx$$

$$\int \frac{1}{\frac{3+y}{y}}\,dy = \int \frac{2}{x}\,dx$$

$$y - 3\ln(3+y) = 2\ln(x) + c_1$$

Which results in

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{-1-\frac{c_1}{3}-\frac{2\ln(x)}{3}}}{3}\right)-1-\frac{c_1}{3}-\frac{2\ln(x)}{3}} - 3$$

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{-1-\frac{c_1}{3}-\frac{2\ln(x)}{3}}}{3}\right)-1-\frac{c_1}{3}-\frac{2\ln(x)}{3}} - 3 \tag{1}$$

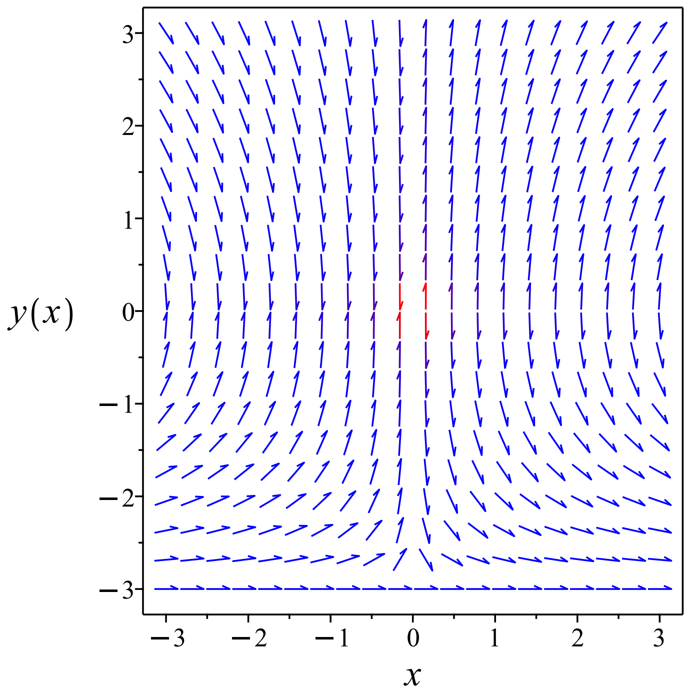


Figure 406: Slope field plot

Verification of solutions

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{-1-\frac{c_1}{3}-\frac{2\ln(x)}{3}}}{3}\right)-1-\frac{c_1}{3}-\frac{2\ln(x)}{3}} - 3$$

Verified OK.

8.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{6+2y}{yx} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 238: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{x}{2}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x}{2}} dx \end{aligned}$$

Which results in

$$S = 2\ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{6+2y}{yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{2}{x} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{3+y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{3+R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R - 3\ln(3+R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\ln(x) = y - 3\ln(3+y) + c_1$$

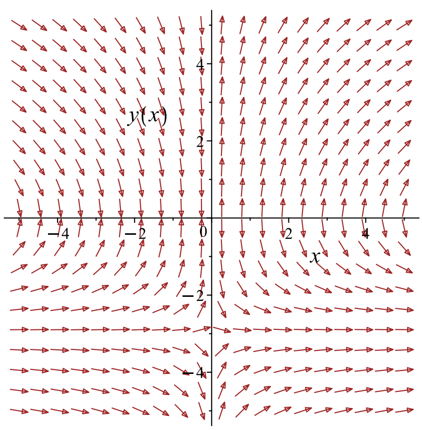
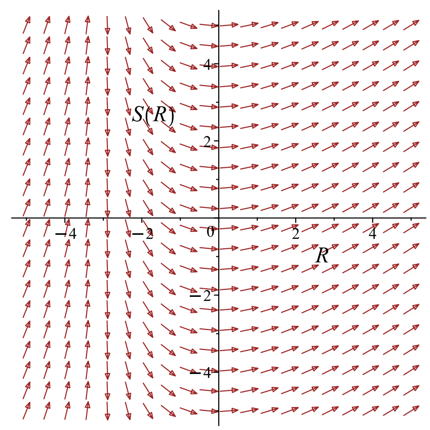
Which simplifies to

$$2\ln(x) = y - 3\ln(3+y) + c_1$$

Which gives

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-1+\frac{c_1}{3}-\frac{2\ln(x)}{3}}}{3}\right)-1+\frac{c_1}{3}-\frac{2\ln(x)}{3}} - 3$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{6+2y}{yx}$ 	$R = y$ $S = 2\ln(x)$	$\frac{dS}{dR} = \frac{R}{3+R}$ 

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{-1+\frac{c_1}{3}-\frac{2\ln(x)}{3}}}{3}\right)-1+\frac{c_1}{3}-\frac{2\ln(x)}{3}} - 3 \tag{1}$$

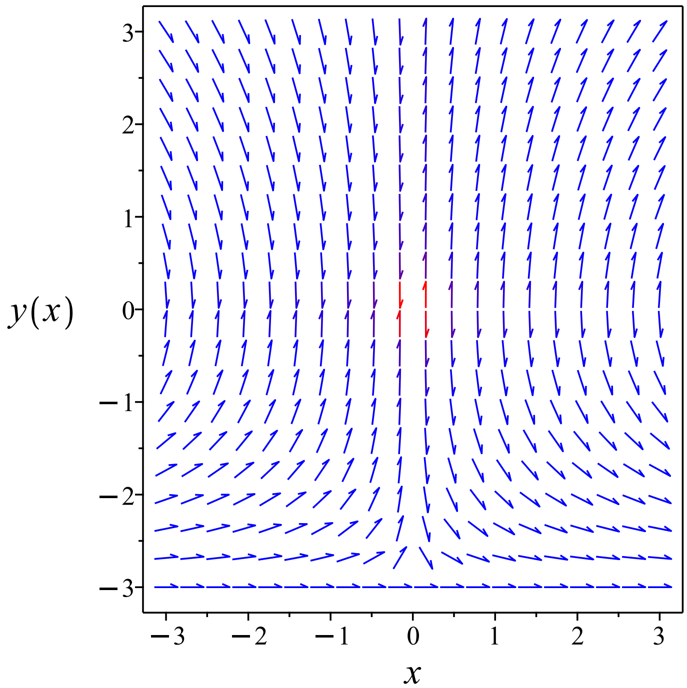


Figure 407: Slope field plot

Verification of solutions

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{-1+\frac{c_1}{3}-\frac{2\ln(x)}{3}}}{3}\right)-1+\frac{c_1}{3}-\frac{2\ln(x)}{3}} - 3$$

Verified OK.

8.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{y}{6 + 2y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{y}{6 + 2y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= \frac{y}{6 + 2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{6+2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{align}
\frac{\partial \phi}{\partial x} &= M \tag{1} \\
\frac{\partial \phi}{\partial y} &= N \tag{2}
\end{align}
$$

Integrating (1) w.r.t. x gives

$$
\begin{align}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \nonumber \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x} \,\mathrm{d}x \nonumber \\
\phi &= -\ln(x) + f(y) \tag{3}
\end{align}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{6+2y}$. Therefore equation (4) becomes

$$
\frac{y}{6+2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y}{6+2y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{y}{6+2y}\right) \mathrm{d}y \\
f(y) &= \frac{y}{2} - \frac{3\ln(3+y)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{y}{2} - \frac{3\ln(3+y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{y}{2} - \frac{3\ln(3+y)}{2}$$

The solution becomes

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{-1-\frac{2c_1}{3}-\frac{2\ln(x)}{3}}}{3}\right)-1-\frac{2c_1}{3}-\frac{2\ln(x)}{3}} - 3$$

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{-1-\frac{2c_1}{3}-\frac{2\ln(x)}{3}}}{3}\right)-1-\frac{2c_1}{3}-\frac{2\ln(x)}{3}} - 3 \tag{1}$$

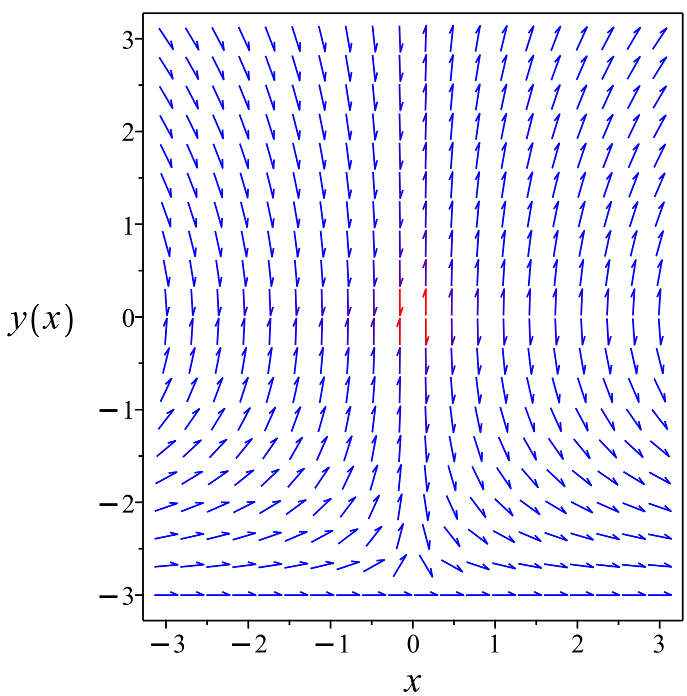


Figure 408: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-1-\frac{2c_1}{3}-\frac{2\ln(x)}{3}}}{3}\right)-1-\frac{2c_1}{3}-\frac{2\ln(x)}{3}} - 3$$

Verified OK.

8.8.4 Maple step by step solution

Let's solve

$2y - y'xy = -6$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{-6-2y} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{-6-2y}dx = \int -\frac{1}{x}dx + c_1$

- Evaluate integral

  $-\frac{y}{2} + \frac{3\ln(3+y)}{2} = -\ln(x) + c_1$

- Solve for y

  $y = \mathrm{e}^{-LambertW\left(-\frac{\mathrm{e}^{-1+\frac{2c_1}{3}-\frac{2\ln(x)}{3}}}{3}\right)-1+\frac{2c_1}{3}-\frac{2\ln(x)}{3}} - 3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(2*(y(x)+3)=x*y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -3\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-1-\frac{2c_1}{3}}}{3x^{\frac{2}{3}}}\right) - 3$$

✓ Solution by Mathematica

Time used: 20.439 (sec). Leaf size: 106

```
DSolve[2*(y[x]+3)==x*y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned}
y(x) &\to -3\left(1 + W\left(\frac{1}{3}\sqrt[3]{-\frac{e^{-3-c_1}}{x^2}}\right)\right) \\
y(x) &\to -3\left(1 + W\left(-\frac{1}{3}\sqrt[3]{-1}\sqrt[3]{-\frac{e^{-3-c_1}}{x^2}}\right)\right) \\
y(x) &\to -3\left(1 + W\left(\frac{1}{3}(-1)^{2/3}\sqrt[3]{-\frac{e^{-3-c_1}}{x^2}}\right)\right) \\
y(x) &\to -3
\end{aligned}$$

8.9 problem 9

Internal problem ID [2041]
Internal file name [`OUTPUT/2041_Sunday_February_25_2024_06_45_49_AM_71951654/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-3y - (3y - x + 2)\, y' = -x}$$

8.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-x + 3y}{3y - x + 2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(-x+3y)(b_3-a_2)}{3y-x+2} - \frac{(-x+3y)^2 a_3}{(3y-x+2)^2} \\ & - \left(\frac{1}{3y-x+2} - \frac{-x+3y}{(3y-x+2)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{3}{3y-x+2} + \frac{-3x+9y}{(3y-x+2)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 + x^2b_2 - x^2b_3 - 6xya_2 + 6xya_3 - 6xyb_2 + 6xyb_3 + 9y^2a_2 - 9y^2a_3 + 9y^2b_2 - 9y^2b_3 - 4xa_2 +}{(-3y+x-2)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & x^2a_2 - x^2a_3 + x^2b_2 - x^2b_3 - 6xya_2 + 6xya_3 - 6xyb_2 + 6xyb_3 + 9y^2a_2 - 9y^2a_3 \\ & +9y^2b_2 - 9y^2b_3 - 4xa_2 + 2xb_2 + 2xb_3 + 6ya_2 - 2ya_3 + 12yb_2 - 2a_1 + 6b_1 + 4b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_2v_1^2 - 6a_2v_1v_2 + 9a_2v_2^2 - a_3v_1^2 + 6a_3v_1v_2 - 9a_3v_2^2 + b_2v_1^2 \\ & - 6b_2v_1v_2 + 9b_2v_2^2 - b_3v_1^2 + 6b_3v_1v_2 - 9b_3v_2^2 - 4a_2v_1 + 6a_2v_2 \\ & - 2a_3v_2 + 2b_2v_1 + 12b_2v_2 + 2b_3v_1 - 2a_1 + 6b_1 + 4b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(a_2 - a_3 + b_2 - b_3) v_1^2 + (-6a_2 + 6a_3 - 6b_2 + 6b_3) v_1 v_2 + (-4a_2 + 2b_2 + 2b_3) v_1 + (9a_2 - 9a_3 + 9b_2 - 9b_3) v_2^2 + (6a_2 - 2a_3 + 12b_2) v_2 - 2a_1 + 6b_1 + 4b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 + 6b_1 + 4b_2 &= 0 \\ -4a_2 + 2b_2 + 2b_3 &= 0 \\ 6a_2 - 2a_3 + 12b_2 &= 0 \\ -6a_2 + 6a_3 - 6b_2 + 6b_3 &= 0 \\ a_2 - a_3 + b_2 - b_3 &= 0 \\ 9a_2 - 9a_3 + 9b_2 - 9b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 3b_1 - 2a_2 \\ a_2 &= a_2 \\ a_3 &= -3a_2 \\ b_1 &= b_1 \\ b_2 &= -a_2 \\ b_3 &= 3a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 3 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\,\xi \\ &= 1 - \left(-\frac{-x + 3y}{3y - x + 2}\right)(3) \\ &= \frac{4x - 12y - 2}{-3y + x - 2} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4x-12y-2}{-3y+x-2}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{4} + \frac{\ln(-2x + 6y + 1)}{8}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x + 3y}{3y - x + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{8x - 24y - 4} \\ S_y &= \frac{1}{4} - \frac{3}{8x - 24y - 4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{4} + \frac{\ln(-2x + 6y + 1)}{8} = -\frac{x}{4} + c_1$$

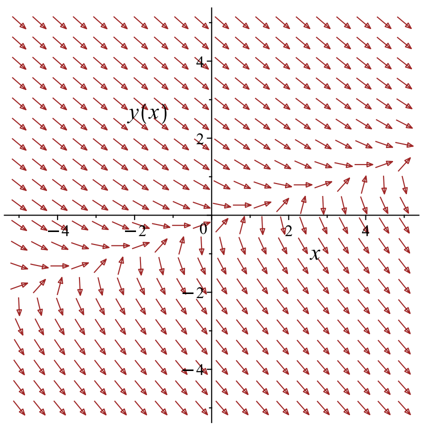
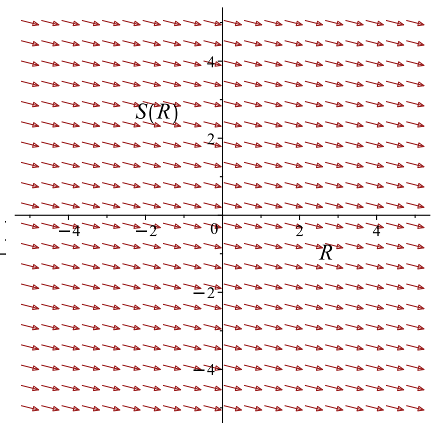
Which simplifies to

$$\frac{y}{4} + \frac{\ln(-2x + 6y + 1)}{8} = -\frac{x}{4} + c_1$$

Which gives

$$y = \frac{\operatorname{LambertW}\left(\frac{e^{\frac{1}{3} - \frac{8x}{3} + 8c_1}}{3}\right)}{2} - \frac{1}{6} + \frac{x}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x+3y}{3y-x+2}$ 	$R = x$ $S = \frac{y}{4} + \frac{\ln(-2x+6y+}{8}$	$\frac{dS}{dR} = -\frac{1}{4}$ 

Summary

The solution(s) found are the following

$$y = \frac{\operatorname{LambertW}\left(\frac{e^{\frac{1}{3}-\frac{8x}{3}+8c_1}}{3}\right)}{2} - \frac{1}{6} + \frac{x}{3} \tag{1}$$

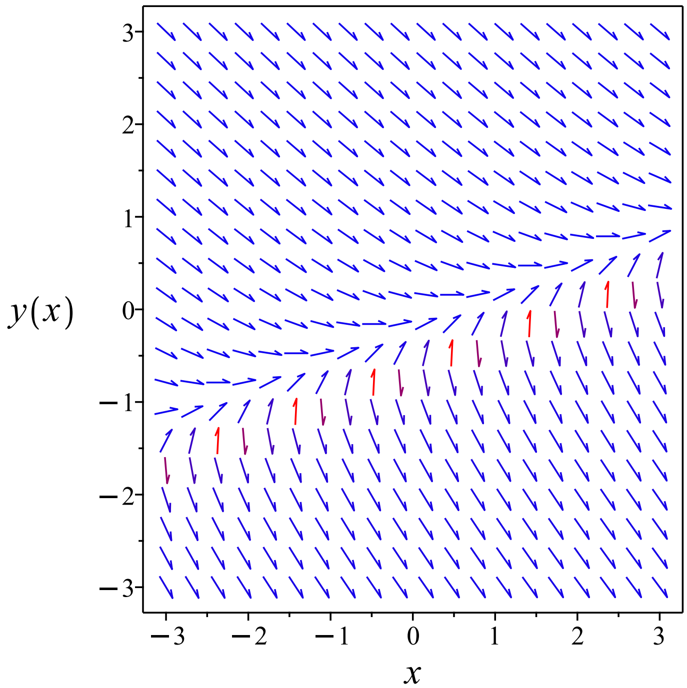


Figure 409: Slope field plot

Verification of solutions

$$y = \frac{\operatorname{LambertW}\left(\frac{e^{\frac{1}{3}-\frac{8x}{3}+8c_1}}{3}\right)}{2} - \frac{1}{6} + \frac{x}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve((x-3*y(x))=(3*y(x)-x+2)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{x}{3} + \frac{\operatorname{LambertW}\left(\frac{c_1 \mathrm{e}^{-\frac{8x}{3}+\frac{1}{3}}}{3}\right)}{2} - \frac{1}{6}$$

✓ Solution by Mathematica

Time used: 4.689 (sec). Leaf size: 43

```
DSolve[(x-3*y[x])==(3*y[x]-x+2)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}\left(3W\left(-e^{-\frac{8x}{3}-1+c_1}\right) + 2x - 1\right)$$
$$y(x) \to \frac{1}{6}(2x - 1)$$

8.10 problem 10

Internal problem ID [2042]
Internal file name [`OUTPUT/2042_Sunday_February_25_2024_06_45_50_AM_85824554/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\sin(x) y - 2\cos(y) - (\cos(x) - 2\sin(y) x + \sin(y)) y' = -\tan(x)}$$

8.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-\cos(x) + 2x\sin(y) - \sin(y))\,\mathrm{d}y = (-\sin(x)\,y + 2\cos(y) - \tan(x$$

$$(\sin(x)\,y + \tan(x) - 2\cos(y))\,\mathrm{d}x + (-\cos(x) + 2x\sin(y) - \sin(y))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \sin(x)\,y + \tan(x) - 2\cos(y) \\ N(x, y) &= -\cos(x) + 2x\sin(y) - \sin(y) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin(x)\,y + \tan(x) - 2\cos(y)) \\ &= \sin(x) + 2\sin(y) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\cos(x) + 2x\sin(y) - \sin(y)) \\ &= \sin(x) + 2\sin(y) \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \sin(x) \, y + \tan(x) - 2\cos(y) \, \mathrm{d}x$$

$$\phi = -\cos(x) \, y - 2x\cos(y) - \ln(\cos(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\cos(x) + 2x\sin(y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\cos(x) + 2x\sin(y) - \sin(y)$. Therefore equation (4) becomes

$$-\cos(x) + 2x\sin(y) - \sin(y) = -\cos(x) + 2x\sin(y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\sin(y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-\sin(y)) \, \mathrm{d}y$$

$$f(y) = \cos(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\cos(x) \, y - 2x\cos(y) - \ln(\cos(x)) + \cos(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\cos(x) \, y - 2x\cos(y) - \ln(\cos(x)) + \cos(y)$$

Summary
The solution(s) found are the following

$$-\cos(x)\, y - 2x\cos(y) - \ln(\cos(x)) + \cos(y) = c_1 \tag{1}$$

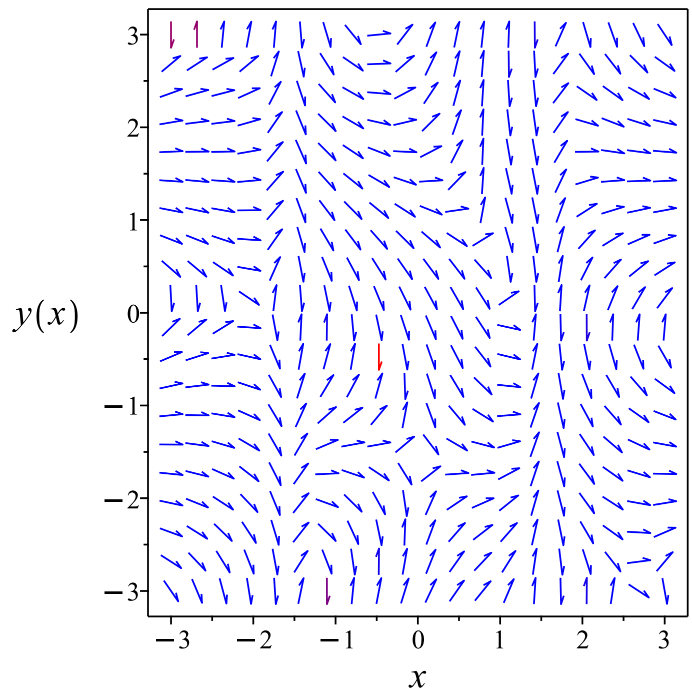


Figure 410: Slope field plot

Verification of solutions

$$-\cos(x)\, y - 2x\cos(y) - \ln(\cos(x)) + \cos(y) = c_1$$

Verified OK.

8.10.2 Maple step by step solution

Let's solve

$\sin(x)\, y - 2\cos(y) - (\cos(x) - 2\sin(y)\, x + \sin(y))\, y' = -\tan(x)$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $\sin(x) + 2\sin(y) = \sin(x) + 2\sin(y)$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (\sin(x)\, y + \tan(x) - 2\cos(y))\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = -\cos(x)\, y - 2x\cos(y) - \ln(\cos(x)) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $-\cos(x) + 2x\sin(y) - \sin(y) = -\cos(x) + 2x\sin(y) + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -\sin(y)$

- Solve for $f_1(y)$

  $f_1(y) = \cos(y)$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = -\cos(x)\, y - 2x\cos(y) - \ln(\cos(x)) + \cos(y)$

- Substitute $F(x, y)$ into the solution of the ODE

  $-\cos(x)\, y - 2x\cos(y) - \ln(\cos(x)) + \cos(y) = c_1$

- Solve for y

  $y = \mathit{RootOf}(2x\cos(\_Z) + \_Z\cos(x) - \cos(\_Z) + \ln(\cos(x)) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 25

```
dsolve((y(x)*sin(x)-2*cos(y(x))+tan(x) )-(cos(x)-2*x*sin(y(x))+sin(y(x)) )*diff(y(x),x)=0,y(
```

$$-\cos(x)\, y(x) - 2x\cos(y(x)) - \ln(\cos(x)) + \cos(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.673 (sec). Leaf size: 29

```
DSolve[(y[x]*Sin[x]-2*Cos[y[x]]+Tan[x] )-(Cos[x]-2*x*Sin[y[x]]+Sin[y[x]] )*y'[x]==0,y[x],x,I
```

$$\text{Solve}[4x\cos(y(x)) - 2\cos(y(x)) + 2y(x)\cos(x) + 2\log(\cos(x)) = c_1, y(x)]$$

8.11 problem 11

Internal problem ID [2043]
Internal file name [`OUTPUT/2043_Sunday_February_25_2024_06_47_00_AM_63282287/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{x^2y - (x^3 + y^3)\, y' = 0}$$

8.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$x^3u(x) - \left(x^3 + u(x)^3\, x^3\right)\left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^4}{(u^3 + 1)\, x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^4}{u^3+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^4}{u^3+1}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u^4}{u^3+1}}\,du = \int -\frac{1}{x}\,dx$$
$$\ln(u) - \frac{1}{3u^3} = -\ln(x) + c_2$$

The solution is

$$\ln(u(x)) - \frac{1}{3u(x)^3} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{y}{x}\right) - \frac{x^3}{3y^3} + \ln(x) - c_2 = 0$$
$$\ln\left(\frac{y}{x}\right) - \frac{x^3}{3y^3} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{y}{x}\right) - \frac{x^3}{3y^3} + \ln(x) - c_2 = 0 \tag{1}$$

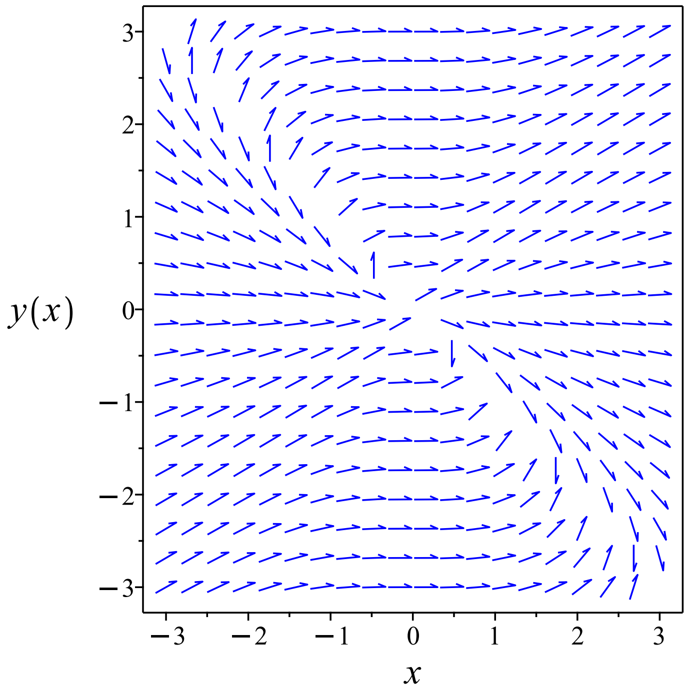


Figure 411: Slope field plot

Verification of solutions

$$\ln\left(\frac{y}{x}\right)-\frac{x^3}{3y^3}+\ln(x)-c_2=0$$

Verified OK.

8.11.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x^2y}{x^3+y^3}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{x^2 y(b_3 - a_2)}{x^3 + y^3} - \frac{x^4 y^2 a_3}{(x^3 + y^3)^2} - \left(\frac{2xy}{x^3 + y^3} - \frac{3x^4 y}{(x^3 + y^3)^2}\right)(x a_2 + y a_3 + a_1) - \left(\frac{x^2}{x^3 + y^3} - \frac{3x^2 y^3}{(x^3 + y^3)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4x^3 y^3 b_2 + 3x^2 y^4 a_2 - 3x^2 y^4 b_3 + 2x\,y^5 a_3 - y^6 b_2 + x^5 b_1 - x^4 y a_1 - 2x^2 y^3 b_1 + 2x\,y^4 a_1}{(x^3 + y^3)^2} = 0$$

Setting the numerator to zero gives

$$4x^3 y^3 b_2 - 3x^2 y^4 a_2 + 3x^2 y^4 b_3 - 2x\,y^5 a_3 + y^6 b_2 - x^5 b_1 + x^4 y a_1 + 2x^2 y^3 b_1 - 2x\,y^4 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-3a_2 v_1^2 v_2^4 - 2a_3 v_1 v_2^5 + 4b_2 v_1^3 v_2^3 + b_2 v_2^6 + 3b_3 v_1^2 v_2^4 + a_1 v_1^4 v_2 - 2a_1 v_1 v_2^4 - b_1 v_1^5 + 2b_1 v_1^2 v_2^3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_1 v_1^5 + a_1 v_1^4 v_2 + 4 b_2 v_1^3 v_2^3 + (-3 a_2 + 3 b_3) v_1^2 v_2^4 + 2 b_1 v_1^2 v_2^3 - 2 a_3 v_1 v_2^5 - 2 a_1 v_1 v_2^4 + b_2 v_2^6 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
b_2 &= 0 \\
-2a_1 &= 0 \\
-2a_3 &= 0 \\
-b_1 &= 0 \\
2b_1 &= 0 \\
4b_2 &= 0 \\
-3a_2 + 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{x^2 y}{x^3 + y^3}\right)(x) \\
&= \frac{y^4}{x^3 + y^3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^4}{x^3+y^3}} dy \end{aligned}$$

Which results in

$$S = \ln(y) - \frac{x^3}{3y^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 y}{x^3 + y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x^2}{y^3} \\ S_y &= \frac{x^3 + y^3}{y^4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(y)\,y^3 - x^3}{3y^3} = c_1$$

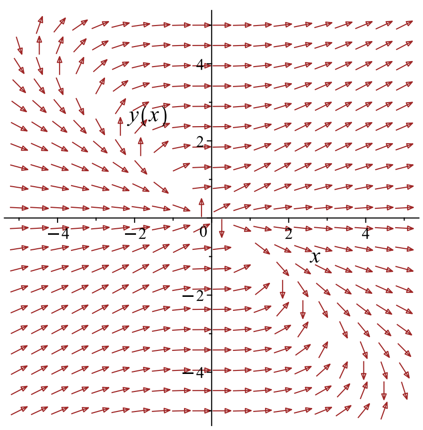
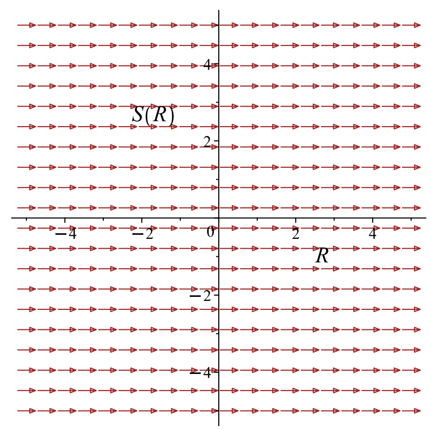
Which simplifies to

$$\frac{3\ln(y)\,y^3 - x^3}{3y^3} = c_1$$

Which gives

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(x^3 \mathrm{e}^{-3c_1}\right)}{3}+c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2 y}{x^3+y^3}$ 	$R = x$ $S = \frac{3\ln(y)\,y^3 - x^3}{3y^3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = e^{\frac{\operatorname{LambertW}\left(x^3 e^{-3c_1}\right)}{3} + c_1} \tag{1}$$

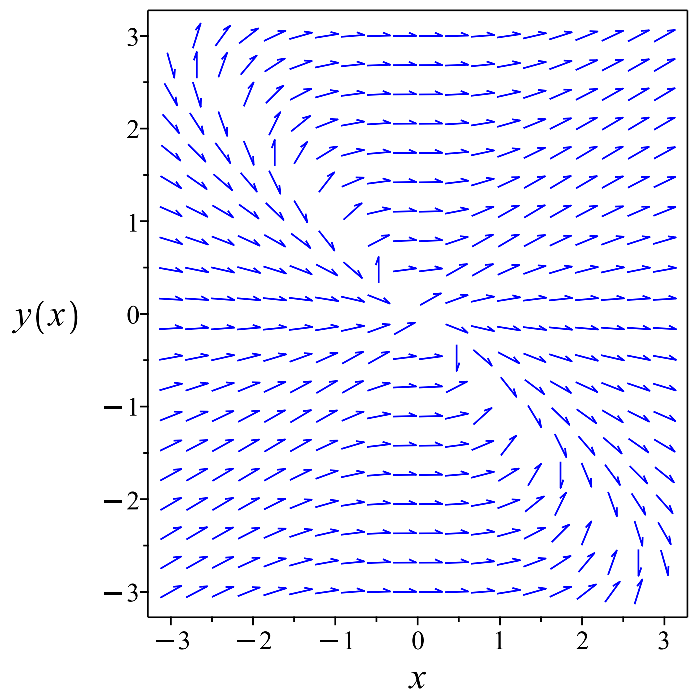


Figure 412: Slope field plot

Verification of solutions

$$
y = e^{\frac{\operatorname{LambertW}\left(x^3 e^{-3c_1}\right)}{3}+c_1}
$$

Verified OK.

8.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}
$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y)=0
$$

Hence

$$
\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}
$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(-x^3 - y^3\right) \mathrm{d}y &= \left(-y\,x^2\right) \mathrm{d}x \\
\left(y\,x^2\right) \mathrm{d}x + \left(-x^3 - y^3\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= y\,x^2 \\
N(x, y) &= -x^3 - y^3
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\,x^2\right) \\
&= x^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^3 - y^3\right) \\
&= -3x^2
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x^3+y^3}\left((x^2) - (-3x^2)\right) \\
&= -\frac{4x^2}{x^3+y^3}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{x^2y}\left((-3x^2) - (x^2)\right) \\
&= -\frac{4}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{4}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-4\ln(y)} \\
&= \frac{1}{y^4}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^4}\left(y\,x^2\right) \\
&= \frac{x^2}{y^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^4}(-x^3 - y^3) \\
&= \frac{-x^3 - y^3}{y^4}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2}{y^3}\right) + \left(\frac{-x^3 - y^3}{y^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2}{y^3}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^3}{3y^3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{x^3}{y^4} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^3 - y^3}{y^4}$. Therefore equation (4) becomes

$$
\frac{-x^3 - y^3}{y^4} = -\frac{x^3}{y^4} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^3}{3y^3} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^3}{3y^3} - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{3c_1}x^3\right)}{3} - c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{3c_1}x^3\right)}{3} - c_1} \tag{1}$$

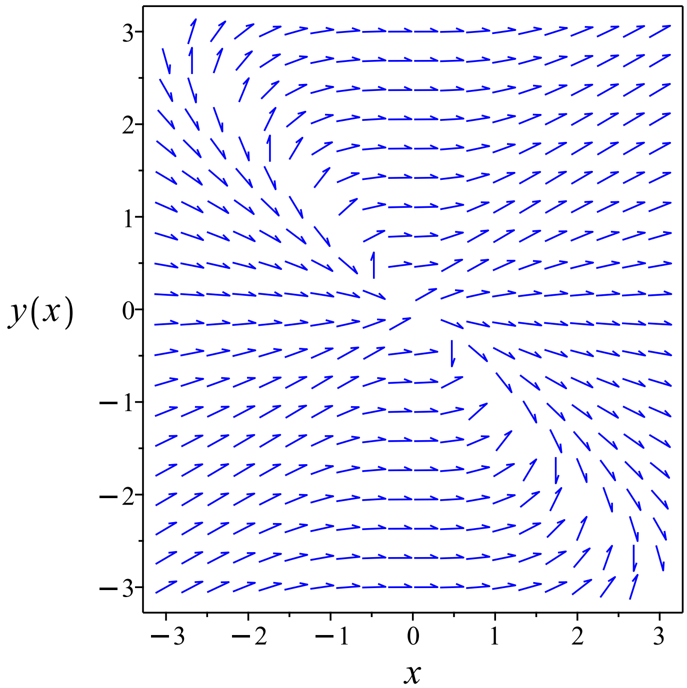


Figure 413: Slope field plot

Verification of solutions

$$y = e^{\frac{\operatorname{LambertW}\left(e^{3c_1}x^3\right)}{3} - c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve((x^2*y(x))-(x^3+y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(\frac{1}{\text{LambertW}\left(c_1 x^3\right)}\right)^{\frac{1}{3}} x$$

✓ Solution by Mathematica

Time used: 7.211 (sec). Leaf size: 80

```
DSolve[(x^2*y[x])-(x^3+y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{\sqrt[3]{W\left(e^{-3c_1}x^3\right)}}$$
$$y(x) \to -\frac{\sqrt[3]{-1}x}{\sqrt[3]{W\left(e^{-3c_1}x^3\right)}}$$
$$y(x) \to \frac{(-1)^{2/3}x}{\sqrt[3]{W\left(e^{-3c_1}x^3\right)}}$$
$$y(x) \to 0$$

8.12 problem 12

Internal problem ID [2044]
Internal file name [`OUTPUT/2044_Sunday_February_25_2024_06_47_06_AM_37991438/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-y'x + y - 2y' - 2y^2 = 0}$$

8.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{2y^2 - y}{2+x} \end{aligned}$$

Where $f(x) = -\frac{1}{2+x}$ and $g(y) = 2y^2 - y$. Integrating both sides gives

$$\frac{1}{2y^2 - y}\,dy = -\frac{1}{2+x}\,dx$$

$$\int \frac{1}{2y^2 - y} dy = \int -\frac{1}{2+x} dx$$
$$\ln(-1+2y) - \ln(y) = -\ln(2+x) + c_1$$

Raising both side to exponential gives

$$e^{\ln(-1+2y)-\ln(y)} = e^{-\ln(2+x)+c_1}$$

Which simplifies to

$$\frac{-1+2y}{y} = \frac{c_2}{2+x}$$

Summary

The solution(s) found are the following

$$y = -\frac{2+x}{-4+c_2-2x} \tag{1}$$

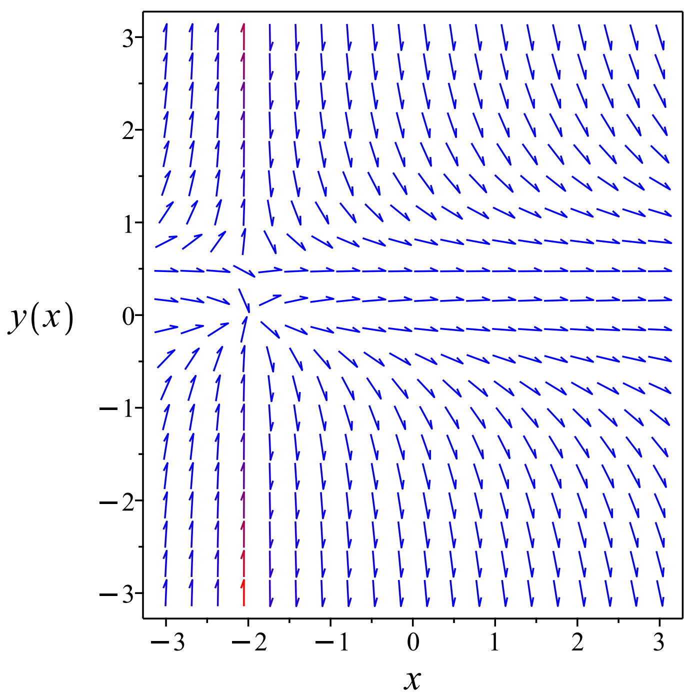


Figure 414: Slope field plot

Verification of solutions

$$y = -\frac{2+x}{-4+c_2-2x}$$

Verified OK.

8.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(-1+2y)}{2+x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 242: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -x - 2 \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-x - 2} dx
\end{aligned}
$$

Which results in

$$
S = -\ln(-x - 2)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{y(-1 + 2y)}{2 + x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{-x-2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y(-1+2y)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(-1+2R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(-1+2R) - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(-x-2) = \ln(2y-1) - \ln(y) + c_1$$

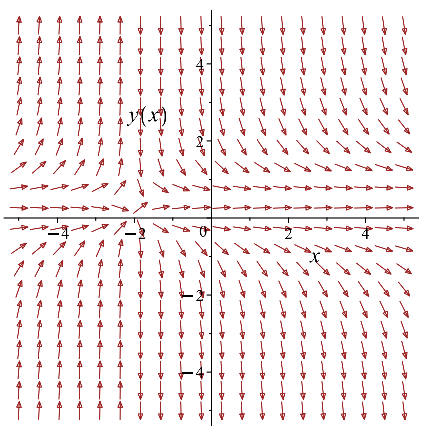
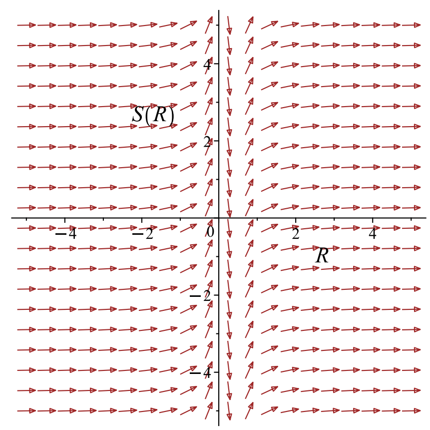
Which simplifies to

$$-\ln(-x-2) = \ln(2y-1) - \ln(y) + c_1$$

Which gives

$$y = \frac{e^{c_1}(2+x)}{2x\,e^{c_1} + 4\,e^{c_1} + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-1+2y)}{2+x}$ 	$R = y$ $S = -\ln(-x-2)$	$\frac{dS}{dR} = \frac{1}{R(-1+2R)}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{c_1}(2+x)}{2x\,e^{c_1} + 4\,e^{c_1} + 1} \tag{1}$$

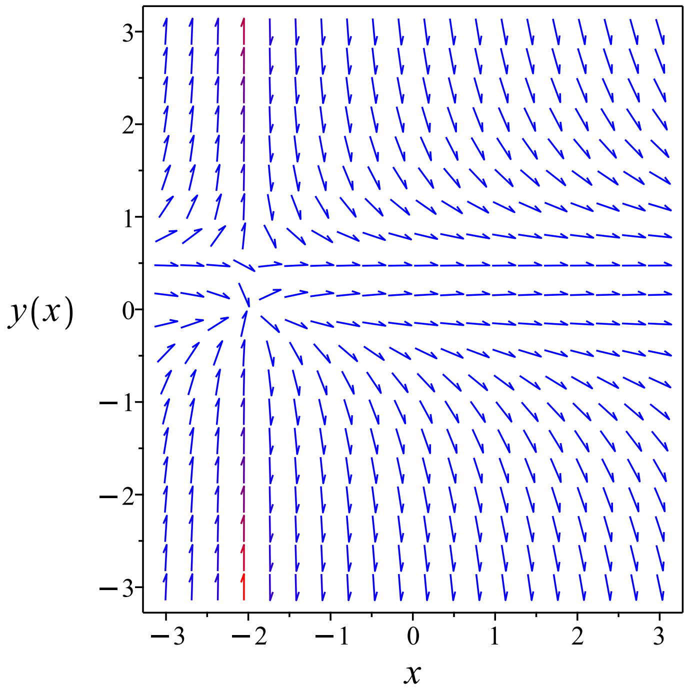


Figure 415: Slope field plot

Verification of solutions

$$y = \frac{e^{c_1}(2+x)}{2x\, e^{c_1} + 4\, e^{c_1} + 1}$$

Verified OK.

8.12.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(-1+2y)}{2+x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2+x}y - \frac{2}{2+x}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{2+x} \\
f_1(x) &= -\frac{2}{2+x} \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y'\frac{1}{y^2} = \frac{1}{(2+x)\,y} - \frac{2}{2+x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= \frac{w(x)}{2+x} - \frac{2}{2+x} \\
w' &= -\frac{w}{2+x} + \frac{2}{2+x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{2+x} \\
q(x) &= \frac{2}{2+x}
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{w(x)}{2+x} = \frac{2}{2+x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2+x}dx} \\ &= 2+x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{2}{2+x}\right) \\ \frac{d}{dx}((2+x)\,w) &= (2+x)\left(\frac{2}{2+x}\right) \\ d((2+x)\,w) &= 2\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}(2+x)\,w &= \int 2\,dx \\ (2+x)\,w &= 2x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = 2+x$ results in

$$w(x) = \frac{2x}{2+x} + \frac{c_1}{2+x}$$

which simplifies to

$$w(x) = \frac{2x+c_1}{2+x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{2x+c_1}{2+x}$$

Or

$$y = \frac{2+x}{2x+c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{2+x}{2x+c_1} \tag{1}$$

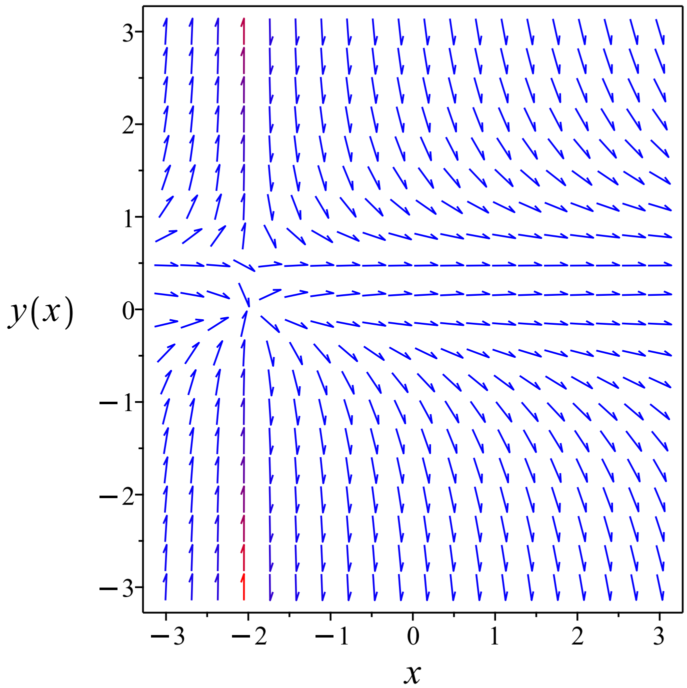


Figure 416: Slope field plot

Verification of solutions

$$y = \frac{2+x}{2x+c_1}$$

Verified OK.

8.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y(-1+2y)}\right) \mathrm{d}y &= \left(\frac{1}{2+x}\right) \mathrm{d}x \\
\left(-\frac{1}{2+x}\right) \mathrm{d}x + \left(-\frac{1}{y(-1+2y)}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{2+x} \\
N(x, y) &= -\frac{1}{y(-1+2y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{2+x}\right) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{y(-1+2y)}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{2+x}\,\mathrm{d}x$$
$$\phi = -\ln(2+x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(-1+2y)}$. Therefore equation (4) becomes

$$-\frac{1}{y(-1+2y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y(-1+2y)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y(-1+2y)}\right)\mathrm{d}y$$
$$f(y) = -\ln(-1+2y) + \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(2+x) - \ln(-1+2y) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(2+x) - \ln(-1+2y) + \ln(y)$$

The solution becomes

$$y = \frac{e^{c_1}(2+x)}{-1 + 2x\,e^{c_1} + 4\,e^{c_1}}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{c_1}(2+x)}{-1 + 2x\,e^{c_1} + 4\,e^{c_1}} \tag{1}$$

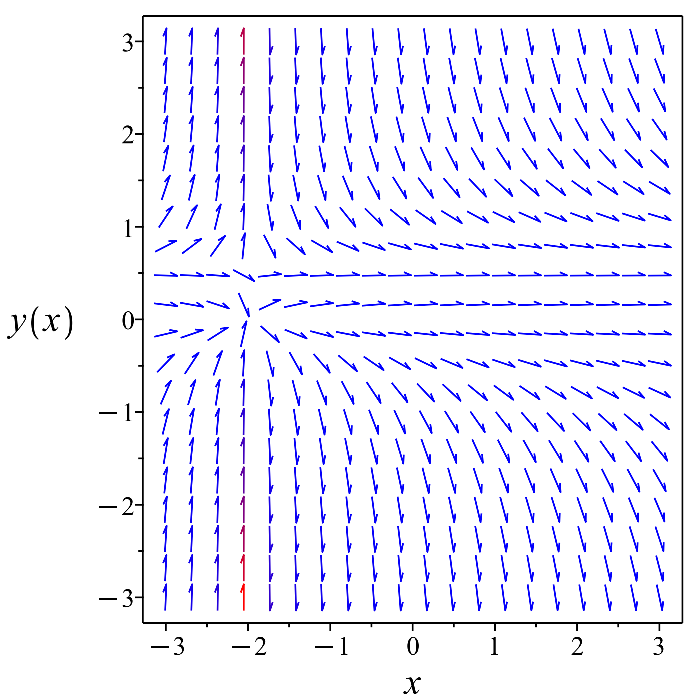


Figure 417: Slope field plot

Verification of solutions

$$y = \frac{e^{c_1}(2+x)}{-1+2x\,e^{c_1}+4\,e^{c_1}}$$

Verified OK.

8.12.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(-1+2y)}{2+x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{2+x} - \frac{2y^2}{2+x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{2+x}$ and $f_2(x) = -\frac{2}{2+x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{2u}{2+x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2}{(2+x)^2} \\ f_1f_2 &= -\frac{2}{(2+x)^2} \\ f_2^2f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{2u''(x)}{2+x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1x + c_2$$

The above shows that

$$u'(x) = c_1$$

Using the above in (1) gives the solution

$$y = \frac{c_1(2 + x)}{2c_1x + 2c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3(2 + x)}{2c_3x + 2}$$

Summary
The solution(s) found are the following

$$y = \frac{c_3(2 + x)}{2c_3x + 2} \tag{1}$$

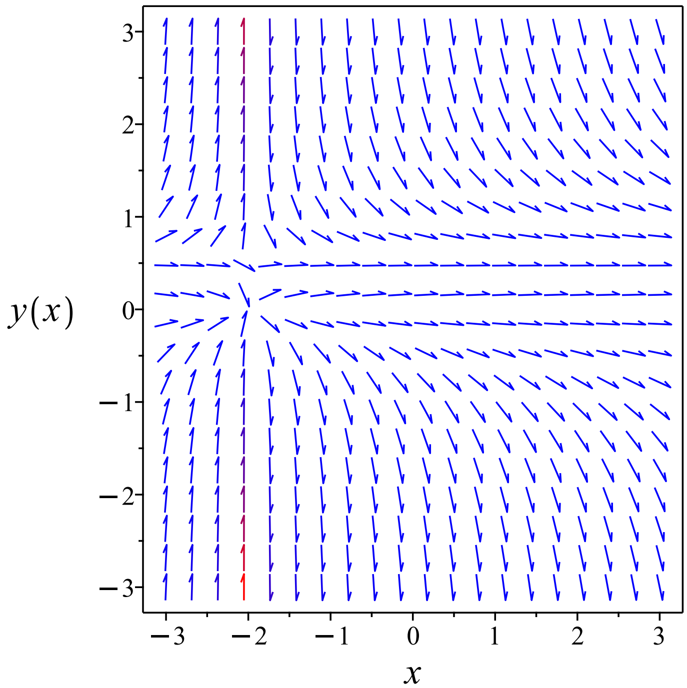


Figure 418: Slope field plot

Verification of solutions

$$y = \frac{c_3(2+x)}{2c_3x+2}$$

Verified OK.

8.12.6 Maple step by step solution

Let's solve

$-y'x + y - 2y' - 2y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y^2-y} = \frac{1}{-x-2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{2y^2-y}dx = \int \frac{1}{-x-2}dx + c_1$

- Evaluate integral

$$\ln(2y-1)-\ln(y)=-\ln(-x-2)+c_1$$

- Solve for y

$$y=\frac{2+x}{2x+e^{c_1}+4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(y(x)-x*diff(y(x),x)=2*(y(x)^2+diff(y(x),x)),y(x), singsol=all)
```

$$y(x)=\frac{2+x}{2x+c_1}$$

✓ Solution by Mathematica

Time used: 0.39 (sec). Leaf size: 32

```
DSolve[y[x]-x*y'[x]==2*(y[x]^2+y'[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{x+2}{2x+4+e^{c_1}}$$
$$y(x)\to 0$$
$$y(x)\to\frac{1}{2}$$

8.13 problem 13

Internal problem ID [2045]
Internal file name [`OUTPUT/2045_Sunday_February_25_2024_06_47_07_AM_20569925/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\tan(y) - (3x+4)\,y' = 0}$$

8.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\tan(y)}{3x+4} \end{aligned}$$

Where $f(x) = \frac{1}{3x+4}$ and $g(y) = \tan(y)$. Integrating both sides gives

$$\frac{1}{\tan(y)}\,dy = \frac{1}{3x+4}\,dx$$

$$\int \frac{1}{\tan(y)}\,dy = \int \frac{1}{3x+4}\,dx$$

$$\ln(\sin(y)) = \frac{\ln(3x+4)}{3} + c_1$$

Raising both side to exponential gives

$$\sin(y) = e^{\frac{\ln(3x+4)}{3}+c_1}$$

Which simplifies to

$$\sin(y) = c_2(3x+4)^{\frac{1}{3}}$$

Which simplifies to

$$y = \arcsin\left(c_2(3x+4)^{\frac{1}{3}} e^{c_1}\right)$$

Summary

The solution(s) found are the following

$$y = \arcsin\left(c_2(3x+4)^{\frac{1}{3}} e^{c_1}\right) \tag{1}$$

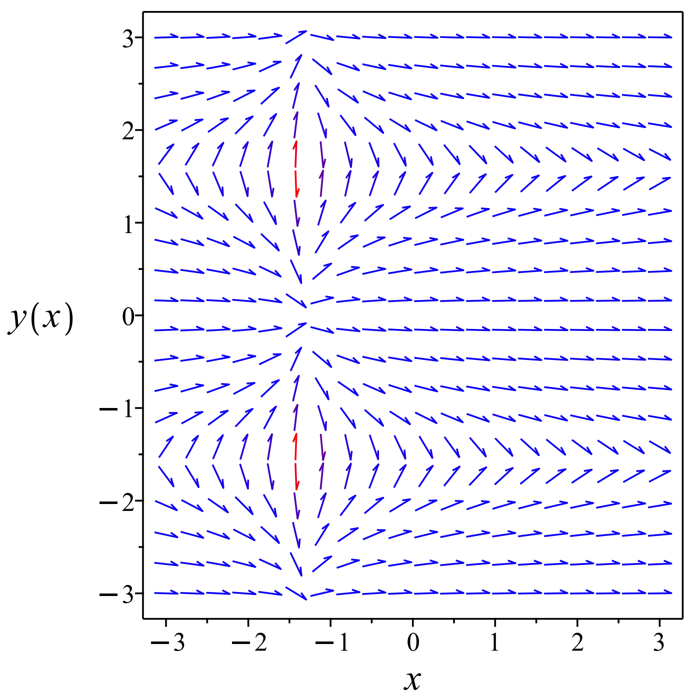


Figure 419: Slope field plot

Verification of solutions

$$y = \arcsin\left(c_2(3x+4)^{\frac{1}{3}} e^{c_1}\right)$$

Verified OK.

8.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\tan(y)}{3x+4} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 245: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 3x+4\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{3x+4} dx\end{aligned}$$

Which results in

$$S = \frac{\ln(3x+4)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\tan(y)}{3x+4}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{3x+4} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cot(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cot(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(\sin(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(3x+4)}{3} = \ln(\sin(y)) + c_1$$

Which simplifies to

$$\frac{\ln(3x+4)}{3} = \ln(\sin(y)) + c_1$$

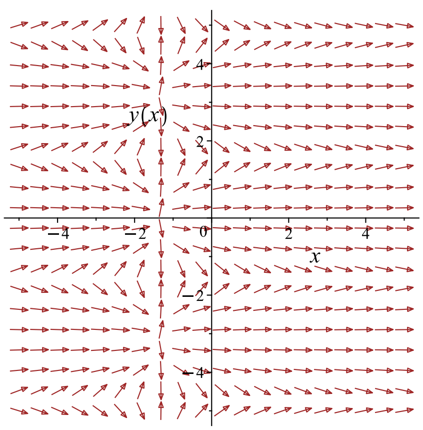
Which gives

$$y = \arcsin\left(\mathrm{e}^{\frac{\ln(3x+4)}{3} - c_1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{\tan(y)}{3x+4}$



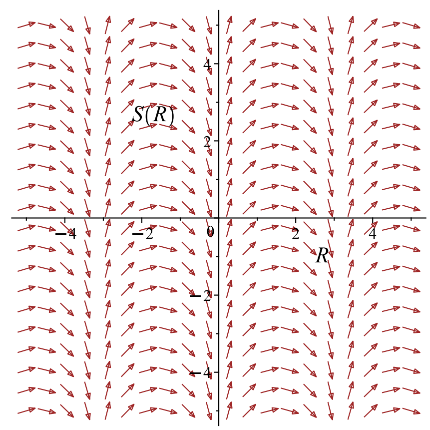
</td><td>

$R = y$

$S = \frac{\ln(3x+4)}{3}$

</td><td>

$\frac{dS}{dR} = \cot(R)$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \arcsin\left(e^{\frac{\ln(3x+4)}{3} - c_1}\right) \tag{1}$$

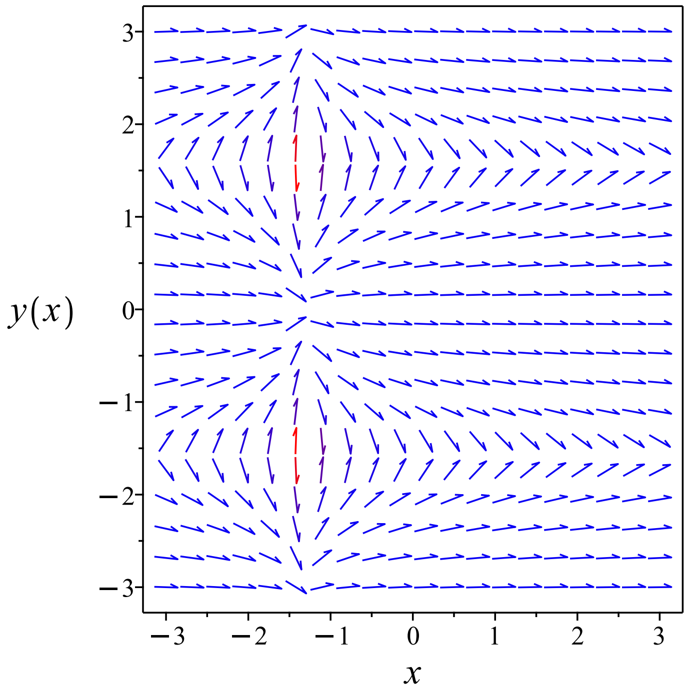


Figure 420: Slope field plot

Verification of solutions

$$y = \arcsin\left(e^{\frac{\ln(3x+4)}{3}-c_1}\right)$$

Verified OK.

8.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{\tan(y)}\right) \mathrm{d}y &= \left(\frac{1}{3x+4}\right) \mathrm{d}x \\
\left(-\frac{1}{3x+4}\right) \mathrm{d}x + \left(\frac{1}{\tan(y)}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{3x+4} \\
N(x, y) &= \frac{1}{\tan(y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{3x+4}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\tan(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{3x+4} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(3x+4)}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\tan(y)}$. Therefore equation (4) becomes

$$
\frac{1}{\tan(y)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{1}{\tan(y)} \\
&= \cot(y)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\cot(y))\,\mathrm{d}y$$
$$f(y) = \ln(\sin(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(3x+4)}{3} + \ln(\sin(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(3x+4)}{3} + \ln(\sin(y))$$

Summary

The solution(s) found are the following

$$\ln(\sin(y)) - \frac{\ln(3x+4)}{3} = c_1 \tag{1}$$

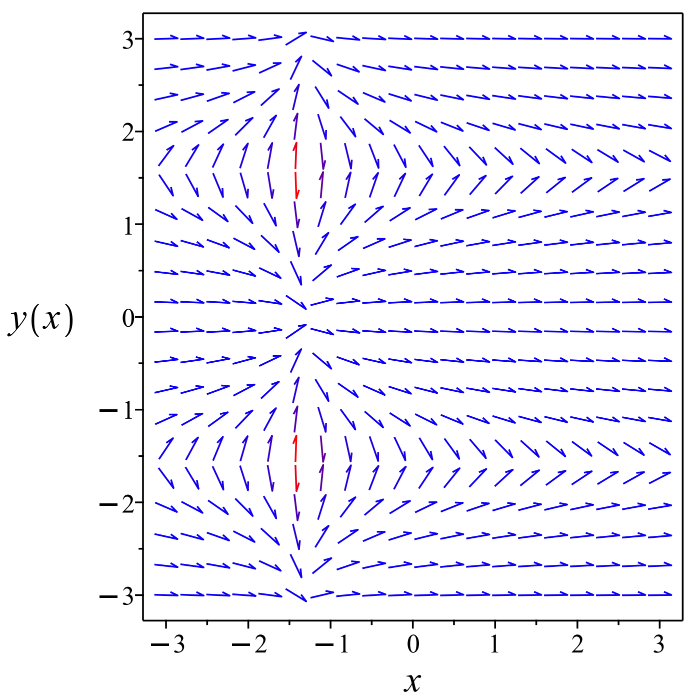


Figure 421: Slope field plot

Verification of solutions

$$\ln(\sin(y)) - \frac{\ln(3x+4)}{3} = c_1$$

Verified OK.

8.13.4 Maple step by step solution

Let's solve

$\tan(y) - (3x+4)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\tan(y)} = \frac{1}{3x+4}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\tan(y)} dx = \int \frac{1}{3x+4} dx + c_1$

- Evaluate integral

  $\ln(\sin(y)) = \frac{\ln(3x+4)}{3} + c_1$

- Solve for y

  $y = \arcsin\left(\mathrm{e}^{\frac{\ln(3x+4)}{3}+c_1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(tan(y(x))=(3*x+4)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \arcsin\left((3x+4)^{\frac{1}{3}} c_1\right)$$

✓ Solution by Mathematica

Time used: 23.059 (sec). Leaf size: 25

```
DSolve[Tan[y[x]]==(3*x+4)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(e^{c_1}\sqrt[3]{3x+4}\right)$$
$$y(x) \to 0$$

8.14 problem 14

Internal problem ID [2046]
Internal file name [`OUTPUT/2046_Sunday_February_25_2024_06_47_09_AM_19711343/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

Unable to solve or complete the solution.

$$\boxed{y' + y \ln(y) \tan(x) - 2y = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
<- symmetry pattern of the form [0, F(x)*G(y)] successful`
```

✓ Solution by Maple

Time used: 1.25 (sec). Leaf size: 25

```
dsolve(diff(y(x),x)+y(x)*ln(y(x))*tan(x)=2*y(x),y(x), singsol=all)
```

$$y(x) = \left(-\frac{\cos(x)}{\sin(x)-1}\right)^{2\cos(x)} e^{\cos(x)c_1}$$

✓ Solution by Mathematica

Time used: 2.002 (sec). Leaf size: 17

```
DSolve[y'[x]+y[x]*Log[y[x]]*Tan[x]==2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2\cos(x)(\coth^{-1}(\sin(x))+c_1)}$$

8.15 problem 15

Internal problem ID [2047]
Internal file name [`OUTPUT/2047_Sunday_February_25_2024_06_47_10_AM_39583957/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{2yx + y^4 + \left(x y^3 - 2x^2\right) y' = 0}$$

8.15.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(y^3 + 2x)}{x\left(y^3 - 2x\right)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{y(y^3+2x)(b_3-a_2)}{x(y^3-2x)} - \frac{y^2(y^3+2x)^2 a_3}{x^2(y^3-2x)^2} \\
&- \left(-\frac{2y}{x(y^3-2x)} + \frac{y(y^3+2x)}{x^2(y^3-2x)} - \frac{2y(y^3+2x)}{x(y^3-2x)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{y^3+2x}{x(y^3-2x)} - \frac{3y^3}{x(y^3-2x)} + \frac{3y^3(y^3+2x)}{x(y^3-2x)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-2x^2y^6b_2 + 2y^8a_3 - x\,y^6b_1 + y^7a_1 + 16x^3y^3b_2 - 4x^2y^4a_2 + 12x^2y^4b_3 + 12x^2y^3b_1 - 4x\,y^4a_1 + 4x^3b_1 - 4x}{x^2(-y^3+2x)^2}
$$
$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&2x^2y^6b_2 - 2y^8a_3 + x\,y^6b_1 - y^7a_1 - 16x^3y^3b_2 + 4x^2y^4a_2 \\
&- 12x^2y^4b_3 - 12x^2y^3b_1 + 4x\,y^4a_1 - 4x^3b_1 + 4x^2ya_1 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2a_3v_2^8 + 2b_2v_1^2v_2^6 - a_1v_2^7 + b_1v_1v_2^6 + 4a_2v_1^2v_2^4 - 16b_2v_1^3v_2^3 \\
&- 12b_3v_1^2v_2^4 + 4a_1v_1v_2^4 - 12b_1v_1^2v_2^3 + 4a_1v_1^2v_2 - 4b_1v_1^3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-16b_2v_1^3v_2^3 - 4b_1v_1^3 + 2b_2v_1^2v_2^6 + (4a_2 - 12b_3)\,v_1^2v_2^4 - 12b_1v_1^2v_2^3 + 4a_1v_1^2v_2 + b_1v_1v_2^6 + 4a_1v_1v_2^4 - 2a_3v_2^8 - a_1v_2^7 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ 4a_1 &= 0 \\ -2a_3 &= 0 \\ -12b_1 &= 0 \\ -4b_1 &= 0 \\ -16b_2 &= 0 \\ 2b_2 &= 0 \\ 4a_2 - 12b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 3b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 3x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y(y^3+2x)}{x\,(y^3-2x)}\right)(3x) \\ &= \frac{-4y^4-4yx}{-y^3+2x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy \\ &= \int \frac{1}{\frac{-4y^4-4yx}{-y^3+2x}}dy\end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} + \frac{\ln(y^3+x)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(y^3+2x)}{x\,(y^3-2x)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{1}{4y^3 + 4x}\\
S_y &= \frac{y^3 - 2x}{4y\,(y^3 + x)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{4x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{4R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{4} + c_1 \tag{4}
$$

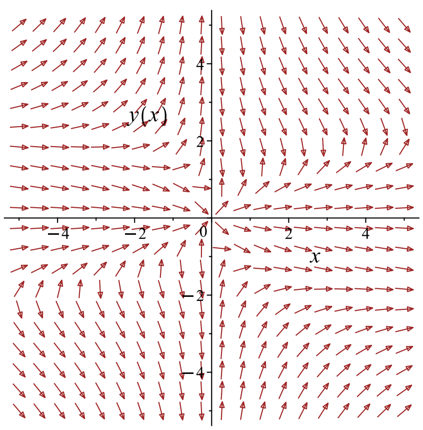
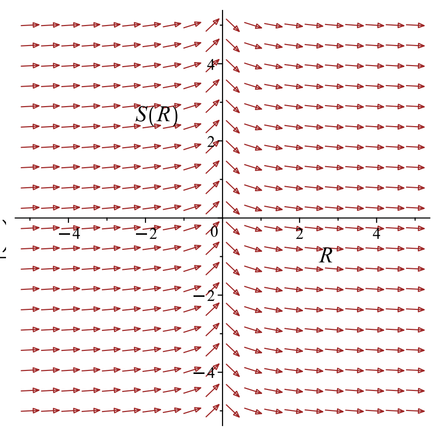
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{\ln(y)}{2} + \frac{\ln(y^3 + x)}{4} = -\frac{\ln(x)}{4} + c_1
$$

Which simplifies to

$$
-\frac{\ln(y)}{2} + \frac{\ln(y^3 + x)}{4} = -\frac{\ln(x)}{4} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y^3+2x)}{x(y^3-2x)}$ 	$R = x$ $S = -\frac{\ln(y)}{2} + \frac{\ln(y^3+x)}{4}$	$\frac{dS}{dR} = -\frac{1}{4R}$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} + \frac{\ln(y^3+x)}{4} = -\frac{\ln(x)}{4} + c_1 \tag{1}$$

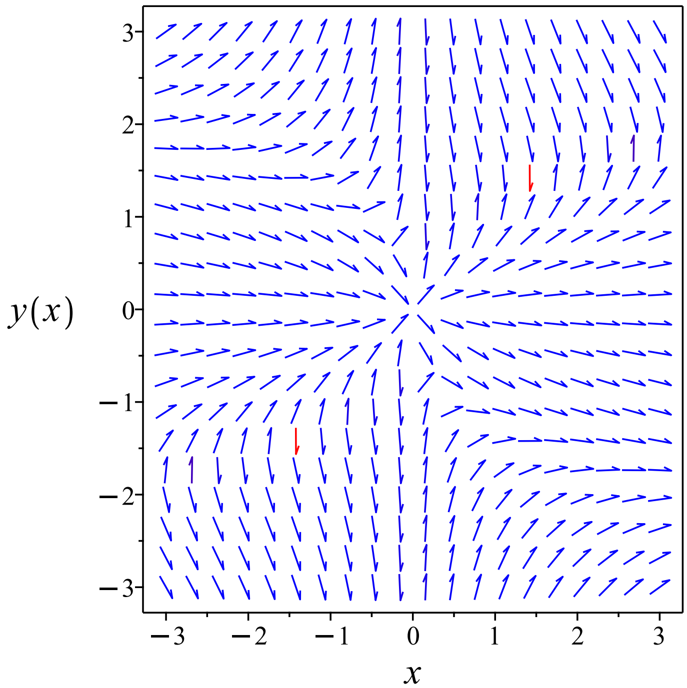


Figure 422: Slope field plot

Verification of solutions

$$-\frac{\ln(y)}{2} + \frac{\ln(y^3 + x)}{4} = -\frac{\ln(x)}{4} + c_1$$

Verified OK.

8.15.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\,y^3 - 2x^2\right)\mathrm{d}y &= \left(-y^4 - 2yx\right)\mathrm{d}x\\ \left(y^4 + 2yx\right)\mathrm{d}x + \left(x\,y^3 - 2x^2\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^4 + 2yx\\ N(x, y) &= x\,y^3 - 2x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^4 + 2yx\right)\\ &= 4y^3 + 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\,y^3 - 2x^2\right)\\ &= y^3 - 4x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{-x\,y^3 + 2x^2}\left(\left(4y^3 + 2x\right) - \left(y^3 - 4x\right)\right) \\
&= \frac{-3y^3 - 6x}{-x\,y^3 + 2x^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^4 + 2yx}\left(\left(y^3 - 4x\right) - \left(4y^3 + 2x\right)\right) \\
&= -\frac{3}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{3}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(y)} \\
&= \frac{1}{y^3}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^3}\left(y^4 + 2yx\right) \\
&= \frac{y^3 + 2x}{y^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^3}(x\,y^3 - 2x^2) \\
&= \frac{x\,y^3 - 2x^2}{y^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y^3 + 2x}{y^2}\right) + \left(\frac{x\,y^3 - 2x^2}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{y^3 + 2x}{y^2}\,\mathrm{d}x \\
\phi &= \frac{(y^3 + x)\,x}{y^2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2(y^3 + x)\,x}{y^3} + 3x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x\,y^3 - 2x^2}{y^3}$. Therefore equation (4) becomes

$$
\frac{x\,y^3 - 2x^2}{y^3} = \frac{x\,y^3 - 2x^2}{y^3} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(y^3 + x)\,x}{y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(y^3 + x)\,x}{y^2}$$

Summary

The solution(s) found are the following

$$\frac{(y^3 + x)\,x}{y^2} = c_1 \tag{1}$$

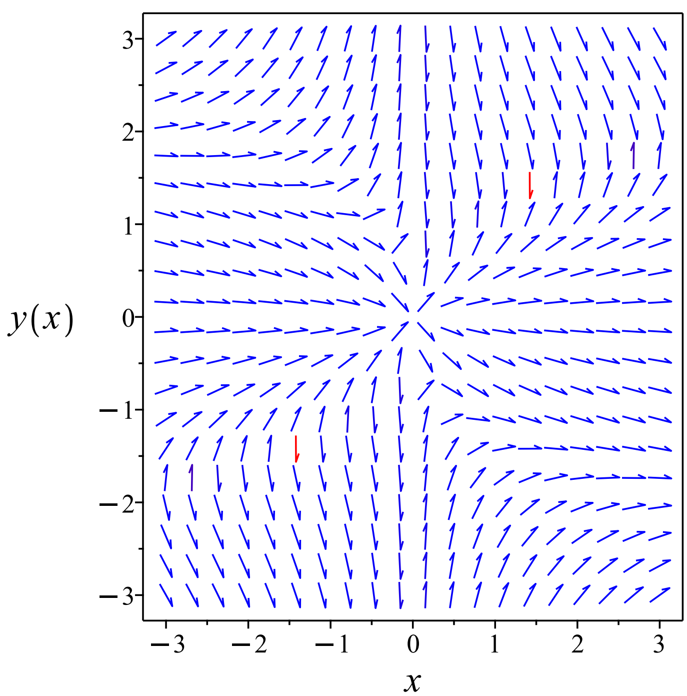


Figure 423: Slope field plot

Verification of solutions

$$\frac{(y^3+x)\,x}{y^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 330

```
dsolve((2*x*y(x)+y(x)^4)+(x*y(x)^3-2*x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\frac{\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{1}{3}}}{2} + \frac{2c_1^2}{\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{1}{3}}} + c_1}{3x}$$

$$y(x) = \frac{\left(-1-i\sqrt{3}\right)\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{2}{3}} + 4c_1\left(i\sqrt{3}\,c_1 - c_1 + \left(-108x^4+12\sqrt{81x^4-12c_1^3}\right.\right.}{12\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{1}{3}} x}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{2}{3}} + 4\left(-i\sqrt{3}\,c_1 - c_1 + \left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x\right.\right.}{12\left(-108x^4+12\sqrt{81x^4-12c_1^3}\,x^2+8c_1^3\right)^{\frac{1}{3}} x}$$

✓ Solution by Mathematica

Time used: 12.491 (sec). Leaf size: 371

```
DSolve[(2*x*y[x]+y[x]^4)+(x*y[x]^3-2*x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\frac{2\sqrt[3]{2}c_1^2}{\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1^3x^4}+2c_1^3}}+2^{2/3}\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1^3x^4}+2c_1^3}+2c_1}{6x}$$

$$y(x) \to \frac{\frac{2\sqrt[3]{2}\left(1+i\sqrt{3}\right)c_1^2}{\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1^3x^4}+2c_1^3}}+2^{2/3}\left(1-i\sqrt{3}\right)\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1^3x^4}+2c_1^3}-4c_1}{12x}$$

$$y(x) \to \frac{\frac{2\sqrt[3]{2}\left(1-i\sqrt{3}\right)c_1^2}{\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1^3x^4}+2c_1^3}}+2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{27x^4+3\sqrt{81x^8+12c_1^3x^4}+2c_1^3}-4c_1}{12x}$$

$$y(x) \to 0$$

8.16 problem 16

Internal problem ID [2048]
Internal file name [`OUTPUT/2048_Sunday_February_25_2024_06_47_11_AM_89686211/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (-2y + 3x)\, y' = 0}$$

8.16.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x + (-2u(x)\, x + 3x)\, (u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u(u-2)}{(2u-3)\, x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u(u-2)}{2u-3}$. Integrating both sides gives

$$\frac{1}{\frac{u(u-2)}{2u-3}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{u(u-2)}{2u-3}}\,du = \int -\frac{2}{x}\,dx$$
$$\frac{3\ln(u)}{2} + \frac{\ln(u-2)}{2} = -2\ln(x) + c_2$$

The above can be written as

$$\begin{aligned}\frac{3\ln(u) + \ln(u-2)}{2} &= -2\ln(x) + c_2 \\ 3\ln(u) + \ln(u-2) &= (2)(-2\ln(x) + c_2) \\ &= -4\ln(x) + 2c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{3\ln(u)+\ln(u-2)} = e^{-4\ln(x)+2c_2}$$

Which simplifies to

$$\begin{aligned}u^3(u-2) &= \frac{2c_2}{x^4} \\ &= \frac{c_3}{x^4}\end{aligned}$$

Which simplifies to

$$u(x) = \text{RootOf}\left(\_Z^4 - 2\_Z^3 - \frac{c_3 e^{2c_2}}{x^4}\right)$$

Therefore the solution y is

$$\begin{aligned}y &= xu \\ &= x\,\text{RootOf}\left(x^4\_Z^4 - 2\_Z^3 x^4 - c_3 e^{2c_2}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\,\text{RootOf}\left(x^4\_Z^4 - 2\_Z^3 x^4 - c_3 e^{2c_2}\right) \tag{1}$$

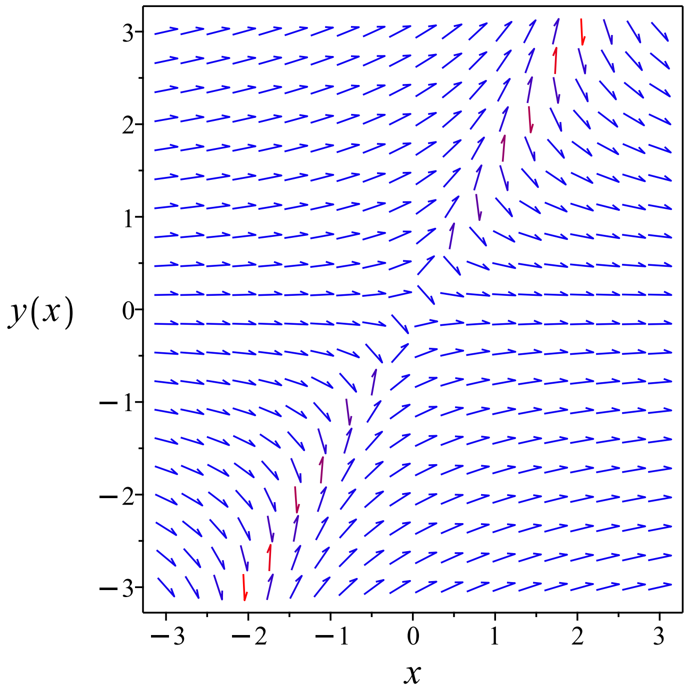


Figure 424: Slope field plot

Verification of solutions

$$y = x\,\text{RootOf}\left(x^4\_Z^4 - 2\_Z^3x^4 - c_3\mathrm{e}^{2c_2}\right)$$

Verified OK.

8.16.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y}{-3x+2y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(b_3 - a_2)}{-3x + 2y} - \frac{y^2 a_3}{(-3x + 2y)^2} - \frac{3y(x a_2 + y a_3 + a_1)}{(-3x + 2y)^2} - \left(\frac{1}{-3x + 2y} - \frac{2y}{(-3x + 2y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{12x^2 b_2 - 12xy b_2 - 2y^2 a_2 - 4y^2 a_3 + 4y^2 b_2 + 2y^2 b_3 + 3x b_1 - 3y a_1}{(3x - 2y)^2} = 0$$

Setting the numerator to zero gives

$$12x^2 b_2 - 12xy b_2 - 2y^2 a_2 - 4y^2 a_3 + 4y^2 b_2 + 2y^2 b_3 + 3x b_1 - 3y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2 v_2^2 - 4a_3 v_2^2 + 12b_2 v_1^2 - 12b_2 v_1 v_2 + 4b_2 v_2^2 + 2b_3 v_2^2 - 3a_1 v_2 + 3b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$12b_2 v_1^2 - 12b_2 v_1 v_2 + 3b_1 v_1 + (-2a_2 - 4a_3 + 4b_2 + 2b_3) v_2^2 - 3a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-3a_1 &= 0 \\
3b_1 &= 0 \\
-12b_2 &= 0 \\
12b_2 &= 0 \\
-2a_2 - 4a_3 + 4b_2 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -2a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y}{-3x + 2y}\right)(x) \\
&= \frac{4yx - 2y^2}{3x - 2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4yx-2y^2}{3x-2y}} dy \end{aligned}$$

Which results in

$$S = \frac{3\ln(y)}{4} + \frac{\ln(y-2x)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{-3x+2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{4x-2y} \\ S_y &= \frac{3}{4y} - \frac{1}{8x-4y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

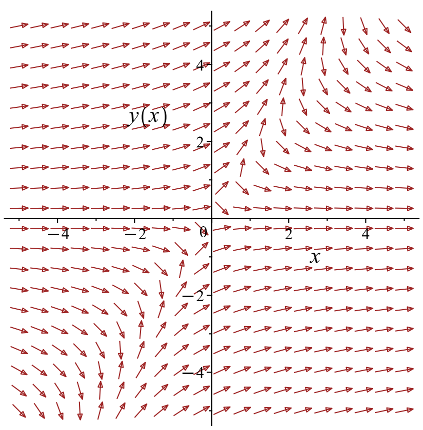
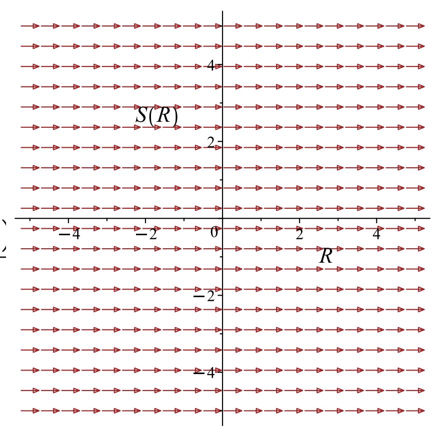
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(y)}{4} + \frac{\ln(-2x+y)}{4} = c_1$$

Which simplifies to

$$\frac{3\ln(y)}{4} + \frac{\ln(-2x+y)}{4} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{-3x+2y}$ 	$R = x$ $S = \frac{3\ln(y)}{4} + \frac{\ln(y-2x)}{4}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{3\ln(y)}{4} + \frac{\ln(-2x+y)}{4} = c_1 \tag{1}$$

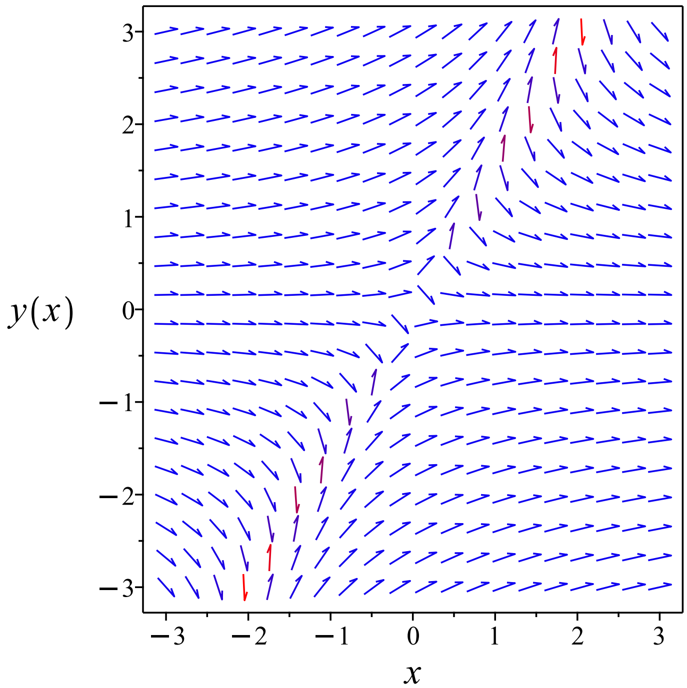


Figure 425: Slope field plot

Verification of solutions

$$\frac{3\ln(y)}{4} + \frac{\ln(-2x+y)}{4} = c_1$$

Verified OK.

8.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x - 2y)\,\mathrm{d}y &= (-y)\,\mathrm{d}x\\ (y)\,\mathrm{d}x + (3x - 2y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y\\ N(x, y) &= 3x - 2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x - 2y)\\ &= 3\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{3x-2y}((1)-(3)) \\ &= -\frac{2}{3x-2y} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y}((3)-(1)) \\ &= \frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int \frac{2}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2\ln(y)} \\ &= y^2 \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= y^2(y) \\ &= y^3 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= y^2(3x-2y) \\ &= (3x-2y)\,y^2 \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(y^3) + \left((3x - 2y)\, y^2\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int y^3 \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x\, y^3 + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 3x\, y^2 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (3x - 2y)\, y^2$. Therefore equation (4) becomes

$$
(3x - 2y)\, y^2 = 3x\, y^2 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -2y^3
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-2y^3\right) \mathrm{d}y \\
f(y) &= -\frac{y^4}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x\,y^3 - \frac{1}{2}y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x\,y^3 - \frac{1}{2}y^4$$

Summary

The solution(s) found are the following

$$xy^3 - \frac{y^4}{2} = c_1 \tag{1}$$

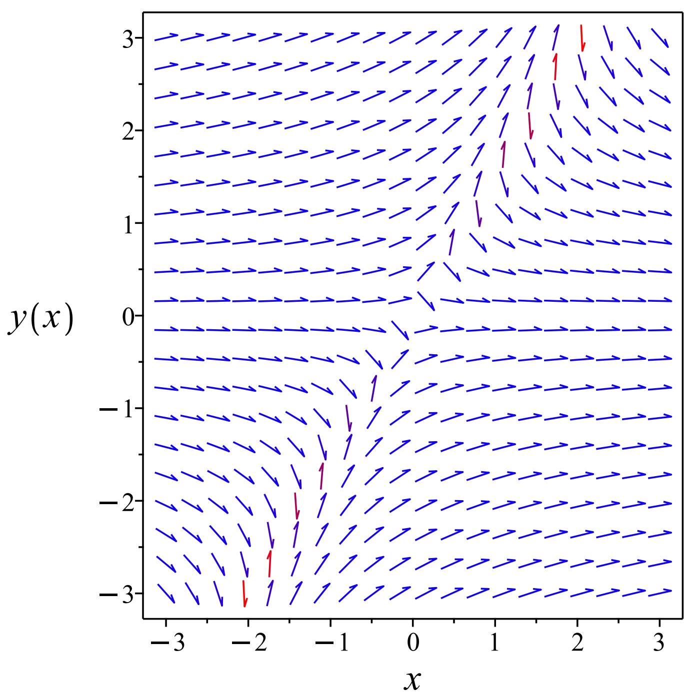


Figure 426: Slope field plot

Verification of solutions

$$xy^3 - \frac{y^4}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((y(x))+(3*x-2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$x - \frac{y(x)}{2} - \frac{c_1}{y(x)^3} = 0$$

✓ Solution by Mathematica

Time used: 60.08 (sec). Leaf size: 1509

```
DSolve[(y[x])+(3*x-2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}\sqrt{x^2+\frac{\sqrt[3]{2}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}{3^{2/3}}+\frac{2\,2^{2/3}e^{2c_1}}{\sqrt[3]{3}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}} -\frac{1}{2}\sqrt{2x^2-\frac{\sqrt[3]{2}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}{3^{2/3}}-\frac{2\,2^{2/3}e^{2c_1}}{\sqrt[3]{3}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}-\frac{\cdots}{\sqrt{x^2+\frac{\sqrt[3]{2}\sqrt[3]{\cdots}}{\cdots}}}} +\frac{x}{2}$$

$$y(x) \to -\frac{1}{2}\sqrt{x^2+\frac{\sqrt[3]{2}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}{3^{2/3}}+\frac{2\,2^{2/3}e^{2c_1}}{\sqrt[3]{3}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}} +\frac{1}{2}\sqrt{2x^2-\frac{\sqrt[3]{2}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}{3^{2/3}}-\frac{2\,2^{2/3}e^{2c_1}}{\sqrt[3]{3}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}-\frac{\cdots}{\sqrt{x^2+\frac{\sqrt[3]{2}\sqrt[3]{\cdots}}{\cdots}}}} +\frac{x}{2}$$

$$y(x) \to \frac{1}{2}\sqrt{x^2+\frac{\sqrt[3]{2}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}{3^{2/3}}+\frac{2\,2^{2/3}e^{2c_1}}{\sqrt[3]{3}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}} -\frac{1}{2}\sqrt{2x^2-\frac{\sqrt[3]{2}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}{3^{2/3}}-\frac{2\,2^{2/3}e^{2c_1}}{\sqrt[3]{3}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}+\frac{\cdots}{\sqrt{x^2+\frac{\sqrt[3]{2}\sqrt[3]{\cdots}}{\cdots}}}} +\frac{x}{2}$$

$$y(x) \to \frac{1}{2}\sqrt{x^2+\frac{\sqrt[3]{2}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}{3^{2/3}}+\frac{2\,2^{2/3}e^{2c_1}}{\sqrt[3]{3}\sqrt[3]{\sqrt{81e^{4c_1}x^4-48e^{6c_1}}+9e^{2c_1}x^2}}}\cdots$$

8.17 problem 17

Internal problem ID [2049]
Internal file name [`OUTPUT/2049_Sunday_February_25_2024_06_47_13_AM_1336657/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{r' - r\cot(\theta) = 0}$$

8.17.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} r' &= F(\theta, r) \\ &= f(\theta)g(r) \\ &= r\cot(\theta) \end{aligned}$$

Where $f(\theta) = \cot(\theta)$ and $g(r) = r$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{r}\,dr &= \cot(\theta)\;d\theta \\
\int \frac{1}{r}\,dr &= \int \cot(\theta)\;d\theta \\
\ln(r) &= \ln(\sin(\theta)) + c_1 \\
r &= \mathrm{e}^{\ln(\sin(\theta))+c_1} \\
&= c_1 \sin(\theta)
\end{aligned}$$

Summary

The solution(s) found are the following

$$r = c_1 \sin(\theta) \tag{1}$$

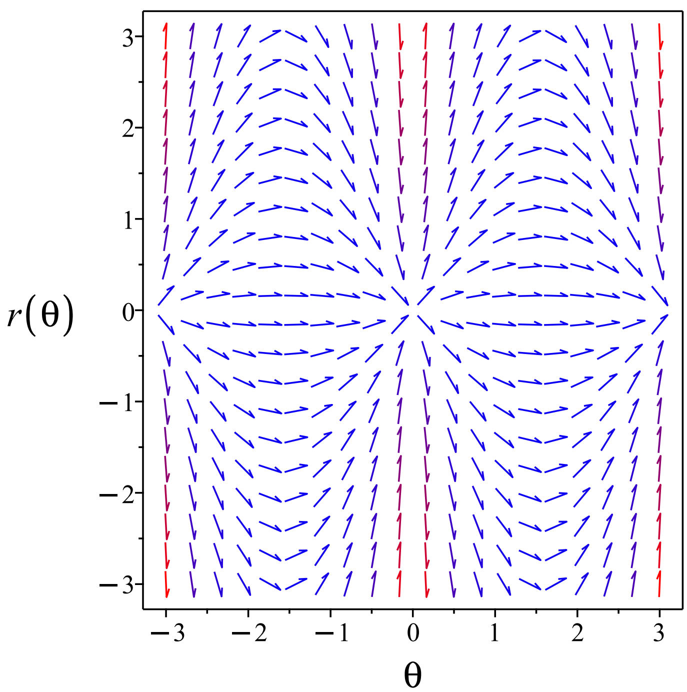


Figure 427: Slope field plot

Verification of solutions

$$r = c_1 \sin(\theta)$$

Verified OK.

8.17.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(\theta)r = q(\theta)$$

Where here

$$\begin{aligned} p(\theta) &= -\cot(\theta) \\ q(\theta) &= 0 \end{aligned}$$

Hence the ode is

$$r' - r\cot(\theta) = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\cot(\theta)d\theta} \\ &= \frac{1}{\sin(\theta)} \end{aligned}$$

Which simplifies to

$$\mu = \csc(\theta)$$

The ode becomes

$$\begin{aligned} \frac{d}{d\theta}\mu r &= 0 \\ \frac{d}{d\theta}(\csc(\theta)\, r) &= 0 \end{aligned}$$

Integrating gives

$$\csc(\theta)\, r = c_1$$

Dividing both sides by the integrating factor $\mu = \csc(\theta)$ results in

$$r = c_1 \sin(\theta)$$

Summary
The solution(s) found are the following

$$r = c_1 \sin(\theta) \tag{1}$$

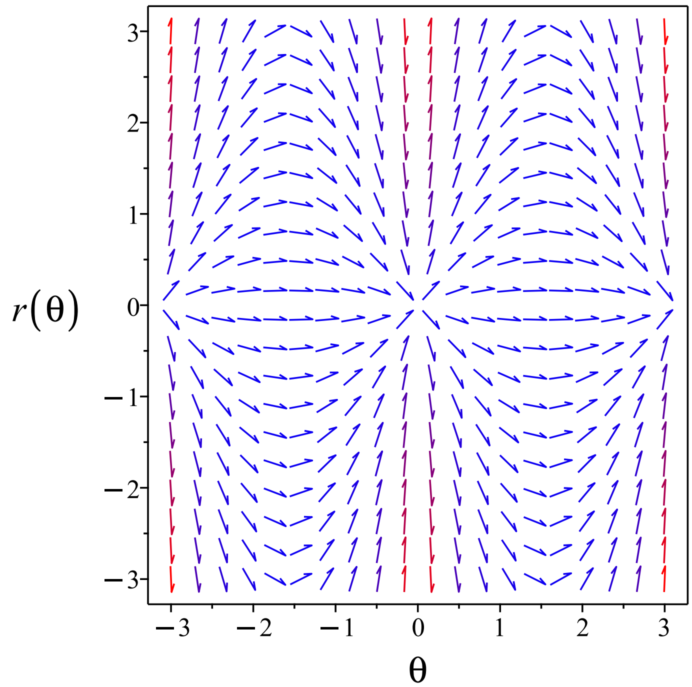


Figure 428: Slope field plot

Verification of solutions

$$r = c_1 \sin(\theta)$$

Verified OK.

8.17.3 Solving as homogeneousTypeD2 ode

Using the change of variables $r = u(\theta)\,\theta$ on the above ode results in new ode in $u(\theta)$

$$u'(\theta)\,\theta + u(\theta) - u(\theta)\,\theta\cot(\theta) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(\theta, u) \\ &= f(\theta)g(u) \\ &= \frac{u(\cot(\theta)\,\theta - 1)}{\theta} \end{aligned}$$

Where $f(\theta) = \frac{\cot(\theta)\theta - 1}{\theta}$ and $g(u) = u$. Integrating both sides gives

$$\frac{1}{u}\, du = \frac{\cot(\theta)\,\theta - 1}{\theta}\, d\theta$$

$$\int \frac{1}{u}\, du = \int \frac{\cot(\theta)\,\theta - 1}{\theta}\, d\theta$$

$$\ln(u) = \ln(\sin(\theta)) - \ln(\theta) + c_2$$

$$u = e^{\ln(\sin(\theta)) - \ln(\theta) + c_2}$$

$$= c_2 e^{\ln(\sin(\theta)) - \ln(\theta)}$$

Which simplifies to

$$u(\theta) = \frac{c_2 \sin(\theta)}{\theta}$$

Therefore the solution r is

$$r = \theta u$$

$$= c_2 \sin(\theta)$$

Summary

The solution(s) found are the following

$$r = c_2 \sin(\theta) \tag{1}$$

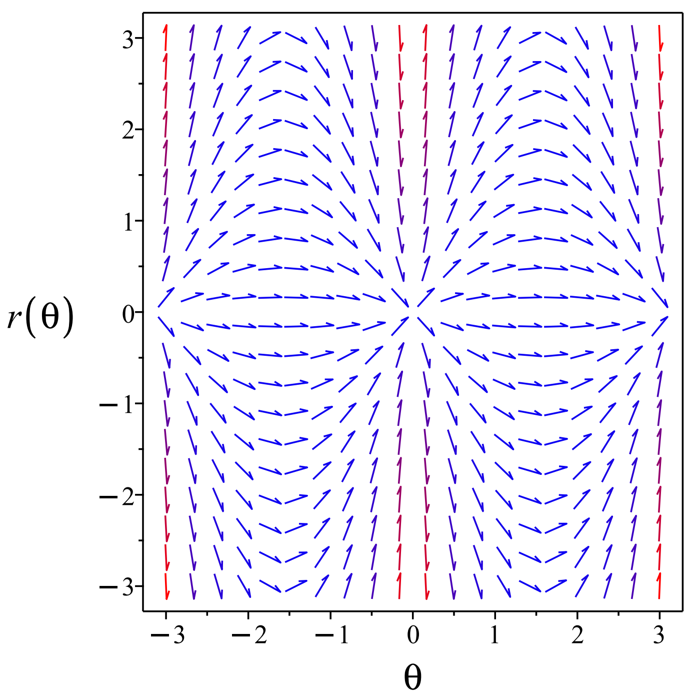


Figure 429: Slope field plot

Verification of solutions

$$r = c_2 \sin(\theta)$$

Verified OK.

8.17.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = r\cot(\theta)$$
$$r' = \omega(\theta, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2\xi_r - \omega_\theta\xi - \omega_r\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 248: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(\theta, r) &= 0 \\ \eta(\theta, r) &= \sin(\theta) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sin(\theta)} dy \end{aligned}$$

Which results in

$$S = \frac{r}{\sin(\theta)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r) S_r}{R_\theta + \omega(\theta, r) R_r} \tag{2}$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = r \cot(\theta)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 1 \\ R_r &= 0 \\ S_\theta &= -\csc(\theta) \cot(\theta) r \\ S_r &= \csc(\theta) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$\csc(\theta) r = c_1$$

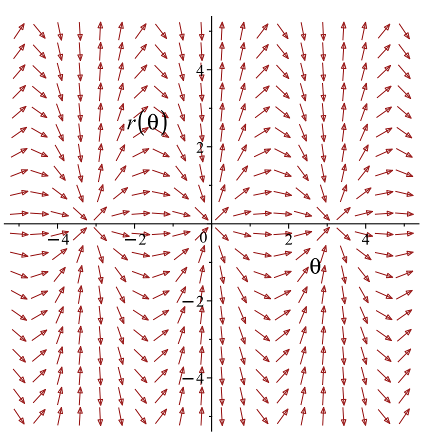
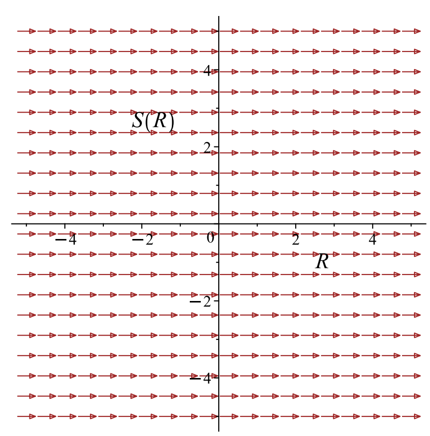
Which simplifies to

$$\csc(\theta) r = c_1$$

Which gives

$$r = \frac{c_1}{\csc(\theta)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{d\theta} = r\cot(\theta)$ 	$R = \theta$ $S = \csc(\theta) r$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$r = \frac{c_1}{\csc(\theta)} \tag{1}$$

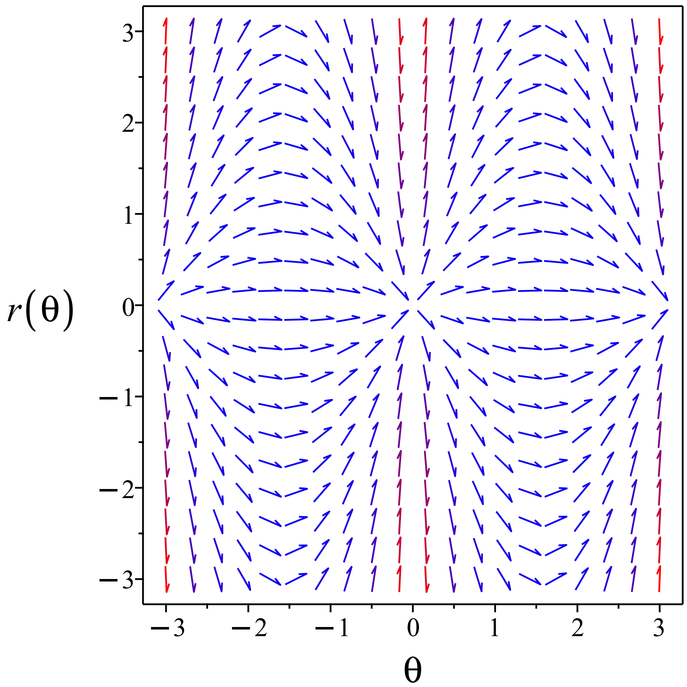


Figure 430: Slope field plot

Verification of solutions

$$r = \frac{c_1}{\csc(\theta)}$$

Verified OK.

8.17.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta, r)\,\mathrm{d}\theta + N(\theta, r)\,\mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{r}\right)\mathrm{d}r = (\cot(\theta))\,\mathrm{d}\theta$$
$$(-\cot(\theta))\,\mathrm{d}\theta + \left(\frac{1}{r}\right)\mathrm{d}r = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(\theta, r) = -\cot(\theta)$$
$$N(\theta, r) = \frac{1}{r}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}(-\cot(\theta)) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(\frac{1}{r}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(\theta, r)$

$$
\frac{\partial \phi}{\partial \theta} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial r} = N \tag{2}
$$

Integrating (1) w.r.t. θ gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta &= \int M \, \mathrm{d}\theta \\
\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta &= \int -\cot(\theta) \, \mathrm{d}\theta \\
\phi &= -\ln(\sin(\theta)) + f(r) \qquad (3)
\end{aligned}
$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$
\frac{\partial \phi}{\partial r} = 0 + f'(r) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \frac{1}{r}$. Therefore equation (4) becomes

$$
\frac{1}{r} = 0 + f'(r) \tag{5}
$$

Solving equation (5) for $f'(r)$ gives

$$
f'(r) = \frac{1}{r}
$$

Integrating the above w.r.t r gives

$$
\begin{aligned}
\int f'(r) \, \mathrm{d}r &= \int \left(\frac{1}{r}\right) \mathrm{d}r \\
f(r) &= \ln(r) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(r)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(\theta)) + \ln(r) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(\theta)) + \ln(r)$$

The solution becomes

$$r = e^{c_1} \sin(\theta)$$

Summary

The solution(s) found are the following

$$r = e^{c_1} \sin(\theta) \tag{1}$$

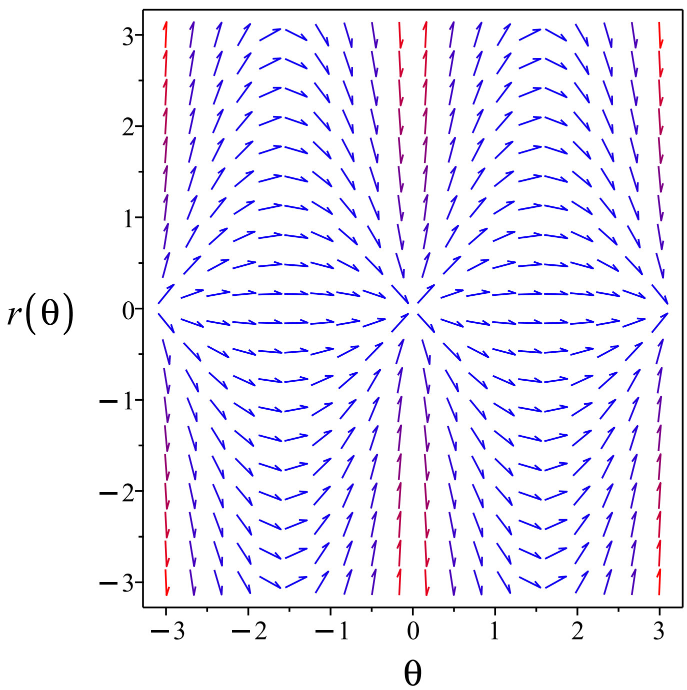


Figure 431: Slope field plot

Verification of solutions

$$r = e^{c_1} \sin(\theta)$$

Verified OK.

8.17.6 Maple step by step solution

Let's solve

$r' - r\cot(\theta) = 0$

- Highest derivative means the order of the ODE is 1

  r'

- Separate variables

  $\frac{r'}{r} = \cot(\theta)$

- Integrate both sides with respect to θ

  $\int \frac{r'}{r} d\theta = \int \cot(\theta) \, d\theta + c_1$

- Evaluate integral

  $\ln(r) = \ln(\sin(\theta)) + c_1$

- Solve for r

  $r = \mathrm{e}^{c_1} \sin(\theta)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(r(theta),theta)=r(theta)*cot(theta),r(theta), singsol=all)
```

$$r(\theta) = c_1 \sin(\theta)$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 15

```
DSolve[r'[\[Theta]]==r[\[Theta]]*Cot[\[Theta]],r[\[Theta]],\[Theta],IncludeSingularSolutions
```

$$r(\theta) \to c_1 \sin(\theta)$$
$$r(\theta) \to 0$$

8.18 problem 18

Internal problem ID [2050]
Internal file name [`OUTPUT/2050_Sunday_February_25_2024_06_47_14_AM_18780331/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(3x+4y)\,y'+y=-2x}$$

8.18.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(3x+4u(x)\,x)\,(u'(x)\,x+u(x))+u(x)\,x=-2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2(2u^2+2u+1)}{x\,(4u+3)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{2u^2+2u+1}{4u+3}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+2u+1}{4u+3}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{2u^2+2u+1}{4u+3}}\,du = \int -\frac{2}{x}\,dx$$
$$\ln\left(2u^2+2u+1\right) + \arctan\left(2u+1\right) = -2\ln\left(x\right) + c_2$$

The solution is

$$\ln\left(2u(x)^2 + 2u(x) + 1\right) + \arctan\left(2u(x)+1\right) + 2\ln\left(x\right) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{2y^2}{x^2} + \frac{2y}{x} + 1\right) + \arctan\left(\frac{2y}{x} + 1\right) + 2\ln\left(x\right) - c_2 = 0$$
$$\ln\left(\frac{2y^2}{x^2} + \frac{2y}{x} + 1\right) + \arctan\left(\frac{x+2y}{x}\right) + 2\ln\left(x\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{2y^2}{x^2} + \frac{2y}{x} + 1\right) + \arctan\left(\frac{x+2y}{x}\right) + 2\ln\left(x\right) - c_2 = 0 \tag{1}$$

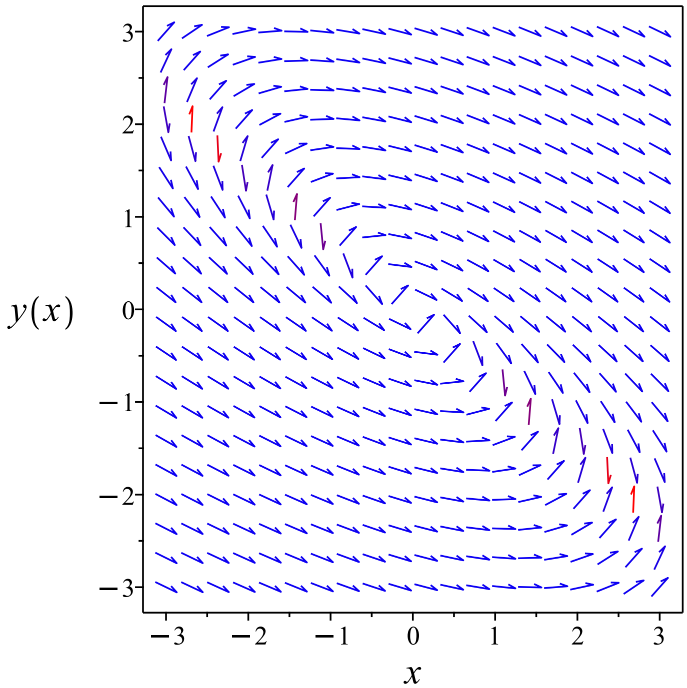


Figure 432: Slope field plot

Verification of solutions

$$\ln \left(\frac{2y^2}{x^2}+\frac{2y}{x}+1\right)+\arctan \left(\frac{x+2y}{x}\right)+2\ln \left(x\right)-c_2=0$$

Verified OK.

8.18.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2x+y}{3x+4y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(2x+y)(b_3-a_2)}{3x+4y} - \frac{(2x+y)^2 a_3}{(3x+4y)^2} - \left(-\frac{2}{3x+4y} + \frac{6x+3y}{(3x+4y)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{1}{3x+4y} + \frac{8x+4y}{(3x+4y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^2a_2 - 4x^2a_3 + 4x^2b_2 - 6x^2b_3 + 16xya_2 - 4xya_3 + 24xyb_2 - 16xyb_3 + 4y^2a_2 + 4y^2a_3 + 16y^2b_2 - 4y^2b_3 -}{(3x+4y)^2} = 0$$

Setting the numerator to zero gives

$$6x^2a_2 - 4x^2a_3 + 4x^2b_2 - 6x^2b_3 + 16xya_2 - 4xya_3 + 24xyb_2 - 16xyb_3 + 4y^2a_2 + 4y^2a_3 + 16y^2b_2 - 4y^2b_3 - 5xb_1 + 5ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$6a_2v_1^2 + 16a_2v_1v_2 + 4a_2v_2^2 - 4a_3v_1^2 - 4a_3v_1v_2 + 4a_3v_2^2 + 4b_2v_1^2 + 24b_2v_1v_2 + 16b_2v_2^2 - 6b_3v_1^2 - 16b_3v_1v_2 - 4b_3v_2^2 + 5a_1v_2 - 5b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(6a_2 - 4a_3 + 4b_2 - 6b_3) v_1^2 + (16a_2 - 4a_3 + 24b_2 - 16b_3) v_1 v_2 \\ - 5b_1 v_1 + (4a_2 + 4a_3 + 16b_2 - 4b_3) v_2^2 + 5a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}5a_1 &= 0\\ -5b_1 &= 0\\ 4a_2 + 4a_3 + 16b_2 - 4b_3 &= 0\\ 6a_2 - 4a_3 + 4b_2 - 6b_3 &= 0\\ 16a_2 - 4a_3 + 24b_2 - 16b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= -2b_2 + b_3\\ a_3 &= -2b_2\\ b_1 &= 0\\ b_2 &= b_2\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi\\ &= y - \left(-\frac{2x + y}{3x + 4y}\right)(x)\\ &= \frac{2x^2 + 4yx + 4y^2}{3x + 4y}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x^2+4yx+4y^2}{3x+4y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + 2yx + 2y^2)}{2} + \frac{\arctan\left(\frac{2x+4y}{2x}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x + y}{3x + 4y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x + y}{2x^2 + 4yx + 4y^2} \\ S_y &= \frac{3x + 4y}{2x^2 + 4yx + 4y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

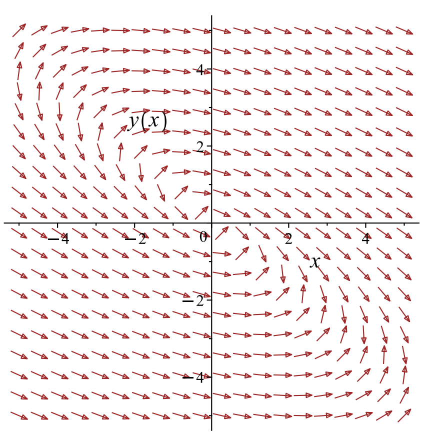
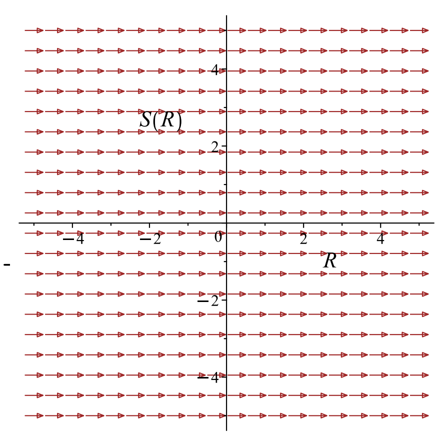
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2y^2+2yx+x^2)}{2} + \frac{\arctan\left(\frac{x+2y}{x}\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(2y^2+2yx+x^2)}{2} + \frac{\arctan\left(\frac{x+2y}{x}\right)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+y}{3x+4y}$ 	$R = x$ $S = \frac{\ln(x^2+2yx+2y^2)}{2}$ -	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln \left(2y^{2}+2yx+x^{2}\right)}{2}+\frac{\arctan \left(\frac{x+2y}{x}\right)}{2} = c_1 \tag{1}$$

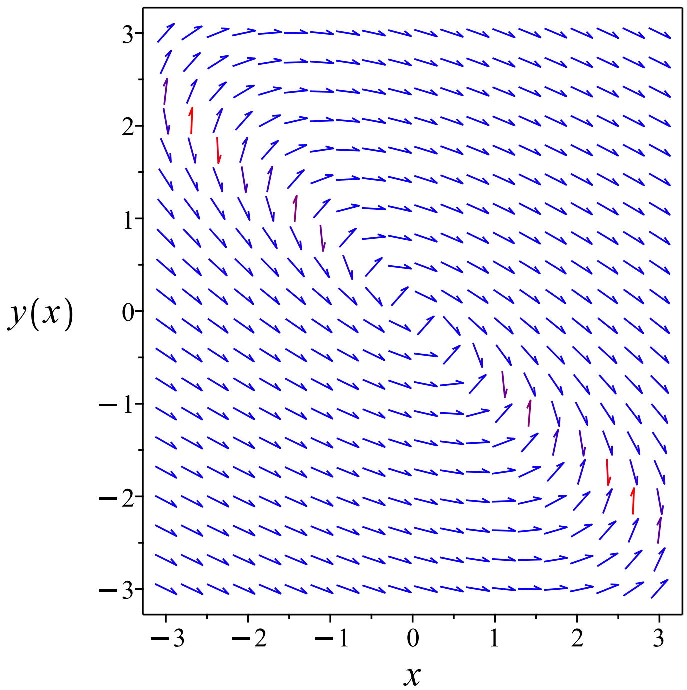


Figure 433: Slope field plot

Verification of solutions

$$\frac{\ln \left(2y^{2}+2yx+x^{2}\right)}{2}+\frac{\arctan \left(\frac{x+2y}{x}\right)}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve((3*x+4*y(x))*diff(y(x),x)+(y(x)+2*x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x(\tan(\mathrm{RootOf}(-\ln(2) + \ln(\sec(\_Z)^2) + \_Z + 2\ln(x) + 2c_1)) - 1)}{2}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 41

```
DSolve[(3*x+4*y[x])*y'[x]+(y[x]+2*x)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\arctan\left(\frac{2y(x)}{x} + 1\right) + \log\left(\frac{2y(x)^2}{x^2} + \frac{2y(x)}{x} + 1\right) = -2\log(x) + c_1, y(x)\right]$$

8.19 problem 19

Internal problem ID [2051]
Internal file name [`OUTPUT/2051_Sunday_February_25_2024_06_47_15_AM_79645682/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{-y^3 + 3y'y^2x = -2x^3 + 3x}$$

8.19.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-2x^3 + y^3 + 3x}{3y^2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 251: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3}{3x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-2x^3 + y^3 + 3x}{3y^2 x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^3}{3x^2} \\ S_y &= \frac{y^2}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-2x^2 + 3}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{-2R^2 + 3}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R^2}{3} + \ln(R) + c_1 \tag{4}$$

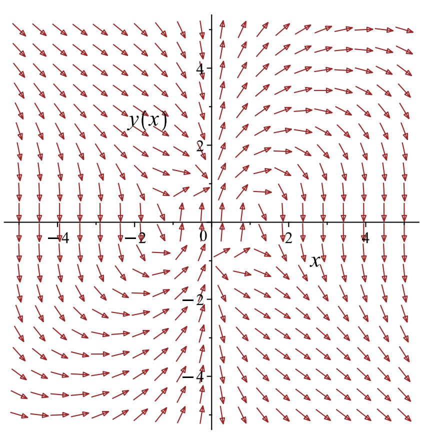
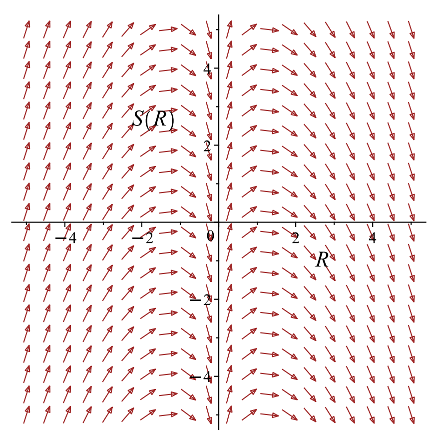
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3}{3x} = -\frac{x^2}{3} + \ln(x) + c_1$$

Which simplifies to

$$\frac{y^3}{3x} = -\frac{x^2}{3} + \ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-2x^3+y^3+3x}{3y^2x}$ 	$R = x$ $S = \frac{y^3}{3x}$	$\frac{dS}{dR} = \frac{-2R^2+3}{3R}$ 

Summary

The solution(s) found are the following

$$\frac{y^3}{3x} = -\frac{x^2}{3} + \ln(x) + c_1 \tag{1}$$

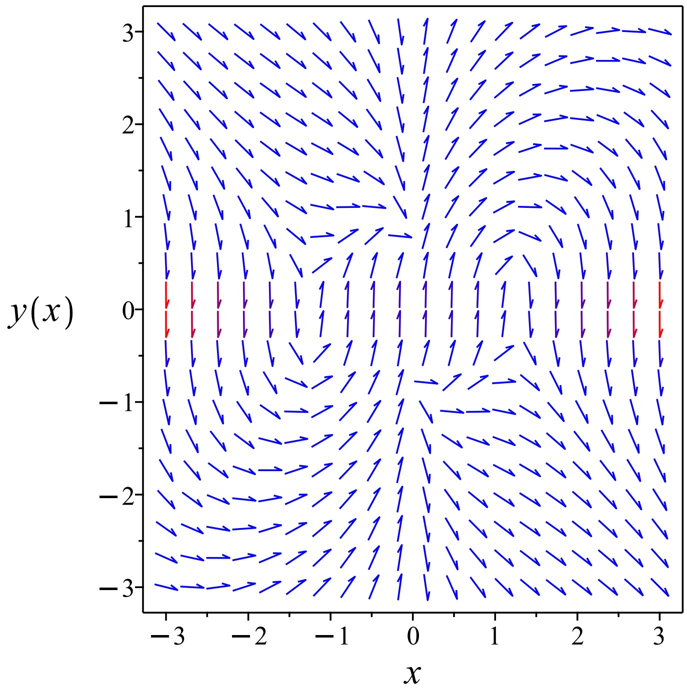


Figure 434: Slope field plot

Verification of solutions

$$\frac{y^3}{3x} = -\frac{x^2}{3} + \ln(x) + c_1$$

Verified OK.

8.19.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-2x^3 + y^3 + 3x}{3y^2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{3x}y + \frac{-2x^3 + 3x}{3x}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{3x} \\ f_1(x) &= \frac{-2x^3+3x}{3x} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = \frac{y^3}{3x} + \frac{-2x^3+3x}{3x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= \frac{w(x)}{3x} + \frac{-2x^3+3x}{3x} \\ w' &= \frac{w}{x} + \frac{-2x^3+3x}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -2x^2+3 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -2x^2 + 3$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-2x^2+3\right) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(-2x^2+3\right) \\ d\left(\frac{w}{x}\right) &= \left(\frac{-2x^2+3}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int \frac{-2x^2+3}{x}\, dx \\ \frac{w}{x} &= -x^2 + 3\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x\left(-x^2 + 3\ln(x)\right) + c_1 x$$

which simplifies to

$$w(x) = x\left(-x^2 + 3\ln(x) + c_1\right)$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = x\left(-x^2 + 3\ln(x) + c_1\right)$$

Solving for y gives

$$\begin{aligned}y(x) &= \left(x\left(-x^2 + 3\ln(x) + c_1\right)\right)^{\frac{1}{3}} \\ y(x) &= \frac{\left(x(-x^2 + 3\ln(x) + c_1)\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2} \\ y(x) &= -\frac{\left(x(-x^2 + 3\ln(x) + c_1)\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(x\left(-x^2+3\ln\left(x\right)+c_1\right)\right)^{\frac{1}{3}} \tag{1}$$
$$y = \frac{\left(x\left(-x^2+3\ln\left(x\right)+c_1\right)\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2} \tag{2}$$
$$y = -\frac{\left(x\left(-x^2+3\ln\left(x\right)+c_1\right)\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2} \tag{3}$$

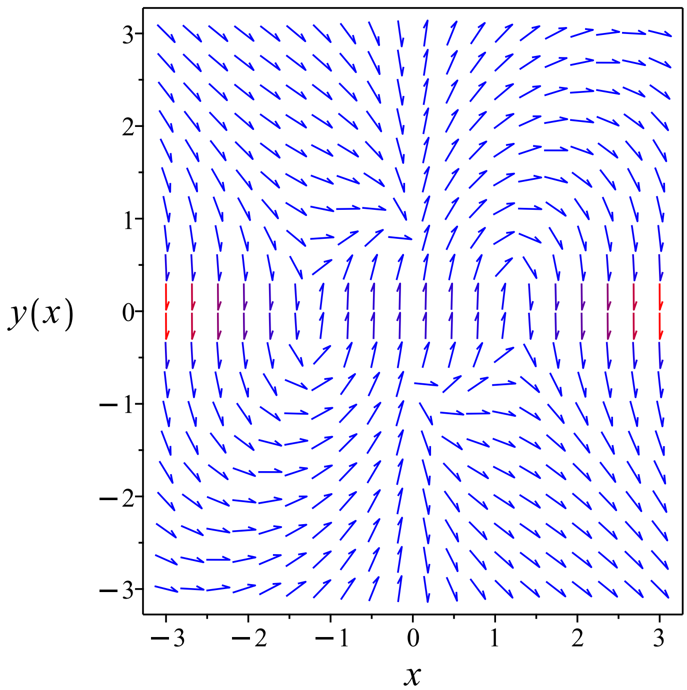


Figure 435: Slope field plot

Verification of solutions

$$y = \left(x\left(-x^2 + 3\ln(x) + c_1\right)\right)^{\frac{1}{3}}$$

Verified OK.

$$y = \frac{\left(x(-x^2 + 3\ln(x) + c_1)\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$

Verified OK.

$$y = -\frac{\left(x(-x^2 + 3\ln(x) + c_1)\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$

Verified OK.

8.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x\,y^2)\,\mathrm{d}y &= (-2x^3 + y^3 + 3x)\,\mathrm{d}x\\ (2x^3 - y^3 - 3x)\,\mathrm{d}x + (3x\,y^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x^3 - y^3 - 3x\\ N(x,y) &= 3x\,y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x^3 - y^3 - 3x)\\ &= -3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x\,y^2)\\ &= 3y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{3x\,y^2}\left((-3y^2) - (3y^2)\right)\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}\left(2x^3 - y^3 - 3x\right) \\
&= \frac{2x^3 - y^3 - 3x}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}\left(3x\,y^2\right) \\
&= \frac{3y^2}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2x^3 - y^3 - 3x}{x^2}\right) + \left(\frac{3y^2}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{2x^3 - y^3 - 3x}{x^2} \, \mathrm{d}x$$
$$\phi = x^2 - 3\ln(x) + \frac{y^3}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{3y^2}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3y^2}{x}$. Therefore equation (4) becomes

$$\frac{3y^2}{x} = \frac{3y^2}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2 - 3\ln(x) + \frac{y^3}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2 - 3\ln(x) + \frac{y^3}{x}$$

Summary

The solution(s) found are the following

$$x^2 - 3\ln(x) + \frac{y^3}{x} = c_1 \tag{1}$$

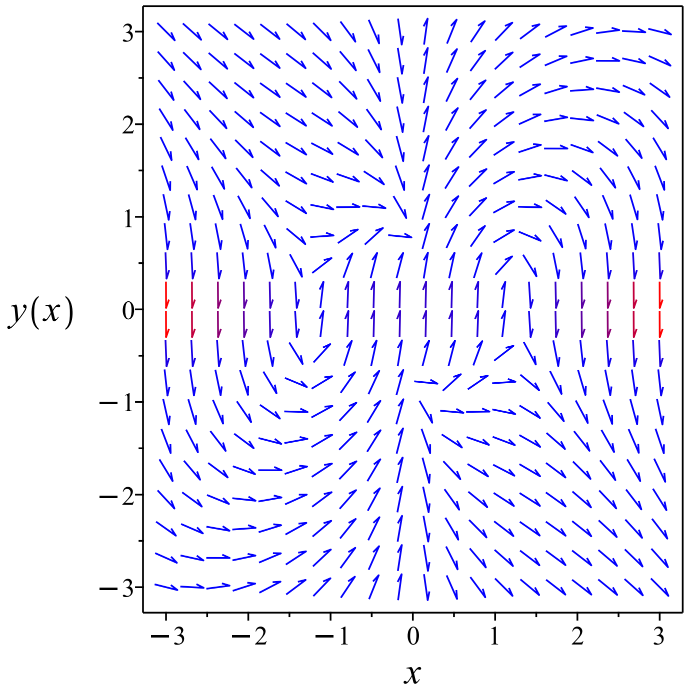


Figure 436: Slope field plot

Verification of solutions

$$x^2 - 3\ln(x) + \frac{y^3}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 75

```
dsolve((2*x^3-y(x)^3-3*x)+(3*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(\left(-x^2 + 3\ln(x) + c_1\right) x\right)^{\frac{1}{3}}$$
$$y(x) = -\frac{\left(\left(-x^2 + 3\ln(x) + c_1\right) x\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$
$$y(x) = \frac{\left(\left(-x^2 + 3\ln(x) + c_1\right) x\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.354 (sec). Leaf size: 80

```
DSolve[(2*x^3-y[x]^3-3*x)+(3*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{x\left(-x^2 + 3\log(x) + c_1\right)}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt[3]{x\left(-x^2 + 3\log(x) + c_1\right)}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{x\left(-x^2 + 3\log(x) + c_1\right)}$$

8.20 problem 20

Internal problem ID [2052]
Internal file name [`OUTPUT/2052_Sunday_February_25_2024_06_47_16_AM_39724273/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y'x - y - \sqrt{x^2+y^2} = 0}$$

8.20.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y+\sqrt{x^2+y^2}}{x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(y+\sqrt{x^2+y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{x^2+y^2}\right)^2 a_3}{x^2} - \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{y+\sqrt{x^2+y^2}}{x^2}\right)(x a_2 + y a_3 + a_1) - \frac{\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)(x b_2 + y b_3 + b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\left(x^2+y^2\right)^{\frac{3}{2}} a_3 + x^3 a_2 - x^3 b_3 + 2x^2 y a_3 + x^2 y b_2 + y^3 a_3 + \sqrt{x^2+y^2}\, x b_1 - \sqrt{x^2+y^2}\, y a_1 + x y b_1 - y^2 a_1}{\sqrt{x^2+y^2}\, x^2} = 0$$

Setting the numerator to zero gives

$$-\left(x^2+y^2\right)^{\frac{3}{2}} a_3 - x^3 a_2 + x^3 b_3 - 2x^2 y a_3 - x^2 y b_2 - y^3 a_3 - \sqrt{x^2+y^2}\, x b_1 + \sqrt{x^2+y^2}\, y a_1 - x y b_1 + y^2 a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-\left(x^2+y^2\right)^{\frac{3}{2}} a_3 + \left(x^2+y^2\right) x b_3 - \left(x^2+y^2\right) y a_3 - x^3 a_2 - x^2 y a_3 - x^2 y b_2 - x\, y^2 b_3 + \left(x^2+y^2\right) a_1 - \sqrt{x^2+y^2}\, x b_1 + \sqrt{x^2+y^2}\, y a_1 - x^2 a_1 - x y b_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-x^3 a_2 + x^3 b_3 - x^2 \sqrt{x^2+y^2}\, a_3 - 2x^2 y a_3 - x^2 y b_2 - \sqrt{x^2+y^2}\, y^2 a_3 - y^3 a_3 - \sqrt{x^2+y^2}\, x b_1 - x y b_1 + \sqrt{x^2+y^2}\, y a_1 + y^2 a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$-v_1^3 a_2 - 2v_1^2 v_2 a_3 - v_1^2 v_3 a_3 - v_2^3 a_3 - v_3 v_2^2 a_3 - v_1^2 v_2 b_2 + v_1^3 b_3 + v_2^2 a_1 + v_3 v_2 a_1 - v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$(b_3 - a_2)\, v_1^3 + (-2a_3 - b_2)\, v_1^2 v_2 - v_1^2 v_3 a_3 - v_1 v_2 b_1 - v_3 v_1 b_1 - v_2^3 a_3 - v_3 v_2^2 a_3 + v_2^2 a_1 + v_3 v_2 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2a_3 - b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y+\sqrt{x^2+y^2}}{x}\right)(x) \\
&= -\sqrt{x^2+y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{x^2+y^2}} dy
\end{aligned}$$

Which results in

$$S = -\ln\left(y+\sqrt{x^2+y^2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + \sqrt{x^2+y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x}{\sqrt{x^2+y^2}\,\left(y+\sqrt{x^2+y^2}\right)} \\
S_y &= -\frac{1}{\sqrt{x^2+y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2\left(\sqrt{x^2+y^2}\,y + x^2 + y^2\right)}{x\sqrt{x^2+y^2}\,\left(y+\sqrt{x^2+y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y + \sqrt{x^2+y^2}\right) = -2\ln(x) + c_1$$

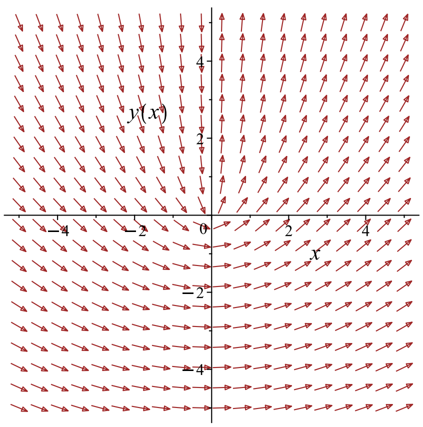
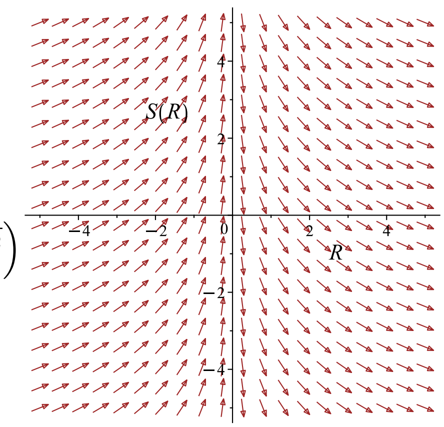
Which simplifies to

$$-\ln\left(y + \sqrt{x^2+y^2}\right) = -2\ln(x) + c_1$$

Which gives

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ 	$R = x$ $S = -\ln\left(y + \sqrt{x^2+y^2}\right)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2} \tag{1}$$

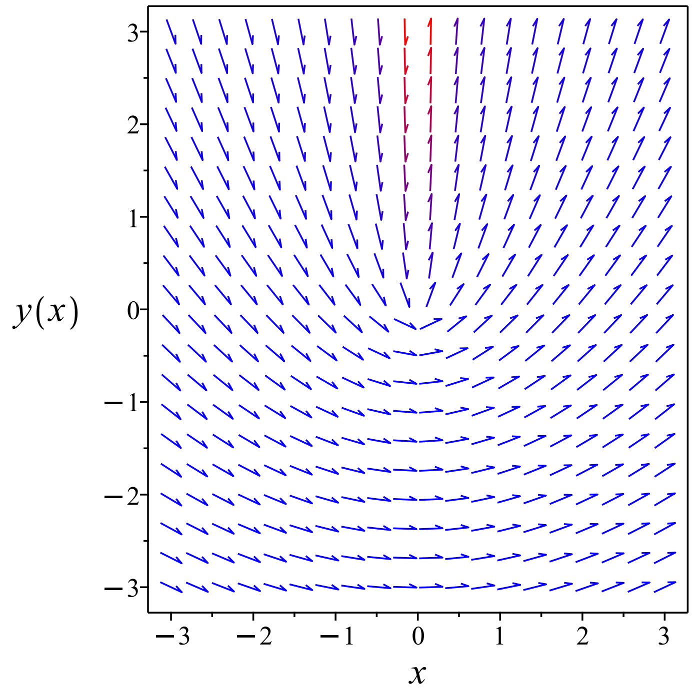


Figure 437: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x*diff(y(x),x)-y(x)-sqrt(x^2+y(x)^2)=0,y(x), singsol=all)
```

$$\frac{-c_1 x^2 + y(x) + \sqrt{x^2 + y(x)^2}}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.349 (sec). Leaf size: 27

```
DSolve[x*y'[x]-y[x]-Sqrt[x^2+y[x]^2]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-c_1} \left(-1 + e^{2c_1} x^2\right)$$

8.21 problem 22

Internal problem ID [2053]
Internal file name [`OUTPUT/2053_Sunday_February_25_2024_06_47_18_AM_59643750/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \cos(x)^2 \cos(y) = 0}$$

8.21.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \cos(y)\cos(x)^2 \end{aligned}$$

Where $f(x) = \cos(x)^2$ and $g(y) = \cos(y)$. Integrating both sides gives

$$\frac{1}{\cos(y)}\, dy = \cos(x)^2 \, dx$$

$$\int \frac{1}{\cos(y)}\, dy = \int \cos(x)^2 \, dx$$

$$\ln(\sec(y) + \tan(y)) = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + c_1$$

Raising both side to exponential gives

$$\sec(y) + \tan(y) = e^{\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + c_1}$$

Which simplifies to

$$\sec(y) + \tan(y) = c_2 e^{\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}}$$

Summary

The solution(s) found are the following

$$y = \arctan\left(\frac{e^{x+2c_1+\frac{\sin(2x)}{2}} c_2^2 - 1}{e^{x+2c_1+\frac{\sin(2x)}{2}} c_2^2 + 1}, \frac{2c_2 e^{\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + c_1}}{e^{x+2c_1+\frac{\sin(2x)}{2}} c_2^2 + 1}\right) \tag{1}$$

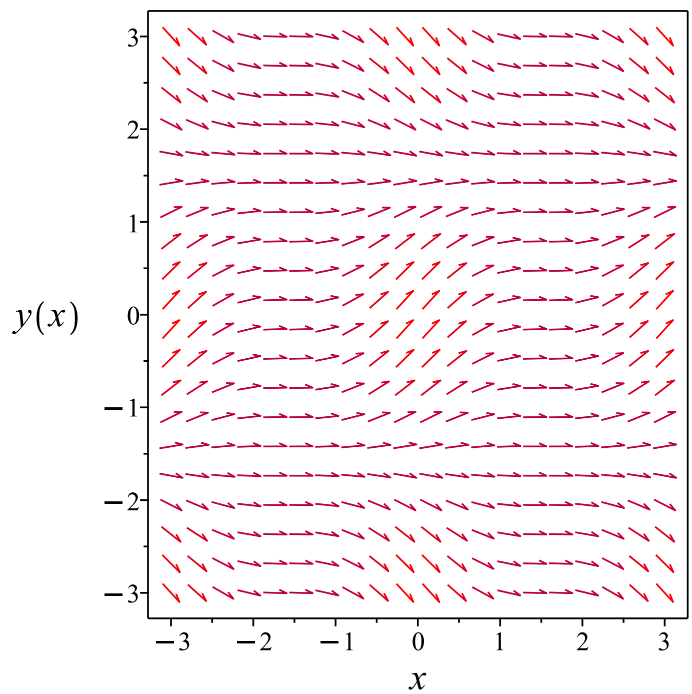


Figure 438: Slope field plot

Verification of solutions

$$y = \arctan\left(\frac{e^{x+2c_1+\frac{\sin(2x)}{2}} c_2^2 - 1}{e^{x+2c_1+\frac{\sin(2x)}{2}} c_2^2 + 1}, \frac{2c_2 e^{\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + c_1}}{e^{x+2c_1+\frac{\sin(2x)}{2}} c_2^2 + 1}\right)$$

Verified OK.

8.21.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \cos(y)\cos(x)^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 253: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{\cos(x)^2} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{\cos(x)^2}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \cos(y)\cos(x)^2
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0\\
R_y &= 1\\
S_x &= \frac{\cos(2x)}{2} + \frac{1}{2}\\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(\sec(R) + \tan(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x}{2} + \frac{\sin(2x)}{4} = \ln(\sec(y) + \tan(y)) + c_1$$

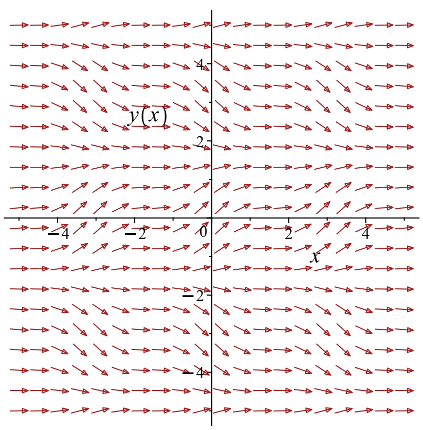
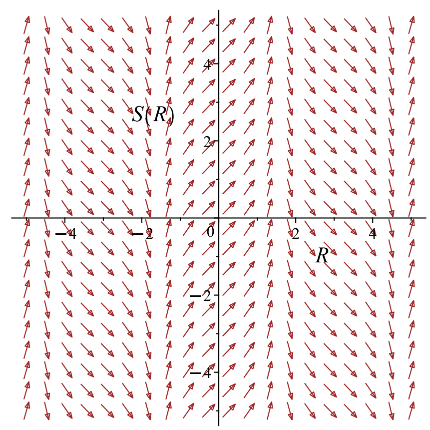
Which simplifies to

$$\frac{x}{2} + \frac{\sin(2x)}{4} = \ln(\sec(y) + \tan(y)) + c_1$$

Which gives

$$y = \arctan\left(\frac{e^{x+\frac{\sin(2x)}{2}-2c_1} - 1}{e^{x+\frac{\sin(2x)}{2}-2c_1} + 1}, \frac{2\,e^{\frac{x}{2}+\frac{\sin(2x)}{4}-c_1}}{e^{x+\frac{\sin(2x)}{2}-2c_1} + 1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \cos(y)\cos(x)^2$ 	$R = y$ $S = \frac{x}{2} + \frac{\sin(2x)}{4}$	$\frac{dS}{dR} = \sec(R)$ 

<u>Summary</u>

The solution(s) found are the following

$$y = \arctan\left(\frac{e^{x+\frac{\sin(2x)}{2}-2c_1} - 1}{e^{x+\frac{\sin(2x)}{2}-2c_1} + 1}, \frac{2\,e^{\frac{x}{2}+\frac{\sin(2x)}{4}-c_1}}{e^{x+\frac{\sin(2x)}{2}-2c_1} + 1}\right) \tag{1}$$

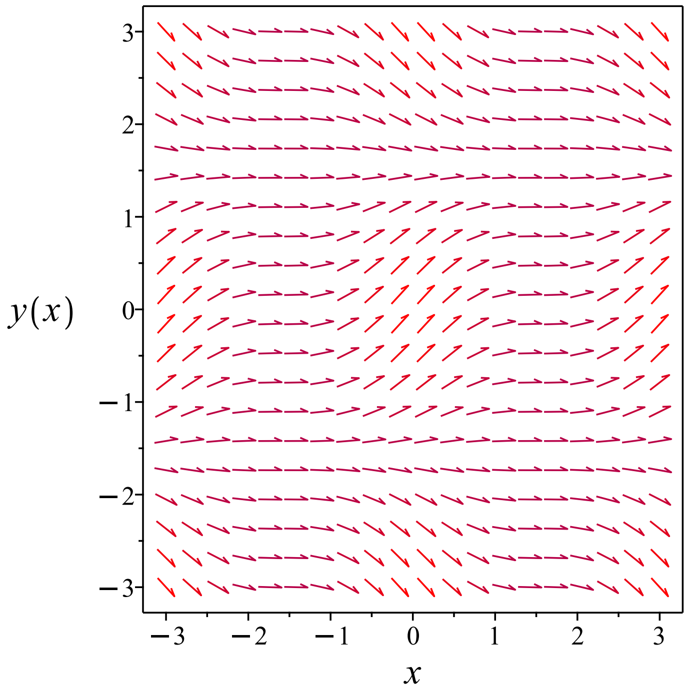


Figure 439: Slope field plot

Verification of solutions

$$y = \arctan\left(\frac{e^{x+\frac{\sin(2x)}{2}-2c_1}-1}{e^{x+\frac{\sin(2x)}{2}-2c_1}+1}, \frac{2\,e^{\frac{x}{2}+\frac{\sin(2x)}{4}-c_1}}{e^{x+\frac{\sin(2x)}{2}-2c_1}+1}\right)$$

Verified OK.

8.21.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{\cos(y)}\right) \mathrm{d}y &= \left(\cos(x)^2\right) \mathrm{d}x \\
\left(-\cos(x)^2\right) \mathrm{d}x + \left(\frac{1}{\cos(y)}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\cos(x)^2 \\
N(x, y) &= \frac{1}{\cos(y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\cos(x)^2\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\cos\left(y\right)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\cos\left(x\right)^{2} \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x}{2} - \frac{\sin\left(2x\right)}{4} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\cos(y)}$. Therefore equation (4) becomes

$$
\frac{1}{\cos\left(y\right)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{1}{\cos\left(y\right)} \\
&= \sec\left(y\right)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\sec(y))\,\mathrm{d}y$$
$$f(y) = \ln(\sec(y) + \tan(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x}{2} - \frac{\sin(2x)}{4} + \ln(\sec(y) + \tan(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x}{2} - \frac{\sin(2x)}{4} + \ln(\sec(y) + \tan(y))$$

Summary

The solution(s) found are the following

$$-\frac{x}{2} - \frac{\sin(2x)}{4} + \ln(\sec(y) + \tan(y)) = c_1 \tag{1}$$

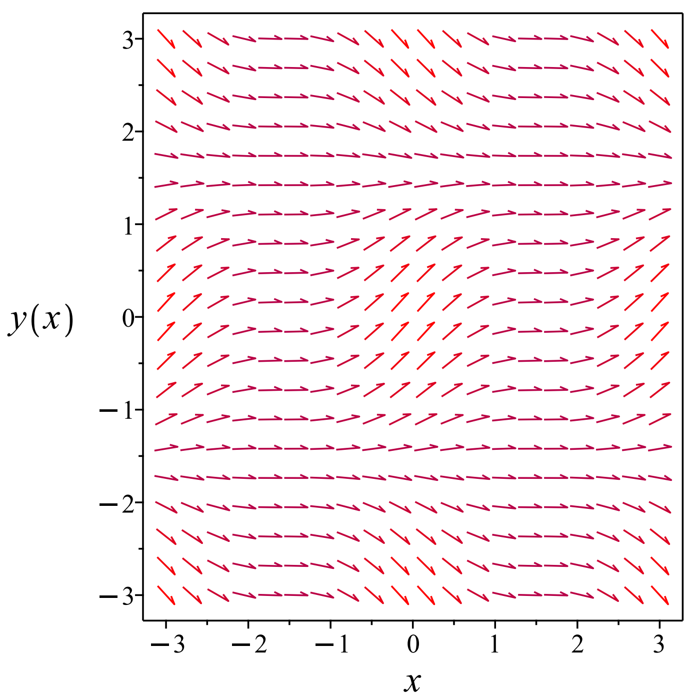


Figure 440: Slope field plot

Verification of solutions

$$-\frac{x}{2}-\frac{\sin(2x)}{4}+\ln(\sec(y)+\tan(y))=c_1$$

Verified OK.

8.21.4 Maple step by step solution

Let's solve

$y'-\cos(x)^2\cos(y)=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(y)}=\cos(x)^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{\cos(y)}dx=\int \cos(x)^2 dx+c_1$

- Evaluate integral

  $\ln(\sec(y)+\tan(y))=\frac{\cos(x)\sin(x)}{2}+\frac{x}{2}+c_1$

- Solve for y

  $y=\arctan\left(\frac{\left(\mathrm{e}^{\frac{\cos(x)\sin(x)}{2}+\frac{x}{2}+c_1}\right)^2-1}{\left(\mathrm{e}^{\frac{\cos(x)\sin(x)}{2}+\frac{x}{2}+c_1}\right)^2+1},\frac{2\,\mathrm{e}^{\frac{\cos(x)\sin(x)}{2}+\frac{x}{2}+c_1}}{\left(\mathrm{e}^{\frac{\cos(x)\sin(x)}{2}+\frac{x}{2}+c_1}\right)^2+1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 69

```
dsolve(diff(y(x),x)=cos(y(x))*cos(x)^2,y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{c_1^2 \mathrm{e}^{x+\frac{\sin(2x)}{2}}-1}{c_1^2 \mathrm{e}^{x+\frac{\sin(2x)}{2}}+1}, \frac{2c_1 \mathrm{e}^{\frac{x}{2}+\frac{\sin(2x)}{4}}}{c_1^2 \mathrm{e}^{x+\frac{\sin(2x)}{2}}+1}\right)$$

✓ Solution by Mathematica

Time used: 0.942 (sec). Leaf size: 41

```
DSolve[y'[x]==Cos[y[x]]*Cos[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\arctan\left(\tanh\left(\frac{1}{8}(2x+\sin(2x)+c_1)\right)\right)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

8.22 problem 23

Internal problem ID [2054]
Internal file name [`OUTPUT/2054_Sunday_February_25_2024_06_47_20_AM_61354485/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (2x + 3y - 1)\, y' = -x}$$

8.22.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{X + x_0 + Y(X) + y_0}{2X + 2x_0 + 3Y(X) + 3y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{X + Y(X)}{2X + 3Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{X+Y}{2X+3Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -X - Y$ and $N = 2X + 3Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u-1}{3u+2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)-1}{3u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-u(X)-1}{3u(X)+2} - u(X)}{X} = 0$$

Or

$$3\left(\frac{d}{dX}u(X)\right)Xu(X) + 2\left(\frac{d}{dX}u(X)\right)X + 3u(X)^2 + 3u(X) + 1 = 0$$

Or

$$1 + X(3u(X)+2)\left(\frac{d}{dX}u(X)\right) + 3u(X)^2 + 3u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{3u^2+3u+1}{X(3u+2)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{3u^2+3u+1}{3u+2}$. Integrating both sides gives

$$\frac{1}{\frac{3u^2+3u+1}{3u+2}}\, du = -\frac{1}{X}\, dX$$

$$\int \frac{1}{\frac{3u^2+3u+1}{3u+2}}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{\ln\left(3u^2+3u+1\right)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{(6u+3)\sqrt{3}}{3}\right)}{3} = -\ln\left(X\right) + c_2$$

The solution is

$$\frac{\ln\left(3u(X)^2+3u(X)+1\right)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{(6u(X)+3)\sqrt{3}}{3}\right)}{3} + \ln\left(X\right) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{3Y(X)^2}{X^2} + \frac{3Y(X)}{X} + 1\right)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{\left(\frac{6Y(X)}{X}+3\right)\sqrt{3}}{3}\right)}{3} + \ln\left(X\right) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{3Y(X)^2}{X^2} + \frac{3Y(X)}{X} + 1\right)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{(2Y(X)+X)\sqrt{3}}{X}\right)}{3} + \ln\left(X\right) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 1$$
$$X = x - 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{3(y-1)^2}{(x+1)^2} + \frac{3y-3}{x+1} + 1\right)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{(x+2y-1)\sqrt{3}}{x+1}\right)}{3} + \ln\left(x+1\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{3(y-1)^2}{(x+1)^2}+\frac{3y-3}{x+1}+1\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{(x+2y-1)\sqrt{3}}{x+1}\right)}{3}+\ln(x+1)-c_2=0 \tag{1}$$

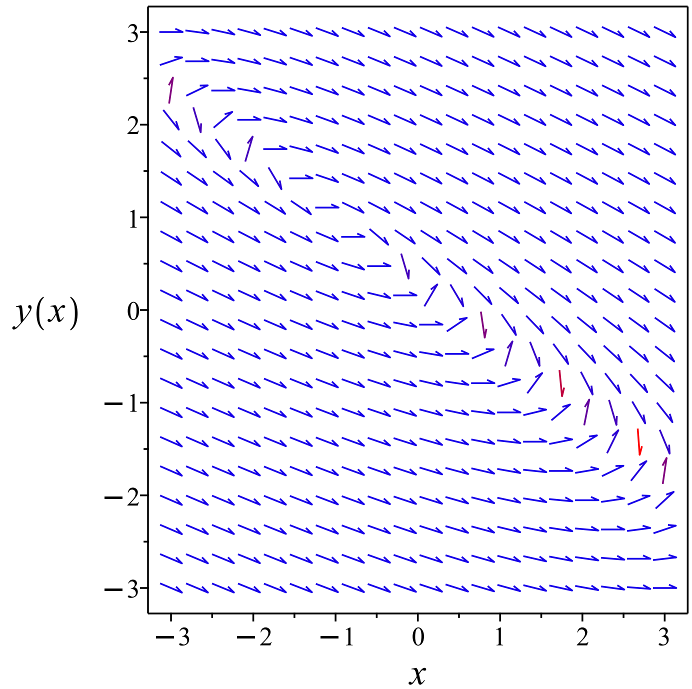


Figure 441: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{3(y-1)^2}{(x+1)^2}+\frac{3y-3}{x+1}+1\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{(x+2y-1)\sqrt{3}}{x+1}\right)}{3}+\ln(x+1)-c_2=0$$

Verified OK.

8.22.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x+y}{2x+3y-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(x+y)(b_3-a_2)}{2x+3y-1} - \frac{(x+y)^2 a_3}{(2x+3y-1)^2} \\ &- \left(-\frac{1}{2x+3y-1} + \frac{2x+2y}{(2x+3y-1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{1}{2x+3y-1} + \frac{3x+3y}{(2x+3y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 + 3x^2b_2 - 2x^2b_3 + 6xya_2 - 2xya_3 + 12xyb_2 - 6xyb_3 + 3y^2a_2 + 9y^2b_2 - 3y^2b_3 - 2xa_2 - xb_1}{(2x+3y-1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &2x^2a_2 - x^2a_3 + 3x^2b_2 - 2x^2b_3 + 6xya_2 - 2xya_3 + 12xyb_2 - 6xyb_3 + 3y^2a_2 + 9y^2b_2 \\ &- 3y^2b_3 - 2xa_2 - xb_1 - 5xb_2 + xb_3 + ya_1 - ya_2 - ya_3 - 6yb_2 - a_1 - b_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2a_2v_1^2 + 6a_2v_1v_2 + 3a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + 3b_2v_1^2 + 12b_2v_1v_2 \\
&+ 9b_2v_2^2 - 2b_3v_1^2 - 6b_3v_1v_2 - 3b_3v_2^2 + a_1v_2 - 2a_2v_1 - a_2v_2 \\
&- a_3v_2 - b_1v_1 - 5b_2v_1 - 6b_2v_2 + b_3v_1 - a_1 - b_1 + b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(2a_2 - a_3 + 3b_2 - 2b_3)\, v_1^2 + (6a_2 - 2a_3 + 12b_2 - 6b_3)\, v_1v_2 + (-2a_2 - b_1 - 5b_2 + b_3)\, v_1 \\
&+ (3a_2 + 9b_2 - 3b_3)\, v_2^2 + (a_1 - a_2 - a_3 - 6b_2)\, v_2 - a_1 - b_1 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-a_1 - b_1 + b_2 &= 0 \\
3a_2 + 9b_2 - 3b_3 &= 0 \\
a_1 - a_2 - a_3 - 6b_2 &= 0 \\
-2a_2 - b_1 - 5b_2 + b_3 &= 0 \\
2a_2 - a_3 + 3b_2 - 2b_3 &= 0 \\
6a_2 - 2a_3 + 12b_2 - 6b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= b_3 \\
a_2 &= -3b_2 + b_3 \\
a_3 &= -3b_2 \\
b_1 &= b_2 - b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -3x - 3y \\
\eta &= x + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x + 1 - \left(-\frac{x+y}{2x+3y-1}\right)(-3x-3y) \\
&= \frac{-x^2 - 3yx - 3y^2 + x + 3y - 1}{2x+3y-1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2-3yx-3y^2+x+3y-1}{2x+3y-1}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln\left(x^2 + 3yx + 3y^2 - x - 3y + 1\right)}{2} + \frac{2\left(-\frac{x}{2} - \frac{1}{2}\right)\sqrt{3}\,\arctan\left(\frac{(6y+3x-3)\sqrt{3}}{3+3x}\right)}{3\,(x+1)}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{x+y}{2x+3y-1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-x-y}{3y^2+(3x-3)\,y+x^2-x+1} \\
S_y &= \frac{-2x-3y+1}{x^2+(3y-1)\,x+3y^2-3y+1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

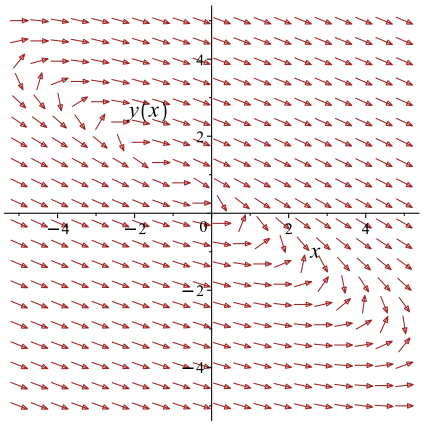
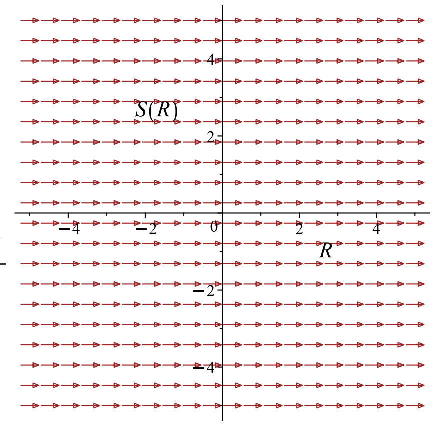
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{\ln\left(x^2+(3y-1)\,x+3y^2-3y+1\right)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(x+2y-1)\sqrt{3}}{x+1}\right)}{3} = c_1
$$

Which simplifies to

$$
-\frac{\ln\left(x^2+(3y-1)\,x+3y^2-3y+1\right)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(x+2y-1)\sqrt{3}}{x+1}\right)}{3} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{2x+3y-1}$ 	$R = x$ $S = -\frac{\ln\left(x^2 + (3y-1)\,x}{2}$	$\frac{dS}{dR} = 0$  $\frac{(2y+x-1)}{x+1}$

Summary

The solution(s) found are the following

$$-\frac{\ln\left(x^2 + (3y-1)\,x + 3y^2 - 3y + 1\right)}{2} - \frac{\sqrt{3}\,\arctan\left(\frac{(x+2y-1)\sqrt{3}}{x+1}\right)}{3} = c_1 \tag{1}$$

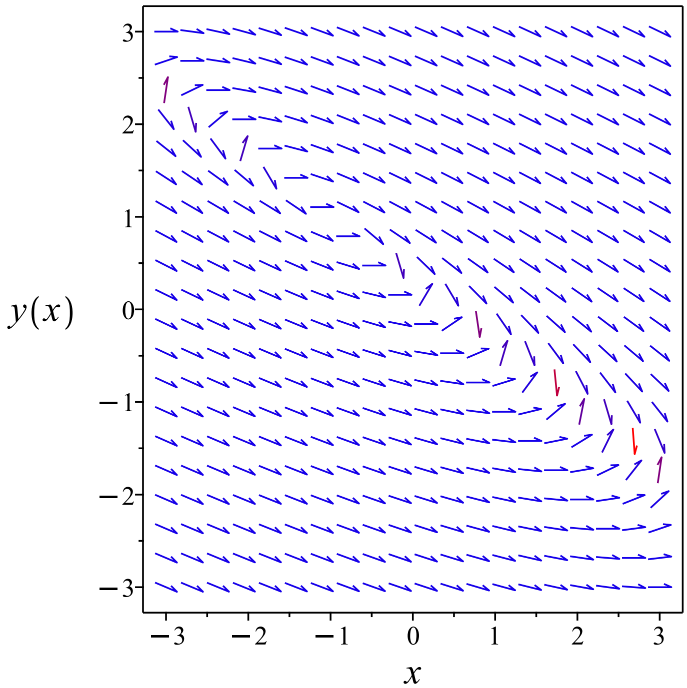


Figure 442: Slope field plot

Verification of solutions

$$-\frac{\ln\left(x^2+(3y-1)\,x+3y^2-3y+1\right)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(x+2y-1)\sqrt{3}}{x+1}\right)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.265 (sec). Leaf size: 51

```
dsolve((x+y(x))+(2*x+3*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{2} - \frac{x}{2} + \frac{\sqrt{3}\,(x+1)\tan\left(\operatorname{RootOf}\left(\sqrt{3}\ln\left(\sec\left(\_Z\right)^2 (x+1)^2\right) - 2\sqrt{3}\ln(2) + 2\sqrt{3}\,c_1 + 2\_Z\right)\right)}{6}$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 73

```
DSolve[(x+y[x])+(2*x+3*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\arctan\left(\frac{\sqrt{3}(y(x)-1)}{3y(x)+2x-1}\right)}{\sqrt{3}} + \frac{1}{2}\log\left(\frac{3(x^2+3y(x)^2+3(x-1)y(x)-x+1)}{(x+1)^2}\right) + \log(x+1) + c_1 = 0, y(x)\right]$$

8.23 problem 24

Internal problem ID [2055]
Internal file name [`OUTPUT/2055_Sunday_February_25_2024_06_47_24_AM_45713068/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _dAlembert]
```

$$\boxed{{\mathrm e}^{\frac{x}{y}}+{\mathrm e}^{\frac{x}{y}} \left(1-\frac{x}{y}\right) y^{\prime} = -1}$$

8.23.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$
{\mathrm e}^{\frac{1}{u(x)}}+{\mathrm e}^{\frac{1}{u(x)}} \left(1-\frac{1}{u \left(x \right)}\right) \left(u^{\prime}\left(x \right) x +u \left(x \right)\right) = -1
$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u \left({\mathrm e}^{-\frac{1}{u}}+u \right)}{x \left(u -1\right)}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u\left(e^{-\frac{1}{u}}+u\right)}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u\left(e^{-\frac{1}{u}}+u\right)}{u-1}} du = -\frac{1}{x} dx$$

$$\int \frac{1}{\frac{u\left(e^{-\frac{1}{u}}+u\right)}{u-1}} du = \int -\frac{1}{x} dx$$

$$\frac{1}{u} + \ln\left(e^{-\frac{1}{u}} + u\right) = -\ln(x) + c_2$$

The solution is

$$\frac{1}{u(x)} + \ln\left(e^{-\frac{1}{u(x)}} + u(x)\right) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{x}{y} + \ln\left(e^{-\frac{x}{y}} + \frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{e^{-\frac{x}{y}}x+y}{x}\right) y + (-c_2 + \ln(x))\, y + x}{y} = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{e^{-\frac{x}{y}}x+y}{x}\right) y + (-c_2 + \ln(x))\, y + x}{y} = 0 \tag{1}$$

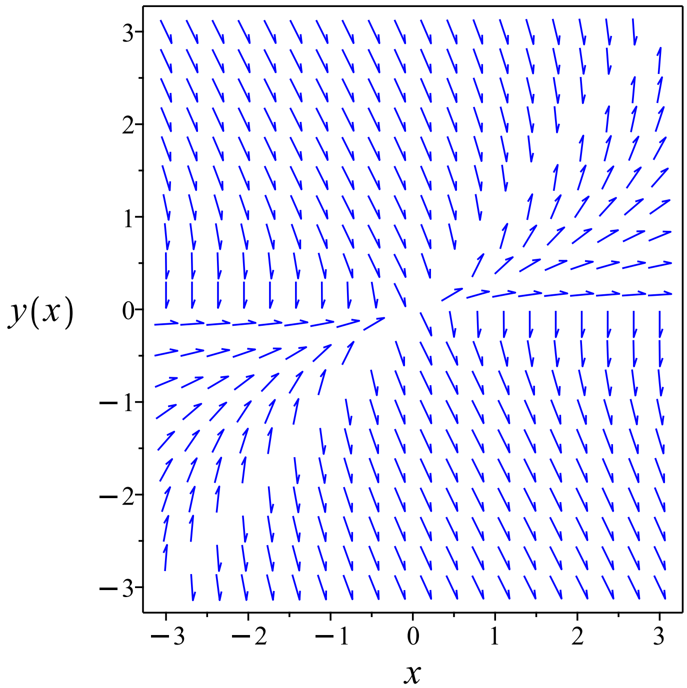


Figure 443: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{e^{-\frac{x}{y}}x+y}{x}\right)y+(-c_2+\ln(x))\,y+x}{y}=0$$

Verified OK.

8.23.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y\left(e^{\frac{x}{y}}+1\right)e^{-\frac{x}{y}}}{-x+y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{y\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}(b_3-a_2)}{-x+y} - \frac{y^2\left(\mathrm{e}^{\frac{x}{y}}+1\right)^2\mathrm{e}^{-\frac{2x}{y}}a_3}{(-x+y)^2} \\
&- \left(-\frac{1}{-x+y} + \frac{\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{-x+y} - \frac{y\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{(-x+y)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{-x+y} + \frac{x}{y(-x+y)} - \frac{\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}x}{y(-x+y)}\right. \\
&\left. + \frac{y\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{(-x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\left(2\,\mathrm{e}^{\frac{2x}{y}}x\,y^2 b_2 - \mathrm{e}^{\frac{2x}{y}}y^3 a_2 - \mathrm{e}^{\frac{2x}{y}}y^3 b_2 + \mathrm{e}^{\frac{2x}{y}}y^3 b_3 + \mathrm{e}^{\frac{2x}{y}}x y b_1 - \mathrm{e}^{\frac{2x}{y}}y^2 a_1 + \mathrm{e}^{\frac{x}{y}}x^3 b_2 - \mathrm{e}^{\frac{x}{y}}x^2 y a_2 + \mathrm{e}^{\frac{x}{y}}x^2 y b_3 + \mathrm{e}^{\frac{x}{y}}x\,y\right.}{(x-y)^2\,y}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-2\,\mathrm{e}^{\frac{2x}{y}}x\,y^2 b_2 + \mathrm{e}^{\frac{2x}{y}}y^3 a_2 + \mathrm{e}^{\frac{2x}{y}}y^3 b_2 - \mathrm{e}^{\frac{2x}{y}}y^3 b_3 - \mathrm{e}^{\frac{2x}{y}}x y b_1 + \mathrm{e}^{\frac{2x}{y}}y^2 a_1 \\
&- \mathrm{e}^{\frac{x}{y}}x^3 b_2 + \mathrm{e}^{\frac{x}{y}}x^2 y a_2 - \mathrm{e}^{\frac{x}{y}}x^2 y b_3 - \mathrm{e}^{\frac{x}{y}}x\,y^2 a_2 + \mathrm{e}^{\frac{x}{y}}x\,y^2 a_3 + \mathrm{e}^{\frac{x}{y}}x\,y^2 b_3 \\
&+ \mathrm{e}^{\frac{x}{y}}y^3 a_2 - 2\,\mathrm{e}^{\frac{x}{y}}y^3 a_3 - \mathrm{e}^{\frac{x}{y}}y^3 b_3 - \mathrm{e}^{\frac{x}{y}}x^2 b_1 + \mathrm{e}^{\frac{x}{y}}x y a_1 - y^3 a_3 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}
&-2\,\mathrm{e}^{\frac{2x}{y}}x\,y^2 b_2 + \mathrm{e}^{\frac{2x}{y}}y^3 a_2 + \mathrm{e}^{\frac{2x}{y}}y^3 b_2 - \mathrm{e}^{\frac{2x}{y}}y^3 b_3 - \mathrm{e}^{\frac{2x}{y}}x y b_1 + \mathrm{e}^{\frac{2x}{y}}y^2 a_1 \\
&- \mathrm{e}^{\frac{x}{y}}x^3 b_2 + \mathrm{e}^{\frac{x}{y}}x^2 y a_2 - \mathrm{e}^{\frac{x}{y}}x^2 y b_3 - \mathrm{e}^{\frac{x}{y}}x\,y^2 a_2 + \mathrm{e}^{\frac{x}{y}}x\,y^2 a_3 + \mathrm{e}^{\frac{x}{y}}x\,y^2 b_3 \\
&+ \mathrm{e}^{\frac{x}{y}}y^3 a_2 - 2\,\mathrm{e}^{\frac{x}{y}}y^3 a_3 - \mathrm{e}^{\frac{x}{y}}y^3 b_3 - \mathrm{e}^{\frac{x}{y}}x^2 b_1 + \mathrm{e}^{\frac{x}{y}}x y a_1 - y^3 a_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \mathrm{e}^{\frac{x}{y}}, \mathrm{e}^{\frac{2x}{y}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \mathrm{e}^{\frac{x}{y}} = v_3, \mathrm{e}^{\frac{2x}{y}} = v_4\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &v_3 v_1^2 v_2 a_2 - v_3 v_1 v_2^2 a_2 + v_3 v_2^3 a_2 + v_4 v_2^3 a_2 + v_3 v_1 v_2^2 a_3 - 2 v_3 v_2^3 a_3 \\ &- v_3 v_1^3 b_2 - 2 v_4 v_1 v_2^2 b_2 + v_4 v_2^3 b_2 - v_3 v_1^2 v_2 b_3 + v_3 v_1 v_2^2 b_3 - v_3 v_2^3 b_3 \\ &- v_4 v_2^3 b_3 + v_3 v_1 v_2 a_1 + v_4 v_2^2 a_1 - v_2^3 a_3 - v_3 v_1^2 b_1 - v_4 v_1 v_2 b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} &-v_3 v_1^3 b_2 + (-b_3 + a_2) v_1^2 v_2 v_3 - v_3 v_1^2 b_1 + (-a_2 + a_3 + b_3) v_1 v_2^2 v_3 \\ &- 2 v_4 v_1 v_2^2 b_2 + v_3 v_1 v_2 a_1 - v_4 v_1 v_2 b_1 + (a_2 - 2a_3 - b_3) v_2^3 v_3 \\ &+ (a_2 + b_2 - b_3) v_2^3 v_4 - v_2^3 a_3 + v_4 v_2^2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2b_2 &= 0 \\ -b_2 &= 0 \\ -b_3 + a_2 &= 0 \\ -a_2 + a_3 + b_3 &= 0 \\ a_2 - 2a_3 - b_3 &= 0 \\ a_2 + b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{y\left(\mathrm{e}^{\frac{x}{y}} + 1\right) \mathrm{e}^{-\frac{x}{y}}}{-x + y}\right)(x) \\
&= \frac{-y^2 \mathrm{e}^{\frac{x}{y}} - yx}{\mathrm{e}^{\frac{x}{y}} x - y\, \mathrm{e}^{\frac{x}{y}}} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y^2 \mathrm{e}^{\frac{x}{y}} - yx}{\mathrm{e}^{\frac{x}{y}} x - y\,\mathrm{e}^{\frac{x}{y}}}} dy
\end{aligned}
$$

Which results in

$$
S = \ln \left(y\,\mathrm{e}^{\frac{x}{y}} + x \right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{y \left(\mathrm{e}^{\frac{x}{y}} + 1 \right) \mathrm{e}^{-\frac{x}{y}}}{-x + y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{\mathrm{e}^{\frac{x}{y}} + 1}{y\,\mathrm{e}^{\frac{x}{y}} + x} \\
S_y &= -\frac{\mathrm{e}^{\frac{x}{y}} (x - y)}{y \left(y\,\mathrm{e}^{\frac{x}{y}} + x \right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(\mathrm{e}^{\frac{x}{y}} y + x\right) = c_1$$

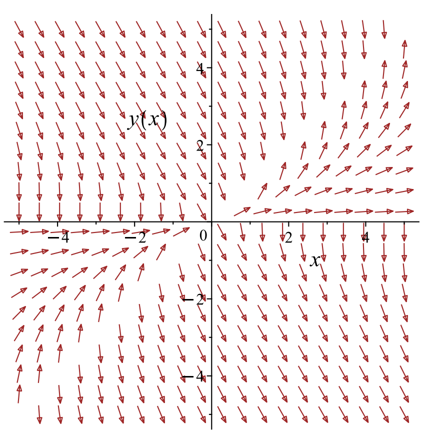
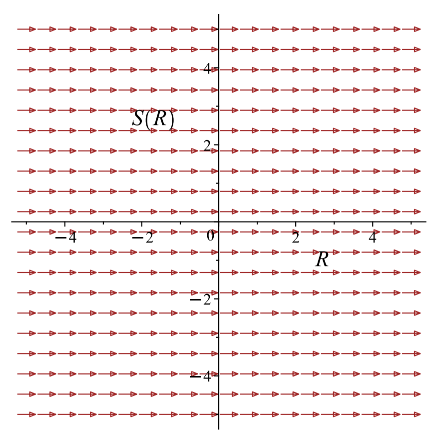
Which simplifies to

$$\ln\left(\mathrm{e}^{\frac{x}{y}} y + x\right) = c_1$$

Which gives

$$y = -\frac{x}{\operatorname{LambertW}\left(-\frac{x}{\mathrm{e}^{c_1 - x}}\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{-x+y}$ 	$R = x$ $S = \ln\left(y\,\mathrm{e}^{\frac{x}{y}} + x\right)$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = -\frac{x}{\text{LambertW}\left(-\frac{x}{e^{c_1 - x}}\right)} \tag{1}$$

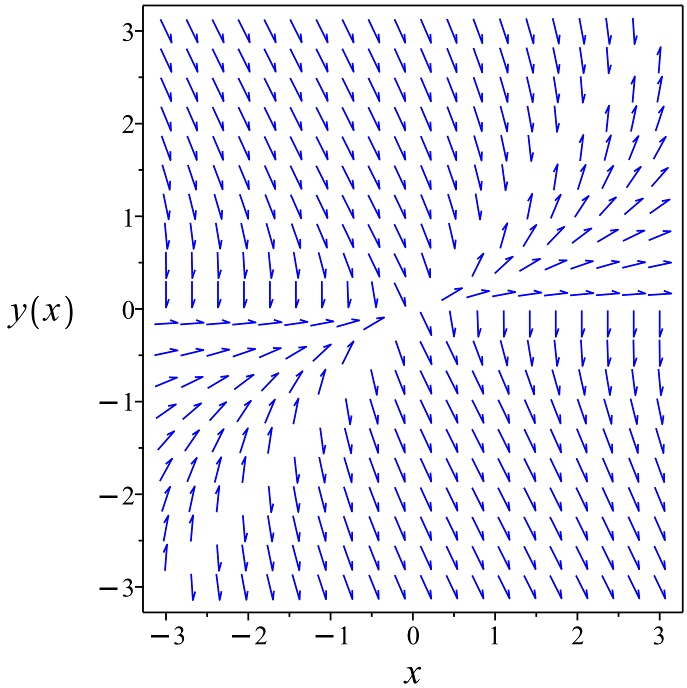


Figure 444: Slope field plot

Verification of solutions

$$y = -\frac{x}{\text{LambertW}\left(-\frac{x}{e^{c_1 - x}}\right)}$$

Verified OK.

8.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned} \left(\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\right) \mathrm{d} y &= \left(-1-\mathrm{e}^{\frac{x}{y}}\right) \mathrm{d} x \\ \left(\mathrm{e}^{\frac{x}{y}}+1\right) \mathrm{d} x+\left(\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\right) \mathrm{d} y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \mathrm{e}^{\frac{x}{y}}+1 \\ N(x, y) &= \mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{\frac{x}{y}}+1\right) \\ &= -\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2} \end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\right)$$
$$= -\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \mathrm{e}^{\frac{x}{y}} + 1\,\mathrm{d}x$$
$$\phi = y\,\mathrm{e}^{\frac{x}{y}} + x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{x}{y}} - \frac{x\,\mathrm{e}^{\frac{x}{y}}}{y} + f'(y) \tag{4}$$
$$= -\frac{\mathrm{e}^{\frac{x}{y}}(x-y)}{y} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$. Therefore equation (4) becomes

$$\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right) = -\frac{\mathrm{e}^{\frac{x}{y}}(x-y)}{y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,e^{\frac{x}{y}} + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,e^{\frac{x}{y}} + x$$

The solution becomes

$$y = -\frac{x}{\text{LambertW}\left(-\frac{x}{-x+c_1}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\text{LambertW}\left(-\frac{x}{-x+c_1}\right)} \tag{1}$$

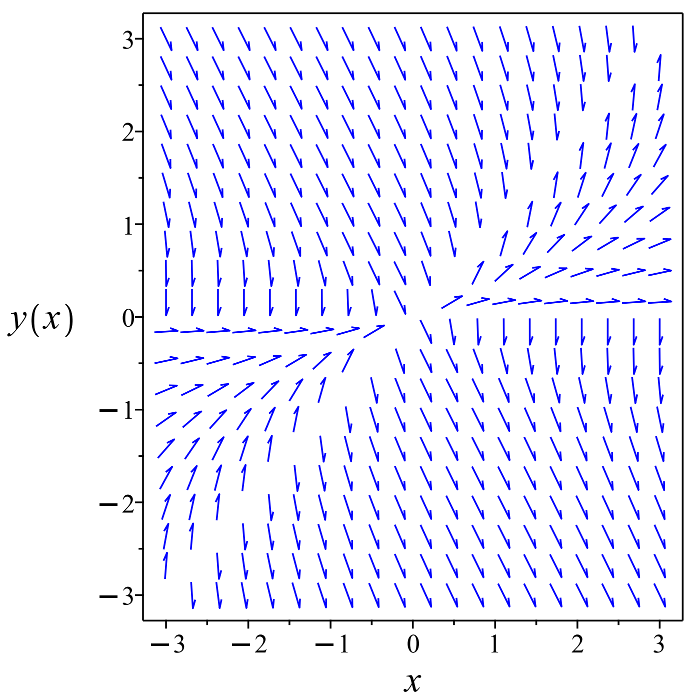


Figure 445: Slope field plot

Verification of solutions

$$y = -\frac{x}{\operatorname{LambertW}\left(-\frac{x}{-x+c_1}\right)}$$

Verified OK.

8.23.4 Maple step by step solution

Let's solve

$\mathrm{e}^{\frac{x}{y}}+\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

  - Evaluate derivatives

    $-\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2} = \frac{\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{y}-\frac{\mathrm{e}^{\frac{x}{y}}}{y}$

  - Simplify

    $-\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2} = -\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int\left(\mathrm{e}^{\frac{x}{y}}+1\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y) = y\,\mathrm{e}^{\frac{x}{y}}+x+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $e^{\frac{x}{y}}\left(1-\frac{x}{y}\right) = e^{\frac{x}{y}} - \frac{x\,e^{\frac{x}{y}}}{y} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right) - e^{\frac{x}{y}} + \frac{x\,e^{\frac{x}{y}}}{y}$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = y\,e^{\frac{x}{y}} + x$

- Substitute $F(x,y)$ into the solution of the ODE

  $y\,e^{\frac{x}{y}} + x = c_1$

- Solve for y

  $y = -\frac{x}{\mathit{LambertW}\left(-\frac{x}{-x+c_1}\right)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 20

```
dsolve((1+exp(x/y(x)))+( exp(x/y(x))*(1-x/y(x)) )*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{\mathrm{LambertW}\left(\frac{xc_1}{c_1x-1}\right)}$$

✓ Solution by Mathematica

Time used: 1.251 (sec). Leaf size: 34

```
DSolve[(1+Exp[x/y[x]])+( Exp[x/y[x]]*(1-x/y[x]) )*y'[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to -\frac{x}{W\left(\frac{x}{x-e^{c_1}}\right)}$$
$$y(x) \to -\frac{x}{W(1)}$$

8.24 problem 25

Internal problem ID [2056]
Internal file name [`OUTPUT/2056_Sunday_February_25_2024_06_47_26_AM_97461569/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cot(x) = -x}$$

8.24.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot(x)$$
$$q(x) = -x$$

Hence the ode is

$$y' + y\cot(x) = -x$$

The integrating factor μ is

$$\mu = e^{\int \cot(x)dx}$$
$$= \sin(x)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(-x)$$
$$\frac{d}{dx}(\sin(x) y) = (\sin(x))(-x)$$
$$d(\sin(x) y) = (-\sin(x) x)\ dx$$

Integrating gives

$$\sin(x) y = \int -\sin(x) x \, dx$$
$$\sin(x) y = x\cos(x) - \sin(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$y = \csc(x)(x\cos(x) - \sin(x)) + \csc(x) c_1$$

which simplifies to

$$y = x\cot(x) - 1 + \csc(x) c_1$$

Summary

The solution(s) found are the following

$$y = x\cot(x) - 1 + \csc(x) c_1 \tag{1}$$

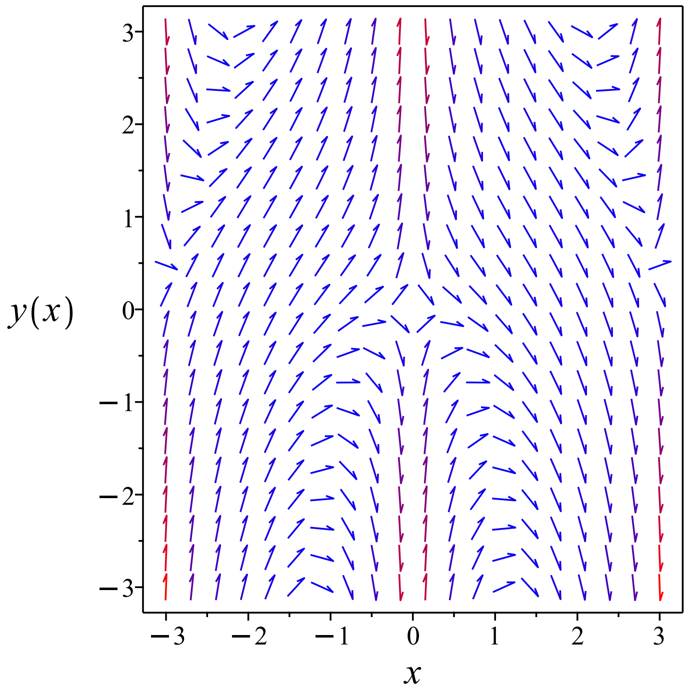


Figure 446: Slope field plot

Verification of solutions

$$y = x\cot(x) - 1 + \csc(x)\, c_1$$

Verified OK.

8.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -x - y\cot(x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 257: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sin(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -x - y\cot(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\, y \\ S_y &= \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\sin(x)\, x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\sin(R)\, R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \cos(R) R - \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(x) y = x \cos(x) - \sin(x) + c_1$$

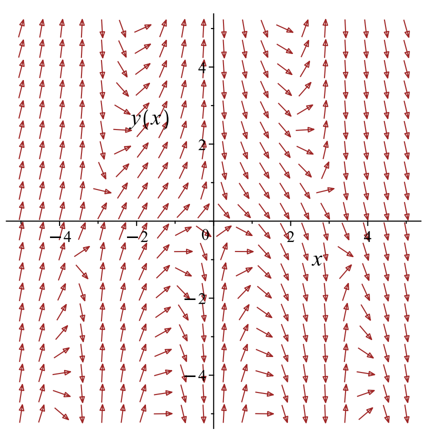
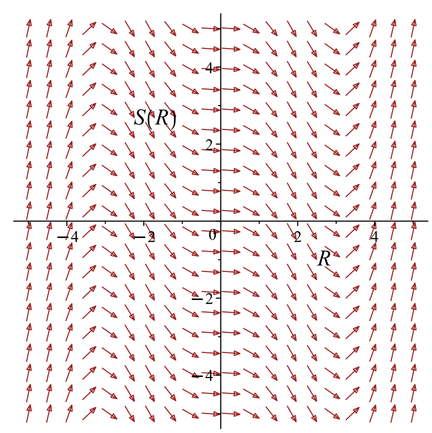
Which simplifies to

$$\sin(x) y = x \cos(x) - \sin(x) + c_1$$

Which gives

$$y = -\frac{-x\cos(x) + \sin(x) - c_1}{\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x - y\cot(x)$ 	$R = x$ $S = \sin(x) y$	$\frac{dS}{dR} = -\sin(R) R$ 

Summary

The solution(s) found are the following

$$y = -\frac{-x\cos(x) + \sin(x) - c_1}{\sin(x)} \tag{1}$$

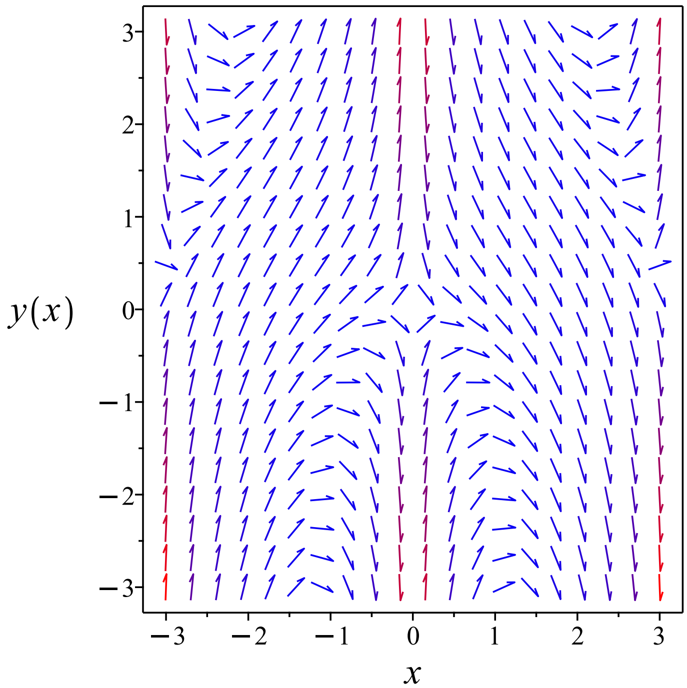


Figure 447: Slope field plot

Verification of solutions

$$y = -\frac{-x\cos(x) + \sin(x) - c_1}{\sin(x)}$$

Verified OK.

8.24.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-x - y \cot(x))\,\mathrm{d}x$$
$$(y \cot(x) + x)\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y \cot(x) + x$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y \cot(x) + x)$$
$$= \cot(x)$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cot(x)) - (0)) \\ &= \cot(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cot(x) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(\sin(x))} \\ &= \sin(x) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sin(x)\,(y\cot(x) + x) \\ &= \sin(x)\,x + \cos(x)\,y \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sin(x)\,(1) \\ &= \sin(x) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\sin(x)\,x + \cos(x)\,y) + (\sin(x))\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \sin(x) x + \cos(x) y \mathrm{d}x$$

$$\phi = \sin(x) y - x\cos(x) + \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$\sin(x) = \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x) y - x\cos(x) + \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x) y - x\cos(x) + \sin(x)$$

The solution becomes

$$y = -\frac{-x\cos(x) + \sin(x) - c_1}{\sin(x)}$$

Summary
The solution(s) found are the following

$$y = -\frac{-x\cos(x) + \sin(x) - c_1}{\sin(x)} \tag{1}$$

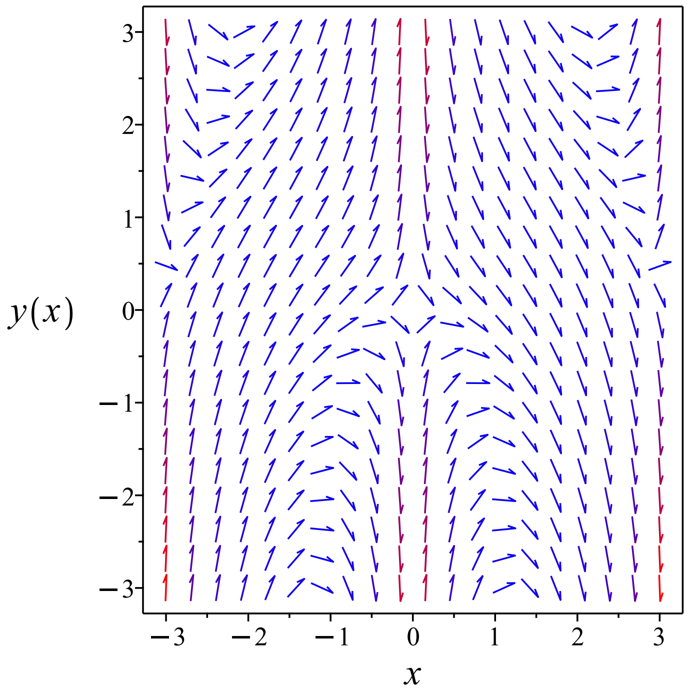


Figure 448: Slope field plot

Verification of solutions

$$y = -\frac{-x\cos(x) + \sin(x) - c_1}{\sin(x)}$$

Verified OK.

8.24.4 Maple step by step solution

Let's solve

$y' + y\cot(x) = -x$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

$$y' = -x - y\cot(x)$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + y\cot(x) = -x$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + y\cot(x)\right) = -\mu(x)\, x$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' + y\cot(x)\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)\cot(x)$$

- Solve to find the integrating factor

$$\mu(x) = \sin(x)$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\mu(x)\, x dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int -\mu(x)\, x dx + c_1$$

- Solve for y

$$y = \frac{\int -\mu(x) x dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \sin(x)$

$$y = \frac{\int -\sin(x) x dx + c_1}{\sin(x)}$$

- Evaluate the integrals on the rhs

$$y = \frac{x\cos(x) - \sin(x) + c_1}{\sin(x)}$$

- Simplify

$$y = x\cot(x) - 1 + \csc(x)\, c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+x+y(x)*cot(x)=0,y(x), singsol=all)
```

$$y(x) = -1 + \cot(x) x + \csc(x) c_1$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 16

```
DSolve[y'[x]+x+y[x]*Cot[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \cot(x) + c_1 \csc(x) - 1$$

8.25 problem 26

Internal problem ID [2057]
Internal file name [`OUTPUT/2057_Sunday_February_25_2024_06_47_26_AM_66896355/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-y'xy = 6-3x}$$

8.25.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-6+3x}{yx} \end{aligned}$$

Where $f(x) = \frac{-6+3x}{x}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = \frac{-6+3x}{x}\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int \frac{-6+3x}{x}\,dx$$

$$\frac{y^2}{2} = 3x - 6\ln(x) + c_1$$

Which results in

$$y = \sqrt{-12\ln(x) + 2c_1 + 6x}$$

$$y = -\sqrt{-12\ln(x) + 2c_1 + 6x}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-12\ln(x) + 2c_1 + 6x} \tag{1}$$

$$y = -\sqrt{-12\ln(x) + 2c_1 + 6x} \tag{2}$$

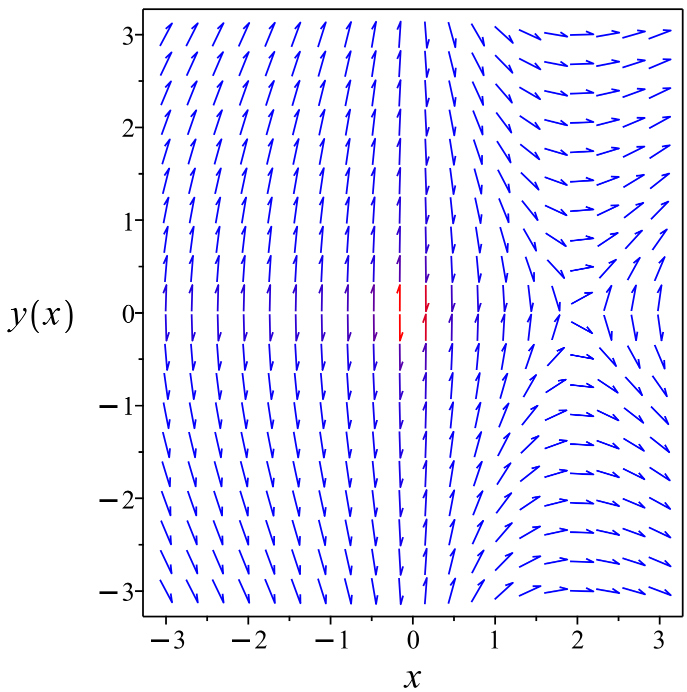


Figure 449: Slope field plot

Verification of solutions

$$y = \sqrt{-12\ln(x) + 2c_1 + 6x}$$

Verified OK.

$$y = -\sqrt{-12\ln(x) + 2c_1 + 6x}$$

Verified OK.

8.25.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-6+3x}{yx} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 260: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{x}{-6+3x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x}{-6+3x}} dx \end{aligned}$$

Which results in

$$S = 3x - 6\ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-6+3x}{yx}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{-6+3x}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = y \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^2}{2} + c_1 \tag{4}
$$

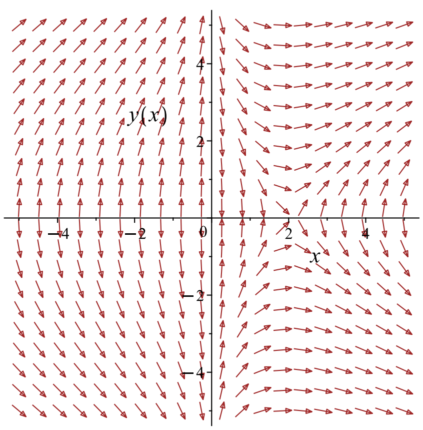
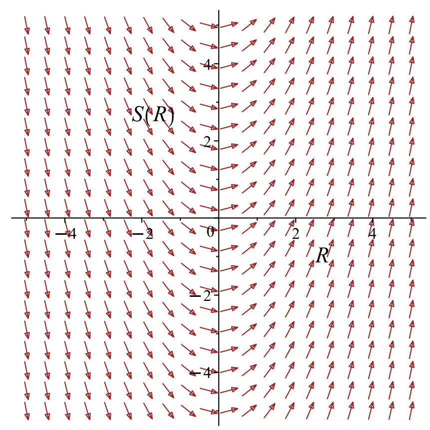
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
3x - 6\ln(x) = \frac{y^2}{2} + c_1
$$

Which simplifies to

$$
3x - 6\ln(x) = \frac{y^2}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-6+3x}{yx}$ 	$R = y$ $S = 3x - 6\ln(x)$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$3x - 6\ln(x) = \frac{y^2}{2} + c_1 \tag{1}$$

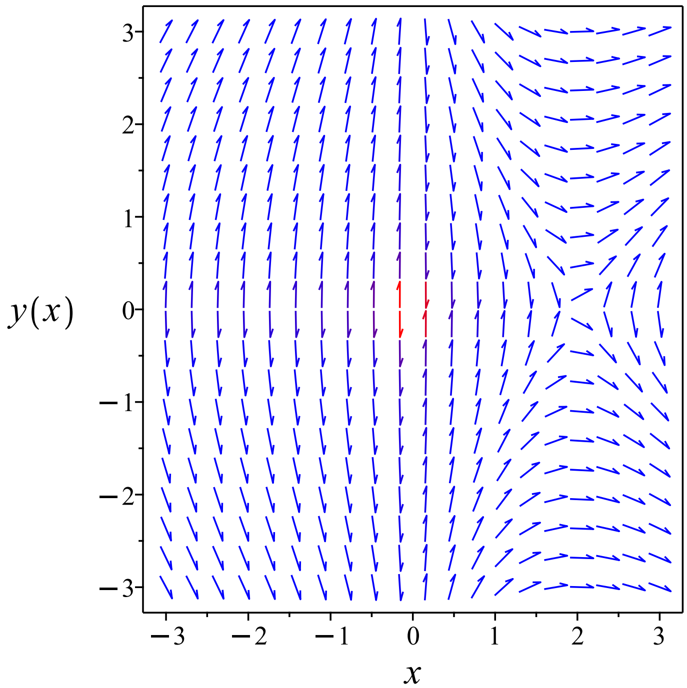


Figure 450: Slope field plot

Verification of solutions

$$3x - 6\ln(x) = \frac{y^2}{2} + c_1$$

Verified OK.

8.25.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{3}\right) \mathrm{d}y &= \left(\frac{-2+x}{x}\right) \mathrm{d}x\\ \left(-\frac{-2+x}{x}\right) \mathrm{d}x + \left(\frac{y}{3}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{-2+x}{x}\\ N(x, y) &= \frac{y}{3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-2+x}{x}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{-2+x}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -x + 2\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{3}$. Therefore equation (4) becomes

$$
\frac{y}{3} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y}{3}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{y}{3}\right) \mathrm{d}y \\
f(y) &= \frac{y^2}{6} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + 2\ln(x) + \frac{y^2}{6} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + 2\ln(x) + \frac{y^2}{6}$$

Summary

The solution(s) found are the following

$$-x + 2\ln(x) + \frac{y^2}{6} = c_1 \tag{1}$$

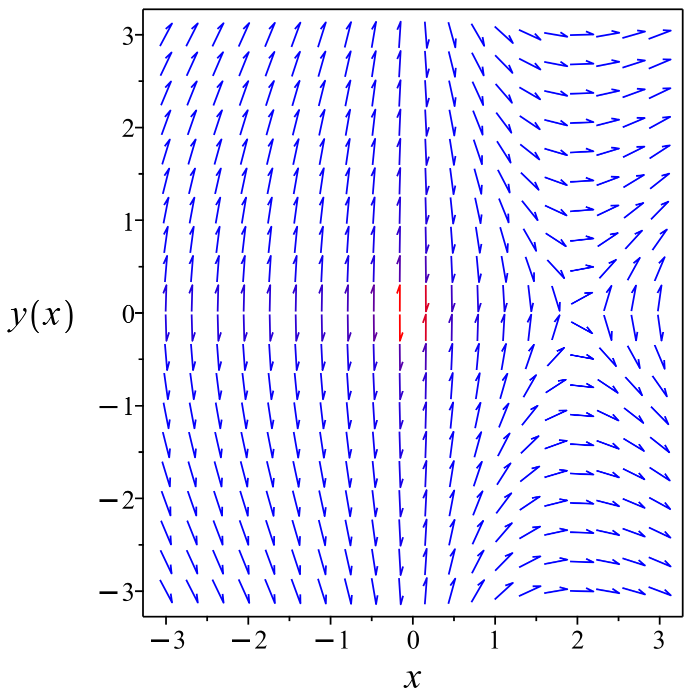


Figure 451: Slope field plot

Verification of solutions

$$-x + 2\ln(x) + \frac{y^2}{6} = c_1$$

Verified OK.

8.25.4 Maple step by step solution

Let's solve

$-y'xy = 6-3x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = -\frac{6-3x}{x}$

- Integrate both sides with respect to x

  $\int yy'dx = \int -\frac{6-3x}{x}dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = 3x - 6\ln(x) + c_1$

- Solve for y

  $\left\{y = \sqrt{-12\ln(x) + 2c_1 + 6x}, y = -\sqrt{-12\ln(x) + 2c_1 + 6x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve(3*(x-2)=x*y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \sqrt{-12\ln(x) + c_1 + 6x}$$
$$y(x) = -\sqrt{-12\ln(x) + c_1 + 6x}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 50

```
DSolve[3*(x-2)==x*y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt{3x - 6\log(x) + c_1}$$
$$y(x) \to \sqrt{2}\sqrt{3x - 6\log(x) + c_1}$$

8.26 problem 27

Internal problem ID [2058]
Internal file name [`OUTPUT/2058_Sunday_February_25_2024_06_47_27_AM_2767112/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{-2yx + \mathrm{e}^{y} + \left(y - x^{2} + \mathrm{e}^{y} x\right) y' = -x}$$

8.26.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y - x^2 + \mathrm{e}^y x\right)\mathrm{d}y &= \left(-x + 2yx - \mathrm{e}^y\right)\mathrm{d}x \\ \left(-2yx + \mathrm{e}^y + x\right)\mathrm{d}x + \left(y - x^2 + \mathrm{e}^y x\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2yx + \mathrm{e}^y + x \\ N(x,y) &= y - x^2 + \mathrm{e}^y x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2yx + \mathrm{e}^y + x\right) \\ &= -2x + \mathrm{e}^y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y - x^2 + \mathrm{e}^y x\right) \\ &= -2x + \mathrm{e}^y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -2yx + \mathrm{e}^y + x \mathrm{d}x \\
\phi &= \mathrm{e}^y x - \left(y - \frac{1}{2}\right) x^2 + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \mathrm{e}^y x - x^2 + f'(y) \qquad (4) \\
&= x(\mathrm{e}^y - x) + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - x^2 + \mathrm{e}^y x$. Therefore equation (4) becomes

$$
y - x^2 + \mathrm{e}^y x = x(\mathrm{e}^y - x) + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int (y) \mathrm{d}y \\
f(y) &= \frac{y^2}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \mathrm{e}^y x - \left(y - \frac{1}{2}\right) x^2 + \frac{y^2}{2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^y x - \left(y - \frac{1}{2}\right) x^2 + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$e^y x - \left(y - \frac{1}{2}\right) x^2 + \frac{y^2}{2} = c_1 \tag{1}$$

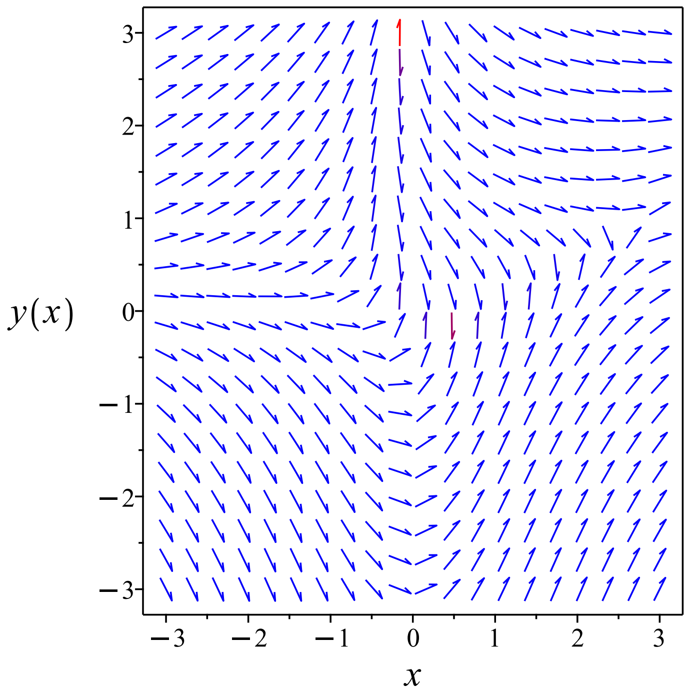


Figure 452: Slope field plot

Verification of solutions

$$e^y x - \left(y - \frac{1}{2}\right) x^2 + \frac{y^2}{2} = c_1$$

Verified OK.

8.26.2 Maple step by step solution

Let's solve

$-2yx + e^{y} + (y - x^{2} + e^{y}x)\, y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $-2x + e^{y} = -2x + e^{y}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (-2yx + e^{y} + x)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = -y\, x^{2} + e^{y}x + \frac{x^{2}}{2} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $y - x^{2} + e^{y}x = -x^{2} + e^{y}x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = y$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^{2}}{2}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = -y\,x^2 + e^y x + \frac{x^2}{2} + \frac{y^2}{2}$$

- Substitute $F(x,y)$ into the solution of the ODE

$$-y\,x^2 + e^y x + \frac{x^2}{2} + \frac{y^2}{2} = c_1$$

- Solve for y

$$y = RootOf\left(2\_Z x^2 - 2x\,e^{\_Z} - \_Z^2 - x^2 + 2c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve((x-2*x*y(x)+exp(y(x)) )+(y(x)-x^2+x*exp(y(x)) )*diff(y(x),x)=0,y(x), singsol=all)
```

$$-x^2 y(x) + e^{y(x)} x + \frac{x^2}{2} + \frac{y(x)^2}{2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.306 (sec). Leaf size: 35

```
DSolve[(x-2*x*y[x]+Exp[y[x]] )+(y[x]-x^2+x*Exp[y[x]] )*y'[x]==0,y[x],x,IncludeSingularSoluti
```

$$\text{Solve}\left[x^2(-y(x)) + \frac{x^2}{2} + x e^{y(x)} + \frac{y(x)^2}{2} = c_1, y(x)\right]$$

8.27 problem 28

Internal problem ID [2059]
Internal file name [`OUTPUT/2059_Sunday_February_25_2024_06_47_28_AM_81968639/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{2y'x - y + \frac{x^2}{y^2} = 0}$$

8.27.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y^3 - x^2}{2x\,y^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 264: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{x^{\frac{3}{2}}}{y^2} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^{\frac{3}{2}}}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3}{3x^{\frac{3}{2}}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^3 - x^2}{2x\,y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^3}{2x^{\frac{5}{2}}} \\ S_y &= \frac{y^2}{x^{\frac{3}{2}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2\sqrt{x}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2\sqrt{R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\sqrt{R} + c_1 \tag{4}$$

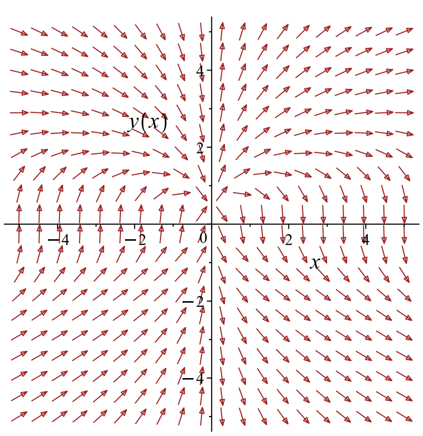
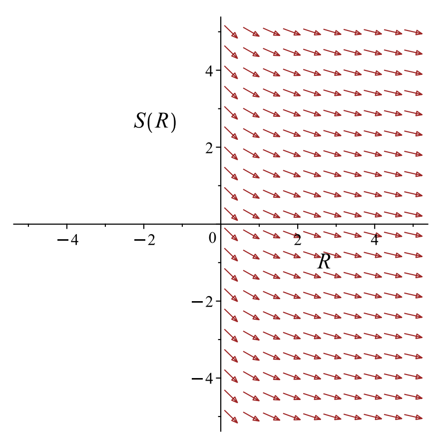
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3}{3x^{\frac{3}{2}}} = -\sqrt{x} + c_1$$

Which simplifies to

$$\frac{y^3}{3x^{\frac{3}{2}}} = -\sqrt{x} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^3 - x^2}{2x\,y^2}$ 	$R = x$ $S = \frac{y^3}{3x^{\frac{3}{2}}}$	$\frac{dS}{dR} = -\frac{1}{2\sqrt{R}}$ 

Summary
The solution(s) found are the following

$$\frac{y^3}{3x^{\frac{3}{2}}} = -\sqrt{x} + c_1 \tag{1}$$

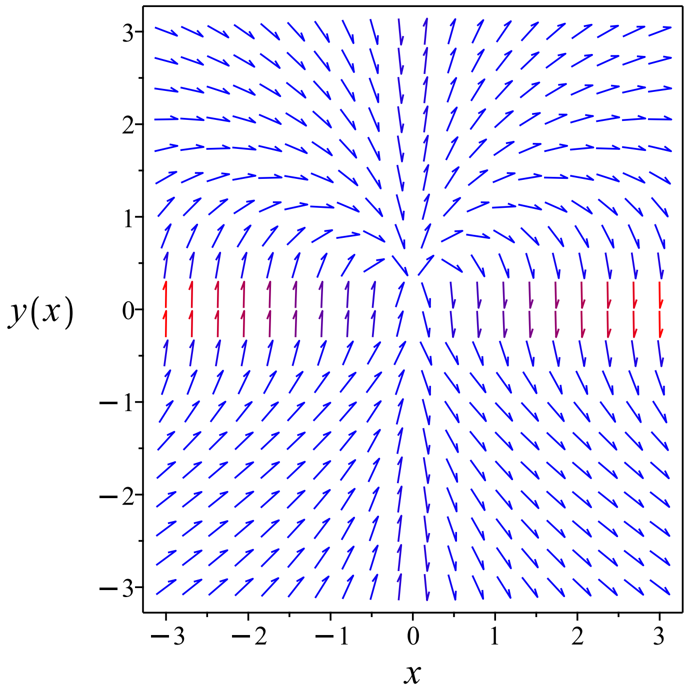


Figure 453: Slope field plot

Verification of solutions

$$\frac{y^3}{3x^{\frac{3}{2}}} = -\sqrt{x} + c_1$$

Verified OK.

8.27.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^3 - x^2}{2x\,y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y - \frac{x}{2}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{2x} \\
f_1(x) &= -\frac{x}{2} \\
n &= -2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$
y'y^2 = \frac{y^3}{2x} - \frac{x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^3
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 3y^2y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= \frac{w(x)}{2x} - \frac{x}{2} \\
w' &= \frac{3w}{2x} - \frac{3x}{2}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{2x} \\
q(x) &= -\frac{3x}{2}
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{2x} = -\frac{3x}{2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{3}{2x}dx} \\
&= \frac{1}{x^{\frac{3}{2}}}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\frac{3x}{2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^{\frac{3}{2}}}\right) &= \left(\frac{1}{x^{\frac{3}{2}}}\right)\left(-\frac{3x}{2}\right) \\
\mathrm{d}\left(\frac{w}{x^{\frac{3}{2}}}\right) &= \left(-\frac{3}{2\sqrt{x}}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{w}{x^{\frac{3}{2}}} &= \int -\frac{3}{2\sqrt{x}}\,\mathrm{d}x \\
\frac{w}{x^{\frac{3}{2}}} &= -3\sqrt{x} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^{\frac{3}{2}}}$ results in

$$w(x) = -3x^2 + c_1 x^{\frac{3}{2}}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = -3x^2 + c_1 x^{\frac{3}{2}}$$

Solving for y gives

$$\begin{aligned}
y(x) &= \left(-x^{\frac{3}{2}}\left(3\sqrt{x} - c_1\right)\right)^{\frac{1}{3}} \\
y(x) &= \frac{\left(x^{\frac{3}{2}}\left(-3\sqrt{x} + c_1\right)\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2} \\
y(x) &= -\frac{\left(x^{\frac{3}{2}}\left(-3\sqrt{x} + c_1\right)\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-x^{\frac{3}{2}}\left(3\sqrt{x}-c_1\right)\right)^{\frac{1}{3}} \tag{1}$$

$$y = \frac{\left(x^{\frac{3}{2}}\left(-3\sqrt{x}+c_1\right)\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2} \tag{2}$$

$$y = -\frac{\left(x^{\frac{3}{2}}\left(-3\sqrt{x}+c_1\right)\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2} \tag{3}$$

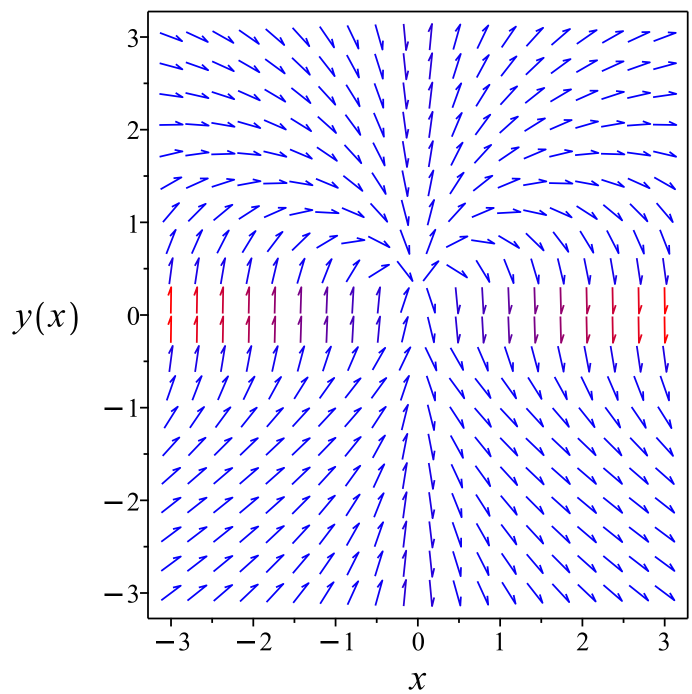


Figure 454: Slope field plot

Verification of solutions

$$y = \left(-x^{\frac{3}{2}}\left(3\sqrt{x}-c_1\right)\right)^{\frac{1}{3}}$$

Verified OK.

$$y = \frac{\left(x^{\frac{3}{2}}\left(-3\sqrt{x}+c_1\right)\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2}$$

Verified OK.

$$y = -\frac{\left(x^{\frac{3}{2}}\left(-3\sqrt{x}+c_1\right)\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2}$$

Verified OK.

8.27.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x\,y^2)\,\mathrm{d}y &= (y^3 - x^2)\,\mathrm{d}x\\ (-y^3 + x^2)\,\mathrm{d}x + (2x\,y^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y^3 + x^2\\ N(x,y) &= 2x\,y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y^3 + x^2)\\ &= -3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x\,y^2)\\ &= 2y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x\,y^2}\left((-3y^2) - (2y^2)\right)\\ &= -\frac{5}{2x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{5}{2x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{5 \ln(x)}{2}} \\
&= \frac{1}{x^{\frac{5}{2}}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^{\frac{5}{2}}}\left(-y^3 + x^2\right) \\
&= \frac{-y^3 + x^2}{x^{\frac{5}{2}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^{\frac{5}{2}}}\left(2x\, y^2\right) \\
&= \frac{2y^2}{x^{\frac{3}{2}}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y^3 + x^2}{x^{\frac{5}{2}}}\right) + \left(\frac{2y^2}{x^{\frac{3}{2}}}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-y^3 + x^2}{x^{\frac{5}{2}}} \, \mathrm{d}x$$

$$\phi = \frac{2y^3 + 6x^2}{3x^{\frac{3}{2}}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2y^2}{x^{\frac{3}{2}}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y^2}{x^{\frac{3}{2}}}$. Therefore equation (4) becomes

$$\frac{2y^2}{x^{\frac{3}{2}}} = \frac{2y^2}{x^{\frac{3}{2}}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2y^3 + 6x^2}{3x^{\frac{3}{2}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{2y^3 + 6x^2}{3x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$\frac{2y^3 + 6x^2}{3x^{\frac{3}{2}}} = c_1 \tag{1}$$

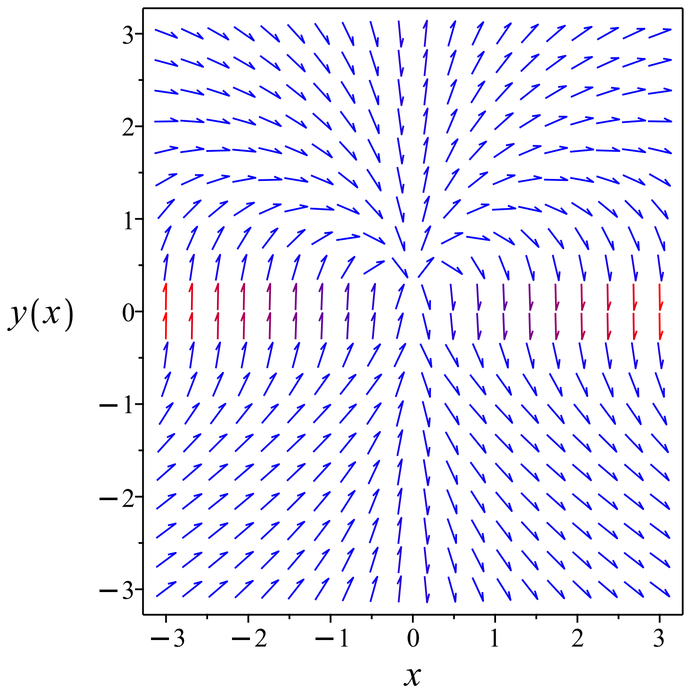


Figure 455: Slope field plot

Verification of solutions

$$\frac{2y^3 + 6x^2}{3x^{\frac{3}{2}}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 72

```
dsolve(2*x*diff(y(x),x)-y(x)+x^2/y(x)^2=0,y(x), singsol=all)
```

$$y(x) = \left(-\left(3\sqrt{x}-c_1\right)x^{\frac{3}{2}}\right)^{\frac{1}{3}}$$
$$y(x) = -\frac{\left(\left(-3\sqrt{x}+c_1\right)x^{\frac{3}{2}}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2}$$
$$y(x) = \frac{\left(\left(-3\sqrt{x}+c_1\right)x^{\frac{3}{2}}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2}$$

✓ Solution by Mathematica

Time used: 3.57 (sec). Leaf size: 80

```
DSolve[2*x*y'[x]-y[x]+x^2/y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{-3x^2+c_1x^{3/2}}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt[3]{-3x^2+c_1x^{3/2}}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{-3x^2+c_1x^{3/2}}$$

8.28 problem 29

Internal problem ID [2060]
Internal file name [`OUTPUT/2060_Sunday_February_25_2024_06_47_29_AM_30995245/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x + y(1 + y^2) = 0}$$

8.28.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(y^2+1)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y(y^2 + 1)$. Integrating both sides gives

$$\frac{1}{y\,(y^2+1)}\,dy = -\frac{1}{x}\,dx$$

$$\int \frac{1}{y\left(y^{2}+1\right)} d y=\int -\frac{1}{x} d x$$

$$\ln \left(y\right)-\frac{\ln \left(y^{2}+1\right)}{2}=-\ln \left(x\right)+c_{1}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln \left(y\right)-\frac{\ln \left(y^{2}+1\right)}{2}}=\mathrm{e}^{-\ln \left(x\right)+c_{1}}$$

Which simplifies to

$$\frac{y}{\sqrt{y^{2}+1}}=\frac{c_{2}}{x}$$

The solution is

$$\frac{y}{\sqrt{1+y^{2}}}=\frac{c_{2}}{x}$$

Summary
The solution(s) found are the following

$$\frac{y}{\sqrt{1+y^{2}}}=\frac{c_{2}}{x} \tag{1}$$

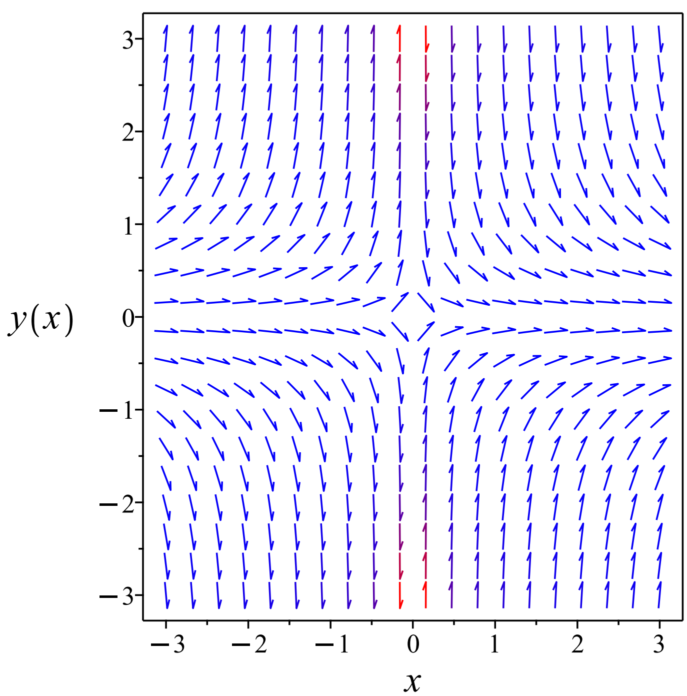


Figure 456: Slope field plot

Verification of solutions

$$\frac{y}{\sqrt{1+y^2}} = \frac{c_2}{x}$$

Verified OK.

8.28.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(y^2+1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 266: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -x \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x} dx \end{aligned}$$

Which results in

$$S = -\ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(y^2 + 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y(y^2 + 1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(R^2 + 1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) - \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

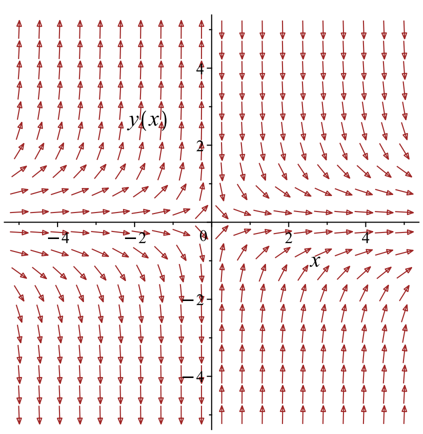
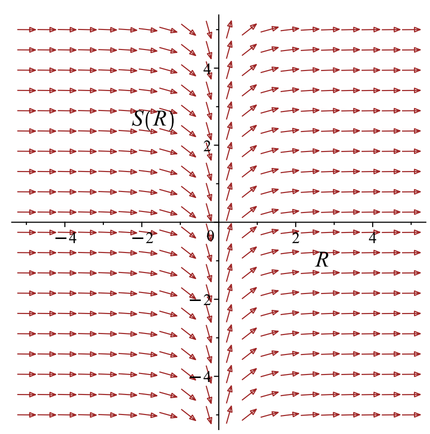
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(x) = \ln(y) - \frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$-\ln(x) = \ln(y) - \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y^2+1)}{x}$ 	$R = y$ $S = -\ln(x)$	$\frac{dS}{dR} = \frac{1}{R(R^2+1)}$ 

Summary

The solution(s) found are the following

$$-\ln(x) = \ln(y) - \frac{\ln(1+y^2)}{2} + c_1 \tag{1}$$

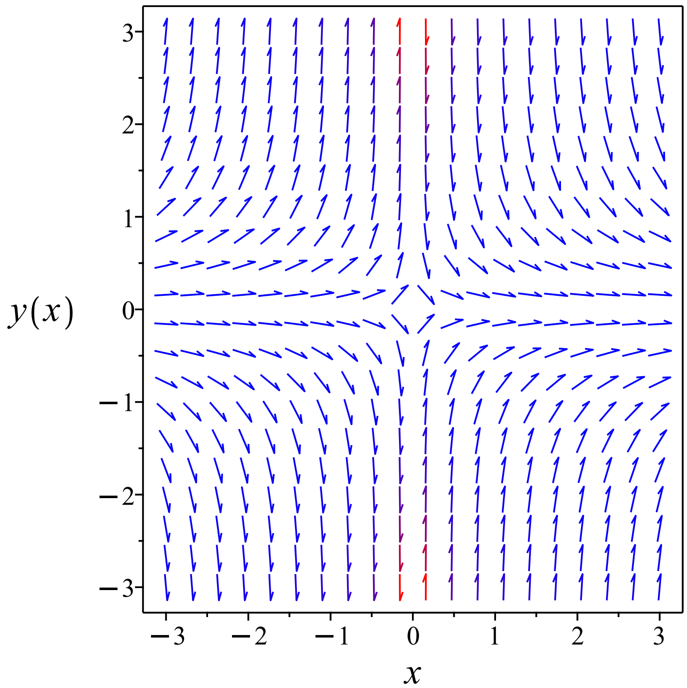


Figure 457: Slope field plot

Verification of solutions

$$-\ln(x) = \ln(y) - \frac{\ln(1+y^2)}{2} + c_1$$

Verified OK.

8.28.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(y^2+1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - \frac{1}{x}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= -\frac{1}{x} \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y' \frac{1}{y^3} = -\frac{1}{x\, y^2} - \frac{1}{x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= -\frac{w(x)}{x} - \frac{1}{x} \\
w' &= \frac{2w}{x} + \frac{2}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2}{x}
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = \frac{2}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{2}{x}\right) \\ \frac{d}{dx}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{2}{x}\right) \\ d\left(\frac{w}{x^2}\right) &= \left(\frac{2}{x^3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int \frac{2}{x^3}\, dx \\ \frac{w}{x^2} &= -\frac{1}{x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = c_1x^2 - 1$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = c_1x^2 - 1$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{\sqrt{c_1x^2 - 1}} \\ y(x) &= -\frac{1}{\sqrt{c_1x^2 - 1}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{c_1 x^2 - 1}} \tag{1}$$
$$y = -\frac{1}{\sqrt{c_1 x^2 - 1}} \tag{2}$$

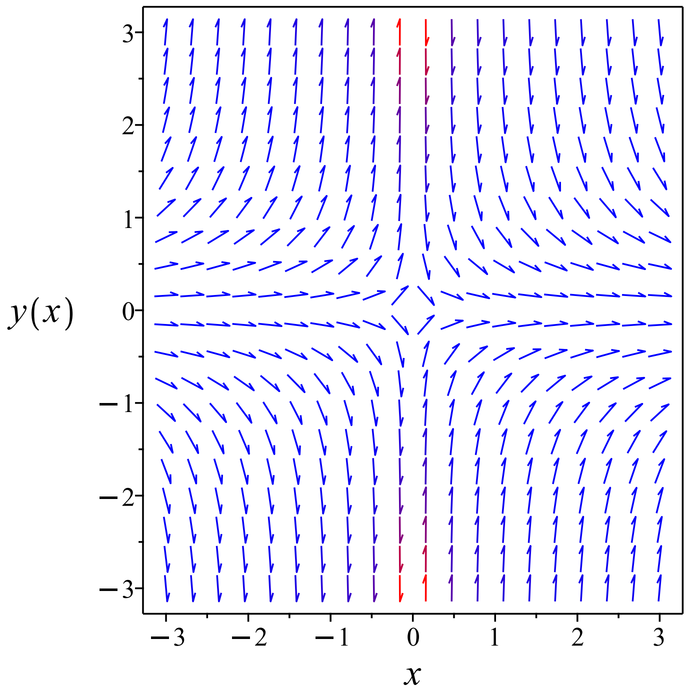


Figure 458: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{c_1 x^2 - 1}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{c_1 x^2 - 1}}$$

Verified OK.

8.28.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y\,(y^2+1)}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{1}{y\,(y^2+1)}\right)\mathrm{d}y &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x} \\
N(x,y) &= -\frac{1}{y\left(y^2+1\right)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y\left(y^2+1\right)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x}\, \mathrm{d}x \\
\phi &= -\ln(x) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(y^2+1)}$. Therefore equation (4) becomes

$$-\frac{1}{y\left(y^2+1\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y\left(y^2+1\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y\left(y^2+1\right)}\right) \mathrm{d}y$$

$$f(y) = -\ln(y) + \frac{\ln\left(y^2+1\right)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y) + \frac{\ln\left(y^2+1\right)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y) + \frac{\ln\left(y^2+1\right)}{2}$$

Summary
The solution(s) found are the following

$$-\ln(x) - \ln(y) + \frac{\ln\left(1+y^2\right)}{2} = c_1 \tag{1}$$

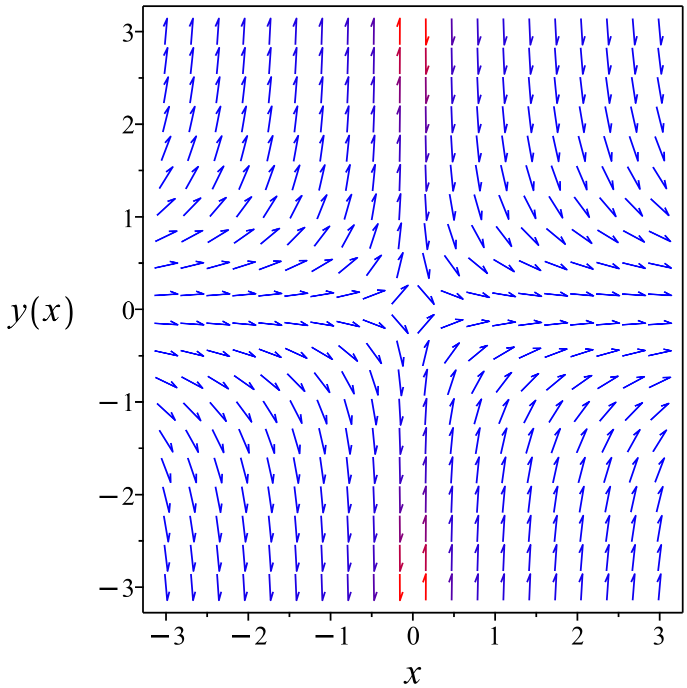


Figure 459: Slope field plot

Verification of solutions

$$-\ln(x) - \ln(y) + \frac{\ln(1+y^2)}{2} = c_1$$

Verified OK.

8.28.5 Maple step by step solution

Let's solve

$y'x + y(1+y^2) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(1+y^2)} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(1+y^2)} dx = \int -\frac{1}{x} dx + c_1$

- Evaluate integral

$$\ln(y) - \frac{\ln(1+y^2)}{2} = -\ln(x) + c_1$$

- Solve for y

$$\left\{y = \frac{e^{c_1}}{\sqrt{x^2 - (e^{c_1})^2}}, y = -\frac{e^{c_1}}{\sqrt{x^2 - (e^{c_1})^2}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x)+y(x)*(y(x)^2+1)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{c_1 x^2 - 1}}$$
$$y(x) = -\frac{1}{\sqrt{c_1 x^2 - 1}}$$

✓ Solution by Mathematica

Time used: 0.438 (sec). Leaf size: 76

```
DSolve[x*y'[x]+y[x]*(y[x]^2+1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{ie^{c_1}}{\sqrt{-x^2 + e^{2c_1}}}$$
$$y(x) \to \frac{ie^{c_1}}{\sqrt{-x^2 + e^{2c_1}}}$$
$$y(x) \to 0$$
$$y(x) \to -i$$
$$y(x) \to i$$

8.29 problem 30

Internal problem ID [2061]
Internal file name [`OUTPUT/2061_Sunday_February_25_2024_06_47_30_AM_12376118/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{\sqrt{x^2+y^2}\,y+yx-y'x^2=0}$$

8.29.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y\left(\sqrt{x^2+y^2}+x\right)}{x^2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{y\left(\sqrt{x^2+y^2}+x\right)(b_3-a_2)}{x^2} - \frac{y^2\left(\sqrt{x^2+y^2}+x\right)^2 a_3}{x^4} \\
&- \left(\frac{y\left(\frac{x}{\sqrt{x^2+y^2}}+1\right)}{x^2} - \frac{2y\left(\sqrt{x^2+y^2}+x\right)}{x^3}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{\sqrt{x^2+y^2}+x}{x^2} + \frac{y^2}{\sqrt{x^2+y^2}\,x^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{\left(x^2+y^2\right)^{\frac{3}{2}} y^2 a_3 + x^5 b_2 + x^3 y^2 a_3 + 2x^3 y^2 b_2 - x^2 y^3 a_2 + x^2 y^3 b_3 + \sqrt{x^2+y^2}\,x^3 b_1 - \sqrt{x^2+y^2}\,x^2 y a_1 + x^4 b_1}{x^4\sqrt{x^2+y^2}}
$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-\left(x^2+y^2\right)^{\frac{3}{2}} y^2 a_3 - x^5 b_2 - x^3 y^2 a_3 - 2x^3 y^2 b_2 + x^2 y^3 a_2 - x^2 y^3 b_3 \\
&- \sqrt{x^2+y^2}\,x^3 b_1 + \sqrt{x^2+y^2}\,x^2 y a_1 - x^4 b_1 + x^3 y a_1 - 2x^2 y^2 b_1 + 2x\,y^3 a_1 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-\left(x^2+y^2\right)^{\frac{3}{2}} y^2 a_3 - \left(x^2+y^2\right) x^3 b_2 + \left(x^2+y^2\right) x^2 y a_2 - x^4 y a_2 \\
&- x^3 y^2 a_3 - x^3 y^2 b_2 - x^2 y^3 b_3 - \left(x^2+y^2\right) x^2 b_1 + 2\left(x^2+y^2\right) x y a_1 \\
&- \sqrt{x^2+y^2}\,x^3 b_1 + \sqrt{x^2+y^2}\,x^2 y a_1 - x^3 y a_1 - x^2 y^2 b_1 = 0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-x^5 b_2 - x^3 y^2 a_3 - 2x^3 y^2 b_2 - x^2\sqrt{x^2+y^2}\,y^2 a_3 + x^2 y^3 a_2 - x^2 y^3 b_3 - \sqrt{x^2+y^2}\,y^4 a_3 \\
&- x^4 b_1 - \sqrt{x^2+y^2}\,x^3 b_1 + x^3 y a_1 + \sqrt{x^2+y^2}\,x^2 y a_1 - 2x^2 y^2 b_1 + 2x\,y^3 a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & v_1^2 v_2^3 a_2 - v_1^3 v_2^2 a_3 - v_1^2 v_3 v_2^2 a_3 - v_3 v_2^4 a_3 - v_1^5 b_2 - 2 v_1^3 v_2^2 b_2 - v_1^2 v_2^3 b_3 \\ & + v_1^3 v_2 a_1 + v_3 v_1^2 v_2 a_1 + 2 v_1 v_2^3 a_1 - v_1^4 b_1 - v_3 v_1^3 b_1 - 2 v_1^2 v_2^2 b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} & -v_1^5 b_2 - v_1^4 b_1 + (-a_3 - 2b_2) v_1^3 v_2^2 + v_1^3 v_2 a_1 - v_3 v_1^3 b_1 + (-b_3 + a_2) v_1^2 v_2^3 \\ & - v_1^2 v_3 v_2^2 a_3 - 2 v_1^2 v_2^2 b_1 + v_3 v_1^2 v_2 a_1 + 2 v_1 v_2^3 a_1 - v_3 v_2^4 a_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ 2a_1 &= 0 \\ -a_3 &= 0 \\ -2b_1 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ -a_3 - 2b_2 &= 0 \\ -b_3 + a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y\left(\sqrt{x^2+y^2}+x\right)}{x^2}\right)(x) \\
&= -\frac{y\sqrt{x^2+y^2}}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{y\sqrt{x^2+y^2}}{x}} dy
\end{aligned}
$$

Which results in

$$S = \frac{x \ln\left(\frac{2x^2+2\sqrt{x^2}\,\sqrt{x^2+y^2}}{y}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y\left(\sqrt{x^2+y^2}+x\right)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}\,x} \\
S_y &= -\frac{x}{y\sqrt{x^2+y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(2)+\ln(x)-\ln(y)+\ln\left(x+\sqrt{x^2+y^2}\right)=c_1$$

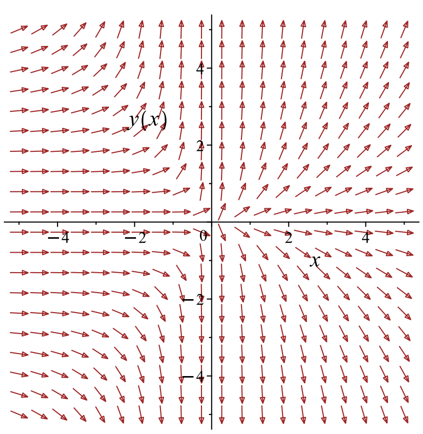
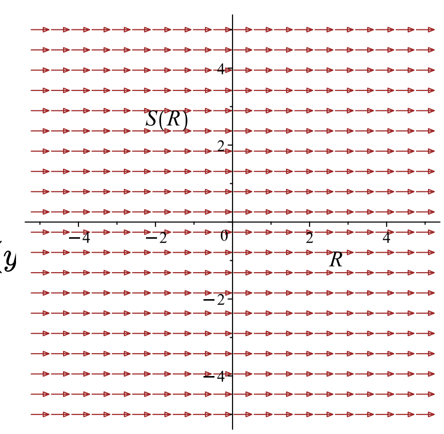
Which simplifies to

$$\ln(2)+\ln(x)-\ln(y)+\ln\left(x+\sqrt{x^2+y^2}\right)=c_1$$

Which gives

$$y=\frac{4\,\mathrm{e}^{c_1}x^2}{-4x^2+\mathrm{e}^{2c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=\frac{y\left(\sqrt{x^2+y^2}+x\right)}{x^2}$ 	$R=x$ $S=\ln(2)+\ln(x)-\ln(y$	$\frac{dS}{dR}=0$ 

Summary
The solution(s) found are the following

$$y=\frac{4\,\mathrm{e}^{c_1}x^2}{-4x^2+\mathrm{e}^{2c_1}} \tag{1}$$

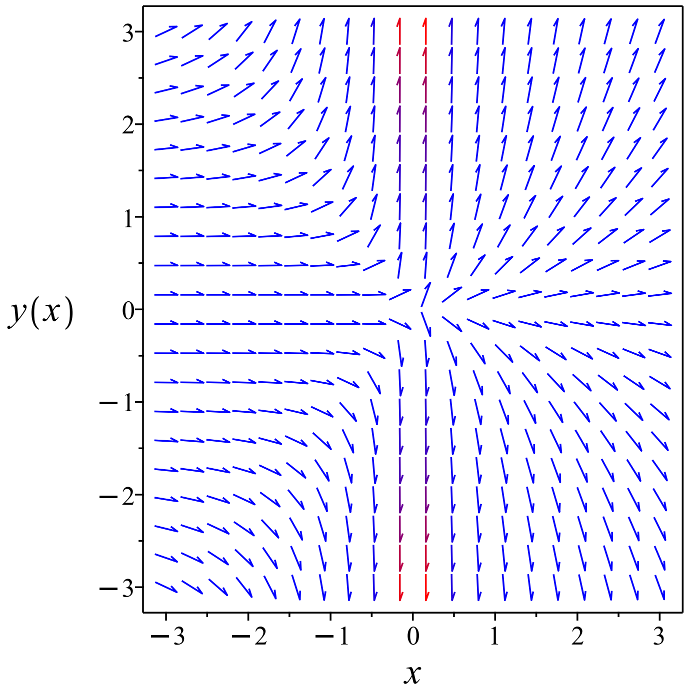


Figure 460: Slope field plot

Verification of solutions

$$y = \frac{4\,e^{c_1}x^2}{-4x^2 + e^{2c_1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 29

```
dsolve(y(x)*sqrt(x^2+y(x)^2)+x*y(x)=x^2*diff(y(x),x),y(x), singsol=all)
```

$$\frac{-c_1 y(x)+\sqrt{x^2+y\left(x\right)^2}\, x+x^2}{y\left(x\right)}=0$$

✓ Solution by Mathematica

Time used: 0.295 (sec). Leaf size: 47

```
DSolve[y[x]*Sqrt[x^2+y[x]^2]+x*y[x]==x^2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -x\sqrt{-\mathrm{sech}^2(\log(x)+c_1)}$$
$$y(x)\to x\sqrt{-\mathrm{sech}^2(\log(x)+c_1)}$$
$$y(x)\to 0$$

8.30 problem 31

Internal problem ID [2062]
Internal file name [`OUTPUT/2062_Sunday_February_25_2024_06_47_32_AM_45381210/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{3\,\mathrm{e}^{x}\tan\left(y\right)-\left(-\mathrm{e}^{x}+1\right)\sec\left(y\right)^{2}y'=0}$$

8.30.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= -\frac{3\,\mathrm{e}^{x}\sin\left(2y\right)}{2\left(\mathrm{e}^{x}-1\right)} \end{aligned}$$

Where $f(x)=-\frac{3\,\mathrm{e}^{x}}{\mathrm{e}^{x}-1}$ and $g(y)=\frac{\sin(2y)}{2}$. Integrating both sides gives

$$\frac{1}{\frac{\sin(2y)}{2}}\,dy=-\frac{3\,\mathrm{e}^{x}}{\mathrm{e}^{x}-1}\,dx$$

$$\int\frac{1}{\frac{\sin(2y)}{2}}\,dy=\int-\frac{3\,\mathrm{e}^{x}}{\mathrm{e}^{x}-1}\,dx$$

$$\ln \left(\csc \left(2y\right)-\cot \left(2y\right)\right) = -3\ln \left({\mathrm e}^{x}-1\right)+c_1$$

Raising both side to exponential gives

$$\csc \left(2y\right)-\cot \left(2y\right) = {\mathrm e}^{-3\ln \left({\mathrm e}^{x}-1\right)+c_1}$$

Which simplifies to

$$\csc \left(2y\right)-\cot \left(2y\right) = \frac{c_2}{\left({\mathrm e}^{x}-1\right)^{3}}$$

Summary

The solution(s) found are the following

$$y = \frac{\arctan \left(\frac{2c_2 {\mathrm e}^{c_1} \left({\mathrm e}^{3x}-3\,{\mathrm e}^{2x}+3\,{\mathrm e}^{x}-1\right)}{{\mathrm e}^{6x}-6\,{\mathrm e}^{5x}+c_2^{2} {\mathrm e}^{2c_1}+15\,{\mathrm e}^{4x}-20\,{\mathrm e}^{3x}+15\,{\mathrm e}^{2x}-6\,{\mathrm e}^{x}+1}, -\frac{-{\mathrm e}^{6x}+6\,{\mathrm e}^{5x}+c_2^{2} {\mathrm e}^{2c_1}-15\,{\mathrm e}^{4x}+20\,{\mathrm e}^{3x}-15\,{\mathrm e}^{2x}+6\,{\mathrm e}^{x}-1}{{\mathrm e}^{6x}-6\,{\mathrm e}^{5x}+c_2^{2} {\mathrm e}^{2c_1}+15\,{\mathrm e}^{4x}-20\,{\mathrm e}^{3x}+15\,{\mathrm e}^{2x}-6\,{\mathrm e}^{x}+1}\right)}{2} \tag{1}$$

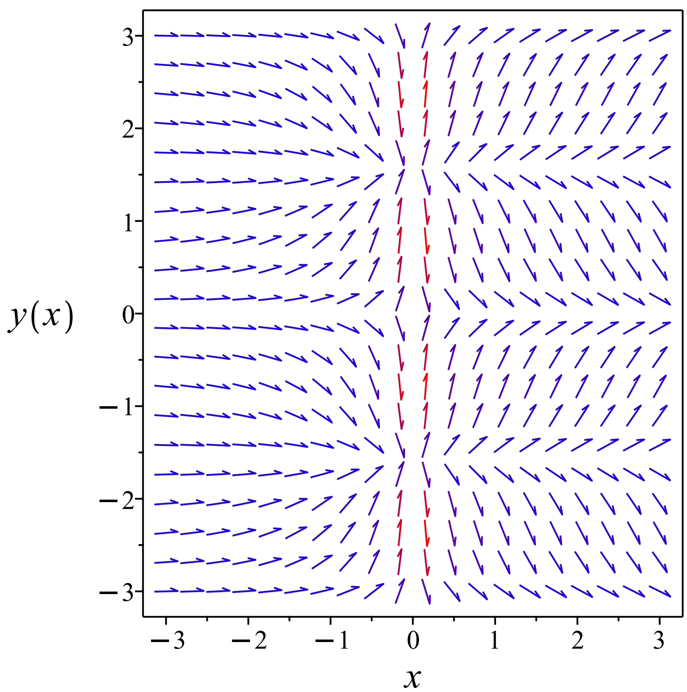


Figure 461: Slope field plot

Verification of solutions

$$y = \frac{\arctan \left(\frac{2c_2 {\mathrm e}^{c_1} \left({\mathrm e}^{3x}-3\,{\mathrm e}^{2x}+3\,{\mathrm e}^{x}-1\right)}{{\mathrm e}^{6x}-6\,{\mathrm e}^{5x}+c_2^{2} {\mathrm e}^{2c_1}+15\,{\mathrm e}^{4x}-20\,{\mathrm e}^{3x}+15\,{\mathrm e}^{2x}-6\,{\mathrm e}^{x}+1}, -\frac{-{\mathrm e}^{6x}+6\,{\mathrm e}^{5x}+c_2^{2} {\mathrm e}^{2c_1}-15\,{\mathrm e}^{4x}+20\,{\mathrm e}^{3x}-15\,{\mathrm e}^{2x}+6\,{\mathrm e}^{x}-1}{{\mathrm e}^{6x}-6\,{\mathrm e}^{5x}+c_2^{2} {\mathrm e}^{2c_1}+15\,{\mathrm e}^{4x}-20\,{\mathrm e}^{3x}+15\,{\mathrm e}^{2x}-6\,{\mathrm e}^{x}+1}\right)}{2}$$

Verified OK.

8.30.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3\,\mathrm{e}^{x}\tan(y)}{(\mathrm{e}^{x}-1)\sec(y)^{2}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 269: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{\mathrm{e}^{-x}(\mathrm{e}^{x}-1)}{3} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{\mathrm{e}^{-x}(\mathrm{e}^{x}-1)}{3}} dx
\end{aligned}
$$

Which results in

$$
S = -3\ln\left(\mathrm{e}^{x}-1\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{3\,\mathrm{e}^{x}\tan(y)}{(\mathrm{e}^{x}-1)\sec(y)^{2}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{3\,\mathrm{e}^x}{\mathrm{e}^x - 1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \sec(y)\csc(y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \sec(R)\csc(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(\tan(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-3\ln(\mathrm{e}^x - 1) = \ln(\tan(y)) + c_1
$$

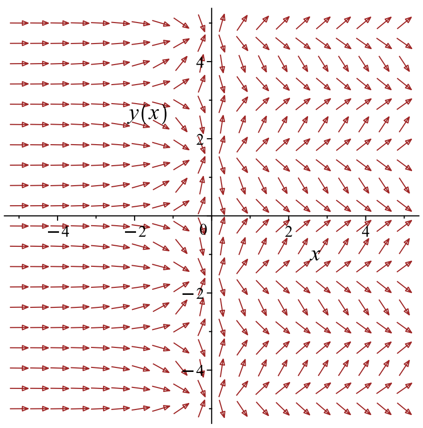
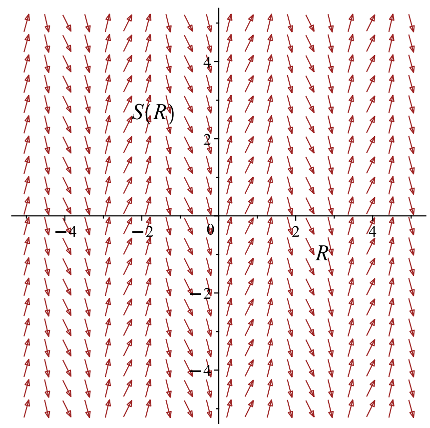
Which simplifies to

$$
-3\ln(\mathrm{e}^x - 1) = \ln(\tan(y)) + c_1
$$

Which gives

$$
y = \arctan\left(\frac{\mathrm{e}^{-c_1}}{(\mathrm{e}^x - 1)^3}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3\,e^{x}\tan(y)}{(e^{x}-1)\sec(y)^{2}}$ 	$R = y$ $S = -3\ln(e^{x} - 1)$	$\frac{dS}{dR} = \sec(R)\csc(R)$ 

Summary

The solution(s) found are the following

$$y = \arctan\left(\frac{e^{-c_1}}{(e^{x} - 1)^{3}}\right) \tag{1}$$

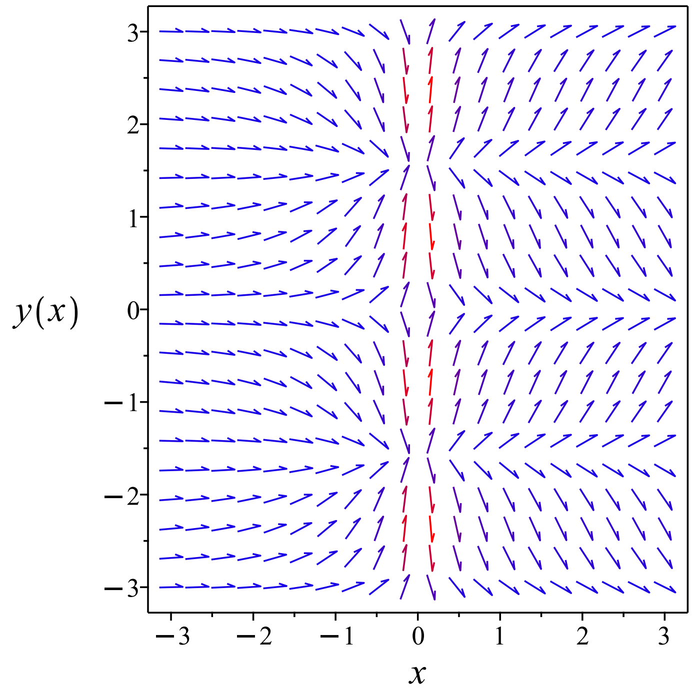


Figure 462: Slope field plot

Verification of solutions

$$y = \arctan\left(\frac{e^{-c_1}}{(e^x - 1)^3}\right)$$

Verified OK.

8.30.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{\sec(y)^2}{3\tan(y)}\right) \mathrm{d}y &= \left(\frac{\mathrm{e}^x}{\mathrm{e}^x - 1}\right) \mathrm{d}x \\ \left(-\frac{\mathrm{e}^x}{\mathrm{e}^x - 1}\right) \mathrm{d}x + \left(-\frac{\sec(y)^2}{3\tan(y)}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{\mathrm{e}^x}{\mathrm{e}^x - 1} \\ N(x, y) &= -\frac{\sec(y)^2}{3\tan(y)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\mathrm{e}^x}{\mathrm{e}^x - 1}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\sec(y)^2}{3\tan(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{\mathrm{e}^x}{\mathrm{e}^x - 1} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(\mathrm{e}^x - 1) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\sec(y)^2}{3\tan(y)}$. Therefore equation (4) becomes

$$
-\frac{\sec(y)^2}{3\tan(y)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -\frac{\sec(y)^2}{3\tan(y)} \\
&= -\frac{\sec(y)\csc(y)}{3}
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{\sec\left(y\right)\csc\left(y\right)}{3}\right)\mathrm{d}y$$

$$f(y) = -\frac{\ln\left(\tan\left(y\right)\right)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln\left(\mathrm{e}^x - 1\right) - \frac{\ln\left(\tan\left(y\right)\right)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln\left(\mathrm{e}^x - 1\right) - \frac{\ln\left(\tan\left(y\right)\right)}{3}$$

Summary

The solution(s) found are the following

$$-\ln\left(\mathrm{e}^x - 1\right) - \frac{\ln\left(\tan\left(y\right)\right)}{3} = c_1 \tag{1}$$

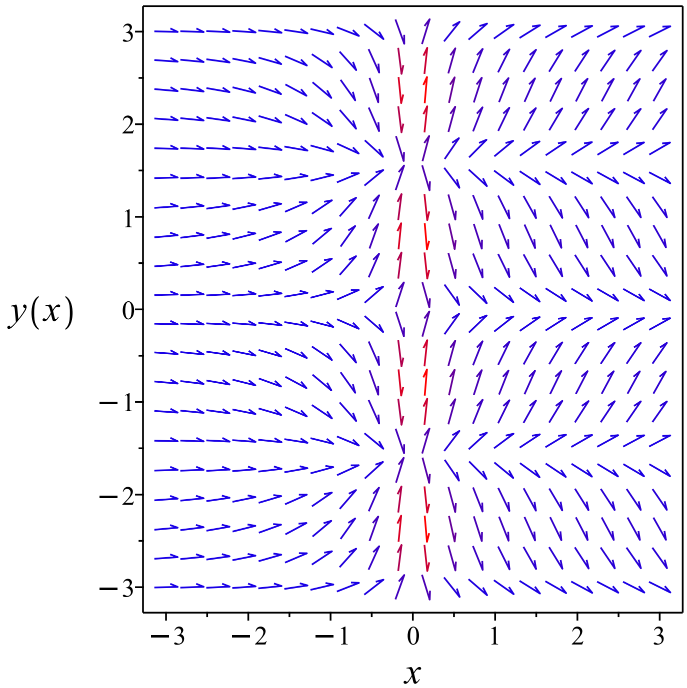


Figure 463: Slope field plot

Verification of solutions

$$-\ln\left(e^{x}-1\right)-\frac{\ln\left(\tan\left(y\right)\right)}{3}=c_1$$

Verified OK.

8.30.4 Maple step by step solution

Let's solve

$3\,e^{x}\tan\left(y\right)-\left(-e^{x}+1\right)\sec\left(y\right)^{2}y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\sec(y)^2}{\tan(y)}=\frac{3\,e^{x}}{-e^{x}+1}$

- Integrate both sides with respect to x

  $\int\frac{y'\sec(y)^2}{\tan(y)}dx=\int\frac{3\,e^{x}}{-e^{x}+1}dx+c_1$

- Evaluate integral

$$\ln(\tan(y)) = -3\ln(-e^x+1)+c_1$$

- Solve for y

$$y = -\arctan\left(\frac{e^{c_1}}{(e^x-1)^3}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 144

```
dsolve(3*exp(x)*tan(y(x))=(1-exp(x))*sec(y(x))^2*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{\arctan\left(\frac{2c_1(e^{3x}-3e^{2x}+3e^x-1)}{e^{6x}-6e^{5x}+15e^{4x}-20e^{3x}+15e^{2x}+c_1^2-6e^x+1}, \frac{e^{6x}-6e^{5x}+15e^{4x}-20e^{3x}+15e^{2x}-c_1^2-6e^x+1}{e^{6x}-6e^{5x}+15e^{4x}-20e^{3x}+15e^{2x}+c_1^2-6e^x+1}\right)}{2}$$

✓ Solution by Mathematica

Time used: 1.19 (sec). Leaf size: 78

```
DSolve[3*Exp[x]*Tan[y[x]]==(1-Exp[x])*Sec[y[x]]^2*y'[x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{1}{2}\arccos(-\tanh(-3\log(2-2e^x)+2c_1))$$
$$y(x) \to \frac{1}{2}\arccos(-\tanh(-3\log(2-2e^x)+2c_1))$$
$$y(x) \to 0$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

8.31 problem 32

Internal problem ID [2063]
Internal file name [`OUTPUT/2063_Sunday_February_25_2024_06_47_34_AM_34548290/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{\sec(y)^2 y' - \tan(y) = 2x\,e^{x}}$$

8.31.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\sec(y)^2\right)\mathrm{d}y&=(\tan(y)+2x\,\mathrm{e}^x)\,\mathrm{d}x\\(-\tan(y)-2x\,\mathrm{e}^x)\,\mathrm{d}x+\left(\sec(y)^2\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\tan(y)-2x\,\mathrm{e}^x\\N(x,y)&=\sec(y)^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-\tan(y)-2x\,\mathrm{e}^x)\\&=-\sec(y)^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(\sec(y)^2\right)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=\cos(y)^2\left(\left(-\tan(y)^2-1\right)-(0)\right)\\&=-1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x} \\ &= \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(-\tan(y) - 2x\,\mathrm{e}^{x}) \\ &= -2x - \mathrm{e}^{-x}\tan(y)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-x}\left(\sec(y)^2\right) \\ &= \sec(y)^2\,\mathrm{e}^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-2x - \mathrm{e}^{-x}\tan(y)\right) + \left(\sec(y)^2\,\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -2x - \mathrm{e}^{-x}\tan(y)\,\mathrm{d}x \\ \phi &= -x^2 + \mathrm{e}^{-x}\tan(y) + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \mathrm{e}^{-x}\left(1+\tan (y)^{2}\right)+f'(y) \qquad (4)\\
&= \sec (y)^{2} \mathrm{e}^{-x}+f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec (y)^{2} \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$
\sec (y)^{2} \mathrm{e}^{-x} = \sec (y)^{2} \mathrm{e}^{-x}+f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -x^{2}+\mathrm{e}^{-x} \tan (y)+c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -x^{2}+\mathrm{e}^{-x} \tan (y)
$$

Summary

The solution(s) found are the following

$$
-x^{2}+\mathrm{e}^{-x} \tan (y) = c_1 \qquad (1)
$$

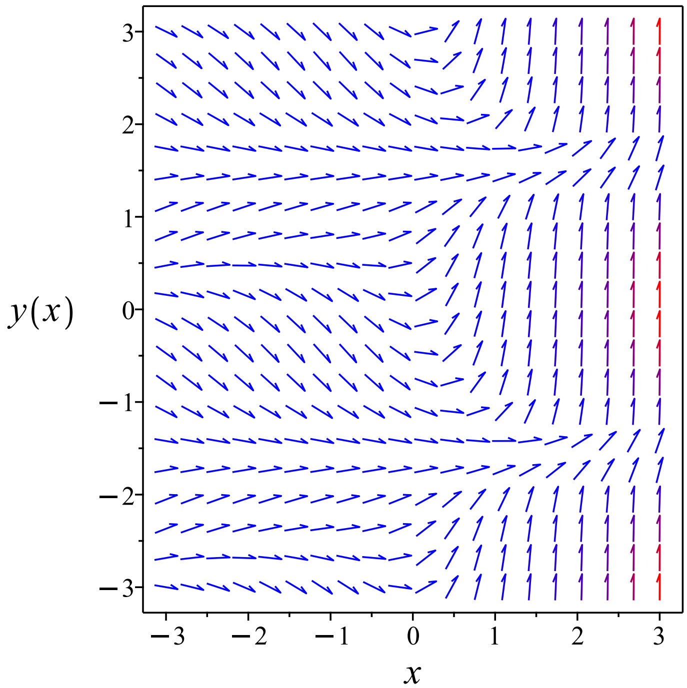


Figure 464: Slope field plot

Verification of solutions

$$-x^2 + e^{-x} \tan(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+y(x)*(x+1)/x, y(x)`      *** Sublevel 2 *
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2*tan(x)*y(x), y(x)`      *** Sublevel
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
```

✗ Solution by Maple

```
dsolve(sec(y(x))^2*diff(y(x),x)=tan(y(x))+2*x*exp(x),y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 10.872 (sec). Leaf size: 64

```
DSolve[Sec[y[x]]^2*y'[x]==Tan[y[x]]+2*x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \arctan\left(e^x\left(x^2+2c_1\right)\right)$$
$$y(x) \rightarrow -\frac{1}{2}\pi e^{-x}\sqrt{e^{2x}}$$
$$y(x) \rightarrow \frac{1}{2}\pi e^{-x}\sqrt{e^{2x}}$$

8.32 problem 33

Internal problem ID [2064]
Internal file name [`OUTPUT/2064_Sunday_February_25_2024_06_47_37_AM_30737233/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{2 \tan \left(y\right) x +3 y^{2}+\left(x^{2} \sec \left(y\right)^{2}+6 y x -y^{2}\right) y^{\prime} = -x^{2}}$$

8.32.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\sec(y)^2 x^2 + 6yx - y^2\right)\mathrm{d}y &= \left(-2x\tan(y) - 3y^2 - x^2\right)\mathrm{d}x\\ \left(2x\tan(y) + x^2 + 3y^2\right)\mathrm{d}x + \left(\sec(y)^2 x^2 + 6yx - y^2\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x\tan(y) + x^2 + 3y^2\\ N(x,y) &= \sec(y)^2 x^2 + 6yx - y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\tan(y) + x^2 + 3y^2\right)\\ &= 2x\sec(y)^2 + 6y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sec(y)^2 x^2 + 6yx - y^2\right)\\ &= 2x\sec(y)^2 + 6y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 2x \tan(y) + x^2 + 3y^2 \mathrm{d}x$$

$$\phi = \frac{x(3x \tan(y) + x^2 + 9y^2)}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x(3x(1 + \tan(y)^2) + 18y)}{3} + f'(y) \tag{4}$$

$$= (6y + x \sec(y)^2) x + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(y)^2 x^2 + 6yx - y^2$. Therefore equation (4) becomes

$$\sec(y)^2 x^2 + 6yx - y^2 = (6y + x \sec(y)^2) x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int (-y^2) \mathrm{d}y$$

$$f(y) = -\frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(3x \tan(y) + x^2 + 9y^2)}{3} - \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(3x\tan(y) + x^2 + 9y^2)}{3} - \frac{y^3}{3}$$

Summary
The solution(s) found are the following

$$\frac{x(3\tan(y)\,x + x^2 + 9y^2)}{3} - \frac{y^3}{3} = c_1 \tag{1}$$

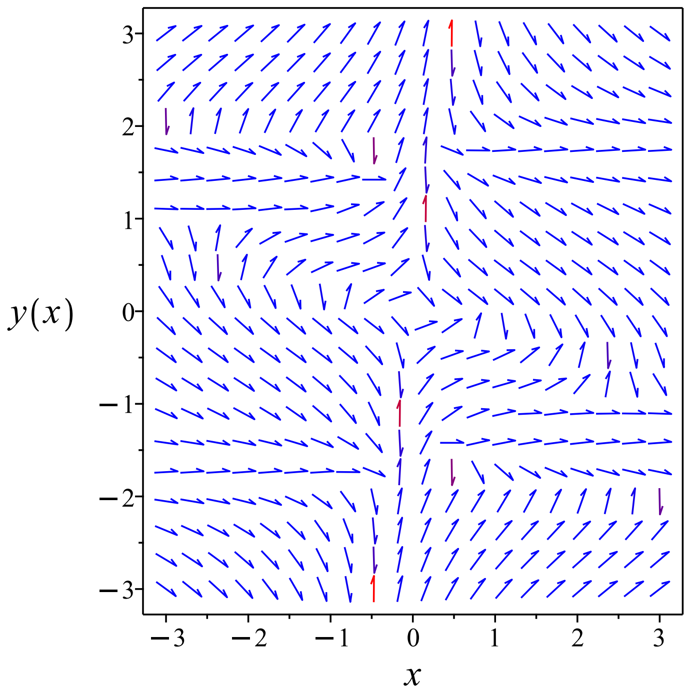


Figure 465: Slope field plot

Verification of solutions

$$\frac{x(3\tan(y)\,x + x^2 + 9y^2)}{3} - \frac{y^3}{3} = c_1$$

Verified OK.

8.32.2 Maple step by step solution

Let's solve

$2\tan(y)x+3y^2+\left(x^2\sec(y)^2+6yx-y^2\right)y'=-x^2$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $2x\left(1+\tan(y)^2\right)+6y=2x\sec(y)^2+6y$

  - Simplify

    $2x\sec(y)^2+6y=2x\sec(y)^2+6y$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(2x\tan(y)+x^2+3y^2)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=x^2\tan(y)+\frac{x^3}{3}+3x\,y^2+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\sec(y)^2x^2+6yx-y^2=x^2\left(1+\tan(y)^2\right)+6yx+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=\sec(y)^2x^2-y^2-x^2\left(1+\tan(y)^2\right)$

- Solve for $f_1(y)$

$f_1(y) = -\frac{y^3}{3}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = x^2\tan(y) + \frac{x^3}{3} + 3x\,y^2 - \frac{y^3}{3}$
- Substitute $F(x,y)$ into the solution of the ODE

  $x^2\tan(y) + \frac{x^3}{3} + 3x\,y^2 - \frac{y^3}{3} = c_1$
- Solve for y

  $y = \mathit{RootOf}\left(-3\tan(\_Z)\,x^2 + \_Z^3 - 9x\,\_Z^2 - x^3 + 3c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve((2*x*tan(y(x))+3*y(x)^2+x^2)+(x^2*sec(y(x))^2+6*x*y(x)-y(x)^2)*diff(y(x),x)=0,y(x), s
```

$$\tan(y(x))\,x^2 + \frac{x^3}{3} + 3xy(x)^2 - \frac{y(x)^3}{3} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.443 (sec). Leaf size: 87

```
DSolve[(2*x*Tan[y[x]]+3*y[x]^2+x^2)+(x^2*Sec[y[x]]^2+6*x*y[x]-y[x]^2)*y'[x]==0,y[x],x,Includ
```

$$\text{Solve}\left[\frac{1}{3}x^3\sec^2(y(x))+\frac{1}{3}x^3\cos(2y(x))\sec^2(y(x))+x^2\sin(2y(x))\sec^2(y(x))\right.$$
$$\left.-\frac{2y(x)^3}{3}+3xy(x)^2\sec^2(y(x))+3xy(x)^2\cos(2y(x))\sec^2(y(x))=c_1,y(x)\right]$$

8.33 problem 35

Internal problem ID [2065]
Internal file name [`OUTPUT/2065_Sunday_February_25_2024_06_48_55_AM_22368202/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y \cos\left(\frac{x}{y}\right) - \left(y + x \cos\left(\frac{x}{y}\right)\right) y' = 0}$$

8.33.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x \cos\left(\frac{1}{u(x)}\right) - \left(u(x)\,x + x \cos\left(\frac{1}{u(x)}\right)\right)(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2}{\left(\cos\left(\frac{1}{u}\right) + u\right)x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2}{\cos(\frac{1}{u})+u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{\cos(\frac{1}{u})+u}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2}{\cos(\frac{1}{u})+u}}\,du = \int -\frac{1}{x}\,dx$$

$$-\ln\left(\frac{1}{u}\right) - \sin\left(\frac{1}{u}\right) = -\ln(x) + c_2$$

The solution is

$$-\ln\left(\frac{1}{u(x)}\right) - \sin\left(\frac{1}{u(x)}\right) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\ln\left(\frac{x}{y}\right) - \sin\left(\frac{x}{y}\right) + \ln(x) - c_2 = 0$$
$$-\ln\left(\frac{x}{y}\right) - \sin\left(\frac{x}{y}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\ln\left(\frac{x}{y}\right) - \sin\left(\frac{x}{y}\right) + \ln(x) - c_2 = 0 \tag{1}$$

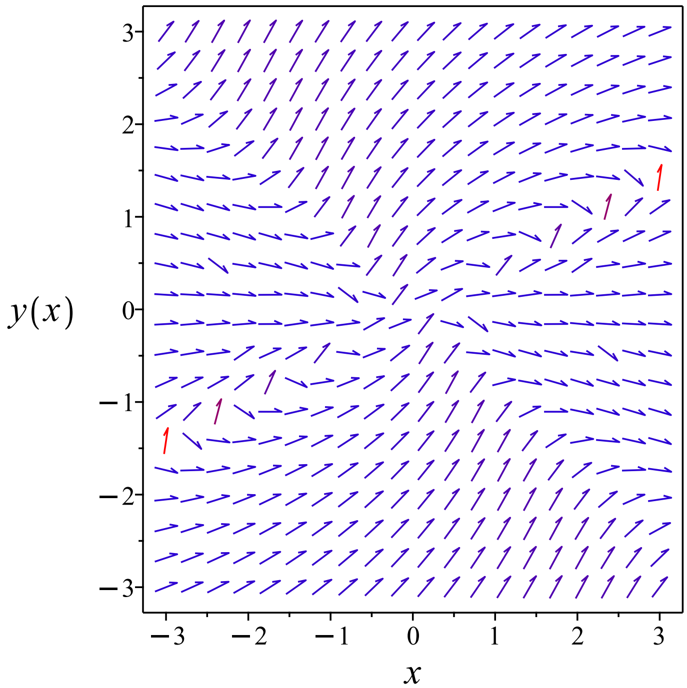


Figure 466: Slope field plot

Verification of solutions

$$-\ln\left(\frac{x}{y}\right) - \sin\left(\frac{x}{y}\right) + \ln(x) - c_2 = 0$$

Verified OK.

8.33.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y\cos\left(\frac{x}{y}\right)}{y + x\cos\left(\frac{x}{y}\right)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y \cos\left(\frac{x}{y}\right)(b_3 - a_2)}{y + x\cos\left(\frac{x}{y}\right)} - \frac{y^2 \cos\left(\frac{x}{y}\right)^2 a_3}{\left(y + x\cos\left(\frac{x}{y}\right)\right)^2} - \left(-\frac{\sin\left(\frac{x}{y}\right)}{y + x\cos\left(\frac{x}{y}\right)} - \frac{y\cos\left(\frac{x}{y}\right)\left(\cos\left(\frac{x}{y}\right) - \frac{x\sin\left(\frac{x}{y}\right)}{y}\right)}{\left(y + x\cos\left(\frac{x}{y}\right)\right)^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{\cos\left(\frac{x}{y}\right)}{y + x\cos\left(\frac{x}{y}\right)} + \frac{x\sin\left(\frac{x}{y}\right)}{y\left(y + x\cos\left(\frac{x}{y}\right)\right)} - \frac{y\cos\left(\frac{x}{y}\right)\left(1 + \frac{x^2\sin\left(\frac{x}{y}\right)}{y^2}\right)}{\left(y + x\cos\left(\frac{x}{y}\right)\right)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\cos\left(\frac{x}{y}\right)^2 xb_1 - \cos\left(\frac{x}{y}\right)^2 ya_1 - 2\cos\left(\frac{x}{y}\right) xyb_2 + \cos\left(\frac{x}{y}\right) y^2a_2 - \cos\left(\frac{x}{y}\right) y^2b_3 + \sin\left(\frac{x}{y}\right) x^2b_2 - \sin\left(\frac{x}{y}\right) x}{\left(y + x\cos\left(\frac{x}{y}\right)\right)^2} = 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-\cos\left(\frac{x}{y}\right)^{2} x b_1+\cos\left(\frac{x}{y}\right)^{2} y a_1+2\cos\left(\frac{x}{y}\right) x y b_2-\cos\left(\frac{x}{y}\right) y^{2} a_2 \\
&+\cos\left(\frac{x}{y}\right) y^{2} b_3-\sin\left(\frac{x}{y}\right) x^{2} b_2+\sin\left(\frac{x}{y}\right) x y a_2-\sin\left(\frac{x}{y}\right) x y b_3 \\
&+\sin\left(\frac{x}{y}\right) y^{2} a_3-\sin\left(\frac{x}{y}\right) x b_1+\sin\left(\frac{x}{y}\right) y a_1+y^{2} b_2=0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
-\frac{y^{2}\left(-2y^{2}b_2+xb_1-ya_1+xb_1\cos\left(\frac{2x}{y}\right)-ya_1\cos\left(\frac{2x}{y}\right)-4\cos\left(\frac{x}{y}\right)xyb_2+2\cos\left(\frac{x}{y}\right)y^{2}a_2-2\cos\left(\frac{x}{y}\right)y^{2}\right.}{2} = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \cos\left(\frac{x}{y}\right), \cos\left(\frac{2x}{y}\right), \sin\left(\frac{x}{y}\right)\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \cos\left(\frac{x}{y}\right)=v_3, \cos\left(\frac{2x}{y}\right)=v_4, \sin\left(\frac{x}{y}\right)=v_5\right\}
$$

The above PDE (6E) now becomes

$$
-\frac{v_2^{2}(-2v_5v_1v_2a_2+2v_3v_2^{2}a_2-2v_5v_2^{2}a_3+2v_5v_1^{2}b_2-4v_3v_1v_2b_2+2v_5v_1v_2b_3-2v_3v_2^{2}b_3-v_2a_1v_4-2v_5v_2a_1+}{2} = 0 \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-v_2^{2}b_2v_5v_1^{2}+2b_2v_3v_1v_2^{3}+(-b_3+a_2)v_5v_1v_2^{3}-\frac{b_1v_1v_2^{2}}{2}-\frac{b_1v_4v_1v_2^{2}}{2}-b_1v_5v_1v_2^{2} \\
&+(b_3-a_2)v_3v_2^{4}+a_3v_5v_2^{4}+b_2v_2^{4}+\frac{a_1v_2^{3}}{2}+\frac{a_1v_4v_2^{3}}{2}+a_1v_5v_2^{3}=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
a_3 &= 0 \\
b_2 &= 0 \\
\frac{a_1}{2} &= 0 \\
-b_1 &= 0 \\
-\frac{b_1}{2} &= 0 \\
-b_2 &= 0 \\
2b_2 &= 0 \\
-b_3 + a_2 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y \cos\left(\frac{x}{y}\right)}{y + x \cos\left(\frac{x}{y}\right)}\right)(x) \\
&= \frac{y^2}{y + x \cos\left(\frac{x}{y}\right)} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{y + x \cos\left(\frac{x}{y}\right)}} dy \end{aligned}$$

Which results in

$$S = -\ln\left(\frac{1}{y}\right) - \sin\left(\frac{x}{y}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y \cos\left(\frac{x}{y}\right)}{y + x \cos\left(\frac{x}{y}\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\cos\left(\frac{x}{y}\right)}{y} \\ S_y &= \frac{y + x \cos\left(\frac{x}{y}\right)}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

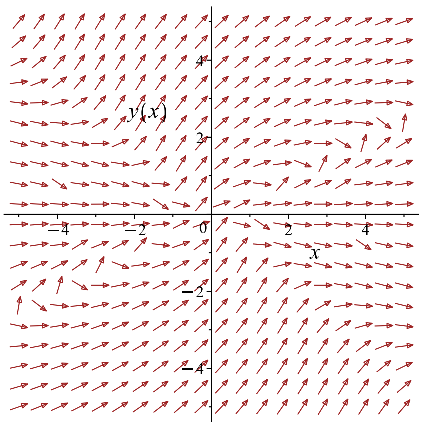
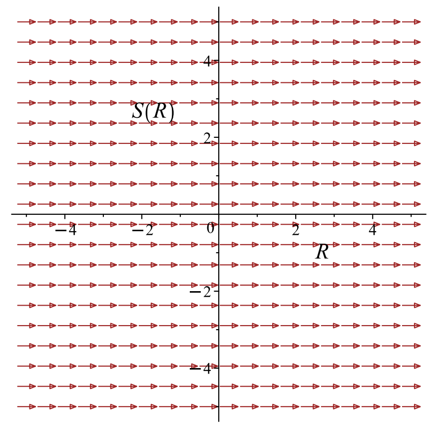
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - \sin\left(\frac{x}{y}\right) = c_1$$

Which simplifies to

$$\ln(y) - \sin\left(\frac{x}{y}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y\cos\left(\frac{x}{y}\right)}{y + x\cos\left(\frac{x}{y}\right)}$ 	$R = x$ $S = \ln(y) - \sin\left(\frac{x}{y}\right)$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\ln(y) - \sin\left(\frac{x}{y}\right) = c_1 \tag{1}$$

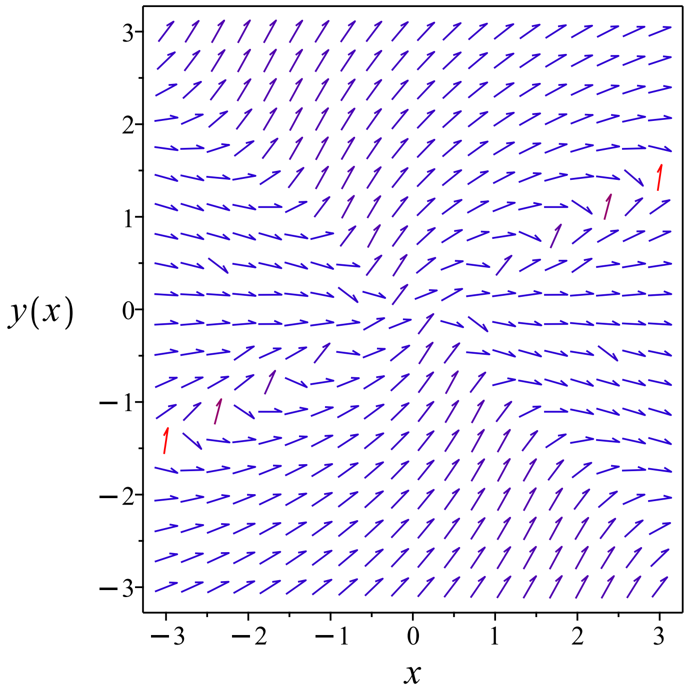


Figure 467: Slope field plot

Verification of solutions

$$\ln(y) - \sin\left(\frac{x}{y}\right) = c_1$$

Verified OK.

8.33.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\left(-y - x\cos\left(\frac{x}{y}\right)\right)\mathrm{d}y = \left(-y\cos\left(\frac{x}{y}\right)\right)\mathrm{d}x
$$

$$
\left(y\cos\left(\frac{x}{y}\right)\right)\mathrm{d}x + \left(-y - x\cos\left(\frac{x}{y}\right)\right)\mathrm{d}y = 0 \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= y\cos\left(\frac{x}{y}\right) \\
N(x,y) &= -y - x\cos\left(\frac{x}{y}\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\cos\left(\frac{x}{y}\right)\right) \\
&= \cos\left(\frac{x}{y}\right) + \frac{x\sin\left(\frac{x}{y}\right)}{y}
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-y - x\cos\left(\frac{x}{y}\right)\right) \\ &= -\cos\left(\frac{x}{y}\right) + \frac{x\sin\left(\frac{x}{y}\right)}{y}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{-y - x\cos\left(\frac{x}{y}\right)}\left(\left(\cos\left(\frac{x}{y}\right) + \frac{x\sin\left(\frac{x}{y}\right)}{y}\right) - \left(-\cos\left(\frac{x}{y}\right) + \frac{x\sin\left(\frac{x}{y}\right)}{y}\right)\right) \\ &= -\frac{2\cos\left(\frac{x}{y}\right)}{y + x\cos\left(\frac{x}{y}\right)}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{\sec\left(\frac{x}{y}\right)}{y}\left(\left(-\cos\left(\frac{x}{y}\right) + \frac{x\sin\left(\frac{x}{y}\right)}{y}\right) - \left(\cos\left(\frac{x}{y}\right) + \frac{x\sin\left(\frac{x}{y}\right)}{y}\right)\right) \\ &= -\frac{2}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y} \\ &= e^{\int -\frac{2}{y}\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2}\left(y\cos\left(\frac{x}{y}\right)\right) \\
&= \frac{\cos\left(\frac{x}{y}\right)}{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}\left(-y - x\cos\left(\frac{x}{y}\right)\right) \\
&= \frac{-y - x\cos\left(\frac{x}{y}\right)}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\cos\left(\frac{x}{y}\right)}{y}\right) + \left(\frac{-y - x\cos\left(\frac{x}{y}\right)}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{\cos\left(\frac{x}{y}\right)}{y}\,\mathrm{d}x \\
\phi &= \sin\left(\frac{x}{y}\right) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x \cos\left(\frac{x}{y}\right)}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-y - x\cos\left(\frac{x}{y}\right)}{y^2}$. Therefore equation (4) becomes

$$\frac{-y - x \cos\left(\frac{x}{y}\right)}{y^2} = -\frac{x \cos\left(\frac{x}{y}\right)}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin\left(\frac{x}{y}\right) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin\left(\frac{x}{y}\right) - \ln(y)$$

Summary

The solution(s) found are the following

$$-\ln(y) + \sin\left(\frac{x}{y}\right) = c_1 \tag{1}$$

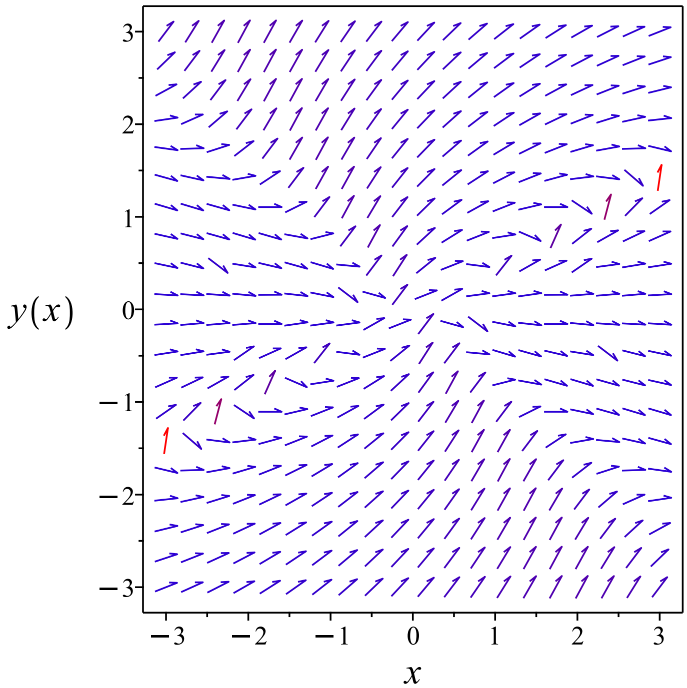


Figure 468: Slope field plot

Verification of solutions

$$-\ln(y) + \sin\left(\frac{x}{y}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve((y(x)*cos(x/y(x)))-(y(x)+x*cos(x/y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{\text{RootOf}\left(-\_Z\,\mathrm{e}^{\sin(\_Z)} + c_1 x\right)}$$

✓ Solution by Mathematica

Time used: 0.185 (sec). Leaf size: 28

```
DSolve[(y[x]*Cos[x/y[x]])-(y[x]+x*Cos[x/y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$\text{Solve}\left[\log\left(\frac{y(x)}{x}\right) - \sin\left(\frac{x}{y(x)}\right) = -\log(x) + c_1, y(x)\right]$$

8.34 problem 36

Internal problem ID [2066]
Internal file name [`OUTPUT/2066_Sunday_February_25_2024_06_48_56_AM_37918308/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y(3x^2+y)-x(x^2-y)\,y'=0}$$

8.34.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(3x^2+y)}{x\left(-x^2+y\right)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 - \frac{y(3x^2+y)\,(b_3-a_2)}{x\,(-x^2+y)} - \frac{y^2(3x^2+y)^2\,a_3}{x^2\,(-x^2+y)^2}\\ &- \left(-\frac{6y}{-x^2+y} + \frac{y(3x^2+y)}{x^2\,(-x^2+y)} - \frac{2y(3x^2+y)}{(-x^2+y)^2}\right)(xa_2+ya_3+a_1)\\ &- \left(-\frac{3x^2+y}{x\,(-x^2+y)} - \frac{y}{x\,(-x^2+y)} + \frac{y(3x^2+y)}{x\,(-x^2+y)^2}\right)(xb_2+yb_3+b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^6b_2 + 6x^4y^2a_3 + 3x^5b_1 - 3x^4ya_1 + 4x^4yb_2 - 8x^3y^2a_2 + 4x^3y^2b_3 + 2x^3yb_1 - 6x^2y^2a_1 - 2x^2y^2b_2 + 2y^4a}{(x^2-y)^2\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}&-2x^6b_2 - 6x^4y^2a_3 - 3x^5b_1 + 3x^4ya_1 - 4x^4yb_2 + 8x^3y^2a_2 - 4x^3y^2b_3\\ &- 2x^3yb_1 + 6x^2y^2a_1 + 2x^2y^2b_2 - 2y^4a_3 + x\,y^2b_1 - y^3a_1 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-6a_3v_1^4v_2^2 - 2b_2v_1^6 + 3a_1v_1^4v_2 + 8a_2v_1^3v_2^2 - 3b_1v_1^5 - 4b_2v_1^4v_2 - 4b_3v_1^3v_2^2\\ &+ 6a_1v_1^2v_2^2 - 2a_3v_2^4 - 2b_1v_1^3v_2 + 2b_2v_1^2v_2^2 - a_1v_2^3 + b_1v_1v_2^2 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-2b_2v_1^6 - 3b_1v_1^5 - 6a_3v_1^4v_2^2 + (3a_1 - 4b_2)\, v_1^4v_2 + (8a_2 - 4b_3)\, v_1^3v_2^2 - 2b_1v_1^3v_2 + (6a_1 + 2b_2)\, v_1^2v_2^2 + b_1v_1v_2^2 - 2a_3v_2^4 - a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_1 &= 0\\
-a_1 &= 0\\
-6a_3 &= 0\\
-2a_3 &= 0\\
-3b_1 &= 0\\
-2b_1 &= 0\\
-2b_2 &= 0\\
3a_1 - 4b_2 &= 0\\
6a_1 + 2b_2 &= 0\\
8a_2 - 4b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= 2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= 2y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 2y - \left(-\frac{y(3x^2+y)}{x\,(-x^2+y)}\right)(x) \\
&= \frac{-y\,x^2 - 3y^2}{x^2 - y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y\,x^2-3y^2}{x^2-y}} dy
\end{aligned}$$

Which results in

$$S = -\ln(y) + \frac{4\ln(x^2+3y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(3x^2+y)}{x\,(-x^2+y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{8x}{3x^2 + 9y} \\
S_y &= -\frac{1}{y} + \frac{4}{x^2 + 3y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{3x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{3R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{3} + c_1 \tag{4}
$$

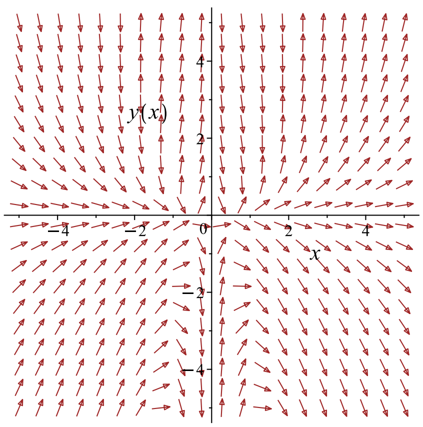
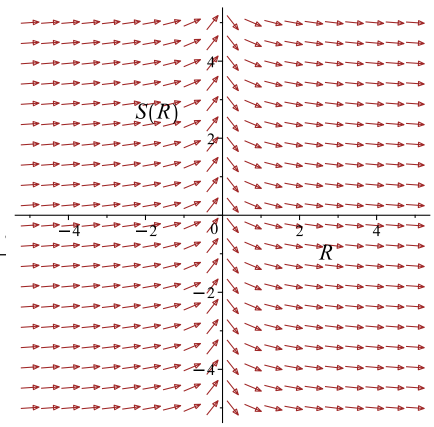
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(y) + \frac{4\ln(x^2 + 3y)}{3} = -\frac{\ln(x)}{3} + c_1
$$

Which simplifies to

$$
-\ln(y) + \frac{4\ln(x^2 + 3y)}{3} = -\frac{\ln(x)}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(3x^2+y)}{x(-x^2+y)}$ 	$R = x$ $S = -\ln(y) + \frac{4\ln\left(x^2 + \right.}{3}$	$\frac{dS}{dR} = -\frac{1}{3R}$ 

Summary

The solution(s) found are the following

$$-\ln(y) + \frac{4\ln(x^2+3y)}{3} = -\frac{\ln(x)}{3} + c_1 \tag{1}$$

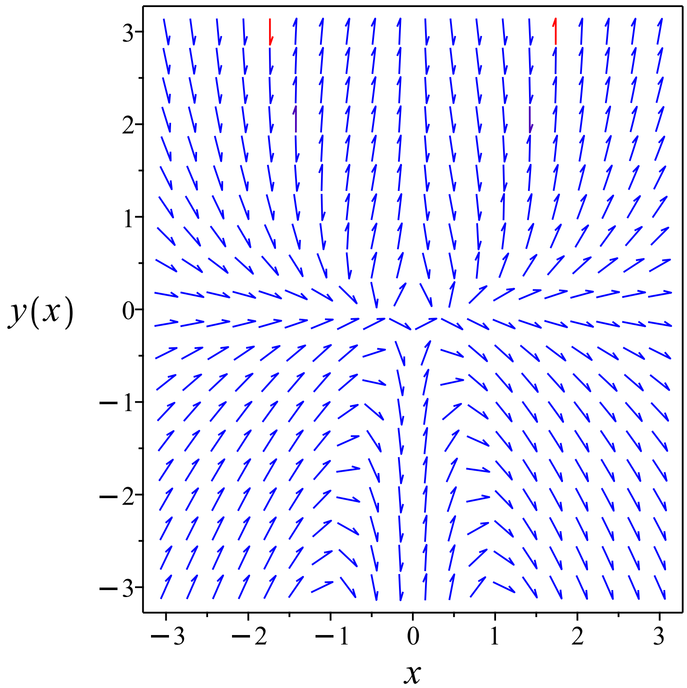


Figure 469: Slope field plot

Verification of solutions

$$-\ln(y) + \frac{4\ln(x^2 + 3y)}{3} = -\frac{\ln(x)}{3} + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 43

```
dsolve(y(x)*(3*x^2+y(x))-x*(x^2-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^2 \left(\operatorname{RootOf}\left(\_Z^4 c_1 - \_Z c_1 + 3x\right)^3 - 1\right)}{3 \operatorname{RootOf}\left(\_Z^4 c_1 - \_Z c_1 + 3x\right)^3}$$

✓ Solution by Mathematica

Time used: 60.121 (sec). Leaf size: 1665

```
DSolve[y[x]*(3*x^2+y[x])-x*(x^2-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

8.35 problem 37

Internal problem ID [2067]
Internal file name [`OUTPUT/2067_Sunday_February_25_2024_06_48_58_AM_17926469/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class C`], _dAlembert]
```

$$\boxed{(2x+3y+2)\,y' = -x}$$

8.35.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{X + x_0}{2X + 2x_0 + 3Y(X) + 3y_0 + 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = -\frac{2}{3}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{X}{2X + 3Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= -\frac{X}{2X+3Y} \qquad (1)
\end{aligned}
$$

An ode of the form $Y'=\frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M=-X$ and $N=2X+3Y$ are both homogeneous and of the same order $n=1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u=\frac{Y}{X}$, or $Y=uX$. Hence

$$
\frac{dY}{dX} = \frac{du}{dX}X + u
$$

Applying the transformation $Y=uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{du}{dX}X + u &= -\frac{1}{3u+2} \\
\frac{du}{dX} &= \frac{-\frac{1}{3u(X)+2} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{-\frac{1}{3u(X)+2} - u(X)}{X} = 0
$$

Or

$$
3\left(\frac{d}{dX}u(X)\right)Xu(X) + 2\left(\frac{d}{dX}u(X)\right)X + 3u(X)^2 + 2u(X) + 1 = 0
$$

Or

$$
1 + X(3u(X)+2)\left(\frac{d}{dX}u(X)\right) + 3u(X)^2 + 2u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{3u^2+2u+1}{X(3u+2)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{3u^2+2u+1}{3u+2}$. Integrating both sides gives

$$\frac{1}{\frac{3u^2+2u+1}{3u+2}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{3u^2+2u+1}{3u+2}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(3u^2+2u+1)}{2} + \frac{\sqrt{2}\arctan\left(\frac{(6u+2)\sqrt{2}}{4}\right)}{2} = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(3u(X)^2+2u(X)+1\right)}{2} + \frac{\sqrt{2}\arctan\left(\frac{(6u(X)+2)\sqrt{2}}{4}\right)}{2} + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{3Y(X)^2}{X^2} + \frac{2Y(X)}{X} + 1\right)}{2} + \frac{\sqrt{2}\arctan\left(\frac{\left(\frac{6Y(X)}{X}+2\right)\sqrt{2}}{4}\right)}{2} + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{3Y(X)^2}{X^2} + \frac{2Y(X)}{X} + 1\right)}{2} + \frac{\sqrt{2}\arctan\left(\frac{(3Y(X)+X)\sqrt{2}}{2X}\right)}{2} + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - \frac{2}{3}$$
$$X = x$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{3\left(y+\frac{2}{3}\right)^2}{x^2} + \frac{2y+\frac{4}{3}}{x} + 1\right)}{2} + \frac{\sqrt{2}\arctan\left(\frac{(3y+2+x)\sqrt{2}}{2x}\right)}{2} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{3\left(y+\frac{2}{3}\right)^2}{x^2}+\frac{2y+\frac{4}{3}}{x}+1\right)}{2}+\frac{\sqrt{2}\,\arctan\left(\frac{(3y+2+x)\sqrt{2}}{2x}\right)}{2}+\ln\left(x\right)-c_2=0 \tag{1}$$

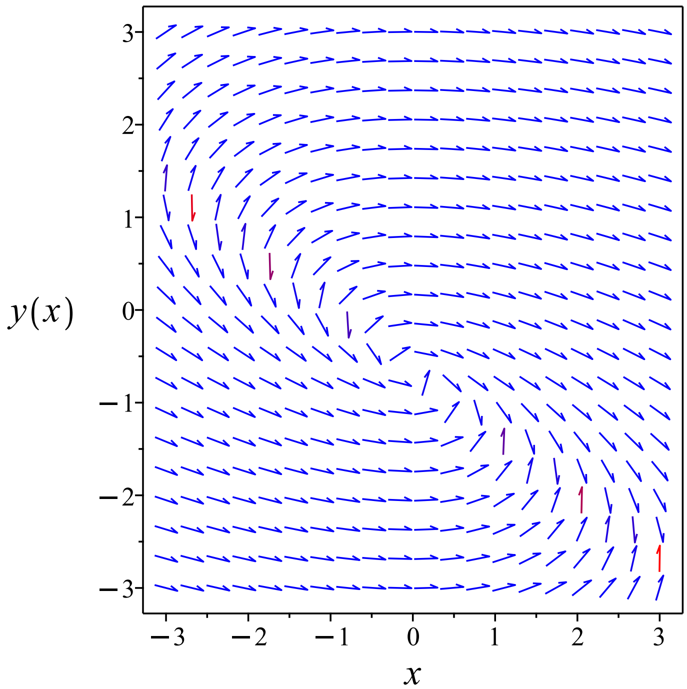


Figure 470: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{3\left(y+\frac{2}{3}\right)^2}{x^2}+\frac{2y+\frac{4}{3}}{x}+1\right)}{2}+\frac{\sqrt{2}\,\arctan\left(\frac{(3y+2+x)\sqrt{2}}{2x}\right)}{2}+\ln\left(x\right)-c_2=0$$

Verified OK.

8.35.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x}{2x+3y+2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{x(b_3 - a_2)}{2x+3y+2} - \frac{x^2 a_3}{(2x+3y+2)^2} \\
&- \left(-\frac{1}{2x+3y+2} + \frac{2x}{(2x+3y+2)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \frac{3x(x b_2 + y b_3 + b_1)}{(2x+3y+2)^2} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x^2a_2 - x^2a_3 + x^2b_2 - 2x^2b_3 + 6xya_2 + 12xyb_2 - 6xyb_3 + 3y^2a_3 + 9y^2b_2 + 4xa_2 - 3xb_1 + 8xb_2 - 2b_3x + }{(2x+3y+2)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&2x^2a_2 - x^2a_3 + x^2b_2 - 2x^2b_3 + 6xya_2 + 12xyb_2 - 6xyb_3 + 3y^2a_3 + 9y^2b_2 \\
&+ 4xa_2 - 3xb_1 + 8xb_2 - 2b_3x + 3ya_1 + 2ya_3 + 12yb_2 + 2a_1 + 4b_2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} 2a_2v_1^2 + 6a_2v_1v_2 - a_3v_1^2 + 3a_3v_2^2 + b_2v_1^2 + 12b_2v_1v_2 + 9b_2v_2^2 - 2b_3v_1^2 - 6b_3v_1v_2 \\ + 3a_1v_2 + 4a_2v_1 + 2a_3v_2 - 3b_1v_1 + 8b_2v_1 + 12b_2v_2 - 2b_3v_1 + 2a_1 + 4b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} (2a_2 - a_3 + b_2 - 2b_3) v_1^2 + (6a_2 + 12b_2 - 6b_3) v_1v_2 + (4a_2 - 3b_1 + 8b_2 - 2b_3) v_1 \\ + (3a_3 + 9b_2) v_2^2 + (3a_1 + 2a_3 + 12b_2) v_2 + 2a_1 + 4b_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 + 4b_2 &= 0 \\ 3a_3 + 9b_2 &= 0 \\ 3a_1 + 2a_3 + 12b_2 &= 0 \\ 6a_2 + 12b_2 - 6b_3 &= 0 \\ 2a_2 - a_3 + b_2 - 2b_3 &= 0 \\ 4a_2 - 3b_1 + 8b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -2b_2 \\ a_2 &= -2b_2 + \frac{3b_1}{2} \\ a_3 &= -3b_2 \\ b_1 &= b_1 \\ b_2 &= b_2 \\ b_3 &= \frac{3b_1}{2} \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = \frac{3x}{2}$$
$$\eta = 1 + \frac{3y}{2}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 + \frac{3y}{2} - \left(-\frac{x}{2x + 3y + 2}\right)\left(\frac{3x}{2}\right) \\
&= \frac{3x^2 + 6yx + 9y^2 + 4x + 12y + 4}{4x + 6y + 4} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2+6yx+9y^2+4x+12y+4}{4x+6y+4}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(3x^2 + 6yx + 9y^2 + 4x + 12y + 4)}{3} + \frac{\sqrt{2}\arctan\left(\frac{(18y+6x+12)\sqrt{2}}{12x}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x}{2x + 3y + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x}{3x^2 + (6y+4)\,x + 9\left(y + \frac{2}{3}\right)^2} \\
S_y &= \frac{4x + 6y + 4}{3x^2 + (6y+4)\,x + 9\left(y + \frac{2}{3}\right)^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

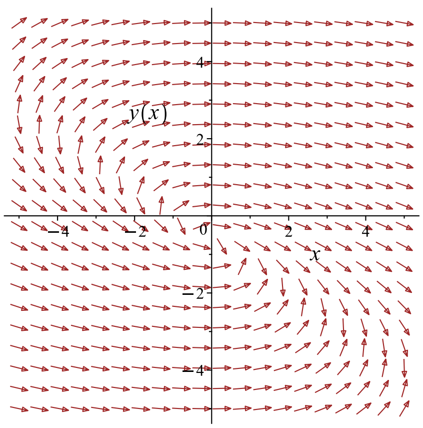
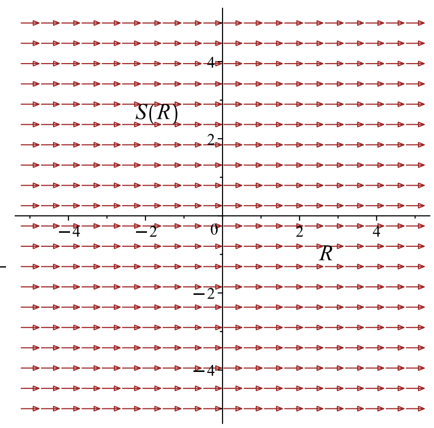
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(3x^2 + (6y+4)\,x + 9\left(y + \frac{2}{3}\right)^2\right)}{3} + \frac{\sqrt{2}\,\arctan\left(\frac{(3y+2+x)\sqrt{2}}{2x}\right)}{3} = c_1$$

Which simplifies to

$$\frac{\ln\left(3x^2+(6y+4)\,x+9\left(y+\frac{2}{3}\right)^2\right)}{3}+\frac{\sqrt{2}\,\arctan\left(\frac{(3y+2+x)\sqrt{2}}{2x}\right)}{3}=c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=-\frac{x}{2x+3y+2}$ 	$R=x$ $S=\frac{\ln\left(3x^2+(6y+4)\,x\right.}{3}$ $\frac{3y+2)\sqrt{2}}{2x}$	$\frac{dS}{dR}=0$ 

Summary

The solution(s) found are the following

$$\frac{\ln\left(3x^2+(6y+4)\,x+9\left(y+\frac{2}{3}\right)^2\right)}{3}+\frac{\sqrt{2}\,\arctan\left(\frac{(3y+2+x)\sqrt{2}}{2x}\right)}{3}=c_1 \tag{1}$$

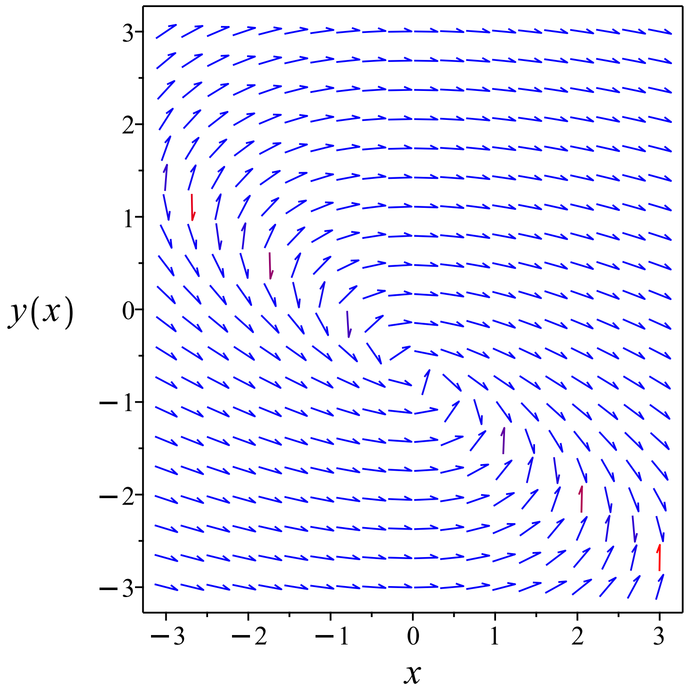


Figure 471: Slope field plot

Verification of solutions

$$\frac{\ln\left(3x^2+(6y+4)x+9\left(y+\frac{2}{3}\right)^2\right)}{3}+\frac{\sqrt{2}\,\arctan\left(\frac{(3y+2+x)\sqrt{2}}{2x}\right)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 52

```
dsolve(x+(2*x+3*y(x)+2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2}{3} + \frac{\sqrt{2}\, x \tan\left(\operatorname{RootOf}\left(\sqrt{2}\, \ln(3) + \sqrt{2}\, \ln\left(\sec(\_Z)^2 x^2\right) + \sqrt{2}\, \ln(2) + 2\sqrt{2}\, c_1 + 2\_Z\right)\right)}{3} - \frac{x}{3}$$

✓ Solution by Mathematica

Time used: 0.118 (sec). Leaf size: 78

```
DSolve[x+(2*x+3*y[x]+2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\sqrt{2}\arctan\left(\frac{-3y(x)+x-2}{\sqrt{2}(3y(x)+2x+2)}\right) = 2\log\left(\frac{3x^2+9y(x)^2+6(x+2)y(x)+4x+4}{3x^2}\right) + 4\log(x) + 3c_1, y(x)\right]$$

8.36 problem 38

Internal problem ID [2068]
Internal file name [`OUTPUT/2068_Sunday_February_25_2024_06_49_01_AM_38888723/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y'x - 5y - \sqrt{y}\, x = 0}$$

8.36.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{5y + x\sqrt{y}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 273: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \sqrt{y}\, x^{\frac{5}{2}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{y}\, x^{\frac{5}{2}}} dy \end{aligned}$$

Which results in

$$S = \frac{2\sqrt{y}}{x^{\frac{5}{2}}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{5y + x\sqrt{y}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{5\sqrt{y}}{x^{\frac{7}{2}}} \\ S_y &= \frac{1}{\sqrt{y}\, x^{\frac{5}{2}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^{\frac{5}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^{\frac{5}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{2}{3R^{\frac{3}{2}}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\sqrt{y}}{x^{\frac{5}{2}}} = -\frac{2}{3x^{\frac{3}{2}}} + c_1$$

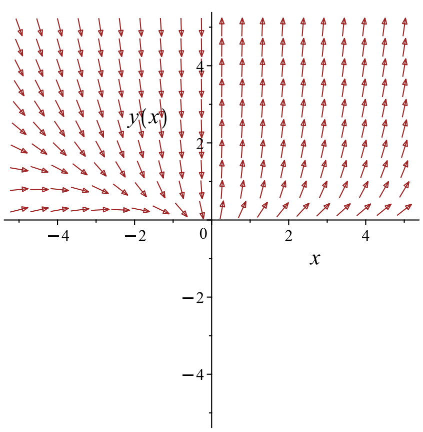
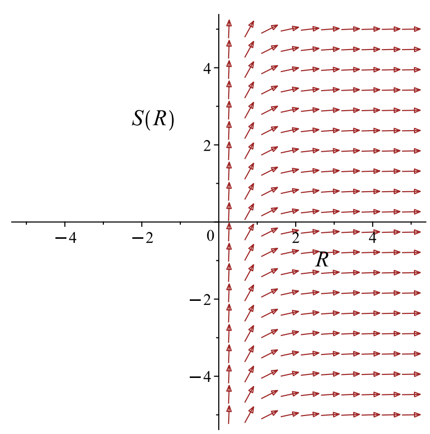
Which simplifies to

$$\frac{2\sqrt{y}}{x^{\frac{5}{2}}} = -\frac{2}{3x^{\frac{3}{2}}} + c_1$$

Which gives

$$y = -\frac{x^{\frac{7}{2}} c_1}{3} + \frac{x^5 c_1^2}{4} + \frac{x^2}{9}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{5y+x\sqrt{y}}{x}$ 	$R = x$ $S = \frac{2\sqrt{y}}{x^{\frac{5}{2}}}$	$\frac{dS}{dR} = \frac{1}{R^{\frac{5}{2}}}$ 

Summary
The solution(s) found are the following

$$y = -\frac{x^{\frac{7}{2}} c_1}{3} + \frac{x^5 c_1^2}{4} + \frac{x^2}{9} \tag{1}$$

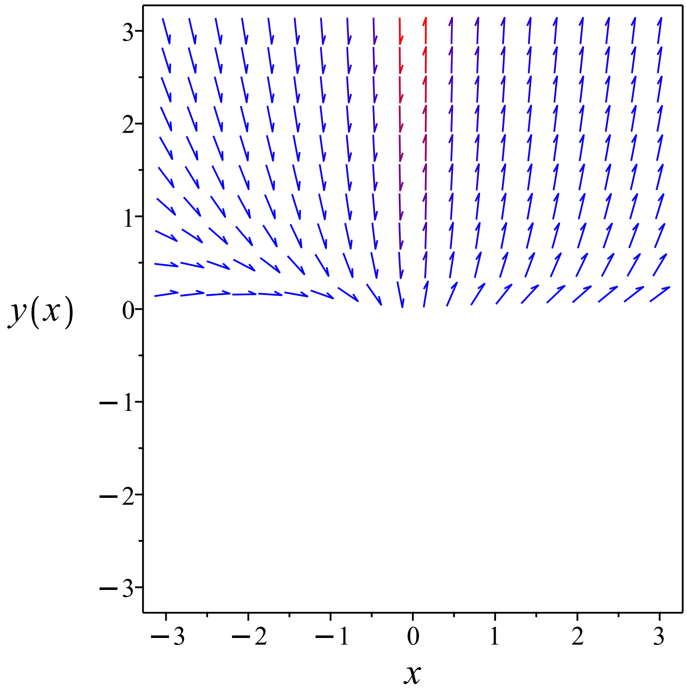


Figure 472: Slope field plot

Verification of solutions

$$y = -\frac{x^{\frac{7}{2}} c_1}{3} + \frac{x^5 c_1^2}{4} + \frac{x^2}{9}$$

Verified OK.

8.36.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{5y + x\sqrt{y}}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{5}{x} y + \sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{5}{x} \\ f_1(x) &= 1 \\ n &= \frac{1}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y'\frac{1}{\sqrt{y}} = \frac{5\sqrt{y}}{x} + 1 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \sqrt{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{2\sqrt{y}}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 2w'(x) &= \frac{5w(x)}{x} + 1 \\ w' &= \frac{5w}{2x} + \frac{1}{2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{5}{2x} \\ q(x) &= \frac{1}{2} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{5w(x)}{2x} = \frac{1}{2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{5}{2x}dx} \\ &= \frac{1}{x^{\frac{5}{2}}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{2}\right) \\ \frac{d}{dx}\left(\frac{w}{x^{\frac{5}{2}}}\right) &= \left(\frac{1}{x^{\frac{5}{2}}}\right)\left(\frac{1}{2}\right) \\ d\left(\frac{w}{x^{\frac{5}{2}}}\right) &= \left(\frac{1}{2x^{\frac{5}{2}}}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{w}{x^{\frac{5}{2}}} &= \int \frac{1}{2x^{\frac{5}{2}}} dx \\ \frac{w}{x^{\frac{5}{2}}} &= -\frac{1}{3x^{\frac{3}{2}}} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^{\frac{5}{2}}}$ results in

$$w(x) = -\frac{x}{3} + x^{\frac{5}{2}}c_1$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$\sqrt{y} = -\frac{x}{3} + x^{\frac{5}{2}}c_1$$

Summary

The solution(s) found are the following

$$\sqrt{y} = -\frac{x}{3} + x^{\frac{5}{2}} c_1 \tag{1}$$

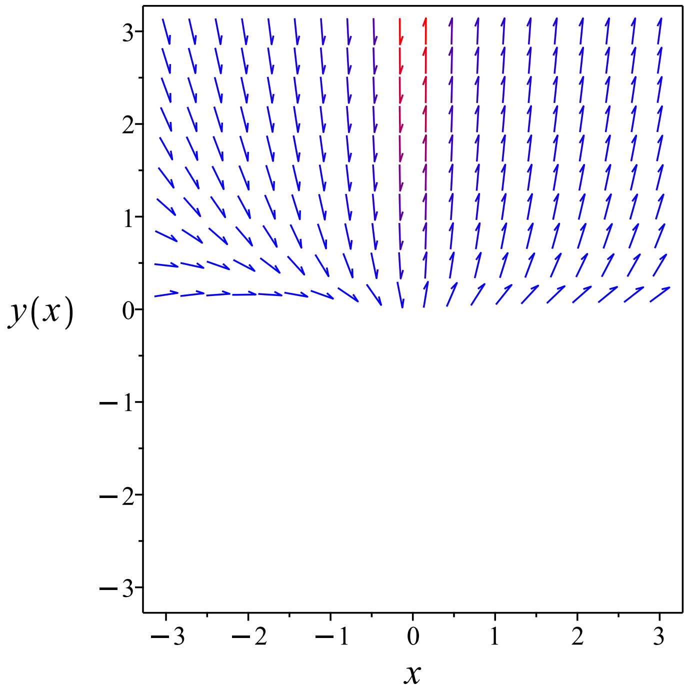


Figure 473: Slope field plot

Verification of solutions

$$\sqrt{y} = -\frac{x}{3} + x^{\frac{5}{2}} c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x)-(5*y(x)+x*sqrt(y(x)))=0,y(x), singsol=all)
```

$$\sqrt{y(x)}+\frac{x}{3}-x^{\frac{5}{2}}c_1=0$$

✓ Solution by Mathematica

Time used: 0.153 (sec). Leaf size: 25

```
DSolve[x*y'[x]-(5*y[x]+x*Sqrt[y[x]])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{1}{9}x^2\left(1-3c_1x^{3/2}\right)^2$$

8.37 problem 39

Internal problem ID [2069]
Internal file name [`OUTPUT/2069_Sunday_February_25_2024_06_49_02_AM_68089545/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x\sqrt{1-y}-\sqrt{1-x^2}\, y'=0}$$

With initial conditions

$$[y(0)=0]$$

8.37.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{x\sqrt{-y+1}}{\sqrt{1-x^2}} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-1<x<1\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x\sqrt{-y+1}}{\sqrt{1-x^2}}\right) \\
&= -\frac{x}{2\sqrt{-y+1}\,\sqrt{1-x^2}}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-1 < x < 1\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

8.37.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{x\sqrt{-y+1}}{\sqrt{1-x^2}}
\end{aligned}$$

Where $f(x) = \frac{x}{\sqrt{1-x^2}}$ and $g(y) = \sqrt{-y+1}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\sqrt{-y+1}}\,dy &= \frac{x}{\sqrt{1-x^2}}\,dx \\
\int \frac{1}{\sqrt{-y+1}}\,dy &= \int \frac{x}{\sqrt{1-x^2}}\,dx \\
-2\sqrt{-y+1} &= -\sqrt{1-x^2} + c_1
\end{aligned}$$

The solution is

$$-2\sqrt{1-y} + \sqrt{1-x^2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-1 - c_1 = 0$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$-2\sqrt{-y+1} + \sqrt{1-x^2} + 1 = 0$$

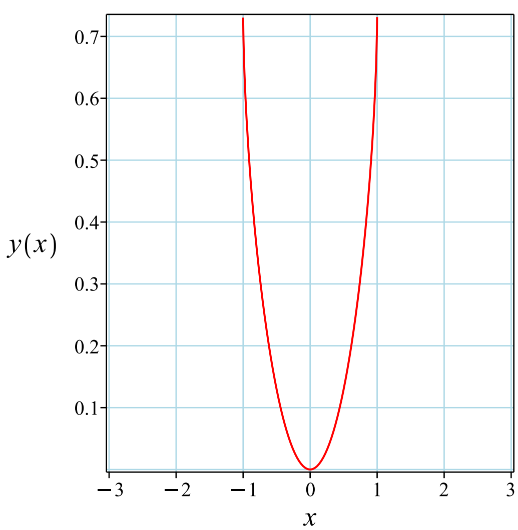
Solving for y from the above gives

$$y = \frac{x^2}{4} - \frac{\sqrt{1-x^2}}{2} + \frac{1}{2}$$

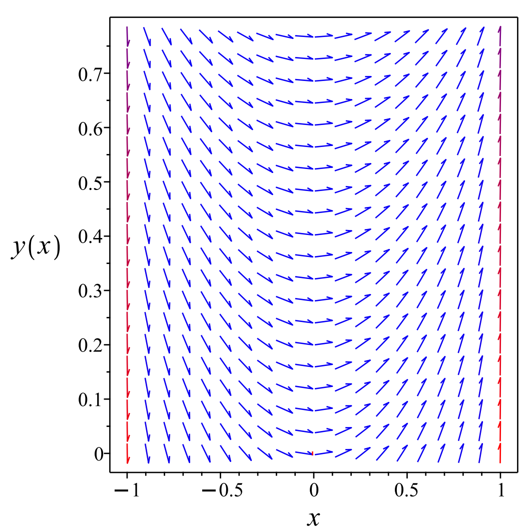
Summary

The solution(s) found are the following

$$y = \frac{x^2}{4} - \frac{\sqrt{1-x^2}}{2} + \frac{1}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2}{4} - \frac{\sqrt{1-x^2}}{2} + \frac{1}{2}$$

Verified OK.

8.37.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x\sqrt{-y+1}}{\sqrt{1-x^2}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 275: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = \frac{\sqrt{1-x^2}}{x}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{\sqrt{1-x^2}}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{(x-1)(x+1)}{\sqrt{1-x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x\sqrt{-y+1}}{\sqrt{1-x^2}}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x}{\sqrt{1-x^2}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{\sqrt{-y+1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{\sqrt{-R+1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -2\sqrt{-R+1} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\sqrt{1-x^2} = -2\sqrt{1-y} + c_1
$$

Which simplifies to

$$
-\sqrt{1-x^2} = -2\sqrt{1-y} + c_1
$$

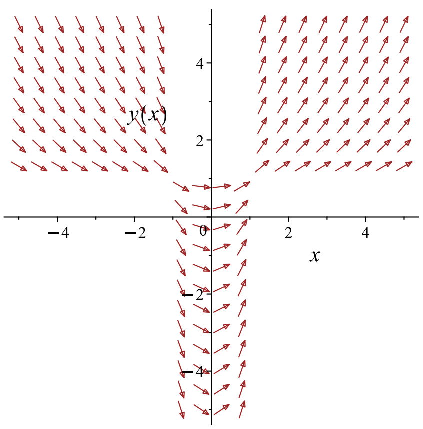
Which gives

$$
y = -\frac{\sqrt{1-x^2}\, c_1}{2} - \frac{c_1^2}{4} + \frac{x^2}{4} + \frac{3}{4}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{x\sqrt{-y+1}}{\sqrt{1-x^2}}$



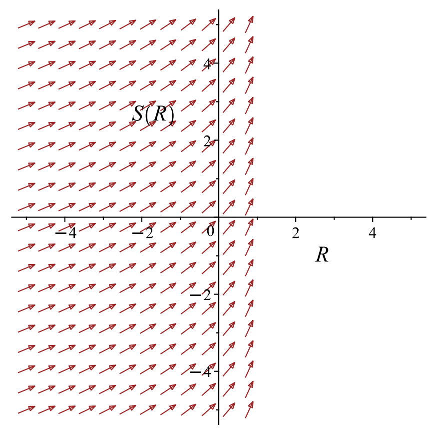
</td><td>

$R = y$

$S = -\sqrt{1-x^2}$

</td><td>

$\frac{dS}{dR} = \frac{1}{\sqrt{-R+1}}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{2}c_1 - \frac{1}{4}c_1^2 + \frac{3}{4}$$

$$c_1 = 1$$

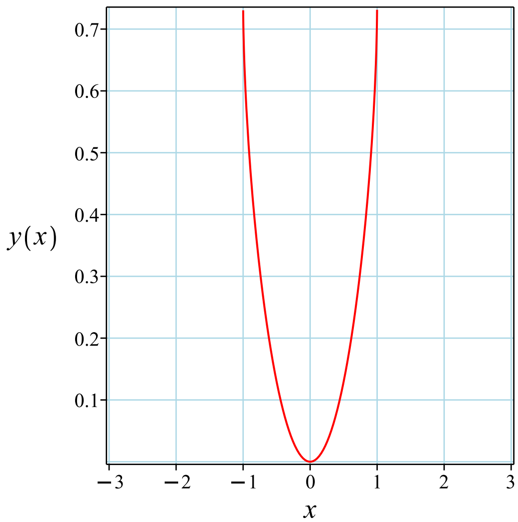
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2}{4} - \frac{\sqrt{1-x^2}}{2} + \frac{1}{2}$$

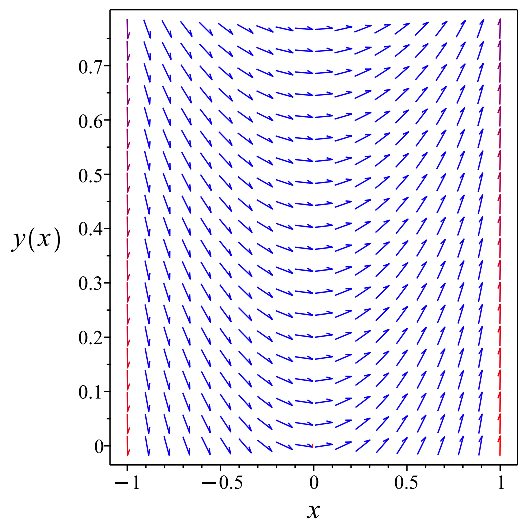
Summary

The solution(s) found are the following

$$y = \frac{x^2}{4} - \frac{\sqrt{1-x^2}}{2} + \frac{1}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2}{4} - \frac{\sqrt{1-x^2}}{2} + \frac{1}{2}$$

Verified OK.

8.37.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\sqrt{-y+1}}\right)\mathrm{d}y = \left(\frac{x}{\sqrt{1-x^2}}\right)\mathrm{d}x$$

$$\left(-\frac{x}{\sqrt{1-x^2}}\right)\mathrm{d}x + \left(\frac{1}{\sqrt{-y+1}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x}{\sqrt{1-x^2}}$$
$$N(x, y) = \frac{1}{\sqrt{-y+1}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{\sqrt{1-x^2}}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{-y+1}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x$$
$$\phi = \sqrt{1-x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{-y+1}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{-y+1}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{-y+1}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{\sqrt{-y+1}} \right) \mathrm{d}y$$
$$f(y) = -2\sqrt{-y+1} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\sqrt{-y+1} + \sqrt{1-x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\sqrt{-y+1} + \sqrt{1-x^2}$$

The solution becomes

$$y = \frac{\sqrt{1-x^2}\, c_1}{2} - \frac{c_1^2}{4} + \frac{x^2}{4} + \frac{3}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{2}c_1 - \frac{1}{4}c_1^2 + \frac{3}{4}$$

$$c_1 = 3$$

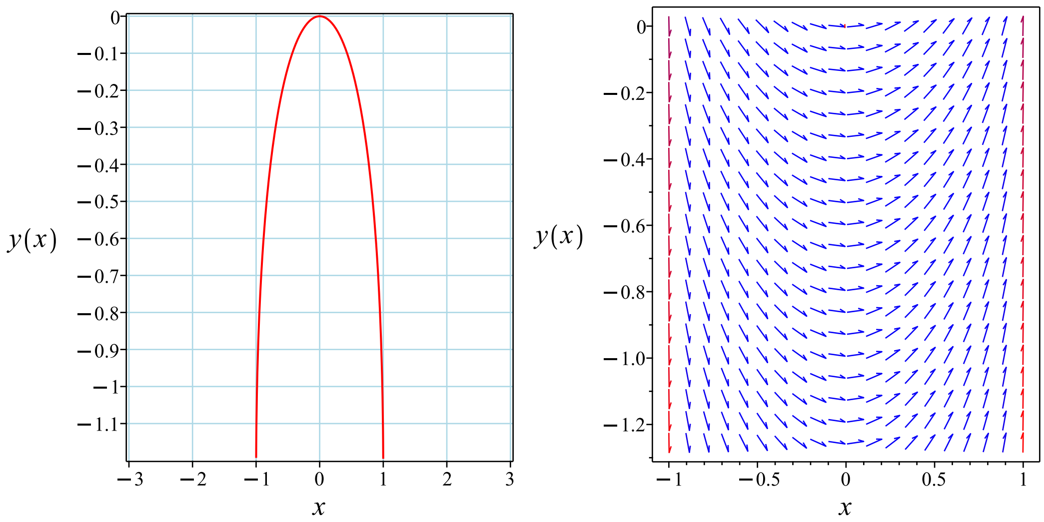
Substituting c_1 found above in the general solution gives

$$y = \frac{3\sqrt{1-x^2}}{2} - \frac{3}{2} + \frac{x^2}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{3\sqrt{1-x^2}}{2} - \frac{3}{2} + \frac{x^2}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{3\sqrt{1-x^2}}{2} - \frac{3}{2} + \frac{x^2}{4}$$

Verified OK.

8.37.5 Maple step by step solution

Let's solve

$\left[x\sqrt{1-y} - \sqrt{1-x^2}\, y' = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y}} = \frac{x}{\sqrt{1-x^2}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y}}dx = \int \frac{x}{\sqrt{1-x^2}}dx + c_1$

- Evaluate integral

  $-2\sqrt{1-y} = \frac{(x-1)(x+1)}{\sqrt{1-x^2}} + c_1$

- Solve for y

$$y = \frac{\sqrt{1-x^2}\, c_1}{2} - \frac{c_1^2}{4} + \frac{x^2}{4} + \frac{3}{4}$$

- Use initial condition $y(0) = 0$

$$0 = \frac{1}{2}c_1 - \frac{1}{4}c_1^2 + \frac{3}{4}$$

- Solve for c_1

$$c_1 = (3, -1)$$

- Substitute $c_1 = (3, -1)$ into general solution and simplify

$$y = \frac{3\sqrt{1-x^2}}{2} - \frac{3}{2} + \frac{x^2}{4}$$

- Solution to the IVP

$$y = \frac{3\sqrt{1-x^2}}{2} - \frac{3}{2} + \frac{x^2}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 22

```
dsolve([x*sqrt(1-y(x))-sqrt(1-x^2)*diff(y(x),x)=0,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{x^2}{4} - \frac{\sqrt{-x^2+1}}{2} + \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 0.231 (sec). Leaf size: 53

```
DSolve[{x*Sqrt[1-y[x]]-Sqrt[1-x^2]*y'[x]==0,{y[0]==0}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{4}\left(x^2 - 2\sqrt{1-x^2} + 2\right)$$
$$y(x) \to \frac{1}{4}\left(x^2 + 6\sqrt{1-x^2} - 6\right)$$

8.38 problem 40

Internal problem ID [2070]
Internal file name [`OUTPUT/2070_Sunday_February_25_2024_06_49_04_AM_84770115/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{yx - y^2 - y'x^2 = 0}$$

With initial conditions

$$[y(1) = 1]$$

8.38.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{y(-x + y)}{x^2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y(-x+y)}{x^2}\right)\\ &= -\frac{-x+y}{x^2} - \frac{y}{x^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.38.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x^2 - u(x)^2\,x^2 - (u'(x)\,x + u(x))\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^2}{x}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u^2}\,du &= -\frac{1}{x}\,dx\\ \int \frac{1}{u^2}\,du &= \int -\frac{1}{x}\,dx\\ -\frac{1}{u} &= -\ln(x) + c_2\end{aligned}$$

The solution is

$$-\frac{1}{u(x)}+\ln(x)-c_2=0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x}{y}+\ln(x)-c_2=0$$
$$-\frac{x}{y}+\ln(x)-c_2=0$$

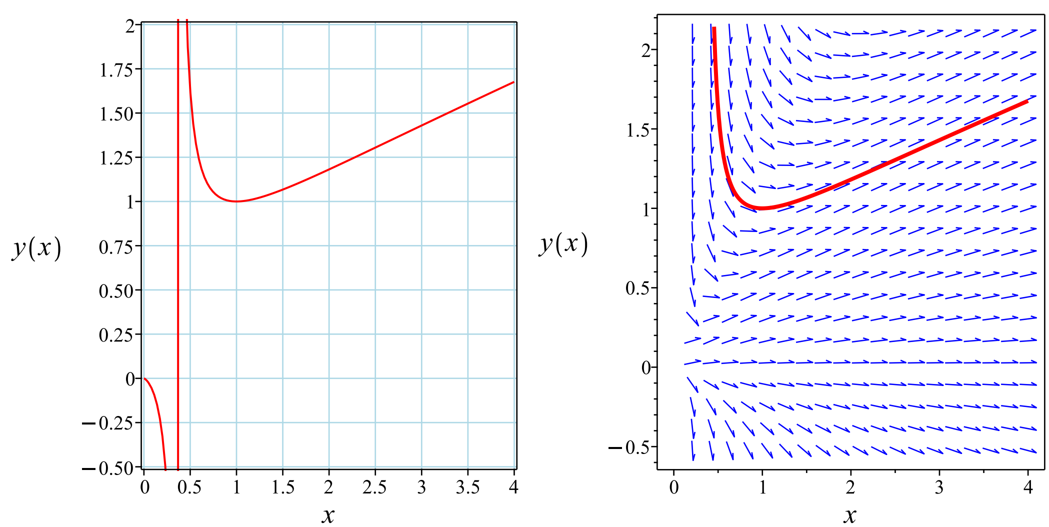
Substituting initial conditions and solving for c_2 gives $c_2=-1$. Hence the solution becomes Solving for y from the above gives

$$y=\frac{x}{\ln(x)+1}$$

Summary

The solution(s) found are the following

$$y=\frac{x}{\ln(x)+1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y=\frac{x}{\ln(x)+1}$$

Verified OK.

8.38.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(-x+y)}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 278: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0 \\ \eta(x,y) &= \frac{y^2}{x}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy\end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(-x+y)}{x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{y} \\
S_y &= \frac{x}{y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{x}{y} = -\ln(x) + c_1
$$

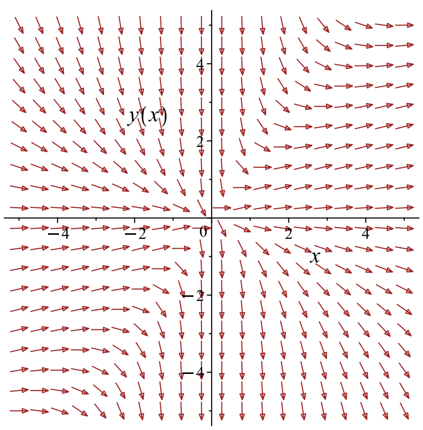
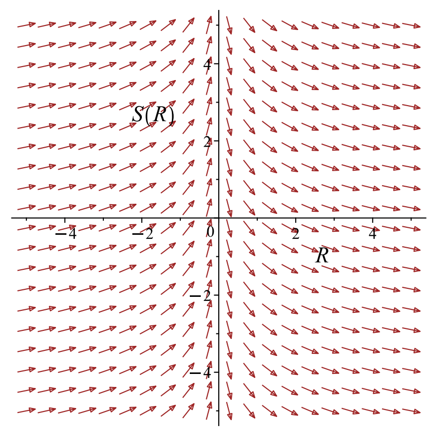
Which simplifies to

$$
-\frac{x}{y} = -\ln(x) + c_1
$$

Which gives

$$
y = \frac{x}{\ln(x) - c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-x+y)}{x^2}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_1}$$

$$c_1 = -1$$

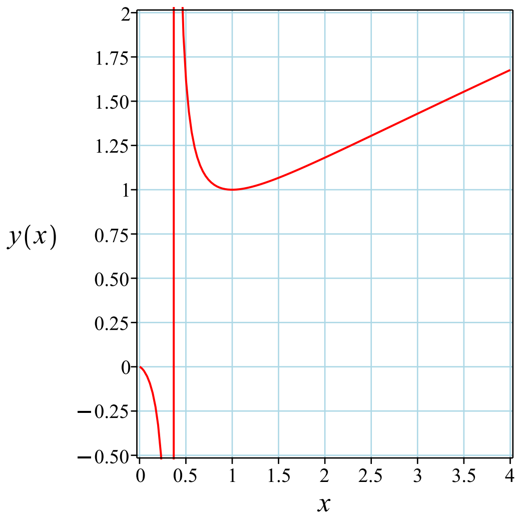
Substituting c_1 found above in the general solution gives

$$y = \frac{x}{\ln(x) + 1}$$

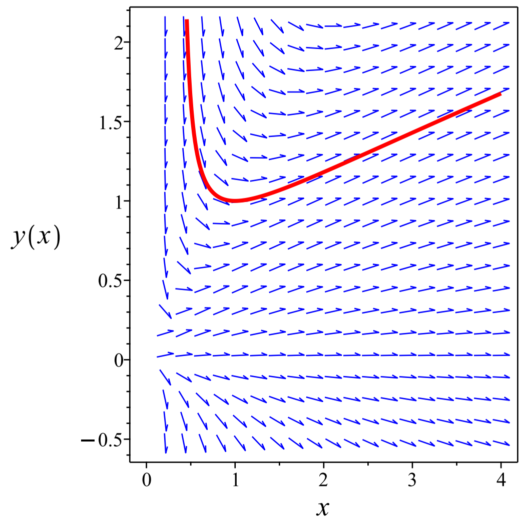
Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x}{\ln(x) + 1}$$

Verified OK.

8.38.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(-x + y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{x} \\
f_1(x) &= -\frac{1}{x^2} \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y' \frac{1}{y^2} = \frac{1}{yx} - \frac{1}{x^2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= \frac{w(x)}{x} - \frac{1}{x^2} \\
w' &= -\frac{w}{x} + \frac{1}{x^2}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{w(x)}{x} = \frac{1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x^2}\right) \\
\frac{d}{dx}(wx) &= (x)\left(\frac{1}{x^2}\right) \\
d(wx) &= \frac{1}{x}\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
wx &= \int \frac{1}{x}\, dx \\
wx &= \ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
w(x) = \frac{\ln(x)}{x} + \frac{c_1}{x}
$$

which simplifies to

$$
w(x) = \frac{\ln(x) + c_1}{x}
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = \frac{\ln(x) + c_1}{x}
$$

Or

$$
y = \frac{x}{\ln(x) + c_1}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = \frac{1}{c_1}
$$

$$
c_1 = 1
$$

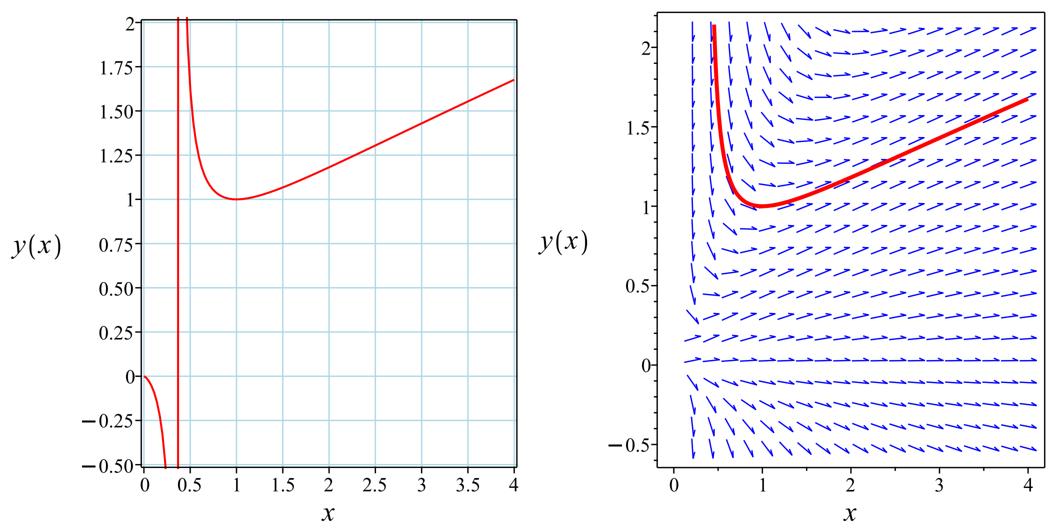
Substituting c_1 found above in the general solution gives

$$y = \frac{x}{\ln(x) + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) + 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) + 1}$$

Verified OK.

8.38.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2\right)\mathrm{d}y&=\left(-yx+y^2\right)\mathrm{d}x\\ \left(yx-y^2\right)\mathrm{d}x+\left(-x^2\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=yx-y^2\\ N(x,y)&=-x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(yx-y^2\right)\\ &=x-2y\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-x^2)$$
$$= -2x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = -y^2 + yx$ and $N = -x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{-y^2 + yx}{xy^2}$$
$$N = -\frac{x}{y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{x}{y^2}\right)\mathrm{d}y = \left(-\frac{yx - y^2}{y^2 x}\right)\mathrm{d}x$$

$$\left(\frac{yx - y^2}{y^2 x}\right)\mathrm{d}x + \left(-\frac{x}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= \frac{yx - y^2}{y^2 x} \\ N(x,y) &= -\frac{x}{y^2} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{yx - y^2}{y^2 x}\right) \\ &= -\frac{1}{y^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{x}{y^2}\right) \\ &= -\frac{1}{y^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{yx - y^2}{y^2 x} \mathrm{d}x \\
\phi &= -\ln(x) + \frac{x}{y} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{y^2}$. Therefore equation (4) becomes

$$
-\frac{x}{y^2} = -\frac{x}{y^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\ln(x) + \frac{x}{y} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\ln(x) + \frac{x}{y}
$$

The solution becomes

$$
y = \frac{x}{\ln(x) + c_1}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_1}$$

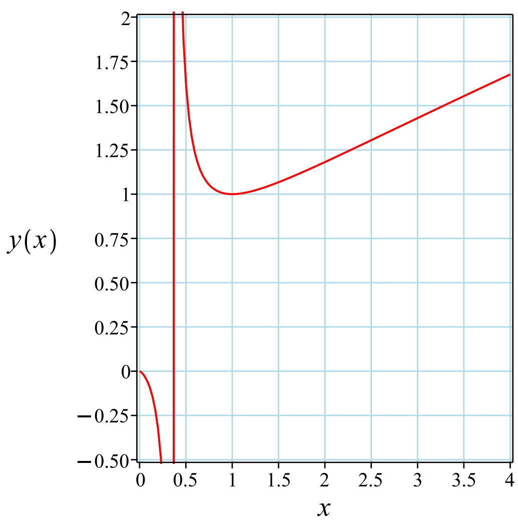
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

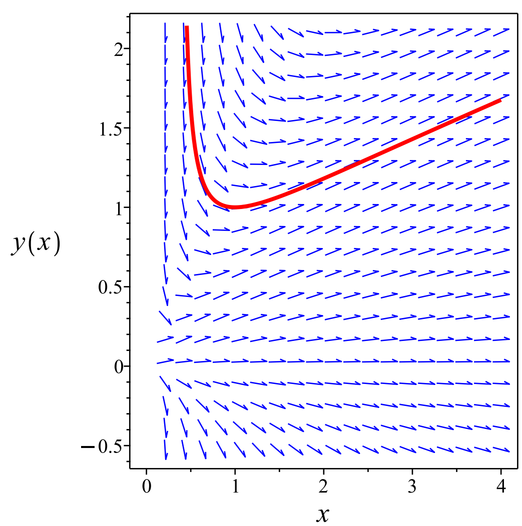
$$y = \frac{x}{\ln(x) + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{\ln(x) + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) + 1}$$

Verified OK.

8.38.6 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(-x+y)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} - \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2}{x^3} \\ f_1 f_2 &= -\frac{1}{x^3} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^2} - \frac{u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + c_2 \ln(x)$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x}{c_1 + c_2 \ln(x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{c_3 + \ln(x)}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_3}$$

$$c_3 = 1$$

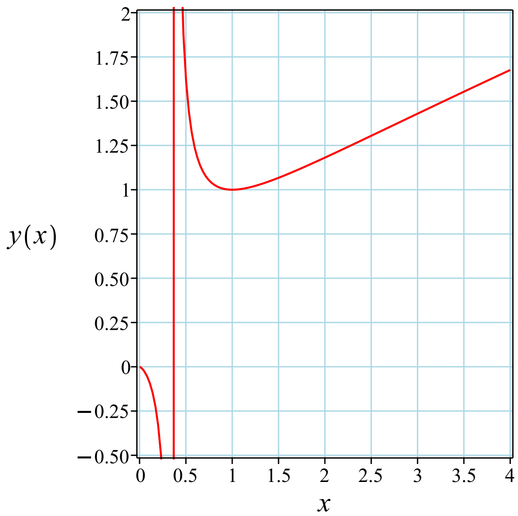
Substituting c_3 found above in the general solution gives

$$y = \frac{x}{\ln(x) + 1}$$

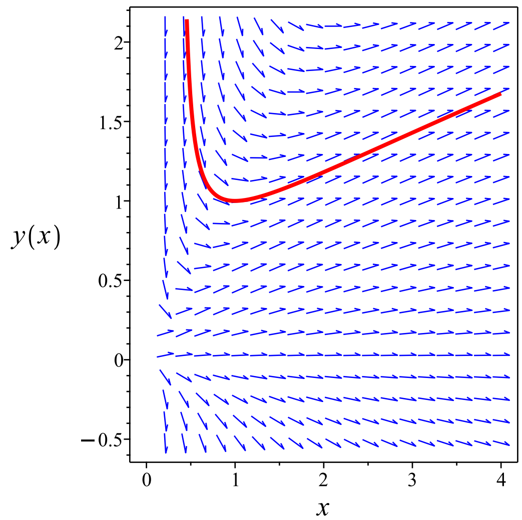
Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([(x*y(x)-y(x)^2)-x^2*diff(y(x),x)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{x}{\ln(x) + 1}$$

✓ Solution by Mathematica

Time used: 0.137 (sec). Leaf size: 13

```
DSolve[{(x*y[x]-y[x]^2)-x^2*y'[x]==0,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{\log(x) + 1}$$

8.39 problem 41

Internal problem ID [2071]
Internal file name [`OUTPUT/2071_Sunday_February_25_2024_06_49_05_AM_51252479/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x\,\mathrm{e}^{-y^2}+yy'=0}$$

With initial conditions

$$[y(0)=0]$$

8.39.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y)\\ &= -\frac{x\,\mathrm{e}^{-y^2}}{y} \end{aligned}$$

$f(x,y)$ is not defined at $y=0$ therefore existence and uniqueness theorem do not apply.

8.39.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x\,\mathrm{e}^{-y^2}}{y} \end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{\mathrm{e}^{-y^2}}{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{\mathrm{e}^{-y^2}}{y}}\,dy &= -x\,dx \\ \int \frac{1}{\frac{\mathrm{e}^{-y^2}}{y}}\,dy &= \int -x\,dx \\ \frac{\mathrm{e}^{y^2}}{2} &= -\frac{x^2}{2} + c_1 \end{aligned}$$

Which results in

$$\begin{aligned} y &= \sqrt{\ln\left(-x^2 + 2c_1\right)} \\ y &= -\sqrt{\ln\left(-x^2 + 2c_1\right)} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\sqrt{\ln(2) + \ln(c_1)}$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{\ln\left(-\left(x-1\right)\left(x+1\right)\right)}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \sqrt{\ln(2) + \ln(c_1)}$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \sqrt{\ln\left(-\left(x-1\right)\left(x+1\right)\right)}$$

Summary

The solution(s) found are the following

$$y = \sqrt{\ln\left(-\left(x-1\right)\left(x+1\right)\right)} \tag{1}$$
$$y = -\sqrt{\ln\left(-\left(x-1\right)\left(x+1\right)\right)} \tag{2}$$

Verification of solutions

$$y = \sqrt{\ln\left(-\left(x-1\right)\left(x+1\right)\right)}$$

Verified OK.

$$y = -\sqrt{\ln\left(-\left(x-1\right)\left(x+1\right)\right)}$$

Verified OK.

8.39.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x\,\mathrm{e}^{-y^2}}{y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 280: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x \,\mathrm{e}^{-y^2}}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{y^2} y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{R^2} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{e^{R^2}}{2} + c_1 \tag{4}$$

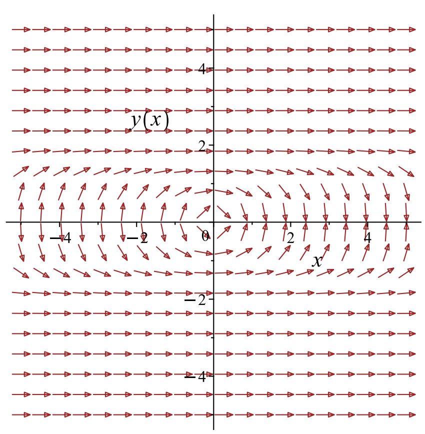
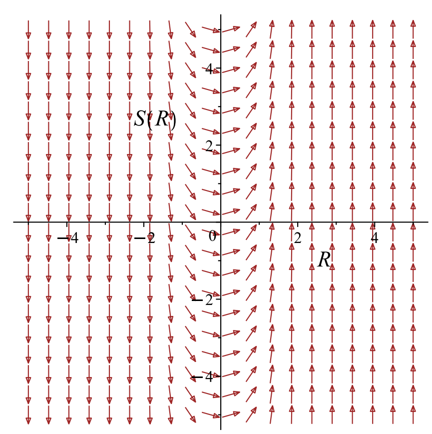
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \frac{e^{y^2}}{2} + c_1$$

Which simplifies to

$$-\frac{x^2}{2} = \frac{e^{y^2}}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{x\,e^{-y^2}}{y}$

</td><td>$R = y$
$S = -\frac{x^2}{2}$</td><td>$\frac{dS}{dR} = e^{R^2} R$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{2} + c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2}{2} = \frac{e^{y^2}}{2} - \frac{1}{2}$$

Solving for y from the above gives

$$y = \sqrt{\ln(1 - x^2)}$$
$$y = -\sqrt{\ln(1 - x^2)}$$

Summary

The solution(s) found are the following

$$y = \sqrt{\ln(1 - x^2)} \tag{1}$$
$$y = -\sqrt{\ln(1 - x^2)} \tag{2}$$

Verification of solutions

$$y = \sqrt{\ln(1 - x^2)}$$

Verified OK.

$$y = -\sqrt{\ln(1 - x^2)}$$

Verified OK.

8.39.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\mathrm{e}^{y^2} y\right) \mathrm{d}y = (x)\, \mathrm{d}x$$
$$(-x)\, \mathrm{d}x + \left(-\mathrm{e}^{y^2} y\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = -\mathrm{e}^{y^2} y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\mathrm{e}^{y^2} y\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\mathrm{e}^{y^2} y$. Therefore equation (4) becomes

$$-\mathrm{e}^{y^2} y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= -\mathrm{e}^{y^2} y \\ &= -\mathrm{e}^{y^2} y \end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y) \, \mathrm{d}y = \int \left(-\mathrm{e}^{y^2} y\right) \mathrm{d}y$$
$$f(y) = -\frac{\mathrm{e}^{y^2}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{e^{y^2}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{e^{y^2}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^2}{2} - \frac{e^{y^2}}{2} = -\frac{1}{2}$$

Solving for y from the above gives

$$y = \sqrt{\ln(1 - x^2)}$$
$$y = -\sqrt{\ln(1 - x^2)}$$

Summary

The solution(s) found are the following

$$y = \sqrt{\ln(1 - x^2)} \tag{1}$$
$$y = -\sqrt{\ln(1 - x^2)} \tag{2}$$

Verification of solutions

$$y = \sqrt{\ln(1 - x^2)}$$

Verified OK.

$$y = -\sqrt{\ln(1 - x^2)}$$

Verified OK.

8.39.5 Maple step by step solution

Let's solve

$\left[x\,\mathrm{e}^{-y^2}+yy'=0, y(0)=0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{\mathrm{e}^{-y^2}}=-x$

- Integrate both sides with respect to x

  $\int \frac{y'y}{\mathrm{e}^{-y^2}}dx=\int -xdx+c_1$

- Evaluate integral

  $\frac{1}{2\,\mathrm{e}^{-y^2}}=-\frac{x^2}{2}+c_1$

- Solve for y

  $\left\{y=\sqrt{\ln\left(-x^2+2c_1\right)}, y=-\sqrt{\ln\left(-x^2+2c_1\right)}\right\}$

- Use initial condition $y(0)=0$

  $0=\sqrt{\ln\left(2c_1\right)}$

- Solve for c_1

  $c_1=\frac{1}{2}$

- Substitute $c_1=\frac{1}{2}$ into general solution and simplify

  $y=\sqrt{\ln\left(1-x^2\right)}$

- Use initial condition $y(0)=0$

  $0=-\sqrt{\ln\left(2c_1\right)}$

- Solve for c_1

  $c_1=\frac{1}{2}$

- Substitute $c_1=\frac{1}{2}$ into general solution and simplify

  $y=-\sqrt{\ln\left(1-x^2\right)}$

- Solutions to the IVP

  $\left\{y=\sqrt{\ln\left(1-x^2\right)}, y=-\sqrt{\ln\left(1-x^2\right)}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 4.5 (sec). Leaf size: 29

```
dsolve([(x*exp(-y(x)^2))+y(x)*diff(y(x),x)=0,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \sqrt{\ln(-x^2+1)}$$
$$y(x) = -\sqrt{\ln(-x^2+1)}$$

✓ Solution by Mathematica

Time used: 2.245 (sec). Leaf size: 35

```
DSolve[{(x*Exp[-y[x]^2])+y[x]*y'[x]==0,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\log(1-x^2)}$$
$$y(x) \to \sqrt{\log(1-x^2)}$$

8.40 problem 42

Internal problem ID [2072]
Internal file name [`OUTPUT/2072_Sunday_February_25_2024_06_49_05_AM_65050180/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\frac{2y^3 - 2y^3x^2 - x + xy^2 \ln(y)}{x y^2} + \frac{(2\ln(x)\, y^3 - y^3x^2 + 2x + xy^2)\, y'}{y^3} = 0}$$

With initial conditions

$$[y(1) = 1]$$

8.40.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{2\ln(x)\,y^3 - y^3x^2 + 2x + x\,y^2}{y^3}\right)\mathrm{d}y = \left(-\frac{2y^3 - 2y^3x^2 - x + x\,y^2\,\mathrm{l}}{x\,y^2}\right.$$

$$\left(\frac{2y^3 - 2y^3x^2 - x + x\,y^2\ln(y)}{x\,y^2}\right)\mathrm{d}x + \left(\frac{2\ln(x)\,y^3 - y^3x^2 + 2x + x\,y^2}{y^3}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{2y^3 - 2y^3x^2 - x + x\,y^2\ln(y)}{x\,y^2}$$
$$N(x, y) = \frac{2\ln(x)\,y^3 - y^3x^2 + 2x + x\,y^2}{y^3}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2y^3 - 2y^3x^2 - x + x\,y^2\ln(y)}{x\,y^2}\right)\\ &= \frac{(-2x^2 + 2)\,y^3 + x\,y^2 + 2x}{x\,y^3}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2\ln(x)\, y^3 - y^3x^2 + 2x + x\, y^2}{y^3}\right) \\
&= \frac{\frac{2y^3}{x} - 2x\, y^3 + 2 + y^2}{y^3}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int \frac{2y^3 - 2y^3x^2 - x + x\, y^2 \ln(y)}{x\, y^2}\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -y\, x^2 + x\ln(y) + 2\ln(x)\, y - \frac{x}{y^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -x^2 + \frac{x}{y} + 2\ln(x) + \frac{2x}{y^3} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2\ln(x)y^3 - y^3x^2 + 2x + x\, y^2}{y^3}$. Therefore equation (4) becomes

$$
\frac{2\ln(x)\, y^3 - y^3x^2 + 2x + x\, y^2}{y^3} = -x^2 + \frac{x}{y} + 2\ln(x) + \frac{2x}{y^3} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -y\,x^{2}+x\ln\left(y\right)+2\ln\left(x\right)y-\frac{x}{y^{2}}+c_{1}$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_{1} = -y\,x^{2}+x\ln\left(y\right)+2\ln\left(x\right)y-\frac{x}{y^{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_{1}$$

$$c_{1} = -2$$

Substituting c_1 found above in the general solution gives

$$-y\,x^{2}+x\ln\left(y\right)+2\ln\left(x\right)y-\frac{x}{y^{2}} = -2$$

The above simplifies to

$$-y^{3}x^{2}+x\,y^{2}\ln\left(y\right)+2\ln\left(x\right)y^{3}+2y^{2}-x = 0$$

Summary

The solution(s) found are the following

$$-y^{3}x^{2}+xy^{2}\ln\left(y\right)+2\ln\left(x\right)y^{3}+2y^{2}-x = 0 \tag{1}$$

Verification of solutions

$$-y^{3}x^{2}+xy^{2}\ln\left(y\right)+2\ln\left(x\right)y^{3}+2y^{2}-x = 0$$

Verified OK.

8.40.2 Maple step by step solution

Let's solve

$$\left[\frac{2y^{3}-2y^{3}x^{2}-x+xy^{2}\ln\left(y\right)}{xy^{2}}+\frac{\left(2\ln\left(x\right)y^{3}-y^{3}x^{2}+2x+xy^{2}\right)y'}{y^{3}} = 0, y\left(1\right)=1\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

  - Evaluate derivatives

    $\frac{6y^2-6x^2y^2+2\ln(y)yx+yx}{x\,y^2} - \frac{2(2y^3-2y^3x^2-x+x\,y^2\ln(y))}{x\,y^3} = \frac{\frac{2y^3}{x}-2x\,y^3+2+y^2}{y^3}$

  - Simplify

    $\frac{(-2x^2+2)y^3+x\,y^2+2x}{x\,y^3} = \frac{\frac{2y^3}{x}-2x\,y^3+2+y^2}{y^3}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int \frac{2y^3-2y^3x^2-x+x\,y^2\ln(y)}{x\,y^2}dx + f_1(y)$

- Evaluate integral

  $F(x,y) = -y\,x^2 + x\ln(y) + 2\ln(x)\,y - \frac{x}{y^2} + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\frac{2\ln(x)y^3-y^3x^2+2x+x\,y^2}{y^3} = -x^2 + \frac{x}{y} + 2\ln(x) + \frac{2x}{y^3} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = \frac{2\ln(x)y^3-y^3x^2+2x+x\,y^2}{y^3} + x^2 - \frac{x}{y} - 2\ln(x) - \frac{2x}{y^3}$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = -y\,x^2 + x\ln(y) + 2\ln(x)\,y - \frac{x}{y^2}$$

- Substitute $F(x,y)$ into the solution of the ODE

$$-y\,x^2 + x\ln(y) + 2\ln(x)\,y - \frac{x}{y^2} = c_1$$

- Solve for y

$$y = \mathrm{e}^{\mathit{RootOf}\left((\mathrm{e}^{\_Z})^3 x^2 - 2\ln(x)(\mathrm{e}^{\_Z})^3 - \_Zx(\mathrm{e}^{\_Z})^2 + c_1(\mathrm{e}^{\_Z})^2 + x\right)}$$

- Use initial condition $y(1) = 1$

$$1 = \mathrm{e}^{\mathit{RootOf}\left((\mathrm{e}^{\_Z})^3 + 1 - \_Z(\mathrm{e}^{\_Z})^2 + c_1(\mathrm{e}^{\_Z})^2\right)}$$

- Solve for c_1

$$c_1 = -2$$

- Substitute $c_1 = -2$ into general solution and simplify

$$y = \mathrm{e}^{\mathit{RootOf}\left(-\mathrm{e}^{3\_Z}x^2 + 2\ln(x)\mathrm{e}^{3\_Z} + \_Zx\,\mathrm{e}^{2\_Z} + 2\,\mathrm{e}^{2\_Z} - x\right)}$$

- Solution to the IVP

$$y = \mathrm{e}^{\mathit{RootOf}\left(-\mathrm{e}^{3\_Z}x^2 + 2\ln(x)\mathrm{e}^{3\_Z} + \_Zx\,\mathrm{e}^{2\_Z} + 2\,\mathrm{e}^{2\_Z} - x\right)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.532 (sec). Leaf size: 40

```
dsolve([((2*y(x)^3-2*x^2*y(x)^3-x+x*y(x)^2*ln(y(x)))/(x*y(x)^2))+( (2*y(x)^3*ln(x)-x^2*y(x)^
```

$$y(x) = e^{\text{RootOf}\left(-x^2 e^{3\_Z}+2\ln(x)e^{3\_Z}+\_Zx\, e^{2\_Z}+2\, e^{2\_Z}-x\right)}$$

✓ Solution by Mathematica

Time used: 0.685 (sec). Leaf size: 30

```
DSolve[{((2*y[x]^3-2*x^2*y[x]^3-x+x*y[x]^2*Log[y[x]])/(x*y[x]^2))+( (2*y[x]^3*Log[x]-x^2*y[x
```

$$\text{Solve}\left[x^2y(x)+\frac{x}{y(x)^2}-x\log(y(x))-2y(x)\log(x)=2,y(x)\right]$$

8.41 problem 43

Internal problem ID [2073]
Internal file name [`OUTPUT/2073_Sunday_February_25_2024_06_49_09_AM_24179419/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y'x - 2y - 2y^3x^4 = 0}$$

With initial conditions

$$[y(1) = 1]$$

8.41.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{2y(x^4y^2+1)}{x} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2y(x^4y^2+1)}{x}\right) \\
&= \frac{2x^4y^2+2}{x} + 4x^3y^2
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.41.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{2y(x^4y^2+1)}{x} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 284: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^3}{x^4}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^3}{x^4}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^4}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y(x^4y^2 + 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2x^3}{y^2} \\ S_y &= \frac{x^4}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x^7 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R^7$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^8}{4} + c_1 \tag{4}$$

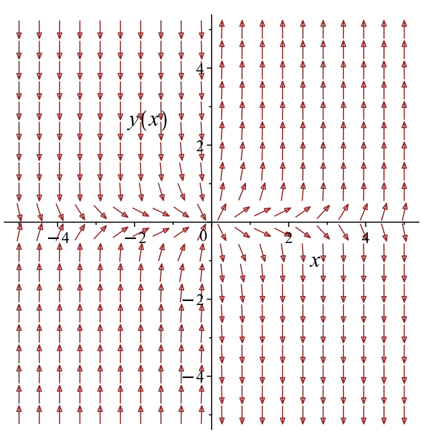
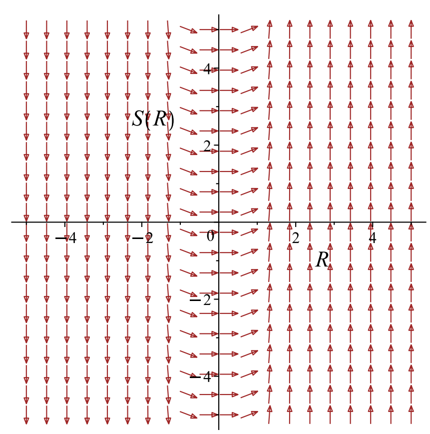
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^4}{2y^2} = \frac{x^8}{4} + c_1$$

Which simplifies to

$$-\frac{x^4}{2y^2} = \frac{x^8}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y(x^4y^2+1)}{x}$ 	$R = x$ $S = -\frac{x^4}{2y^2}$	$\frac{dS}{dR} = 2R^7$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = \frac{1}{4} + c_1$$

$$c_1 = -\frac{3}{4}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^4}{2y^2} = \frac{x^8}{4} - \frac{3}{4}$$

The above simplifies to

$$-x^8y^2 - 2x^4 + 3y^2 = 0$$

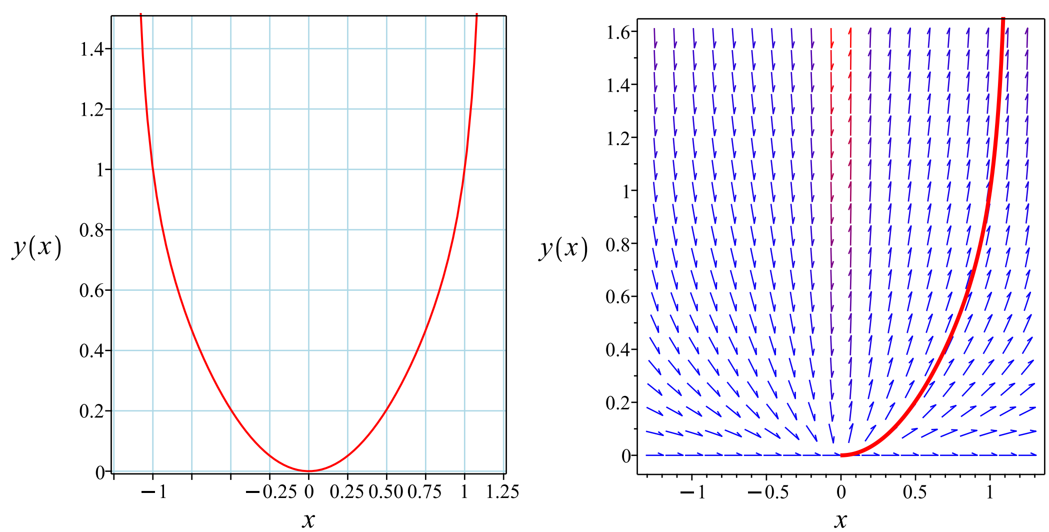
Solving for y from the above gives

$$y = \frac{2x^2}{\sqrt{-2x^8 + 6}}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^2}{\sqrt{-2x^8 + 6}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2x^2}{\sqrt{-2x^8 + 6}}$$

Verified OK.

8.41.3 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{2y(x^4y^2+1)}{x}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = \frac{2}{x}y + 2x^3y^3 \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{2}{x} \\
f_1(x) &= 2x^3 \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y'\frac{1}{y^3} = \frac{2}{x\,y^2} + 2x^3 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2} \tag{5}
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= \frac{2w(x)}{x} + 2x^3 \\
w' &= -\frac{4w}{x} - 4x^3
\end{aligned}
\tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{4}{x} \\
q(x) &= -4x^3
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{4w(x)}{x} = -4x^3
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{4}{x} dx} \\
&= x^4
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-4x^3\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^4 w\right) &= \left(x^4\right)\left(-4x^3\right) \\
\mathrm{d}\left(x^4 w\right) &= \left(-4x^7\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^4 w &= \int -4x^7\,\mathrm{d}x \\
x^4 w &= -\frac{x^8}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^4$ results in

$$
w(x) = -\frac{x^4}{2} + \frac{c_1}{x^4}
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = -\frac{x^4}{2} + \frac{c_1}{x^4}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{2} + c_1$$

$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{y^2} = -\frac{x^8 - 3}{2x^4}$$

The above simplifies to

$$x^8y^2 + 2x^4 - 3y^2 = 0$$

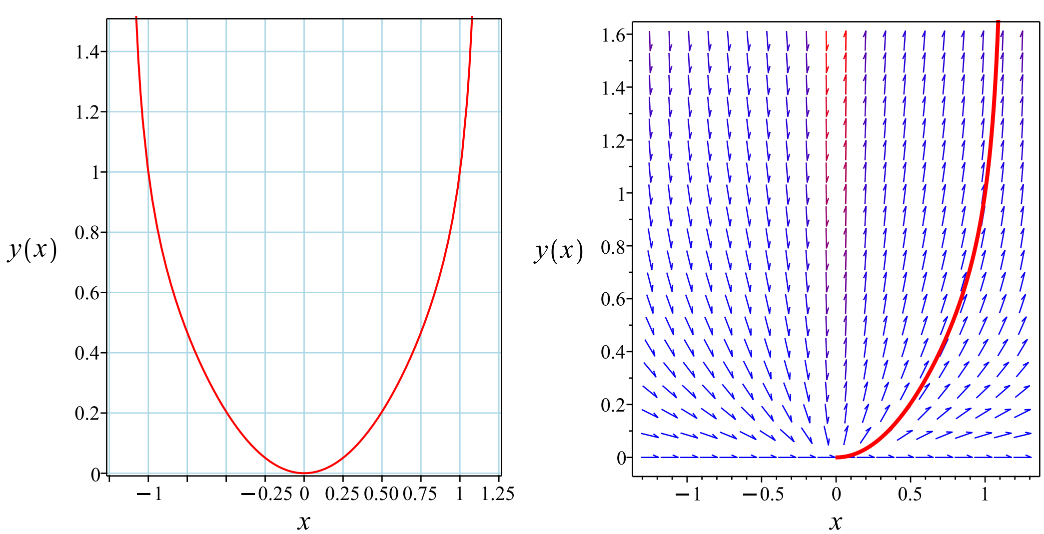
Solving for y from the above gives

$$y = \frac{2x^2}{\sqrt{-2x^8 + 6}}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^2}{\sqrt{-2x^8 + 6}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2x^2}{\sqrt{-2x^8 + 6}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 18

```
dsolve([x*diff(y(x),x)+(-2*y(x)-2*x^4*y(x)^3)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{2x^2}{\sqrt{-2x^8 + 6}}$$

✓ Solution by Mathematica

Time used: 0.225 (sec). Leaf size: 25

```
DSolve[{x*y'[x]+(-2*y[x]-2*x^4*y[x]^3)==0,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{\sqrt{2}x^2}{\sqrt{3 - x^8}}$$

8.42 problem 44

Internal problem ID [2074]
Internal file name [`OUTPUT/2074_Sunday_February_25_2024_06_49_10_AM_3623622/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{(-2x^2 - 3yx)\, y' + y^2 = 0}$$

With initial conditions

$$[y(1) = 1]$$

8.42.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y^2}{x(2x + 3y)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\left\{-\infty \leq x < 0, 0 < x < -\frac{3}{2}, -\frac{3}{2} < x \leq \infty\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\left\{y < -\frac{2}{3} \vee -\frac{2}{3} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^2}{x\,(2x+3y)}\right)\\ &= \frac{2y}{x\,(2x+3y)} - \frac{3y^2}{x\,(2x+3y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\left\{-\infty \leq x < 0, 0 < x < -\frac{3}{2}, -\frac{3}{2} < x \leq \infty\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < -\frac{2}{3} \vee -\frac{2}{3} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.42.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\left(-2x^2 - 3u(x)\,x^2\right)\left(u'(x)\,x + u(x)\right) + u(x)^2\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= -\frac{2u(u+1)}{(3u+2)\,x}\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u(u+1)}{3u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u(u+1)}{3u+2}}\,du = -\frac{2}{x}\,dx$$

$$\int \frac{1}{\frac{u(u+1)}{3u+2}}\,du = \int -\frac{2}{x}\,dx$$

$$2\ln(u) + \ln(u+1) = -2\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u)+\ln(u+1)} = e^{-2\ln(x)+c_2}$$

Which simplifies to

$$u^2(u+1) = \frac{c_3}{x^2}$$

The solution is

$$u(x)^2 \left(u(x)+1\right) = \frac{c_3}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2\left(1+\frac{y}{x}\right)}{x^2} = \frac{c_3}{x^2}$$
$$\frac{y^2(x+y)}{x^3} = \frac{c_3}{x^2}$$

Which simplifies to

$$\frac{y^2(x+y)}{x} = c_3$$

Substituting initial conditions and solving for c_3 gives $c_3 = 2$. Hence the solution be-

Summary

The solution(s) found are the following

comes

$$\frac{y^2(x+y)}{x} = 2 \tag{1}$$

Verification of solutions

$$\frac{y^2(x+y)}{x} = 2$$

Verified OK.

8.42.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y^2}{x(2x+3y)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{y^2 (b_3 - a_2)}{x (2x + 3y)} - \frac{y^4 a_3}{x^2 (2x + 3y)^2} \\ &- \left(-\frac{y^2}{x^2 (2x + 3y)} - \frac{2y^2}{x (2x + 3y)^2}\right) (x a_2 + y a_3 + a_1) \\ &- \left(\frac{2y}{x (2x + 3y)} - \frac{3y^2}{x (2x + 3y)^2}\right) (x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^4 b_2 + 8x^3 y b_2 + 2x^2 y^2 a_2 + 6x^2 y^2 b_2 - 2x^2 y^2 b_3 + 4x\, y^3 a_3 + 2y^4 a_3 - 4x^2 y b_1 + 4x\, y^2 a_1 - 3x\, y^2 b_1 + 3y^3 a_1}{x^2 (2x + 3y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &4x^4 b_2 + 8x^3 y b_2 + 2x^2 y^2 a_2 + 6x^2 y^2 b_2 - 2x^2 y^2 b_3 + 4x\, y^3 a_3 \\ &+ 2y^4 a_3 - 4x^2 y b_1 + 4x\, y^2 a_1 - 3x\, y^2 b_1 + 3y^3 a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1^2v_2^2 + 4a_3v_1v_2^3 + 2a_3v_2^4 + 4b_2v_1^4 + 8b_2v_1^3v_2 + 6b_2v_1^2v_2^2 - 2b_3v_1^2v_2^2 + 4a_1v_1v_2^2 + 3a_1v_2^3 - 4b_1v_1^2v_2 - 3b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$4b_2v_1^4 + 8b_2v_1^3v_2 + (2a_2 + 6b_2 - 2b_3)\, v_1^2v_2^2 - 4b_1v_1^2v_2 + 4a_3v_1v_2^3 + (4a_1 - 3b_1)\, v_1v_2^2 + 2a_3v_2^4 + 3a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 3a_1 &= 0 \\ 2a_3 &= 0 \\ 4a_3 &= 0 \\ -4b_1 &= 0 \\ 4b_2 &= 0 \\ 8b_2 &= 0 \\ 4a_1 - 3b_1 &= 0 \\ 2a_2 + 6b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y^2}{x\,(2x + 3y)}\right)(x) \\
&= \frac{2yx + 2y^2}{2x + 3y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2yx + 2y^2}{2x + 3y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln (x + y)}{2} + \ln (y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2}{x(2x + 3y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2x + 2y} \\ S_y &= \frac{1}{2x + 2y} + \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x + y)}{2} + \ln(y) = \frac{\ln(x)}{2} + c_1$$

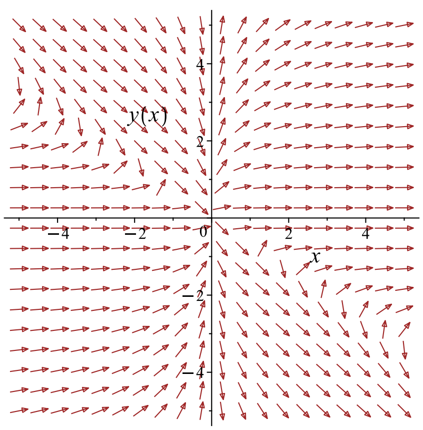
Which simplifies to

$$\frac{\ln(x + y)}{2} + \ln(y) = \frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{y^2}{x(2x+3y)}$



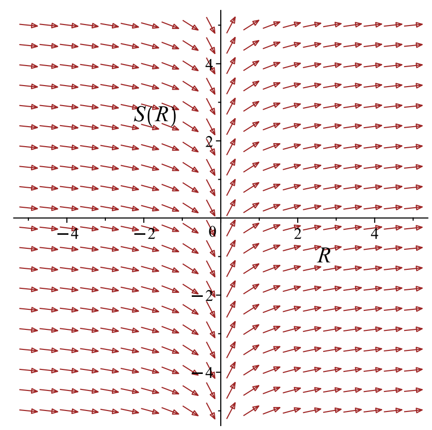
</td><td>

$R = x$

$S = \frac{\ln(x+y)}{2} + \ln(y)$

</td><td>

$\frac{dS}{dR} = \frac{1}{2R}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln(2)}{2} = c_1$$

$$c_1 = \frac{\ln(2)}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(x+y)}{2} + \ln(y) = \frac{\ln(x)}{2} + \frac{\ln(2)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln(x+y)}{2} + \ln(y) = \frac{\ln(x)}{2} + \frac{\ln(2)}{2} \tag{1}$$

Verification of solutions

$$\frac{\ln(x+y)}{2} + \ln(y) = \frac{\ln(x)}{2} + \frac{\ln(2)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.61 (sec). Leaf size: 86

```
dsolve([(-2*x^2-3*x*y(x))*diff(y(x),x)+y(x)^2=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\left(-x^3+3\sqrt{3}\,\sqrt{-2x^4+27x^2}+27x\right)^{\frac{2}{3}}-x\left(-x^3+3\sqrt{3}\,\sqrt{-2x^4+27x^2}+27x\right)^{\frac{1}{3}}+x^2}{3\left(-x^3+3\sqrt{3}\,\sqrt{-2x^4+27x^2}+27x\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 60.281 (sec). Leaf size: 77

```
DSolve[{(-2*x^2-3*x*y[x])*y'[x]+y[x]^2==0,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x)\to \frac{1}{3}\left(\frac{x^2}{\sqrt[3]{-x^3+3\sqrt{81x^2-6x^4}+27x}}+\sqrt[3]{-x^3+3\sqrt{81x^2-6x^4}+27x}-x\right)$$

8.43 problem 45

Internal problem ID [2075]
Internal file name [`OUTPUT/2075_Sunday_February_25_2024_06_49_12_AM_1738676/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x - 4y = x^4}$$

With initial conditions

$$[y(1) = 0]$$

8.43.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{4}{x}$$
$$q(x) = x^3$$

Hence the ode is

$$y' - \frac{4y}{x} = x^3$$

The domain of $p(x) = -\frac{4}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = x^3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

8.43.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{4}{x} dx} \\ &= \frac{1}{x^4}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(x^3\right) \\ \frac{d}{dx}\left(\frac{y}{x^4}\right) &= \left(\frac{1}{x^4}\right)\left(x^3\right) \\ d\left(\frac{y}{x^4}\right) &= \frac{1}{x}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^4} &= \int \frac{1}{x}\, dx \\ \frac{y}{x^4} &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^4}$ results in

$$y = \ln(x)\, x^4 + c_1 x^4$$

which simplifies to

$$y = x^4(\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

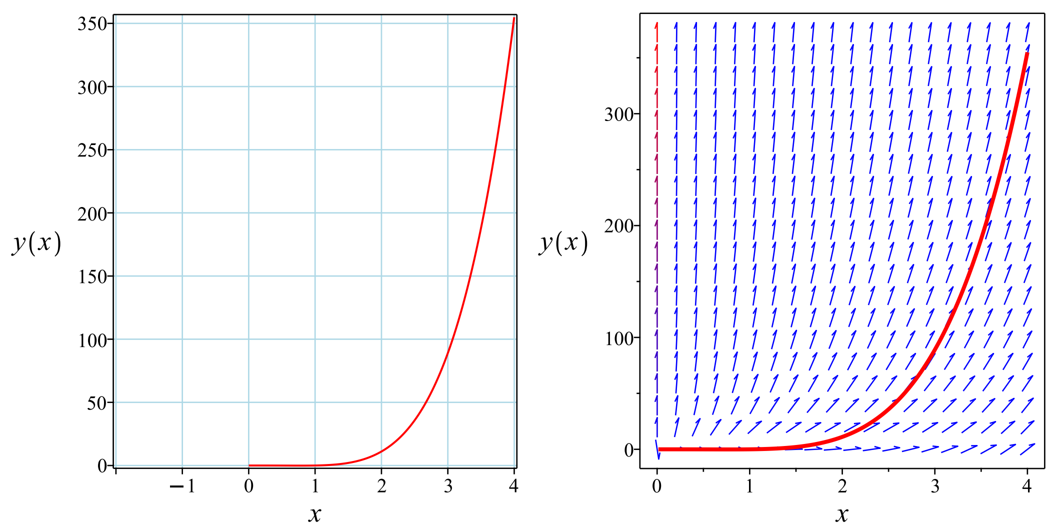
Substituting c_1 found above in the general solution gives

$$y = \ln(x)\, x^4$$

Summary

The solution(s) found are the following

$$y = \ln(x)\, x^4 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \ln(x)\, x^4$$

Verified OK.

8.43.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x^4 + 4y}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 286: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = x^4 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^4} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^4 + 4y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{4y}{x^5} \\
S_y &= \frac{1}{x^4}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{x^4} = \ln(x) + c_1
$$

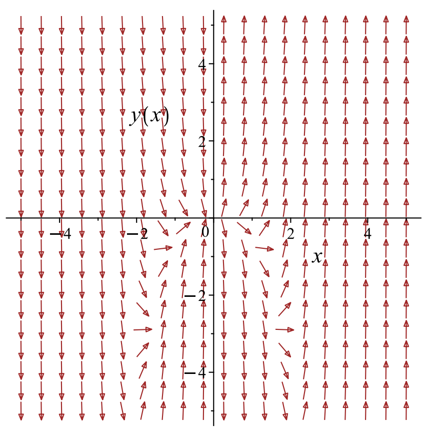
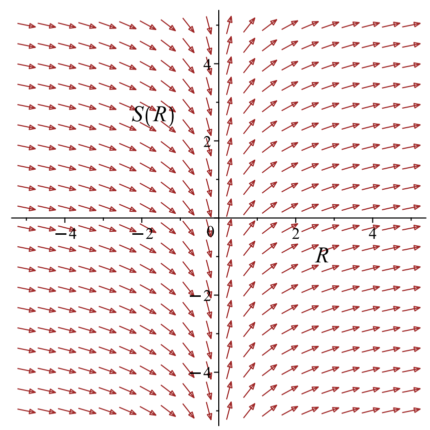
Which simplifies to

$$
\frac{y}{x^4} = \ln(x) + c_1
$$

Which gives

$$
y = x^4(\ln(x) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^4+4y}{x}$ 	$R = x$ $S = \frac{y}{x^4}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

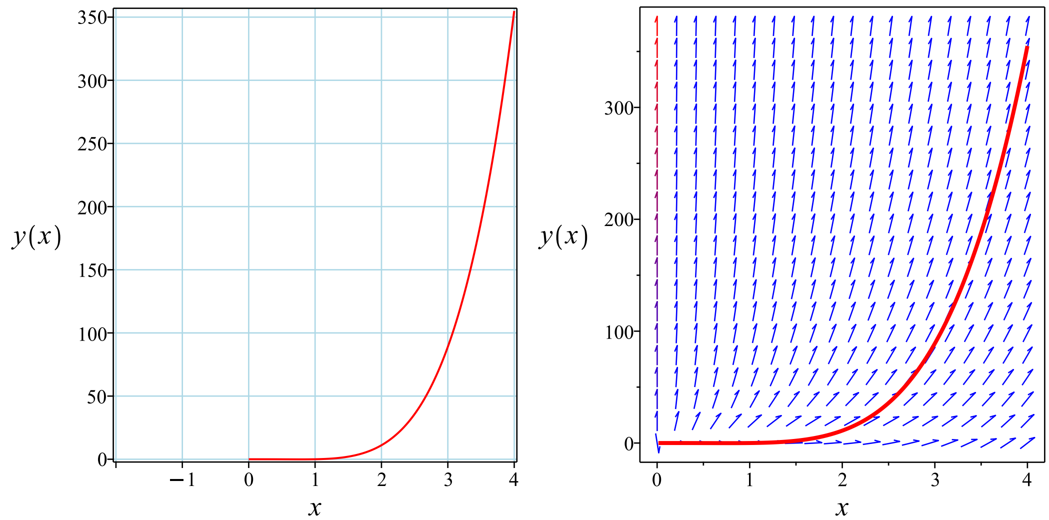
Substituting c_1 found above in the general solution gives

$$y = \ln(x)\, x^4$$

Summary

The solution(s) found are the following

$$y = \ln(x)\, x^4 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \ln(x)\, x^4$$

Verified OK.

8.43.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(x^4 + 4y\right)\mathrm{d}x\\ \left(-x^4 - 4y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^4 - 4y\\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^4 - 4y\right)\\ &= -4\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-4) - (1))\\ &= -\frac{5}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{5}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-5\ln(x)}\\ &= \frac{1}{x^5}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^5}\left(-x^4 - 4y\right)\\ &= \frac{-x^4 - 4y}{x^5}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^5}(x)\\ &= \frac{1}{x^4}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-x^4 - 4y}{x^5}\right) + \left(\frac{1}{x^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{-x^4 - 4y}{x^5} \,\mathrm{d}x$$

$$\phi = \frac{y}{x^4} - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^4} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^4}$. Therefore equation (4) becomes

$$\frac{1}{x^4} = \frac{1}{x^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x^4} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{x^4} - \ln(x)$$

The solution becomes

$$y = x^4(\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

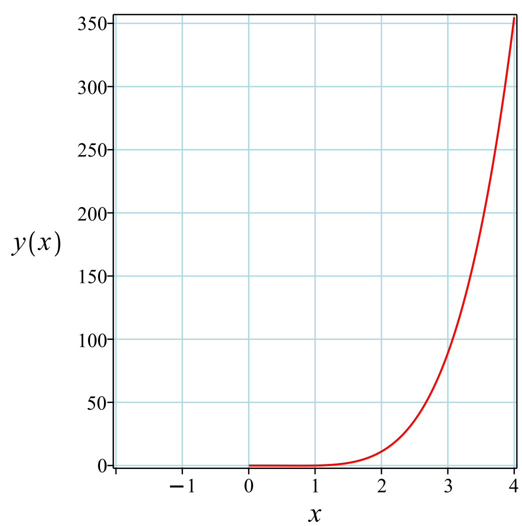
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

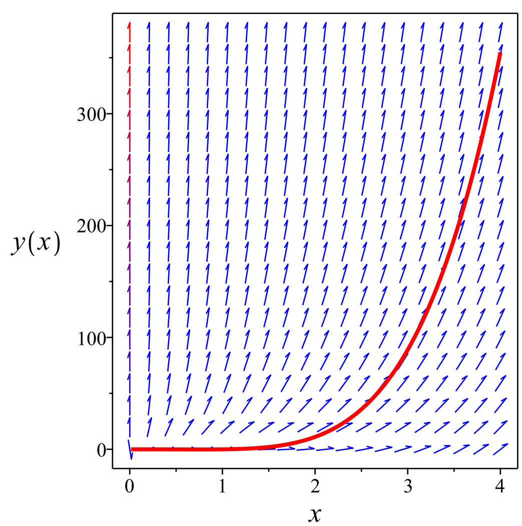
$$y = \ln(x)\, x^4$$

Summary
The solution(s) found are the following

$$y = \ln(x)\, x^4 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(x)\, x^4$$

Verified OK.

8.43.5 Maple step by step solution

Let's solve

$[y'x - 4y = x^4, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{4y}{x} + x^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{4y}{x} = x^3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{4y}{x}\right) = \mu(x)\, x^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{4y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{4\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^4}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^3 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^3 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^3 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^4}$

  $y = x^4\left(\int \frac{1}{x} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^4(\ln(x) + c_1)$

- Use initial condition $y(1) = 0$

  $0 = c_1$

- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \ln(x) x^4$
- Solution to the IVP

  $y = \ln(x) x^4$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([x*diff(y(x),x)=x^4+4*y(x),y(1) = 0],y(x), singsol=all)
```

$$y(x) = x^4 \ln(x)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 11

```
DSolve[{x*y'[x]==x^4+4*y[x],{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^4 \log(x)$$

8.44 problem 46

Internal problem ID [2076]
Internal file name [`OUTPUT/2076_Sunday_February_25_2024_06_49_12_AM_95011642/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y'x + y - y^6x^3 = 0}$$

With initial conditions

$$[y(1) = 1]$$

8.44.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y(x^3y^5 - 1)}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(x^3y^5 - 1)}{x}\right) \\ &= \frac{x^3y^5 - 1}{x} + 5x^2y^5\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.44.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \frac{y(x^3y^5 - 1)}{x} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 289: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^5 y^6
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^5 y^6} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{5x^5 y^5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(x^3 y^5 - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^6 y^5} \\ S_y &= \frac{1}{x^5 y^6} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

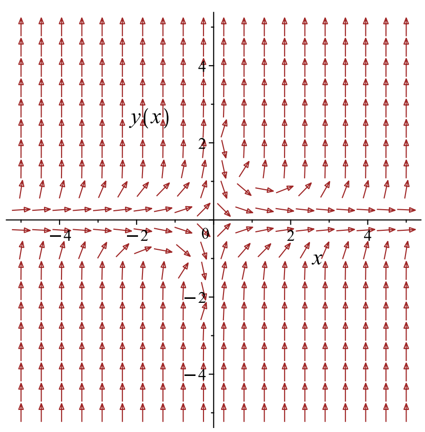
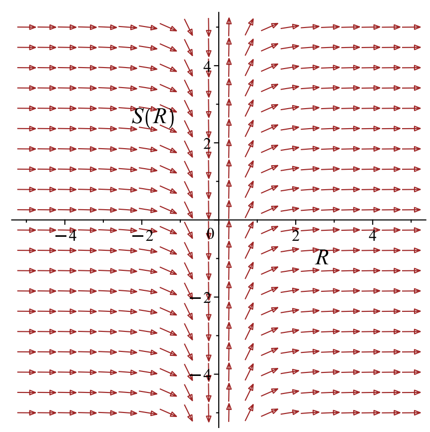
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{5x^5y^5} = -\frac{1}{2x^2} + c_1$$

Which simplifies to

$$-\frac{1}{5x^5y^5} = -\frac{1}{2x^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x^3y^5-1)}{x}$ 	$R = x$ $S = -\frac{1}{5x^5y^5}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{5} = -\frac{1}{2} + c_1$$

$$c_1 = \frac{3}{10}$$

Substituting c_1 found above in the general solution gives

$$-\frac{1}{5x^5y^5} = \frac{3x^2 - 5}{10x^2}$$

The above simplifies to

$$-3x^5y^5 + 5x^3y^5 - 2 = 0$$

Summary

The solution(s) found are the following

$$-2 + \left(-3x^5 + 5x^3\right) y^5 = 0 \tag{1}$$

Verification of solutions

$$-2 + \left(-3x^5 + 5x^3\right) y^5 = 0$$

Verified OK.

8.44.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(x^3y^5 - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + x^2y^6 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= x^2 \\
n &= 6
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^6$ gives

$$
y' \frac{1}{y^6} = -\frac{1}{x\, y^5} + x^2 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^5}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{5}{y^6} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{5} &= -\frac{w(x)}{x} + x^2 \\
w' &= \frac{5w}{x} - 5x^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{5}{x} \\
q(x) &= -5x^2
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{5w(x)}{x} = -5x^2
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{5}{x}dx} \\
&= \frac{1}{x^5}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-5x^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^5}\right) &= \left(\frac{1}{x^5}\right)\left(-5x^2\right) \\
\mathrm{d}\left(\frac{w}{x^5}\right) &= \left(-\frac{5}{x^3}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x^5} &= \int -\frac{5}{x^3}\,\mathrm{d}x \\
\frac{w}{x^5} &= \frac{5}{2x^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^5}$ results in

$$
w(x) = \frac{5}{2}x^3 + c_1 x^5
$$

Replacing w in the above by $\frac{1}{y^5}$ using equation (5) gives the final solution.

$$
\frac{1}{y^5} = \frac{5}{2}x^3 + c_1 x^5
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = \frac{5}{2} + c_1
$$

$$
c_1 = -\frac{3}{2}
$$

Substituting c_1 found above in the general solution gives

$$
\frac{1}{y^5} = \frac{5}{2}x^3 - \frac{3}{2}x^5
$$

The above simplifies to

$$3x^5y^5 - 5x^3y^5 + 2 = 0$$

Summary
The solution(s) found are the following

$$2 + (3x^5 - 5x^3)\,y^5 = 0 \tag{1}$$

Verification of solutions

$$2 + (3x^5 - 5x^3)\,y^5 = 0$$

Verified OK.

8.44.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(x^3y^6 - y\right)\mathrm{d}x\\ \left(-x^3y^6 + y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -x^3y^6 + y\\ N(x,y) &= x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3y^6 + y\right)\\ &= -6x^3y^5 + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}\left(\left(-6x^3y^5 + 1\right) - (1)\right)\\ &= -6x^2y^5\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{-x^3y^6 + y}\left((1) - \left(-6x^3y^5 + 1\right)\right) \\
&= -\frac{6x^3y^4}{x^3y^5 - 1}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (-6x^3y^5 + 1)}{x\left(-x^3y^6 + y\right) - y\left(x\right)} \\
&= -\frac{6}{yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{6}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{6}{t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-6\ln(t)} \\
&= \frac{1}{t^6}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{y^6x^6}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^6 x^6}\left(-x^3 y^6 + y\right) \\
&= \frac{-x^3 y^5 + 1}{x^6 y^5}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^6 x^6}(x) \\
&= \frac{1}{x^5 y^6}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^3 y^5 + 1}{x^6 y^5}\right) + \left(\frac{1}{x^5 y^6}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-x^3 y^5 + 1}{x^6 y^5}\,\mathrm{d}x \\
\phi &= \frac{5x^3 y^5 - 2}{10 x^5 y^5} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{5}{2x^2 y} - \frac{5x^3y^5 - 2}{2x^5y^6} + f'(y) \qquad (4)\\
&= \frac{1}{x^5y^6} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^5y^6}$. Therefore equation (4) becomes

$$
\frac{1}{x^5y^6} = \frac{1}{x^5y^6} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{5x^3y^5 - 2}{10x^5y^5} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{5x^3y^5 - 2}{10x^5y^5}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
\frac{3}{10} = c_1
$$

$$
c_1 = \frac{3}{10}
$$

Substituting c_1 found above in the general solution gives

$$\frac{5x^3y^5-2}{10x^5y^5}=\frac{3}{10}$$

The above simplifies to

$$-3x^5y^5+5x^3y^5-2=0$$

Summary

The solution(s) found are the following

$$-2+\left(-3x^5+5x^3\right)y^5=0 \tag{1}$$

Verification of solutions

$$-2+\left(-3x^5+5x^3\right)y^5=0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.781 (sec). Leaf size: 54

```
dsolve([x*diff(y(x),x)+y(x)=x^3*y(x)^6,y(1) = 1],y(x), singsol=all)
```

$$y(x)=\frac{\left(-\sqrt{5}-1+i\sqrt{10-2\sqrt{5}}\right)2^{\frac{1}{5}}\left(-x^2(3x^2-5)^4\right)^{\frac{1}{5}}}{12x^3-20x}$$

✓ Solution by Mathematica

Time used: 0.473 (sec). Leaf size: 26

```
DSolve[{x*y'[x]+y[x]==x^3*y[x]^6,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[5]{2}}{\sqrt[5]{5x^3 - 3x^5}}$$

8.45 problem 48

Internal problem ID [2077]
Internal file name [`OUTPUT/2077_Sunday_February_25_2024_06_49_13_AM_45591139/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Bernoulli]
```

$$\boxed{x' - x - x^2 \mathrm{e}^{\theta} = 0}$$

With initial conditions

$$[x(0) = 2]$$

8.45.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} x' &= f(\theta, x) \\ &= x + x^2 \mathrm{e}^{\theta} \end{aligned}$$

The θ domain of $f(\theta, x)$ when $x = 2$ is

$$\{-\infty < \theta < \infty\}$$

And the point $\theta_0 = 0$ is inside this domain. The x domain of $f(\theta, x)$ when $\theta = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(x + x^2 \mathrm{e}^{\theta}\right)\\ &= 2\,\mathrm{e}^{\theta} x + 1\end{aligned}$$

The θ domain of $\frac{\partial f}{\partial x}$ when $x = 2$ is

$$\{-\infty < \theta < \infty\}$$

And the point $\theta_0 = 0$ is inside this domain. The x domain of $\frac{\partial f}{\partial x}$ when $\theta = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. Therefore solution exists and is unique.

8.45.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}x' &= x + x^2 \mathrm{e}^{\theta}\\ x' &= \omega(\theta, x)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_x - \xi_\theta) - \omega^2 \xi_x - \omega_\theta \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 291: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, x) &= 0 \\
\eta(\theta, x) &= x^2 e^{-\theta}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial x}\right) S(\theta, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2 \mathrm{e}^{-\theta}} dy \end{aligned}$$

Which results in

$$S = -\frac{\mathrm{e}^{\theta}}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_{\theta} + \omega(\theta, x) S_x}{R_{\theta} + \omega(\theta, x) R_x} \tag{2}$$

Where in the above $R_{\theta}, R_x, S_{\theta}, S_x$ are all partial derivatives and $\omega(\theta, x)$ is the right hand side of the original ode given by

$$\omega(\theta, x) = x + x^2 \mathrm{e}^{\theta}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_{\theta} &= 1 \\ R_x &= 0 \\ S_{\theta} &= -\frac{\mathrm{e}^{\theta}}{x} \\ S_x &= \frac{\mathrm{e}^{\theta}}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{2\theta} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, x coordinates. This results in

$$-\frac{e^{\theta}}{x} = \frac{e^{2\theta}}{2} + c_1$$

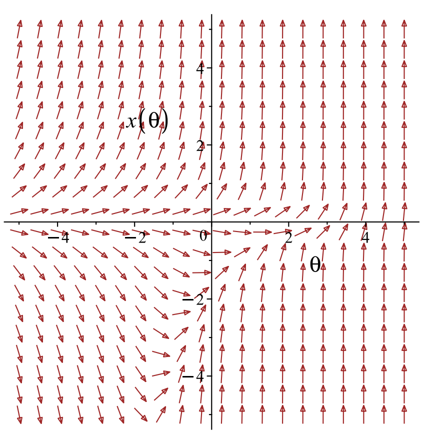
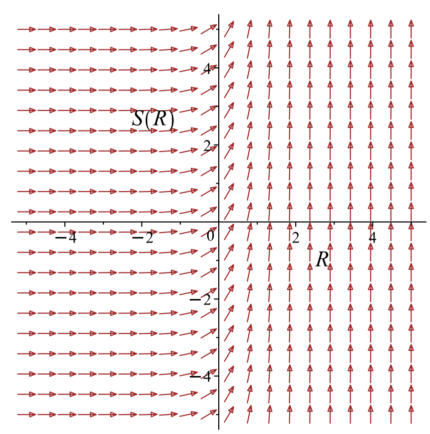
Which simplifies to

$$-\frac{e^{\theta}}{x} = \frac{e^{2\theta}}{2} + c_1$$

Which gives

$$x = -\frac{2\,e^{\theta}}{e^{2\theta} + 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{d\theta} = x + x^2 e^{\theta}$ 	$R = \theta$ $S = -\frac{e^{\theta}}{x}$	$\frac{dS}{dR} = e^{2R}$ 

Initial conditions are used to solve for c_1. Substituting $\theta = 0$ and $x = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{2}{2c_1 + 1}$$

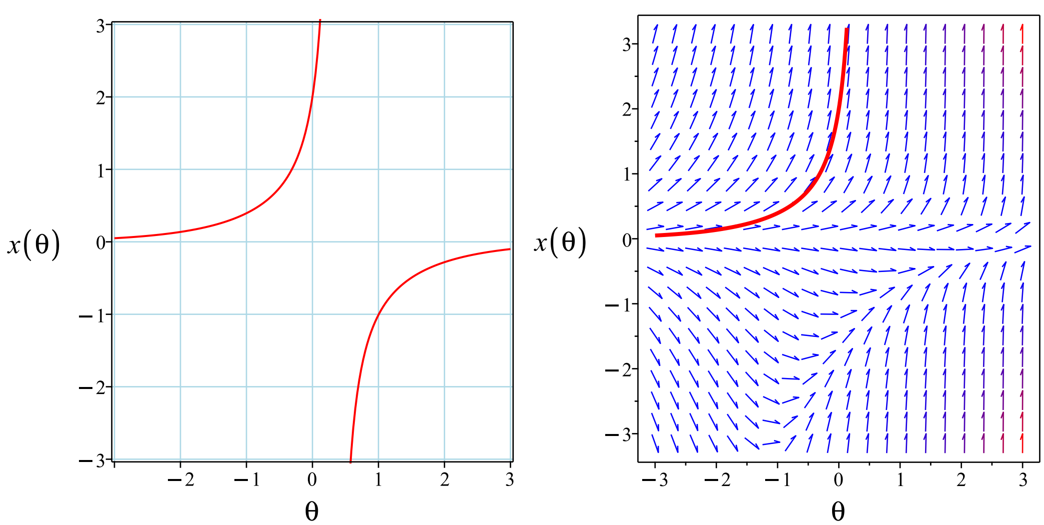
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$x = -\frac{2\,\mathrm{e}^{\theta}}{\mathrm{e}^{2\theta} - 2}$$

Summary
The solution(s) found are the following

$$x = -\frac{2\,\mathrm{e}^{\theta}}{\mathrm{e}^{2\theta} - 2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -\frac{2\,\mathrm{e}^{\theta}}{\mathrm{e}^{2\theta} - 2}$$

Verified OK.

8.45.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} x' &= F(\theta, x) \\ &= x + x^2 e^{\theta} \end{aligned}$$

This is a Bernoulli ODE.

$$x' = x + e^{\theta} x^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$x' = f_0(\theta)x + f_1(\theta)x^n \tag{2}$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_0(\theta)x^{1-n} + f_1(\theta) \tag{3}$$

The next step is use the substitution $w = x^{1-n}$ in equation (3) which generates a new ODE in $w(\theta)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(\theta)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(\theta) &= 1 \\ f_1(\theta) &= e^{\theta} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $x^n = x^2$ gives

$$x' \frac{1}{x^2} = \frac{1}{x} + e^{\theta} \tag{4}$$

Let

$$\begin{aligned} w &= x^{1-n} \\ &= \frac{1}{x} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t θ gives

$$w' = -\frac{1}{x^2} x' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(\theta) &= w(\theta) + \mathrm{e}^{\theta} \\ w' &= -w - \mathrm{e}^{\theta} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(\theta)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(\theta) + p(\theta)w(\theta) = q(\theta)$$

Where here

$$\begin{aligned} p(\theta) &= 1 \\ q(\theta) &= -\mathrm{e}^{\theta} \end{aligned}$$

Hence the ode is

$$w'(\theta) + w(\theta) = -\mathrm{e}^{\theta}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1 d\theta} \\ &= \mathrm{e}^{\theta} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}(\mu w) &= (\mu)\left(-\mathrm{e}^{\theta}\right) \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathrm{e}^{\theta} w\right) &= \left(\mathrm{e}^{\theta}\right)\left(-\mathrm{e}^{\theta}\right) \\ \mathrm{d}\left(\mathrm{e}^{\theta} w\right) &= \left(-\mathrm{e}^{2\theta}\right) \mathrm{d}\theta \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{\theta} w &= \int -\mathrm{e}^{2\theta} \,\mathrm{d}\theta \\ \mathrm{e}^{\theta} w &= -\frac{\mathrm{e}^{2\theta}}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\theta}$ results in

$$w(\theta) = -\frac{\mathrm{e}^{-\theta}\mathrm{e}^{2\theta}}{2} + c_1\mathrm{e}^{-\theta}$$

which simplifies to

$$w(\theta) = -\frac{e^{\theta}}{2} + c_1 e^{-\theta}$$

Replacing w in the above by $\frac{1}{x}$ using equation (5) gives the final solution.

$$\frac{1}{x} = -\frac{e^{\theta}}{2} + c_1 e^{-\theta}$$

Or

$$x = \frac{1}{-\frac{e^{\theta}}{2} + c_1 e^{-\theta}}$$

Which is simplified to

$$x = -\frac{2 e^{\theta}}{e^{2\theta} - 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $\theta = 0$ and $x = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{2}{-1 + 2c_1}$$

$$c_1 = 1$$

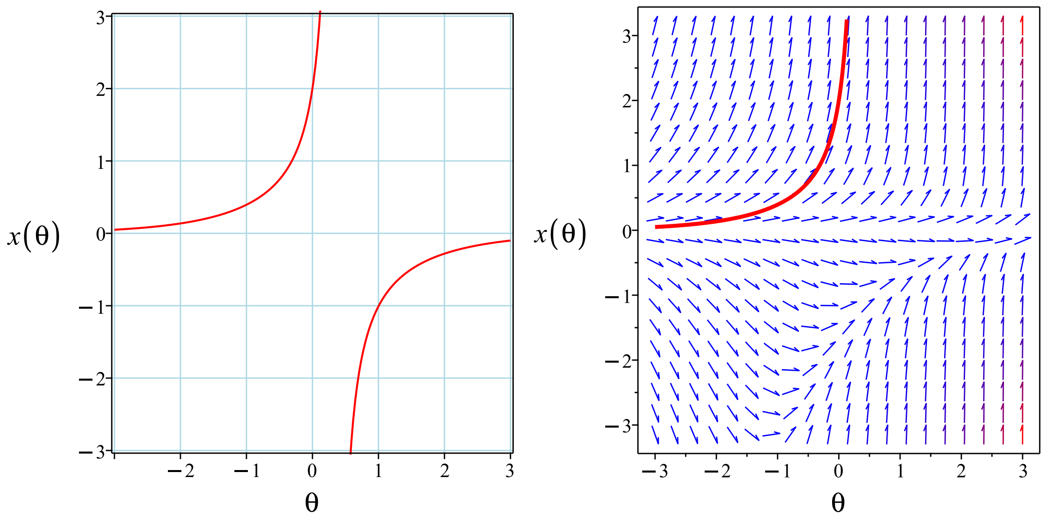
Substituting c_1 found above in the general solution gives

$$x = -\frac{2 e^{\theta}}{e^{2\theta} - 2}$$

Summary

The solution(s) found are the following

$$x = -\frac{2 e^{\theta}}{e^{2\theta} - 2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -\frac{2\,e^{\theta}}{e^{2\theta} - 2}$$

Verified OK.

8.45.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(\theta, x) \\ &= x + x^2 e^{\theta} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$x' = x + x^2 e^{\theta}$$

With Riccati ODE standard form

$$x' = f_0(\theta) + f_1(\theta)x + f_2(\theta)x^2$$

Shows that $f_0(\theta) = 0$, $f_1(\theta) = 1$ and $f_2(\theta) = e^{\theta}$. Let

$$\begin{aligned} x &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{e^{\theta} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(\theta) - (f_2' + f_1f_2)\,u'(\theta) + f_2^2f_0u(\theta) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= e^{\theta} \\ f_1f_2 &= e^{\theta} \\ f_2^2f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$e^{\theta}u''(\theta) - 2\,e^{\theta}u'(\theta) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(\theta) = c_1 + c_2e^{2\theta}$$

The above shows that

$$u'(\theta) = 2c_2e^{2\theta}$$

Using the above in (1) gives the solution

$$x = -\frac{2c_2e^{2\theta}e^{-\theta}}{c_1 + c_2e^{2\theta}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$x = -\frac{2\,e^{\theta}}{c_3 + e^{2\theta}}$$

Initial conditions are used to solve for c_3. Substituting $\theta = 0$ and $x = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{2}{c_3 + 1}$$

$$c_3 = -2$$

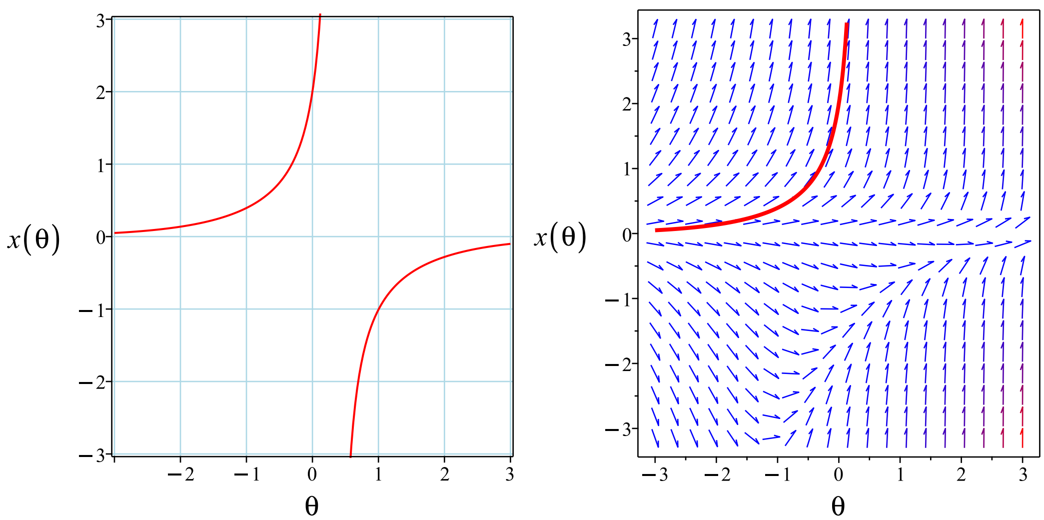
Substituting c_3 found above in the general solution gives

$$x = -\frac{2\,e^{\theta}}{e^{2\theta}-2}$$

Summary

The solution(s) found are the following

$$x = -\frac{2\,e^{\theta}}{e^{2\theta}-2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -\frac{2\,e^{\theta}}{e^{2\theta}-2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 16

```
dsolve([diff(x(theta),theta)=x(theta)+x(theta)^2*exp(theta),x(0) = 2],x(theta), singsol=all)
```

$$x(\theta) = -\frac{2\,\mathrm{e}^{\theta}}{\mathrm{e}^{2\theta}-2}$$

✓ Solution by Mathematica

Time used: 0.217 (sec). Leaf size: 19

```
DSolve[{x'[\[Theta]]==x[\[Theta]]+x[\[Theta]]^2*Exp[\[Theta]],{x[0]==2}},x[\[Theta]],\[Theta
```

$$x(\theta) \to -\frac{2e^{\theta}}{e^{2\theta}-2}$$

8.46 problem 49

Internal problem ID [2078]
Internal file name [`OUTPUT/2078_Sunday_February_25_2024_06_49_14_AM_20919100/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - 2y'xy = -x^2}$$

With initial conditions

$$[y(2) = 0]$$

8.46.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{x^2 + y^2}{2xy} \end{aligned}$$

$f(x, y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

8.46.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\, x^2 - 2(u'(x)\, x + u(x))\, x^2 u(x) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u^2 - 1}{2ux} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2-1}{u}}\, du &= -\frac{1}{2x}\, dx \\ \int \frac{1}{\frac{u^2-1}{u}}\, du &= \int -\frac{1}{2x}\, dx \\ \frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} &= -\frac{\ln(x)}{2} + c_2 \end{aligned}$$

The above can be written as

$$\begin{aligned} \left(\frac{1}{2}\right) (\ln(u-1) + \ln(u+1)) &= -\frac{\ln(x)}{2} + 2c_2 \\ \ln(u-1) + \ln(u+1) &= (2) \left(-\frac{\ln(x)}{2} + 2c_2\right) \\ &= -\ln(x) + 4c_2 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-\ln(x)+2c_2}$$

Which simplifies to

$$\begin{aligned} u^2 - 1 &= \frac{2c_2}{x} \\ &= \frac{c_3}{x} \end{aligned}$$

The solution is

$$u(x)^2 - 1 = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$
$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$

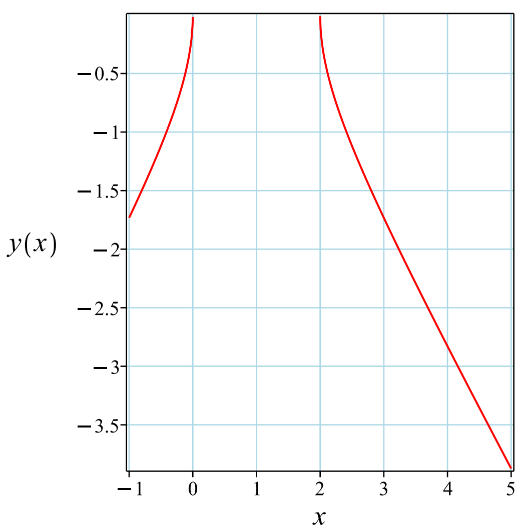
Substituting initial conditions and solving for c_3 gives $c_3 = -2$. Hence the solution becomes Solving for y from the above gives

$$y = \sqrt{x(-2+x)}$$
$$y = -\sqrt{x(-2+x)}$$

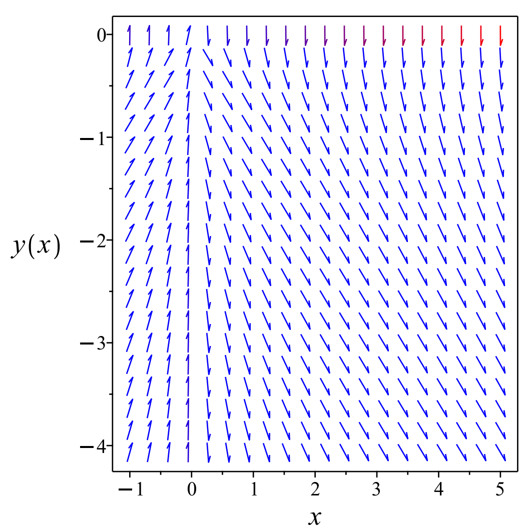
Summary

The solution(s) found are the following

$$y = \sqrt{x(-2+x)} \tag{1}$$
$$y = -\sqrt{x(-2+x)} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{x(-2+x)}$$

Verified OK.

$$y = -\sqrt{x(-2+x)}$$

Verified OK.

8.46.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y^2}{2xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 293: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x}{y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{y^2}{2x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x^2 + y^2}{2xy}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^2}{2x^2} \\
S_y &= \frac{y}{x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R}{2} + c_1 \tag{4}
$$

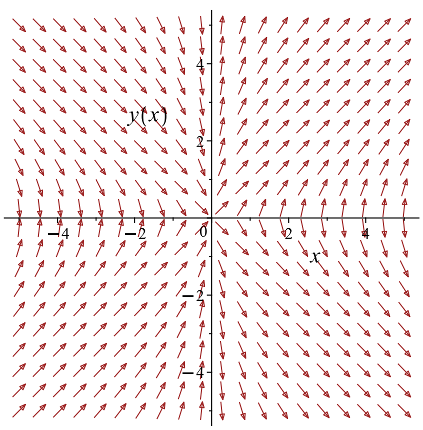
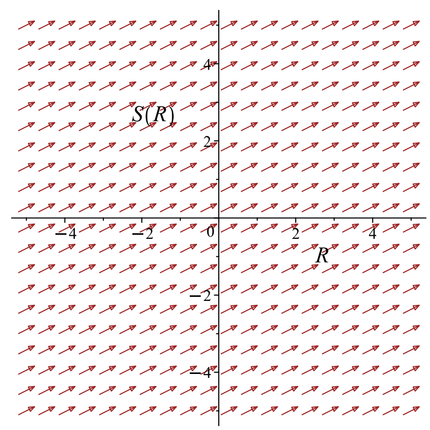
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y^2}{2x} = \frac{x}{2} + c_1
$$

Which simplifies to

$$
\frac{y^2}{2x} = \frac{x}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 1 + c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$\frac{y^2}{2x} = -1 + \frac{x}{2}$$

The above simplifies to

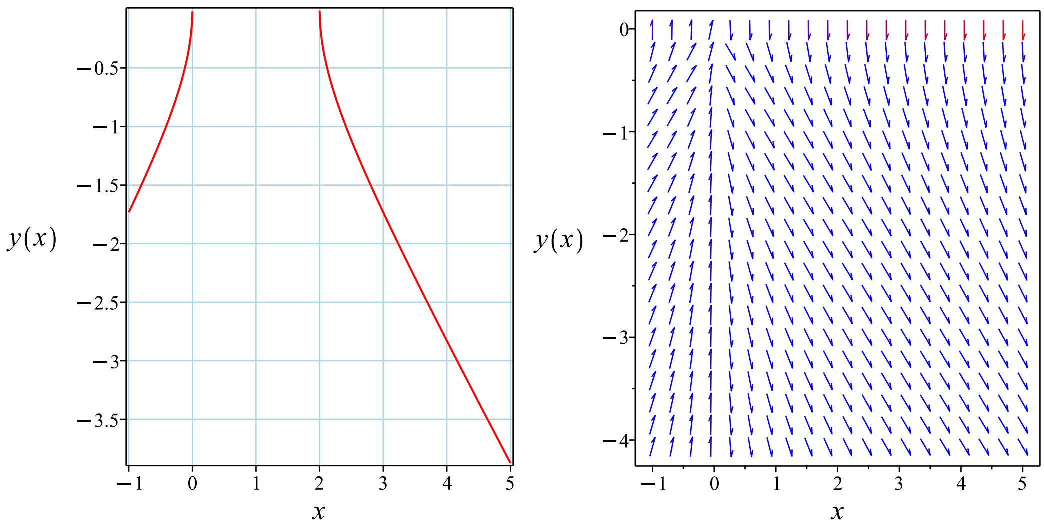
$$-x^2 + y^2 + 2x = 0$$

Solving for y from the above gives

$$y = \sqrt{x(-2+x)}$$
$$y = -\sqrt{x(-2+x)}$$

Summary
The solution(s) found are the following

$$y = \sqrt{x(-2+x)} \tag{1}$$
$$y = -\sqrt{x(-2+x)} \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{x(-2+x)}$$

Verified OK.

$$y = -\sqrt{x(-2+x)}$$

Verified OK.

8.46.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2+y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{2x} \\ f_1(x) &= \frac{x}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{2x} + \frac{x}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{2x} + \frac{x}{2} \\ w' &= \frac{w}{x} + x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = x$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = x$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(x)$$
$$\frac{d}{dx}\left(\frac{w}{x}\right) = \left(\frac{1}{x}\right)(x)$$
$$d\left(\frac{w}{x}\right) = dx$$

Integrating gives

$$\frac{w}{x} = \int dx$$
$$\frac{w}{x} = x + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x + x^2$$

which simplifies to

$$w(x) = x(x + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x(x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 2c_1 + 4$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$y^2 = x(-2 + x)$$

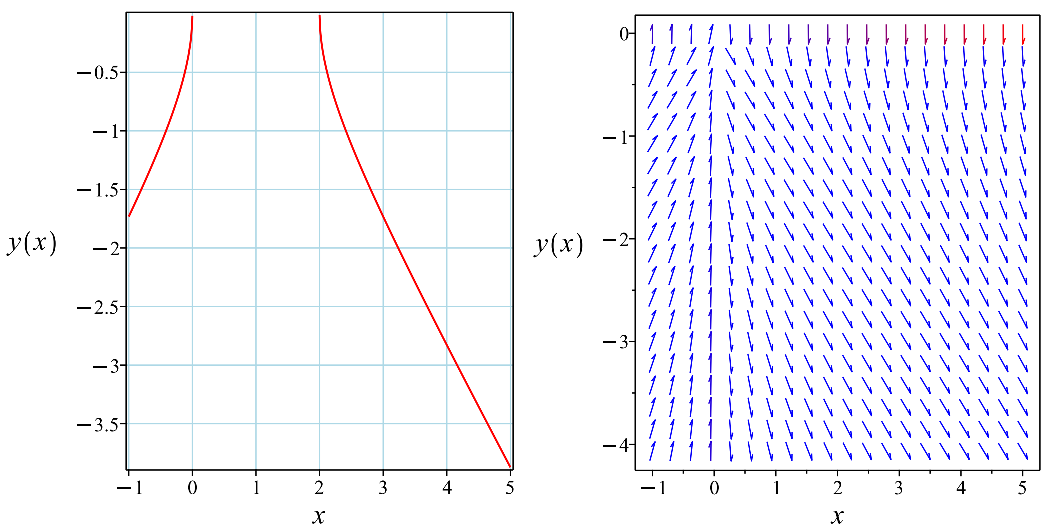
Solving for y from the above gives

$$y = \sqrt{x(-2 + x)}$$
$$y = -\sqrt{x(-2 + x)}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x(-2 + x)} \tag{1}$$
$$y = -\sqrt{x(-2 + x)} \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{x(-2 + x)}$$

Verified OK.

$$y = -\sqrt{x(-2 + x)}$$

Verified OK.

8.46.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2yx)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x\\ \left(x^2 + y^2\right)\mathrm{d}x + (-2yx)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^2 + y^2\\ N(x,y) &= -2yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 + y^2) \\ &= 2y \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2yx) \\ &= -2y \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{2yx}((2y) - (-2y)) \\ &= -\frac{2}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^2}(x^2 + y^2) \\ &= \frac{x^2 + y^2}{x^2} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-2yx) \\
&= -\frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2+y^2}{x^2}\right) + \left(-\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2+y^2}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x - \frac{y^2}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$
-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{y^2}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$x - \frac{y^2}{x} = 2$$

The above simplifies to

$$x^2 - y^2 - 2x = 0$$

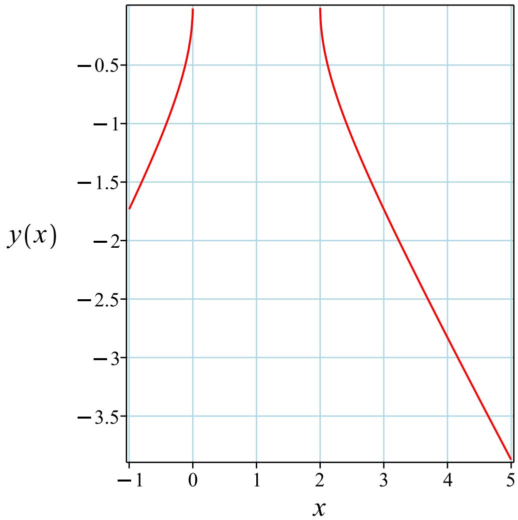
Solving for y from the above gives

$$y = \sqrt{x(-2 + x)}$$
$$y = -\sqrt{x(-2 + x)}$$

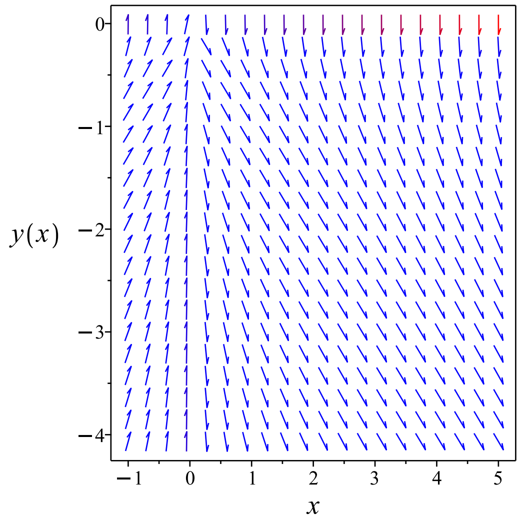
Summary

The solution(s) found are the following

$$y = \sqrt{x(-2 + x)} \tag{1}$$
$$y = -\sqrt{x(-2 + x)} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{x(-2+x)}$$

Verified OK.

$$y = -\sqrt{x(-2+x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 27

```
dsolve([(x^2+y(x)^2)=2*x*y(x)*diff(y(x),x),y(2) = 0],y(x), singsol=all)
```

$$y(x) = \sqrt{x(-2+x)}$$
$$y(x) = -\sqrt{x(-2+x)}$$

✓ Solution by Mathematica

Time used: 0.182 (sec). Leaf size: 36

```
DSolve[{(x^2+y[x]^2)==2*x*y[x]*y'[x],{y[2]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x-2}\sqrt{x}$$
$$y(x) \to \sqrt{x-2}\sqrt{x}$$

8.47 problem 50

Internal problem ID [2079]
Internal file name [`OUTPUT/2079_Sunday_February_25_2024_06_49_20_AM_84792875/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{3yx + (3x^2 + y^2)\, y' = 0}$$

With initial conditions

$$[y(0) = 1]$$

8.47.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{3yx}{3x^2 + y^2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3yx}{3x^2+y^2}\right)\\ &= -\frac{3x}{3x^2+y^2} + \frac{6y^2x}{(3x^2+y^2)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.47.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$3u(x)\,x^2 + \left(3x^2 + u(x)^2\,x^2\right)\left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= -\frac{u(u^2+6)}{x\,(u^2+3)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2+6)}{u^2+3}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u(u^2+6)}{u^2+3}}\,du &= -\frac{1}{x}\,dx\\ \int \frac{1}{\frac{u(u^2+6)}{u^2+3}}\,du &= \int -\frac{1}{x}\,dx\\ \frac{\ln(u)}{2} + \frac{\ln(u^2+6)}{4} &= -\ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{\frac{\ln(u)}{2}+\frac{\ln\left(u^2+6\right)}{4}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u}\left(u^2+6\right)^{\frac{1}{4}} = \frac{c_3}{x}$$

The solution is

$$\sqrt{u(x)}\left(u(x)^2+6\right)^{\frac{1}{4}} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y}{x}}\left(\frac{y^2}{x^2}+6\right)^{\frac{1}{4}} = \frac{c_3}{x}$$
$$\sqrt{\frac{y}{x}}\left(\frac{y^2+6x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3}{x}$$

Initial conditions are used to solve for c_3. Substituting $x=0$ and $y=1$ in the above solution gives an equation to solve for the constant of integration. Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

8.47.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{3yx}{3x^2+y^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{3yx(b_3 - a_2)}{3x^2 + y^2} - \frac{9y^2x^2a_3}{(3x^2 + y^2)^2} - \left(-\frac{3y}{3x^2 + y^2} + \frac{18y\,x^2}{(3x^2 + y^2)^2}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{3x}{3x^2 + y^2} + \frac{6y^2x}{(3x^2 + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{18x^4b_2 - 18y^2x^2a_3 + 3x^2y^2b_2 + 6x\,y^3a_2 - 6x\,y^3b_3 + 3y^4a_3 + y^4b_2 + 9x^3b_1 - 9x^2ya_1 - 3x\,y^2b_1 + 3y^3a_1}{(3x^2 + y^2)^2} = 0$$

Setting the numerator to zero gives

$$18x^4b_2 - 18y^2x^2a_3 + 3x^2y^2b_2 + 6x\,y^3a_2 - 6x\,y^3b_3 + 3y^4a_3 + y^4b_2 + 9x^3b_1 - 9x^2ya_1 - 3x\,y^2b_1 + 3y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$6a_2v_1v_2^3 - 18a_3v_1^2v_2^2 + 3a_3v_2^4 + 18b_2v_1^4 + 3b_2v_1^2v_2^2 + b_2v_2^4 - 6b_3v_1v_2^3 - 9a_1v_1^2v_2 + 3a_1v_2^3 + 9b_1v_1^3 - 3b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} 18b_2v_1^4 + 9b_1v_1^3 + (-18a_3 + 3b_2)\, v_1^2v_2^2 - 9a_1v_1^2v_2 \\ + (6a_2 - 6b_3)\, v_1v_2^3 - 3b_1v_1v_2^2 + (3a_3 + b_2)\, v_2^4 + 3a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -9a_1 &= 0 \\ 3a_1 &= 0 \\ -3b_1 &= 0 \\ 9b_1 &= 0 \\ 18b_2 &= 0 \\ 6a_2 - 6b_3 &= 0 \\ -18a_3 + 3b_2 &= 0 \\ 3a_3 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{3yx}{3x^2+y^2}\right)(x) \\
&= \frac{6y\,x^2+y^3}{3x^2+y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{6y\,x^2+y^3}{3x^2+y^2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(6x^2+y^2)}{4} + \frac{\ln(y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3yx}{3x^2+y^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{3x}{6x^2+y^2}\\
S_y &= \frac{3x^2+y^2}{6y\,x^2+y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

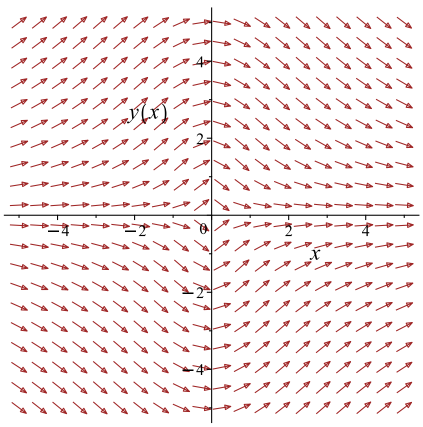
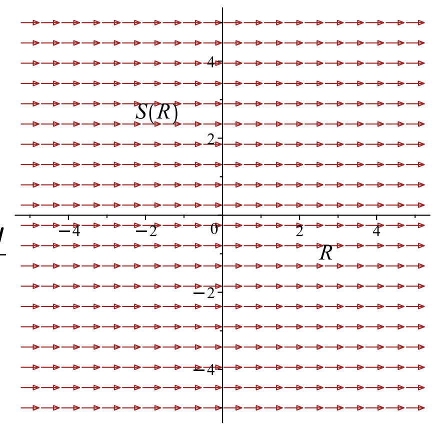
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y^2+6x^2)}{4} + \frac{\ln(y)}{2} = c_1
$$

Which simplifies to

$$
\frac{\ln(y^2+6x^2)}{4} + \frac{\ln(y)}{2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3yx}{3x^2+y^2}$ 	$R = x$ $S = \frac{\ln(6x^2+y^2)}{4} + \frac{\ln(y}{2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(6x^2+y^2)}{4} + \frac{\ln(y)}{2} = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln(y^2+6x^2)}{4} + \frac{\ln(y)}{2} = 0 \tag{1}$$

Verification of solutions

$$\frac{\ln(y^2+6x^2)}{4} + \frac{\ln(y)}{2} = 0$$

Verified OK.

8.47.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x^2+y^2)\,\mathrm{d}y&=(-3yx)\,\mathrm{d}x\\ (3yx)\,\mathrm{d}x+(3x^2+y^2)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3yx\\ N(x,y)&=3x^2+y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3yx)\\ &= 3x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x^2 + y^2)\\ &= 6x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{3x^2 + y^2}((3x) - (6x))\\ &= -\frac{3x}{3x^2 + y^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{3yx}((6x) - (3x))\\ &= \frac{1}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int \frac{1}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\mu = e^{\ln(y)}$$
$$= y$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= y(3yx) \\ &= 3x\,y^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= y\left(3x^2 + y^2\right) \\ &= 3y\,x^2 + y^3\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(3x\,y^2\right) + \left(3y\,x^2 + y^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 3x\,y^2\,\mathrm{d}x\end{aligned}$$
$$\phi = \frac{3x^2y^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3y\,x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y\,x^2 + y^3$. Therefore equation (4) becomes

$$3y\,x^2 + y^3 = 3y\,x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^3)\,\mathrm{d}y$$

$$f(y) = \frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3}{2}x^2y^2 + \frac{1}{4}y^4$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{4} = c_1$$

$$c_1 = \frac{1}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{3}{2}x^2y^2 + \frac{1}{4}y^4 = \frac{1}{4}$$

Summary

The solution(s) found are the following

$$\frac{3x^2y^2}{2} + \frac{y^4}{4} = \frac{1}{4} \tag{1}$$

Verification of solutions

$$\frac{3x^2y^2}{2} + \frac{y^4}{4} = \frac{1}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 4.016 (sec). Leaf size: 21

```
dsolve([(3*x*y(x))+(3*x^2+y(x)^2)*diff(y(x),x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \sqrt{-3x^2 + \sqrt{9x^4 + 1}}$$

✓ Solution by Mathematica

Time used: 8.701 (sec). Leaf size: 26

```
DSolve[{(3*x*y[x])+(3*x^2+y[x]^2)*y'[x]==0,{y[0]==1}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \sqrt{\sqrt{9x^4+1}-3x^2}$$

8.48 problem 51

Internal problem ID [2080]
Internal file name [`OUTPUT/2080_Sunday_February_25_2024_06_49_22_AM_35808696/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{2y + y' = 3\,e^{2x}}$$

With initial conditions

$$[y(0) = 1]$$

8.48.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= 3\,e^{2x} \end{aligned}$$

Hence the ode is

$$2y + y' = 3\,\mathrm{e}^{2x}$$

The domain of $p(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 3\,\mathrm{e}^{2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.48.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 2dx}\\ &= \mathrm{e}^{2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(3\,\mathrm{e}^{2x}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x}y\right) &= \left(\mathrm{e}^{2x}\right)\left(3\,\mathrm{e}^{2x}\right)\\ \mathrm{d}\left(\mathrm{e}^{2x}y\right) &= \left(3\,\mathrm{e}^{4x}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{2x}y &= \int 3\,\mathrm{e}^{4x}\,\mathrm{d}x\\ \mathrm{e}^{2x}y &= \frac{3\,\mathrm{e}^{4x}}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$y = \frac{3\,\mathrm{e}^{-2x}\mathrm{e}^{4x}}{4} + c_1\mathrm{e}^{-2x}$$

which simplifies to

$$y = \frac{3\,\mathrm{e}^{2x}}{4} + c_1\mathrm{e}^{-2x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{3}{4} + c_1$$

$$c_1 = \frac{1}{4}$$

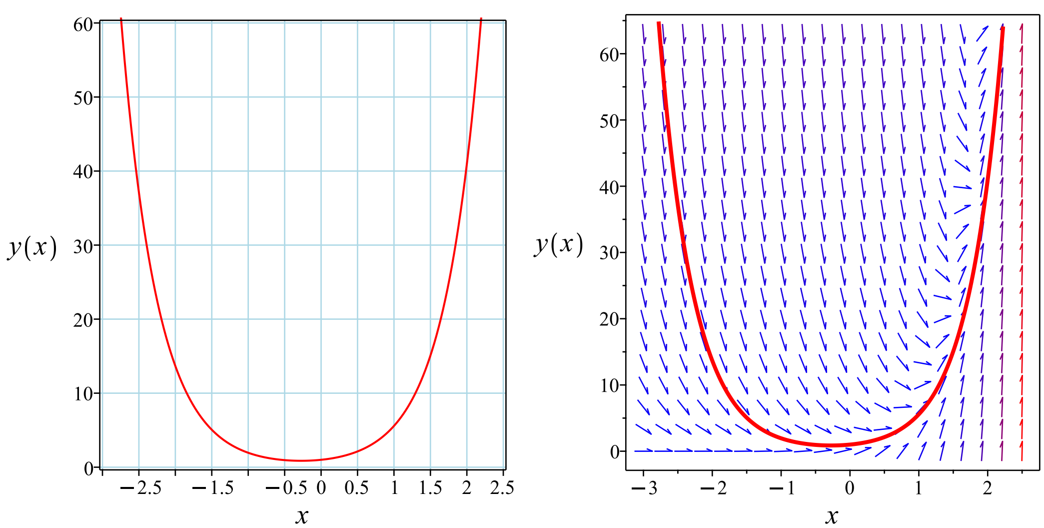
Substituting c_1 found above in the general solution gives

$$y = \frac{3\,\mathrm{e}^{2x}}{4} + \frac{\mathrm{e}^{-2x}}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{3\,\mathrm{e}^{2x}}{4} + \frac{\mathrm{e}^{-2x}}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{3\,\mathrm{e}^{2x}}{4} + \frac{\mathrm{e}^{-2x}}{4}$$

Verified OK.

8.48.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -2y + 3\,\mathrm{e}^{2x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 295: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \mathrm{e}^{-2x} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$S = \int \frac{1}{\eta} dy$$
$$= \int \frac{1}{\mathrm{e}^{-2x}} dy$$

Which results in

$$S = \mathrm{e}^{2x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2y + 3\,\mathrm{e}^{2x}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = 2\,\mathrm{e}^{2x} y$$
$$S_y = \mathrm{e}^{2x}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3\,\mathrm{e}^{4x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3\,\mathrm{e}^{4R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3\,\mathrm{e}^{4R}}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{2x} = \frac{3\,\mathrm{e}^{4x}}{4} + c_1$$

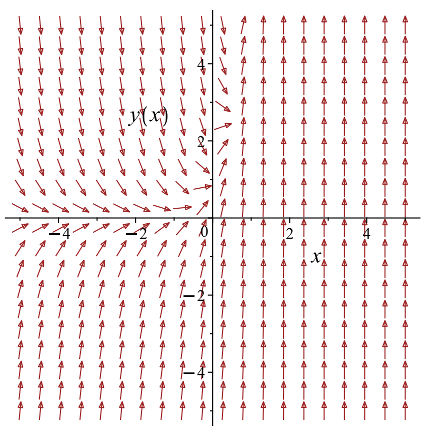
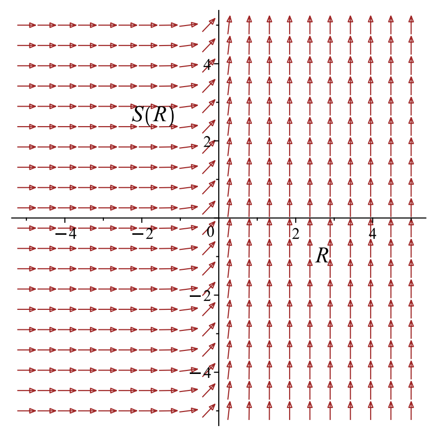
Which simplifies to

$$y\,\mathrm{e}^{2x} = \frac{3\,\mathrm{e}^{4x}}{4} + c_1$$

Which gives

$$y = \frac{(3\,\mathrm{e}^{4x} + 4c_1)\,\mathrm{e}^{-2x}}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2y + 3\,\mathrm{e}^{2x}$ 	$R = x$ $S = \mathrm{e}^{2x} y$	$\frac{dS}{dR} = 3\,\mathrm{e}^{4R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{3}{4} + c_1$$

$$c_1 = \frac{1}{4}$$

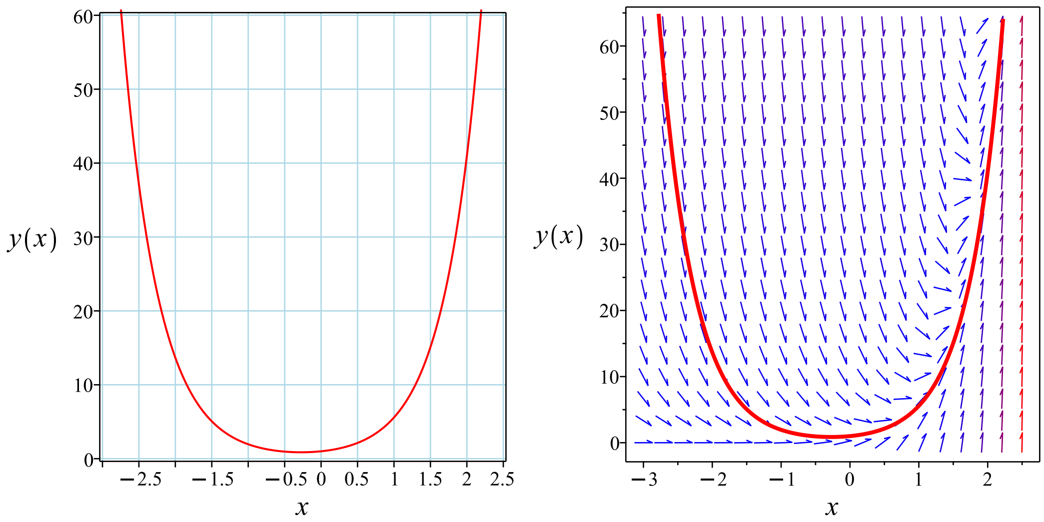
Substituting c_1 found above in the general solution gives

$$y = \frac{3\,\mathrm{e}^{2x}}{4} + \frac{\mathrm{e}^{-2x}}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{3\,\mathrm{e}^{2x}}{4} + \frac{\mathrm{e}^{-2x}}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{3\,e^{2x}}{4} + \frac{e^{-2x}}{4}$$

Verified OK.

8.48.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= \left(-2y + 3\,\mathrm{e}^{2x}\right)\mathrm{d}x \\ \left(2y - 3\,\mathrm{e}^{2x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 2y - 3\,\mathrm{e}^{2x} \\ N(x, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y - 3\,\mathrm{e}^{2x}\right) \\ &= 2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((2) - (0)) \\ &= 2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int 2 \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2x}\\ &= \mathrm{e}^{2x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{2x}\left(2y - 3\,\mathrm{e}^{2x}\right)\\ &= \left(2y - 3\,\mathrm{e}^{2x}\right)\mathrm{e}^{2x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{2x}(1)\\ &= \mathrm{e}^{2x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\left(2y - 3\,\mathrm{e}^{2x}\right)\mathrm{e}^{2x}\right) + \left(\mathrm{e}^{2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \left(2y - 3\,\mathrm{e}^{2x}\right)\mathrm{e}^{2x}\,\mathrm{d}x\end{aligned}$$

$$\phi = \mathrm{e}^{2x} y - \frac{3\,\mathrm{e}^{4x}}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x}$. Therefore equation (4) becomes

$$\mathrm{e}^{2x} = \mathrm{e}^{2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{2x} y - \frac{3\,\mathrm{e}^{4x}}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{2x} y - \frac{3\,\mathrm{e}^{4x}}{4}$$

The solution becomes

$$y = \frac{(3\,\mathrm{e}^{4x} + 4c_1)\,\mathrm{e}^{-2x}}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{3}{4} + c_1$$

$$c_1 = \frac{1}{4}$$

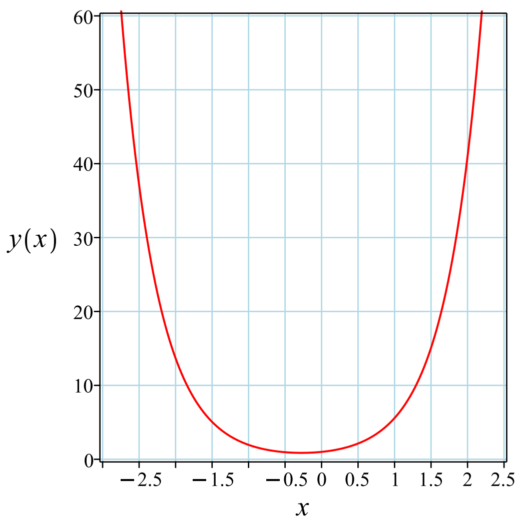
Substituting c_1 found above in the general solution gives

$$y = \frac{3\,e^{2x}}{4} + \frac{e^{-2x}}{4}$$

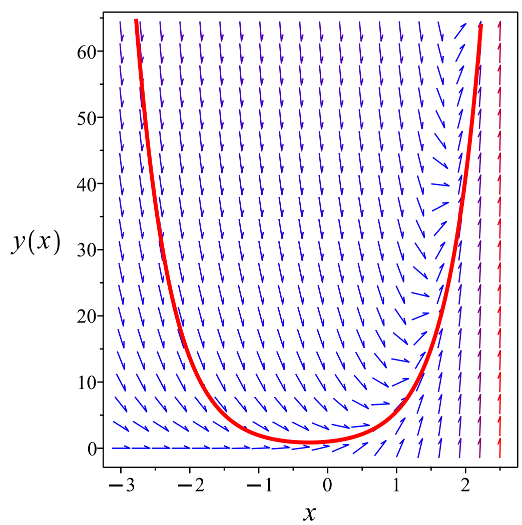
Summary

The solution(s) found are the following

$$y = \frac{3\,e^{2x}}{4} + \frac{e^{-2x}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3\,e^{2x}}{4} + \frac{e^{-2x}}{4}$$

Verified OK.

8.48.5 Maple step by step solution

Let's solve

$[2y + y' = 3\,e^{2x}, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -2y + 3\,\mathrm{e}^{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $2y + y' = 3\,\mathrm{e}^{2x}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(2y + y') = 3\mu(x)\,\mathrm{e}^{2x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(2y + y') = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{2x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 3\mu(x)\,\mathrm{e}^{2x} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 3\mu(x)\,\mathrm{e}^{2x} dx + c_1$
- Solve for y

  $y = \frac{\int 3\mu(x)\mathrm{e}^{2x} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{2x}$

  $y = \frac{\int 3(\mathrm{e}^{2x})^2 dx + c_1}{\mathrm{e}^{2x}}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{3(\mathrm{e}^{2x})^2}{4} + c_1}{\mathrm{e}^{2x}}$
- Simplify

  $y = \frac{3\,\mathrm{e}^{2x}}{4} + c_1 \mathrm{e}^{-2x}$
- Use initial condition $y(0) = 1$

  $1 = \frac{3}{4} + c_1$
- Solve for c_1

  $c_1 = \frac{1}{4}$
- Substitute $c_1 = \frac{1}{4}$ into general solution and simplify

$$y = \frac{3\,e^{2x}}{4} + \frac{e^{-2x}}{4}$$

- Solution to the IVP

$$y = \frac{3\,e^{2x}}{4} + \frac{e^{-2x}}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)+2*y(x)=3*exp(2*x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{3\,e^{2x}}{4} + \frac{e^{-2x}}{4}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 23

```
DSolve[{y'[x]+2*y[x]==3*Exp[2*x],{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-2x}\left(3e^{4x} + 1\right)$$

8.49 problem 52

Internal problem ID [2081]
Internal file name [`OUTPUT/2081_Sunday_February_25_2024_06_49_23_AM_85019945/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{4xy^2 + \left(x^2+1\right) y' = 0}$$

With initial conditions

$$[y(0) = 1]$$

8.49.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{4x\,y^2}{x^2+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{4x\,y^2}{x^2+1}\right) \\
&= -\frac{8xy}{x^2+1}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.49.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= -\frac{4x\,y^2}{x^2+1}
\end{aligned}$$

Where $f(x) = -\frac{4x}{x^2+1}$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y^2}\,dy &= -\frac{4x}{x^2+1}\,dx \\
\int \frac{1}{y^2}\,dy &= \int -\frac{4x}{x^2+1}\,dx \\
-\frac{1}{y} &= -2\ln\left(x^2+1\right) + c_1
\end{aligned}$$

Which results in

$$y = \frac{1}{2\ln\left(x^2+1\right) - c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_1}$$

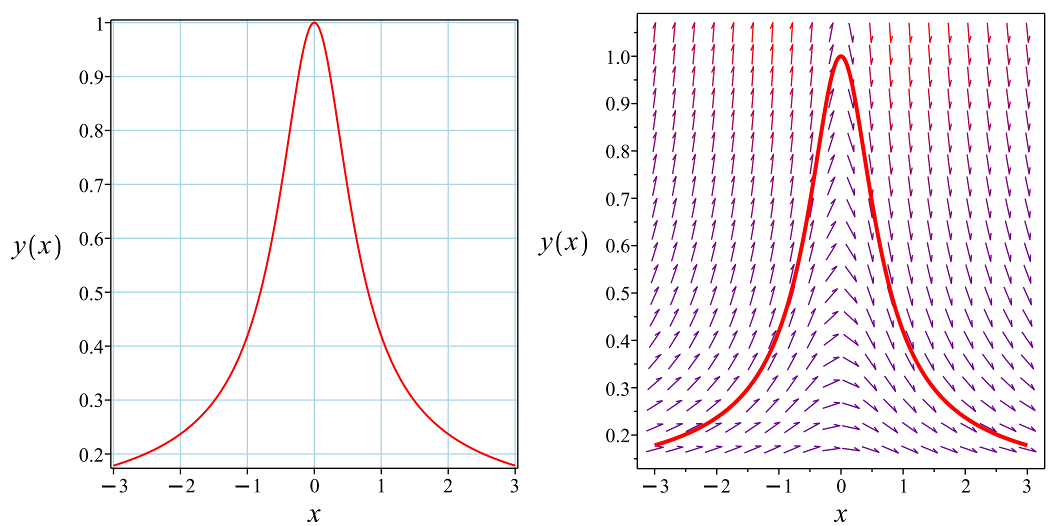
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2\ln(x^2+1)+1}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{2\ln(x^2+1)+1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{2\ln(x^2+1)+1}$$

Verified OK.

8.49.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{4x\,y^2}{x^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 298: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= -\frac{x^2+1}{4x}\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{-\frac{x^2+1}{4x}} dx\end{aligned}$$

Which results in

$$S = -2\ln\left(x^2+1\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{4x\,y^2}{x^2+1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{4x}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-2\ln\left(x^2+1\right) = -\frac{1}{y} + c_1
$$

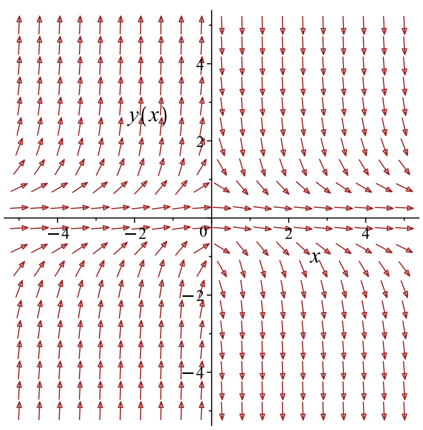
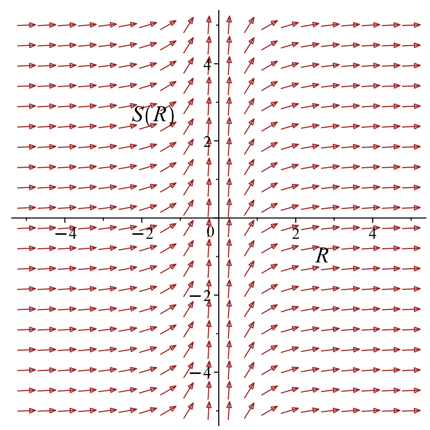
Which simplifies to

$$
-2\ln\left(x^2+1\right) = -\frac{1}{y} + c_1
$$

Which gives

$$
y = \frac{1}{2\ln\left(x^2+1\right) + c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4x\,y^2}{x^2+1}$ 	$R = y$ $S = -2\ln\left(x^2+1\right)$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{c_1}$$

$$c_1 = 1$$

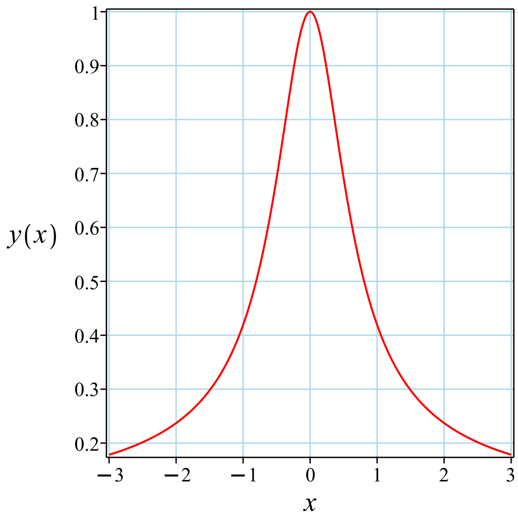
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2\ln\left(x^2+1\right)+1}$$

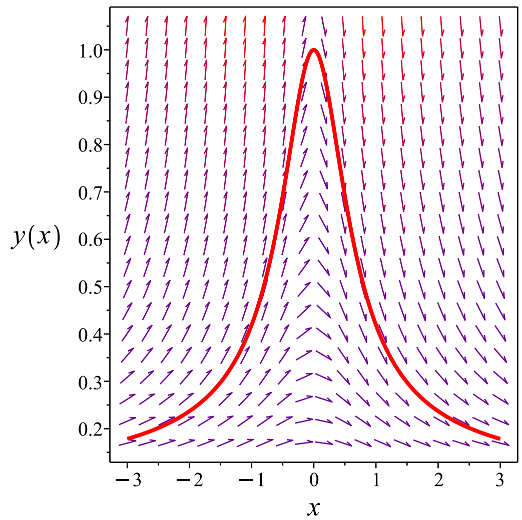
Summary

The solution(s) found are the following

$$y = \frac{1}{2\ln\left(x^2+1\right)+1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{2\ln(x^2+1)+1}$$

Verified OK.

8.49.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{4y^2}\right)\mathrm{d}y &= \left(\frac{x}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(-\frac{1}{4y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{x}{x^2+1}\\ N(x,y) &= -\frac{1}{4y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{4y^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -\frac{x}{x^2+1}\,\mathrm{d}x$$
$$\phi=-\frac{\ln(x^2+1)}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=-\frac{1}{4y^2}$. Therefore equation (4) becomes

$$-\frac{1}{4y^2}=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-\frac{1}{4y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int\left(-\frac{1}{4y^2}\right)\mathrm{d}y$$
$$f(y)=\frac{1}{4y}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} + \frac{1}{4y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} + \frac{1}{4y}$$

The solution becomes

$$y = \frac{1}{2\ln(x^2+1) + 4c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{4c_1}$$

$$c_1 = \frac{1}{4}$$

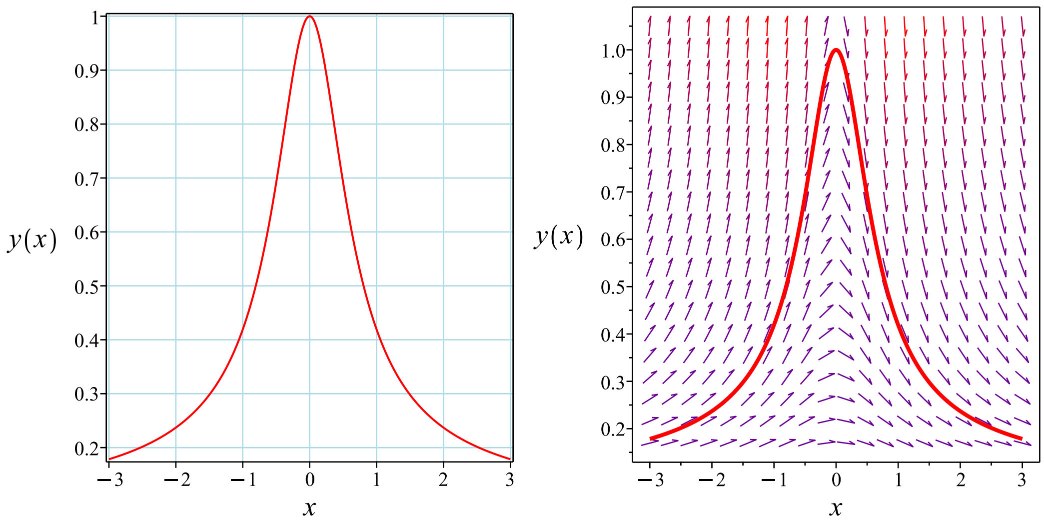
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2\ln(x^2+1) + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2\ln(x^2+1) + 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{2\ln\left(x^2+1\right)+1}$$

Verified OK.

8.49.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= -\frac{4x\,y^2}{x^2+1}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{4x\,y^2}{x^2+1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = -\frac{4x}{x^2+1}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{-\frac{4xu}{x^2+1}}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= -\frac{4}{x^2+1} + \frac{8x^2}{(x^2+1)^2} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= 0
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{4xu''(x)}{x^2+1} - \left(-\frac{4}{x^2+1} + \frac{8x^2}{(x^2+1)^2}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \ln\left(x^2+1\right) c_2$$

The above shows that

$$u'(x) = \frac{2xc_2}{x^2+1}$$

Using the above in (1) gives the solution

$$y = \frac{c_2}{2c_1 + 2\ln(x^2+1) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{2c_3 + 2\ln(x^2+1)}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2c_3}$$

$$c_3 = \frac{1}{2}$$

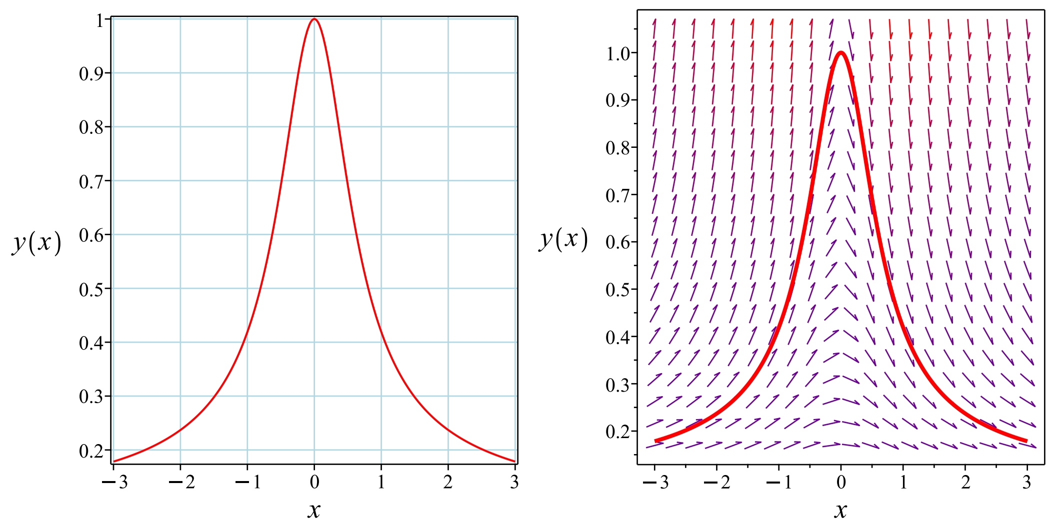
Substituting c_3 found above in the general solution gives

$$y = \frac{1}{2\ln(x^2+1)+1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2\ln(x^2+1)+1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{2\ln(x^2+1)+1}$$

Verified OK.

8.49.6 Maple step by step solution

Let's solve

$\left[4 x y^{2}+\left(x^{2}+1\right) y^{\prime}=0, y(0)=1\right]$

- Highest derivative means the order of the ODE is 1

  $y^{\prime}$

- Separate variables

  $\frac{y^{\prime}}{y^{2}}=-\frac{4 x}{x^{2}+1}$

- Integrate both sides with respect to x

  $\int \frac{y^{\prime}}{y^{2}} d x=\int-\frac{4 x}{x^{2}+1} d x+c_{1}$

- Evaluate integral

  $-\frac{1}{y}=-2 \ln \left(x^{2}+1\right)+c_{1}$

- Solve for y

  $y=\frac{1}{2 \ln \left(x^{2}+1\right)-c_{1}}$

- Use initial condition $y(0)=1$

  $1=-\frac{1}{c_{1}}$

- Solve for c_{1}

  $c_{1}=-1$

- Substitute $c_{1}=-1$ into general solution and simplify

  $y=\frac{1}{2 \ln \left(x^{2}+1\right)+1}$

- Solution to the IVP

  $y=\frac{1}{2 \ln \left(x^{2}+1\right)+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 16

```
dsolve([4*x*y(x)^2+(x^2+1)*diff(y(x),x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{1 + 2\ln(x^2 + 1)}$$

✓ Solution by Mathematica

Time used: 0.156 (sec). Leaf size: 17

```
DSolve[{4*x*y[x]^2+(x^2+1)*y'[x]==0,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2\log(x^2 + 1) + 1}$$

8.50 problem 53

Internal problem ID [2082]
Internal file name [`OUTPUT/2082_Sunday_February_25_2024_06_49_24_AM_91766949/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 53.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-2y-(x-2y+1)\,y'=-x-3}$$

With initial conditions

$$[y(0)=2]$$

8.50.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{-x+2y-3}{-x+2y-1} \end{aligned}$$

The x domain of $f(x,y)$ when $y=2$ is

$$\{x<3 \lor 3<x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\left\{y < \frac{1}{2} \vee \frac{1}{2} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-x + 2y - 3}{-x + 2y - 1}\right) \\ &= \frac{2}{-x + 2y - 1} - \frac{2(-x + 2y - 3)}{(-x + 2y - 1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{x < 3 \vee 3 < x\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\left\{y < \frac{1}{2} \vee \frac{1}{2} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

8.50.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{-x + 2y - 3}{-x + 2y - 1} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(-x+2y-3)(b_3-a_2)}{-x+2y-1} - \frac{(-x+2y-3)^2 a_3}{(-x+2y-1)^2} \\ &- \left(-\frac{1}{-x+2y-1} + \frac{-x+2y-3}{(-x+2y-1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(\frac{2}{-x+2y-1} - \frac{2(-x+2y-3)}{(-x+2y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 - x^2b_2 - x^2b_3 - 4xya_2 - 4xya_3 + 4xyb_2 + 4xyb_3 + 4y^2a_2 + 4y^2a_3 - 4y^2b_2 - 4y^2b_3 + 2xa_2 + \cdots}{(x-2y+1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^2a_2 - x^2a_3 + x^2b_2 + x^2b_3 + 4xya_2 + 4xya_3 - 4xyb_2 - 4xyb_3 - 4y^2a_2 \\ &- 4y^2a_3 + 4y^2b_2 + 4y^2b_3 - 2xa_2 - 6xa_3 - 2xb_2 + 4xb_3 + 8ya_2 \\ &+ 14ya_3 - 4yb_2 - 12yb_3 + 2a_1 - 3a_2 - 9a_3 - 4b_1 + b_2 + 3b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a_2v_1^2 + 4a_2v_1v_2 - 4a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 - 4a_3v_2^2 + b_2v_1^2 - 4b_2v_1v_2 \\ &+ 4b_2v_2^2 + b_3v_1^2 - 4b_3v_1v_2 + 4b_3v_2^2 - 2a_2v_1 + 8a_2v_2 - 6a_3v_1 + 14a_3v_2 \\ &- 2b_2v_1 - 4b_2v_2 + 4b_3v_1 - 12b_3v_2 + 2a_1 - 3a_2 - 9a_3 - 4b_1 + b_2 + 3b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_2 - a_3 + b_2 + b_3)\, v_1^2 + (4a_2 + 4a_3 - 4b_2 - 4b_3)\, v_1 v_2 \\ + (-2a_2 - 6a_3 - 2b_2 + 4b_3)\, v_1 + (-4a_2 - 4a_3 + 4b_2 + 4b_3)\, v_2^2 \\ + (8a_2 + 14a_3 - 4b_2 - 12b_3)\, v_2 + 2a_1 - 3a_2 - 9a_3 - 4b_1 + b_2 + 3b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-4a_2 - 4a_3 + 4b_2 + 4b_3 &= 0\\ -2a_2 - 6a_3 - 2b_2 + 4b_3 &= 0\\ -a_2 - a_3 + b_2 + b_3 &= 0\\ 4a_2 + 4a_3 - 4b_2 - 4b_3 &= 0\\ 8a_2 + 14a_3 - 4b_2 - 12b_3 &= 0\\ 2a_1 - 3a_2 - 9a_3 - 4b_1 + b_2 + 3b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -5b_2 + 2b_1\\ a_2 &= b_2\\ a_3 &= -2b_2\\ b_1 &= b_1\\ b_2 &= b_2\\ b_3 &= -2b_2\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 2\\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\, \xi\\ &= 1 - \left(\frac{-x + 2y - 3}{-x + 2y - 1}\right)(2)\\ &= \frac{-x + 2y - 5}{x - 2y + 1}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x+2y-5}{x-2y+1}} dy \end{aligned}$$

Which results in

$$S = -y - 2 \ln(-x + 2y - 5)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x + 2y - 3}{-x + 2y - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2}{x - 2y + 5} \\ S_y &= -1 + \frac{4}{x - 2y + 5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-y - 2\ln(-x + 2y - 5) = -x + c_1$$

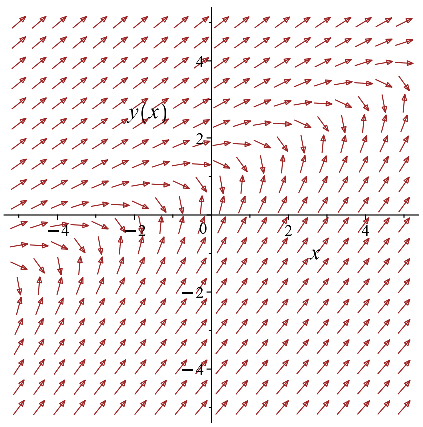
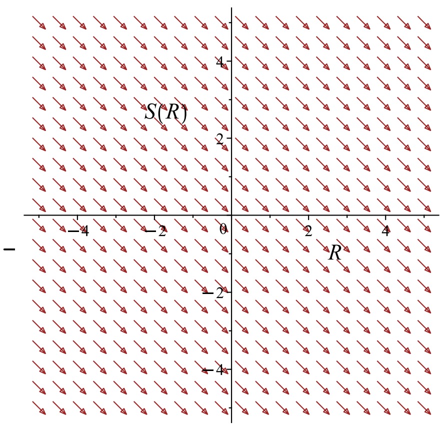
Which simplifies to

$$-y - 2\ln(-x + 2y - 5) = -x + c_1$$

Which gives

$$y = 2\,\text{LambertW}\left(\frac{e^{\frac{x}{4} - \frac{c_1}{2} - \frac{5}{4}}}{4}\right) + \frac{x}{2} + \frac{5}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+2y-3}{-x+2y-1}$ 	$R = x$ $S = -y - 2\ln\left(-x + 2y -\right.$	$\frac{dS}{dR} = -1$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 2\,\text{LambertW}\left(\frac{e^{-\frac{c_1}{2}-\frac{5}{4}}}{4}\right) + \frac{5}{2}$$

$$c_1 = -2i\pi - 2$$

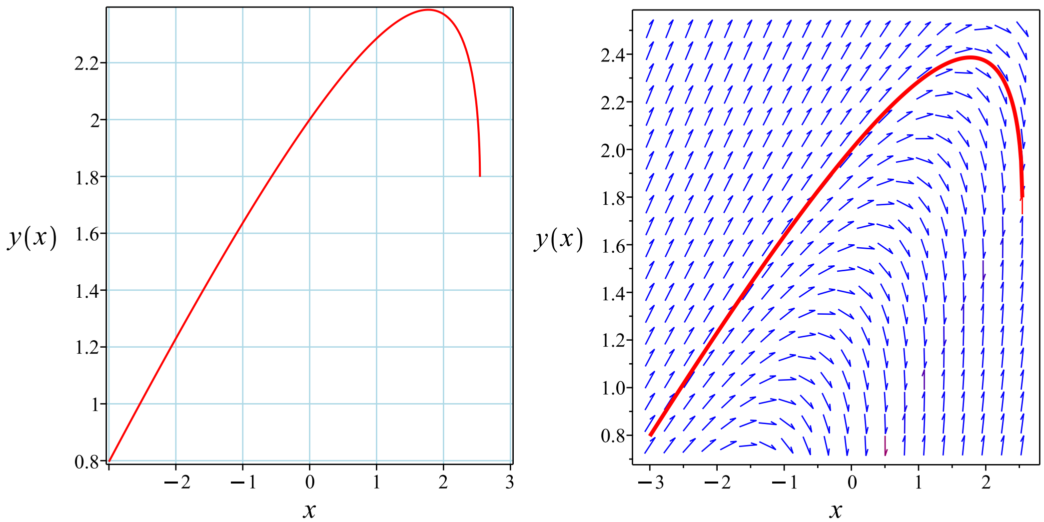
Substituting c_1 found above in the general solution gives

$$y = 2\,\text{LambertW}\left(-\frac{e^{\frac{x}{4}-\frac{1}{4}}}{4}\right) + \frac{x}{2} + \frac{5}{2}$$

Summary

The solution(s) found are the following

$$y = 2\,\text{LambertW}\left(-\frac{e^{\frac{x}{4}-\frac{1}{4}}}{4}\right) + \frac{x}{2} + \frac{5}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2\,\text{LambertW}\left(-\frac{e^{\frac{x}{4}-\frac{1}{4}}}{4}\right) + \frac{x}{2} + \frac{5}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 1/2, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.25 (sec). Leaf size: 20

```
dsolve([(x-2*y(x)+3)=(x-2*y(x)+1)*diff(y(x),x),y(0) = 2],y(x), singsol=all)
```

$$y(x) = \frac{5}{2} + \frac{x}{2} + 2\,\mathrm{LambertW}\left(-\frac{e^{\frac{x}{4}-\frac{1}{4}}}{4}\right)$$

✓ Solution by Mathematica

Time used: 4.657 (sec). Leaf size: 28

```
DSolve[{(x-2*y[x]+3)==(x-2*y[x]+1)*y'[x],{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(4W\left(-\frac{1}{4}e^{\frac{x-1}{4}}\right) + x + 5\right)$$

8.51 problem 54

Internal problem ID [2083]
Internal file name [`OUTPUT/2083_Sunday_February_25_2024_06_49_25_AM_72858164/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 54.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2 + (x^3 - 2yx)\, y' = 0}$$

With initial conditions

$$[y(2) = 1]$$

8.51.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y^2}{x\,(-x^2 + 2y)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2+\frac{y^2(b_3-a_2)}{x\left(-x^2+2y\right)}-\frac{y^4 a_3}{x^2\left(-x^2+2y\right)^2} \\ & -\left(-\frac{y^2}{x^2\left(-x^2+2y\right)}+\frac{2y^2}{\left(-x^2+2y\right)^2}\right)\left(x a_2+y a_3+a_1\right) \\ & -\left(\frac{2y}{x\left(-x^2+2y\right)}-\frac{2y^2}{x\left(-x^2+2y\right)^2}\right)\left(x b_2+y b_3+b_1\right)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^6 b_2-2x^4 y b_2-2x^3 y^2 a_2+x^3 y^2 b_3-3x^2 y^3 a_3+2x^3 y b_1-3x^2 y^2 a_1+2x^2 y^2 b_2+y^4 a_3-2x\,y^2 b_1+2y^3 a_1}{x^2\left(x^2-2y\right)^2}=0$$

Setting the numerator to zero gives

$$\begin{aligned} & x^6 b_2-2x^4 y b_2-2x^3 y^2 a_2+x^3 y^2 b_3-3x^2 y^3 a_3+2x^3 y b_1 \\ & -3x^2 y^2 a_1+2x^2 y^2 b_2+y^4 a_3-2x\,y^2 b_1+2y^3 a_1=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & b_2 v_1^6-2a_2 v_1^3 v_2^2-3a_3 v_1^2 v_2^3-2b_2 v_1^4 v_2+b_3 v_1^3 v_2^2-3a_1 v_1^2 v_2^2 \\ & +a_3 v_2^4+2b_1 v_1^3 v_2+2b_2 v_1^2 v_2^2+2a_1 v_2^3-2b_1 v_1 v_2^2=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &b_2 v_1^6 - 2b_2 v_1^4 v_2 + (-2a_2 + b_3) v_1^3 v_2^2 + 2b_1 v_1^3 v_2 - 3a_3 v_1^2 v_2^3 \\ &+ (-3a_1 + 2b_2) v_1^2 v_2^2 - 2b_1 v_1 v_2^2 + a_3 v_2^4 + 2a_1 v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_3 &= 0 \\ b_2 &= 0 \\ 2a_1 &= 0 \\ -3a_3 &= 0 \\ -2b_1 &= 0 \\ 2b_1 &= 0 \\ -2b_2 &= 0 \\ -3a_1 + 2b_2 &= 0 \\ -2a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 2y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 2y - \left(\frac{y^2}{x\,(-x^2+2y)}\right)(x) \\
&= \frac{2y\,x^2 - 3y^2}{x^2 - 2y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2y\,x^2-3y^2}{x^2-2y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{2} + \frac{\ln(-2x^2+3y)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2}{x\,(-x^2+2y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x}{6x^2 - 9y} \\
S_y &= \frac{x^2 - 2y}{2y\,x^2 - 3y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R)}{3} + c_1 \tag{4}$$

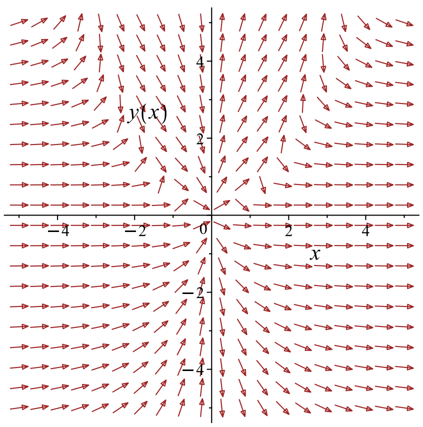
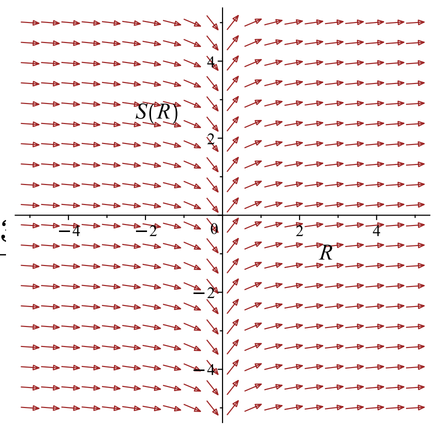
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln (y)}{2} + \frac{\ln (-2x^2 + 3y)}{6} = \frac{\ln (x)}{3} + c_1$$

Which simplifies to

$$\frac{\ln (y)}{2} + \frac{\ln (-2x^2 + 3y)}{6} = \frac{\ln (x)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x(-x^2+2y)}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(-2x^2 +}{6}$	$\frac{dS}{dR} = \frac{1}{3R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln(5)}{6} + \frac{i\pi}{6} = \frac{\ln(2)}{3} + c_1$$

$$c_1 = -\frac{\ln(2)}{3} + \frac{\ln(5)}{6} + \frac{i\pi}{6}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(y)}{2} + \frac{\ln(-2x^2 + 3y)}{6} = \frac{\ln(x)}{3} - \frac{\ln(2)}{3} + \frac{\ln(5)}{6} + \frac{i\pi}{6}$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{2} + \frac{\ln(-2x^2 + 3y)}{6} = \frac{\ln(x)}{3} - \frac{\ln(2)}{3} + \frac{\ln(5)}{6} + \frac{i\pi}{6} \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{2} + \frac{\ln(-2x^2 + 3y)}{6} = \frac{\ln(x)}{3} - \frac{\ln(2)}{3} + \frac{\ln(5)}{6} + \frac{i\pi}{6}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 4.031 (sec). Leaf size: 255

```
dsolve([y(x)^2+(x^3-2*x*y(x))*diff(y(x),x)=0,y(2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{x^2 \left(-\frac{\sqrt{10}\left(\sqrt{\frac{\left(20x^3+20\sqrt{x^6-20}\right)^{\frac{2}{3}}+20}{x\left(20x^3+20\sqrt{x^6-20}\right)^{\frac{1}{3}}}} + \sqrt{\frac{4\sqrt{10}\,x\left(20x^3+20\sqrt{x^6-20}\right)^{\frac{1}{3}} - \sqrt{\frac{\left(20x^3+20\sqrt{x^6-20}\right)^{\frac{2}{3}}+20}{x\left(20x^3+20\sqrt{x^6-20}\right)^{\frac{1}{3}}}}\left(20x^3+20\sqrt{x^6-20}\right)^{\frac{2}{3}} - 20\sqrt{\frac{\left(20x^3+20\sqrt{x^6-2\cdots}\right.}{x\left(20x^3+20\sqrt{x^6\cdots}\right.}}}{x\left(20x^3+20\sqrt{x^6-20}\right)^{\frac{1}{3}}\sqrt{\frac{\left(20x^3+20\sqrt{x^6-20}\right)^{\frac{2}{3}}+20}{x\left(20x^3+20\sqrt{x^6-20}\right)^{\frac{1}{3}}}}}}\right.\cdots}{20} \cdots\right.}{12}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y[x]^2+(x^3-2*x*y[x])*y'[x]==0,{y[2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

Timed out

8.52 problem 55

Internal problem ID [2084]
Internal file name [`OUTPUT/2084_Sunday_February_25_2024_06_49_30_AM_29081219/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 55.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2yx - 2y + x(x-1)\, y' = -1}$$

With initial conditions

$$[y(2) = 2]$$

8.52.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = -\frac{1}{x\,(x-1)}$$

Hence the ode is

$$y' + \frac{2y}{x} = -\frac{1}{x(x-1)}$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = -\frac{1}{x(x-1)}$ is

$$\{-\infty \le x < 0, 0 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

8.52.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(-\frac{1}{x(x-1)}\right) \\ \frac{d}{dx}(y\,x^2) &= (x^2)\left(-\frac{1}{x(x-1)}\right) \\ d(y\,x^2) &= \left(-\frac{x}{x-1}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^2 &= \int -\frac{x}{x-1}\, dx \\ y\,x^2 &= -x - \ln(x-1) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{-x - \ln(x-1)}{x^2} + \frac{c_1}{x^2}$$

which simplifies to

$$y = \frac{-x - \ln(x - 1) + c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{c_1}{4} - \frac{1}{2}$$

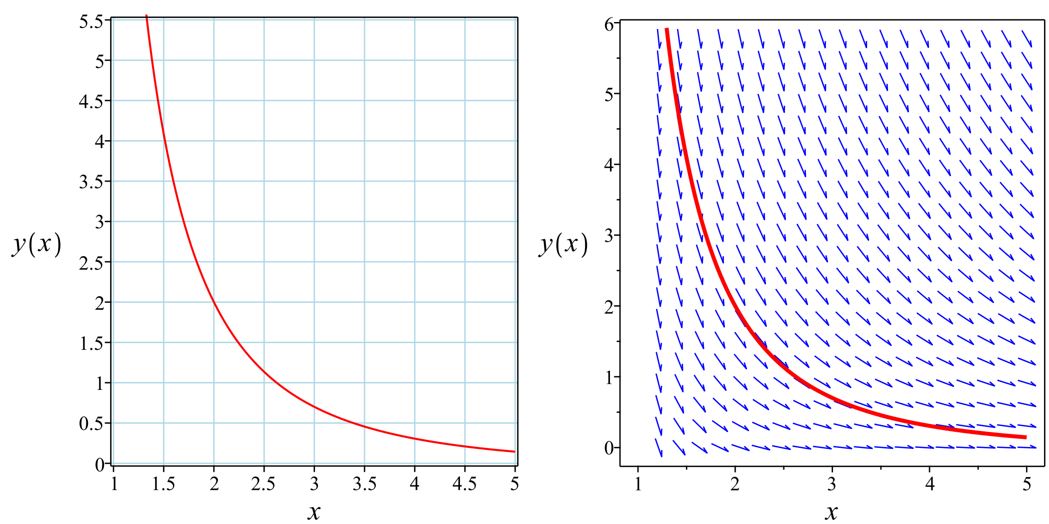
$$c_1 = 10$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-x - \ln(x - 1) + 10}{x^2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{-x - \ln(x - 1) + 10}{x^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-x - \ln(x - 1) + 10}{x^2}$$

Verified OK.

8.52.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2yx - 2y + 1}{x(x-1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 301: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x^2}} dy
\end{aligned}
$$

Which results in

$$
S = y\,x^2
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2yx - 2y + 1}{x\,(x-1)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2yx \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x}{x-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R}{R-1}$$

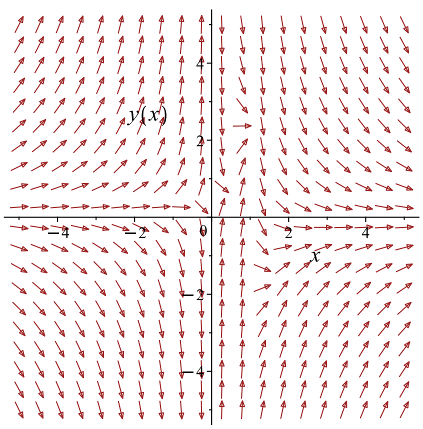
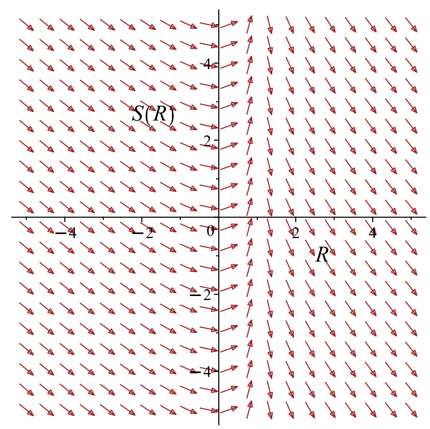
The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R - \ln(R-1) + c_1 \tag{4}$$

Which gives

$$y = -\frac{x + \ln(x-1) - c_1}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2yx-2y+1}{x(x-1)}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = -\frac{R}{R-1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{c_1}{4} - \frac{1}{2}$$

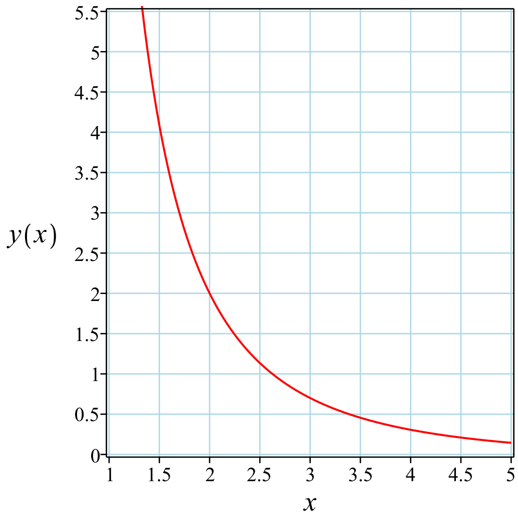
$$c_1 = 10$$

Substituting c_1 found above in the general solution gives

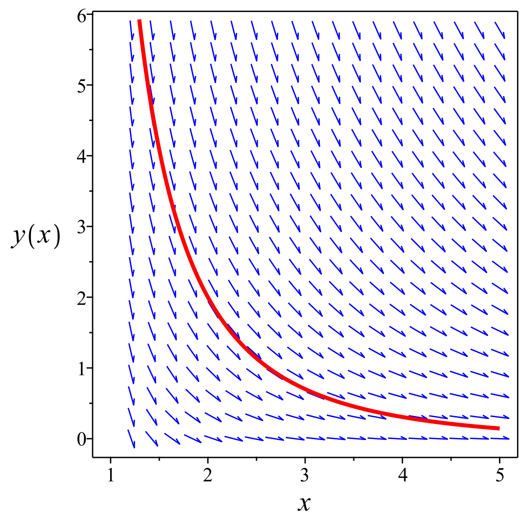
$$y = \frac{-x - \ln(x-1) + 10}{x^2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{-x - \ln(x-1) + 10}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-x - \ln(x-1) + 10}{x^2}$$

Verified OK.

8.52.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x(x-1))\,\mathrm{d}y &= (-2yx + 2y - 1)\,\mathrm{d}x \\
(2yx - 2y + 1)\,\mathrm{d}x + (x(x-1))\,\mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= 2yx - 2y + 1 \\
N(x,y) &= x(x-1)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2yx - 2y + 1) \\
&= 2x - 2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(x-1)) \\
&= 2x - 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\,(x-1)}((2x-2) - (2x-1)) \\
&= -\frac{1}{x\,(x-1)}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{1}{x(x-1)} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)-\ln(x-1)}\\ &= \frac{x}{x-1}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{x}{x-1}(2yx-2y+1)\\ &= \frac{x(2yx-2y+1)}{x-1}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{x}{x-1}(x(x-1))\\ &= x^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{x(2yx-2y+1)}{x-1}\right) + \left(x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{x(2yx - 2y + 1)}{x - 1} \mathrm{d}x$$
$$\phi = y x^2 + x + \ln(x - 1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y x^2 + x + \ln(x - 1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y x^2 + x + \ln(x - 1)$$

The solution becomes

$$y = -\frac{x + \ln(x - 1) - c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{c_1}{4} - \frac{1}{2}$$

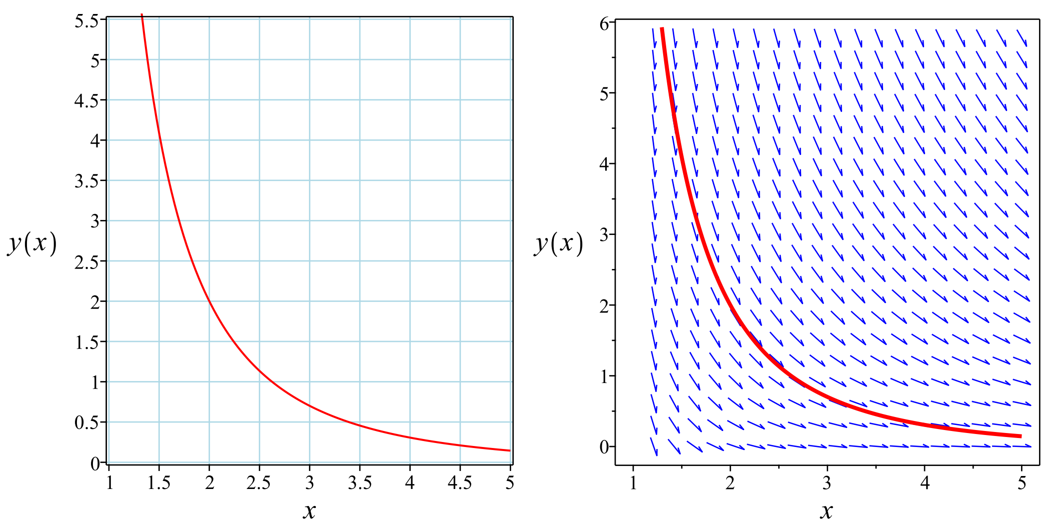
$$c_1 = 10$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-x - \ln (x - 1) + 10}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{-x - \ln (x - 1) + 10}{x^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{-x - \ln (x - 1) + 10}{x^2}$$

Verified OK.

8.52.5 Maple step by step solution

Let's solve

$[2yx - 2y + x(x-1)\,y' = -1, y(2) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} - \frac{1}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = -\frac{1}{x(x-1)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = -\frac{\mu(x)}{x(x-1)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\frac{\mu(x)}{x(x-1)}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\frac{\mu(x)}{x(x-1)}dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{\mu(x)}{x(x-1)}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int -\frac{x}{x-1}dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{-x - \ln(x-1) + c_1}{x^2}$

- Use initial condition $y(2) = 2$

$$2 = \frac{c_1}{4} - \frac{1}{2}$$

- Solve for c_1

$$c_1 = 10$$

- Substitute $c_1 = 10$ into general solution and simplify

$$y = \frac{-x - \ln(x-1) + 10}{x^2}$$

- Solution to the IVP

$$y = \frac{-x - \ln(x-1) + 10}{x^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve([(2*x*y(x)-2*y(x)+1)+x*(x-1)*diff(y(x),x)=0,y(2) = 2],y(x), singsol=all)
```

$$y(x) = \frac{-x - \ln(x-1) + 10}{x^2}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 17

```
DSolve[{(2*x*y[x]-2*y[x]+1)+x*(x-1)*y'[x]==0,{y[2]==2}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{x + \log(x-1) - 10}{x^2}$$

8.53 problem 56

Internal problem ID [2085]
Internal file name [`OUTPUT/2085_Sunday_February_25_2024_06_49_31_AM_58712087/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 56.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y^3 + 2x^2y + (-3x^3 - 2xy^2)\, y' = 0}$$

With initial conditions

$$[y(1) = 1]$$

8.53.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y(2x^2+y^2)}{x\,(3x^2+2y^2)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(2x^2 + y^2)}{x\,(3x^2 + 2y^2)}\right) \\
&= \frac{2x^2 + y^2}{x\,(3x^2 + 2y^2)} + \frac{2y^2}{x\,(3x^2 + 2y^2)} - \frac{4y^2(2x^2 + y^2)}{x\,(3x^2 + 2y^2)^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.53.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^3\,x^3 + 2x^3u(x) + \left(-3x^3 - 2x^3u(x)^2\right)\left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u(u^2 + 1)}{(2u^2 + 3)\,x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2+1)}{2u^2+3}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+1)}{2u^2+3}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u(u^2+1)}{2u^2+3}}\,du = \int -\frac{1}{x}\,dx$$

$$3\ln(u) - \frac{\ln(u^2 + 1)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{3\ln(u)-\frac{\ln(u^2+1)}{2}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u^3}{\sqrt{u^2+1}} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)^3}{\sqrt{u(x)^2+1}} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^3}{x^3\sqrt{\frac{y^2}{x^2}+1}} = \frac{c_3}{x}$$
$$\frac{y^3}{x^3\sqrt{\frac{x^2+y^2}{x^2}}} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{y^3}{x^2\sqrt{\frac{x^2+y^2}{x^2}}} = c_3$$

Substituting initial conditions and solving for c_3 gives $c_3 = \frac{\sqrt{2}}{2}$. Hence the solution be-

Summary

The solution(s) found are the following

comes

$$\frac{y^3}{x^2\sqrt{\frac{x^2+y^2}{x^2}}} = \frac{\sqrt{2}}{2} \tag{1}$$

Verification of solutions

$$\frac{y^3}{x^2\sqrt{\frac{x^2+y^2}{x^2}}} = \frac{\sqrt{2}}{2}$$

Verified OK.

8.53.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y(2x^2+y^2)}{x\left(3x^2+2y^2\right)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{y(2x^2+y^2)(b_3-a_2)}{x\left(3x^2+2y^2\right)} - \frac{y^2\left(2x^2+y^2\right)^2 a_3}{x^2\left(3x^2+2y^2\right)^2} \\
&- \left(\frac{4y}{3x^2+2y^2} - \frac{y(2x^2+y^2)}{x^2\left(3x^2+2y^2\right)} - \frac{6y(2x^2+y^2)}{\left(3x^2+2y^2\right)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(\frac{2x^2+y^2}{x\left(3x^2+2y^2\right)} + \frac{2y^2}{x\left(3x^2+2y^2\right)} - \frac{4y^2(2x^2+y^2)}{x\left(3x^2+2y^2\right)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{3x^6 b_2 + 2x^4 y^2 a_3 + 7x^4 y^2 b_2 - 2x^3 y^3 a_2 + 2x^3 y^3 b_3 + x^2 y^4 a_3 + 2x^2 y^4 b_2 + y^6 a_3 - 6x^5 b_1 + 6x^4 y a_1 - 5x^3 y^2 b_1 +}{\left(3x^2+2y^2\right)^2 x^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&3x^6 b_2 + 2x^4 y^2 a_3 + 7x^4 y^2 b_2 - 2x^3 y^3 a_2 + 2x^3 y^3 b_3 + x^2 y^4 a_3 + 2x^2 y^4 b_2 \\
&+ y^6 a_3 - 6x^5 b_1 + 6x^4 y a_1 - 5x^3 y^2 b_1 + 5x^2 y^3 a_1 - 2x\, y^4 b_1 + 2y^5 a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2v_1^3v_2^3 + 2a_3v_1^4v_2^2 + a_3v_1^2v_2^4 + a_3v_2^6 + 3b_2v_1^6 + 7b_2v_1^4v_2^2 + 2b_2v_1^2v_2^4 + 2b_3v_1^3v_2^3 + 6a_1v_1^4v_2 + 5a_1v_1^2v_2^3 + 2a_1v_2^5 - 6b_1v_1^5 - 5b_1v_1^3v_2^2 - 2b_1v_1v_2^4 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$3b_2v_1^6 - 6b_1v_1^5 + (2a_3 + 7b_2)\,v_1^4v_2^2 + 6a_1v_1^4v_2 + (-2a_2 + 2b_3)\,v_1^3v_2^3 - 5b_1v_1^3v_2^2 + (a_3 + 2b_2)\,v_1^2v_2^4 + 5a_1v_1^2v_2^3 - 2b_1v_1v_2^4 + a_3v_2^6 + 2a_1v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_3 &= 0\\
2a_1 &= 0\\
5a_1 &= 0\\
6a_1 &= 0\\
-6b_1 &= 0\\
-5b_1 &= 0\\
-2b_1 &= 0\\
3b_2 &= 0\\
-2a_2 + 2b_3 &= 0\\
a_3 + 2b_2 &= 0\\
2a_3 + 7b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{y(2x^2 + y^2)}{x\,(3x^2 + 2y^2)}\right)(x) \\
&= \frac{y\,x^2 + y^3}{3x^2 + 2y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y\,x^2 + y^3}{3x^2 + 2y^2}} dy
\end{aligned}
$$

Which results in

$$S = 3\ln(y) - \frac{\ln(x^2+y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2x^2+y^2)}{x(3x^2+2y^2)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x}{x^2+y^2} \\
S_y &= \frac{3}{y} - \frac{y}{x^2+y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

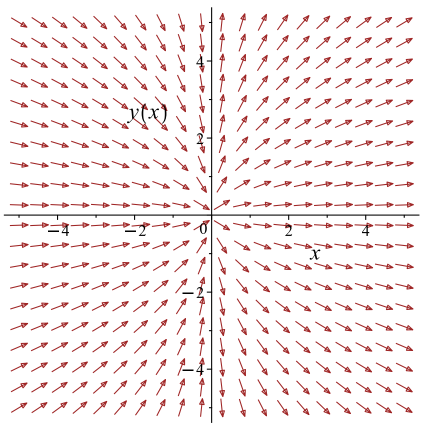
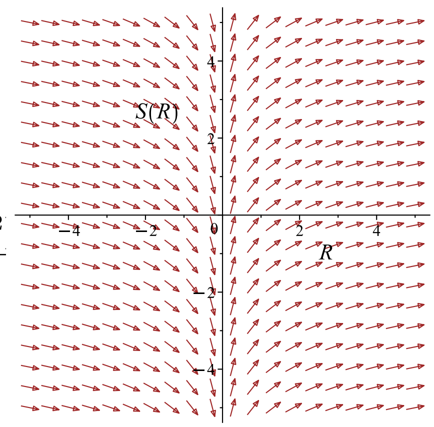
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$3\ln(y) - \frac{\ln(x^2+y^2)}{2} = \ln(x) + c_1$$

Which simplifies to

$$3\ln(y) - \frac{\ln(x^2+y^2)}{2} = \ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2x^2+y^2)}{x(3x^2+2y^2)}$ 	$R = x$ $S = 3\ln(y) - \frac{\ln(x^2+y^2)}{2}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\ln(2)}{2} = c_1$$

$$c_1 = -\frac{\ln(2)}{2}$$

Substituting c_1 found above in the general solution gives

$$3\ln(y) - \frac{\ln(x^2+y^2)}{2} = \ln(x) - \frac{\ln(2)}{2}$$

<u>Summary</u>

The solution(s) found are the following

$$3\ln(y) - \frac{\ln(x^2+y^2)}{2} = \ln(x) - \frac{\ln(2)}{2} \tag{1}$$

Verification of solutions

$$3\ln(y) - \frac{\ln(x^2+y^2)}{2} = \ln(x) - \frac{\ln(2)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 2.406 (sec). Leaf size: 62

```
dsolve([(y(x)^3+2*x^2*y(x))+(-3*x^3-2*x*y(x)^2)*diff(y(x),x)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{\frac{\left(54x^4+6\sqrt{3}\,\sqrt{27x^8-2x^6}\right)^{\frac{2}{3}}+6x^2}{\left(54x^4+6\sqrt{3}\,\sqrt{27x^8-2x^6}\right)^{\frac{1}{3}}}}}{6}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(y[x]^3+2*x^2*y[x])+(-3*x^3-2*x*y[x]^2)*y'[x]==0,{y[1]==1}},y[x],x,IncludeSingularSo
```

Timed out

8.54 problem 57

Internal problem ID [2086]
Internal file name [`OUTPUT/2086_Sunday_February_25_2024_06_49_44_AM_72532284/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 12, page 46
Problem number: 57.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2(x^2+1)\,y'-(2y^2-1)\,xy=0}$$

With initial conditions

$$[y(0)=1]$$

8.54.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{(2y^2-1)\,xy}{2x^2+2} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{(2y^2-1)\,xy}{2x^2+2}\right)\\ &= \frac{2x\,y^2}{x^2+1} + \frac{(2y^2-1)\,x}{2x^2+2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.54.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{x(2y^3-y)}{2x^2+2}\end{aligned}$$

Where $f(x) = \frac{x}{2x^2+2}$ and $g(y) = 2y^3 - y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{2y^3-y}\,dy &= \frac{x}{2x^2+2}\,dx\\ \int \frac{1}{2y^3-y}\,dy &= \int \frac{x}{2x^2+2}\,dx\\ -\ln(y) + \frac{\ln(2y^2-1)}{2} &= \frac{\ln(x^2+1)}{4} + c_1\end{aligned}$$

Raising both side to exponential gives

$$e^{-\ln(y)+\frac{\ln(2y^2-1)}{2}} = e^{\frac{\ln(x^2+1)}{4}+c_1}$$

Which simplifies to

$$\frac{\sqrt{2y^2-1}}{y} = c_2 (x^2+1)^{\frac{1}{4}}$$

The solution is

$$\frac{\sqrt{2y^2-1}}{y} = c_2 (x^2+1)^{\frac{1}{4}}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2$$

$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$\frac{\sqrt{2y^2-1}}{y} = (x^2+1)^{\frac{1}{4}}$$

The above simplifies to

$$-(x^2+1)^{\frac{1}{4}} y + \sqrt{2y^2-1} = 0$$

Summary

The solution(s) found are the following

$$-(x^2+1)^{\frac{1}{4}} y + \sqrt{2y^2-1} = 0 \tag{1}$$

Verification of solutions

$$-(x^2+1)^{\frac{1}{4}} y + \sqrt{2y^2-1} = 0$$

Verified OK.

8.54.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{(2y^2-1)\,xy}{2x^2+2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 304: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{2x^2+2}{x} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{2x^2+2}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(x^2+1)}{4}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{(2y^2-1)\,xy}{2x^2+2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x}{2x^2 + 2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2y^3 - y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R^3 - R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + \frac{\ln(2R^2 - 1)}{2} + c_1 \tag{4}$$

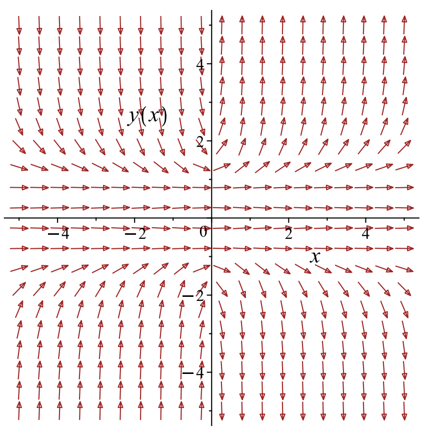
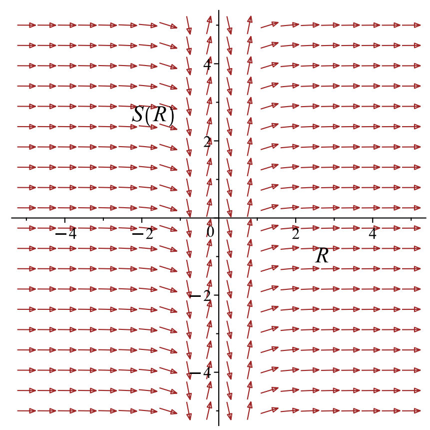
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2 + 1)}{4} = -\ln(y) + \frac{\ln(2y^2 - 1)}{2} + c_1$$

Which simplifies to

$$\frac{\ln(x^2 + 1)}{4} = -\ln(y) + \frac{\ln(2y^2 - 1)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(2y^2-1)xy}{2x^2+2}$ 	$R = y$ $S = \frac{\ln(x^2+1)}{4}$	$\frac{dS}{dR} = \frac{1}{2R^3-R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(x^2+1)}{4} = -\ln(y) + \frac{\ln(2y^2-1)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+1)}{4} = -\ln(y) + \frac{\ln(2y^2-1)}{2} \tag{1}$$

Verification of solutions

$$\frac{\ln(x^2+1)}{4} = -\ln(y) + \frac{\ln(2y^2-1)}{2}$$

Verified OK.

8.54.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{(2y^2-1)\,xy}{2x^2+2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{x}{2\,(x^2+1)}y + \frac{x}{x^2+1}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{x}{2\,(x^2+1)} \\ f_1(x) &= \frac{x}{x^2+1} \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y'\frac{1}{y^3} = -\frac{x}{2\,(x^2+1)\,y^2} + \frac{x}{x^2+1} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
-\frac{w'(x)}{2} &= -\frac{xw(x)}{2\left(x^2+1\right)} + \frac{x}{x^2+1} \\
w' &= \frac{xw}{x^2+1} - \frac{2x}{x^2+1} \qquad (7)
\end{aligned}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{x}{x^2+1} \\
q(x) &= -\frac{2x}{x^2+1}
\end{aligned}$$

Hence the ode is

$$w'(x) - \frac{xw(x)}{x^2+1} = -\frac{2x}{x^2+1}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{x}{x^2+1}dx} \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{2x}{x^2+1}\right) \\
\frac{d}{dx}\left(\frac{w}{\sqrt{x^2+1}}\right) &= \left(\frac{1}{\sqrt{x^2+1}}\right)\left(-\frac{2x}{x^2+1}\right) \\
d\left(\frac{w}{\sqrt{x^2+1}}\right) &= \left(-\frac{2x}{\left(x^2+1\right)^{\frac{3}{2}}}\right) dx
\end{aligned}$$

Integrating gives

$$\frac{w}{\sqrt{x^2+1}} = \int -\frac{2x}{(x^2+1)^{\frac{3}{2}}}\,\mathrm{d}x$$
$$\frac{w}{\sqrt{x^2+1}} = \frac{2}{\sqrt{x^2+1}} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x^2+1}}$ results in

$$w(x) = 2 + \sqrt{x^2+1}\,c_1$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = 2 + \sqrt{x^2+1}\,c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + 2$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{y^2} = -\sqrt{x^2+1} + 2$$

The above simplifies to

$$\sqrt{x^2+1}\,y^2 - 2y^2 + 1 = 0$$

Summary

The solution(s) found are the following

$$\sqrt{x^2+1}\,y^2 - 2y^2 + 1 = 0 \tag{1}$$

Verification of solutions

$$\sqrt{x^2+1}\,y^2 - 2y^2 + 1 = 0$$

Verified OK.

8.54.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{2}{(2y^2-1)\,y}\right)\mathrm{d}y&=\left(\frac{x}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{x}{x^2+1}\right)\mathrm{d}x+\left(\frac{2}{(2y^2-1)\,y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x}{x^2+1} \\
N(x,y) &= \frac{2}{(2y^2-1)\,y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2}{(2y^2-1)\,y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{x}{x^2+1} \,\mathrm{d}x \\
\phi &= -\frac{\ln(x^2+1)}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2}{(2y^2-1)y}$. Therefore equation (4) becomes

$$\frac{2}{(2y^2-1)\,y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= \frac{2}{(2y^2-1)\,y} \\ &= \frac{2}{2y^3-y} \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y)\,\mathrm{d}y &= \int \left(\frac{2}{2y^3-y}\right) \mathrm{d}y \\ f(y) &= -2\ln(y) + \ln\left(2y^2-1\right) + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} - 2\ln(y) + \ln\left(2y^2-1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} - 2\ln(y) + \ln\left(2y^2-1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-\frac{\ln(x^2+1)}{2} - 2\ln(y) + \ln\left(2y^2-1\right) = 0$$

Summary
The solution(s) found are the following

$$-\frac{\ln (x^2+1)}{2} - 2\ln (y) + \ln (2y^2-1) = 0 \tag{1}$$

Verification of solutions

$$-\frac{\ln (x^2+1)}{2} - 2\ln (y) + \ln (2y^2-1) = 0$$

Verified OK.

8.54.6 Maple step by step solution

Let's solve

$[2(x^2+1)\,y' - (2y^2-1)\,xy = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(2y^2-1)y} = \frac{x}{2(x^2+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{(2y^2-1)y}dx = \int \frac{x}{2(x^2+1)}dx + c_1$

- Evaluate integral

  $-\ln (y) + \frac{\ln(2y^2-1)}{2} = \frac{\ln(x^2+1)}{4} + c_1$

- Use initial condition $y(0) = 1$

  $0 = c_1$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $-\ln (y) + \frac{\ln(2y^2-1)}{2} = \frac{\ln(x^2+1)}{4}$

- Solution to the IVP

  $-\ln (y) + \frac{\ln(2y^2-1)}{2} = \frac{\ln(x^2+1)}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 17

```
dsolve([2*(1+x^2)*diff(y(x),x)=(2*y(x)^2-1)*x*y(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{2-\sqrt{x^2+1}}}$$

✓ Solution by Mathematica

Time used: 4.797 (sec). Leaf size: 32

```
DSolve[{2*(1+x^2)*y'[x]==(2*y[x]^2-1)*x*y[x],{y[0]==1}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{\sqrt{-\sqrt{x^2+1}-2}}{\sqrt{x^2-3}}$$

9 Exercise 17, page 78

9.1 problem 15

Internal problem ID [2087]
Internal file name [`OUTPUT/2087_Sunday_February_25_2024_06_49_46_AM_77842853/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y = 0}$$

9.1.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}
\int \frac{1}{y} dy &= x + c_1 \\
\ln(y) &= x + c_1 \\
y &= e^{x+c_1} \\
y &= c_1 e^x
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x \tag{1}$$

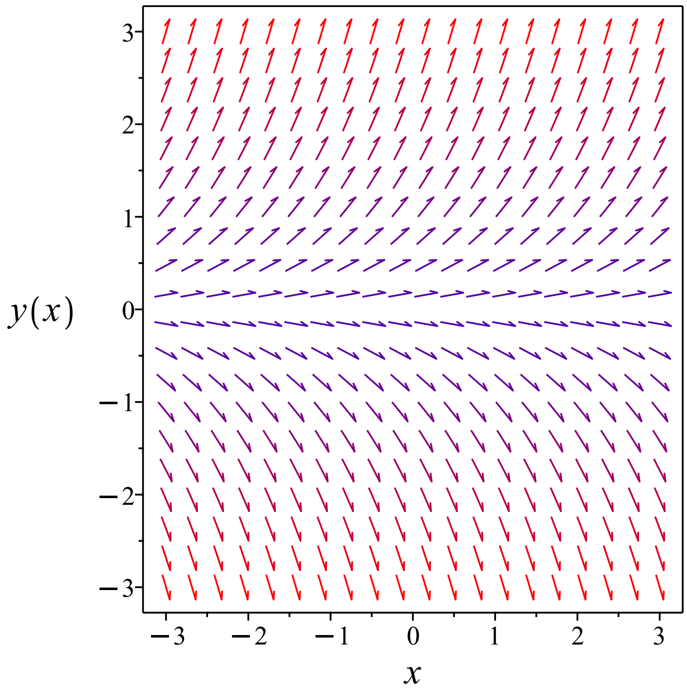


Figure 505: Slope field plot

Verification of solutions

$$y = c_1 e^x$$

Verified OK.

9.1.2 Maple step by step solution

Let's solve

$y' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 1 dx + c_1$

- Evaluate integral

$$\ln(y) = x + c_1$$

- Solve for y

$$y = e^{x+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x c_1$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 16

```
DSolve[y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x$$
$$y(x) \to 0$$

9.2 problem 16

Internal problem ID [2088]
Internal file name [`OUTPUT/2088_Sunday_February_25_2024_06_49_46_AM_29145436/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y = 0}$$

9.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x} \tag{1}$$

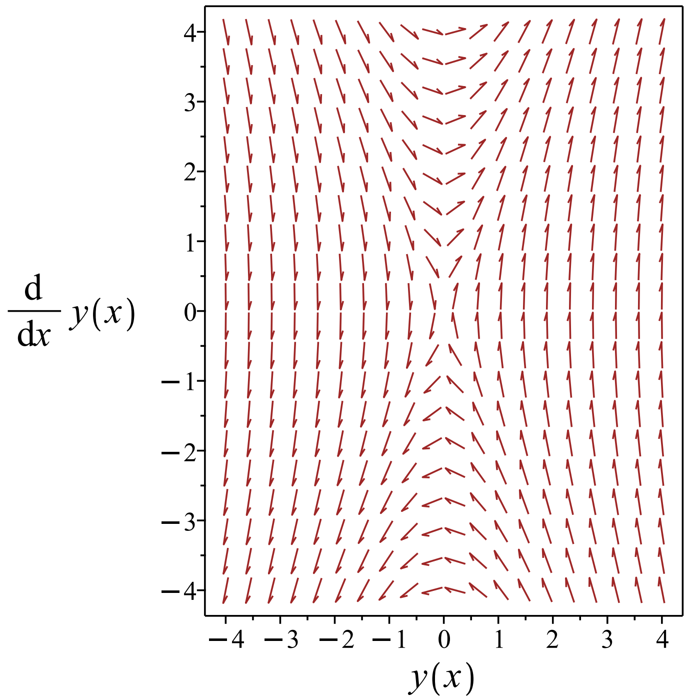


Figure 506: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

Verified OK.

9.2.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - 4yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - 4yy')\, dx = 0$$
$$\frac{y'^2}{2} - 2y^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{4y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{4y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{4y^2+2c_1}}dy = \int dx$$
$$\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4} = x+c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4}} = e^{x+c_2}$$

Which simplifies to

$$\sqrt{2y+\sqrt{4y^2+2c_1}} = c_3 e^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{4y^2+2c_1}}dy = \int dx$$
$$-\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4} = x+c_4$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4}} = e^{x+c_4}$$

Which simplifies to

$$\frac{1}{\sqrt{2y+\sqrt{4y^2+2c_1}}} = c_5 e^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(e^{4x}c_3^4-2c_1\right)e^{-2x}}{4c_3^2} \tag{1}$$
$$y = -\frac{\left(2c_1c_5^4e^{4x}-1\right)e^{-2x}}{4c_5^2} \tag{2}$$

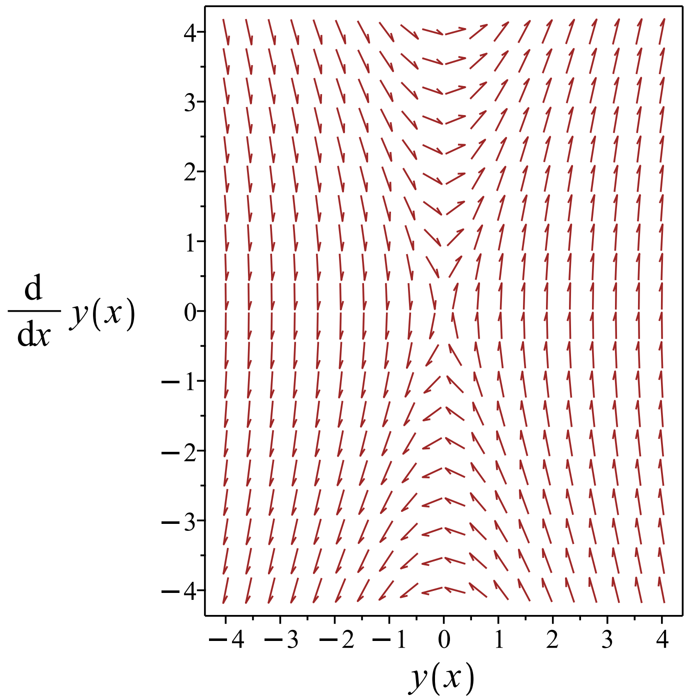


Figure 507: Slope field plot

Verification of solutions

$$y = \frac{\left(e^{4x} c_3^4 - 2c_1\right) e^{-2x}}{4c_3^2}$$

Verified OK.

$$y = -\frac{\left(2c_1 c_5^4 e^{4x} - 1\right) e^{-2x}}{4c_5^2}$$

Verified OK.

9.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 308: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-2x} \int \frac{1}{\mathrm{e}^{-4x}}\, dx \\ &= \mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4} \tag{1}$$

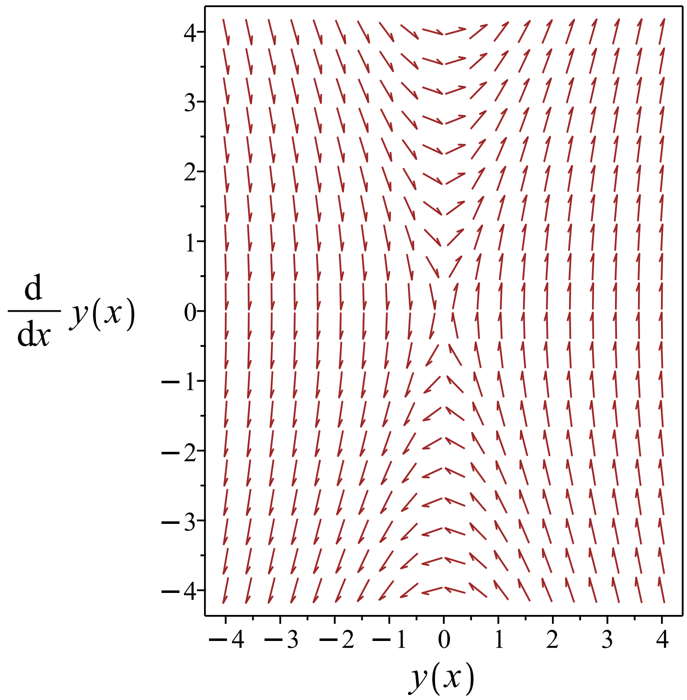


Figure 508: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}$$

Verified OK.

9.2.4 Maple step by step solution

Let's solve

$y'' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r - 2)(r + 2) = 0$

- Roots of the characteristic polynomial

$$r = (-2, 2)$$

- 1st solution of the ODE

$$y_1(x) = e^{-2x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{2x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-2x} + c_2 e^{2x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + e^{-2x} c_2$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 22

```
DSolve[y''[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_1 e^{4x} + c_2\right)$$

9.3 problem 17

Internal problem ID [2089]
Internal file name [`OUTPUT/2089_Sunday_February_25_2024_06_49_46_AM_27920706/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 7y' + 12y = 0}$$

9.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 7, C = 12$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 7\lambda \,\mathrm{e}^{\lambda x} + 12\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 7\lambda + 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 7, C = 12$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-7}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{7^2 - (4)(1)(12)} \\
&= -\frac{7}{2} \pm \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= -\frac{7}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{7}{2} - \frac{1}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= -3 \\
\lambda_2 &= -4
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-3)x} + c_2 e^{(-4)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-4x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-4x} \tag{1}
$$

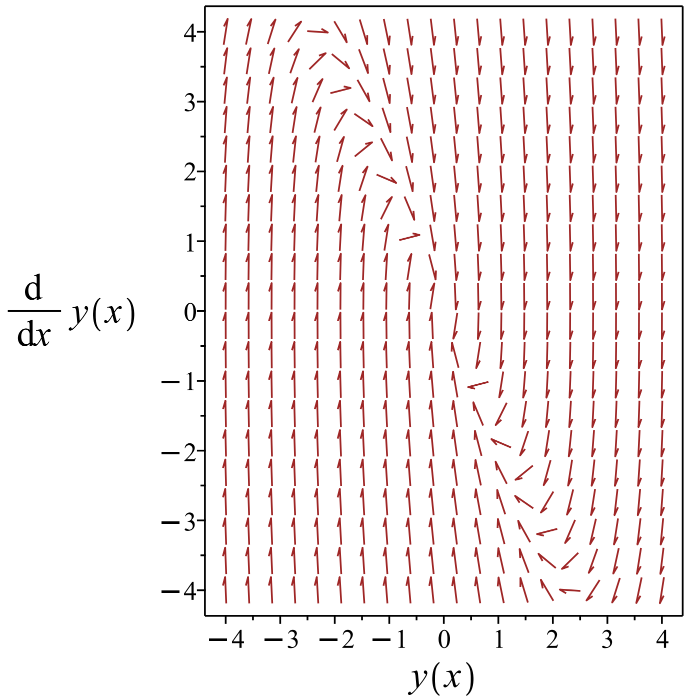


Figure 509: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + c_2 e^{-4x}$$

Verified OK.

9.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 7y' + 12y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 7 \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 310: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7}{1}\, dx} \\ &= z_1 e^{-\frac{7x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{7x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-4x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-7x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-4x}\right) + c_2\left(\mathrm{e}^{-4x}(\mathrm{e}^{x})\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-4x} + \mathrm{e}^{-3x} c_2 \tag{1}$$

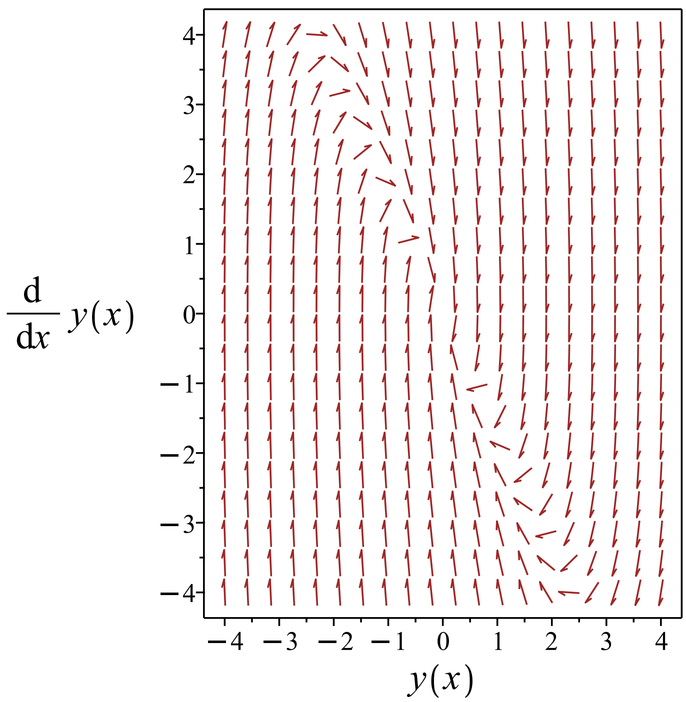


Figure 510: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-4x} + \mathrm{e}^{-3x} c_2$$

Verified OK.

9.3.3 Maple step by step solution

Let's solve

$y'' + 7y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 7r + 12 = 0$

- Factor the characteristic polynomial

  $(r+4)(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -3)$

- 1st solution of the ODE

  $y_1(x) = e^{-4x}$

- 2nd solution of the ODE

  $y_2(x) = e^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-4x} + e^{-3x} c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+7*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-3x} + c_2 e^{-4x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[y''[x]+7*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-4x}(c_2 e^x + c_1)$$

9.4 problem 18

Internal problem ID [2090]
Internal file name [`OUTPUT/2090_Sunday_February_25_2024_06_49_47_AM_3712173/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 3y' + 2y = 0}$$

9.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(2)} \\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} \tag{1}$$

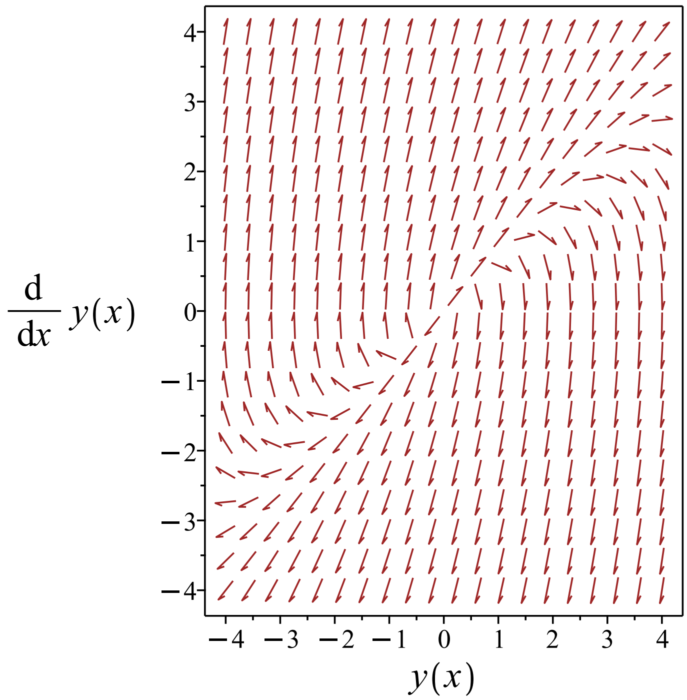


Figure 511: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{2x} + c_2 e^{x}$$

Verified OK.

9.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 312: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^x + c_2\mathrm{e}^{2x} \tag{1}
$$

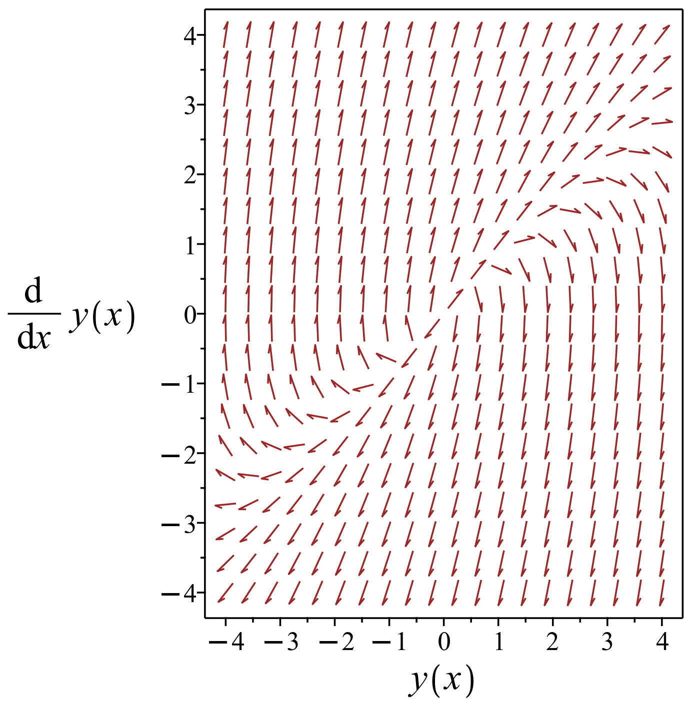


Figure 512: Slope field plot

Verification of solutions

$$
y = c_1\mathrm{e}^x + c_2\mathrm{e}^{2x}
$$

Verified OK.

9.4.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

  $y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + c_2 e^{x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[y''[x]-3*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x (c_2 e^x + c_1)$$

9.5 problem 19

Internal problem ID [2091]
Internal file name [`OUTPUT/2091_Sunday_February_25_2024_06_49_47_AM_41321270/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 7y' + 6y = 0}$$

9.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -7, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 7\lambda\, \mathrm{e}^{\lambda x} + 6\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 7\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -7, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{7}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-7^2 - (4)(1)(6)} \\ &= \frac{7}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{7}{2} + \frac{5}{2}$$
$$\lambda_2 = \frac{7}{2} - \frac{5}{2}$$

Which simplifies to

$$\lambda_1 = 6$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(6)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 e^{6x} + c_2 e^{x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{6x} + c_2 e^{x} \tag{1}$$

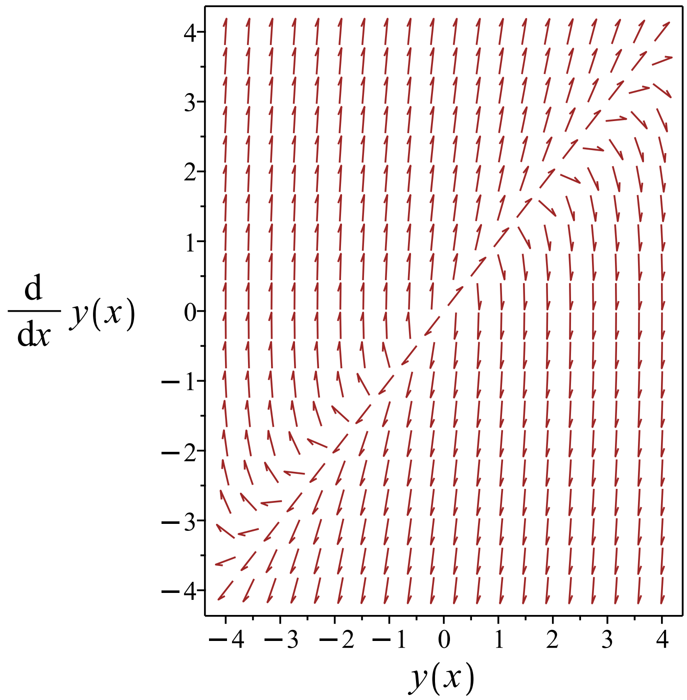


Figure 513: Slope field plot

Verification of solutions

$$y = c_1 e^{6x} + c_2 e^{x}$$

Verified OK.

9.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 7y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$
$$B = -7 \tag{3}$$
$$C = 6$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 314: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{5x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-7}{1}\,dx} \\
&= z_1 e^{\frac{7x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{7x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-7}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{7x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x} + \frac{c_2 \mathrm{e}^{6x}}{5} \tag{1}
$$

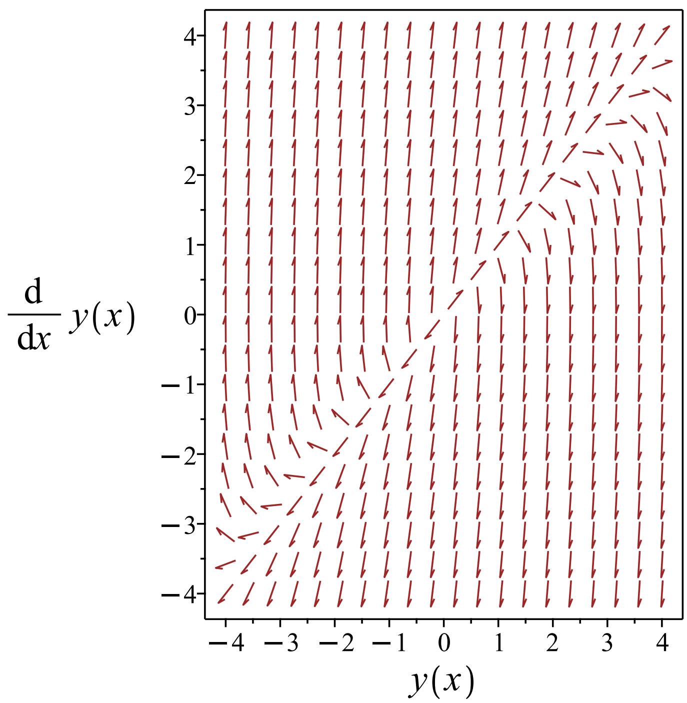


Figure 514: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{x} + \frac{c_2 \mathrm{e}^{6x}}{5}
$$

Verified OK.

9.5.3 Maple step by step solution

Let's solve

$y'' - 7y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 7r + 6 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-6) = 0$

- Roots of the characteristic polynomial

  $r = (1, 6)$

- 1st solution of the ODE

  $y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{6x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 e^{6x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-7*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{6x} + c_2 \mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[y''[x]-7*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}\left(c_2 e^{5x} + c_1\right)$$

9.6 problem 20

Internal problem ID [2092]
Internal file name [`OUTPUT/2092_Sunday_February_25_2024_06_49_47_AM_62988296/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' + 3y' - 2y = 0}$$

9.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 3, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} - 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + 3\lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 3, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{3^2 - (4)\,(2)\,(-2)}\\ &= -\frac{3}{4} \pm \frac{5}{4}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{4} + \frac{5}{4}\\ \lambda_2 &= -\frac{3}{4} - \frac{5}{4}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{1}{2}\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-2x} \tag{1}$$

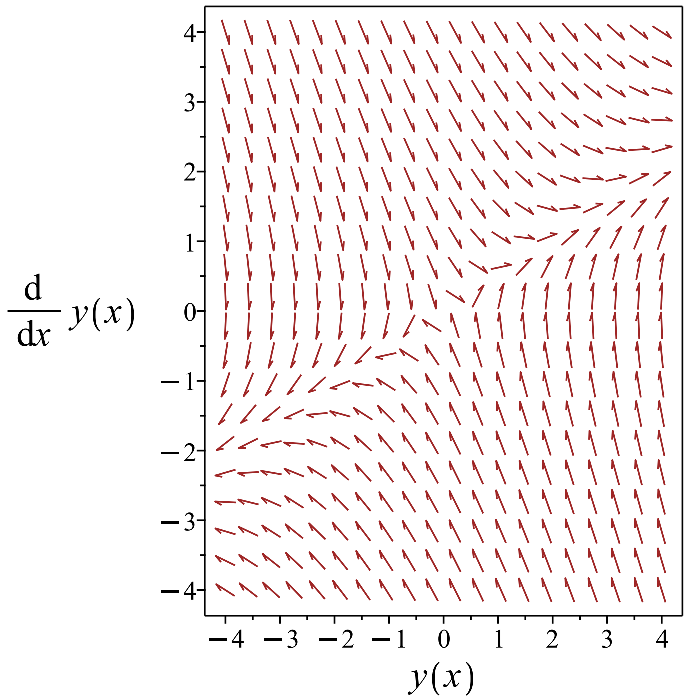


Figure 515: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{-2x}$$

Verified OK.

9.6.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + 3y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 3 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{16} \tag{6}$$

Comparing the above to (5) shows that

$$s = 25$$
$$t = 16$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 316: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{2}\,dx} \\ &= z_1 e^{-\frac{3x}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{3x}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{2\,\mathrm{e}^{\frac{5x}{2}}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-2x}\right) + c_2\left(e^{-2x}\left(\frac{2\,e^{\frac{5x}{2}}}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-2x} + \frac{2c_2 e^{\frac{x}{2}}}{5} \tag{1}
$$

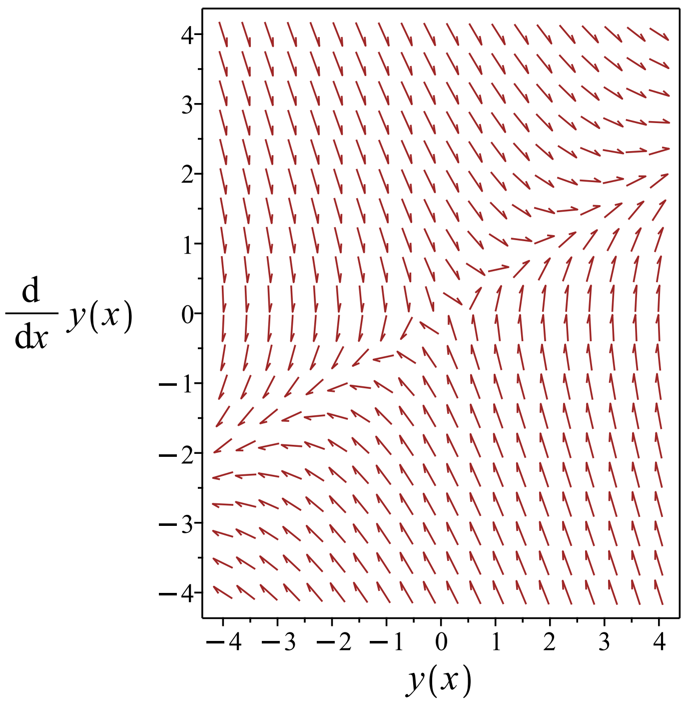


Figure 516: Slope field plot

Verification of solutions

$$
y = c_1 e^{-2x} + \frac{2c_2 e^{\frac{x}{2}}}{5}
$$

Verified OK.

9.6.3 Maple step by step solution

Let's solve

$2y'' + 3y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{2} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2} - y = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{3}{2}r - 1 = 0$

- Factor the characteristic polynomial

  $\frac{(r+2)(2r-1)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(-2, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 e^{\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(2*diff(y(x),x$2)+3*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{\frac{5x}{2}} + c_2\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 24

```
DSolve[2*y''[x]+3*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_1 e^{5x/2} + c_2\right)$$

9.7 problem 21

Internal problem ID [2093]
Internal file name [`OUTPUT/2093_Sunday_February_25_2024_06_49_47_AM_92437445/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - y = 0}$$

9.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda \,\mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(-1)} \\ &= 1 \pm \sqrt{2}\end{aligned}$$

Hence

$$\lambda_1 = 1 + \sqrt{2}$$
$$\lambda_2 = 1 - \sqrt{2}$$

Which simplifies to

$$\lambda_1 = 1 + \sqrt{2}$$
$$\lambda_2 = 1 - \sqrt{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(1+\sqrt{2}\right)x} + c_2 e^{\left(1-\sqrt{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{x\left(1+\sqrt{2}\right)} + c_2 \mathrm{e}^{\left(1-\sqrt{2}\right)x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x\left(1+\sqrt{2}\right)} + c_2 \mathrm{e}^{\left(1-\sqrt{2}\right)x} \tag{1}$$

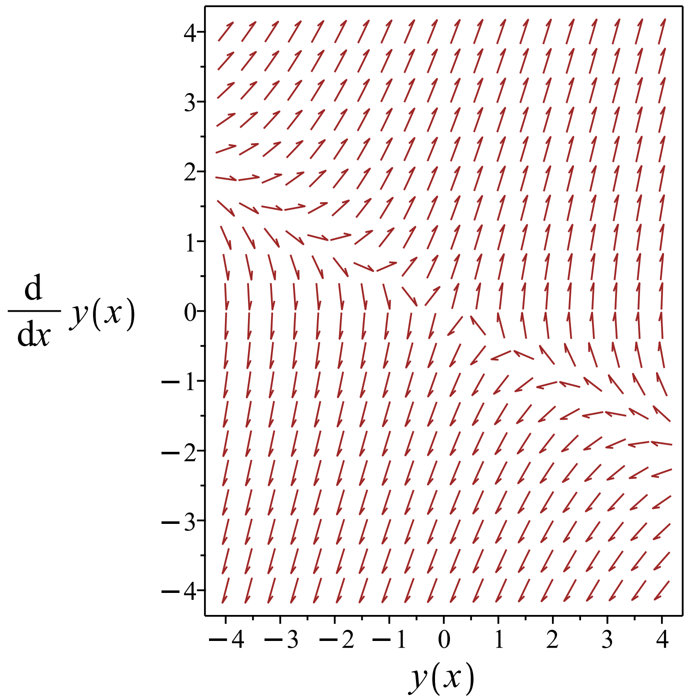


Figure 517: Slope field plot

Verification of solutions

$$y = c_1 e^{x(1+\sqrt{2})} + c_2 e^{(1-\sqrt{2})x}$$

Verified OK.

9.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 318: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\sqrt{2}\,x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x\left(\sqrt{2}-1\right)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x\left(\sqrt{2}-1\right)}\right) + c_2\left(\mathrm{e}^{-x\left(\sqrt{2}-1\right)}\left(\frac{\sqrt{2}\,\mathrm{e}^{2\sqrt{2}\,x}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x\left(\sqrt{2}-1\right)} + \frac{c_2\sqrt{2}\,\mathrm{e}^{x\left(1+\sqrt{2}\right)}}{4} \tag{1}
$$

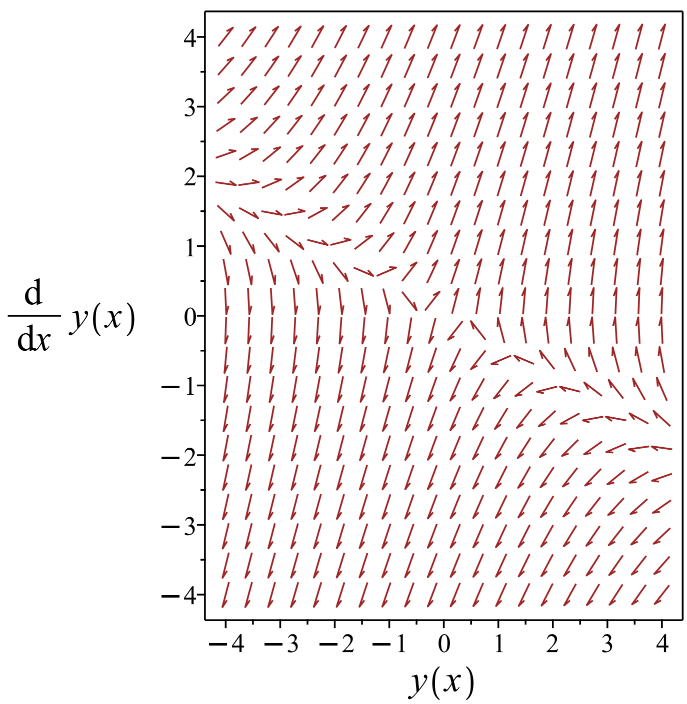


Figure 518: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x\left(\sqrt{2}-1\right)} + \frac{c_2\sqrt{2}\,\mathrm{e}^{x\left(1+\sqrt{2}\right)}}{4}
$$

Verified OK.

9.7.3 Maple step by step solution

Let's solve

$y'' - 2y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r - 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{8})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \sqrt{2}, 1 + \sqrt{2})$

- 1st solution of the ODE

  $y_1(x) = e^{(1-\sqrt{2})x}$

- 2nd solution of the ODE

  $y_2(x) = e^{x(1+\sqrt{2})}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{(1-\sqrt{2})x} + c_2 e^{x(1+\sqrt{2})}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{x\left(1+\sqrt{2}\right)} + c_2 e^{-x\left(\sqrt{2}-1\right)}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 34

```
DSolve[y''[x]-2*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x-\sqrt{2}x}\left(c_2 e^{2\sqrt{2}x} + c_1\right)$$

9.8 problem 22

Internal problem ID [2094]
Internal file name [`OUTPUT/2094_Sunday_February_25_2024_06_49_48_AM_3236419/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - 2y = 0}$$

9.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} - 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -2$ into the above gives

$$\lambda_{1,2} = \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(-2)}$$
$$= 1 \pm \sqrt{3}$$

Hence

$$\lambda_1 = 1 + \sqrt{3}$$
$$\lambda_2 = 1 - \sqrt{3}$$

Which simplifies to

$$\lambda_1 = 1 + \sqrt{3}$$
$$\lambda_2 = 1 - \sqrt{3}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(1+\sqrt{3}\right)x} + c_2 e^{\left(1-\sqrt{3}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(1+\sqrt{3}\right)x} + c_2 \mathrm{e}^{\left(1-\sqrt{3}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(1+\sqrt{3}\right)x} + c_2 \mathrm{e}^{\left(1-\sqrt{3}\right)x} \tag{1}$$

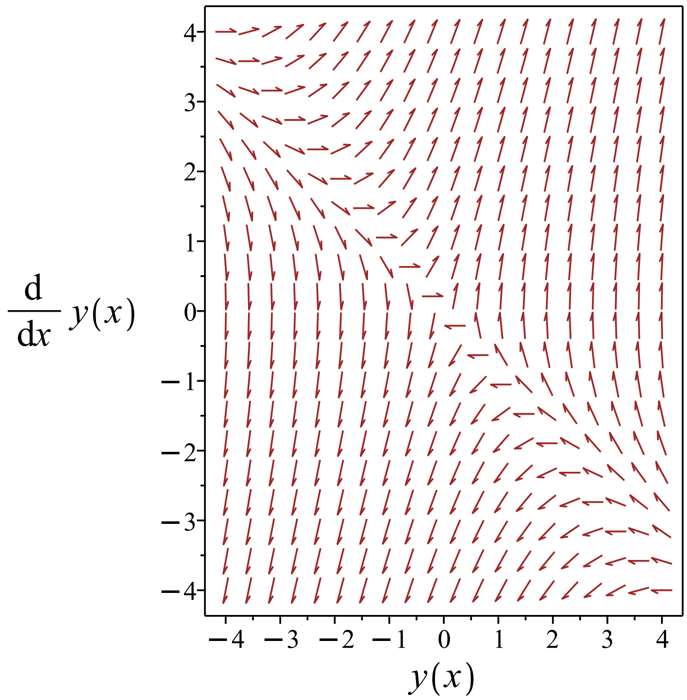


Figure 519: Slope field plot

Verification of solutions

$$y = c_1 e^{\left(1+\sqrt{3}\right)x} + c_2 e^{\left(1-\sqrt{3}\right)x}$$

Verified OK.

9.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 320: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\sqrt{3}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\, dx} \\ &= z_1 e^{x} \\ &= z_1 (\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\left(\sqrt{3}-1\right)x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\, x}}{6}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\left(\sqrt{3}-1\right)x}\right) + c_2 \left(\mathrm{e}^{-\left(\sqrt{3}-1\right)x} \left(\frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,x}}{6}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\left(\sqrt{3}-1\right)x} + \frac{c_2 \sqrt{3}\,\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{6} \tag{1}
$$

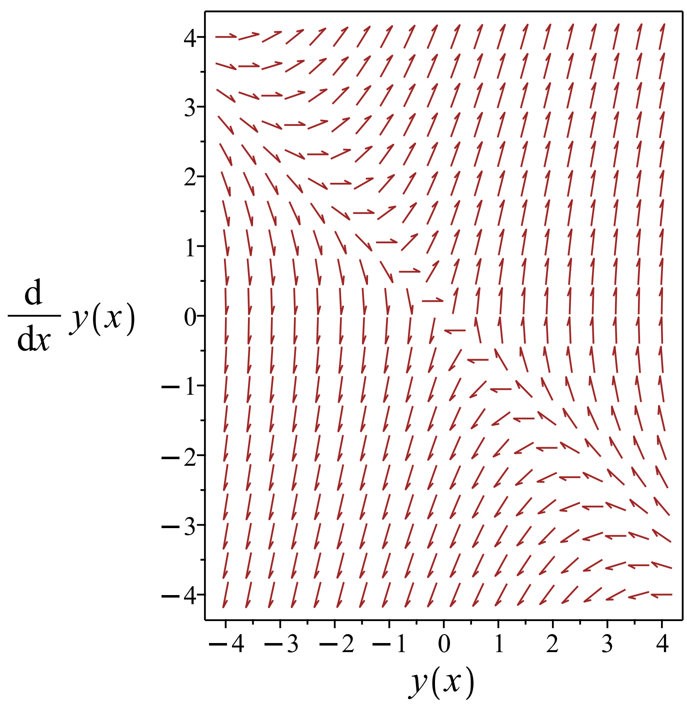


Figure 520: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{-\left(\sqrt{3}-1\right)x} + \frac{c_2 \sqrt{3}\,\mathrm{e}^{\left(1+\sqrt{3}\right)x}}{6}
$$

Verified OK.

9.8.3 Maple step by step solution

Let's solve

$y'' - 2y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r - 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm \left(\sqrt{12}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(1 - \sqrt{3}, 1 + \sqrt{3}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\left(1-\sqrt{3}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = e^{\left(1+\sqrt{3}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\left(1-\sqrt{3}\right)x} + c_2 e^{\left(1+\sqrt{3}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{\left(1+\sqrt{3}\right)x} + c_2 \mathrm{e}^{-\left(\sqrt{3}-1\right)x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 34

```
DSolve[y''[x]-2*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x-\sqrt{3}x}\left(c_2 e^{2\sqrt{3}x} + c_1\right)$$

9.9 problem 23

Internal problem ID [2095]
Internal file name [`OUTPUT/2095_Sunday_February_25_2024_06_49_48_AM_47304410/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' + y = 0}$$

9.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-3^2 - (4)(1)(1)} \\
&= \frac{3}{2} \pm \frac{\sqrt{5}}{2}
\end{aligned}
$$

Hence

$$
\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}
$$

$$
\lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}
$$

Which simplifies to

$$
\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}
$$

$$
\lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)x} \tag{1}
$$

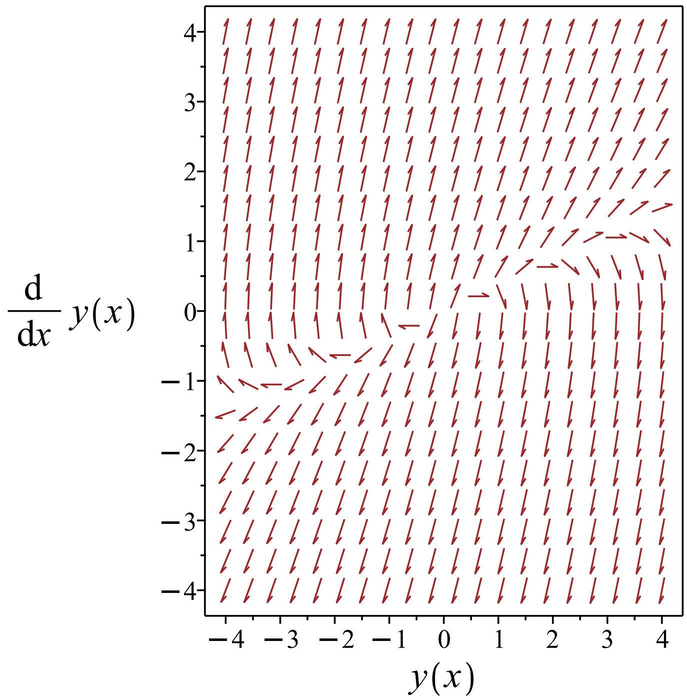


Figure 521: Slope field plot

Verification of solutions

$$y = c_1 e^{\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)x}$$

Verified OK.

9.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{5z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 322: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{5}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x\sqrt{5}}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{(\sqrt{5}-3)x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\sqrt{5}\,\mathrm{e}^{x\sqrt{5}}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{(\sqrt{5}-3)x}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{(\sqrt{5}-3)x}{2}} \left(\frac{\sqrt{5}\,\mathrm{e}^{x\sqrt{5}}}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{(\sqrt{5}-3)x}{2}} + \frac{c_2 \sqrt{5}\,\mathrm{e}^{\frac{(3+\sqrt{5})x}{2}}}{5} \tag{1}
$$

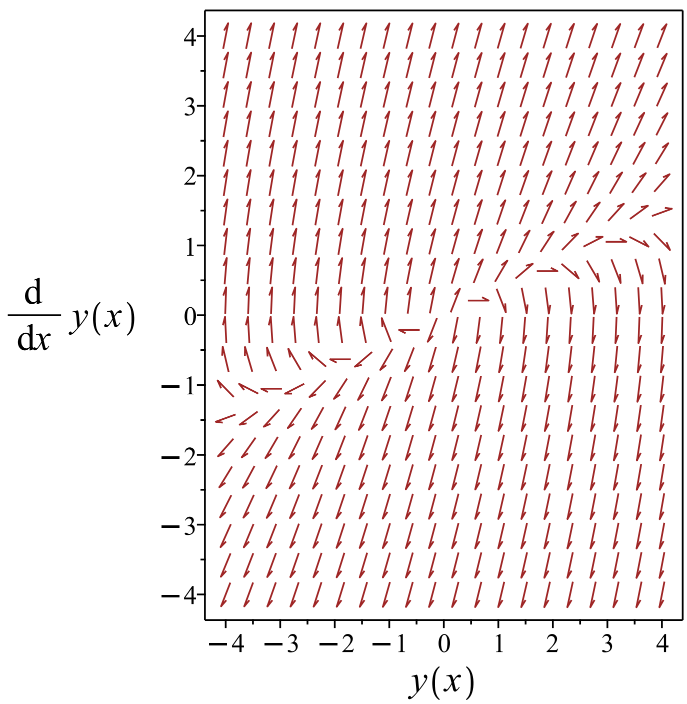


Figure 522: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{(\sqrt{5}-3)x}{2}} + \frac{c_2 \sqrt{5}\,\mathrm{e}^{\frac{(3+\sqrt{5})x}{2}}}{5}
$$

Verified OK.

9.9.3 Maple step by step solution

Let's solve

$y'' - 3y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 3r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{3 \pm (\sqrt{5})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{3}{2} - \frac{\sqrt{5}}{2}, \frac{3}{2} + \frac{\sqrt{5}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = e^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{(3+\sqrt{5})x}{2}} + c_2 e^{-\frac{(\sqrt{5}-3)x}{2}}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 35

```
DSolve[y''[x]-3*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{1}{2}(\sqrt{5}-3)x}\left(c_2 e^{\sqrt{5}x} + c_1\right)$$

9.10 problem 24

Internal problem ID [2096]
Internal file name [`OUTPUT/2096_Sunday_February_25_2024_06_49_48_AM_44556157/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y''+2y'-y=0}$$

9.10.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=2,B=2,C=-1$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 \mathrm{e}^{\lambda x}+2\lambda\,\mathrm{e}^{\lambda x}-\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2+2\lambda-1=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A = 2, B = 2, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(2)} \pm \frac{1}{(2)(2)}\sqrt{2^2 - (4)(2)(-1)} \\ &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = \frac{\sqrt{3}}{2} - \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)x} + c_2 e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)x} \tag{1}$$

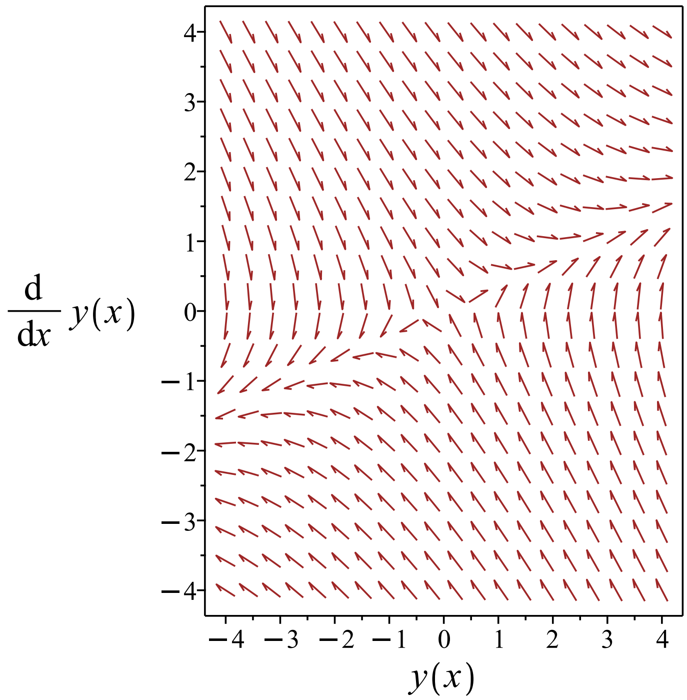


Figure 523: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x}$$

Verified OK.

9.10.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + 2y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 324: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{\sqrt{3}\, x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{2}\, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{\left(1+\sqrt{3}\right) x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{2}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{\left(y_1\right)^2}\, dx \\ &= y_1 \left(\frac{\sqrt{3}\, \mathrm{e}^{\sqrt{3}\, x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{\left(1+\sqrt{3}\right)x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{\left(1+\sqrt{3}\right)x}{2}}\left(\frac{\sqrt{3}\,\mathrm{e}^{\sqrt{3}\,x}}{3}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{\left(1+\sqrt{3}\right)x}{2}} + \frac{c_2\sqrt{3}\,\mathrm{e}^{\frac{\left(\sqrt{3}-1\right)x}{2}}}{3} \tag{1}$$

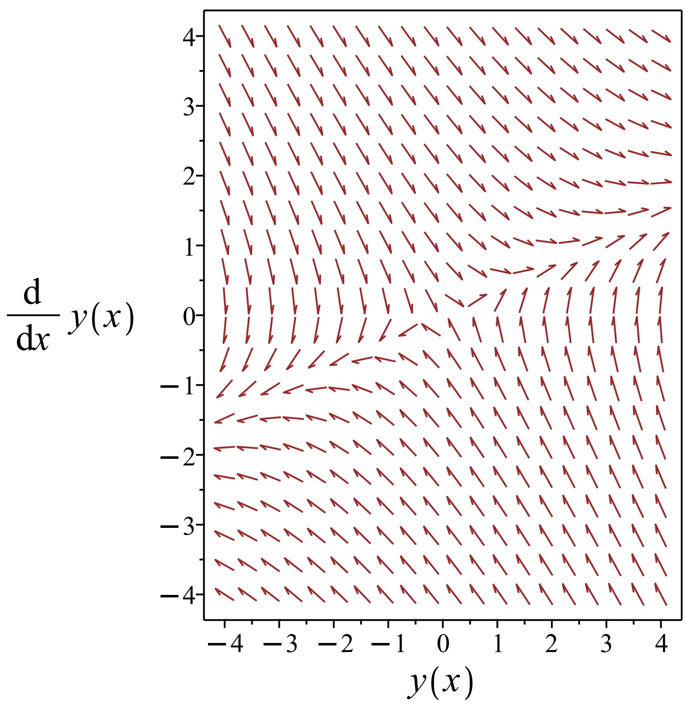


Figure 524: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{\left(1+\sqrt{3}\right)x}{2}} + \frac{c_2\sqrt{3}\,\mathrm{e}^{\frac{\left(\sqrt{3}-1\right)x}{2}}}{3}$$

Verified OK.

9.10.3 Maple step by step solution

Let's solve

$2y'' + 2y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' + \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - \frac{y}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 + r - \frac{1}{2} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm\left(\sqrt{3}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(2*diff(y(x),x$2)+2*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{\frac{(\sqrt{3}-1)x}{2}} + c_2 \mathrm{e}^{-\frac{(1+\sqrt{3})x}{2}}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 35

```
DSolve[2*y''[x]+2*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{1}{2}(1+\sqrt{3})x}\left(c_2 e^{\sqrt{3}x} + c_1\right)$$

9.11 problem 25

Internal problem ID [2097]
Internal file name [`OUTPUT/2097_Sunday_February_25_2024_06_49_48_AM_70265156/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 25.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{2y''' - y'' - 2y' + y = 0}$$

The characteristic equation is

$$2\lambda^3 - \lambda^2 - 2\lambda + 1 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= \frac{1}{2}\\ \lambda_3 &= -1\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + c_3 e^{\frac{x}{2}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= e^{-x}\\ y_2 &= e^{x}\\ y_3 &= e^{\frac{x}{2}}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{\frac{x}{2}} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{\frac{x}{2}}$$

Verified OK.

9.11.1 Maple step by step solution

Let's solve

$2y''' - y'' - 2y' + y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{y''}{2} + y' - \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{2} - y' + \frac{y}{2} = 0$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \frac{y_3(x)}{2} + y_2(x) - \frac{y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{y_3(x)}{2} + y_2(x) - \frac{y_1(x)}{2}\right]$

- Define vector

$$\overrightarrow{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \left(c_3 e^{2x} + 4c_2 e^{\frac{3x}{2}} + c_1\right) e^{-x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(2*diff(y(x),x$3)-diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_3 e^{2x} + c_1 e^{\frac{3x}{2}} + c_2\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 30

```
DSolve[2*y'''[x]-y''[x]-2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x/2} + c_2 e^{-x} + c_3 e^x$$

9.12 problem 26

Internal problem ID [2098]
Internal file name [`OUTPUT/2098_Sunday_February_25_2024_06_49_49_AM_8061279/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 26.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' - 4y' + 12y = 0}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + c_3 \mathrm{e}^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{2x}$$
$$y_3 = \mathrm{e}^{3x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{3x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{3x}$$

Verified OK.

9.12.1 Maple step by step solution

Let's solve

$y''' - 3y'' - 4y' + 12y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 3y_3(x) + 4y_2(x) - 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3y_3(x) + 4y_2(x) - 12y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(4c_3 e^{5x} + 9c_2 e^{4x} + 9c_1) e^{-2x}}{36}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)-4*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{4x} + c_3\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 29

```
DSolve[y'''[x]-3*y''[x]-4*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(e^{4x}(c_3 e^x + c_2) + c_1\right)$$

9.13 problem 27

Internal problem ID [2099]
Internal file name [`OUTPUT/2099_Sunday_February_25_2024_06_49_49_AM_2528552/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 27.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 4y'' + y' + 6y = 0}$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{2x}$$
$$y_3 = e^{3x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x}$$

Verified OK.

9.13.1 Maple step by step solution

Let's solve

$y''' - 4y'' + y' + 6y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 4y_3(x) - y_2(x) - 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 4y_3(x) - y_2(x) - 6y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{4} + \frac{c_3 e^{3x}}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)+diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{3x} + c_2 e^{2x} + c_3 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 29

```
DSolve[y'''[x]-4*y''[x]+y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(e^{3x}(c_3 e^{x} + c_2) + c_1\right)$$

9.14 problem 28

Internal problem ID [2100]
Internal file name [`OUTPUT/2100_Sunday_February_25_2024_06_49_49_AM_63143157/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 28.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 6y'' + 8y = 0}$$

The characteristic equation is

$$\lambda^4 - 6\lambda^2 + 8 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= \sqrt{2}\\ \lambda_2 &= -\sqrt{2}\\ \lambda_3 &= 2\\ \lambda_4 &= -2\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{\sqrt{2}\, x} c_3 + \mathrm{e}^{-\sqrt{2}\, x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{-2x}\\ y_2 &= \mathrm{e}^{2x}\\ y_3 &= \mathrm{e}^{\sqrt{2}\, x}\\ y_4 &= \mathrm{e}^{-\sqrt{2}\, x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{2x} + e^{\sqrt{2}\,x} c_3 + e^{-\sqrt{2}\,x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{2x} + e^{\sqrt{2}\,x} c_3 + e^{-\sqrt{2}\,x} c_4$$

Verified OK.

9.14.1 Maple step by step solution

Let's solve

$y'''' - 6y'' + 8y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 6y_3(x) - 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 6y_3(x) - 8y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & 0 & 6 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & 0 & 6 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[\sqrt{2}, \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right], \left[-\sqrt{2}, \begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\sqrt{2}, \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{\sqrt{2}\,x} \cdot \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\sqrt{2}, \begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_4 = \mathrm{e}^{-\sqrt{2}\,x} \cdot \begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + c_4 \overrightarrow{y}_4$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 \mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \mathrm{e}^{\sqrt{2}\,x} c_3 \cdot \begin{bmatrix} \frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix} + \mathrm{e}^{-\sqrt{2}\,x} c_4 \cdot \begin{bmatrix} -\frac{\sqrt{2}}{4} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_1 \mathrm{e}^{-2x}}{8} + \frac{c_2 \mathrm{e}^{2x}}{8} + \frac{\sqrt{2}\,\mathrm{e}^{\sqrt{2}\,x} c_3}{4} - \frac{\sqrt{2}\,\mathrm{e}^{-\sqrt{2}\,x} c_4}{4}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$4)-6*diff(y(x),x$2)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + e^{-2x} c_2 + c_3 e^{\sqrt{2}\, x} + c_4 e^{-\sqrt{2}\, x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 47

```
DSolve[y''''[x]-6*y''[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} + c_3 e^{-2x} + c_4 e^{2x}$$

9.15 problem 29

Internal problem ID [2101]
Internal file name [`OUTPUT/2101_Sunday_February_25_2024_06_49_49_AM_85120480/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 29.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 7y' + 6y = 0}$$

The characteristic equation is

$$\lambda^3 - 7\lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{-3x} c_2 + c_3 e^{2x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{-3x}$$
$$y_3 = e^{2x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} + e^{-3x} c_2 + c_3 e^{2x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{x} + e^{-3x} c_2 + c_3 e^{2x}$$

Verified OK.

9.15.1 Maple step by step solution

Let's solve

$y''' - 7y' + 6y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 7y_2(x) - 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 7y_2(x) - 6y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = e^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(9c_3 e^{5x} + 36c_2 e^{4x} + 4c_1) e^{-3x}}{36}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-7*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{5x} + c_3 e^{4x} + c_2\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 28

```
DSolve[y'''[x]-7*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-3x} + c_2 e^{x} + c_3 e^{2x}$$

9.16 problem 30

Internal problem ID [2102]
Internal file name [OUTPUT/2102_Sunday_February_25_2024_06_49_49_AM_33154996/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 30.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 11y' - 6y = 0}$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} + c_3 \mathrm{e}^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{x}$$
$$y_2 = \mathrm{e}^{2x}$$
$$y_3 = \mathrm{e}^{3x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

Verified OK.

9.16.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 11y' - 6y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 6y_3(x) - 11y_2(x) + 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 6y_3(x) - 11y_2(x) + 6y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^{x} + \frac{c_2 e^{2x}}{4} + \frac{c_3 e^{3x}}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+11*diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{3x} + c_2 e^{2x} + c_3 e^{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 25

```
DSolve[y'''[x]-6*y''[x]+11*y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(e^x(c_3 e^x + c_2) + c_1)$$

9.17 problem 31

Internal problem ID [2103]
Internal file name [OUTPUT/2103_Sunday_February_25_2024_06_49_49_AM_62047249/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 31.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 4y'' - 17y' + 60y = 0}$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 - 17\lambda + 60 = 0$$

The roots of the above equation are

$$\lambda_1 = 5$$
$$\lambda_2 = -4$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-4x} + c_2 e^{3x} + c_3 e^{5x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-4x}$$
$$y_2 = e^{3x}$$
$$y_3 = e^{5x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-4x} + c_2 \mathrm{e}^{3x} + c_3 \mathrm{e}^{5x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-4x} + c_2 \mathrm{e}^{3x} + c_3 \mathrm{e}^{5x}$$

Verified OK.

9.17.1 Maple step by step solution

Let's solve

$y''' - 4y'' - 17y' + 60y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

- ○ Define new variable $y_1(x)$

  $y_1(x) = y$

- ○ Define new variable $y_2(x)$

  $y_2(x) = y'$

- ○ Define new variable $y_3(x)$

  $y_3(x) = y''$

- ○ Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 4y_3(x) + 17y_2(x) - 60y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 4y_3(x) + 17y_2(x) - 60y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & 17 & 4 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & 17 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-4x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 5, \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{5x} \cdot \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-4x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix} + c_2 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + c_3 e^{5x} \cdot \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(144 c_3 e^{9x} + 400 c_2 e^{7x} + 225 c_1) e^{-4x}}{3600}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)-17*diff(y(x),x)+60*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 \mathrm{e}^{9x} + c_1 \mathrm{e}^{7x} + c_3\right) \mathrm{e}^{-4x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[y'''[x]-4*y''[x]-17*y'[x]+60*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-4x}\left(c_2 e^{7x} + c_3 e^{9x} + c_1\right)$$

9.18 problem 32

Internal problem ID [2104]
Internal file name [`OUTPUT/2104_Sunday_February_25_2024_06_49_50_AM_15871838/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 32.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 9y'' + 23y' - 15y = 0}$$

The characteristic equation is

$$\lambda^3 - 9\lambda^2 + 23\lambda - 15 = 0$$

The roots of the above equation are

$$\lambda_1 = 5$$
$$\lambda_2 = 1$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 e^{3x} + c_3 e^{5x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{3x}$$
$$y_3 = e^{5x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{5x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{5x}$$

Verified OK.

9.18.1 Maple step by step solution

Let's solve

$y''' - 9y'' + 23y' - 15y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 9y_3(x) - 23y_2(x) + 15y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 9y_3(x) - 23y_2(x) + 15y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 15 & -23 & 9 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 15 & -23 & 9 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{5x} \cdot \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + c_3 e^{5x} \cdot \begin{bmatrix} \frac{1}{25} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^{x} + \frac{c_2 e^{3x}}{9} + \frac{c_3 e^{5x}}{25}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-9*diff(y(x),x$2)+23*diff(y(x),x)-15*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{3x} + c_2 e^{5x} + c_3 e^{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 28

```
DSolve[y'''[x]-9*y''[x]+23*y'[x]-15*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(c_2 e^{2x} + c_3 e^{4x} + c_1\right)$$

9.19 problem 33

Internal problem ID [2105]
Internal file name [`OUTPUT/2105_Sunday_February_25_2024_06_49_50_AM_72884397/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 33.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + y''' - 7y'' - y' + 6y = 0}$$

The characteristic equation is

$$\lambda^4 + \lambda^3 - 7\lambda^2 - \lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -3$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{-3x} c_3 + e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{-3x}$$
$$y_4 = e^{2x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + e^{-3x} c_3 + e^{2x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + e^{-3x} c_3 + e^{2x} c_4$$

Verified OK.

9.19.1 Maple step by step solution

Let's solve

$y'''' + y''' - 7y'' - y' + 6y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ○ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -y_4(x) + 7y_3(x) + y_2(x) - 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -y_4(x) + 7y_3(x) + y_2(x) - 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 1 & 7 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 1 & 7 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} -1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{2x} c_4 \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\left(27 c_2 e^{2x} - 27 e^{4x} c_3 - \frac{27 e^{5x} c_4}{8} + c_1\right) e^{-3x}}{27}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$4)+diff(y(x),x$3)-7*diff(y(x),x$2)-diff(y(x),x)+6*y(x)=0,y(x), singsol=al
```

$$y(x) = \left(c_1 \mathrm{e}^{5x} + c_4 \mathrm{e}^{4x} + c_3 \mathrm{e}^{2x} + c_2\right) \mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 36

```
DSolve[y''''[x]+y'''[x]-7*y''[x]-y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-3x} + c_2 e^{-x} + c_3 e^x + c_4 e^{2x}$$

9.20 problem 34

Internal problem ID [2106]
Internal file name [`OUTPUT/2106_Sunday_February_25_2024_06_49_50_AM_28597167/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 34.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{2y'''' - 3y''' - 20y'' + 27y' + 18y = 0}$$

The characteristic equation is

$$2\lambda^4 - 3\lambda^3 - 20\lambda^2 + 27\lambda + 18 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -\frac{1}{2}$$
$$\lambda_4 = -3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{2x} + c_3 \mathrm{e}^{3x} + \mathrm{e}^{-\frac{x}{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-3x}$$
$$y_2 = \mathrm{e}^{2x}$$
$$y_3 = \mathrm{e}^{3x}$$
$$y_4 = \mathrm{e}^{-\frac{x}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-3x} + c_2 e^{2x} + c_3 e^{3x} + e^{-\frac{x}{2}} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-3x} + c_2 e^{2x} + c_3 e^{3x} + e^{-\frac{x}{2}} c_4$$

Verified OK.

9.20.1 Maple step by step solution

Let's solve

$2y'''' - 3y''' - 20y'' + 27y' + 18y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = \frac{3y'''}{2} + 10y'' - \frac{27y'}{2} - 9y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - \frac{3y'''}{2} - 10y'' + \frac{27y'}{2} + 9y = 0$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = \frac{3y_4(x)}{2} + 10y_3(x) - \frac{27y_2(x)}{2} - 9y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \frac{3y_4(x)}{2} + 10y_3(x) - \frac{27y_2(x)}{2} - 9y_1(x)\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & -\frac{27}{2} & 10 & \frac{3}{2} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & -\frac{27}{2} & 10 & \frac{3}{2} \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} -\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_4 = e^{3x} \cdot \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + c_4 \overrightarrow{y}_4$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 e^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{-\frac{x}{2}} \cdot \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + e^{3x} c_4 \cdot \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\left(216 c_2 e^{\frac{5x}{2}} - \frac{27 c_3 e^{5x}}{8} - e^{6x} c_4 + c_1\right) e^{-3x}}{27}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(2*diff(y(x),x$4)-3*diff(y(x),x$3)-20*diff(y(x),x$2)+27*diff(y(x),x)+18*y(x)=0,y(x), s
```

$$y(x) = \left(c_1 e^{6x} + c_2 e^{5x} + c_4 e^{\frac{5x}{2}} + c_3\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 40

```
DSolve[2*y''''[x]-3*y'''[x]-20*y''[x]+27*y'[x]+18*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \rightarrow e^{-3x}\left(c_1 e^{5x/2} + c_3 e^{5x} + c_4 e^{6x} + c_2\right)$$

9.21 problem 35

Internal problem ID [2107]
Internal file name [`OUTPUT/2107_Sunday_February_25_2024_06_49_50_AM_3166317/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 35.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{12y'''' - 4y''' - 3y'' + y' = 0}$$

The characteristic equation is

$$12\lambda^4 - 4\lambda^3 - 3\lambda^2 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = \frac{1}{3}$$
$$\lambda_3 = -\frac{1}{2}$$
$$\lambda_4 = \frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 \mathrm{e}^{\frac{x}{3}} + \mathrm{e}^{-\frac{x}{2}} c_3 + c_4 \mathrm{e}^{\frac{x}{2}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{\frac{x}{3}}$$
$$y_3 = e^{-\frac{x}{2}}$$
$$y_4 = e^{\frac{x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{\frac{x}{3}} + e^{-\frac{x}{2}} c_3 + c_4 e^{\frac{x}{2}} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^{\frac{x}{3}} + e^{-\frac{x}{2}} c_3 + c_4 e^{\frac{x}{2}}$$

Verified OK.

9.21.1 Maple step by step solution

Let's solve

$12y'''' - 4y''' - 3y'' + y' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = \frac{y'''}{3} + \frac{y''}{4} - \frac{y'}{12}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - \frac{y'''}{3} - \frac{y''}{4} + \frac{y'}{12} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

$y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = \frac{y_4(x)}{3} + \frac{y_3(x)}{4} - \frac{y_2(x)}{12}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \frac{y_4(x)}{3} + \frac{y_3(x)}{4} - \frac{y_2(x)}{12}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{12} & \frac{1}{4} & \frac{1}{3} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{12} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2}, \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[\frac{1}{3}, \begin{bmatrix} 27 \\ 9 \\ 3 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \left[\begin{array}{c}-8 \\ 4 \\ -2 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c}-8 \\ 4 \\ -2 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c}1 \\ 0 \\ 0 \\ 0\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \left[\begin{array}{c}1 \\ 0 \\ 0 \\ 0\end{array}\right]$$

- Consider eigenpair

$$\left[\frac{1}{3}, \left[\begin{array}{c}27 \\ 9 \\ 3 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = e^{\frac{x}{3}} \cdot \begin{bmatrix} 27 \\ 9 \\ 3 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} \frac{1}{2}, \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_4 = e^{\frac{x}{2}} \cdot \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + c_4 \overrightarrow{y}_4$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 e^{-\frac{x}{2}} \cdot \begin{bmatrix} -8 \\ 4 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{\frac{x}{3}} \cdot \begin{bmatrix} 27 \\ 9 \\ 3 \\ 1 \end{bmatrix} + c_4 e^{\frac{x}{2}} \cdot \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \left(8\,e^{x} c_4 + 27 c_3 e^{\frac{5x}{6}} + c_2 e^{\frac{x}{2}} - 8 c_1\right) e^{-\frac{x}{2}}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(12*diff(y(x),x$4)-4*diff(y(x),x$3)-3*diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 e^x + c_3 e^{\frac{5x}{6}} + c_1 e^{\frac{x}{2}} + c_4\right) e^{-\frac{x}{2}}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 39

```
DSolve[12*y''''[x]-4*y'''[x]-3*y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(3c_1 e^{5x/6} + 2c_3 e^x - 2c_2\right) + c_4$$

9.22 problem 36

Internal problem ID [2108]
Internal file name [`OUTPUT/2108_Sunday_February_25_2024_06_49_50_AM_2911826/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 36.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_x]]`

$$\boxed{y''' - 4y'' + 3y' = 0}$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 3$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^x + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = e^{3x}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^x + c_3 e^{3x} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^x + c_3 e^{3x}$$

Verified OK.

9.22.1 Maple step by step solution

Let's solve

$y''' - 4y'' + 3y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 4y_3(x) - 3y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 4y_3(x) - 3y_2(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 4 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_2 e^x + \frac{c_3 e^{3x}}{9} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)+3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{3x} + c_3 e^{x}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 25

```
DSolve[y'''[x]-4*y''[x]+3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + \frac{1}{3} c_2 e^{3x} + c_3$$

9.23 problem 37

Internal problem ID [2109]
Internal file name [`OUTPUT/2109_Sunday_February_25_2024_06_49_50_AM_23139370/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 37.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{4y''' + 2y'' - 4y' + y = 0}$$

The characteristic equation is

$$4\lambda^3 + 2\lambda^2 - 4\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = \frac{\sqrt{3}}{2} - \frac{1}{2}$$
$$\lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)x} + c_3 \mathrm{e}^{\frac{x}{2}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x}$$
$$y_2 = \mathrm{e}^{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)x}$$
$$y_3 = \mathrm{e}^{\frac{x}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x} + c_2 e^{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)x} + c_3 e^{\frac{x}{2}} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x} + c_2 e^{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)x} + c_3 e^{\frac{x}{2}}$$

Verified OK.

9.23.1 Maple step by step solution

Let's solve

$4y''' + 2y'' - 4y' + y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{y''}{2} + y' - \frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{y''}{2} - y' + \frac{y}{4} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -\frac{y_3(x)}{2} + y_2(x) - \frac{y_1(x)}{4}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -\frac{y_3(x)}{2} + y_2(x) - \frac{y_1(x)}{4}\right]$

- Define vector

$$\overrightarrow{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{4} & 1 & -\frac{1}{2} \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{4} & 1 & -\frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}-\frac{\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{\sqrt{3}}{2}-\frac{1}{2}, \begin{bmatrix} \frac{1}{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{3}}{2}-\frac{1}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{2}-\frac{\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{2} = \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[\frac{\sqrt{3}}{2}-\frac{1}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)^{2}} \\ \frac{1}{\frac{\sqrt{3}}{2}-\frac{1}{2}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{3} = \mathrm{e}^{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)^{2}} \\ \frac{1}{\frac{\sqrt{3}}{2}-\frac{1}{2}} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_{1}\overrightarrow{y}_{1} + c_{2}\overrightarrow{y}_{2} + c_{3}\overrightarrow{y}_{3}$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_{1}\mathrm{e}^{\frac{x}{2}} \cdot \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right] + c_{2}\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\sqrt{3}}{2}} \\ 1 \end{array}\right] + c_{3}\mathrm{e}^{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)^{2}} \\ \frac{1}{\frac{\sqrt{3}}{2}-\frac{1}{2}} \\ 1 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -2c_{2}\left(\sqrt{3}-2\right)\mathrm{e}^{-\frac{\left(1+\sqrt{3}\right)x}{2}} + 2c_{3}\left(2+\sqrt{3}\right)\mathrm{e}^{\frac{\left(\sqrt{3}-1\right)x}{2}} + 4c_{1}\mathrm{e}^{\frac{x}{2}}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(4*diff(y(x),x$3)+2*diff(y(x),x$2)-4*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{(\sqrt{3}-1)x}{2}} + c_3 e^{-\frac{(1+\sqrt{3})x}{2}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 50

```
DSolve[4*y'''[x]+2*y''[x]-4*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{1}{2}(1+\sqrt{3})x} + c_2 e^{\frac{1}{2}(\sqrt{3}-1)x} + c_3 e^{x/2}$$

9.24 problem 38

Internal problem ID [2110]
Internal file name [OUTPUT/2110_Sunday_February_25_2024_06_49_51_AM_31092352/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 38.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 5y'' - 2y' + 24y = 0}$$

The characteristic equation is

$$\lambda^3 - 5\lambda^2 - 2\lambda + 24 = 0$$

The roots of the above equation are

$$\lambda_1 = 3$$
$$\lambda_2 = 4$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{3x} + e^{4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{3x}$$
$$y_3 = e^{4x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{3x} + \mathrm{e}^{4x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{3x} + \mathrm{e}^{4x} c_3$$

Verified OK.

9.24.1 Maple step by step solution

Let's solve

$y''' - 5y'' - 2y' + 24y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 5y_3(x) + 2y_2(x) - 24y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 5y_3(x) + 2y_2(x) - 24y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & 2 & 5 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & 2 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{4x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + e^{4x} c_3 \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(9c_3 e^{6x} + 16c_2 e^{5x} + 36c_1) e^{-2x}}{144}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-5*diff(y(x),x$2)-2*diff(y(x),x)+24*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_3 \mathrm{e}^{6x} + c_1 \mathrm{e}^{5x} + c_2\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 29

```
DSolve[y'''[x]-5*y''[x]-2*y'[x]+24*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(e^{5x}(c_3 e^x + c_2) + c_1\right)$$

9.25 problem 39

Internal problem ID [2111]
Internal file name [`OUTPUT/2111_Sunday_February_25_2024_06_49_51_AM_99366000/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 39.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 2y''' - 7y'' - 8y' + 12y = 0}$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 - 7\lambda^2 - 8\lambda + 12 = 0$$

The roots of the above equation are

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= -3 \\ \lambda_4 &= -2 \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-3x} c_3 + \mathrm{e}^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-2x} \\ y_2 &= \mathrm{e}^{x} \\ y_3 &= \mathrm{e}^{-3x} \\ y_4 &= \mathrm{e}^{2x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-3x} c_3 + \mathrm{e}^{2x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-3x} c_3 + \mathrm{e}^{2x} c_4$$

Verified OK.

9.25.1 Maple step by step solution

Let's solve

$y'''' + 2y''' - 7y'' - 8y' + 12y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -2y_4(x) + 7y_3(x) + 8y_2(x) - 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -2y_4(x) + 7y_3(x) + 8y_2(x) - 12y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & 8 & 7 & -2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & 8 & 7 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} -2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{2x} c_4 \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{e^{-3x}\left(\frac{27c_2 e^{x}}{8} - 27\,e^{4x}c_3 - \frac{27\,e^{5x}c_4}{8} + c_1\right)}{27}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)-7*diff(y(x),x$2)-8*diff(y(x),x)+12*y(x)=0,y(x), sings
```

$$y(x) = \left(c_1 \mathrm{e}^{5x} + c_4 \mathrm{e}^{4x} + c_2 \mathrm{e}^{x} + c_3\right) \mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 35

```
DSolve[y''''[x]+2*y'''[x]-7*y''[x]-8*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-3x}\left(c_2 e^x + e^{4x}(c_4 e^x + c_3) + c_1\right)$$

9.26 problem 40

Internal problem ID [2112]
Internal file name [`OUTPUT/2112_Sunday_February_25_2024_06_49_51_AM_97293734/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 40.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} - 3y'''' - 5y''' + 15y'' + 4y' - 12y = 0}$$

The characteristic equation is

$$\lambda^5 - 3\lambda^4 - 5\lambda^3 + 15\lambda^2 + 4\lambda - 12 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$
$$\lambda_4 = -2$$
$$\lambda_5 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + e^{2x} c_4 + e^{3x} c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{-2x}$$
$$y_3 = e^{x}$$
$$y_4 = e^{2x}$$
$$y_5 = e^{3x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + e^{2x} c_4 + e^{3x} c_5 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{x} + e^{2x} c_4 + e^{3x} c_5$$

Verified OK.

9.26.1 Maple step by step solution

Let's solve

$y^{(5)} - 3y'''' - 5y''' + 15y'' + 4y' - 12y = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = 3y_5(x) + 5y_4(x) - 15y_3(x) - 4y_2(x) + 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = 3y_5(x) + 5y_4(x) - 15y_3(x)$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 12 & -4 & -15 & 5 & 3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 12 & -4 & -15 & 5 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{2x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_5 = e^{3x} \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4 + c_5\vec{y}_5$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{2x} c_4 \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + e^{3x} c_5 \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{e^{-2x}\left(c_4 e^{4x} + \frac{16\, e^{5x} c_5}{81} + 16 c_2 e^{x} + 16 c_3 e^{3x} + c_1\right)}{16}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$5)-3*diff(y(x),x$4)-5*diff(y(x),x$3)+15*diff(y(x),x$2)+4*diff(y(x),x)-12*
```

$$y(x) = \left(c_1 e^{5x} + c_2 e^{4x} + c_5 e^{3x} + c_4 e^{x} + c_3\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
DSolve[y'''''[x]-3*y''''[x]-5*y'''[x]+15*y''[x]+4*y'[x]-12*y[x]==0,y[x],x,IncludeSingularSol
```

$$y(x) \to e^{-2x}\left(c_2 e^{x} + e^{3x}(e^{x}(c_5 e^{x} + c_4) + c_3) + c_1\right)$$

9.27 problem 41

Internal problem ID [2113]
Internal file name [`OUTPUT/2113_Sunday_February_25_2024_06_49_51_AM_52999642/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 41.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} + y'''' - 13y''' - 13y'' + 36y' + 36y = 0}$$

The characteristic equation is

$$\lambda^5 + \lambda^4 - 13\lambda^3 - 13\lambda^2 + 36\lambda + 36 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= 3 \\
\lambda_3 &= -3 \\
\lambda_4 &= -2 \\
\lambda_5 &= -1
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x} + e^{-3x} c_3 + e^{2x} c_4 + e^{3x} c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{-x} \\
y_2 &= e^{-2x} \\
y_3 &= e^{-3x} \\
y_4 &= e^{2x} \\
y_5 &= e^{3x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-x} + c_2 e^{-2x} + e^{-3x} c_3 + e^{2x} c_4 + e^{3x} c_5 \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{-x} + c_2 e^{-2x} + e^{-3x} c_3 + e^{2x} c_4 + e^{3x} c_5
$$

Verified OK.

9.27.1 Maple step by step solution

Let's solve

$y^{(5)} + y'''' - 13y''' - 13y'' + 36y' + 36y = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = -y_5(x) + 13y_4(x) + 13y_3(x) - 36y_2(x) - 36y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = -y_5(x) + 13y_4(x) + 13y_3(x$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -36 & -36 & 13 & 13 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -36 & -36 & 13 & 13 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \left[\begin{array}{c}\frac{1}{81} \\ -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1\end{array}\right]\right], \left[-2, \left[\begin{array}{c}\frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1\end{array}\right]\right], \left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1 \\ -1 \\ 1\end{array}\right]\right], \left[2, \left[\begin{array}{c}\frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]\right], \left[3, \left[\begin{array}{c}\frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-3, \left[\begin{array}{c}\frac{1}{81} \\ -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-3x} \cdot \left[\begin{array}{c}\frac{1}{81} \\ -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c}\frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{2x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_5 = e^{3x} \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4 + c_5 \vec{y}_5$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-3x} \cdot \begin{bmatrix} \frac{1}{81} \\ -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{-2x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + e^{2x} c_4 \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + e^{3x} c_5 \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(16\, e^{6x} c_5 + 81\, e^{5x} c_4 + 1296 c_3 e^{2x} + 81 c_2 e^{x} + 16 c_1\right) e^{-3x}}{1296}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$5)+diff(y(x),x$4)-13*diff(y(x),x$3)-13*diff(y(x),x$2)+36*diff(y(x),x)+36*
```

$$y(x) = \left(c_1 e^{6x} + c_2 e^{5x} + c_5 e^{2x} + c_3 e^{x} + c_4\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
DSolve[y'''''[x]+y''''[x]-13*y'''[x]-13*y''[x]+36*y'[x]+36*y[x]==0,y[x],x,IncludeSingularSol
```

$$y(x) \to e^{-3x}\left(e^{x}\left(c_3 e^{x} + e^{4x}(c_5 e^{x} + c_4) + c_2\right) + c_1\right)$$

9.28 problem 42

Internal problem ID [2114]
Internal file name [OUTPUT/2114_Sunday_February_25_2024_06_49_51_AM_62526698/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 42.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} + 3y'''' - 15y''' - 19y'' + 30y' = 0}$$

The characteristic equation is

$$\lambda^5 + 3\lambda^4 - 15\lambda^3 - 19\lambda^2 + 30\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 3$$
$$\lambda_4 = -5$$
$$\lambda_5 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{-2x} + c_3 e^x + e^{3x} c_4 + e^{-5x} c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= e^{-2x} \\
y_3 &= e^{x} \\
y_4 &= e^{3x} \\
y_5 &= e^{-5x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-2x} + c_3 e^{x} + e^{3x} c_4 + e^{-5x} c_5 \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^{-2x} + c_3 e^{x} + e^{3x} c_4 + e^{-5x} c_5$$

Verified OK.

9.28.1 Maple step by step solution

Let's solve

$y^{(5)} + 3y'''' - 15y''' - 19y'' + 30y' = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE
  $y_5'(x) = -3y_5(x) + 15y_4(x) + 19y_3(x) - 30y_2(x)$
  Convert linear ODE into a system of first order ODEs
  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = -3y_5(x) + 15y_4(x) + 19y_3($

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -30 & 19 & 15 & -3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -30 & 19 & 15 & -3 \end{bmatrix}$$

- Rewrite the system as
  $\vec{y}'(x) = A \cdot \vec{y}(x)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \left[\begin{array}{c} \frac{1}{625} \\ -\frac{1}{125} \\ \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{array}\right]\right], \left[-2, \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-5, \left[\begin{array}{c} \frac{1}{625} \\ -\frac{1}{125} \\ \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-5x} \cdot \left[\begin{array}{c} \frac{1}{625} \\ -\frac{1}{125} \\ \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_5 = e^{3x} \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4 + c_5 \vec{y}_5$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-5x} \cdot \begin{bmatrix} \frac{1}{625} \\ -\frac{1}{125} \\ \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix} + c_2 e^{-2x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + e^{x} c_4 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{3x} c_5 \cdot \begin{bmatrix} \frac{1}{81} \\ \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} c_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \left(e^{6x} c_4 + \frac{e^{8x} c_5}{81} + \frac{c_2 e^{3x}}{16} + c_3 e^{5x} + \frac{c_1}{625}\right) e^{-5x}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$5)+3*diff(y(x),x$4)-15*diff(y(x),x$3)-19*diff(y(x),x$2)+30*diff(y(x),x)=0
```

$$y(x) = \left(c_2 \mathrm{e}^{8x} + c_5 \mathrm{e}^{6x} + c_1 \mathrm{e}^{5x} + c_4 \mathrm{e}^{3x} + c_3\right) \mathrm{e}^{-5x}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 47

```
DSolve[y'''''[x]+3*y''''[x]-15*y'''[x]-19*y''[x]+30*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\frac{1}{5}c_1 e^{-5x} - \frac{1}{2}c_2 e^{-2x} + c_3 e^x + \frac{1}{3}c_4 e^{3x} + c_5$$

9.29 problem 43

Internal problem ID [2115]
Internal file name [`OUTPUT/2115_Sunday_February_25_2024_06_49_51_AM_99215506/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 43.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 3y'' - 4y = 0}$$

The characteristic equation is

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{x}$$
$$y_3 = \mathrm{e}^{2ix}$$
$$y_4 = \mathrm{e}^{-2ix}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + e^{2ix} c_3 + e^{-2ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + e^{2ix} c_3 + e^{-2ix} c_4$$

Verified OK.

9.29.1 Maple step by step solution

Let's solve

$y'''' + 3y'' - 4y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -3y_3(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -3y_3(x) + 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -3 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -3 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2x) - \mathrm{I}\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1\mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2\mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{c_3\sin(2x)}{8} - \frac{c_4\cos(2x)}{8} \\ -\frac{c_3\cos(2x)}{4} + \frac{c_4\sin(2x)}{4} \\ \frac{c_3\sin(2x)}{2} + \frac{c_4\cos(2x)}{2} \\ c_3\cos(2x) - c_4\sin(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x} - \frac{c_4\cos(2x)}{8} - \frac{c_3\sin(2x)}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)+3*diff(y(x),x$2)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} c_1 + c_2 \mathrm{e}^{x} + c_3 \sin(2x) + c_4 \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 34

```
DSolve[y''''[x]+3*y''[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^{-x} + c_4 e^{x} + c_1 \cos(2x) + c_2 \sin(2x)$$

9.30 problem 44

Internal problem ID [2116]
Internal file name [`OUTPUT/2116_Sunday_February_25_2024_06_49_51_AM_3545591/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 17, page 78
Problem number: 44.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} + 3y''' + 2y' = 0}$$

The characteristic equation is

$$\lambda^5 + 3\lambda^3 + 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i\sqrt{2}$$
$$\lambda_3 = -i\sqrt{2}$$
$$\lambda_4 = i$$
$$\lambda_5 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + \mathrm{e}^{-i\sqrt{2}\,x} c_2 + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{i\sqrt{2}\,x} c_4 + \mathrm{e}^{ix} c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{-i\sqrt{2}\,x}$$
$$y_3 = e^{-ix}$$
$$y_4 = e^{i\sqrt{2}\,x}$$
$$y_5 = e^{ix}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^{-i\sqrt{2}\,x}c_2 + e^{-ix}c_3 + e^{i\sqrt{2}\,x}c_4 + e^{ix}c_5 \tag{1}$$

Verification of solutions

$$y = c_1 + e^{-i\sqrt{2}\,x}c_2 + e^{-ix}c_3 + e^{i\sqrt{2}\,x}c_4 + e^{ix}c_5$$

Verified OK.

9.30.1 Maple step by step solution

Let's solve

$y^{(5)} + 3y''' + 2y' = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = -3y_4(x) - 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = -3y_4(x) - 2y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & -3 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & -3 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-I, \begin{bmatrix} 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix}\right], \left[I, \begin{bmatrix} 1 \\ I \\ -1 \\ -I \\ 1 \end{bmatrix}\right], \left[-I\sqrt{2}, \begin{bmatrix} \frac{1}{4} \\ -\frac{I}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[I\sqrt{2}, \begin{bmatrix} \frac{1}{4} \\ \frac{I}{4}\sqrt{2} \\ -\frac{1}{2} \\ -\frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \cos(x) - I\sin(x) \\ -I(\cos(x) - I\sin(x)) \\ -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} \cos(x) \\ -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{2}, \left[\begin{array}{c}\frac{1}{4}\\ -\frac{\mathrm{I}}{4}\sqrt{2}\\ -\frac{1}{2}\\ \frac{\mathrm{I}}{2}\sqrt{2}\\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}\sqrt{2}\,x}\cdot\left[\begin{array}{c}\frac{1}{4}\\ -\frac{\mathrm{I}}{4}\sqrt{2}\\ -\frac{1}{2}\\ \frac{\mathrm{I}}{2}\sqrt{2}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)\right)\cdot\left[\begin{array}{c}\frac{1}{4}\\ -\frac{\mathrm{I}}{4}\sqrt{2}\\ -\frac{1}{2}\\ \frac{\mathrm{I}}{2}\sqrt{2}\\ 1\end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c}\frac{\cos\left(\sqrt{2}\,x\right)}{4}-\frac{\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{4}\\ -\frac{\mathrm{I}}{4}\left(\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2}\\ -\frac{\cos\left(\sqrt{2}\,x\right)}{2}+\frac{\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{2}\\ \frac{\mathrm{I}}{2}\left(\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2}\\ \cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_4(x) = \left[\begin{array}{c} \frac{\cos\left(\sqrt{2}\, x\right)}{4} \\ -\frac{\sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{4} \\ -\frac{\cos\left(\sqrt{2}\, x\right)}{2} \\ \frac{\sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2} \\ \cos\left(\sqrt{2}\, x\right) \end{array}\right], \vec{y}_5(x) = \left[\begin{array}{c} -\frac{\sin\left(\sqrt{2}\, x\right)}{4} \\ -\frac{\sqrt{2}\, \cos\left(\sqrt{2}\, x\right)}{4} \\ \frac{\sin\left(\sqrt{2}\, x\right)}{2} \\ \frac{\sqrt{2}\, \cos\left(\sqrt{2}\, x\right)}{2} \\ -\sin\left(\sqrt{2}\, x\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x) + c_5 \vec{y}_5(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \left[\begin{array}{c} -\frac{c_5 \sin\left(\sqrt{2}\, x\right)}{4} + \frac{c_4 \cos\left(\sqrt{2}\, x\right)}{4} - c_3 \sin(x) + c_2 \cos(x) + c_1 \\ -\frac{c_5 \sqrt{2}\, \cos\left(\sqrt{2}\, x\right)}{4} - \frac{c_4 \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{4} - c_3 \cos(x) - c_2 \sin(x) \\ \frac{c_5 \sin\left(\sqrt{2}\, x\right)}{2} - \frac{c_4 \cos\left(\sqrt{2}\, x\right)}{2} + c_3 \sin(x) - c_2 \cos(x) \\ \frac{c_5 \sqrt{2}\, \cos\left(\sqrt{2}\, x\right)}{2} + \frac{c_4 \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2} + c_3 \cos(x) + c_2 \sin(x) \\ -c_5 \sin\left(\sqrt{2}\, x\right) + c_4 \cos\left(\sqrt{2}\, x\right) - c_3 \sin(x) + c_2 \cos(x) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_5 \sin\left(\sqrt{2}\, x\right)}{4} + \frac{c_4 \cos\left(\sqrt{2}\, x\right)}{4} - c_3 \sin(x) + c_2 \cos(x) + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$5)+3*diff(y(x),x$3)+2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 \sin\left(\sqrt{2}\,x\right) + c_3 \cos\left(\sqrt{2}\,x\right) + c_4 \sin(x) + c_5 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 52

```
DSolve[y'''''[x]+3*y'''[x]+2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -c_4 \cos(x) - \frac{c_2 \cos\left(\sqrt{2}x\right)}{\sqrt{2}} + c_3 \sin(x) + \frac{c_1 \sin\left(\sqrt{2}x\right)}{\sqrt{2}} + c_5$$

10 Exercise 18, page 82

10.1 problem 1

Internal problem ID [2117]
Internal file name [`OUTPUT/2117_Sunday_February_25_2024_06_49_52_AM_63034211/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + y = 0}$$

10.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x e^x \tag{1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^x + c_2 x e^x \tag{1}$$

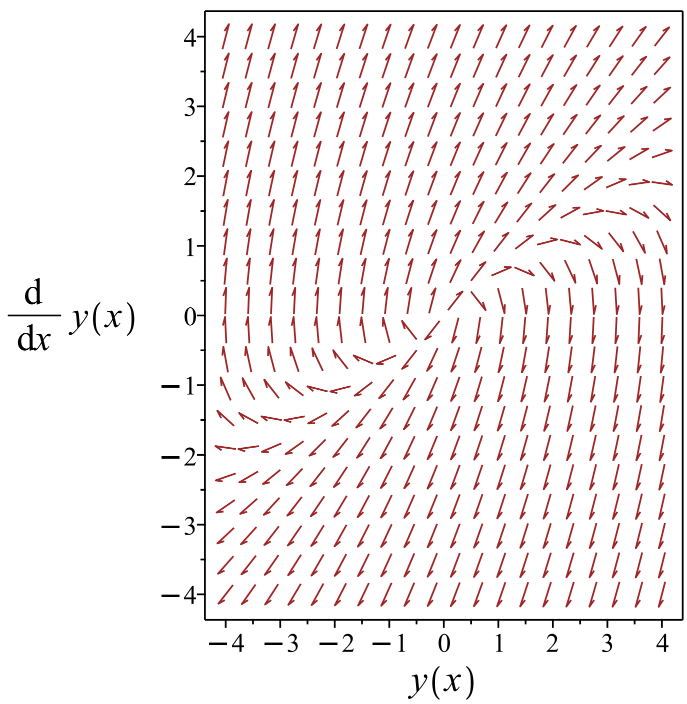


Figure 525: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^x + c_2 x e^x$$

Verified OK.

10.1.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2}+p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2\, dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{-x} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \tag{1}$$

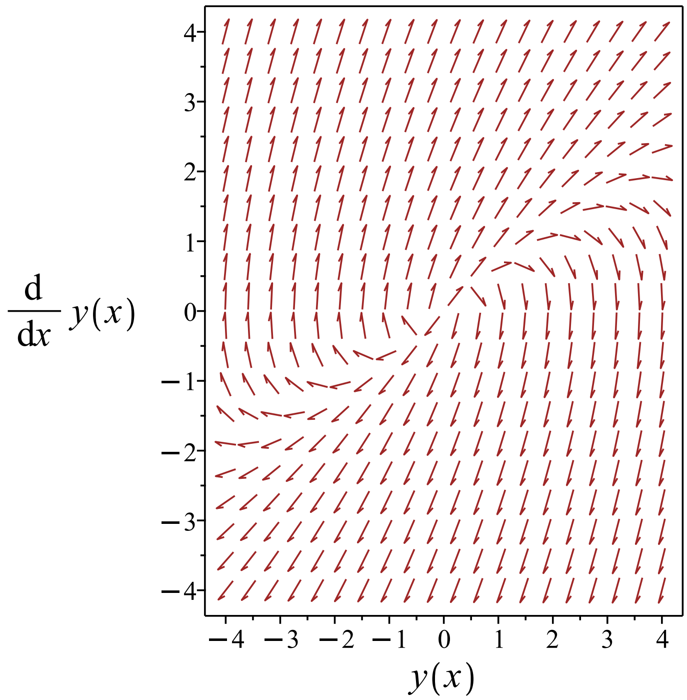


Figure 526: Slope field plot

Verification of solutions

$$y = c_1 x\,e^x + c_2 e^x$$

Verified OK.

10.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 346: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x \tag{1}$$

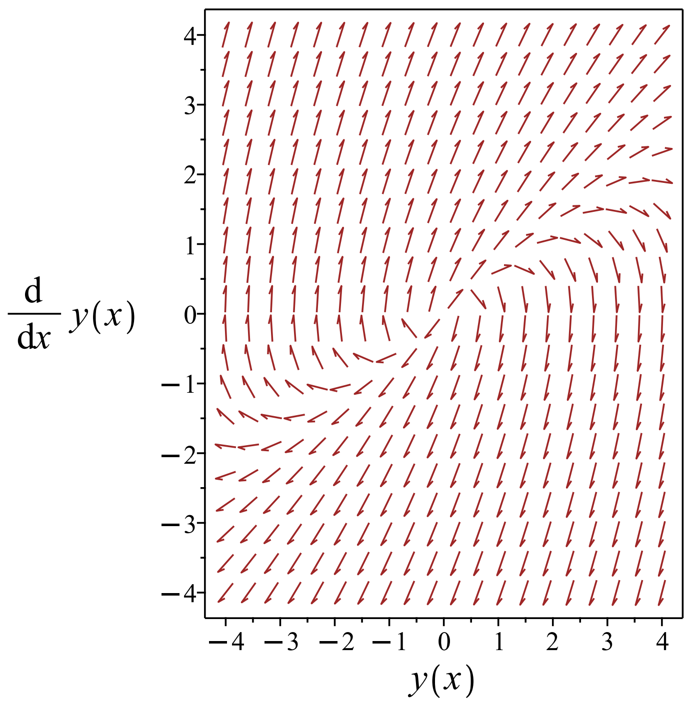


Figure 527: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x$$

Verified OK.

10.1.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 x\,e^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 16

```
DSolve[y''[x]-2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2x + c_1)$$

10.2 problem 2

Internal problem ID [2118]
Internal file name [`OUTPUT/2118_Sunday_February_25_2024_06_49_52_AM_95497789/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = 0}$$

10.2.1 Solving as second order ode quadrature ode

Integrating twice gives the solution

$$y = c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

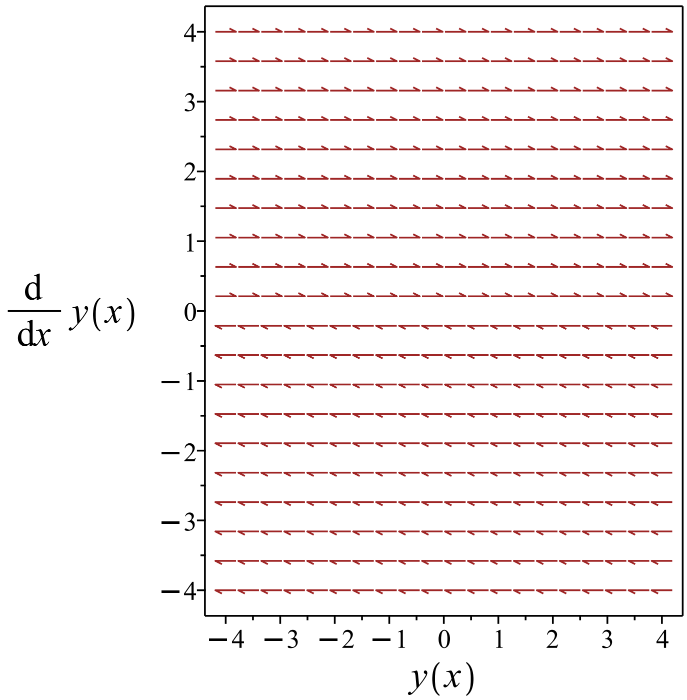


Figure 528: Slope field plot

Verification of solutions

$$y = c_1x + c_2$$

Verified OK.

10.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)}$$
$$= 0$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_2 x + c_1 \tag{1}$$

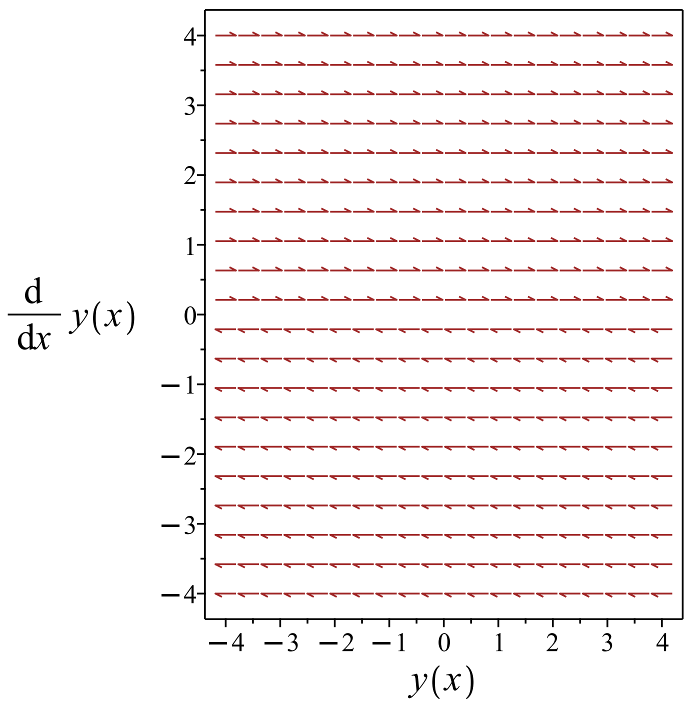


Figure 529: Slope field plot

<u>Verification of solutions</u>

$$y = c_2 x + c_1$$

Verified OK.

10.2.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' = 0$$

Integrating the above w.r.t x gives

$$\int y'y''dx = 0$$
$$\frac{y'^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{c_1}\,\sqrt{2} \tag{1}$$
$$y' = -\sqrt{c_1}\,\sqrt{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$y = \int \sqrt{c_1}\,\sqrt{2}\,\mathrm{d}x$$
$$= x\sqrt{c_1}\,\sqrt{2} + c_2$$

Solving equation (2)

Integrating both sides gives

$$y = \int -\sqrt{c_1}\,\sqrt{2}\,\mathrm{d}x$$
$$= -x\sqrt{c_1}\,\sqrt{2} + c_3$$

Summary
The solution(s) found are the following

$$y = x\sqrt{c_1}\,\sqrt{2} + c_2 \tag{1}$$
$$y = -x\sqrt{c_1}\,\sqrt{2} + c_3 \tag{2}$$

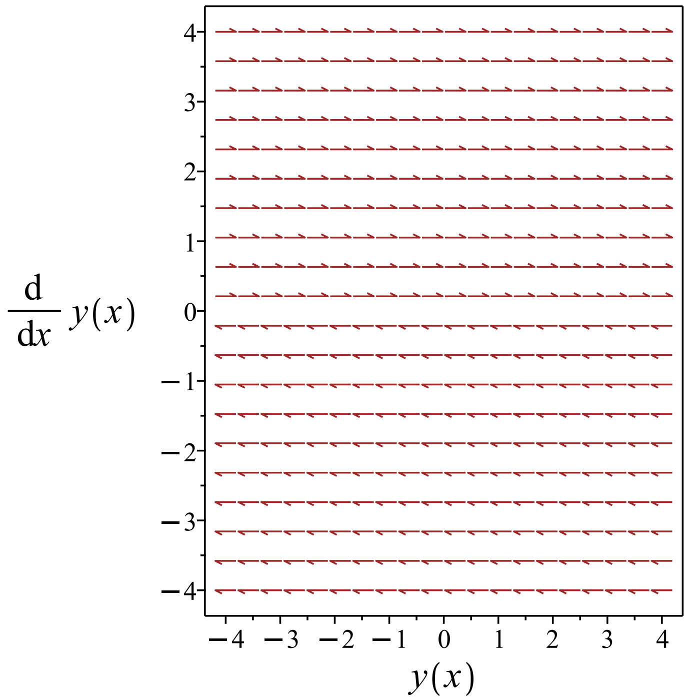


Figure 530: Slope field plot

Verification of solutions

$$y = x\sqrt{c_1}\,\sqrt{2} + c_2$$

Verified OK.

$$y = -x\sqrt{c_1}\,\sqrt{2} + c_3$$

Verified OK.

10.2.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = 0$$
$$y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int c_1 \, dx$$
$$= c_1 x + c_2$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

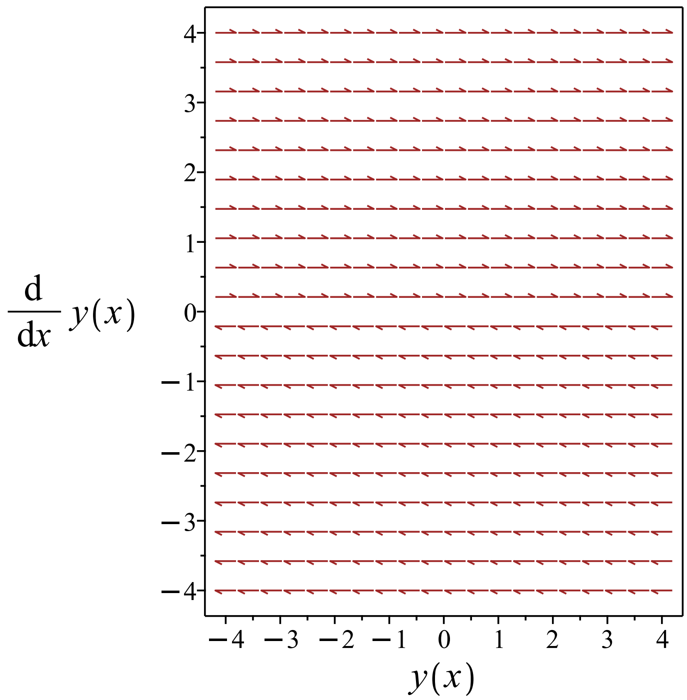


Figure 531: Slope field plot

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

10.2.5 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$p(x) = \int 0 \, dx$$
$$= c_1$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1$$

Integrating both sides gives

$$y = \int c_1 \, dx$$
$$= c_1 x + c_2$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 \tag{1}$$

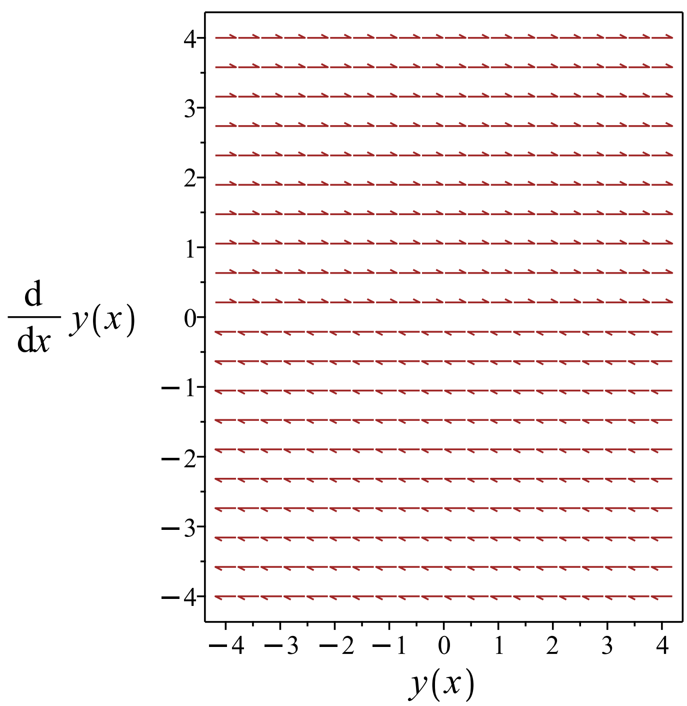


Figure 532: Slope field plot

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

10.2.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 348: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= 1 \int \frac{1}{1}\, dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 \tag{1}$$

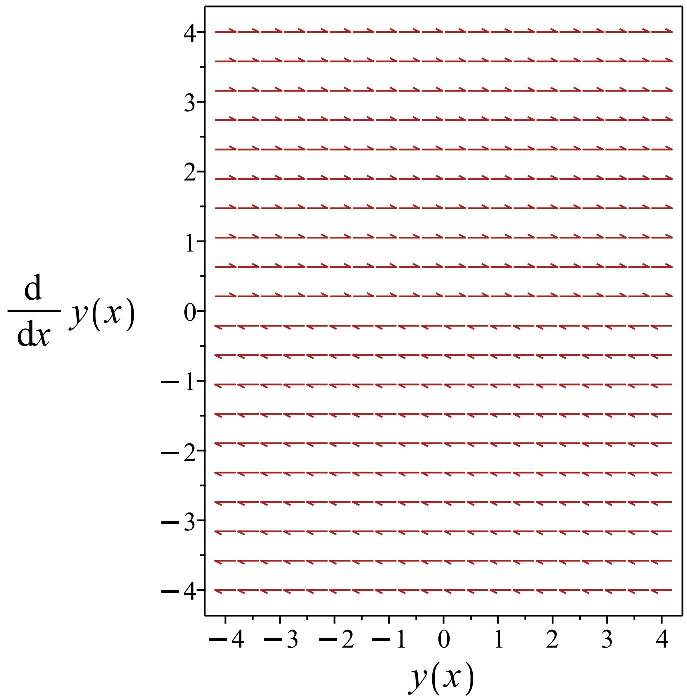


Figure 533: Slope field plot

Verification of solutions

$$y = c_2x + c_1$$

Verified OK.

10.2.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1\\
q(x) &= 0\\
r(x) &= 0\\
s(x) &= 0
\end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y' = c_1$$

We now have a first order ode to solve which is

$$y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \, \mathrm{d}x \\ &= c_1x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1x + c_2 \tag{1}$$

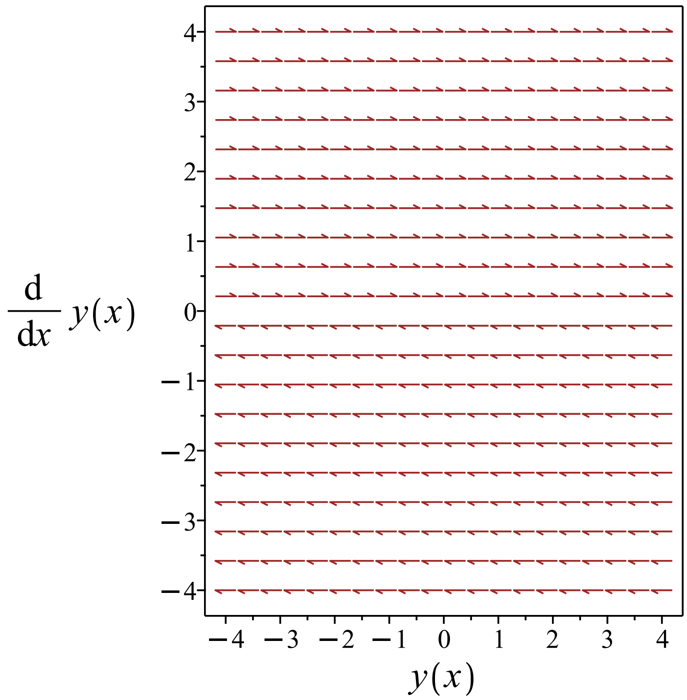


Figure 534: Slope field plot

Verification of solutions

$$y = c_1 x + c_2$$

Verified OK.

10.2.8 Maple step by step solution

Let's solve

$y'' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

$r = 0$

- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_2 x + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x + c_1$$

10.3 problem 3

Internal problem ID [2119]
Internal file name [OUTPUT/2119_Sunday_February_25_2024_06_49_52_AM_47823928/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{2y''' + y'' - 4y' - 3y = 0}$$

The characteristic equation is

$$2\lambda^3 + \lambda^2 - 4\lambda - 3 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{3}{2}$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{\frac{3x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = x\,\mathrm{e}^{-x}$$
$$y_3 = \mathrm{e}^{\frac{3x}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{\frac{3x}{2}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{\frac{3x}{2}} c_3$$

Verified OK.

10.3.1 Maple step by step solution

Let's solve

$2y''' + y'' - 4y' - 3y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{y''}{2} + 2y' + \frac{3y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{y''}{2} - 2y' - \frac{3y}{2} = 0$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -\frac{y_3(x)}{2} + 2y_2(x) + \frac{3y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -\frac{y_3(x)}{2} + 2y_2(x) + \frac{3y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 2 & -\frac{1}{2} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 2 & -\frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[\frac{3}{2}, \begin{bmatrix} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\vec{y}_1(x) = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A\cdot\overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda\cdot I)\cdot\overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 2 & -\frac{1}{2} \end{array}\right] - (-1)\cdot\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p} = \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -1

$$\overrightarrow{y}_2(x) = \mathrm{e}^{-x}\cdot\left(x\cdot\left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right] + \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

$$\left[\frac{3}{2}, \begin{bmatrix} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{\frac{3x}{2}} \cdot \begin{bmatrix} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + e^{\frac{3x}{2}} c_3 \cdot \begin{bmatrix} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(4c_3 e^{\frac{5x}{2}} + 9c_2 x + 9c_1 + 9c_2\right) e^{-x}}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(2*diff(y(x),x$3)+diff(y(x),x$2)-4*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{\frac{5x}{2}} + c_3 x + c_2\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 28

```
DSolve[2*y'''[x]+y''[x]-4*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_1 e^{5x/2} + c_3 x + c_2\right)$$

10.4 problem 4

Internal problem ID [2120]
Internal file name [OUTPUT/2120_Sunday_February_25_2024_06_49_53_AM_1079900/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 4.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' + 3y' - y = 0}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 x e^x + x^2 e^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x e^x$$
$$y_3 = x^2 e^x$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x} + x^2 \mathrm{e}^{x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x} + x^2 \mathrm{e}^{x} c_3$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+3*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x} \left(c_3 x^2 + c_2 x + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 21

```
DSolve[y'''[x]-3*y''[x]+3*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x (x(c_3 x + c_2) + c_1)$$

10.5 problem 5

Internal problem ID [2121]
Internal file name [`OUTPUT/2121_Sunday_February_25_2024_06_49_53_AM_37930531/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 5.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _quadrature]]
```

$$\boxed{y'''' = 0}$$

The characteristic equation is

$$\lambda^4 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_4x^3 + c_3x^2 + c_2x + c_1$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = x^2$$
$$y_4 = x^3$$

Summary
The solution(s) found are the following

$$y = c_4x^3 + c_3x^2 + c_2x + c_1 \tag{1}$$

Verification of solutions

$$y = c_4x^3 + c_3x^2 + c_2x + c_1$$

Verified OK.

10.5.1 Maple step by step solution

Let's solve

$y'''' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ○ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 0$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 0]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$4)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{6}c_1x^3 + \frac{1}{2}c_2x^2 + c_3x + c_4$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 22

```
DSolve[y''''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x(c_4x + c_3) + c_2) + c_1$$

10.6 problem 6

Internal problem ID [2122]
Internal file name [`OUTPUT/2122_Sunday_February_25_2024_06_49_53_AM_5763094/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 6.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' - y' - y = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + c_3 \mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = x\,\mathrm{e}^{-x}$$
$$y_3 = \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + x e^{-x} c_2 + c_3 e^{x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + x e^{-x} c_2 + c_3 e^{x}$$

Verified OK.

10.6.1 Maple step by step solution

Let's solve

$y''' + y'' - y' - y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_3(x) + y_2(x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) + y_2(x) + y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\vec{y}_1(x) = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\vec{y}_2(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\mathrm{e}^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$

- Simplify equation

  $\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$

- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

  $(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

  $$\left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 1 & -1\end{bmatrix}-(-1)\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

  $$\vec{p}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue -1

  $$\vec{y}_2(x)=\mathrm{e}^{-x}\cdot\left(x\cdot\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\right)$$

- Consider eigenpair

  $$\left[1,\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_3=\mathrm{e}^{x}\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = (c_2(x+1) + c_1) e^{-x} + c_3 e^{x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_3 x + c_2) e^{-x} + e^{x} c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y'''[x]+y''[x]-y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} \left(c_2 x + c_3 e^{2x} + c_1\right)$$

10.7 problem 7

Internal problem ID [2123]
Internal file name [`OUTPUT/2123_Sunday_February_25_2024_06_49_53_AM_50343556/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 7.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{4y''' - 3y' + y = 0}$$

The characteristic equation is

$$4\lambda^3 - 3\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = \frac{1}{2}$$
$$\lambda_3 = \frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{\frac{x}{2}} + x\, e^{\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{\frac{x}{2}}$$
$$y_3 = e^{\frac{x}{2}} x$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{\frac{x}{2}} + x\,\mathrm{e}^{\frac{x}{2}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{\frac{x}{2}} + x\,\mathrm{e}^{\frac{x}{2}} c_3$$

Verified OK.

10.7.1 Maple step by step solution

Let's solve

$4y''' - 3y' + y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{3y'}{4} - \frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y'}{4} + \frac{y}{4} = 0$

☐ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

  $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

  $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

  $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = \frac{3y_2(x)}{4} - \frac{y_1(x)}{4}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{3y_2(x)}{4} - \frac{y_1(x)}{4}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_2(x) = \mathrm{e}^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = \frac{1}{2}$ is the eigenvalue, an

$$\vec{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_3(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue $\frac{1}{2}$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_3(x) = e^{\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix} \right)$$

- First component of the vector is the solution to the ODE

$$y = 4\left(\left((-2 + x) c_3 + c_2\right) e^{\frac{3x}{2}} + \frac{c_1}{4}\right) e^{-x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(4*diff(y(x),x$3)-3*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}\left((c_3 x + c_2) e^{\frac{3x}{2}} + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 29

```
DSolve[4*y'''[x]-3*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(e^{3x/2}(c_2x + c_1) + c_3\right)$$

10.8 problem 8

Internal problem ID [2124]
Internal file name [`OUTPUT/2124_Monday_February_26_2024_09_17_44_AM_67136864/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 8.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{4y^{(5)} - 3y''' - y'' = 0}$$

The characteristic equation is

$$4\lambda^5 - 3\lambda^3 - \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 1$$
$$\lambda_4 = -\frac{1}{2}$$
$$\lambda_5 = -\frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + c_3 \mathrm{e}^{x} + \mathrm{e}^{-\frac{x}{2}} c_4 + x\,\mathrm{e}^{-\frac{x}{2}} c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= \mathrm{e}^{x} \\
y_4 &= \mathrm{e}^{-\frac{x}{2}} \\
y_5 &= x\,\mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 x + c_1 + c_3 \mathrm{e}^{x} + \mathrm{e}^{-\frac{x}{2}} c_4 + x\,\mathrm{e}^{-\frac{x}{2}} c_5 \tag{1}
$$

Verification of solutions

$$
y = c_2 x + c_1 + c_3 \mathrm{e}^{x} + \mathrm{e}^{-\frac{x}{2}} c_4 + x\,\mathrm{e}^{-\frac{x}{2}} c_5
$$

Verified OK.

10.8.1 Maple step by step solution

Let's solve

$4y^{(5)} - 3y''' - y'' = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

- Isolate 5th derivative

  $y^{(5)} = \frac{3y'''}{4} + \frac{y''}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y^{(5)} - \frac{3y'''}{4} - \frac{y''}{4} = 0$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Define new variable $y_5(x)$

  $y_5(x) = y''''$
- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = \frac{3y_4(x)}{4} + \frac{y_3(x)}{4}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = \frac{3y_4(x)}{4} + \frac{y_3(x)}{4}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2}, \left[\begin{array}{c} 16 \\ -8 \\ 4 \\ -2 \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-\frac{1}{2}, \left[\begin{array}{c} 16 \\ -8 \\ 4 \\ -2 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue $-\frac{1}{2}$

$$\vec{y}_1(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} 16 \\ -8 \\ 4 \\ -2 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -\frac{1}{2}$ is the eigenvalue, a

$$\vec{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue $-\frac{1}{2}$

$$\left(\left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{array}\right] - -\frac{1}{2} \cdot \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} 16 \\ -8 \\ 4 \\ -2 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \left[\begin{array}{c} 32 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue $-\frac{1}{2}$

$$\overrightarrow{y}_2(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \left(x \cdot \left[\begin{array}{c} 16 \\ -8 \\ 4 \\ -2 \\ 1 \end{array}\right] + \left[\begin{array}{c} 32 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_5 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs
  $$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3 + c_4\vec{y}_4 + c_5\vec{y}_5$$
- Substitute solutions into the general solution
  $$\vec{y} = c_1 e^{-\frac{x}{2}} \cdot \begin{bmatrix} 16 \\ -8 \\ 4 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{-\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} 16 \\ -8 \\ 4 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 32 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + c_5 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
- First component of the vector is the solution to the ODE
  $$y = 16((2+x)\, c_2 + c_1)\, e^{-\frac{x}{2}} + c_5 e^{x} + c_3$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(4*diff(y(x),x$5)-3*diff(y(x),x$3)-diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = (c_5 x + c_4)\, e^{-\frac{x}{2}} + c_2 x + c_3 e^{x} + c_1$$

✓ Solution by Mathematica

Time used: 0.196 (sec). Leaf size: 36

```
DSolve[4*y'''''[x]-3*y'''[x]-y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 4e^{-x/2}(c_2(x+4)+c_1)+c_3e^x+c_5x+c_4$$

10.9 problem 9

Internal problem ID [2125]
Internal file name [`OUTPUT/2125_Monday_February_26_2024_09_17_45_AM_5854370/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 7y'' + 16y' - 12y = 0}$$

The characteristic equation is

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

The roots of the above equation are

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + c_2 x\, e^{2x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = x\, e^{2x}$$
$$y_3 = e^{3x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 x\, e^{2x} + c_3 e^{3x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} + c_2 x\, e^{2x} + c_3 e^{3x}$$

Verified OK.

10.9.1 Maple step by step solution

Let's solve

$y''' - 7y'' + 16y' - 12y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ∘ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 7y_3(x) - 16y_2(x) + 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 7y_3(x) - 16y_2(x) + 12y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & -16 & 7 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & -16 & 7 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

$$\vec{y}_1(x) = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$$\vec{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\mathrm{e}^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$

- Simplify equation

  $\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$

- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

  $(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

  $$\left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 12 & -16 & 7\end{bmatrix}-2\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

  $$\vec{p}=\begin{bmatrix}-\frac{1}{8}\\ 0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue 2

  $$\vec{y}_2(x)=\mathrm{e}^{2x}\cdot\left(x\cdot\begin{bmatrix}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{bmatrix}+\begin{bmatrix}-\frac{1}{8}\\ 0\\ 0\end{bmatrix}\right)$$

- Consider eigenpair

  $$\left[3,\begin{bmatrix}\frac{1}{9}\\ \frac{1}{3}\\ 1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{((2x-1)c_2+2c_1)e^{2x}}{8} + \frac{c_3 e^{3x}}{9}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-7*diff(y(x),x$2)+16*diff(y(x),x)-12*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_3 x + c_2) e^{2x} + c_1 e^{3x}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 24

```
DSolve[y'''[x]-7*y''[x]+16*y'[x]-12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(c_2 x + c_3 e^x + c_1)$$

10.10 problem 10

Internal problem ID [2126]
Internal file name [`OUTPUT/2126_Monday_February_26_2024_09_17_45_AM_53084472/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 10.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{4y''' - 8y'' + 5y' - y = 0}$$

The characteristic equation is

$$4\lambda^3 - 8\lambda^2 + 5\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = \frac{1}{2}$$
$$\lambda_3 = \frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{\frac{x}{2}} + x\,\mathrm{e}^{\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{x}$$
$$y_2 = \mathrm{e}^{\frac{x}{2}}$$
$$y_3 = \mathrm{e}^{\frac{x}{2}} x$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{\frac{x}{2}} + x\, e^{\frac{x}{2}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{\frac{x}{2}} + x\, e^{\frac{x}{2}} c_3$$

Verified OK.

10.10.1 Maple step by step solution

Let's solve

$4y''' - 8y'' + 5y' - y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = 2y'' - \frac{5y'}{4} + \frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - 2y'' + \frac{5y'}{4} - \frac{y}{4} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 2y_3(x) - \frac{5y_2(x)}{4} + \frac{y_1(x)}{4}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2y_3(x) - \frac{5y_2(x)}{4} + \frac{y_1(x)}{4}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & -\frac{5}{4} & 2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & -\frac{5}{4} & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_1(x) = e^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = \frac{1}{2}$ is the eigenvalue, an

$\vec{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue $\frac{1}{2}$

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & -\frac{5}{4} & 2 \end{array}\right] - \frac{1}{2} \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \vec{p} = \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p} = \left[\begin{array}{c} -8 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_2(x) = \mathrm{e}^{\frac{x}{2}} \cdot \left(x \cdot \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right] + \left[\begin{array}{c} -8 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix}\right) + c_3 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = ((4x - 8) c_2 + 4c_1) e^{\frac{x}{2}} + c_3 e^x$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(4*diff(y(x),x$3)-8*diff(y(x),x$2)+5*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_3 x + c_2) e^{\frac{x}{2}} + e^x c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 30

```
DSolve[4*y'''[x]-8*y''[x]+5*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x/2}\left(c_2 x + c_3 e^{x/2} + c_1\right)$$

10.11 problem 11

Internal problem ID [2127]
Internal file name [`OUTPUT/2127_Monday_February_26_2024_09_17_45_AM_71406617/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 11.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - y = 0}$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{-ix} c_3 + e^{ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{-ix}$$
$$y_4 = e^{ix}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + e^{-ix} c_3 + e^{ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + e^{-ix} c_3 + e^{ix} c_4$$

Verified OK.

10.11.1 Maple step by step solution

Let's solve

$y'''' - y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -c_3\sin(x) - c_4\cos(x) \\ -c_3\cos(x) + c_4\sin(x) \\ c_3\sin(x) + c_4\cos(x) \\ c_3\cos(x) - c_4\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - c_4\cos(x) - c_3\sin(x)$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$4)-y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + c_2e^x + c_3 \sin(x) + c_4 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 30

```
DSolve[y''''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1e^x + c_3e^{-x} + c_2 \cos(x) + c_4 \sin(x)$$

10.12 problem 12

Internal problem ID [2128]
Internal file name [OUTPUT/2128_Monday_February_26_2024_09_17_45_AM_67772220/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 12.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 8y = 0}$$

The characteristic equation is

$$\lambda^3 - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = i\sqrt{3} - 1$$
$$\lambda_3 = -i\sqrt{3} - 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + e^{\left(-i\sqrt{3}-1\right)x} c_2 + e^{\left(i\sqrt{3}-1\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = e^{\left(-i\sqrt{3}-1\right)x}$$
$$y_3 = e^{\left(i\sqrt{3}-1\right)x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + e^{\left(-i\sqrt{3}-1\right)x} c_2 + e^{\left(i\sqrt{3}-1\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} + e^{\left(-i\sqrt{3}-1\right)x} c_2 + e^{\left(i\sqrt{3}-1\right)x} c_3$$

Verified OK.

10.12.1 Maple step by step solution

Let's solve

$y''' - 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 8y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{3}-1, \begin{bmatrix} \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^2} \\ \frac{1}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{3}-1, \begin{bmatrix} \frac{1}{\left(\mathrm{I}\sqrt{3}-1\right)^2} \\ \frac{1}{\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{3}-1, \begin{bmatrix} \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^2} \\ \frac{1}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-I\sqrt{3}-1\right)x}\cdot\left[\begin{array}{c}\frac{1}{\left(-I\sqrt{3}-1\right)^{2}}\\ \frac{1}{-I\sqrt{3}-1}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-x}\cdot\left(\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)\right)\cdot\left[\begin{array}{c}\frac{1}{\left(-I\sqrt{3}-1\right)^{2}}\\ \frac{1}{-I\sqrt{3}-1}\\ 1\end{array}\right]$$

- Simplify expression

$$e^{-x}\cdot\left[\begin{array}{c}\frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{\left(-I\sqrt{3}-1\right)^{2}}\\ \frac{\cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)}{-I\sqrt{3}-1}\\ \cos\left(\sqrt{3}\,x\right)-I\sin\left(\sqrt{3}\,x\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,x\right)}{8}-\frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{8}\\ -\frac{\cos\left(\sqrt{3}\,x\right)}{4}+\frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{4}\\ \cos\left(\sqrt{3}\,x\right)\end{array}\right],\vec{y}_{3}(x)=e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{8}+\frac{\sin\left(\sqrt{3}\,x\right)}{8}\\ \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{4}+\frac{\sin\left(\sqrt{3}\,x\right)}{4}\\ -\sin\left(\sqrt{3}\,x\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_{1}e^{2x}\cdot\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]+c_{2}e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,x\right)}{8}-\frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{8}\\ -\frac{\cos\left(\sqrt{3}\,x\right)}{4}+\frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{4}\\ \cos\left(\sqrt{3}\,x\right)\end{array}\right]+c_{3}e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{8}+\frac{\sin\left(\sqrt{3}\,x\right)}{8}\\ \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{4}+\frac{\sin\left(\sqrt{3}\,x\right)}{4}\\ -\sin\left(\sqrt{3}\,x\right)\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=-\frac{e^{-x}\left(c_{3}\sqrt{3}+c_{2}\right)\cos\left(\sqrt{3}\,x\right)}{8}-\frac{e^{-x}\left(\sqrt{3}\,c_{2}-c_{3}\right)\sin\left(\sqrt{3}\,x\right)}{8}+\frac{c_{1}e^{2x}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$3)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + c_2 e^{-x} \sin\left(\sqrt{3}\, x\right) + c_3 e^{-x} \cos\left(\sqrt{3}\, x\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
DSolve[y'''[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_1 e^{3x} + c_2 \cos\left(\sqrt{3}x\right) + c_3 \sin\left(\sqrt{3}x\right)\right)$$

10.13 problem 13

Internal problem ID [2129]
Internal file name [OUTPUT/2129_Monday_February_26_2024_09_17_45_AM_66620466/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 3y = 0}$$

10.13.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + 3 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 3$ into the above gives

$$\lambda_{1,2} = \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(3)}$$
$$= 1 \pm i\sqrt{2}$$

Hence

$$\lambda_1 = 1 + i\sqrt{2}$$
$$\lambda_2 = 1 - i\sqrt{2}$$

Which simplifies to

$$\lambda_1 = 1 + i\sqrt{2}$$
$$\lambda_2 = 1 - i\sqrt{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)$$

Summary

The solution(s) found are the following

$$y = e^{x}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) \tag{1}$$

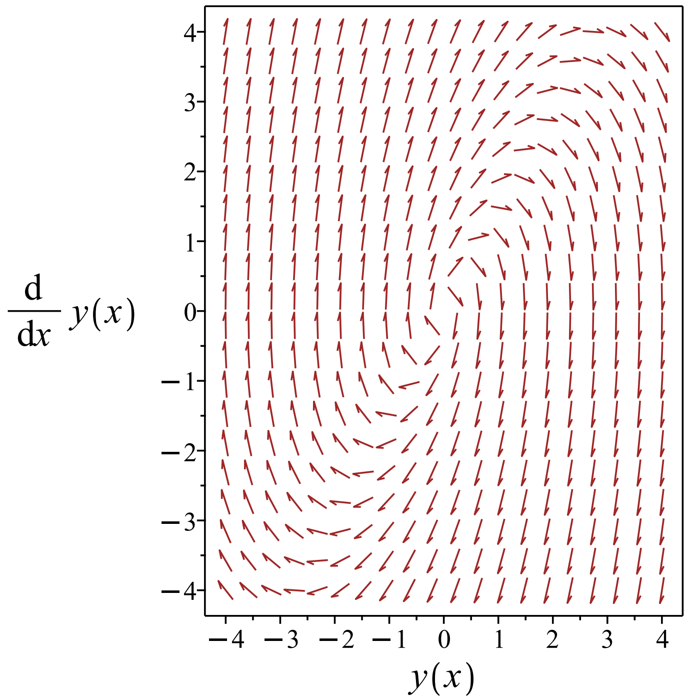


Figure 535: Slope field plot

Verification of solutions

$$y = e^{x}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)$$

Verified OK.

10.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 359: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{2}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\, dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x} \cos\left(\sqrt{2}\, x\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^x \cos\left(\sqrt{2}\,x\right)\right) + c_2\left(e^x \cos\left(\sqrt{2}\,x\right)\left(\frac{\sqrt{2}\,\tan\left(\sqrt{2}\,x\right)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^x \cos\left(\sqrt{2}\,x\right) + \frac{c_2 \sin\left(\sqrt{2}\,x\right) e^x \sqrt{2}}{2} \tag{1}
$$

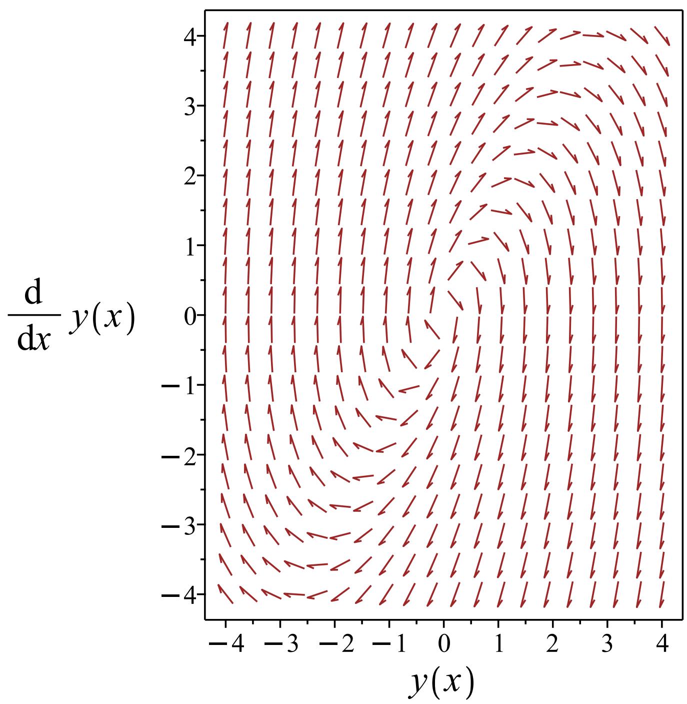


Figure 536: Slope field plot

Verification of solutions

$$
y = c_1 e^x \cos\left(\sqrt{2}\,x\right) + \frac{c_2 \sin\left(\sqrt{2}\,x\right) e^x \sqrt{2}}{2}
$$

Verified OK.

10.13.3 Maple step by step solution

Let's solve

$y'' - 2y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 3 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

  $r = \left(1 - \mathrm{I}\sqrt{2}, 1 + \mathrm{I}\sqrt{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x \cos\left(\sqrt{2}\, x\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^x \sin\left(\sqrt{2}\, x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x \cos\left(\sqrt{2}\, x\right) + c_2 \mathrm{e}^x \sin\left(\sqrt{2}\, x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x} \left(c_1 \sin\left(\sqrt{2}\, x\right) + c_2 \cos\left(\sqrt{2}\, x\right)\right)$$

✓ Solution by Mathematica

Time used: 0.13 (sec). Leaf size: 32

```
DSolve[y''[x]-2*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x} \left(c_2 \cos\left(\sqrt{2}x\right) + c_1 \sin\left(\sqrt{2}x\right)\right)$$

10.14 problem 14

Internal problem ID [2130]
Internal file name [`OUTPUT/2130_Monday_February_26_2024_09_17_46_AM_14734510/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 14.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + y'' - 20y = 0}$$

The characteristic equation is

$$\lambda^4 + \lambda^2 - 20 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -2 \\
\lambda_3 &= i\sqrt{5} \\
\lambda_4 &= -i\sqrt{5}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{ix\sqrt{5}} c_3 + \mathrm{e}^{-ix\sqrt{5}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= \mathrm{e}^{2x} \\
y_3 &= \mathrm{e}^{ix\sqrt{5}} \\
y_4 &= \mathrm{e}^{-ix\sqrt{5}}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{2x} + e^{ix\sqrt{5}} c_3 + e^{-ix\sqrt{5}} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{2x} + e^{ix\sqrt{5}} c_3 + e^{-ix\sqrt{5}} c_4$$

Verified OK.

10.14.1 Maple step by step solution

Let's solve

$y'''' + y'' - 20y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -y_3(x) + 20y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -y_3(x) + 20y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 20 & 0 & -1 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 20 & 0 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{5}, \begin{bmatrix} -\frac{\mathrm{I}}{25}\sqrt{5} \\ -\frac{1}{5} \\ \frac{\mathrm{I}}{5}\sqrt{5} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{5}, \begin{bmatrix} \frac{\mathrm{I}}{25}\sqrt{5} \\ -\frac{1}{5} \\ -\frac{\mathrm{I}}{5}\sqrt{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{5}, \begin{bmatrix} -\frac{\mathrm{I}}{25}\sqrt{5} \\ -\frac{1}{5} \\ \frac{\mathrm{I}}{5}\sqrt{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x\sqrt{5}} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{25}\sqrt{5} \\ -\frac{1}{5} \\ \frac{\mathrm{I}}{5}\sqrt{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos \left(x\sqrt{5}\right)-\mathrm{I}\sin \left(x\sqrt{5}\right)\right)\cdot \left[\begin{array}{c}-\frac{\mathrm{I}}{25}\sqrt{5} \\ -\frac{1}{5} \\ \frac{\mathrm{I}}{5}\sqrt{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c}-\frac{\mathrm{I}}{25}\left(\cos \left(x\sqrt{5}\right)-\mathrm{I}\sin \left(x\sqrt{5}\right)\right)\sqrt{5} \\ -\frac{\cos \left(x\sqrt{5}\right)}{5}+\frac{\mathrm{I}\sin \left(x\sqrt{5}\right)}{5} \\ \frac{\mathrm{I}}{5}\left(\cos \left(x\sqrt{5}\right)-\mathrm{I}\sin \left(x\sqrt{5}\right)\right)\sqrt{5} \\ \cos \left(x\sqrt{5}\right)-\mathrm{I}\sin \left(x\sqrt{5}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x)=\left[\begin{array}{c}-\frac{\sqrt{5}\,\sin \left(x\sqrt{5}\right)}{25} \\ -\frac{\cos \left(x\sqrt{5}\right)}{5} \\ \frac{\sqrt{5}\,\sin \left(x\sqrt{5}\right)}{5} \\ \cos \left(x\sqrt{5}\right) \end{array}\right], \vec{y}_{4}(x)=\left[\begin{array}{c}-\frac{\sqrt{5}\,\cos \left(x\sqrt{5}\right)}{25} \\ \frac{\sin \left(x\sqrt{5}\right)}{5} \\ \frac{\sqrt{5}\,\cos \left(x\sqrt{5}\right)}{5} \\ -\sin \left(x\sqrt{5}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}+c_{3}\vec{y}_{3}(x)+c_{4}\vec{y}_{4}(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_{1}\mathrm{e}^{-2x}\cdot \left[\begin{array}{c}-\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]+c_{2}\mathrm{e}^{2x}\cdot \left[\begin{array}{c}\frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]+\left[\begin{array}{c}-\frac{c_{4}\sqrt{5}\,\cos \left(x\sqrt{5}\right)}{25}-\frac{c_{3}\sqrt{5}\,\sin \left(x\sqrt{5}\right)}{25} \\ \frac{c_{4}\sin \left(x\sqrt{5}\right)}{5}-\frac{c_{3}\cos \left(x\sqrt{5}\right)}{5} \\ \frac{c_{4}\sqrt{5}\,\cos \left(x\sqrt{5}\right)}{5}+\frac{c_{3}\sqrt{5}\,\sin \left(x\sqrt{5}\right)}{5} \\ -c_{4}\sin \left(x\sqrt{5}\right)+c_{3}\cos \left(x\sqrt{5}\right) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=-\frac{c_{1}\mathrm{e}^{-2x}}{8}+\frac{c_{2}\mathrm{e}^{2x}}{8}-\frac{c_{4}\sqrt{5}\,\cos \left(x\sqrt{5}\right)}{25}-\frac{c_{3}\sqrt{5}\,\sin \left(x\sqrt{5}\right)}{25}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$4)+diff(y(x),x$2)-20*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + e^{-2x} c_2 + c_3 \sin\left(\sqrt{5}\, x\right) + c_4 \cos\left(\sqrt{5}\, x\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 44

```
DSolve[y''''[x]+y''[x]-20*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^{-2x} + c_4 e^{2x} + c_1 \cos\left(\sqrt{5} x\right) + c_2 \sin\left(\sqrt{5} x\right)$$

10.15 problem 15

Internal problem ID [2131]
Internal file name [`OUTPUT/2131_Monday_February_26_2024_09_17_46_AM_71634107/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 15.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 5y'' + 6y = 0}$$

The characteristic equation is

$$\lambda^4 + 5\lambda^2 + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$
$$\lambda_3 = i\sqrt{3}$$
$$\lambda_4 = -i\sqrt{3}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-i\sqrt{2}\,x} c_1 + \mathrm{e}^{i\sqrt{2}\,x} c_2 + \mathrm{e}^{i\sqrt{3}\,x} c_3 + \mathrm{e}^{-i\sqrt{3}\,x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-i\sqrt{2}\,x}$$
$$y_2 = \mathrm{e}^{i\sqrt{2}\,x}$$
$$y_3 = \mathrm{e}^{i\sqrt{3}\,x}$$
$$y_4 = \mathrm{e}^{-i\sqrt{3}\,x}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-i\sqrt{2}\,x} c_1 + \mathrm{e}^{i\sqrt{2}\,x} c_2 + \mathrm{e}^{i\sqrt{3}\,x} c_3 + \mathrm{e}^{-i\sqrt{3}\,x} c_4 \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-i\sqrt{2}\,x} c_1 + \mathrm{e}^{i\sqrt{2}\,x} c_2 + \mathrm{e}^{i\sqrt{3}\,x} c_3 + \mathrm{e}^{-i\sqrt{3}\,x} c_4$$

Verified OK.

10.15.1 Maple step by step solution

Let's solve

$y'''' + 5y'' + 6y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -5y_3(x) - 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_4(x) = y_3'(x)\,, y_4'(x) = -5y_3(x) - 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 0 & -5 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 0 & -5 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{3}, \begin{bmatrix} -\frac{\mathrm{I}}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{\mathrm{I}}{3}\sqrt{3} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ -\frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{\mathrm{I}}{9}\sqrt{3} \\ -\frac{1}{3} \\ -\frac{\mathrm{I}}{3}\sqrt{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-I\sqrt{2}\,x} \cdot \begin{bmatrix} -\frac{I}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right)\right) \cdot \begin{bmatrix} -\frac{I}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{I}{4}\left(\cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2} \\ -\frac{\cos\left(\sqrt{2}\,x\right)}{2} + \frac{I\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{I}{2}\left(\cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2} \\ \cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = \begin{bmatrix} -\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{4} \\ -\frac{\cos\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} \\ \cos\left(\sqrt{2}\,x\right) \end{bmatrix}, \vec{y}_2(x) = \begin{bmatrix} -\frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{4} \\ \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ -\sin\left(\sqrt{2}\,x\right) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I\sqrt{3}, \begin{bmatrix} -\frac{I}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{I}{3}\sqrt{3} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-I\sqrt{3}\,x} \cdot \begin{bmatrix} -\frac{I}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{I}{3}\sqrt{3} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(\sqrt{3}\,x\right) - I\sin\left(\sqrt{3}\,x\right)\right) \cdot \begin{bmatrix} -\frac{I}{9}\sqrt{3} \\ -\frac{1}{3} \\ \frac{I}{3}\sqrt{3} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{I}{9}\left(\cos\left(\sqrt{3}\,x\right) - I\sin\left(\sqrt{3}\,x\right)\right)\sqrt{3} \\ -\frac{\cos\left(\sqrt{3}\,x\right)}{3} + \frac{I\sin\left(\sqrt{3}\,x\right)}{3} \\ \frac{I}{3}\left(\cos\left(\sqrt{3}\,x\right) - I\sin\left(\sqrt{3}\,x\right)\right)\sqrt{3} \\ \cos\left(\sqrt{3}\,x\right) - I\sin\left(\sqrt{3}\,x\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{9} \\ -\frac{\cos\left(\sqrt{3}\,x\right)}{3} \\ \frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{3} \\ \cos\left(\sqrt{3}\,x\right) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{9} \\ \frac{\sin\left(\sqrt{3}\,x\right)}{3} \\ \frac{\cos\left(\sqrt{3}\,x\right)\sqrt{3}}{3} \\ -\sin\left(\sqrt{3}\,x\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} -\frac{c_4\sqrt{3}\cos(\sqrt{3}\,x)}{9} - \frac{c_3\sqrt{3}\sin(\sqrt{3}\,x)}{9} - \frac{c_2\sqrt{2}\cos(\sqrt{2}\,x)}{4} - \frac{c_1\sqrt{2}\sin(\sqrt{2}\,x)}{4} \\ \frac{c_4\sin(\sqrt{3}\,x)}{3} - \frac{c_3\cos(\sqrt{3}\,x)}{3} + \frac{c_2\sin(\sqrt{2}\,x)}{2} - \frac{c_1\cos(\sqrt{2}\,x)}{2} \\ \frac{c_4\sqrt{3}\cos(\sqrt{3}\,x)}{3} + \frac{c_3\sqrt{3}\sin(\sqrt{3}\,x)}{3} + \frac{c_2\sqrt{2}\cos(\sqrt{2}\,x)}{2} + \frac{c_1\sqrt{2}\sin(\sqrt{2}\,x)}{2} \\ -c_4\sin(\sqrt{3}\,x) + c_3\cos(\sqrt{3}\,x) - c_2\sin(\sqrt{2}\,x) + c_1\cos(\sqrt{2}\,x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_4\sqrt{3}\cos(\sqrt{3}\,x)}{9} - \frac{c_3\sqrt{3}\sin(\sqrt{3}\,x)}{9} - \frac{c_2\sqrt{2}\cos(\sqrt{2}\,x)}{4} - \frac{c_1\sqrt{2}\sin(\sqrt{2}\,x)}{4}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$4)+5*diff(y(x),x$2)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sin\left(\sqrt{2}\,x\right) + c_2\cos\left(\sqrt{2}\,x\right) + c_3\sin\left(\sqrt{3}\,x\right) + c_4\cos\left(\sqrt{3}\,x\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 50

```
DSolve[y''''[x]+5*y''[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3\cos\left(\sqrt{2}x\right) + c_1\cos\left(\sqrt{3}x\right) + c_4\sin\left(\sqrt{2}x\right) + c_2\sin\left(\sqrt{3}x\right)$$

10.16 problem 16

Internal problem ID [2132]
Internal file name [`OUTPUT/2132_Monday_February_26_2024_09_17_47_AM_23103942/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 16.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 4y''' + 6y'' - 8y' + 8y = 0}$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 8\lambda + 8 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2}\\ \lambda_3 &= 2\\ \lambda_4 &= 2\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-i\sqrt{2}\,x}c_1 + c_2\mathrm{e}^{2x} + c_3x\,\mathrm{e}^{2x} + \mathrm{e}^{i\sqrt{2}\,x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{-i\sqrt{2}\,x}\\ y_2 &= \mathrm{e}^{2x}\\ y_3 &= x\,\mathrm{e}^{2x}\\ y_4 &= \mathrm{e}^{i\sqrt{2}\,x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-i\sqrt{2}\,x}c_1 + c_2 e^{2x} + c_3 x\, e^{2x} + e^{i\sqrt{2}\,x}c_4 \tag{1}$$

Verification of solutions

$$y = e^{-i\sqrt{2}\,x}c_1 + c_2 e^{2x} + c_3 x\, e^{2x} + e^{i\sqrt{2}\,x}c_4$$

Verified OK.

10.16.1 Maple step by step solution

Let's solve

$y'''' - 4y''' + 6y'' - 8y' + 8y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 4y_4(x) - 6y_3(x) + 8y_2(x) - 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 4y_4(x) - 6y_3(x) + 8y_2(x) - 8y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & 8 & -6 & 4 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & 8 & -6 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ -\frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

$$\overrightarrow{y}_1(x) = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

  $$\overrightarrow{y}_2(x) = e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

  $$\lambda\, e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = \left(e^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $$\lambda\, e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = e^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

  $$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

  $$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

  $$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

  $$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & 8 & -6 & 4 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{y}_2(x) = e^{2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I\sqrt{2}, \begin{bmatrix} -\frac{I}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{-I\sqrt{2}\,x} \cdot \begin{bmatrix} -\frac{I}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right)\right) \cdot \begin{bmatrix} -\frac{I}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{I}{4}\left(\cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2} \\ -\frac{\cos\left(\sqrt{2}\,x\right)}{2} + \frac{I\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{I}{2}\left(\cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2} \\ \cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{4} \\ -\frac{\cos\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} \\ \cos\left(\sqrt{2}\,x\right) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{4} \\ \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ -\sin\left(\sqrt{2}\,x\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs
  $\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$
- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} -\frac{c_4\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{4} - \frac{c_3\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{4} \\ \frac{c_4\sin\left(\sqrt{2}\,x\right)}{2} - \frac{c_3\cos\left(\sqrt{2}\,x\right)}{2} \\ \frac{c_4\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} + \frac{c_3\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} \\ -c_4\sin\left(\sqrt{2}\,x\right) + c_3\cos\left(\sqrt{2}\,x\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE
  $y = -\frac{c_4\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{4} - \frac{c_3\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{4} + \frac{\left(\left(x-\frac{1}{2}\right)c_2+c_1\right)e^{2x}}{8}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$3)+6*diff(y(x),x$2)-8*diff(y(x),x)+8*y(x)=0,y(x), singso
```

$$y(x) = c_4\cos\left(\sqrt{2}\,x\right) + c_3\sin\left(\sqrt{2}\,x\right) + e^{2x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 41

```
DSolve[y''''[x]-4*y'''[x]+6*y''[x]-8*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{2x}(c_4x + c_3) + c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right)$$

10.17 problem 17

Internal problem ID [2133]
Internal file name [`OUTPUT/2133_Monday_February_26_2024_09_17_47_AM_96573020/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 17.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' - 2y''' - 6y' + 2y = 0}$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 - 6\lambda + 2 = 0$$

The roots of the above equation are

$$\lambda_1 = \text{RootOf}\left(\_Z^4 - 2\_Z^3 - 6\_Z + 2, \text{index} = 1\right)$$
$$\lambda_2 = \text{RootOf}\left(\_Z^4 - 2\_Z^3 - 6\_Z + 2, \text{index} = 2\right)$$
$$\lambda_3 = \text{RootOf}\left(\_Z^4 - 2\_Z^3 - 6\_Z + 2, \text{index} = 3\right)$$
$$\lambda_4 = \text{RootOf}\left(\_Z^4 - 2\_Z^3 - 6\_Z + 2, \text{index} = 4\right)$$

Therefore the homogeneous solution is

$y_h(x) = \mathrm{e}^{\text{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\text{index}=2)x} c_1 + \mathrm{e}^{\text{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\text{index}=1)x} c_2 + \mathrm{e}^{\text{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\text{index}}$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{\text{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\text{index}=2)x}$$
$$y_2 = \mathrm{e}^{\text{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\text{index}=1)x}$$
$$y_3 = \mathrm{e}^{\text{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\text{index}=4)x}$$
$$y_4 = \mathrm{e}^{\text{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\text{index}=3)x}$$

Summary

The solution(s) found are the following

$$y = e^{\mathrm{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\mathrm{index}=2)x} c_1 + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\mathrm{index}=1)x} c_2 + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\mathrm{index}=4)x} c_3 + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\mathrm{index}=3)x} c_4 \tag{1}$$

Verification of solutions

$$y = e^{\mathrm{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\mathrm{index}=2)x} c_1 + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\mathrm{index}=1)x} c_2 + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\mathrm{index}=4)x} c_3 + e^{\mathrm{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\mathrm{index}=3)x} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$3)-6*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sum_{\_a=1}^{4} e^{\mathrm{RootOf}(\_Z^4-2\_Z^3-6\_Z+2,\mathrm{index}=\_a)x} \_C\_a$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 114

```
DSolve[y''''[x]-2*y'''[x]-6*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \exp\left(x\mathrm{Root}\left[\#1^4 - 2\#1^3 - 6\#1 + 2\&, 1\right]\right) + c_3 \exp\left(x\mathrm{Root}\left[\#1^4 - 2\#1^3 - 6\#1 + 2\&, 3\right]\right) + c_4 \exp\left(x\mathrm{Root}\left[\#1^4 - 2\#1^3 - 6\#1 + 2\&, 4\right]\right) + c_2 \exp\left(x\mathrm{Root}\left[\#1^4 - 2\#1^3 - 6\#1 + 2\&, 2\right]\right)$$

10.18 problem 18

Internal problem ID [2134]
Internal file name [`OUTPUT/2134_Monday_February_26_2024_09_17_47_AM_94047950/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 18.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + y''' - 3y'' - 4y' - 4y = 0}$$

The characteristic equation is

$$\lambda^4 + \lambda^3 - 3\lambda^2 - 4\lambda - 4 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -2 \\
\lambda_3 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2} \\
\lambda_4 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{2x}$$
$$y_3 = \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$
$$y_4 = \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{2x} + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}c_3 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}c_4 \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{2x} + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}c_3 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}c_4$$

Verified OK.

10.18.1 Maple step by step solution

Let's solve

$y'''' + y''' - 3y'' - 4y' - 4y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -y_4(x) + 3y_3(x) + 4y_2(x) + 4y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -y_4(x) + 3y_3(x) + 4y_2(x) + 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 4 & 3 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 4 & 3 & -1 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) - I\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right], \vec{y}_{4}(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + e^{-\frac{x}{2}} c_3 \cdot \begin{bmatrix} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} + e^{-\frac{x}{2}} c_4 \cdot \Bigg[\cdots$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(c_2 e^{4x} + 8c_3 \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{\frac{3x}{2}} - 8c_4 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{\frac{3x}{2}} - c_1\right) e^{-2x}}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$4)+diff(y(x),x$3)-3*diff(y(x),x$2)-4*diff(y(x),x)-4*y(x)=0,y(x), singsol=
```

$$y(x) = \left(e^{4x} c_1 + c_3 e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + c_4 e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 64

```
DSolve[y''''[x]+y'''[x]-3*y''[x]-4*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x} \left(c_4 e^{4x} + c_2 e^{3x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{3x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) + c_3\right)$$

10.19 problem 19

Internal problem ID [2135]
Internal file name [`OUTPUT/2135_Monday_February_26_2024_09_17_47_AM_75656588/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 19.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{2y''' - 3y'' + 10y' - 15y = 0}$$

The characteristic equation is

$$2\lambda^3 - 3\lambda^2 + 10\lambda - 15 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{3}{2}$$
$$\lambda_2 = i\sqrt{5}$$
$$\lambda_3 = -i\sqrt{5}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{ix\sqrt{5}}c_1 + \mathrm{e}^{-ix\sqrt{5}}c_2 + \mathrm{e}^{\frac{3x}{2}}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{ix\sqrt{5}}$$
$$y_2 = \mathrm{e}^{-ix\sqrt{5}}$$
$$y_3 = \mathrm{e}^{\frac{3x}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{ix\sqrt{5}}c_1 + e^{-ix\sqrt{5}}c_2 + e^{\frac{3x}{2}}c_3 \tag{1}$$

Verification of solutions

$$y = e^{ix\sqrt{5}}c_1 + e^{-ix\sqrt{5}}c_2 + e^{\frac{3x}{2}}c_3$$

Verified OK.

10.19.1 Maple step by step solution

Let's solve

$2y''' - 3y'' + 10y' - 15y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{3y''}{2} - 5y' + \frac{15y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y''}{2} + 5y' - \frac{15y}{2} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

  $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

  $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

  $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = \frac{3y_3(x)}{2} - 5y_2(x) + \frac{15y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{3y_3(x)}{2} - 5y_2(x) + \frac{15y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{15}{2} & -5 & \frac{3}{2} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{15}{2} & -5 & \frac{3}{2} \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{3}{2}, \begin{bmatrix} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{5}, \begin{bmatrix} -\frac{1}{5} \\ \frac{\mathrm{I}}{5}\sqrt{5} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{5}, \begin{bmatrix} -\frac{1}{5} \\ -\frac{\mathrm{I}}{5}\sqrt{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\frac{3}{2}, \begin{bmatrix} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{\frac{3x}{2}} \cdot \begin{bmatrix} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{5}, \left[\begin{array}{c} -\frac{1}{5} \\ \frac{\mathrm{I}}{5}\sqrt{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x\sqrt{5}} \cdot \left[\begin{array}{c} -\frac{1}{5} \\ \frac{\mathrm{I}}{5}\sqrt{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(x\sqrt{5}\right) - \mathrm{I}\sin\left(x\sqrt{5}\right)\right) \cdot \left[\begin{array}{c} -\frac{1}{5} \\ \frac{\mathrm{I}}{5}\sqrt{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{\cos\left(x\sqrt{5}\right)}{5} + \frac{\mathrm{I}\sin\left(x\sqrt{5}\right)}{5} \\ \frac{\mathrm{I}}{5}\left(\cos\left(x\sqrt{5}\right) - \mathrm{I}\sin\left(x\sqrt{5}\right)\right)\sqrt{5} \\ \cos\left(x\sqrt{5}\right) - \mathrm{I}\sin\left(x\sqrt{5}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \left[\begin{array}{c} -\frac{\cos\left(x\sqrt{5}\right)}{5} \\ \frac{\sqrt{5}\,\sin\left(x\sqrt{5}\right)}{5} \\ \cos\left(x\sqrt{5}\right) \end{array}\right], \vec{y}_3(x) = \left[\begin{array}{c} \frac{\sin\left(x\sqrt{5}\right)}{5} \\ \frac{\sqrt{5}\,\cos\left(x\sqrt{5}\right)}{5} \\ -\sin\left(x\sqrt{5}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{\frac{3x}{2}} \cdot \left[\begin{array}{c} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{array}\right] + \left[\begin{array}{c} \frac{c_3\sin\left(x\sqrt{5}\right)}{5} - \frac{c_2\cos\left(x\sqrt{5}\right)}{5} \\ \frac{c_3\sqrt{5}\,\cos\left(x\sqrt{5}\right)}{5} + \frac{c_2\sin\left(x\sqrt{5}\right)\sqrt{5}}{5} \\ -c_3\sin\left(x\sqrt{5}\right) + c_2\cos\left(x\sqrt{5}\right) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{4c_1 e^{\frac{3x}{2}}}{9} + \frac{c_3 \sin\left(x\sqrt{5}\right)}{5} - \frac{c_2 \cos\left(x\sqrt{5}\right)}{5}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(2*diff(y(x),x$3)-3*diff(y(x),x$2)+10*diff(y(x),x)-15*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{3x}{2}} + c_2 \sin\left(\sqrt{5}\, x\right) + c_3 \cos\left(\sqrt{5}\, x\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 38

```
DSolve[2*y'''[x]-3*y''[x]+10*y'[x]-15*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^{3x/2} + c_1 \cos\left(\sqrt{5}x\right) + c_2 \sin\left(\sqrt{5}x\right)$$

10.20 problem 20

Internal problem ID [2136]
Internal file name [`OUTPUT/2136_Monday_February_26_2024_09_17_47_AM_79654824/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 20.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{2y''' - 3y'' + 11y' - 40y = 0}$$

The characteristic equation is

$$2\lambda^3 - 3\lambda^2 + 11\lambda - 40 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= \frac{5}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{31}}{2}\\ \lambda_3 &= -\frac{1}{2} + \frac{i\sqrt{31}}{2}\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{\left(-\frac{1}{2}+\frac{i\sqrt{31}}{2}\right)x} c_1 + c_2 e^{\frac{5x}{2}} + e^{\left(-\frac{1}{2}-\frac{i\sqrt{31}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= e^{\left(-\frac{1}{2}+\frac{i\sqrt{31}}{2}\right)x}\\ y_2 &= e^{\frac{5x}{2}}\\ y_3 &= e^{\left(-\frac{1}{2}-\frac{i\sqrt{31}}{2}\right)x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{\left(-\frac{1}{2}+\frac{i\sqrt{31}}{2}\right)x} c_1 + c_2 e^{\frac{5x}{2}} + e^{\left(-\frac{1}{2}-\frac{i\sqrt{31}}{2}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = e^{\left(-\frac{1}{2}+\frac{i\sqrt{31}}{2}\right)x} c_1 + c_2 e^{\frac{5x}{2}} + e^{\left(-\frac{1}{2}-\frac{i\sqrt{31}}{2}\right)x} c_3$$

Verified OK.

10.20.1 Maple step by step solution

Let's solve

$2y''' - 3y'' + 11y' - 40y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{3y''}{2} - \frac{11y'}{2} + 20y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y''}{2} + \frac{11y'}{2} - 20y = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \frac{3y_3(x)}{2} - \frac{11y_2(x)}{2} + 20y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{3y_3(x)}{2} - \frac{11y_2(x)}{2} + 20y_1(x)\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 20 & -\frac{11}{2} & \frac{3}{2} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 20 & -\frac{11}{2} & \frac{3}{2} \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{5}{2}, \begin{bmatrix} \frac{4}{25} \\ \frac{2}{5} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{31}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\mathrm{I}\sqrt{31}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{31}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{31}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\frac{5}{2}, \begin{bmatrix} \frac{4}{25} \\ \frac{2}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{\frac{5x}{2}} \cdot \begin{bmatrix} \frac{4}{25} \\ \frac{2}{5} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{31}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{31}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{31}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{31}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{31}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{31}\,x}{2}\right)}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{31}}{2}} \\ \cos\left(\frac{\sqrt{31}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{31}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=\mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{15\cos\left(\frac{\sqrt{31}\,x}{2}\right)}{128}-\frac{\sin\left(\frac{\sqrt{31}\,x}{2}\right)\sqrt{31}}{128} \\ -\frac{\cos\left(\frac{\sqrt{31}\,x}{2}\right)}{16}+\frac{\sin\left(\frac{\sqrt{31}\,x}{2}\right)\sqrt{31}}{16} \\ \cos\left(\frac{\sqrt{31}\,x}{2}\right) \end{array}\right], \vec{y}_{3}(x)=\mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{31}\,x}{2}\right)\sqrt{31}}{128}+\frac{15\sin\left(\frac{\sqrt{31}\,x}{2}\right)}{128} \\ \frac{\cos\left(\frac{\sqrt{31}\,x}{2}\right)\sqrt{31}}{16}+\frac{\sin\left(\frac{\sqrt{31}\,x}{2}\right)}{16} \\ -\sin\left(\frac{\sqrt{31}\,x}{2}\right) \end{array}\right]\right.$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{\frac{5x}{2}} \cdot \begin{bmatrix} \frac{4}{25} \\ \frac{2}{5} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} -\frac{15\cos\left(\frac{\sqrt{31}\,x}{2}\right)}{128} - \frac{\sin\left(\frac{\sqrt{31}\,x}{2}\right)\sqrt{31}}{128} \\ -\frac{\cos\left(\frac{\sqrt{31}\,x}{2}\right)}{16} + \frac{\sin\left(\frac{\sqrt{31}\,x}{2}\right)\sqrt{31}}{16} \\ \cos\left(\frac{\sqrt{31}\,x}{2}\right) \end{bmatrix} + \mathrm{e}^{-\frac{x}{2}} c_3 \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{31}\,x}{2}\right)\sqrt{31}}{128} + \frac{15\sin \\ \frac{\cos\left(\frac{\sqrt{31}\,x}{2}\right)\sqrt{31}}{16} + \frac{\sin(\\ -\sin\left(\frac{\sqrt{31}\,x}{2}\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{15\left(\frac{c_3\sqrt{31}}{15}+c_2\right)\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{31}\,x}{2}\right)}{128} - \frac{\mathrm{e}^{-\frac{x}{2}}\left(\sqrt{31}\,c_2 - 15c_3\right)\sin\left(\frac{\sqrt{31}\,x}{2}\right)}{128} + \frac{4c_1\mathrm{e}^{\frac{5x}{2}}}{25}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(2*diff(y(x),x$3)-3*diff(y(x),x$2)+11*diff(y(x),x)-40*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\mathrm{e}^{\frac{5x}{2}} + c_2\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{31}\,x}{2}\right) + c_3\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{31}\,x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 50

```
DSolve[2*y'''[x]-3*y''[x]+11*y'[x]-40*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(c_3e^{3x} + c_2\cos\left(\frac{\sqrt{31}x}{2}\right) + c_1\sin\left(\frac{\sqrt{31}x}{2}\right)\right)$$

10.21 problem 21

Internal problem ID [2137]
Internal file name [OUTPUT/2137_Monday_February_26_2024_09_17_48_AM_17396529/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 21.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 3y''' + 4y'' - 12y' + 16y = 0}$$

The characteristic equation is

$$\lambda^4 - 3\lambda^3 + 4\lambda^2 - 12\lambda + 16 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{2} - \frac{i\sqrt{15}}{2}$$
$$\lambda_2 = -\frac{1}{2} + \frac{i\sqrt{15}}{2}$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + c_2 x\, e^{2x} + e^{\left(-\frac{1}{2} - \frac{i\sqrt{15}}{2}\right)x} c_3 + e^{\left(-\frac{1}{2} + \frac{i\sqrt{15}}{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{2x} \\
y_2 &= x\,\mathrm{e}^{2x} \\
y_3 &= \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{15}}{2}\right)x} \\
y_4 &= \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{15}}{2}\right)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{2x} + c_2x\,\mathrm{e}^{2x} + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{15}}{2}\right)x}c_3 + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{15}}{2}\right)x}c_4 \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{2x} + c_2x\,\mathrm{e}^{2x} + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{15}}{2}\right)x}c_3 + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{15}}{2}\right)x}c_4
$$

Verified OK.

10.21.1 Maple step by step solution

Let's solve

$y'''' - 3y''' + 4y'' - 12y' + 16y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 3y_4(x) - 4y_3(x) + 12y_2(x) - 16y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 3y_4(x) - 4y_3(x) + 12y_2(x) - 16y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 12 & -4 & 3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 12 & -4 & 3 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{15}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\mathrm{I}\sqrt{15}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{15}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{15}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{15}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

$$\vec{y}_1(x) = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$$\vec{y}_2(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 12 & -4 & 3 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \left[\begin{array}{c} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 2

$$\overrightarrow{y}_2(x) = \mathrm{e}^{2x} \cdot \left(x \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{15}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{15}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{15}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{15}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{15}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{15}}{2}} \\ \cos\left(\frac{\sqrt{15}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{15}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{11\cos\left(\frac{\sqrt{15}\,x}{2}\right)}{128} - \frac{3\sin\left(\frac{\sqrt{15}\,x}{2}\right)\sqrt{15}}{128} \\ -\frac{7\cos\left(\frac{\sqrt{15}\,x}{2}\right)}{32} - \frac{\sin\left(\frac{\sqrt{15}\,x}{2}\right)\sqrt{15}}{32} \\ -\frac{\cos\left(\frac{\sqrt{15}\,x}{2}\right)}{8} + \frac{\sin\left(\frac{\sqrt{15}\,x}{2}\right)\sqrt{15}}{8} \\ \cos\left(\frac{\sqrt{15}\,x}{2}\right) \end{array}\right], \vec{y}_4(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{3\cos\left(\frac{\sqrt{15}\,x}{2}\right)\sqrt{15}}{128} - \frac{11\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{128} \\ -\frac{\cos\left(\frac{\sqrt{15}\,x}{2}\right)\sqrt{15}}{32} + \frac{7\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{32} \\ \frac{\cos\left(\frac{\sqrt{15}\,x}{2}\right)\sqrt{15}}{8} + \frac{\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{8} \\ -\sin\left(\frac{\sqrt{15}\,x}{2}\right) \end{array}\right]\right.$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{2x} \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right] + c_2 e^{2x} \cdot \left(x \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{array}\right]\right) + e^{-\frac{x}{2}} c_3 \cdot \left[\begin{array}{c} \frac{11\cos\left(\frac{\sqrt{15}\,x}{2}\right)}{128} - \frac{3\sin\left(\frac{\sqrt{15}\,x}{2}\right)\sqrt{15}}{128} \\ -\frac{7\cos\left(\frac{\sqrt{15}\,x}{2}\right)}{32} - \frac{\sin\left(\frac{\sqrt{15}\,x}{2}\right)\sqrt{15}}{32} \\ -\frac{\cos\left(\frac{\sqrt{15}\,x}{2}\right)}{8} + \frac{\sin\left(\frac{\sqrt{15}\,x}{2}\right)\sqrt{15}}{8} \\ \cos\left(\frac{\sqrt{15}\,x}{2}\right) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{11\left(-\frac{3c_4\sqrt{15}}{11}+c_3\right)\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{15}\,x}{2}\right)}{128} - \frac{3\left(\sqrt{15}\,c_3+\frac{11c_4}{3}\right)\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{128} + \frac{\left(\left(x-\frac{1}{2}\right)c_2+c_1\right)\mathrm{e}^{2x}}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$4)-3*diff(y(x),x$3)+4*diff(y(x),x$2)-12*diff(y(x),x)+16*y(x)=0,y(x), sing
```

$$y(x) = c_4\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{15}\,x}{2}\right) + c_3\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{15}\,x}{2}\right) + \mathrm{e}^{2x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 57

```
DSolve[y''''[x]-3*y'''[x]+4*y''[x]-12*y'[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^{-x/2}\left(e^{5x/2}(c_4x + c_3) + c_2\cos\left(\frac{\sqrt{15}x}{2}\right) + c_1\sin\left(\frac{\sqrt{15}x}{2}\right)\right)$$

10.22 problem 22

Internal problem ID [2138]
Internal file name [OUTPUT/2138_Monday_February_26_2024_09_17_48_AM_64086885/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 22.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{4y''' + 12y'' - 3y' + 14y = 0}$$

The characteristic equation is

$$4\lambda^3 + 12\lambda^2 - 3\lambda + 14 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{7}{2}$$
$$\lambda_2 = \frac{1}{4} - \frac{i\sqrt{15}}{4}$$
$$\lambda_3 = \frac{i\sqrt{15}}{4} + \frac{1}{4}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{\left(\frac{1}{4} - \frac{i\sqrt{15}}{4}\right)x} c_1 + e^{-\frac{7x}{2}} c_2 + e^{\left(\frac{i\sqrt{15}}{4} + \frac{1}{4}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{\left(\frac{1}{4} - \frac{i\sqrt{15}}{4}\right)x}$$
$$y_2 = e^{-\frac{7x}{2}}$$
$$y_3 = e^{\left(\frac{i\sqrt{15}}{4} + \frac{1}{4}\right)x}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{\left(\frac{1}{4}-\frac{i\sqrt{15}}{4}\right)x} c_1 + \mathrm{e}^{-\frac{7x}{2}} c_2 + \mathrm{e}^{\left(\frac{i\sqrt{15}}{4}+\frac{1}{4}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\left(\frac{1}{4}-\frac{i\sqrt{15}}{4}\right)x} c_1 + \mathrm{e}^{-\frac{7x}{2}} c_2 + \mathrm{e}^{\left(\frac{i\sqrt{15}}{4}+\frac{1}{4}\right)x} c_3$$

Verified OK.

10.22.1 Maple step by step solution

Let's solve

$4y''' + 12y'' - 3y' + 14y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -3y'' + \frac{3y'}{4} - \frac{7y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + 3y'' - \frac{3y'}{4} + \frac{7y}{2} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -3y_3(x) + \frac{3y_2(x)}{4} - \frac{7y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -3y_3(x) + \frac{3y_2(x)}{4} - \frac{7y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{7}{2} & \frac{3}{4} & -3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{7}{2} & \frac{3}{4} & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{7}{2}, \begin{bmatrix} \frac{4}{49} \\ -\frac{2}{7} \\ 1 \end{bmatrix}\right], \left[\frac{1}{4} - \frac{\mathrm{I}\sqrt{15}}{4}, \begin{bmatrix} \frac{1}{\left(\frac{1}{4}-\frac{\mathrm{I}\sqrt{15}}{4}\right)^2} \\ \frac{1}{\frac{1}{4}-\frac{\mathrm{I}\sqrt{15}}{4}} \\ 1 \end{bmatrix}\right], \left[\frac{\mathrm{I}\sqrt{15}}{4} + \frac{1}{4}, \begin{bmatrix} \frac{1}{\left(\frac{\mathrm{I}\sqrt{15}}{4}+\frac{1}{4}\right)^2} \\ \frac{1}{\frac{\mathrm{I}\sqrt{15}}{4}+\frac{1}{4}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{7}{2}, \begin{bmatrix} \frac{4}{49} \\ -\frac{2}{7} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-\frac{7x}{2}} \cdot \begin{bmatrix} \frac{4}{49} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{4}-\frac{I\sqrt{15}}{4}, \left[\begin{array}{c}\frac{1}{\left(\frac{1}{4}-\frac{I\sqrt{15}}{4}\right)^{2}} \\ \frac{1}{\frac{1}{4}-\frac{I\sqrt{15}}{4}} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(\frac{1}{4}-\frac{I\sqrt{15}}{4}\right)x}\cdot\left[\begin{array}{c}\frac{1}{\left(\frac{1}{4}-\frac{I\sqrt{15}}{4}\right)^{2}} \\ \frac{1}{\frac{1}{4}-\frac{I\sqrt{15}}{4}} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{\frac{x}{4}}\cdot\left(\cos\left(\frac{\sqrt{15}\,x}{4}\right)-I\sin\left(\frac{\sqrt{15}\,x}{4}\right)\right)\cdot\left[\begin{array}{c}\frac{1}{\left(\frac{1}{4}-\frac{I\sqrt{15}}{4}\right)^{2}} \\ \frac{1}{\frac{1}{4}-\frac{I\sqrt{15}}{4}} \\ 1\end{array}\right]$$

- Simplify expression

$$e^{\frac{x}{4}}\cdot\left[\begin{array}{c}\frac{\cos\left(\frac{\sqrt{15}\,x}{4}\right)-I\sin\left(\frac{\sqrt{15}\,x}{4}\right)}{\left(\frac{1}{4}-\frac{I\sqrt{15}}{4}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{15}\,x}{4}\right)-I\sin\left(\frac{\sqrt{15}\,x}{4}\right)}{\frac{1}{4}-\frac{I\sqrt{15}}{4}} \\ \cos\left(\frac{\sqrt{15}\,x}{4}\right)-I\sin\left(\frac{\sqrt{15}\,x}{4}\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=e^{\frac{x}{4}}\cdot\left[\begin{array}{c}-\frac{7\cos\left(\frac{\sqrt{15}\,x}{4}\right)}{8}+\frac{\sin\left(\frac{\sqrt{15}\,x}{4}\right)\sqrt{15}}{8} \\ \frac{\cos\left(\frac{\sqrt{15}\,x}{4}\right)}{4}+\frac{\sin\left(\frac{\sqrt{15}\,x}{4}\right)\sqrt{15}}{4} \\ \cos\left(\frac{\sqrt{15}\,x}{4}\right)\end{array}\right], \vec{y}_{3}(x)=e^{\frac{x}{4}}\cdot\left[\begin{array}{c}\frac{\cos\left(\frac{\sqrt{15}\,x}{4}\right)\sqrt{15}}{8}+\frac{7\sin\left(\frac{\sqrt{15}\,x}{4}\right)}{8} \\ \frac{\cos\left(\frac{\sqrt{15}\,x}{4}\right)\sqrt{15}}{4}-\frac{\sin\left(\frac{\sqrt{15}\,x}{4}\right)}{4} \\ -\sin\left(\frac{\sqrt{15}\,x}{4}\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-\frac{7x}{2}} \cdot \begin{bmatrix} \frac{4}{49} \\ -\frac{2}{7} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{\frac{x}{4}} \cdot \begin{bmatrix} -\frac{7\cos\left(\frac{\sqrt{15}\,x}{4}\right)}{8} + \frac{\sin\left(\frac{\sqrt{15}\,x}{4}\right)\sqrt{15}}{8} \\ \frac{\cos\left(\frac{\sqrt{15}\,x}{4}\right)}{4} + \frac{\sin\left(\frac{\sqrt{15}\,x}{4}\right)\sqrt{15}}{4} \\ \cos\left(\frac{\sqrt{15}\,x}{4}\right) \end{bmatrix} + c_3 \mathrm{e}^{\frac{x}{4}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{15}\,x}{4}\right)\sqrt{15}}{8} + \frac{7\sin\left(\frac{\sqrt{15}}{4}\right.}{8} \\ \frac{\cos\left(\frac{\sqrt{15}\,x}{4}\right)\sqrt{15}}{4} - \frac{\sin\left(\frac{\sqrt{15}}{4}\right.}{4} \\ -\sin\left(\frac{\sqrt{15}\,x}{4}\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{4\left(-\frac{343\left(-\frac{\sqrt{15}\,c_3}{7}+c_2\right)\mathrm{e}^{\frac{15x}{4}}\cos\left(\frac{\sqrt{15}\,x}{4}\right)}{32} + \frac{49\,\mathrm{e}^{\frac{15x}{4}}\left(\sqrt{15}\,c_2+7c_3\right)\sin\left(\frac{\sqrt{15}\,x}{4}\right)}{32} + c_1\right)\mathrm{e}^{-\frac{7x}{2}}}{49}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(4*diff(y(x),x$3)+12*diff(y(x),x$2)-3*diff(y(x),x)+14*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 \mathrm{e}^{\frac{15x}{4}} \sin\left(\frac{\sqrt{15}\,x}{4}\right) + c_3 \mathrm{e}^{\frac{15x}{4}} \cos\left(\frac{\sqrt{15}\,x}{4}\right) + c_1\right) \mathrm{e}^{-\frac{7x}{2}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 58

```
DSolve[4*y'''[x]+12*y''[x]-3*y'[x]+14*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^{-7x/2} + c_2 e^{x/4} \cos\left(\frac{\sqrt{15}x}{4}\right) + c_1 e^{x/4} \sin\left(\frac{\sqrt{15}x}{4}\right)$$

10.23 problem 23

Internal problem ID [2139]
Internal file name [OUTPUT/2139_Monday_February_26_2024_09_17_48_AM_47063498/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 18, page 82
Problem number: 23.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} - y'''' + 6y''' - 6y'' + 8y' - 8y = 0}$$

The characteristic equation is

$$\lambda^5 - \lambda^4 + 6\lambda^3 - 6\lambda^2 + 8\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2i$$
$$\lambda_3 = -2i$$
$$\lambda_4 = i\sqrt{2}$$
$$\lambda_5 = -i\sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{-i\sqrt{2}\,x} c_2 + e^{i\sqrt{2}\,x} c_3 + e^{2ix} c_4 + e^{-2ix} c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{x} \\
y_2 &= \mathrm{e}^{-i\sqrt{2}\,x} \\
y_3 &= \mathrm{e}^{i\sqrt{2}\,x} \\
y_4 &= \mathrm{e}^{2ix} \\
y_5 &= \mathrm{e}^{-2ix}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{x} + \mathrm{e}^{-i\sqrt{2}\,x}c_2 + \mathrm{e}^{i\sqrt{2}\,x}c_3 + \mathrm{e}^{2ix}c_4 + \mathrm{e}^{-2ix}c_5 \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{x} + \mathrm{e}^{-i\sqrt{2}\,x}c_2 + \mathrm{e}^{i\sqrt{2}\,x}c_3 + \mathrm{e}^{2ix}c_4 + \mathrm{e}^{-2ix}c_5
$$

Verified OK.

10.23.1 Maple step by step solution

Let's solve

$y^{(5)} - y'''' + 6y''' - 6y'' + 8y' - 8y = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = y_5(x) - 6y_4(x) + 6y_3(x) - 8y_2(x) + 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = y_5(x) - 6y_4(x) + 6y_3(x) -$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 8 & -8 & 6 & -6 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 8 & -8 & 6 & -6 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix}\frac{1}{16}\\-\frac{\mathrm{I}}{8}\\-\frac{1}{4}\\\frac{\mathrm{I}}{2}\\1\end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix}\frac{1}{16}\\\frac{\mathrm{I}}{8}\\-\frac{1}{4}\\-\frac{\mathrm{I}}{2}\\1\end{bmatrix}\right], \left[-\mathrm{I}\sqrt{2}, \begin{bmatrix}\frac{1}{4}\\-\frac{\mathrm{I}}{4}\sqrt{2}\\-\frac{1}{2}\\\frac{\mathrm{I}}{2}\sqrt{2}\\1\end{bmatrix}\right], \left[\mathrm{I}\sqrt{2}, \begin{bmatrix}\frac{1}{4}\\\frac{\mathrm{I}}{4}\sqrt{2}\\-\frac{1}{2}\\-\frac{\mathrm{I}}{2}\sqrt{2}\\1\end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix}\frac{1}{16}\\-\frac{\mathrm{I}}{8}\\-\frac{1}{4}\\\frac{\mathrm{I}}{2}\\1\end{bmatrix}\right]$$

- Solution from eigenpair

$$
e^{-2 I x} \cdot \left[\begin{array}{c} \frac{1}{16} \\ -\frac{I}{8} \\ -\frac{1}{4} \\ \frac{I}{2} \\ 1 \end{array}\right]
$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$
(\cos(2x) - I\sin(2x)) \cdot \left[\begin{array}{c} \frac{1}{16} \\ -\frac{I}{8} \\ -\frac{1}{4} \\ \frac{I}{2} \\ 1 \end{array}\right]
$$

- Simplify expression

$$
\left[\begin{array}{c} \frac{\cos(2x)}{16} - \frac{I\sin(2x)}{16} \\ -\frac{I}{8}(\cos(2x) - I\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{I\sin(2x)}{4} \\ \frac{I}{2}(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \end{array}\right]
$$

- Both real and imaginary parts are solutions to the homogeneous system

$$
\left[\vec{y}_2(x) = \left[\begin{array}{c} \frac{\cos(2x)}{16} \\ -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{array}\right], \vec{y}_3(x) = \left[\begin{array}{c} -\frac{\sin(2x)}{16} \\ -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{array}\right]\right]
$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{2}, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}\sqrt{2}\,x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(\sqrt{2}\,x\right) - \mathrm{I}\sin\left(\sqrt{2}\,x\right)\right) \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{\mathrm{I}}{4}\sqrt{2} \\ -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} \frac{\cos\left(\sqrt{2}\,x\right)}{4} - \frac{\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{4} \\ -\frac{\mathrm{I}}{4}\left(\cos\left(\sqrt{2}\,x\right) - \mathrm{I}\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2} \\ -\frac{\cos\left(\sqrt{2}\,x\right)}{2} + \frac{\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{\mathrm{I}}{2}\left(\cos\left(\sqrt{2}\,x\right) - \mathrm{I}\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2} \\ \cos\left(\sqrt{2}\,x\right) - \mathrm{I}\sin\left(\sqrt{2}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_4(x) = \left[\begin{array}{c} \frac{\cos(\sqrt{2}\,x)}{4} \\ -\frac{\sqrt{2}\,\sin(\sqrt{2}\,x)}{4} \\ -\frac{\cos(\sqrt{2}\,x)}{2} \\ \frac{\sqrt{2}\,\sin(\sqrt{2}\,x)}{2} \\ \cos(\sqrt{2}\,x) \end{array}\right], \vec{y}_5(x) = \left[\begin{array}{c} -\frac{\sin(\sqrt{2}\,x)}{4} \\ -\frac{\sqrt{2}\,\cos(\sqrt{2}\,x)}{4} \\ \frac{\sin(\sqrt{2}\,x)}{2} \\ \frac{\sqrt{2}\,\cos(\sqrt{2}\,x)}{2} \\ -\sin(\sqrt{2}\,x) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + c_5\vec{y}_5(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^x \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right] + \left[\begin{array}{c} \frac{c_2\cos(2x)}{16} - \frac{c_3\sin(2x)}{16} + \frac{c_4\cos(\sqrt{2}\,x)}{4} - \frac{c_5\sin(\sqrt{2}\,x)}{4} \\ -\frac{c_2\sin(2x)}{8} - \frac{c_3\cos(2x)}{8} - \frac{c_4\sqrt{2}\,\sin(\sqrt{2}\,x)}{4} - \frac{c_5\sqrt{2}\,\cos(\sqrt{2}\,x)}{4} \\ -\frac{c_2\cos(2x)}{4} + \frac{c_3\sin(2x)}{4} - \frac{c_4\cos(\sqrt{2}\,x)}{2} + \frac{c_5\sin(\sqrt{2}\,x)}{2} \\ \frac{c_2\sin(2x)}{2} + \frac{c_3\cos(2x)}{2} + \frac{c_4\sqrt{2}\,\sin(\sqrt{2}\,x)}{2} + \frac{c_5\sqrt{2}\,\cos(\sqrt{2}\,x)}{2} \\ c_2\cos(2x) - c_3\sin(2x) + c_4\cos(\sqrt{2}\,x) - c_5\sin(\sqrt{2}\,x) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^x - \frac{c_5\sin(\sqrt{2}\,x)}{4} + \frac{c_4\cos(\sqrt{2}\,x)}{4} - \frac{c_3\sin(2x)}{16} + \frac{c_2\cos(2x)}{16}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$5)-diff(y(x),x$4)+6*diff(y(x),x$3)-6*diff(y(x),x$2)+8*diff(y(x),x)-8*y(x)
```

$$y(x) = \mathrm{e}^{x} c_1 + c_2 \sin\left(\sqrt{2}\, x\right) + c_3 \cos\left(\sqrt{2}\, x\right) + c_4 \sin\left(2x\right) + c_5 \cos\left(2x\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 48

```
DSolve[y'''''[x]-y''''[x]+6*y'''[x]-6*y''[x]+8*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to c_5 e^{x} + c_1 \cos(2x) + c_3 \cos\left(\sqrt{2}x\right) + c_2 \sin(2x) + c_4 \sin\left(\sqrt{2}x\right)$$

11 Exercise 19, page 86

11.1 problem 1

Internal problem ID [2140]
Internal file name [`OUTPUT/2140_Monday_February_26_2024_09_17_48_AM_784968/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 4y = 3\cos(x)}$$

11.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = 3\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(x) - 5A_2 \sin(x) = 3\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{5}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-2x}\right) + \left(-\frac{3\cos(x)}{5}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{3\cos(x)}{5} \tag{1}$$

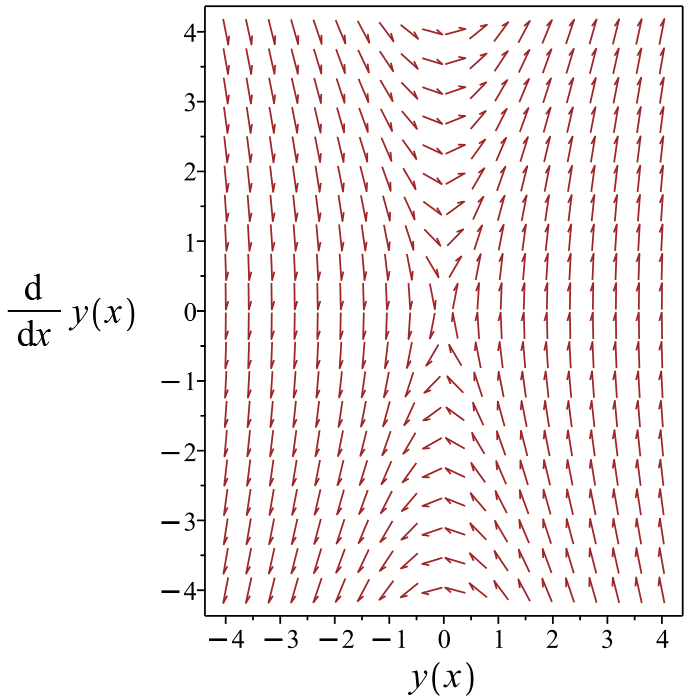


Figure 537: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{3\cos(x)}{5}$$

Verified OK.

11.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 370: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-2x} \int \frac{1}{\mathrm{e}^{-4x}}\,dx \\ &= \mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
3 \cos (x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos (x), \sin (x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2x}}{4}, \mathrm{e}^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos (x) + A_2 \sin (x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(x) - 5A_2 \sin(x) = 3\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{5}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}\right) + \left(-\frac{3\cos(x)}{5}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} - \frac{3\cos(x)}{5} \tag{1}$$

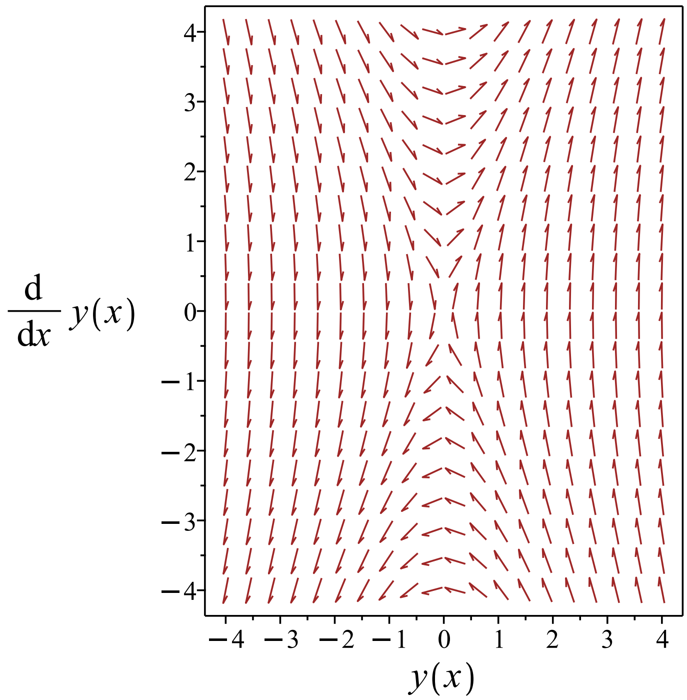


Figure 538: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} - \frac{3\cos(x)}{5}$$

Verified OK.

11.1.3 Maple step by step solution

Let's solve

$y'' - 4y = 3\cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

$r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 3\cos(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{2x} \\ -2\,e^{-2x} & 2\,e^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{3\,e^{-2x} \left(\int e^{2x} \cos(x) dx\right)}{4} + \frac{3\,e^{2x} \left(\int e^{-2x} \cos(x) dx\right)}{4}$

  ◦ Compute integrals

  $y_p(x) = -\frac{3\cos(x)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{2x} - \frac{3\cos(x)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)-4*y(x)=3*cos(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{2x} + e^{-2x} c_1 - \frac{3\cos(x)}{5}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 28

```
DSolve[y''[x]-4*y[x]==3*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3\cos(x)}{5} + c_1 e^{2x} + c_2 e^{-2x}$$

11.2 problem 2

Internal problem ID [2141]
Internal file name [`OUTPUT/2141_Monday_February_26_2024_09_17_49_AM_99659885/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \sin(2x)}$$

11.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(2x) - 3A_2 \sin(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\sin(2x)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\sin(2x)}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{\sin(2x)}{3} \tag{1}$$

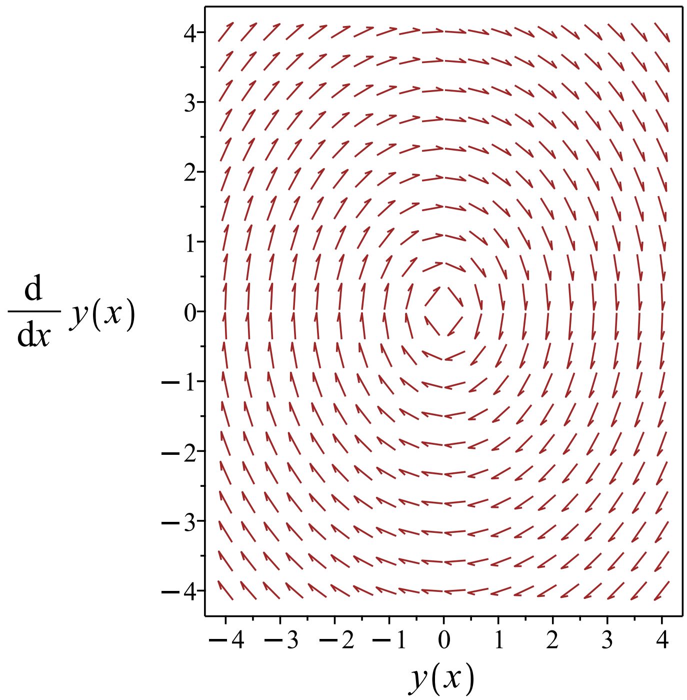


Figure 539: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{\sin(2x)}{3}$$

Verified OK.

11.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 372: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(2x) - 3A_2 \sin(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\sin(2x)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\sin(2x)}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{\sin(2x)}{3} \tag{1}$$

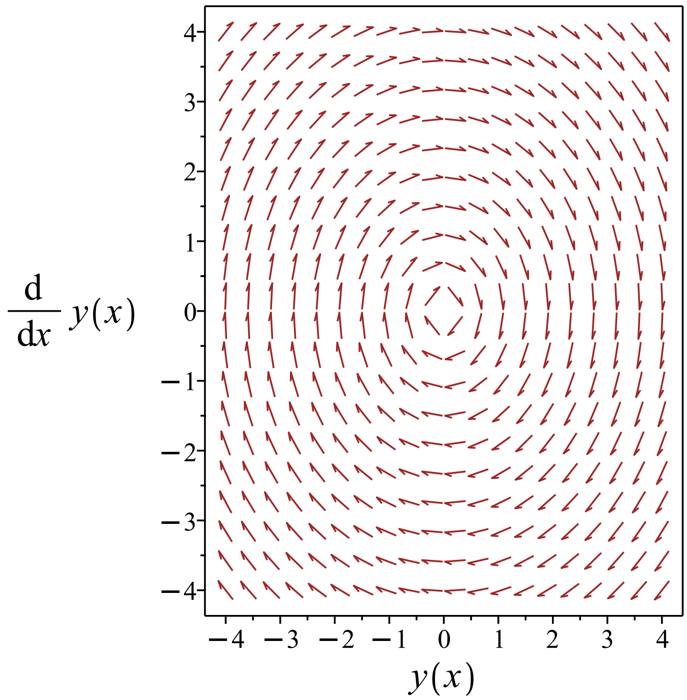


Figure 540: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{\sin(2x)}{3}$$

Verified OK.

11.2.3 Maple step by step solution

Let's solve

$y'' + y = \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(2x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \sin(2x)\sin(x)\,dx\right) + \sin(x)\left(\int \cos(x)\sin(2x)\,dx\right)$

  ◦ Compute integrals

  $y_p(x) = -\frac{\sin(2x)}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \frac{\sin(2x)}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+y(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(3c_1 - 2\sin(x))\cos(x)}{3} + \sin(x)\, c_2$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 23

```
DSolve[y''[x]+y[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \sin(x) + \cos(x)\left(-\frac{2\sin(x)}{3} + c_1\right)$$

11.3 problem 3

Internal problem ID [2142]
Internal file name [`OUTPUT/2142_Monday_February_26_2024_09_17_49_AM_22856276/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' - 2y = e^x}$$

11.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-2)} \\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,e^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{-2x}\right) + \left(\frac{x\,e^x}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-2x} + \frac{x\,e^x}{3} \tag{1}$$

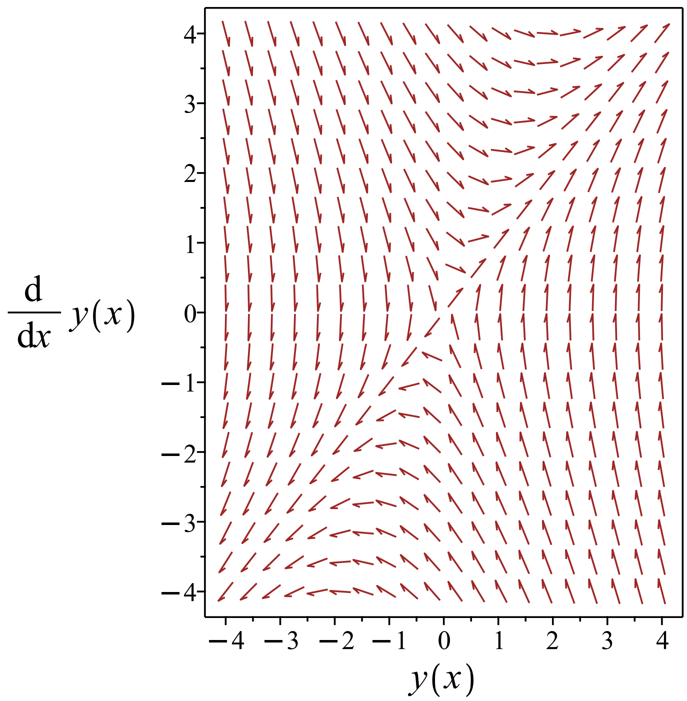


Figure 541: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-2x} + \frac{x\,e^x}{3}$$

Verified OK.

11.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= -2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{9z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 374: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{e^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-2x}\right) + c_2 \left(e^{-2x} \left(\frac{e^{3x}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{c_2 e^{x}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \frac{\mathrm{e}^{x}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{x}}{3} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{x}}{3} \\ -2\,\mathrm{e}^{-2x} & \frac{\mathrm{e}^{x}}{3} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(\frac{\mathrm{e}^{x}}{3}\right) - \left(\frac{\mathrm{e}^{x}}{3}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-2x}\mathrm{e}^{x}$$

Which simplifies to

$$W = e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2x}}{3}}{e^{-x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{3x}}{3}dx$$

Hence

$$u_1 = -\frac{e^{3x}}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x}e^{x}}{e^{-x}}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{-2x}e^{3x}}{9} + \frac{x\,e^{x}}{3}$$

Which simplifies to

$$y_p(x) = \frac{(3x-1)\,e^{x}}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-2x} + \frac{c_2e^{x}}{3}\right) + \left(\frac{(3x-1)\,e^{x}}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} + \frac{(3x-1)\, e^{x}}{9} \tag{1}$$

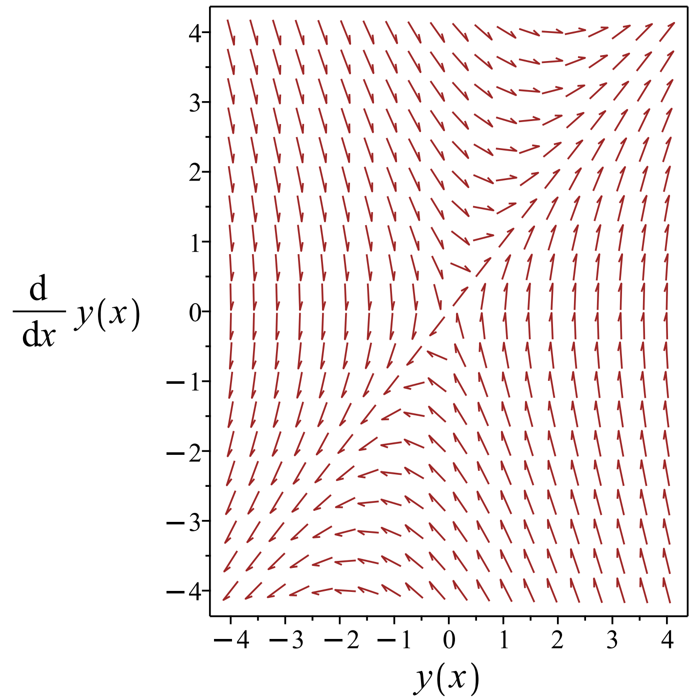


Figure 542: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} + \frac{(3x-1)\, e^{x}}{9}$$

Verified OK.

11.3.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\left(\mathrm{e}^{3x}\left(\int 1 dx\right) - \left(\int \mathrm{e}^{3x} dx\right)\right)\mathrm{e}^{-2x}}{3}$

  ◦ Compute integrals

  $y_p(x) = \frac{(3x-1)\mathrm{e}^{x}}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + \frac{(3x-1)\mathrm{e}^{x}}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{-2x}((x + 3c_1) e^{3x} + 3c_2)}{3}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 29

```
DSolve[y''[x]+y'[x]-2*y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-2x} + e^x \left(\frac{x}{3} - \frac{1}{9} + c_2\right)$$

11.4 problem 4

Internal problem ID [2143]
Internal file name [`OUTPUT/2143_Monday_February_26_2024_09_17_50_AM_57356254/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y' + 2y = \mathrm{e}^{-2x}}$$

11.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = \mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\
&= -\frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-2x} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-2x} = e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -e^{-2x} x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2 e^{-2x}) + (-e^{-2x} x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} x \tag{1}$$

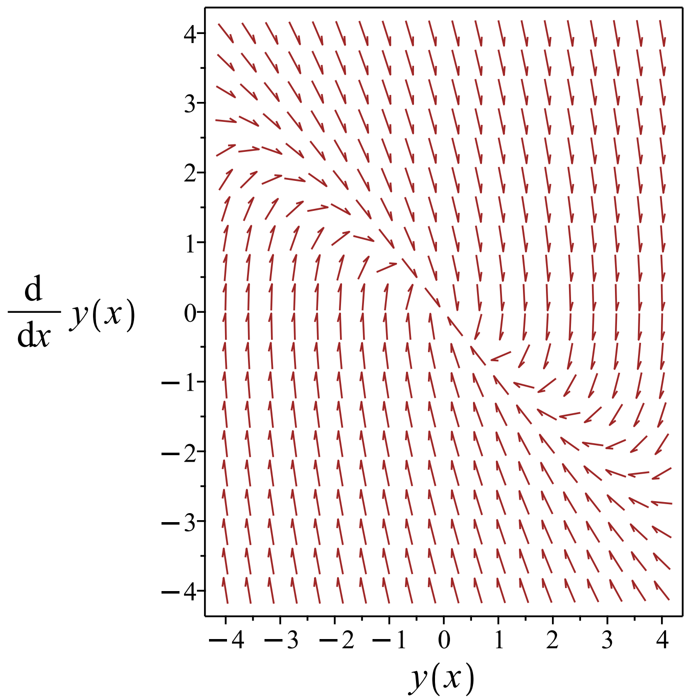


Figure 543: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} x$$

Verified OK.

11.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 376: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^{x})\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-2x} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-2x} = e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -e^{-2x} x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \left(-e^{-2x} x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-x} - e^{-2x} x \tag{1}$$

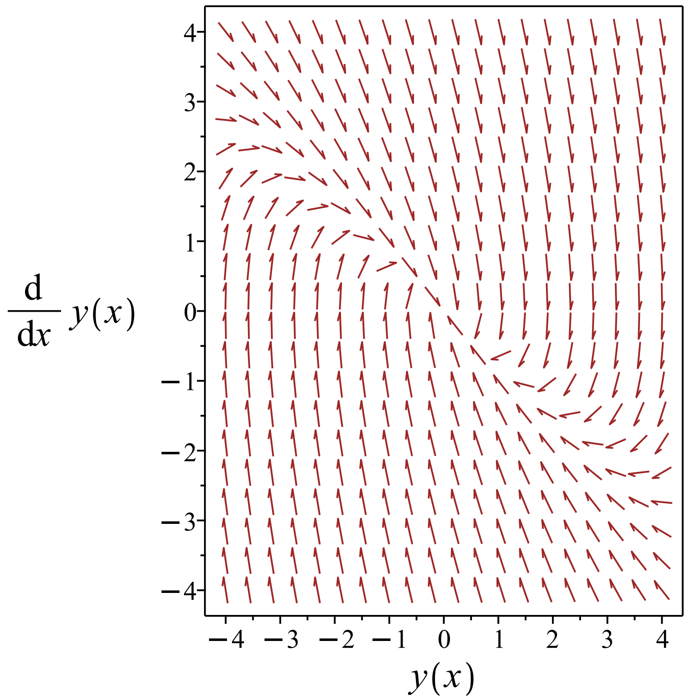


Figure 544: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} - e^{-2x} x$$

Verified OK.

11.4.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = e^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

$r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \mathrm{e}^{-2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-2x}\left(\int 1 dx\right) + \mathrm{e}^{-x}\left(\int \mathrm{e}^{-x} dx\right)$

  - Compute integrals

    $y_p(x) = \mathrm{e}^{-2x}(-x - 1)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \mathrm{e}^{-2x}(-x - 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=exp(-2*x),y(x), singsol=all)
```

$$y(x) = -\mathrm{e}^{-x}\left(\mathrm{e}^{-x}(x + c_1 + 1) - c_2\right)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 24

```
DSolve[y''[x]+3*y'[x]+2*y[x]==Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(-x + c_2 e^x - 1 + c_1)$$

11.5 problem 5

Internal problem ID [2144]
Internal file name [`OUTPUT/2144_Monday_February_26_2024_09_17_50_AM_97496273/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''+y'+y=\sin(x)}$$

11.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=1, B=1, C=1, f(x)=\sin(x)$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'+y=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=1, C=1$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+\lambda \mathrm{e}^{\lambda x}+\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+\lambda+1=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{1^2-(4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2}+\frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2}+\frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha \pm i\beta$$

Where $\alpha=-\frac{1}{2}$ and $\beta=\frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y=e^{\alpha x}(c_1 \cos(\beta x)+c_2 \sin(\beta x))$$

Which becomes

$$y=e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right)+c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + (-\cos(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) - \cos(x) \tag{1}$$

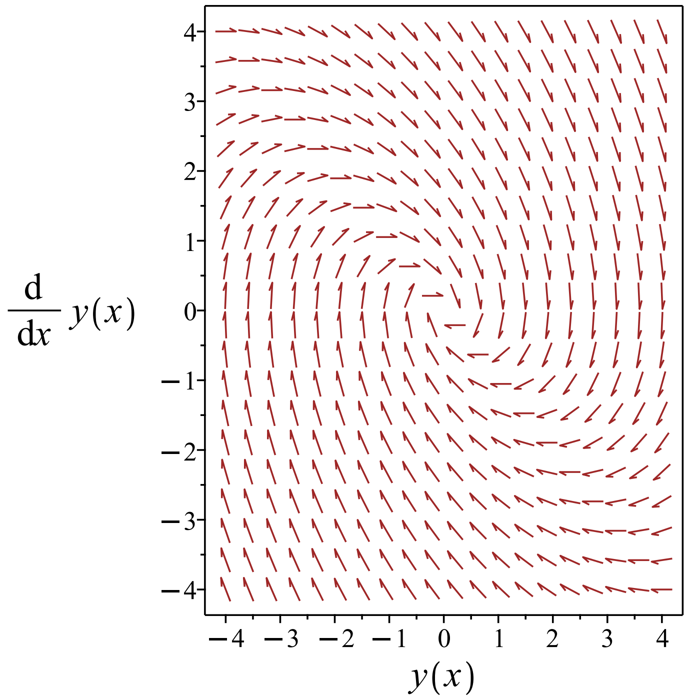


Figure 545: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) - \cos(x)$$

Verified OK.

11.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 378: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'+y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\cos\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}c_1+\frac{2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x),\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right),\frac{2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1\cos(x)+A_2\sin(x)$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\sin(x)+A_2\cos(x)=\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1=-1,A_2=0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=-\cos(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}\right) + (-\cos(x))
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} - \cos(x) \tag{1}
$$

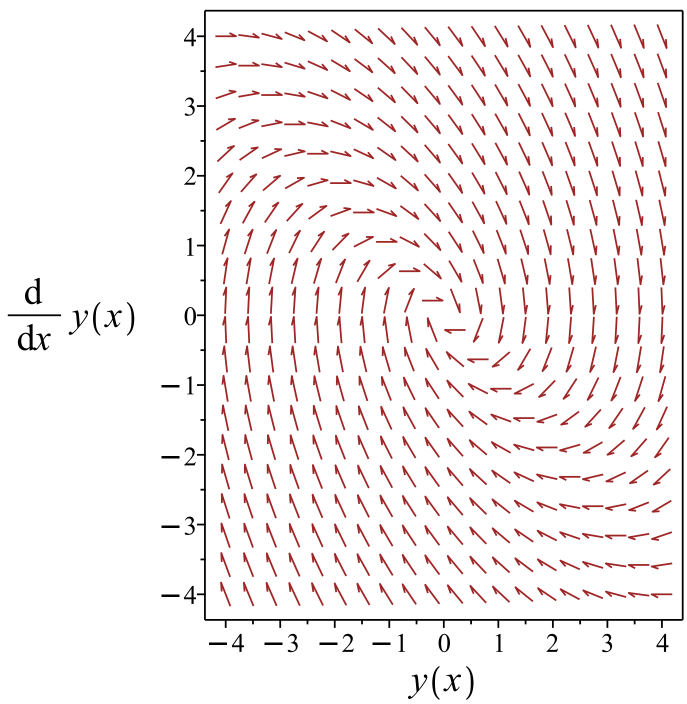


Figure 546: Slope field plot

<u>Verification of solutions</u>

$$
y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} - \cos(x)
$$

Verified OK.

11.5.3 Maple step by step solution

Let's solve

$y'' + y' + y = \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} c_1 + \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2} \end{bmatrix}$

- ○ Compute Wronskian

$$W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{2\,e^{-\frac{x}{2}}\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}\sin(x)\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}\sin(x)\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$$

- Compute integrals

$$y_p(x) = -\cos(x)$$

- Substitute particular solution into general solution to ODE

$$y = \cos\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}c_1 + \sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}c_2 - \cos(x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=sin(x),y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)c_2 + e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)c_1 - \cos(x)$$

✓ Solution by Mathematica

Time used: 1.174 (sec). Leaf size: 53

```
DSolve[y''[x]+y'[x]+y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(-e^{x/2}\cos(x) + c_2\cos\left(\frac{\sqrt{3}x}{2}\right) + c_1\sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

11.6 problem 6

Internal problem ID [2145]
Internal file name [`OUTPUT/2145_Monday_February_26_2024_09_17_51_AM_38707999/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' + y = x^2}$$

11.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3x^2 + A_2x + 2xA_3 + A_1 + A_2 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -2, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + (x^2 - 2x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \right) + x^2 - 2x \tag{1}$$

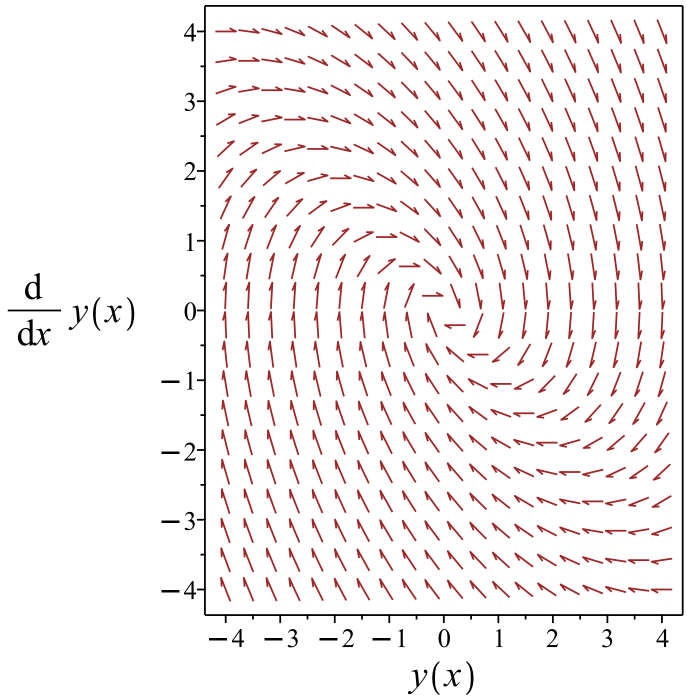


Figure 547: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \right) + x^2 - 2x$$

Verified OK.

11.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 380: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_3 x^2 + A_2 x + 2x A_3 + A_1 + A_2 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -2, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}\right) + \left(x^2 - 2x\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} + x^2 - 2x \tag{1}$$

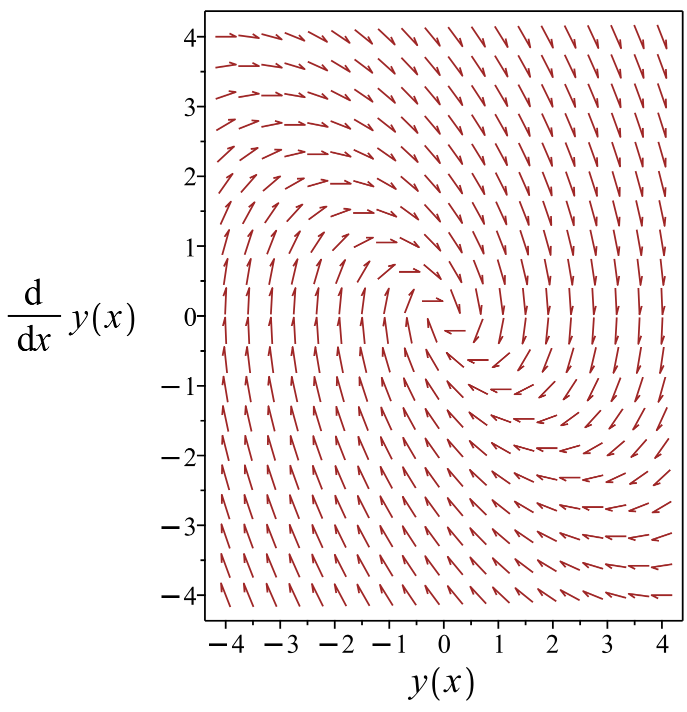


Figure 548: Slope field plot

Verification of solutions

$$y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3} + x^2 - 2x$$

Verified OK.

11.6.3 Maple step by step solution

Let's solve

$y'' + y' + y = x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} c_1 + \sin\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^2\right]$

- ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \sqrt{3}}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\,e^{-\frac{x}{2}} \sqrt{3} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\int x^2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right) \left(\int x^2 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) dx\right)\right)}{3}$

- Compute integrals

  $y_p(x) = x(-2 + x)$

- Substitute particular solution into general solution to ODE

  $y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_2 + x(-2 + x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=x^2,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + x^2 - 2x$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 54

```
DSolve[y''[x]+y'[x]+y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}\left(e^{x/2}(x-2)x + c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

11.7 problem 7

Internal problem ID [2146]
Internal file name [`OUTPUT/2146_Monday_February_26_2024_09_17_51_AM_54095803/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''+3y'+2y = x\,\mathrm{e}^{-x}}$$

11.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = x\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y''+3y'+2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}, x^2e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x\,e^{-x} + A_2x^2e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1e^{-x} + 2A_2e^{-x} + 2A_2x\,e^{-x} = x\,e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,e^{-x} + \frac{x^2e^{-x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^{-x} + c_2e^{-2x}) + \left(-x\,e^{-x} + \frac{x^2e^{-x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} - x\,e^{-x} + \frac{x^2 e^{-x}}{2} \tag{1}$$

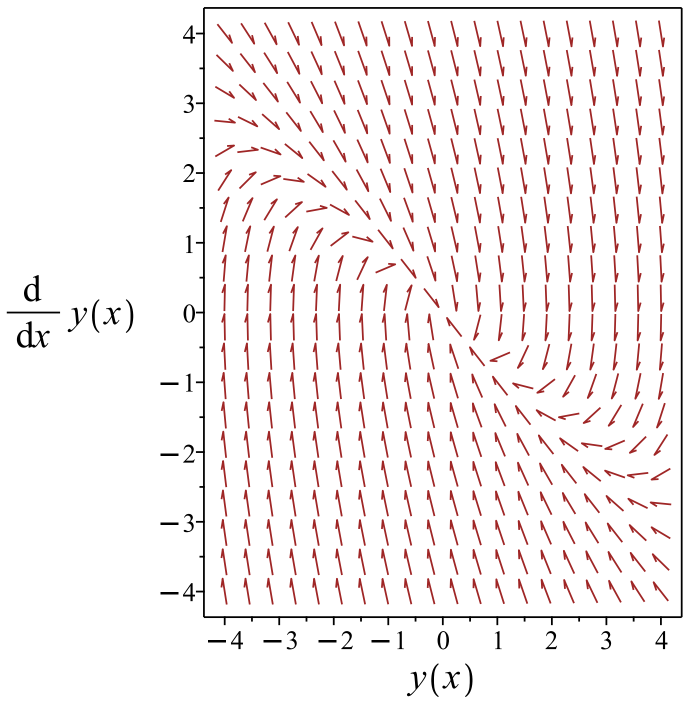


Figure 549: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} - x\,e^{-x} + \frac{x^2 e^{-x}}{2}$$

Verified OK.

11.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 382: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3}{1} \, dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}, x^2 e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-x} + A_2 x^2 e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{-x} + 2A_2 e^{-x} + 2A_2 x\,e^{-x} = x\,e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,e^{-x} + \frac{x^2 e^{-x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \left(-x\,e^{-x} + \frac{x^2 e^{-x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-x} - x\,e^{-x} + \frac{x^2 e^{-x}}{2} \tag{1}$$

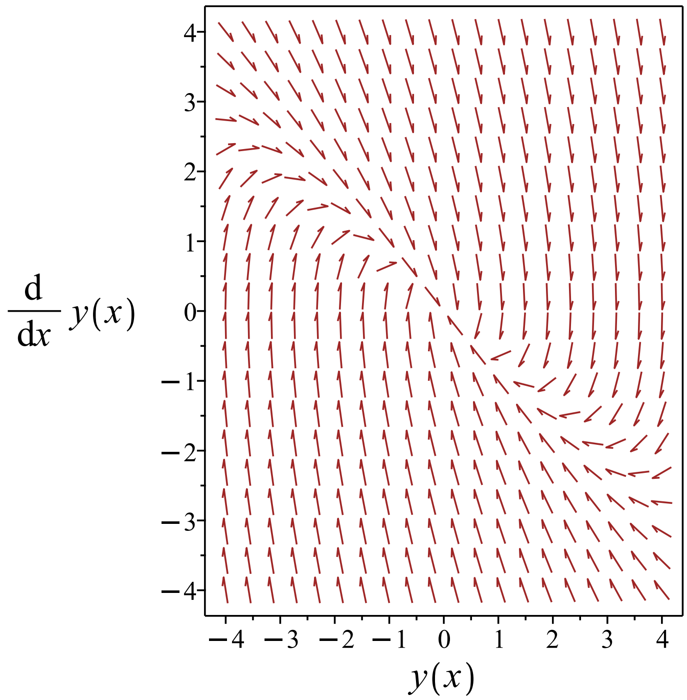


Figure 550: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} - x\,e^{-x} + \frac{x^2 e^{-x}}{2}$$

Verified OK.

11.7.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = x\,e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

$r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\,e^{-x}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$

- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-2x} \left(\int x\,e^{x} dx\right) + e^{-x} \left(\int x dx\right)$

- ◦ Compute integrals

  $y_p(x) = e^{-x} \left(1 + \frac{1}{2}x^2 - x\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + e^{-x} \left(1 + \frac{1}{2}x^2 - x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=x*exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{(-2\,\mathrm{e}^{-x}c_1 + x^2 + 2c_2 - 2x)\,\mathrm{e}^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 35

```
DSolve[y''[x]+3*y'[x]+2*y[x]==x*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-2x}\left(e^x\left(x^2 - 2x + 2 + 2c_2\right) + 2c_1\right)$$

11.8 problem 8

Internal problem ID [2147]
Internal file name [`OUTPUT/2147_Monday_February_26_2024_09_17_52_AM_21072012/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 8.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _with_linear_symmetries]]`

$$\boxed{y'''' - y = e^x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y = 0$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{-ix} c_3 + e^{ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{-ix}$$
$$y_4 = e^{ix}$$

Now the particular solution to the given ODE is found

$$y'''' - y = e^{x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{ix}, e^{-x}, e^{-ix}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{x} = e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2\mathrm{e}^x + \mathrm{e}^{-ix}c_3 + \mathrm{e}^{ix}c_4\right) + \left(\frac{x\,\mathrm{e}^x}{4}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1\mathrm{e}^{-x} + c_2\mathrm{e}^x + \mathrm{e}^{-ix}c_3 + \mathrm{e}^{ix}c_4 + \frac{x\,\mathrm{e}^x}{4} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1\mathrm{e}^{-x} + c_2\mathrm{e}^x + \mathrm{e}^{-ix}c_3 + \mathrm{e}^{ix}c_4 + \frac{x\,\mathrm{e}^x}{4}$$

Verified OK.

11.8.1 Maple step by step solution

Let's solve

$y'''' - y = \mathrm{e}^x$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = e^x + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = e^x + y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\sin(x) & -\cos(x) \\ e^{-x} & e^{x} & -\cos(x) & \sin(x) \\ -e^{-x} & e^{x} & \sin(x) & \cos(x) \\ e^{-x} & e^{x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\sin(x) & -\cos(x) \\ e^{-x} & e^{x} & -\cos(x) & \sin(x) \\ -e^{-x} & e^{x} & \sin(x) & \cos(x) \\ e^{-x} & e^{x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} \\ \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{e^{-x}}{8}+\frac{(-3+2x)e^{x}}{8}+\frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ -\frac{e^{-x}}{8}+\frac{(2x-1)e^{x}}{8}+\frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ \frac{e^{-x}}{8}+\frac{(1+2x)e^{x}}{8}-\frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ -\frac{e^{-x}}{8}+\frac{(2x+3)e^{x}}{8}-\frac{\cos(x)}{4}+\frac{\sin(x)}{4} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} \frac{e^{-x}}{8}+\frac{(-3+2x)e^{x}}{8}+\frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ -\frac{e^{-x}}{8}+\frac{(2x-1)e^{x}}{8}+\frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ \frac{e^{-x}}{8}+\frac{(1+2x)e^{x}}{8}-\frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ -\frac{e^{-x}}{8}+\frac{(2x+3)e^{x}}{8}-\frac{\cos(x)}{4}+\frac{\sin(x)}{4} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = \frac{(-8c_1+1)e^{-x}}{8} + \frac{(2x+8c_2-3)e^{x}}{8} + \frac{(-8c_4+2)\cos(x)}{8} + \frac{(-8c_3+2)\sin(x)}{8}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$4)-y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = c_4 \mathrm{e}^{-x} + \frac{(4c_2 + x)\,\mathrm{e}^{x}}{4} + \cos(x)\,c_1 + c_3 \sin(x)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 45

```
DSolve[y''''[x]-y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x x}{4} - \frac{3e^x}{8} + c_1 e^x + c_3 e^{-x} + c_2 \cos(x) + c_4 \sin(x)$$

11.9 problem 9

Internal problem ID [2148]
Internal file name [`OUTPUT/2148_Monday_February_26_2024_09_17_52_AM_85505114/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y = x + e^{2x}}$$

11.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = x + e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x+\mathrm{e}^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{2x}\},\{1,x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x},\mathrm{e}^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{2x}\},\{1,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{2x} + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\mathrm{e}^{2x} - 4A_2 - 4A_3 x = x + \mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}, A_2 = 0, A_3 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{2x}}{4} - \frac{x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{2x} + c_2\mathrm{e}^{-2x}\right) + \left(\frac{x\,\mathrm{e}^{2x}}{4} - \frac{x}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x\,e^{2x}}{4} - \frac{x}{4} \tag{1}$$

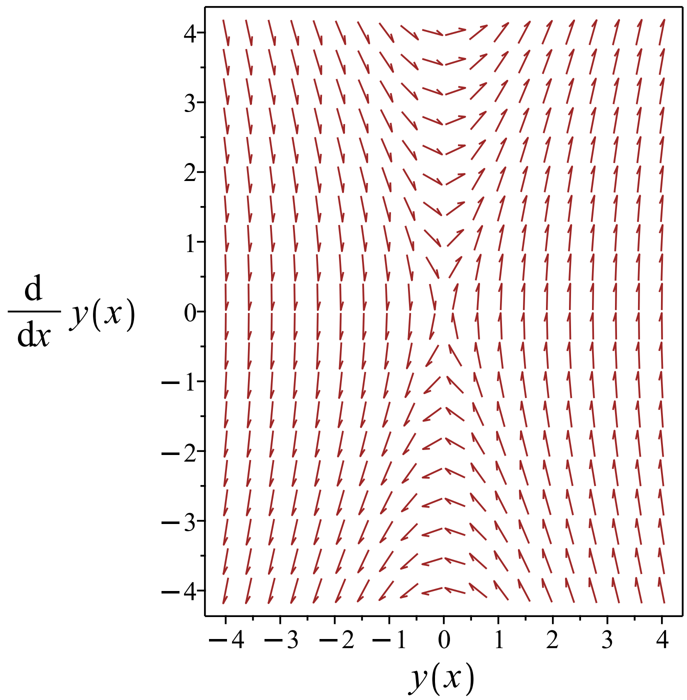


Figure 551: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x\,e^{2x}}{4} - \frac{x}{4}$$

Verified OK.

11.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 385: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{-2x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= e^{-2x} \int \frac{1}{e^{-4x}}\,dx \\ &= e^{-2x}\left(\frac{e^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-2x}\right) + c_2\left(e^{-2x}\left(\frac{e^{4x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \frac{\mathrm{e}^{2x}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{2x}}{4} \\ \frac{d}{dx}(\mathrm{e}^{-2x}) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{2x}}{4} \\ -2\,\mathrm{e}^{-2x} & \frac{\mathrm{e}^{2x}}{2} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-2x})\left(\frac{\mathrm{e}^{2x}}{2}\right) - \left(\frac{\mathrm{e}^{2x}}{4}\right)(-2\,\mathrm{e}^{-2x})$$

Which simplifies to

$$W = \mathrm{e}^{-2x}\mathrm{e}^{2x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2x}(x+e^{2x})}{4}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{2x}(x + e^{2x})}{4} dx$$

Hence

$$u_1 = -\frac{x\, e^{2x}}{8} + \frac{e^{2x}}{16} - \frac{e^{4x}}{16}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x}(x + e^{2x})}{1} dx$$

Which simplifies to

$$u_2 = \int \left(e^{-2x}x + 1\right) dx$$

Hence

$$u_2 = x - \frac{e^{-2x}x}{2} - \frac{e^{-2x}}{4}$$

Which simplifies to

$$u_1 = \frac{(1 - 2x)\, e^{2x}}{16} - \frac{e^{4x}}{16}$$

$$u_2 = \frac{(-1 - 2x)\, e^{-2x}}{4} + x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{(1 - 2x)\, e^{2x}}{16} - \frac{e^{4x}}{16}\right) e^{-2x} + \frac{\left(\frac{(-1-2x)e^{-2x}}{4} + x\right) e^{2x}}{4}$$

Which simplifies to

$$y_p(x) = \frac{(4x-1)\,\mathrm{e}^{2x}}{16} - \frac{x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}\right) + \left(\frac{(4x-1)\,\mathrm{e}^{2x}}{16} - \frac{x}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4} + \frac{(4x-1)\,\mathrm{e}^{2x}}{16} - \frac{x}{4} \tag{1}$$

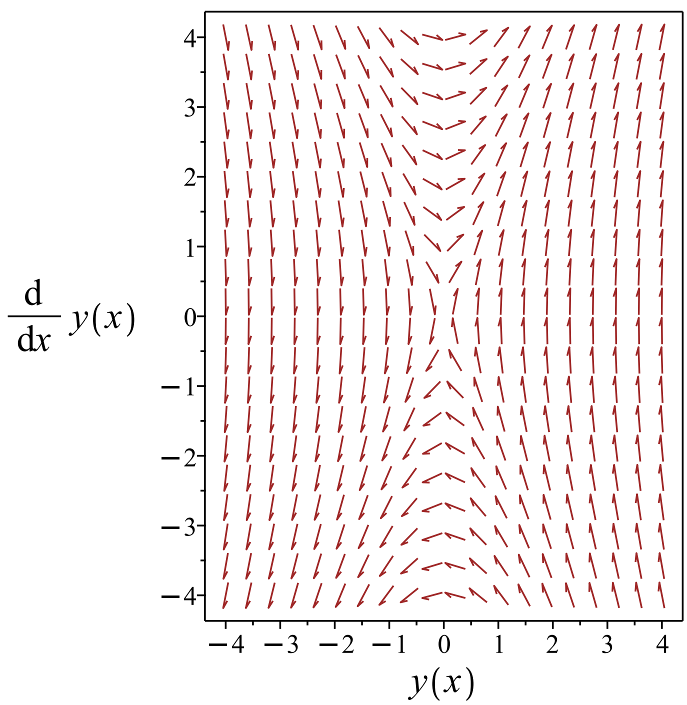


Figure 552: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4} + \frac{(4x-1)\,\mathrm{e}^{2x}}{16} - \frac{x}{4}$$

Verified OK.

11.9.3 Maple step by step solution

Let's solve

$y'' - 4y = x + e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x + e^{2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{2x} \\ -2\,e^{-2x} & 2\,e^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-2x}\left(\int e^{2x}(x+e^{2x})\,dx\right)}{4} + \frac{e^{2x}\left(\int (e^{-2x}x+1)\,dx\right)}{4}$$

- Compute integrals

$$y_p(x) = \frac{(4x-1)e^{2x}}{16} - \frac{x}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-2x} + c_2 e^{2x} + \frac{(4x-1)e^{2x}}{16} - \frac{x}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-4*y(x)=x+exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(4x + 16c_2 - 1)\,e^{2x}}{16} + e^{-2x}c_1 - \frac{x}{4}$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 36

```
DSolve[y''[x]-4*y[x]==x+Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{4} + e^{2x}\left(\frac{x}{4} - \frac{1}{16} + c_1\right) + c_2 e^{-2x}$$

11.10 problem 10

Internal problem ID [2149]
Internal file name [`OUTPUT/2149_Monday_February_26_2024_09_17_52_AM_2751523/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 9y = e^{3x} + \sin(3x)}$$

11.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -9, f(x) = e^{3x} + \sin(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-9)} \\ &= \pm 3\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3 \\ \lambda_2 &= -3\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(3)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 e^{3x} + e^{-3x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + e^{-3x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^{3x} + \sin(3x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{3x}\}, \{\cos(3x), \sin(3x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{e^{-3x}, e^{3x}\}
$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\,e^{3x}\}, \{\cos(3x), \sin(3x)\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 x\,e^{3x} + A_2 \cos(3x) + A_3 \sin(3x)
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
6A_1 e^{3x} - 18A_2 \cos(3x) - 18A_3 \sin(3x) = e^{3x} + \sin(3x)
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = \frac{1}{6}, A_2 = 0, A_3 = -\frac{1}{18}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = \frac{x\,e^{3x}}{6} - \frac{\sin(3x)}{18}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{3x} + e^{-3x} c_2\right) + \left(\frac{x\,e^{3x}}{6} - \frac{\sin(3x)}{18}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} + e^{-3x} c_2 + \frac{x\,e^{3x}}{6} - \frac{\sin(3x)}{18} \tag{1}$$

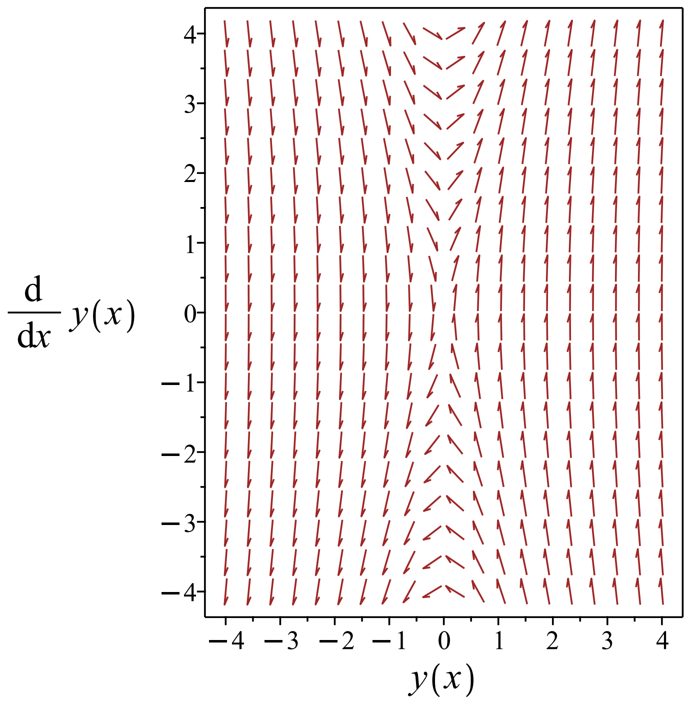


Figure 553: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + e^{-3x} c_2 + \frac{x\,e^{3x}}{6} - \frac{\sin(3x)}{18}$$

Verified OK.

11.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \qquad (3) \\
C &= -9
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5) \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 9z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 387: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \mathrm{e}^{-3x}\end{aligned}$$

Which simplifies to

$$y_1 = e^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= e^{-3x} \int \frac{1}{e^{-6x}}\,dx \\
&= e^{-3x} \left(\frac{e^{6x}}{6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-3x}\right) + c_2\left(e^{-3x}\left(\frac{e^{6x}}{6}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-3x} + \frac{c_2 e^{3x}}{6}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-3x}$$
$$y_2 = \frac{\mathrm{e}^{3x}}{6}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \frac{\mathrm{e}^{3x}}{6} \\ \frac{d}{dx}\left(\mathrm{e}^{-3x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{3x}}{6}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \frac{\mathrm{e}^{3x}}{6} \\ -3\,\mathrm{e}^{-3x} & \frac{\mathrm{e}^{3x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-3x}\right)\left(\frac{\mathrm{e}^{3x}}{2}\right) - \left(\frac{\mathrm{e}^{3x}}{6}\right)\left(-3\,\mathrm{e}^{-3x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-3x}\mathrm{e}^{3x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{3x}(e^{3x}+\sin(3x))}{6}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{3x}(e^{3x} + \sin(3x))}{6} dx$$

Hence

$$u_1 = -\frac{e^{6x}}{36} + \frac{\cos(3x)\, e^{3x}}{36} - \frac{\sin(3x)\, e^{3x}}{36}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-3x}(e^{3x} + \sin(3x))}{1} dx$$

Which simplifies to

$$u_2 = \int \left(1 + \sin(3x)\, e^{-3x}\right) dx$$

Hence

$$u_2 = x - \frac{\cos(3x)\, e^{-3x}}{6} - \frac{\sin(3x)\, e^{-3x}}{6}$$

Which simplifies to

$$u_1 = -\frac{e^{3x}(e^{3x} - \cos(3x) + \sin(3x))}{36}$$

$$u_2 = \frac{(-\cos(3x) - \sin(3x))\, e^{-3x}}{6} + x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{3x}(e^{3x} - \cos(3x) + \sin(3x))\, e^{-3x}}{36} + \frac{\left(\frac{(-\cos(3x)-\sin(3x))e^{-3x}}{6} + x\right) e^{3x}}{6}$$

Which simplifies to

$$y_p(x) = \frac{(-1+6x)\,\mathrm{e}^{3x}}{36} - \frac{\sin(3x)}{18}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{3x}}{6}\right) + \left(\frac{(-1+6x)\,\mathrm{e}^{3x}}{36} - \frac{\sin(3x)}{18}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{3x}}{6} + \frac{(-1+6x)\,\mathrm{e}^{3x}}{36} - \frac{\sin(3x)}{18} \tag{1}$$

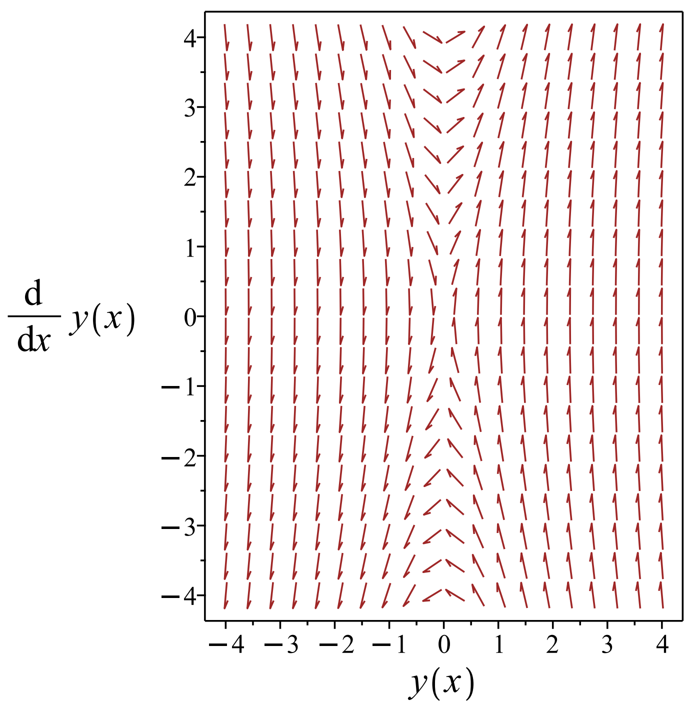


Figure 554: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{3x}}{6} + \frac{(-1+6x)\,\mathrm{e}^{3x}}{36} - \frac{\sin(3x)}{18}$$

Verified OK.

11.10.3 Maple step by step solution

Let's solve

$y'' - 9y = e^{3x} + \sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r-3)(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + c_2 e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{3x} + \sin(3x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & e^{3x} \\ -3\,e^{-3x} & 3\,e^{3x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 6$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-3x}\left(\int e^{3x}\left(e^{3x}+\sin(3x)\right)dx\right)}{6} + \frac{e^{3x}\left(\int \left(1+\sin(3x)e^{-3x}\right)dx\right)}{6}$$

- Compute integrals

$$y_p(x) = \frac{(-1+6x)e^{3x}}{36} - \frac{\sin(3x)}{18}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-3x} + c_2 e^{3x} + \frac{(-1+6x)e^{3x}}{36} - \frac{\sin(3x)}{18}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)-9*y(x)=exp(3*x)+sin(3*x),y(x), singsol=all)
```

$$y(x) = \frac{(6x + 36c_2 - 1)\, e^{3x}}{36} + c_1 e^{-3x} - \frac{\sin(3x)}{18}$$

✓ Solution by Mathematica

Time used: 0.247 (sec). Leaf size: 39

```
DSolve[y''[x]-9*y[x]==Exp[3*x]+Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{18}\sin(3x) + e^{3x}\left(\frac{x}{6} - \frac{1}{36} + c_1\right) + c_2 e^{-3x}$$

11.11 problem 11

Internal problem ID [2150]
Internal file name [`OUTPUT/2150_Monday_February_26_2024_09_17_53_AM_10331552/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y' - 6y = x^3}$$

11.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -6, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-6)} \\ &= \frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{5}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(3)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{3x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_4x^3 - 6A_3x^2 - 3x^2A_4 - 6A_2x - 2xA_3 + 6xA_4 - 6A_1 - A_2 + 2A_3 = x^3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{13}{216}, A_2 = -\frac{7}{36}, A_3 = \frac{1}{12}, A_4 = -\frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{6}x^3 + \frac{1}{12}x^2 - \frac{7}{36}x + \frac{13}{216}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{3x} + c_2e^{-2x}\right) + \left(-\frac{1}{6}x^3 + \frac{1}{12}x^2 - \frac{7}{36}x + \frac{13}{216}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{3x} + c_2 e^{-2x} - \frac{x^3}{6} + \frac{x^2}{12} - \frac{7x}{36} + \frac{13}{216} \tag{1}$$

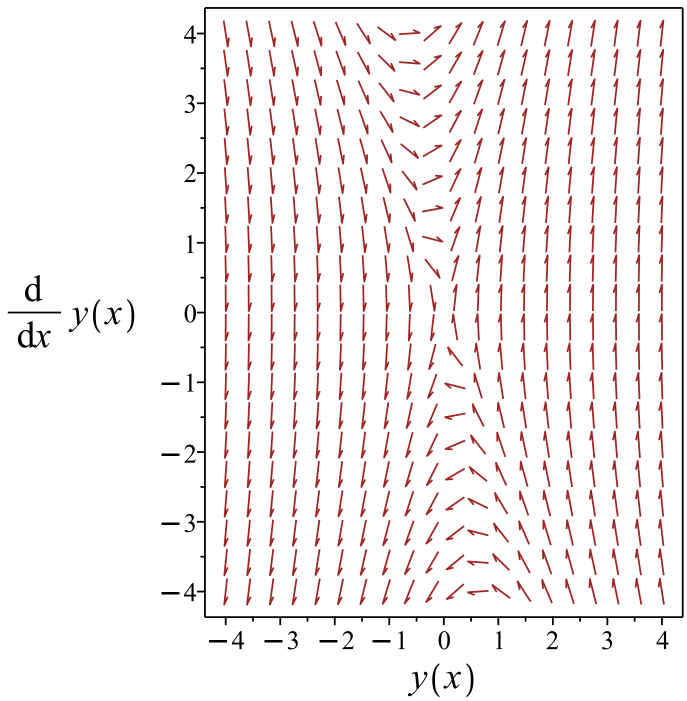


Figure 555: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{-2x} - \frac{x^3}{6} + \frac{x^2}{12} - \frac{7x}{36} + \frac{13}{216}$$

Verified OK.

11.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 389: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{5x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{3x}}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{3x}}{5}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4 x^3 + A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_4x^3 - 6A_3x^2 - 3x^2A_4 - 6A_2x - 2xA_3 + 6xA_4 - 6A_1 - A_2 + 2A_3 = x^3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{13}{216}, A_2 = -\frac{7}{36}, A_3 = \frac{1}{12}, A_4 = -\frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{6}x^3 + \frac{1}{12}x^2 - \frac{7}{36}x + \frac{13}{216}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{3x}}{5}\right) + \left(-\frac{1}{6}x^3 + \frac{1}{12}x^2 - \frac{7}{36}x + \frac{13}{216}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{3x}}{5} - \frac{x^3}{6} + \frac{x^2}{12} - \frac{7x}{36} + \frac{13}{216} \tag{1}$$

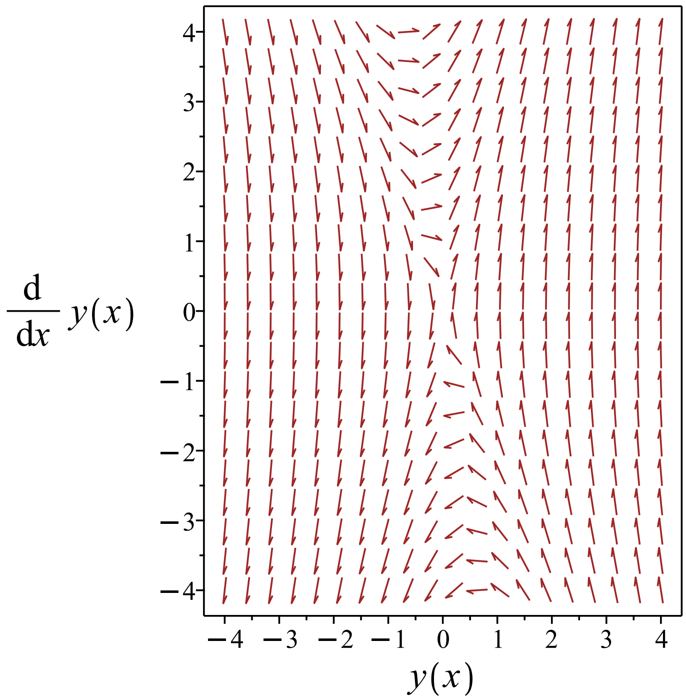


Figure 556: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{3x}}{5} - \frac{x^3}{6} + \frac{x^2}{12} - \frac{7x}{36} + \frac{13}{216}$$

Verified OK.

11.11.3 Maple step by step solution

Let's solve

$y'' - y' - 6y = x^3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 3) = 0$

- Roots of the characteristic polynomial

$r = (-2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^3\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{3x} \\ -2\,\mathrm{e}^{-2x} & 3\,\mathrm{e}^{3x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 5\,\mathrm{e}^{x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\left(\mathrm{e}^{5x}\left(\int \mathrm{e}^{-3x} x^3 dx\right) - \left(\int \mathrm{e}^{2x} x^3 dx\right)\right)\mathrm{e}^{-2x}}{5}$

  ◦ Compute integrals

  $y_p(x) = -\frac{1}{6}x^3 + \frac{1}{12}x^2 - \frac{7}{36}x + \frac{13}{216}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{3x} - \frac{x^3}{6} + \frac{x^2}{12} - \frac{7x}{36} + \frac{13}{216}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-6*y(x)=x^3,y(x), singsol=all)
```

$$y(x) = -\frac{e^{-2x}\left(\left(x^3 - \frac{1}{2}x^2 + \frac{7}{6}x - \frac{13}{36}\right)e^{2x} - 6c_2 e^{5x} - 6c_1\right)}{6}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 41

```
DSolve[y''[x]-y'[x]-6*y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{216}\left(-36x^3 + 18x^2 - 42x + 13\right) + c_1 e^{-2x} + c_2 e^{3x}$$

11.12 problem 12

Internal problem ID [2151]
Internal file name [`OUTPUT/2151_Monday_February_26_2024_09_17_53_AM_55166218/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{-2y'' + 3y = x\,\mathrm{e}^{x}}$$

11.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = -2, B = 0, C = 3, f(x) = x\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$-2y'' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=-2, B=0, C=3$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$-2\lambda^2 \mathrm{e}^{\lambda x}+3\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$-2\lambda^2+3=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=-2, B=0, C=3$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{0}{(2)\,(-2)}\pm\frac{1}{(2)\,(-2)}\sqrt{0^2-(4)\,(-2)\,(3)}\\&=\pm-\frac{\sqrt{6}}{2}\end{aligned}$$

Hence

$$\lambda_1=+-\frac{\sqrt{6}}{2}$$
$$\lambda_2=--\frac{\sqrt{6}}{2}$$

Which simplifies to

$$\lambda_1=-\frac{\sqrt{6}}{2}$$
$$\lambda_2=\frac{\sqrt{6}}{2}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{\left(-\frac{\sqrt{6}}{2}\right)x}+c_2e^{\left(\frac{\sqrt{6}}{2}\right)x}\end{aligned}$$

Or

$$y=c_1\mathrm{e}^{-\frac{\sqrt{6}\,x}{2}}+c_2\mathrm{e}^{\frac{\sqrt{6}\,x}{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-\frac{\sqrt{6}\,x}{2}} + c_2 \mathrm{e}^{\frac{\sqrt{6}\,x}{2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^x, \mathrm{e}^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{\sqrt{6}\,x}{2}}, \mathrm{e}^{\frac{\sqrt{6}\,x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^x + A_2 \mathrm{e}^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \mathrm{e}^x + A_1 x\,\mathrm{e}^x + A_2 \mathrm{e}^x = x\,\mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,\mathrm{e}^x + 4\,\mathrm{e}^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-\frac{\sqrt{6}\,x}{2}} + c_2 \mathrm{e}^{\frac{\sqrt{6}\,x}{2}}\right) + (x\,\mathrm{e}^x + 4\,\mathrm{e}^x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-\frac{\sqrt{6}\, x}{2}} + c_2 e^{\frac{\sqrt{6}\, x}{2}} + x\, e^{x} + 4\, e^{x} \tag{1}$$

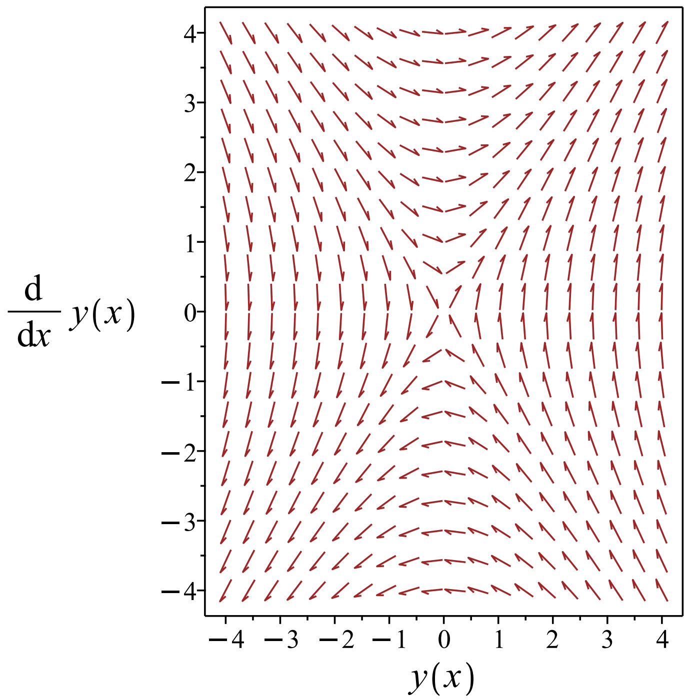


Figure 557: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{\sqrt{6}\, x}{2}} + c_2 e^{\frac{\sqrt{6}\, x}{2}} + x\, e^{x} + 4\, e^{x}$$

Verified OK.

11.12.2 Solving using Kovacic algorithm

Writing the ode as

$$-2y'' + 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2 \\ B &= 0 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{3z(x)}{2} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 391: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{3}{2}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{\sqrt{6}\,x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \mathrm{e}^{-\frac{\sqrt{6}\,x}{2}}\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{\sqrt{6}\, x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-\frac{\sqrt{6}\, x}{2}} \int \frac{1}{\mathrm{e}^{-\sqrt{6}\, x}}\, dx \\
&= \mathrm{e}^{-\frac{\sqrt{6}\, x}{2}} \left(\frac{\mathrm{e}^{\sqrt{6}\, x}\sqrt{6}}{6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{\sqrt{6}\, x}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{\sqrt{6}\, x}{2}} \left(\frac{\mathrm{e}^{\sqrt{6}\, x}\sqrt{6}}{6}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$-2y'' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{\sqrt{6}\, x}{2}} + \frac{c_2 \sqrt{6}\, \mathrm{e}^{\frac{\sqrt{6}\, x}{2}}}{6}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{6}\,\mathrm{e}^{\frac{\sqrt{6}\,x}{2}}}{6}, \mathrm{e}^{-\frac{\sqrt{6}\,x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{x} + A_2\mathrm{e}^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1\mathrm{e}^{x} + A_1 x\,\mathrm{e}^{x} + A_2\mathrm{e}^{x} = x\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,\mathrm{e}^{x} + 4\,\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-\frac{\sqrt{6}\,x}{2}} + \frac{c_2\sqrt{6}\,\mathrm{e}^{\frac{\sqrt{6}\,x}{2}}}{6}\right) + \left(x\,\mathrm{e}^{x} + 4\,\mathrm{e}^{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-\frac{\sqrt{6}\,x}{2}} + \frac{c_2\sqrt{6}\,\mathrm{e}^{\frac{\sqrt{6}\,x}{2}}}{6} + x\,\mathrm{e}^{x} + 4\,\mathrm{e}^{x} \tag{1}$$

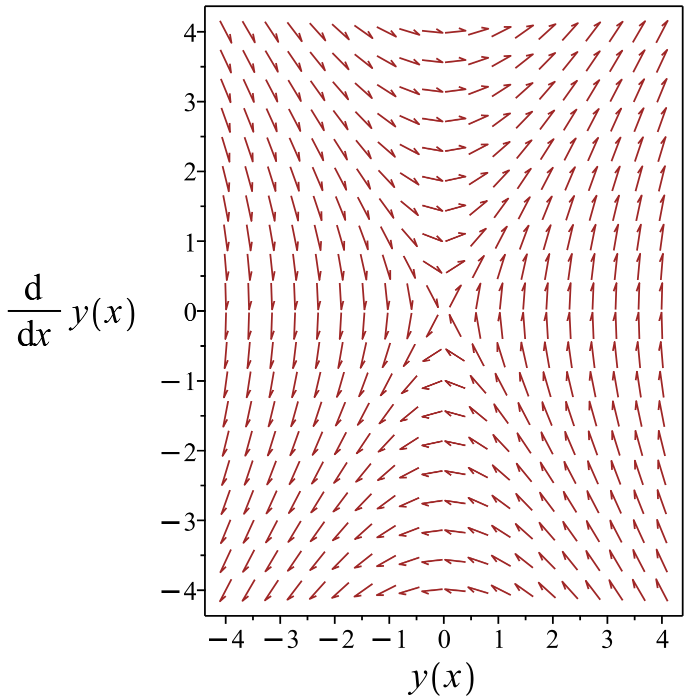


Figure 558: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{\sqrt{6}\,x}{2}} + \frac{c_2\sqrt{6}\,e^{\frac{\sqrt{6}\,x}{2}}}{6} + x\,e^x + 4\,e^x$$

Verified OK.

11.12.3 Maple step by step solution

Let's solve

$-2y'' + 3y = x\,e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y}{2} - \frac{x\,e^x}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y}{2} = -\frac{x\,e^x}{2}$

- Characteristic polynomial of homogeneous ODE

$r^2 - \frac{3}{2} = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{6}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{\sqrt{6}\, x}{2}}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{\frac{\sqrt{6}\, x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-\frac{\sqrt{6}\, x}{2}} + c_2 e^{\frac{\sqrt{6}\, x}{2}} + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -\frac{x\, e^x}{2}\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{\sqrt{6}\, x}{2}} & e^{\frac{\sqrt{6}\, x}{2}} \\ -\frac{\sqrt{6}\, e^{-\frac{\sqrt{6}\, x}{2}}}{2} & \frac{\sqrt{6}\, e^{\frac{\sqrt{6}\, x}{2}}}{2} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \sqrt{6}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{\sqrt{6}\left(e^{-\frac{\sqrt{6}\, x}{2}} \left(\int x\, e^{\frac{x\left(2+\sqrt{6}\right)}{2}} dx\right) - e^{\frac{\sqrt{6}\, x}{2}} \left(\int x\, e^{-\frac{x\left(-2+\sqrt{6}\right)}{2}} dx\right)\right)}{12}$
  - Compute integrals

    $y_p(x) = e^x (x + 4)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-\frac{\sqrt{6}\, x}{2}} + c_2 e^{\frac{\sqrt{6}\, x}{2}} + e^x (x + 4)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x$2)+3*y(x)=x*exp(x),y(x), singsol=all)
```

$$y(x) = e^{\frac{\sqrt{6}\,x}{2}} c_2 + e^{-\frac{\sqrt{6}\,x}{2}} c_1 + (x+4)\,e^{x}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 54

```
DSolve[y''[x]-3*y'[x]+3*y[x]==x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(x + c_2 e^{x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) + 1\right)$$

11.13 problem 13

Internal problem ID [2152]
Internal file name [`OUTPUT/2152_Monday_February_26_2024_09_17_54_AM_86533897/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sin(x)\, x}$$

11.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sin(x)\, x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\, x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(x), \sin(x)\, x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x \cos(x) + A_2 \sin(x)\, x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-2A_1 \sin(x) + 3A_1 x \cos(x) + 3A_2 \sin(x)\, x + 2A_2 \cos(x) + 3A_3 \cos(x) + 3A_4 \sin(x)\\ &= \sin(x)\, x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{3}, A_3 = -\frac{2}{9}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)\, x}{3} - \frac{2\cos(x)}{9}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{\sin(x)\, x}{3} - \frac{2\cos(x)}{9}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(x) x}{3} - \frac{2\cos(x)}{9} \tag{1}$$

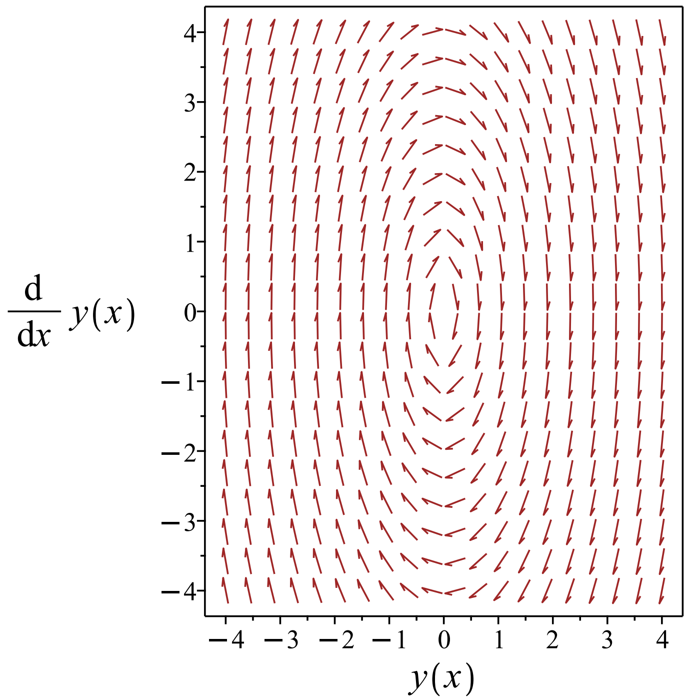


Figure 559: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(x) x}{3} - \frac{2\cos(x)}{9}$$

Verified OK.

11.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 393: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\
&= \cos(2x) \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\, x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(x), \sin(x)\, x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\cos(x) + A_2\sin(x)\, x + A_3\cos(x) + A_4\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sin(x) + 3A_1 x\cos(x) + 3A_2\sin(x)\, x + 2A_2\cos(x) + 3A_3\cos(x) + 3A_4\sin(x) = \sin(x)\, x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{3}, A_3 = -\frac{2}{9}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)\, x}{3} - \frac{2\cos(x)}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(\frac{\sin(x)\, x}{3} - \frac{2\cos(x)}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\sin(x)\,x}{3} - \frac{2\cos(x)}{9} \tag{1}$$

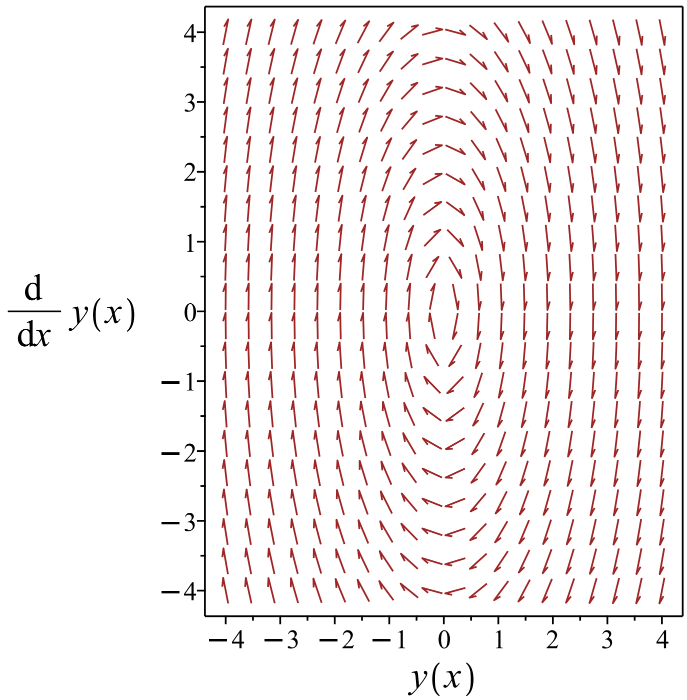


Figure 560: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\sin(x)\,x}{3} - \frac{2\cos(x)}{9}$$

Verified OK.

11.13.3 Maple step by step solution

Let's solve

$y'' + 4y = \sin(x)\,x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\, x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)\sin(x) x dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x)\sin(x) x dx\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{\sin(x) x}{3} - \frac{2\cos(x)}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(x) x}{3} - \frac{2\cos(x)}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+4*y(x)=x*sin(x),y(x), singsol=all)
```

$$y(x) = \sin(2x) c_2 + c_1 \cos(2x) - \frac{2\cos(x)}{9} + \frac{x\sin(x)}{3}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 33

```
DSolve[y''[x]+4*y[x]==x*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3} x \sin(x) + c_1 \cos(2x) + \cos(x) \left(-\frac{2}{9} + 2c_2 \sin(x)\right)$$

11.14 problem 14

Internal problem ID [2153]
Internal file name [`OUTPUT/2153_Monday_February_26_2024_09_17_54_AM_42651828/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 14.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_y]]`

$$\boxed{y''' - 4y'' = x^2 + 8}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 4$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + e^{4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= e^{4x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' - 4y'' = x^2 + 8
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2 + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x, e^{4x}\}
$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x, x^2, x^3\}]
$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2, x^3, x^4\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_3 x^4 + A_2 x^3 + A_1 x^2
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-48x^2 A_3 - 24x A_2 + 24x A_3 - 8A_1 + 6A_2 = x^2 + 8
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{65}{64}, A_2 = -\frac{1}{48}, A_3 = -\frac{1}{48}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{48}x^4 - \frac{1}{48}x^3 - \frac{65}{64}x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x + c_1 + \mathrm{e}^{4x} c_3\right) + \left(-\frac{1}{48}x^4 - \frac{1}{48}x^3 - \frac{65}{64}x^2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 + \mathrm{e}^{4x} c_3 - \frac{x^4}{48} - \frac{x^3}{48} - \frac{65x^2}{64} \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + \mathrm{e}^{4x} c_3 - \frac{x^4}{48} - \frac{x^3}{48} - \frac{65x^2}{64}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^2+4*_b(_a)+8, _b(_a)`   *** Sublevel
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)=x^2+8,y(x), singsol=all)
```

$$y(x) = -\frac{65x^2}{64} - \frac{x^3}{48} - \frac{x^4}{48} + \frac{e^{4x}c_1}{16} + c_2x + c_3$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 41

```
DSolve[y'''[x]-4*y''[x]==x^2+8,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{192}\left(-4x^4 - 4x^3 - 195x^2 + 12c_1e^{4x}\right) + c_3x + c_2$$

11.15 problem 15

Internal problem ID [2154]
Internal file name [`OUTPUT/2154_Monday_February_26_2024_09_17_54_AM_43091232/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y' + y = e^{x} \sin(3x)}$$

11.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = e^{x} \sin(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x} \sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x} \cos(3x), e^{x} \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x} \cos(3x) + A_2 e^{x} \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^{x} \cos(3x) - 9A_1 e^{x} \sin(3x) - 6A_2 e^{x} \sin(3x) + 9A_2 e^{x} \cos(3x) = e^{x} \sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{13}, A_2 = -\frac{2}{39}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{x} \cos(3x)}{13} - \frac{2\, e^{x} \sin(3x)}{39}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + \left(-\frac{e^{x}\cos(3x)}{13} - \frac{2\,e^{x}\sin(3x)}{39}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{-\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) - \frac{e^{x}\cos(3x)}{13} - \frac{2\,e^{x}\sin(3x)}{39} \tag{1}$$

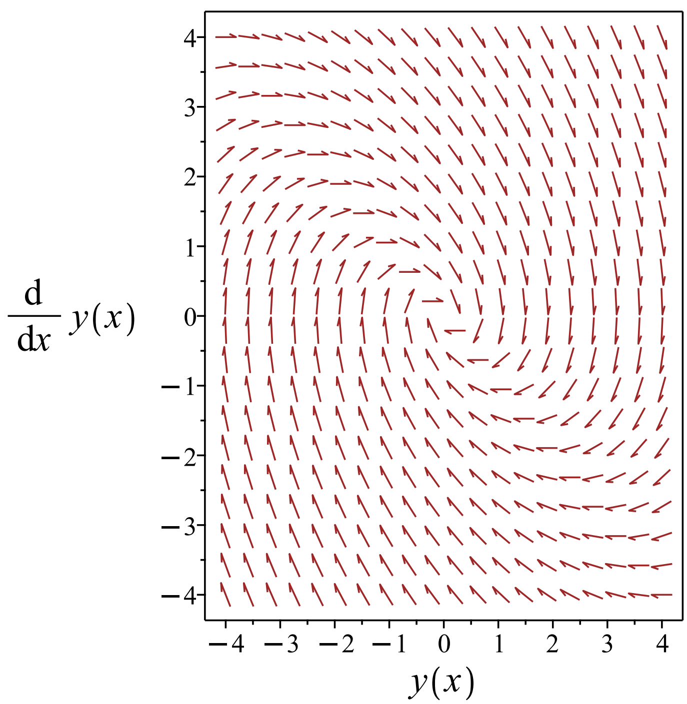


Figure 561: Slope field plot

<u>Verification of solutions</u>

$$y = e^{-\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) - \frac{e^{x}\cos(3x)}{13} - \frac{2\,e^{x}\sin(3x)}{39}$$

Verified OK.

11.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 395: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x \sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(3x), e^x \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \frac{2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x \cos(3x) + A_2 e^x \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^x \cos(3x) - 9A_1 e^x \sin(3x) - 6A_2 e^x \sin(3x) + 9A_2 e^x \cos(3x) = e^x \sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{13}, A_2 = -\frac{2}{39}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^x \cos(3x)}{13} - \frac{2 e^x \sin(3x)}{39}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}\right) + \left(-\frac{e^x \cos(3x)}{13} - \frac{2 e^x \sin(3x)}{39}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} - \frac{e^x \cos(3x)}{13} - \frac{2 e^x \sin(3x)}{39} \tag{1}$$

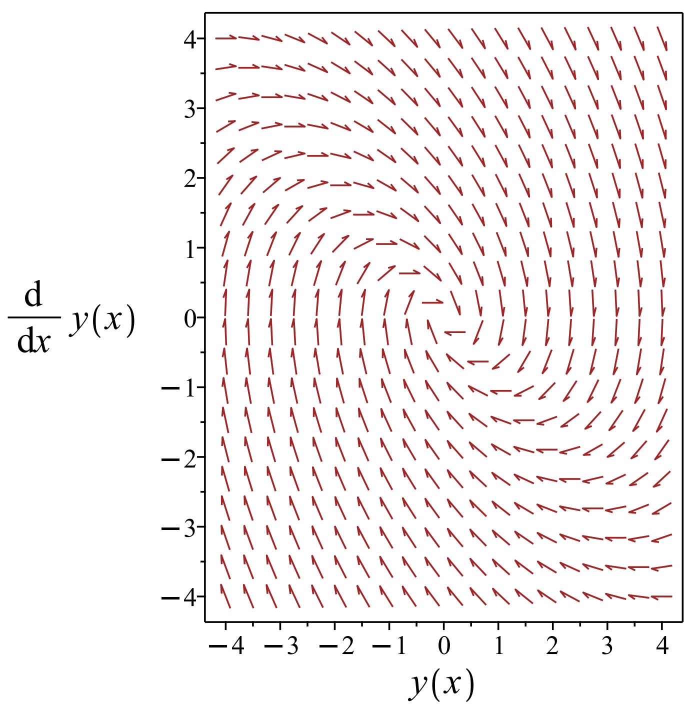


Figure 562: Slope field plot

Verification of solutions

$$y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} - \frac{e^x \cos(3x)}{13} - \frac{2\,e^x \sin(3x)}{39}$$

Verified OK.

11.15.3 Maple step by step solution

Let's solve

$y'' + y' + y = e^x \sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^x \sin(3x)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\, e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2 e^{-\frac{x}{2}} \sqrt{3} \left(\cos\left(\frac{\sqrt{3}\, x}{2}\right) \left(\int e^{\frac{3x}{2}} \sin(3x) \sin\left(\frac{\sqrt{3}\, x}{2}\right) dx\right) - \sin\left(\frac{\sqrt{3}\, x}{2}\right) \left(\int e^{\frac{3x}{2}} \sin(3x) \cos\left(\frac{\sqrt{3}\, x}{2}\right) dx\right)\right)}{3}$

- Compute integrals

  $y_p(x) = -\frac{e^x (2 \sin(3x) + 3 \cos(3x))}{39}$

- Substitute particular solution into general solution to ODE

  $y = \cos\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} c_1 + \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} c_2 - \frac{e^x (2 \sin(3x) + 3 \cos(3x))}{39}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=exp(x)*sin(3*x),y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) c_2 - \frac{e^x \left(\cos(3x) + \frac{2 \sin(3x)}{3}\right)}{13}$$

✓ Solution by Mathematica

Time used: 1.715 (sec). Leaf size: 70

```
DSolve[y''[x]+y'[x]+y[x]==Exp[x]*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{39}e^x \sin(3x) - \frac{1}{13}e^x \cos(3x) + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

11.16 problem 16

Internal problem ID [2155]
Internal file name [`OUTPUT/2155_Monday_February_26_2024_09_17_55_AM_22492681/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 16.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - 3y'' + 4y' - 12y = x + e^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' + 4y' - 12y = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 4\lambda - 12 = 0$$

The roots of the above equation are

$$\lambda_1 = 3$$
$$\lambda_2 = 2i$$
$$\lambda_3 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{3x} + e^{2ix} c_2 + e^{-2ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{3x}$$
$$y_2 = e^{2ix}$$
$$y_3 = e^{-2ix}$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' + 4y' - 12y = x + e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x}, e^{-2ix}, e^{2ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 e^{2x} + 4A_3 - 12A_2 - 12A_3 x = x + e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = -\frac{1}{36}, A_3 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{2x}}{8} - \frac{1}{36} - \frac{x}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + e^{2ix} c_2 + e^{-2ix} c_3\right) + \left(-\frac{e^{2x}}{8} - \frac{1}{36} - \frac{x}{12}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{3x} + e^{2ix} c_2 + e^{-2ix} c_3 - \frac{e^{2x}}{8} - \frac{1}{36} - \frac{x}{12} \tag{1}$$

Verification of solutions

$$y = c_1 e^{3x} + e^{2ix} c_2 + e^{-2ix} c_3 - \frac{e^{2x}}{8} - \frac{1}{36} - \frac{x}{12}$$

Verified OK.

11.16.1 Maple step by step solution

Let's solve

$y''' - 3y'' + 4y' - 12y = x + e^{2x}$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = x + e^{2x} + 3y_3(x) - 4y_2(x) + 12y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x + e^{2x} + 3y_3(x) - 4y_2(x) + 12y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & -4 & 3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x + e^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x + e^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & -4 & 3 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_2(x) = \begin{bmatrix} -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \overrightarrow{y}_3(x) = \begin{bmatrix} \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(x) = \begin{bmatrix} \frac{e^{3x}}{9} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ \frac{e^{3x}}{3} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ e^{3x} & \cos(2x) & -\sin(2x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} \frac{e^{3x}}{9} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ \frac{e^{3x}}{3} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ e^{3x} & \cos(2x) & -\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{9} & -\frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} \frac{4e^{3x}}{13} + \frac{9\cos(2x)}{13} - \frac{6\sin(2x)}{13} & \frac{\sin(2x)}{2} & \frac{e^{3x}}{13} - \frac{\cos(2x)}{13} - \frac{3\sin(2x)}{26} \\ \frac{12e^{3x}}{13} - \frac{18\sin(2x)}{13} - \frac{12\cos(2x)}{13} & \cos(2x) & \frac{3e^{3x}}{13} + \frac{2\sin(2x)}{13} - \frac{3\cos(2x)}{13} \\ \frac{36e^{3x}}{13} - \frac{36\cos(2x)}{13} + \frac{24\sin(2x)}{13} & -2\sin(2x) & \frac{9e^{3x}}{13} + \frac{4\cos(2x)}{13} + \frac{6\sin(2x)}{13} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot \vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot \left(\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c} -\frac{e^{2x}}{8}+\frac{10\,e^{3x}}{117}-\frac{x}{12}-\frac{1}{36}+\frac{7\cos(2x)}{104}+\frac{\sin(2x)}{26} \\ -\frac{e^{2x}}{4}+\frac{10\,e^{3x}}{39}-\frac{1}{12}+\frac{\cos(2x)}{13}-\frac{7\sin(2x)}{52} \\ -\frac{e^{2x}}{2}+\frac{10\,e^{3x}}{13}-\frac{7\cos(2x)}{26}-\frac{2\sin(2x)}{13} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+\left[\begin{array}{c} -\frac{e^{2x}}{8}+\frac{10\,e^{3x}}{117}-\frac{x}{12}-\frac{1}{36}+\frac{7\cos(2x)}{104}+\frac{\sin(2x)}{26} \\ -\frac{e^{2x}}{4}+\frac{10\,e^{3x}}{39}-\frac{1}{12}+\frac{\cos(2x)}{13}-\frac{7\sin(2x)}{52} \\ -\frac{e^{2x}}{2}+\frac{10\,e^{3x}}{13}-\frac{7\cos(2x)}{26}-\frac{2\sin(2x)}{13} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=-\frac{1}{36}+\frac{(-26c_2+7)\cos(2x)}{104}+\frac{(10+13c_1)e^{3x}}{117}+\frac{(13c_3+2)\sin(2x)}{52}-\frac{x}{12}-\frac{e^{2x}}{8}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+4*diff(y(x),x)-12*y(x)=x+exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(936c_1 + 18)\cos(2x)}{936} + \frac{(936c_3 + 27)\sin(2x)}{936} + c_2 e^{3x} - \frac{x}{12} - \frac{e^{2x}}{8} - \frac{1}{36}$$

✓ Solution by Mathematica

Time used: 0.337 (sec). Leaf size: 45

```
DSolve[y'''[x]-3*y''[x]+4*y'[x]-12*y[x]==x+Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{72}\left(-6x - 9e^{2x} + 72c_3 e^{3x} - 2\right) + c_1 \cos(2x) + c_2 \sin(2x)$$

11.17 problem 17

Internal problem ID [2156]
Internal file name [`OUTPUT/2156_Monday_February_26_2024_09_17_55_AM_18371274/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 17.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' - 4y'' + y' - 4y = \sin(x)\, e^{4x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' + y' - 4y = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + \lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 4$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{4x} + c_2 e^{-ix} + e^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{4x}$$
$$y_2 = e^{-ix}$$
$$y_3 = e^{ix}$$

Now the particular solution to the given ODE is found

$$y''' - 4y'' + y' - 4y = \sin(x) e^{4x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x) e^{4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) e^{4x}, \sin(x) e^{4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{ix}, e^{4x}, e^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) e^{4x} + A_2 \sin(x) e^{4x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-16A_1 \sin(x) e^{4x} - 8A_1 \cos(x) e^{4x} + 16A_2 \cos(x) e^{4x} - 8A_2 \sin(x) e^{4x} = \sin(x) e^{4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{20}, A_2 = -\frac{1}{40}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, e^{4x}}{20} - \frac{\sin(x)\, e^{4x}}{40}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{4x} + c_2 e^{-ix} + e^{ix} c_3\right) + \left(-\frac{\cos(x)\, e^{4x}}{20} - \frac{\sin(x)\, e^{4x}}{40}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 e^{-ix} + e^{ix} c_3 - \frac{\cos(x)\, e^{4x}}{20} - \frac{\sin(x)\, e^{4x}}{40} \tag{1}$$

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^{-ix} + e^{ix} c_3 - \frac{\cos(x)\, e^{4x}}{20} - \frac{\sin(x)\, e^{4x}}{40}$$

Verified OK.

11.17.1 Maple step by step solution

Let's solve

$y''' - 4y'' + y' - 4y = \sin(x)\, e^{4x}$

- Highest derivative means the order of the ODE is 3

  y'''

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = \sin(x)\,\mathrm{e}^{4x} + 4y_3(x) - y_2(x) + 4y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \sin(x)\,\mathrm{e}^{4x} + 4y_3(x) - y_2(x) + 4y_1(x)]$

- Define vector

$$\overrightarrow{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -1 & 4 \end{bmatrix} \cdot \overrightarrow{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \sin(x)\,\mathrm{e}^{4x} \end{bmatrix}$$

- Define the forcing function

$$\overrightarrow{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \sin(x)\,\mathrm{e}^{4x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -1 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[4, \left[\begin{array}{c} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{4x} \cdot \left[\begin{array}{c} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \left[\begin{array}{c} -1 \\ I \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \left[\begin{array}{c} -1 \\ I \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \left[\begin{array}{c} -1 \\ I \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \left[\begin{array}{c} -\cos(x) \\ \sin(x) \\ \cos(x) \end{array}\right], \vec{y}_3(x) = \left[\begin{array}{c} \sin(x) \\ \cos(x) \\ -\sin(x) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{4x}}{16} & -\cos(x) & \sin(x) \\ \frac{e^{4x}}{4} & \sin(x) & \cos(x) \\ e^{4x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{4x}}{16} & -\cos(x) & \sin(x) \\ \frac{e^{4x}}{4} & \sin(x) & \cos(x) \\ e^{4x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{16} & -1 & 0 \\ \frac{1}{4} & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{4x}}{17} + \frac{16\cos(x)}{17} - \frac{4\sin(x)}{17} & \sin(x) & \frac{e^{4x}}{17} - \frac{\cos(x)}{17} - \frac{4\sin(x)}{17} \\ \frac{4e^{4x}}{17} - \frac{16\sin(x)}{17} - \frac{4\cos(x)}{17} & \cos(x) & \frac{4e^{4x}}{17} + \frac{\sin(x)}{17} - \frac{4\cos(x)}{17} \\ \frac{16e^{4x}}{17} - \frac{16\cos(x)}{17} + \frac{4\sin(x)}{17} & -\sin(x) & \frac{16e^{4x}}{17} + \frac{\cos(x)}{17} + \frac{4\sin(x)}{17} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(-17\sin(x)-34\cos(x)+40)e^{4x}}{680} - \frac{3\cos(x)}{340} - \frac{7\sin(x)}{680} \\ \frac{(-34\sin(x)-153\cos(x)+160)e^{4x}}{680} - \frac{7\cos(x)}{680} + \frac{3\sin(x)}{340} \\ \frac{(17\sin(x)-646\cos(x)+640)e^{4x}}{680} + \frac{3\cos(x)}{340} + \frac{7\sin(x)}{680} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} \frac{(-17\sin(x)-34\cos(x)+40)e^{4x}}{680} - \frac{3\cos(x)}{340} - \frac{7\sin(x)}{680} \\ \frac{(-34\sin(x)-153\cos(x)+160)e^{4x}}{680} - \frac{7\cos(x)}{680} + \frac{3\sin(x)}{340} \\ \frac{(17\sin(x)-646\cos(x)+640)e^{4x}}{680} + \frac{3\cos(x)}{340} + \frac{7\sin(x)}{680} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(85c_1-68\cos(x)-34\sin(x)+80)e^{4x}}{1360} + \frac{(-3-340c_2)\cos(x)}{340} + \frac{\sin(x)(680c_3-7)}{680}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)+diff(y(x),x)-4*y(x)=exp(4*x)*sin(x),y(x), singsol=all
```

$$y(x) = \frac{(40c_3 - 2\cos(x) - \sin(x))\,e^{4x}}{40} + \cos(x)\,c_1 + \sin(x)\,c_2$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 44

```
DSolve[y'''[x]-4*y''[x]+y'[x]-4*y[x]==Exp[4*x]*Sin[x],y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_3 e^{4x} + \left(-\frac{e^{4x}}{20} + c_1\right)\cos(x) + \left(-\frac{e^{4x}}{40} + c_2\right)\sin(x)$$

11.18 problem 18

Internal problem ID [2157]
Internal file name [`OUTPUT/2157_Monday_February_26_2024_09_17_55_AM_86779602/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 4y = e^{2x}x^3}$$

11.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = e^{2x}x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(4)^2 - (4)\,(1)\,(4)}\\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 e^{-2x} + c_2 e^{-2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2x} + x\, e^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x} x^3$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{2x}, x^2 e^{2x}, e^{2x} x^3, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x} x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x \, e^{2x} + A_2 x^2 e^{2x} + A_3 e^{2x} x^3 + A_4 e^{2x}$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&8A_1 e^{2x} + 16A_1 x \, e^{2x} + 2A_2 e^{2x} + 16A_2 x \, e^{2x} + 16A_2 x^2 e^{2x} \\ &+ 16A_3 e^{2x} x^3 + 24A_3 e^{2x} x^2 + 6A_3 e^{2x} x + 16A_4 e^{2x} = e^{2x} x^3\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{9}{128}, A_2 = -\frac{3}{32}, A_3 = \frac{1}{16}, A_4 = -\frac{3}{128}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{9x \, e^{2x}}{128} - \frac{3x^2 e^{2x}}{32} + \frac{e^{2x} x^3}{16} - \frac{3 \, e^{2x}}{128}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + x \, e^{-2x} c_2\right) + \left(\frac{9x \, e^{2x}}{128} - \frac{3x^2 e^{2x}}{32} + \frac{e^{2x} x^3}{16} - \frac{3 \, e^{2x}}{128}\right)\end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_2 x + c_1) + \frac{9x \, e^{2x}}{128} - \frac{3x^2 e^{2x}}{32} + \frac{e^{2x} x^3}{16} - \frac{3 \, e^{2x}}{128}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_2 x + c_1) + \frac{9x \, e^{2x}}{128} - \frac{3x^2 e^{2x}}{32} + \frac{e^{2x} x^3}{16} - \frac{3 \, e^{2x}}{128} \tag{1}$$

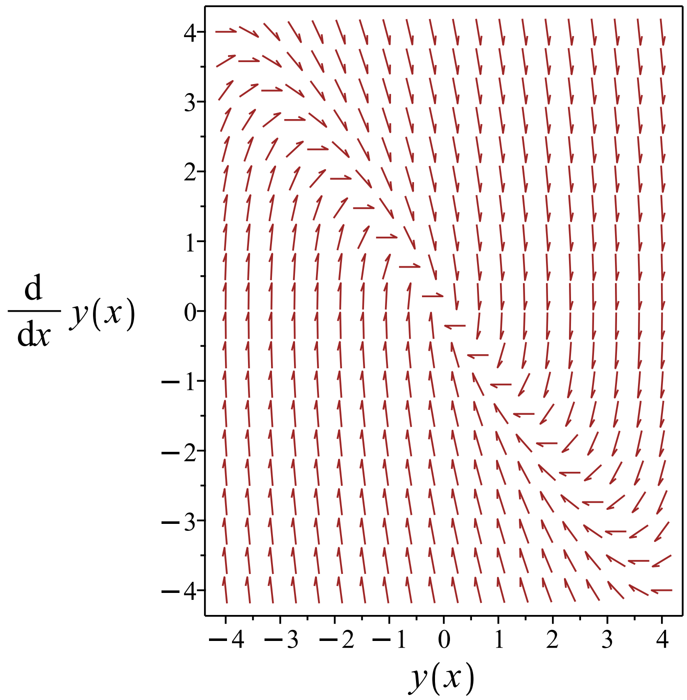


Figure 563: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) + \frac{9x\, e^{2x}}{128} - \frac{3x^2 e^{2x}}{32} + \frac{e^{2x}x^3}{16} - \frac{3\, e^{2x}}{128}$$

Verified OK.

11.18.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 4\, dx} \\ &= e^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = e^{4x}x^3$$
$$(e^{2x}y)'' = e^{4x}x^3$$

Integrating once gives

$$(e^{2x}y)' = \frac{(32x^3 - 24x^2 + 12x - 3)\,e^{4x}}{128} + c_1$$

Integrating again gives

$$(e^{2x}y) = \frac{(8x^3 - 12x^2 + 9x - 3)\,e^{4x}}{128} + c_1x + c_2$$

Hence the solution is

$$y = \frac{\frac{(8x^3-12x^2+9x-3)\,e^{4x}}{128} + c_1x + c_2}{e^{2x}}$$

Or

$$y = \frac{e^{2x}x^3}{16} - \frac{3x^2e^{2x}}{32} + \frac{9x\,e^{2x}}{128} - \frac{3\,e^{2x}}{128} + c_1x\,e^{-2x} + c_2e^{-2x}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{e^{2x}x^3}{16} - \frac{3x^2e^{2x}}{32} + \frac{9x\,e^{2x}}{128} - \frac{3\,e^{2x}}{128} + c_1x\,e^{-2x} + c_2e^{-2x} \tag{1}$$

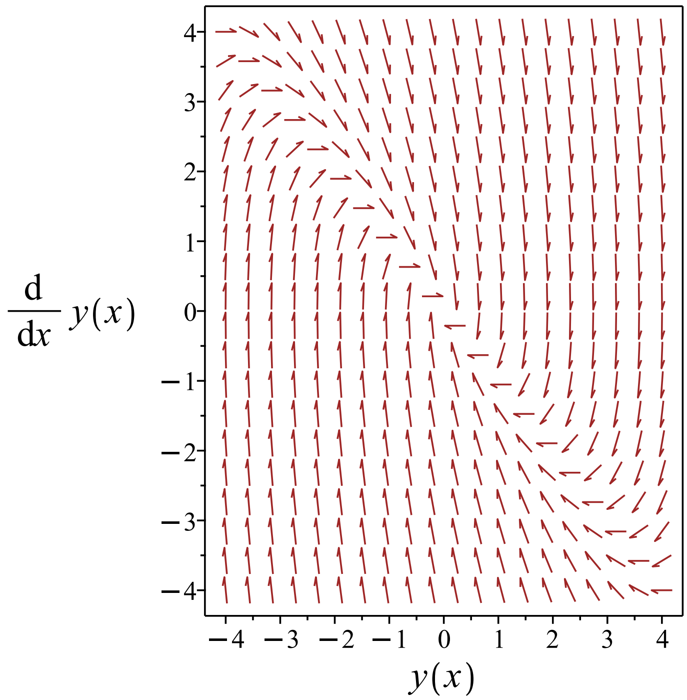


Figure 564: Slope field plot

Verification of solutions

$$y = \frac{e^{2x}x^3}{16} - \frac{3x^2e^{2x}}{32} + \frac{9x\,e^{2x}}{128} - \frac{3\,e^{2x}}{128} + c_1x\,e^{-2x} + c_2e^{-2x}$$

Verified OK.

11.18.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 399: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{\mathrm{e}^{-4x}}{\left(y_1\right)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{2x} x^3
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{2x}, x^2\mathrm{e}^{2x}, \mathrm{e}^{2x}x^3, \mathrm{e}^{2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2x}x, \mathrm{e}^{-2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 x\,\mathrm{e}^{2x} + A_2 x^2 \mathrm{e}^{2x} + A_3 \mathrm{e}^{2x} x^3 + A_4 \mathrm{e}^{2x}
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&8A_1\mathrm{e}^{2x} + 16A_1 x\,\mathrm{e}^{2x} + 2A_2\mathrm{e}^{2x} + 16A_2 x\,\mathrm{e}^{2x} + 16A_2 x^2\mathrm{e}^{2x} \\
&+ 16A_3\mathrm{e}^{2x}x^3 + 24A_3\mathrm{e}^{2x}x^2 + 6A_3\mathrm{e}^{2x}x + 16A_4\mathrm{e}^{2x} = \mathrm{e}^{2x}x^3
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{9}{128}, A_2 = -\frac{3}{32}, A_3 = \frac{1}{16}, A_4 = -\frac{3}{128}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{9x\,\mathrm{e}^{2x}}{128} - \frac{3x^2\mathrm{e}^{2x}}{32} + \frac{\mathrm{e}^{2x}x^3}{16} - \frac{3\,\mathrm{e}^{2x}}{128}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x}c_2\right) + \left(\frac{9x\,\mathrm{e}^{2x}}{128} - \frac{3x^2\mathrm{e}^{2x}}{32} + \frac{\mathrm{e}^{2x}x^3}{16} - \frac{3\,\mathrm{e}^{2x}}{128}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2x + c_1) + \frac{9x\,\mathrm{e}^{2x}}{128} - \frac{3x^2\mathrm{e}^{2x}}{32} + \frac{\mathrm{e}^{2x}x^3}{16} - \frac{3\,\mathrm{e}^{2x}}{128}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(c_2x + c_1) + \frac{9x\,\mathrm{e}^{2x}}{128} - \frac{3x^2\mathrm{e}^{2x}}{32} + \frac{\mathrm{e}^{2x}x^3}{16} - \frac{3\,\mathrm{e}^{2x}}{128} \tag{1}$$

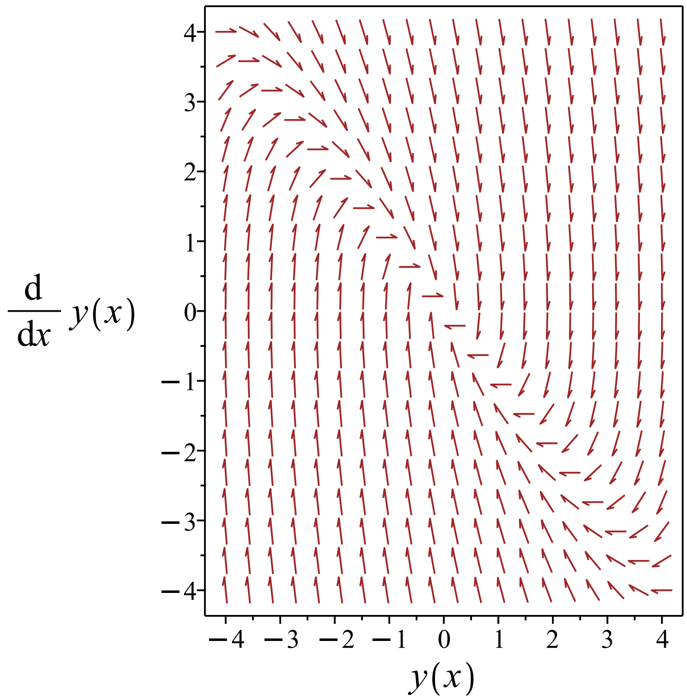


Figure 565: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) + \frac{9x\,e^{2x}}{128} - \frac{3x^2e^{2x}}{32} + \frac{e^{2x}x^3}{16} - \frac{3\,e^{2x}}{128}$$

Verified OK.

11.18.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = e^{2x}x^3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 2)^2 = 0$

- Root of the characteristic polynomial

$r = -2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = \mathrm{e}^{-2x} x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{2x} x^3\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-2x} x \\ -2\,\mathrm{e}^{-2x} & -2\,\mathrm{e}^{-2x} x + \mathrm{e}^{-2x} \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \mathrm{e}^{-2x} \left(-\left(\int x^4 \mathrm{e}^{4x} dx\right) + \left(\int \mathrm{e}^{4x} x^3 dx\right) x\right)$

  ○ Compute integrals

  $y_p(x) = \frac{(8x^3 - 12x^2 + 9x - 3)\,\mathrm{e}^{2x}}{128}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2 + \frac{(8x^3 - 12x^2 + 9x - 3)\,\mathrm{e}^{2x}}{128}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=x^3*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(8x^3 - 12x^2 + 9x - 3)\,\mathrm{e}^{2x}}{128} + \mathrm{e}^{-2x}(c_1x + c_2)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 43

```
DSolve[y''[x]+4*y'[x]+4*y[x]==x^3*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{128}e^{2x}\left(8x^3 - 12x^2 + 9x - 3\right) + e^{-2x}(c_2x + c_1)$$

11.19 problem 19

Internal problem ID [2158]
Internal file name [OUTPUT/2158_Monday_February_26_2024_09_17_56_AM_43331283/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 19.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 2y'' + y' - 2y = x\,\mathrm{e}^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 2y'' + y' - 2y = 0$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 + \lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{2x}$$
$$y_2 = \mathrm{e}^{-ix}$$
$$y_3 = \mathrm{e}^{ix}$$

Now the particular solution to the given ODE is found

$$y''' - 2y'' + y' - 2y = x\,\mathrm{e}^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{2x}, \mathrm{e}^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{ix}, \mathrm{e}^{2x}, \mathrm{e}^{-ix}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{2x}, x^2\mathrm{e}^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{2x} + A_2 x^2 \mathrm{e}^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1\mathrm{e}^{2x} + 8A_2\mathrm{e}^{2x} + 10A_2 x\,\mathrm{e}^{2x} = x\,\mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{25}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4x\,\mathrm{e}^{2x}}{25} + \frac{x^2 \mathrm{e}^{2x}}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3\right) + \left(-\frac{4x\,\mathrm{e}^{2x}}{25} + \frac{x^2 \mathrm{e}^{2x}}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3 - \frac{4x\,\mathrm{e}^{2x}}{25} + \frac{x^2 \mathrm{e}^{2x}}{10} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3 - \frac{4x\,\mathrm{e}^{2x}}{25} + \frac{x^2 \mathrm{e}^{2x}}{10}$$

Verified OK.

11.19.1 Maple step by step solution

Let's solve

$y''' - 2y'' + y' - 2y = x\,\mathrm{e}^{2x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = x\,\mathrm{e}^{2x} + 2y_3(x) - y_2(x) + 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = x\,\mathrm{e}^{2x} + 2y_3(x) - y_2(x) + 2y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x\,\mathrm{e}^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-I, \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}\right], \left[I, \begin{bmatrix} -1 \\ -I \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{2x}}{4} & -\cos(x) & \sin(x) \\ \frac{e^{2x}}{2} & \sin(x) & \cos(x) \\ e^{2x} & \cos(x) & -\sin(x) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{2x}}{4} & -\cos(x) & \sin(x) \\ \frac{e^{2x}}{2} & \sin(x) & \cos(x) \\ e^{2x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & -1 & 0 \\ \frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{2x}}{5} + \frac{4\cos(x)}{5} - \frac{2\sin(x)}{5} & \sin(x) & \frac{e^{2x}}{5} - \frac{\cos(x)}{5} - \frac{2\sin(x)}{5} \\ \frac{2\,e^{2x}}{5} - \frac{4\sin(x)}{5} - \frac{2\cos(x)}{5} & \cos(x) & \frac{2\,e^{2x}}{5} + \frac{\sin(x)}{5} - \frac{2\cos(x)}{5} \\ \frac{4\,e^{2x}}{5} - \frac{4\cos(x)}{5} + \frac{2\sin(x)}{5} & -\sin(x) & \frac{4\,e^{2x}}{5} + \frac{\cos(x)}{5} + \frac{2\sin(x)}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

  - Take the derivative of the particular solution

$\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\overrightarrow{v}'(x)=\overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x)=\int_0^x\frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x\frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x)=\left[\begin{array}{c}\frac{(25x^2-40x+22)e^{2x}}{250}-\frac{11\cos(x)}{125}-\frac{2\sin(x)}{125}\\ \frac{(25x^2-15x+2)e^{2x}}{125}-\frac{2\cos(x)}{125}+\frac{11\sin(x)}{125}\\ \frac{(50x^2+20x-11)e^{2x}}{125}+\frac{11\cos(x)}{125}+\frac{2\sin(x)}{125}\end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x)=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2(x)+c_3\overrightarrow{y}_3(x)+\left[\begin{array}{c}\frac{(25x^2-40x+22)e^{2x}}{250}-\frac{11\cos(x)}{125}-\frac{2\sin(x)}{125}\\ \frac{(25x^2-15x+2)e^{2x}}{125}-\frac{2\cos(x)}{125}+\frac{11\sin(x)}{125}\\ \frac{(50x^2+20x-11)e^{2x}}{125}+\frac{11\cos(x)}{125}+\frac{2\sin(x)}{125}\end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y=\frac{(50x^2+125c_1-80x+44)e^{2x}}{500}+\frac{(-500c_2-44)\cos(x)}{500}+\frac{(500c_3-8)\sin(x)}{500}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)+diff(y(x),x)-2*y(x)=x*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(5x^2 + 50c_3 - 8x)\,\mathrm{e}^{2x}}{50} + \cos(x)\,c_1 + \sin(x)\,c_2$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 39

```
DSolve[y'''[x]-2*y''[x]+y'[x]-2*y[x]==x*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{250}e^{2x}\left(25x^2 - 40x + 22 + 250c_3\right) + c_1\cos(x) + c_2\sin(x)$$

11.20 problem 20

Internal problem ID [2159]
Internal file name [`OUTPUT/2159_Monday_February_26_2024_09_17_56_AM_44492199/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 20.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 2n^2y'' + n^4y = \sin(kx)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2n^2y'' + n^4y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2n^2 + n^4 = 0$$

The roots of the above equation are

$$\lambda_1 = in$$
$$\lambda_2 = -in$$
$$\lambda_3 = in$$
$$\lambda_4 = -in$$

Therefore the homogeneous solution is

$$y_h(x) = e^{inx} c_1 + x\,e^{inx} c_2 + e^{-inx} c_3 + x\,e^{-inx} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{inx} \\ y_2 &= x\,e^{inx} \\ y_3 &= e^{-inx} \\ y_4 &= x\,e^{-inx} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 2n^2 y'' + n^4 y = \sin(kx)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(kx)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(kx), \sin(kx)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{inx}, x\,e^{-inx}, e^{inx}, e^{-inx}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(kx) + A_2 \sin(kx)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 k^4 \cos(kx) + A_2 k^4 \sin(kx) + 2n^2\left(-A_1 k^2 \cos(kx) - A_2 k^2 \sin(kx)\right) + n^4 (A_1 \cos(kx) + A_2 \sin(kx)) = \sin(kx)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{(k-n)^2 (k+n)^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(kx)}{(k-n)^2 (k+n)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{inx} c_1 + x\, e^{inx} c_2 + e^{-inx} c_3 + x\, e^{-inx} c_4\right) + \left(\frac{\sin(kx)}{(k-n)^2 (k+n)^2}\right) \end{aligned}$$

Which simplifies to

$$y = (c_2 x + c_1) e^{inx} + (c_4 x + c_3) e^{-inx} + \frac{\sin(kx)}{(k-n)^2 (k+n)^2}$$

Summary

The solution(s) found are the following

$$y = (c_2 x + c_1) e^{inx} + (c_4 x + c_3) e^{-inx} + \frac{\sin(kx)}{(k-n)^2 (k+n)^2} \tag{1}$$

Verification of solutions

$$y = (c_2 x + c_1) e^{inx} + (c_4 x + c_3) e^{-inx} + \frac{\sin(kx)}{(k-n)^2 (k+n)^2}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$4)+2*n^2*diff(y(x),x$2)+n^4*y(x)=sin(k*x),y(x), singsol=all)
```

$$y(x) = \frac{\sin(kx)}{(k-n)^2 (k+n)^2} + c_1 \cos(nx) + c_2 \sin(nx) + c_3 \cos(nx)\, x + c_4 \sin(nx)\, x$$

✓ Solution by Mathematica

Time used: 0.457 (sec). Leaf size: 69

```
DSolve[y''''[x]+2*n^2*y''[x]+n^4*y[x]==Sin[k*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(c_2x + c_1)(k^2-n^2)^2 \cos(nx) + (c_4x + c_3)(k^2-n^2)^2 \sin(nx) + \sin(kx)}{(k-n)^2(k+n)^2}$$

11.21 problem 21

Internal problem ID [2160]
Internal file name [`OUTPUT/2160_Monday_February_26_2024_09_17_56_AM_55353319/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__linear__constant__coeff", "linear__second__order__ode__solved__by__an__integrating__factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2ny' + n^2y = 5\cos(6x)}$$

11.21.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2n, C = n^2, f(x) = 5\cos(6x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2ny' + n^2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2n, C = n^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2n\lambda\, e^{\lambda x} + n^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2n\lambda + n^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2n, C = n^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2n}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2n)^2 - (4)(1)(n^2)} \\ &= -n\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = n$. Therefore the solution is

$$y = c_1 e^{-nx} + c_2 x\, e^{-nx} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-nx} + c_2 x\, e^{-nx}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\cos(6x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(6x), \sin(6x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-nx}, e^{-nx}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(6x) + A_2 \sin(6x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-36A_1 \cos(6x) - 36A_2 \sin(6x) + 2n(-6A_1 \sin(6x) + 6A_2 \cos(6x)) \\ + n^2(A_1 \cos(6x) + A_2 \sin(6x)) = 5\cos(6x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5n^2 - 180}{(n^2 + 36)^2}, A_2 = \frac{60n}{(n^2 + 36)^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{(5n^2 - 180)\cos(6x)}{(n^2 + 36)^2} + \frac{60n \sin(6x)}{(n^2 + 36)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-nx} + c_2 x\, e^{-nx}\right) + \left(\frac{(5n^2 - 180)\cos(6x)}{(n^2 + 36)^2} + \frac{60n \sin(6x)}{(n^2 + 36)^2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-nx}(c_2 x + c_1) + \frac{(5n^2 - 180)\cos(6x)}{(n^2 + 36)^2} + \frac{60n \sin(6x)}{(n^2 + 36)^2}$$

Summary

The solution(s) found are the following

$$y = e^{-nx}(c_2 x + c_1) + \frac{(5n^2 - 180)\cos(6x)}{(n^2 + 36)^2} + \frac{60n \sin(6x)}{(n^2 + 36)^2} \tag{1}$$

Verification of solutions

$$y = e^{-nx}(c_2 x + c_1) + \frac{(5n^2 - 180)\cos(6x)}{(n^2 + 36)^2} + \frac{60n \sin(6x)}{(n^2 + 36)^2}$$

Verified OK.

11.21.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2n$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 2n\, dx} \\ &= \mathrm{e}^{nx} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 5\,\mathrm{e}^{nx}\cos(6x) \\ (\mathrm{e}^{nx}y)'' &= 5\,\mathrm{e}^{nx}\cos(6x) \end{aligned}$$

Integrating once gives

$$(\mathrm{e}^{nx}y)' = \frac{5\,\mathrm{e}^{nx}(\cos(6x)\, n + 6\sin(6x))}{n^2 + 36} + c_1$$

Integrating again gives

$$(\mathrm{e}^{nx}y) = \frac{5(\cos(6x)\, n^2 + 12n\sin(6x) - 36\cos(6x))\,\mathrm{e}^{nx} + c_1 x(n^2+36)^2}{(n^2+36)^2} + c_2$$

Hence the solution is

$$y = \frac{\frac{5(\cos(6x)n^2 + 12n\sin(6x) - 36\cos(6x))\mathrm{e}^{nx} + c_1 x(n^2+36)^2}{(n^2+36)^2} + c_2}{\mathrm{e}^{nx}}$$

Or

$$\begin{aligned} y = {} & \frac{160n^2\cos(x)^6}{(n^2+36)^2} + \frac{1920n\cos(x)^5\sin(x)}{(n^2+36)^2} - \frac{240n^2\cos(x)^4}{(n^2+36)^2} \\ & - \frac{5760\cos(x)^6}{(n^2+36)^2} - \frac{1920n\cos(x)^3\sin(x)}{(n^2+36)^2} + \frac{90n^2\cos(x)^2}{(n^2+36)^2} \\ & + \frac{8640\cos(x)^4}{(n^2+36)^2} + \frac{360n\cos(x)\sin(x)}{(n^2+36)^2} - \frac{5n^2}{(n^2+36)^2} - \frac{3240\cos(x)^2}{(n^2+36)^2} \\ & + \left(\frac{n^4 x\,\mathrm{e}^{-nx}}{(n^2+36)^2} + \frac{72n^2 x\,\mathrm{e}^{-nx}}{(n^2+36)^2} + \frac{1296x\,\mathrm{e}^{-nx}}{(n^2+36)^2}\right) c_1 + c_2\mathrm{e}^{-nx} + \frac{180}{(n^2+36)^2} \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y = \frac{160n^2 \cos(x)^6}{(n^2+36)^2} + \frac{1920n \cos(x)^5 \sin(x)}{(n^2+36)^2} - \frac{240n^2 \cos(x)^4}{(n^2+36)^2} \\
- \frac{5760 \cos(x)^6}{(n^2+36)^2} - \frac{1920n \cos(x)^3 \sin(x)}{(n^2+36)^2} + \frac{90n^2 \cos(x)^2}{(n^2+36)^2} \\
+ \frac{8640 \cos(x)^4}{(n^2+36)^2} + \frac{360n \cos(x) \sin(x)}{(n^2+36)^2} - \frac{5n^2}{(n^2+36)^2} - \frac{3240 \cos(x)^2}{(n^2+36)^2} \\
+ \left(\frac{n^4 x\, e^{-nx}}{(n^2+36)^2} + \frac{72n^2 x\, e^{-nx}}{(n^2+36)^2} + \frac{1296x\, e^{-nx}}{(n^2+36)^2}\right) c_1 + c_2 e^{-nx} + \frac{180}{(n^2+36)^2} \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y = \frac{160n^2 \cos(x)^6}{(n^2+36)^2} + \frac{1920n \cos(x)^5 \sin(x)}{(n^2+36)^2} - \frac{240n^2 \cos(x)^4}{(n^2+36)^2} \\
- \frac{5760 \cos(x)^6}{(n^2+36)^2} - \frac{1920n \cos(x)^3 \sin(x)}{(n^2+36)^2} + \frac{90n^2 \cos(x)^2}{(n^2+36)^2} \\
+ \frac{8640 \cos(x)^4}{(n^2+36)^2} + \frac{360n \cos(x) \sin(x)}{(n^2+36)^2} - \frac{5n^2}{(n^2+36)^2} - \frac{3240 \cos(x)^2}{(n^2+36)^2} \\
+ \left(\frac{n^4 x\, e^{-nx}}{(n^2+36)^2} + \frac{72n^2 x\, e^{-nx}}{(n^2+36)^2} + \frac{1296x\, e^{-nx}}{(n^2+36)^2}\right) c_1 + c_2 e^{-nx} + \frac{180}{(n^2+36)^2}
\end{align*}
$$

Verified OK.

11.21.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' + 2ny' + n^2 y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 2n \tag{3} \\
C &= n^2
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 402: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2n}{1}\,dx} \\
&= z_1 e^{-nx} \\
&= z_1\left(\mathrm{e}^{-nx}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-nx}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2n}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2nx}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-nx}\right) + c_2\left(\mathrm{e}^{-nx}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2ny' + n^2 y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-nx} + c_2 x\,\mathrm{e}^{-nx}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
5\cos(6x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(6x), \sin(6x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{-nx}, \mathrm{e}^{-nx}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\cos(6x) + A_2\sin(6x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-36A_1\cos(6x) - 36A_2\sin(6x) + 2n(-6A_1\sin(6x) + 6A_2\cos(6x)) \\
&\quad + n^2(A_1\cos(6x) + A_2\sin(6x)) = 5\cos(6x)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5n^2-180}{\left(n^2+36\right)^2}, A_2 = \frac{60n}{\left(n^2+36\right)^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{(5n^2-180)\cos(6x)}{\left(n^2+36\right)^2} + \frac{60n\sin(6x)}{\left(n^2+36\right)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-nx} + c_2 x\,\mathrm{e}^{-nx}\right) + \left(\frac{(5n^2-180)\cos(6x)}{\left(n^2+36\right)^2} + \frac{60n\sin(6x)}{\left(n^2+36\right)^2}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-nx}(c_2 x + c_1) + \frac{(5n^2-180)\cos(6x)}{\left(n^2+36\right)^2} + \frac{60n\sin(6x)}{\left(n^2+36\right)^2}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-nx}(c_2 x + c_1) + \frac{(5n^2-180)\cos(6x)}{\left(n^2+36\right)^2} + \frac{60n\sin(6x)}{\left(n^2+36\right)^2} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-nx}(c_2 x + c_1) + \frac{(5n^2-180)\cos(6x)}{\left(n^2+36\right)^2} + \frac{60n\sin(6x)}{\left(n^2+36\right)^2}$$

Verified OK.

11.21.4 Maple step by step solution

Let's solve

$y'' + 2ny' + n^2y = 5\cos(6x)$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

  $n^2 + 2nr + r^2 = 0$
- Factor the characteristic polynomial

  $(n + r)^2 = 0$
- Root of the characteristic polynomial

  $r = -n$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-nx}$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-nx}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-nx} + c_2 x\,e^{-nx} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 5\cos(6x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-nx} & x\,e^{-nx} \\ -n\,e^{-nx} & e^{-nx} - xn\,e^{-nx} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{-2nx}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = 5\,e^{-nx}\left(-\left(\int x\cos(6x)\,e^{nx}dx\right) + x\left(\int e^{nx}\cos(6x)\,dx\right)\right)$
  - Compute integrals

    $y_p(x) = \frac{5\cos(6x)n^2 + 60n\sin(6x) - 180\cos(6x)}{(n^2+36)^2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-nx} + c_2 x\,e^{-nx} + \frac{5\cos(6x)n^2 + 60n\sin(6x) - 180\cos(6x)}{(n^2+36)^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(diff(y(x),x$2)+2*n*diff(y(x),x)+n^2*y(x)=5*cos(6*x),y(x), singsol=all)
```

$$y(x) = \frac{(n^2+36)^2 (c_1 x + c_2) e^{-nx} + 5\cos(6x) n^2 + 60\sin(6x) n - 180\cos(6x)}{(n^2+36)^2}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 57

```
DSolve[y''[x]+2*n*y''[x]+n^2*y[x]==5*Cos[6*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5\cos(6x)}{n^2 - 72n - 36} + c_1 e^{\frac{nx}{\sqrt{-2n-1}}} + c_2 e^{-\frac{nx}{\sqrt{-2n-1}}}$$

11.22 problem 22

Internal problem ID [2161]
Internal file name [`OUTPUT/2161_Monday_February_26_2024_09_17_57_AM_79266106/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 9y = (1 + \sin(3x))\cos(2x)}$$

11.22.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = (1 + \sin(3x))\cos(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(1 + \sin(3x)) \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}, \{\cos(5x), \sin(5x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 \cos(2x) + A_4 \sin(2x) + A_5 \cos(5x) + A_6 \sin(5x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}8A_1 \cos(x) + 8A_2 \sin(x) + 5A_3 \cos(2x) + 5A_4 \sin(2x) - 16A_5 \cos(5x) - 16A_6 \sin(5x) \\ = (1 + \sin(3x)) \cos(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{16}, A_3 = \frac{1}{5}, A_4 = 0, A_5 = 0, A_6 = -\frac{1}{32}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)}{16} + \frac{\cos(2x)}{5} - \frac{\sin(5x)}{32}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(\frac{\sin(x)}{16} + \frac{\cos(2x)}{5} - \frac{\sin(5x)}{32}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\sin(x)}{16} + \frac{\cos(2x)}{5} - \frac{\sin(5x)}{32} \tag{1}
$$

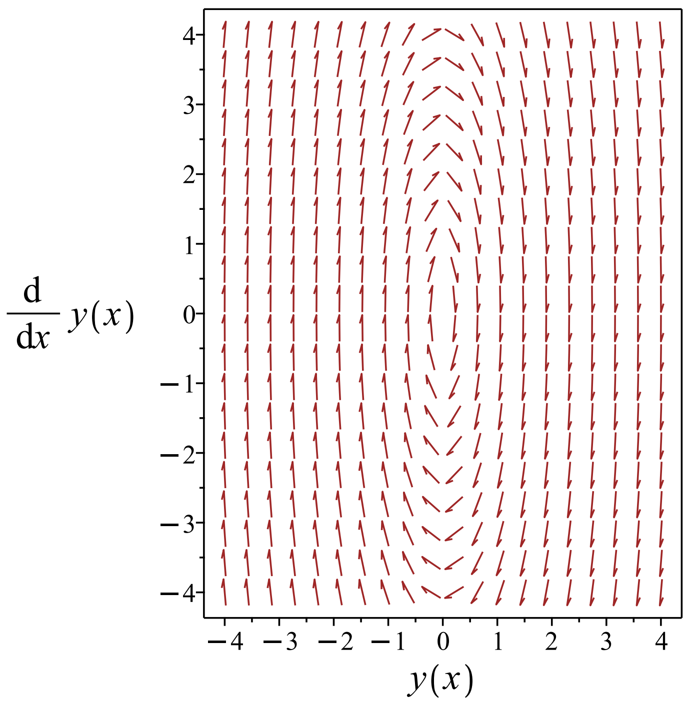


Figure 566: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\sin(x)}{16} + \frac{\cos(2x)}{5} - \frac{\sin(5x)}{32}
$$

Verified OK.

11.22.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 404: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(3x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\ &= \cos(3x) \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(1 + \sin(3x)) \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}, \{\cos(5x), \sin(5x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 \cos(2x) + A_4 \sin(2x) + A_5 \cos(5x) + A_6 \sin(5x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(x) + 8A_2 \sin(x) + 5A_3 \cos(2x) + 5A_4 \sin(2x) - 16A_5 \cos(5x) - 16A_6 \sin(5x) \\ = (1 + \sin(3x)) \cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{16}, A_3 = \frac{1}{5}, A_4 = 0, A_5 = 0, A_6 = -\frac{1}{32}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)}{16} + \frac{\cos(2x)}{5} - \frac{\sin(5x)}{32}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(\frac{\sin(x)}{16} + \frac{\cos(2x)}{5} - \frac{\sin(5x)}{32}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{\sin(x)}{16} + \frac{\cos(2x)}{5} - \frac{\sin(5x)}{32} \tag{1}
$$

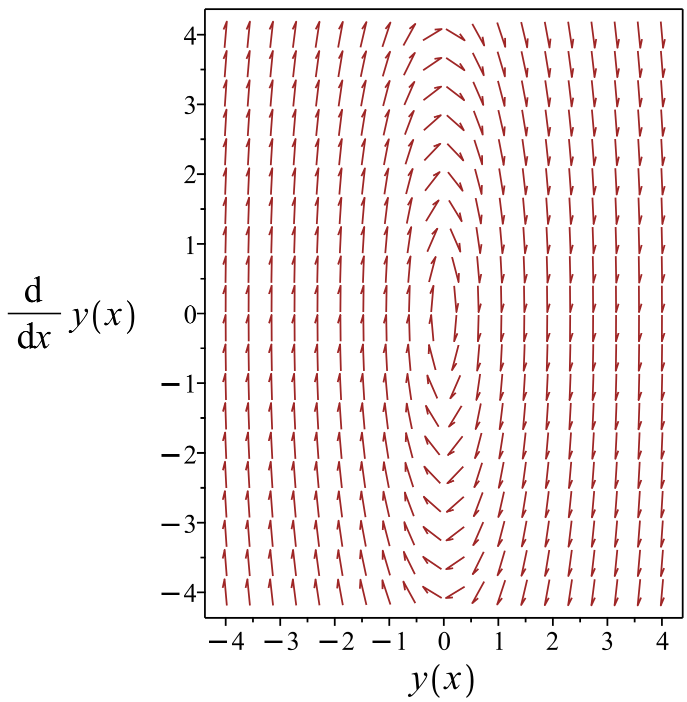


Figure 567: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{\sin(x)}{16} + \frac{\cos(2x)}{5} - \frac{\sin(5x)}{32}
$$

Verified OK.

11.22.3 Maple step by step solution

Let's solve

$y'' + 9y = (1 + \sin(3x))\cos(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = (1 + \sin(3x))\cos(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(3x)\left(\int \sin(3x)(1+\sin(3x))\cos(2x)dx\right)}{3} + \frac{\sin(3x)\left(\int \cos(3x)(1+\sin(3x))\cos(2x)dx\right)}{3}$$

- Compute integrals

$$y_p(x) = -\frac{1}{5} - \frac{\sin(x)\cos(x)^4}{2} + \frac{(16+15\sin(x))\cos(x)^2}{40} + \frac{\sin(x)}{32}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{1}{5} - \frac{\sin(x)\cos(x)^4}{2} + \frac{(16+15\sin(x))\cos(x)^2}{40} + \frac{\sin(x)}{32}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+9*y(x)=(1+sin(3*x))*cos(2*x),y(x), singsol=all)
```

$$y(x) = \sin(3x)\, c_2 + \cos(3x)\, c_1 + \frac{\cos(2x)}{5} + \frac{\sin(x)}{16} - \frac{\sin(5x)}{32}$$

✓ Solution by Mathematica

Time used: 0.266 (sec). Leaf size: 42

```
DSolve[y''[x]+9*y[x]==(1+Sin[3*x])*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(x)}{16} - \frac{1}{32}\sin(5x) + \frac{1}{5}\cos(2x) + c_1\cos(3x) + c_2\sin(3x)$$

11.23 problem 23

Internal problem ID [2162]
Internal file name [`OUTPUT/2162_Monday_February_26_2024_09_17_58_AM_45790329/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y' + 5y = 2x - e^{-4x} + \sin(2x)}$$

11.23.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 5, f(x) = 2x - e^{-4x} + \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda\, \mathrm{e}^{\lambda x} + 5\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(5)}\\ &= -2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x - e^{-4x} + \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-4x}\}, \{1, x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}\cos(x), e^{-2x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-4x} + A_2 + A_3 x + A_4 \cos(2x) + A_5 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}5A_1 e^{-4x} &+ A_4 \cos(2x) + A_5 \sin(2x) + 4A_3 - 8A_4 \sin(2x) \\ &+ 8A_5 \cos(2x) + 5A_2 + 5A_3 x = 2x - e^{-4x} + \sin(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = -\frac{8}{25}, A_3 = \frac{2}{5}, A_4 = -\frac{8}{65}, A_5 = \frac{1}{65}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{-4x}}{5} - \frac{8}{25} + \frac{2x}{5} - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(e^{-2x}(c_1 \cos(x) + c_2 \sin(x))\right) + \left(-\frac{e^{-4x}}{5} - \frac{8}{25} + \frac{2x}{5} - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^{-4x}}{5} - \frac{8}{25} + \frac{2x}{5} - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65} \tag{1}$$

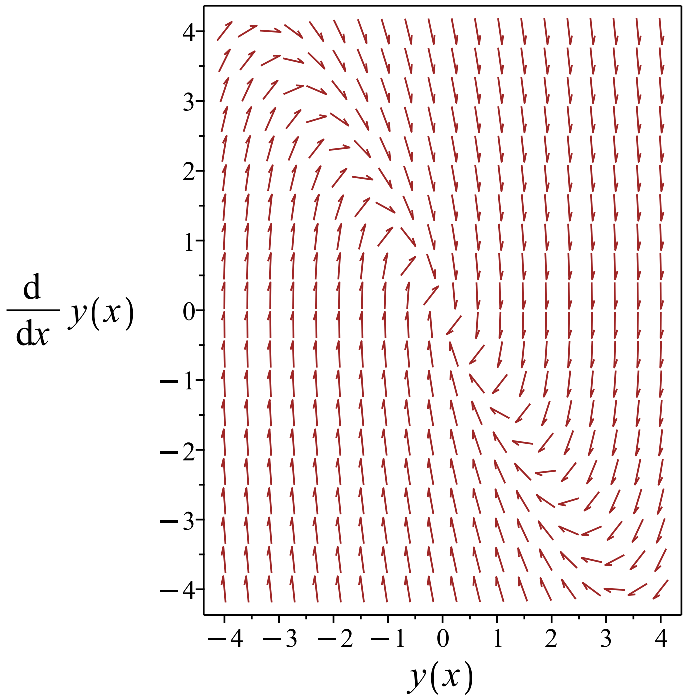


Figure 568: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^{-4x}}{5} - \frac{8}{25} + \frac{2x}{5} - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65}$$

Verified OK.

11.23.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 406: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{4}{1} \, dx} \\ &= z_1 e^{-2x} \\ &= z_1 \left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x} \cos(x)\right) + c_2 \left(\mathrm{e}^{-2x} \cos(x) (\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, \mathrm{e}^{-2x} c_1 + \sin(x)\, \mathrm{e}^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x - \mathrm{e}^{-4x} + \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-4x}\}, \{1, x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}\cos(x), e^{-2x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-4x} + A_2 + A_3 x + A_4 \cos(2x) + A_5 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&5A_1 e^{-4x} + A_4 \cos(2x) + A_5 \sin(2x) + 4A_3 - 8A_4 \sin(2x) \\ &+ 8A_5 \cos(2x) + 5A_2 + 5A_3 x = 2x - e^{-4x} + \sin(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = -\frac{8}{25}, A_3 = \frac{2}{5}, A_4 = -\frac{8}{65}, A_5 = \frac{1}{65}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{-4x}}{5} - \frac{8}{25} + \frac{2x}{5} - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(\cos(x) e^{-2x} c_1 + \sin(x) e^{-2x} c_2\right) + \left(-\frac{e^{-4x}}{5} - \frac{8}{25} + \frac{2x}{5} - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65}\right)\end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^{-4x}}{5} - \frac{8}{25} + \frac{2x}{5} - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65}$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^{-4x}}{5} - \frac{8}{25} + \frac{2x}{5} - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65} \tag{1}$$

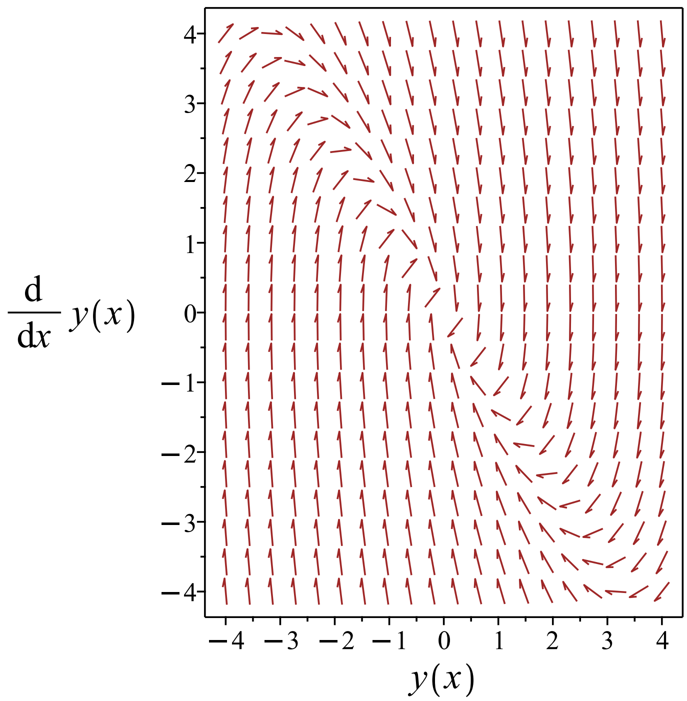


Figure 569: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{e^{-4x}}{5} - \frac{8}{25} + \frac{2x}{5} - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65}$$

Verified OK.

11.23.3 Maple step by step solution

Let's solve

$y'' + 4y' + 5y = 2x - e^{-4x} + \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4) \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - \mathrm{I}, -2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-2x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x)\,\mathrm{e}^{-2x} c_1 + \sin(x)\,\mathrm{e}^{-2x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2x - \mathrm{e}^{-4x} + \sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} \cos(x) & \mathrm{e}^{-2x} \sin(x) \\ -2\,\mathrm{e}^{-2x} \cos(x) - \mathrm{e}^{-2x} \sin(x) & -2\,\mathrm{e}^{-2x} \sin(x) + \mathrm{e}^{-2x} \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{-2x}\left(-\cos(x)\left(\int (\sin(x)(2x + \sin(2x))\,\mathrm{e}^{2x} - \mathrm{e}^{-2x} \sin(x))\,dx\right) + \sin(x)\left(\int (\cos(x)(2x\right.\right.$

  - Compute integrals

    $y_p(x) = -\frac{\mathrm{e}^{-4x}}{5} - \frac{8}{25} + \frac{2x}{5} - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65}$

- Substitute particular solution into general solution to ODE

  $y = \sin(x)\,\mathrm{e}^{-2x} c_2 + \cos(x)\,\mathrm{e}^{-2x} c_1 - \frac{8\cos(2x)}{65} + \frac{\sin(2x)}{65} - \frac{\mathrm{e}^{-4x}}{5} + \frac{2x}{5} - \frac{8}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+5*y(x)=2*x-exp(-4*x)+sin(2*x),y(x), singsol=all)
```

$$y(x) = c_2 e^{-2x} \sin(x) + e^{-2x} \cos(x) c_1 + \frac{\sin(2x)}{65} + \frac{2x}{5} - \frac{8}{25} - \frac{8\cos(2x)}{65} - \frac{e^{-4x}}{5}$$

✓ Solution by Mathematica

Time used: 0.784 (sec). Leaf size: 59

```
DSolve[y''[x]+4*y'[x]+5*y[x]==2*x-Exp[-4*x]+Sin[2*x],y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{2x}{5} - \frac{e^{-4x}}{5} + \frac{1}{65}\sin(2x) - \frac{8}{65}\cos(2x) + c_2 e^{-2x}\cos(x) + c_1 e^{-2x}\sin(x) - \frac{8}{25}$$

11.24 problem 24

Internal problem ID [2163]
Internal file name [OUTPUT/2163_Monday_February_26_2024_09_17_58_AM_88498068/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 24.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 2y'' = (2x^2 + x)\,\mathrm{e}^{-2x} + 5\cos(3x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y'' = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -2$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + \mathrm{e}^{-2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= \mathrm{e}^{-2x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' + 2y'' = \left(2x^2 + x\right) \mathrm{e}^{-2x} + 5\cos\left(3x\right)
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\left(2x^2 + x\right) \mathrm{e}^{-2x} + 5\cos\left(3x\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos\left(3x\right), \sin\left(3x\right)\}, \{x^2\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x, \mathrm{e}^{-2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x, \mathrm{e}^{-2x}\}
$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{\cos\left(3x\right), \sin\left(3x\right)\}, \{x^2\mathrm{e}^{-2x}, x^3\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1\cos\left(3x\right) + A_2\sin\left(3x\right) + A_3x^2\mathrm{e}^{-2x} + A_4x^3\mathrm{e}^{-2x} + A_5\mathrm{e}^{-2x}x
$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&27A_1\sin\left(3x\right) - 27A_2\cos\left(3x\right) - 8A_3\mathrm{e}^{-2x} + 8A_3x\,\mathrm{e}^{-2x} + 6A_4\mathrm{e}^{-2x} - 24A_4x\,\mathrm{e}^{-2x} \\
&\quad + 12A_4x^2\mathrm{e}^{-2x} + 4A_5\mathrm{e}^{-2x} - 18A_1\cos\left(3x\right) - 18A_2\sin\left(3x\right) = \left(2x^2 + x\right)\mathrm{e}^{-2x} + 5\cos\left(3x\right)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{10}{117}, A_2 = -\frac{5}{39}, A_3 = \frac{5}{8}, A_4 = \frac{1}{6}, A_5 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{10\cos(3x)}{117} - \frac{5\sin(3x)}{39} + \frac{5x^2 e^{-2x}}{8} + \frac{x^3 e^{-2x}}{6} + e^{-2x} x$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_2 x + c_1 + e^{-2x} c_3\right) + \left(-\frac{10\cos(3x)}{117} - \frac{5\sin(3x)}{39} + \frac{5x^2 e^{-2x}}{8} + \frac{x^3 e^{-2x}}{6} + e^{-2x} x\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + e^{-2x} c_3 - \frac{10\cos(3x)}{117} - \frac{5\sin(3x)}{39} + \frac{5x^2 e^{-2x}}{8} + \frac{x^3 e^{-2x}}{6} + e^{-2x} x \quad (1)$$

Verification of solutions

$$y = c_2 x + c_1 + e^{-2x} c_3 - \frac{10\cos(3x)}{117} - \frac{5\sin(3x)}{39} + \frac{5x^2 e^{-2x}}{8} + \frac{x^3 e^{-2x}}{6} + e^{-2x} x$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 2*exp(-2*_a)*_a^2+exp(-2*_a)*_a+5*cos(3
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)=(2*x^2+x)*exp(-2*x)+5*cos(3*x),y(x), singsol=all)
```

$$y(x) = \frac{(8x^3 + 30x^2 + 12c_1 + 48x + 33)\,\mathrm{e}^{-2x}}{48} + c_2x + c_3 - \frac{10\cos(3x)}{117} - \frac{5\sin(3x)}{39}$$

✓ Solution by Mathematica

Time used: 0.974 (sec). Leaf size: 56

```
DSolve[y'''[x]+2*y''[x]==(2*x^2+x)*Exp[-2*x]+5*Cos[3*x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{48}e^{-2x}\left(8x^3 + 30x^2 + 48x + 33 + 12c_1\right) - \frac{5}{39}\sin(3x) - \frac{10}{117}\cos(3x) + c_3x + c_2$$

11.25 problem 25

Internal problem ID [2164]
Internal file name [`OUTPUT/2164_Monday_February_26_2024_09_17_59_AM_89032562/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y = 8 \sin(x)^2}$$

11.25.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 8 \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x \cos(2x), x \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x \cos(2x) + A_3 x \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_2 \sin(2x) + 4A_3 \cos(2x) + 4A_1 = 8 \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 - x \sin(2x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2x) + c_2 \sin(2x)) + (1 - x \sin(2x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + c_2 \sin(2x) + 1 - x \sin(2x) \tag{1}
$$

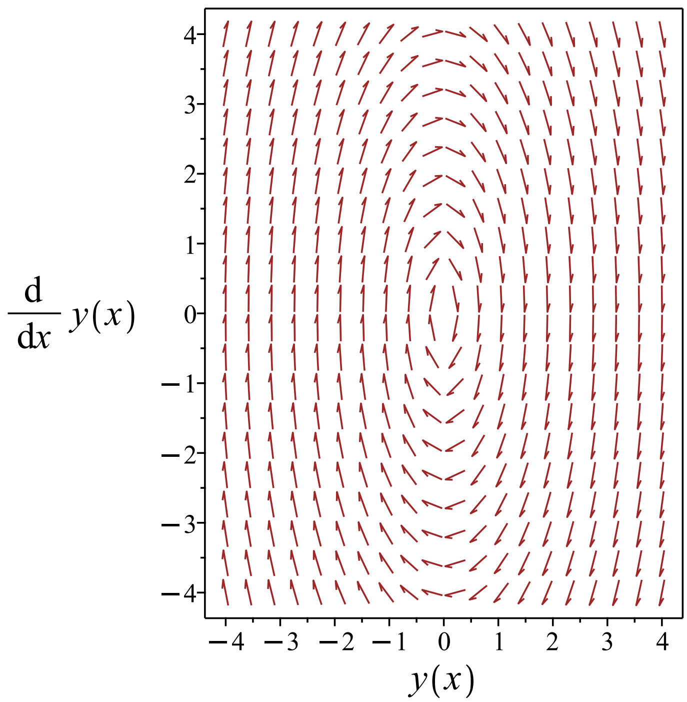


Figure 570: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + c_2 \sin(2x) + 1 - x \sin(2x)
$$

Verified OK.

11.25.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 408: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x \cos(2x) + A_3 x \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_2 \sin(2x) + 4A_3 \cos(2x) + 4A_1 = 8\sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 - x\sin(2x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + (1 - x \sin(2x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + 1 - x \sin(2x) \tag{1}
$$

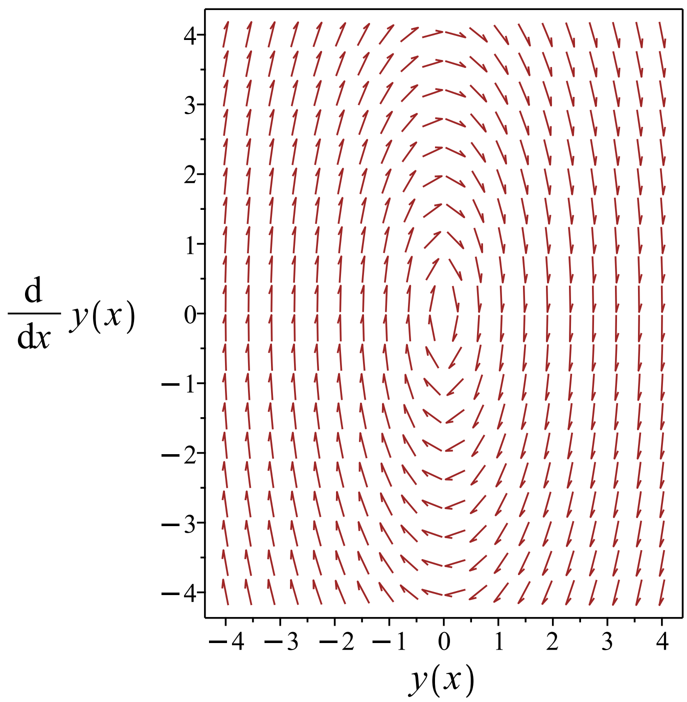


Figure 571: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + 1 - x \sin(2x)
$$

Verified OK.

11.25.3 Maple step by step solution

Let's solve

$y'' + 4y = 8 \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 8 \sin(x)^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(2x)\left(\int (-\sin(4x) + 2\sin(2x))\, dx\right) + 4\sin(2x)\left(\int \cos(2x)\sin(x)^2\, dx\right)$$

- Compute integrals

$$y_p(x) = -\frac{\cos(2x)}{4} + 1 - x\sin(2x)$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(2x) + c_2\sin(2x) - \frac{\cos(2x)}{4} + 1 - x\sin(2x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+4*y(x)=8*sin(x)^2,y(x), singsol=all)
```

$$y(x) = (c_1 - 1)\cos(2x) + 1 + (c_2 - x)\sin(2x)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 27

```
DSolve[y''[x]+4*y[x]==8*Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-1 + c_1)\cos(2x) + (-x + c_2)\sin(2x) + 1$$

11.26 problem 26

Internal problem ID [2165]
Internal file name [`OUTPUT/2165_Monday_February_26_2024_09_18_00_AM_27506519/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 26.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 4y = 5\sin(3x)\,\mathrm{e}^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y = 0$$

The characteristic equation is

$$\lambda^4 + 4 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1 - i\\ \lambda_2 &= 1 + i\\ \lambda_3 &= -1 - i\\ \lambda_4 &= -1 + i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{(-1-i)x}c_1 + e^{(-1+i)x}c_2 + e^{(1-i)x}c_3 + e^{(1+i)x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{(-1-i)x}$$
$$y_2 = e^{(-1+i)x}$$
$$y_3 = e^{(1-i)x}$$
$$y_4 = e^{(1+i)x}$$

Now the particular solution to the given ODE is found

$$y'''' + 4y = 5\sin(3x)e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\sin(3x)e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\cos(3x), \sin(3x)e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{(-1-i)x}, e^{(-1+i)x}, e^{(1-i)x}, e^{(1+i)x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1e^{2x}\cos(3x) + A_2\sin(3x)e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-115A_1e^{2x}\cos(3x) + 120A_1e^{2x}\sin(3x) - 115A_2\sin(3x)e^{2x} - 120A_2\cos(3x)e^{2x} = 5\sin(3x)e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{24}{1105}, A_2 = -\frac{23}{1105}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{24\,\mathrm{e}^{2x} \cos(3x)}{1105} - \frac{23 \sin(3x)\,\mathrm{e}^{2x}}{1105}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{(-1-i)x} c_1 + \mathrm{e}^{(-1+i)x} c_2 + \mathrm{e}^{(1-i)x} c_3 + \mathrm{e}^{(1+i)x} c_4\right) + \left(\frac{24\,\mathrm{e}^{2x} \cos(3x)}{1105} - \frac{23 \sin(3x)\,\mathrm{e}^{2x}}{1105}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{(-1-i)x} c_1 + \mathrm{e}^{(-1+i)x} c_2 + \mathrm{e}^{(1-i)x} c_3 + \mathrm{e}^{(1+i)x} c_4 + \frac{24\,\mathrm{e}^{2x} \cos(3x)}{1105} - \frac{23 \sin(3x)\,\mathrm{e}^{2x}}{1105} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{(-1-i)x} c_1 + \mathrm{e}^{(-1+i)x} c_2 + \mathrm{e}^{(1-i)x} c_3 + \mathrm{e}^{(1+i)x} c_4 + \frac{24\,\mathrm{e}^{2x} \cos(3x)}{1105} - \frac{23 \sin(3x)\,\mathrm{e}^{2x}}{1105}$$

Verified OK.

11.26.1 Maple step by step solution

Let's solve

$y'''' + 4y = 5 \sin(3x)\,\mathrm{e}^{2x}$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 5\sin(3x)\,e^{2x} - 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_4(x) = y_3'(x)\,, y_4'(x) = 5\sin(3x)\,e^{2x} - 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5\sin(3x)\,e^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5\sin(3x)\,e^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1-\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[-1+\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}-\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[1-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[1+\mathrm{I}, \left[\begin{array}{c} -\frac{1}{4}-\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-\mathrm{I})x} \cdot \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-x} \cdot \left[\begin{array}{c} \left(\frac{1}{4}+\frac{\mathrm{I}}{4}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ -\frac{\mathrm{I}}{2}(\cos(x) - \mathrm{I}\sin(x)) \\ \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{-x} \cdot \left[\begin{array}{c} \frac{\sin(x)}{4} + \frac{\cos(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ \cos(x) \end{array}\right], \vec{y}_2(x) = e^{-x} \cdot \left[\begin{array}{c} \frac{\cos(x)}{4} - \frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ -\sin(x) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - I, \left[\begin{array}{c} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(1-I)x} \cdot \left[\begin{array}{c} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x} \cdot (\cos(x) - I\sin(x)) \cdot \left[\begin{array}{c} -\frac{1}{4} + \frac{I}{4} \\ \frac{I}{2} \\ \frac{1}{2} + \frac{I}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{x} \cdot \left[\begin{array}{c} \left(-\frac{1}{4} + \frac{I}{4}\right)(\cos(x) - I\sin(x)) \\ \frac{I}{2}(\cos(x) - I\sin(x)) \\ \left(\frac{1}{2} + \frac{I}{2}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^x \cdot \begin{bmatrix} -\frac{\cos(x)}{4} + \frac{\sin(x)}{4} \\ \frac{\sin(x)}{2} \\ \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = e^x \cdot \begin{bmatrix} \frac{\sin(x)}{4} + \frac{\cos(x)}{4} \\ \frac{\cos(x)}{2} \\ \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x}\left(\frac{\sin(x)}{4} + \frac{\cos(x)}{4}\right) & e^{-x}\left(\frac{\cos(x)}{4} - \frac{\sin(x)}{4}\right) & e^x\left(-\frac{\cos(x)}{4} + \frac{\sin(x)}{4}\right) & e^x\left(\frac{\sin(x)}{4} + \frac{\cos(x)}{4}\right) \\ -\frac{\sin(x)e^{-x}}{2} & -\frac{\cos(x)e^{-x}}{2} & \frac{\sin(x)e^x}{2} & \frac{\cos(x)e^x}{2} \\ e^{-x}\left(-\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & e^{-x}\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & e^x\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right)e^x \\ \cos(x)e^{-x} & -\sin(x)e^{-x} & \cos(x)e^x & -\sin(x)e^x \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x}\left(\frac{\sin(x)}{4} + \frac{\cos(x)}{4}\right) & e^{-x}\left(\frac{\cos(x)}{4} - \frac{\sin(x)}{4}\right) & e^x\left(-\frac{\cos(x)}{4} + \frac{\sin(x)}{4}\right) & e^x\left(\frac{\sin(x)}{4} + \frac{\cos(x)}{4}\right) \\ -\frac{\sin(x)e^{-x}}{2} & -\frac{\cos(x)e^{-x}}{2} & \frac{\sin(x)e^x}{2} & \frac{\cos(x)e^x}{2} \\ e^{-x}\left(-\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & e^{-x}\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & e^x\left(\frac{\cos(x)}{2} + \frac{\sin(x)}{2}\right) & \left(\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right)e^x \\ \cos(x)e^{-x} & -\sin(x)e^{-x} & \cos(x)e^x & -\sin(x)e^x \end{bmatrix}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\cos(x)(e^x+e^{-x})}{2} & \frac{e^{-x}(-\cos(x)+\sin(x))}{4} + \frac{e^x(\cos(x)+\sin(x))}{4} \\ \frac{(-\cos(x)-\sin(x))e^{-x}}{2} + \frac{e^x(\cos(x)-\sin(x))}{2} & \frac{\cos(x)(e^x+e^{-x})}{2} & \frac{e^{-x}(-}{} \\ -\sin(x)(e^x - e^{-x}) & \frac{(-\cos(x)-\sin(x))e^{-x}}{2} + \frac{e^x(\cos(x)-\sin(x))}{2} \\ e^{-x}(\cos(x) - \sin(x)) - e^x(\cos(x) + \sin(x)) & -\sin(x)(e^x - e^{-x}) & \frac{(-\cos}{} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{\left(96\cos(x)^3-92\cos(x)^2\sin(x)-72\cos(x)+23\sin(x)\right)e^{2x}}{1105} + \frac{3(11\sin(x)+23\cos(x))e^{-x}}{520} - \frac{21\left(\cos(x)-\frac{11\sin(x)}{7}\right.}{136} \\ \frac{\left(-84\cos(x)^3-472\cos(x)^2\sin(x)+63\cos(x)+118\sin(x)\right)e^{2x}}{1105} + \frac{3(-17\sin(x)-6\cos(x))e^{-x}}{260} + \frac{3\left(\frac{9\sin(x)}{2}+\cos(\right.}{34} \\ \frac{\left(-1584\cos(x)^3-692\cos(x)^2\sin(x)+1188\cos(x)+173\sin(x)\right)e^{2x}}{1105} + \frac{3(23\sin(x)-11\cos(x))e^{-x}}{260} + \frac{33e^{x}\left(\cos(x)+\right.}{68} \\ \frac{\left(-5244\cos(x)^3+3368\cos(x)^2\sin(x)+3933\cos(x)-842\sin(x)\right)e^{2x}}{1105} + \frac{3(-6\sin(x)+17\cos(x))e^{-x}}{130} + \frac{27\left(\cos(x)-\frac{2}{}\right.}{34} \end{array}\right.$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} \frac{\left(96\cos(x)^3-92\cos(x)^2\sin(x)-72\cos(x)+23\sin(x)\right)\mathrm{e}^{2} \cdots}{1105} \\ \frac{\left(-84\cos(x)^3-472\cos(x)^2\sin(x)+63\cos(x)+118\sin(x) \cdots}{1105} \\ \frac{\left(-1584\cos(x)^3-692\cos(x)^2\sin(x)+1188\cos(x)+173\sin(\cdots}{1105} \\ \frac{\left(-5244\cos(x)^3+3368\cos(x)^2\sin(x)+3933\cos(x)-842\sin(\cdots}{1105} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((69+130c_1+130c_2)\cos(x)+130\sin(x)\left(c_1-c_2+\frac{33}{130}\right)\right)\mathrm{e}^{-x}}{520} + \frac{\left(96\cos(x)^3-92\cos(x)^2\sin(x)-72\cos(x)+23\sin(x)\right)\mathrm{e}^{2x}}{1105} - \frac{((c \cdots}{}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$4)+4*y(x)=5*exp(2*x)*sin(3*x),y(x), singsol=all)
```

$$y(x) = (\cos(x)c_3 + c_4\sin(x))\,\mathrm{e}^{-x} + \frac{\mathrm{e}^{2x}(24\cos(3x) - 23\sin(3x))}{1105} + \mathrm{e}^{x}(\cos(x)c_1 + \sin(x)c_2)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 64

```
DSolve[y''''[x]+4*y[x]==5*Exp[2*x]*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2x}(24\cos(3x) - 23\sin(3x))}{1105} + c_1 e^{-x}\cos(x) + c_4 e^{x}\cos(x) + c_2 e^{-x}\sin(x) + c_3 e^{x}\sin(x)$$

11.27 problem 27

Internal problem ID [2166]
Internal file name [`OUTPUT/2166_Monday_February_26_2024_09_18_00_AM_96670024/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 5y' - 6y = e^{3x}}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

11.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -5 \\ q(x) &= -6 \\ F &= e^{3x} \end{aligned}$$

Hence the ode is

$$y'' - 5y' - 6y = e^{3x}$$

The domain of $p(x) = -5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = e^{3x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.27.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = -6, f(x) = e^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 5\lambda e^{\lambda x} - 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-5^2 - (4)(1)(-6)} \\ &= \frac{5}{2} \pm \frac{7}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{5}{2} + \frac{7}{2} \\ \lambda_2 &= \frac{5}{2} - \frac{7}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 6 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(6)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{6x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{6x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{6x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 e^{3x} = e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{3x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{6x} + c_2 e^{-x}\right) + \left(-\frac{e^{3x}}{12}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{6x} + c_2 e^{-x} - \frac{e^{3x}}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 - \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 6c_1 e^{6x} - c_2 e^{-x} - \frac{e^{3x}}{4}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 6c_1 - c_2 - \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{10}{21}$$
$$c_2 = \frac{45}{28}$$

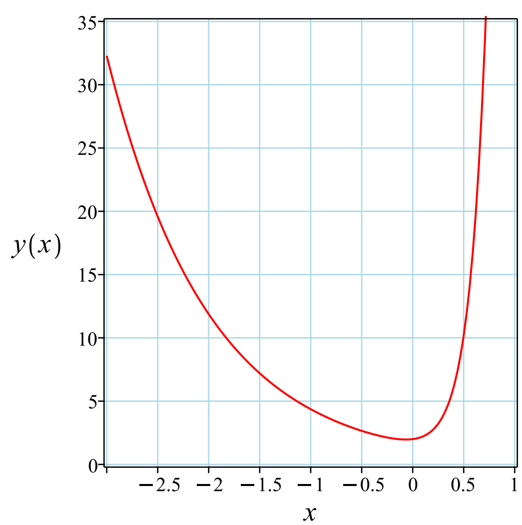
Substituting these values back in above solution results in

$$y = \frac{10\, e^{6x}}{21} + \frac{45\, e^{-x}}{28} - \frac{e^{3x}}{12}$$

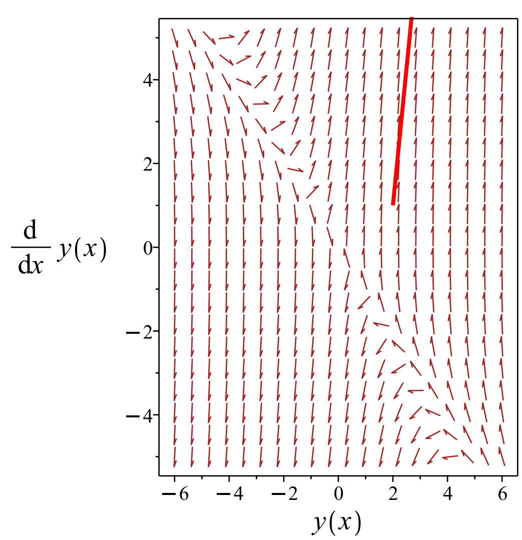
Summary

The solution(s) found are the following

$$y = \frac{10\, e^{6x}}{21} + \frac{45\, e^{-x}}{28} - \frac{e^{3x}}{12} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$$

Verified OK.

11.27.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' - 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 49 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{49z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 411: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{49}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{7x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx} \\
&= z_1 e^{\frac{5x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{5x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{7x}}{7}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{7x}}{7}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' - 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{6x}}{7}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{6x}}{7}, \mathrm{e}^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 \mathrm{e}^{3x} = \mathrm{e}^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{3x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{6x}}{7}\right) + \left(-\frac{e^{3x}}{12}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + \frac{c_2 e^{6x}}{7} - \frac{e^{3x}}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + \frac{c_2}{7} - \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + \frac{6c_2 e^{6x}}{7} - \frac{e^{3x}}{4}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -c_1 + \frac{6c_2}{7} - \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{45}{28} \\ c_2 &= \frac{10}{3} \end{aligned}$$

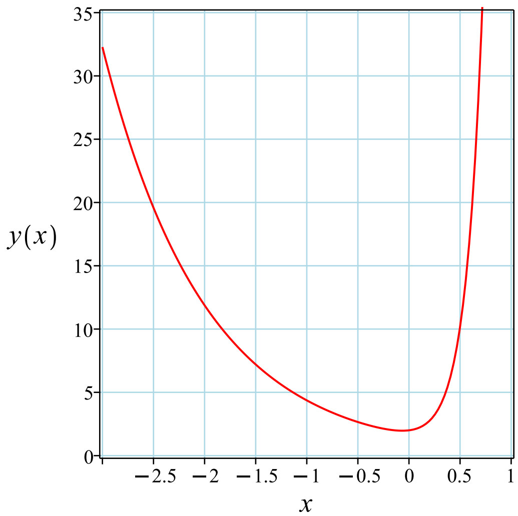
Substituting these values back in above solution results in

$$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$$

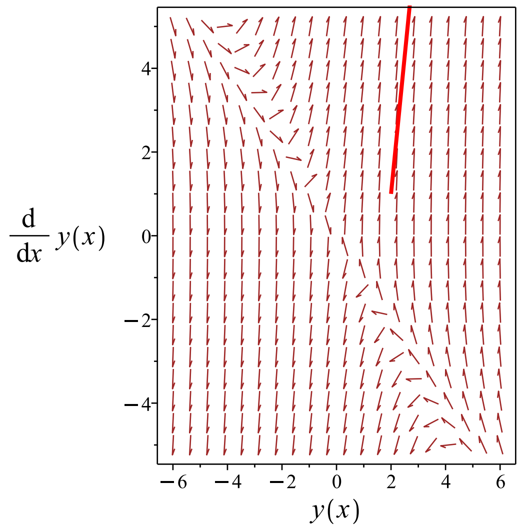
Summary

The solution(s) found are the following

$$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$$

Verified OK.

11.27.4 Maple step by step solution

Let's solve

$$\left[y'' - 5y' - 6y = e^{3x}, y(0) = 2, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r - 6 = 0$

- Factor the characteristic polynomial

$(r+1)(r-6)=0$

- Roots of the characteristic polynomial

  $r=(-1,6)$
- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{6x}$
- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{6x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=\mathrm{e}^{3x}\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x))=\begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{6x} \\ -\mathrm{e}^{-x} & 6\,\mathrm{e}^{6x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x))=7\,\mathrm{e}^{5x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=-\frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{4x}dx\right)}{7}+\frac{\mathrm{e}^{6x}\left(\int \mathrm{e}^{-3x}dx\right)}{7}$
  - Compute integrals

    $y_p(x)=-\frac{\mathrm{e}^{3x}}{12}$

- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{6x}-\frac{\mathrm{e}^{3x}}{12}$

□ Check validity of solution $y=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{6x}-\frac{\mathrm{e}^{3x}}{12}$

  - Use initial condition $y(0)=2$

    $2=c_1+c_2-\frac{1}{12}$
  - Compute derivative of the solution

$$y' = -c_1 e^{-x} + 6c_2 e^{6x} - \frac{e^{3x}}{4}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $$1 = -c_1 + 6c_2 - \frac{1}{4}$$

- Solve for c_1 and c_2

  $$\left\{c_1 = \frac{45}{28}, c_2 = \frac{10}{21}\right\}$$

- Substitute constant values into general solution and simplify

  $$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$$

- Solution to the IVP

  $$y = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)-5*diff(y(x),x)-6*y(x)=exp(3*x),y(0) = 2, D(y)(0) = 1],y(x), singsol=a
```

$$y(x) = \frac{10\,e^{6x}}{21} + \frac{45\,e^{-x}}{28} - \frac{e^{3x}}{12}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 30

```
DSolve[{y''[x]-5*y'[x]-6*y[x]==Exp[3*x],{y[0]==2,y'[0]==1}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{84}e^{-x}\left(-7e^{4x}+40e^{7x}+135\right)$$

11.28 problem 28

Internal problem ID [2167]
Internal file name [`OUTPUT/2167_Monday_February_26_2024_09_18_00_AM_25607207/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 12 \cos(x)^2}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 0, y'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\right]$$

11.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 4 \\ F &= 12 \cos(x)^2 \end{aligned}$$

Hence the ode is

$$y'' + 4y = 12\cos(x)^2$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. The domain of $F = 12\cos(x)^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

11.28.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 12\cos(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12 \cos(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x\cos(2x) + A_3 x\sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_2\sin(2x) + 4A_3\cos(2x) + 4A_1 = 12\cos(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{2}, A_2 = 0, A_3 = \frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3}{2} + \frac{3x\sin(2x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(2x) + c_2\sin(2x)) + \left(\frac{3}{2} + \frac{3x\sin(2x)}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{3}{2} + \frac{3x \sin(2x)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \frac{\pi}{2}$ in the above gives

$$0 = -c_1 + \frac{3}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x) + \frac{3 \sin(2x)}{2} + 3x \cos(2x)$$

substituting $y' = \frac{\pi}{2}$ and $x = \frac{\pi}{2}$ in the above gives

$$\frac{\pi}{2} = -2c_2 - \frac{3\pi}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3}{2}$$
$$c_2 = -\pi$$

Substituting these values back in above solution results in

$$y = \frac{3}{2} + \frac{3\cos(2x)}{2} - \sin(2x)\,\pi + \frac{3x \sin(2x)}{2}$$

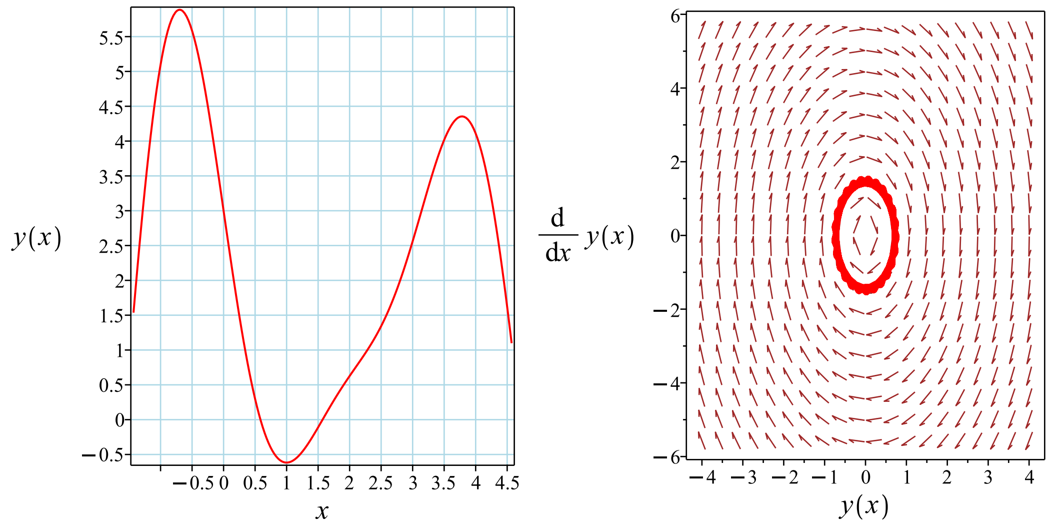
Which simplifies to

$$y = \frac{(3x - 2\pi)\sin(2x)}{2} + \frac{3\cos(2x)}{2} + \frac{3}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(3x - 2\pi)\sin(2x)}{2} + \frac{3\cos(2x)}{2} + \frac{3}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{(3x - 2\pi)\sin(2x)}{2} + \frac{3\cos(2x)}{2} + \frac{3}{2}$$

Verified OK.

11.28.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 413: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(2x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\
&= \cos(2x) \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\cos(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x\cos(2x) + A_3 x\sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_2 \sin(2x) + 4A_3 \cos(2x) + 4A_1 = 12 \cos(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{2}, A_2 = 0, A_3 = \frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3}{2} + \frac{3x \sin(2x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{3}{2} + \frac{3x \sin(2x)}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{3}{2} + \frac{3x \sin(2x)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \frac{\pi}{2}$ in the above gives

$$0 = -c_1 + \frac{3}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2x) + c_2 \cos(2x) + \frac{3 \sin(2x)}{2} + 3x \cos(2x)$$

substituting $y' = \frac{\pi}{2}$ and $x = \frac{\pi}{2}$ in the above gives

$$\frac{\pi}{2} = -c_2 - \frac{3\pi}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3}{2}$$
$$c_2 = -2\pi$$

Substituting these values back in above solution results in

$$y = \frac{3}{2} + \frac{3\cos(2x)}{2} - \sin(2x)\pi + \frac{3x\sin(2x)}{2}$$

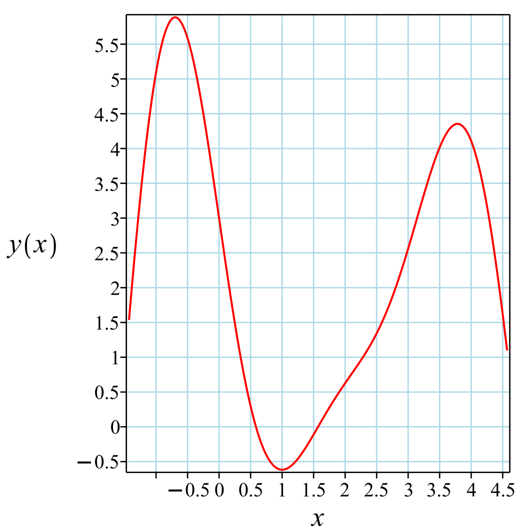
Which simplifies to

$$y = \frac{(3x - 2\pi)\sin(2x)}{2} + \frac{3\cos(2x)}{2} + \frac{3}{2}$$

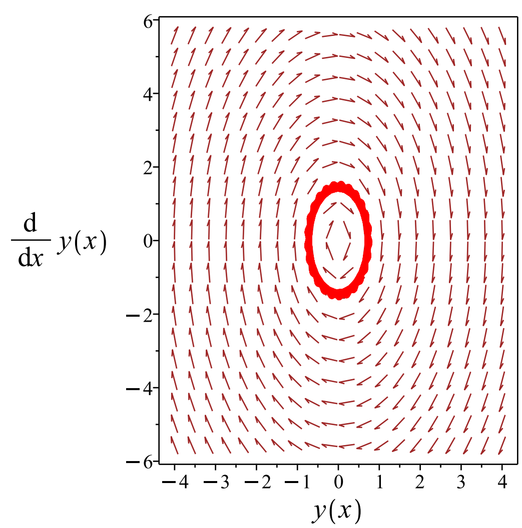
Summary

The solution(s) found are the following

$$y = \frac{(3x - 2\pi)\sin(2x)}{2} + \frac{3\cos(2x)}{2} + \frac{3}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(3x - 2\pi)\sin(2x)}{2} + \frac{3\cos(2x)}{2} + \frac{3}{2}$$

Verified OK.

11.28.4 Maple step by step solution

Let's solve

$$\left[y''+4y=12\cos(x)^2, y\left(\frac{\pi}{2}\right)=0, y'\Big|_{\left\{x=\frac{\pi}{2}\right\}}=\frac{\pi}{2}\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+4=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x)=\cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\sin(2x)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(2x)+c_2\sin(2x)+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=12\cos(x)^2\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x))=\begin{bmatrix}\cos(2x) & \sin(2x)\\ -2\sin(2x) & 2\cos(2x)\end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x))=2$

- ∘ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -6\cos(2x)\left(\int \sin(2x)\cos(x)^2\,dx\right) + 6\sin(2x)\left(\int \cos(2x)\cos(x)^2\,dx\right)$$

- Compute integrals

  $$y_p(x) = 3\cos(x)(\sin(x)x + \cos(x))$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\cos(2x) + c_2\sin(2x) + 3\cos(x)(\sin(x)x + \cos(x))$$

☐ Check validity of solution $y = c_1\cos(2x) + c_2\sin(2x) + 3\cos(x)(\sin(x)x + \cos(x))$

- Use initial condition $y\left(\frac{\pi}{2}\right) = 0$

  $$0 = -c_1$$

- Compute derivative of the solution

  $$y' = -2c_1\sin(2x) + 2c_2\cos(2x) - 3\sin(x)(\sin(x)x + \cos(x)) + 3\cos(x)^2 x$$

- Use the initial condition $y'\Big|_{\left\{x=\frac{\pi}{2}\right\}} = \frac{\pi}{2}$

  $$\frac{\pi}{2} = -2c_2 - \frac{3\pi}{2}$$

- Solve for c_1 and c_2

  $$\{c_1 = 0, c_2 = -\pi\}$$

- Substitute constant values into general solution and simplify

  $$y = 3\left(\cos(x) + \sin(x)\left(x - \frac{2\pi}{3}\right)\right)\cos(x)$$

- Solution to the IVP

  $$y = 3\left(\cos(x) + \sin(x)\left(x - \frac{2\pi}{3}\right)\right)\cos(x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve([diff(y(x),x$2)+4*y(x)=12*cos(x)^2,y(1/2*Pi) = 0, D(y)(1/2*Pi) = 1/2*Pi],y(x), singso
```

$$y(x) = \frac{(3x-2\pi)\sin(2x)}{2} + \frac{3\cos(2x)}{2} + \frac{3}{2}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 23

```
DSolve[{y''[x]+4*y[x]==12*Cos[x]^2,{y[Pi/2]==0,y'[Pi/2]==Pi/2}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \cos(x)((3x-2\pi)\sin(x)+3\cos(x))$$

11.29 problem 29

Internal problem ID [2168]
Internal file name [`OUTPUT/2168_Monday_February_26_2024_09_18_01_AM_5667081/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = x\,\mathrm{e}^{-x}}$$

With initial conditions

$$\left[y(0) = \frac{1}{9}, y'(0) = 0\right]$$

11.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= 2 \\ F &= x\,\mathrm{e}^{-x} \end{aligned}$$

Hence the ode is

$$y'' - 3y' + 2y = x\,\mathrm{e}^{-x}$$

The domain of $p(x) = -3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = x\,\mathrm{e}^{-x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.29.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = x\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(2)} \\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{-x} + A_2 e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 e^{-x} + 6A_1 x\,e^{-x} + 6A_2 e^{-x} = x\,e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = \frac{5}{36}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,e^{-x}}{6} + \frac{5\,e^{-x}}{36}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{x}\right) + \left(\frac{x\,e^{-x}}{6} + \frac{5\,e^{-x}}{36}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{x} + \frac{x\,e^{-x}}{6} + \frac{5\,e^{-x}}{36} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{9}$ and $x = 0$ in the above gives

$$\frac{1}{9} = c_1 + c_2 + \frac{5}{36} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} + c_2 e^{x} + \frac{e^{-x}}{36} - \frac{x\,e^{-x}}{6}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = 2c_1 + c_2 + \frac{1}{36} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = -\frac{1}{36}$$

Substituting these values back in above solution results in

$$y = -\frac{e^{x}}{36} + \frac{x\,e^{-x}}{6} + \frac{5\,e^{-x}}{36}$$

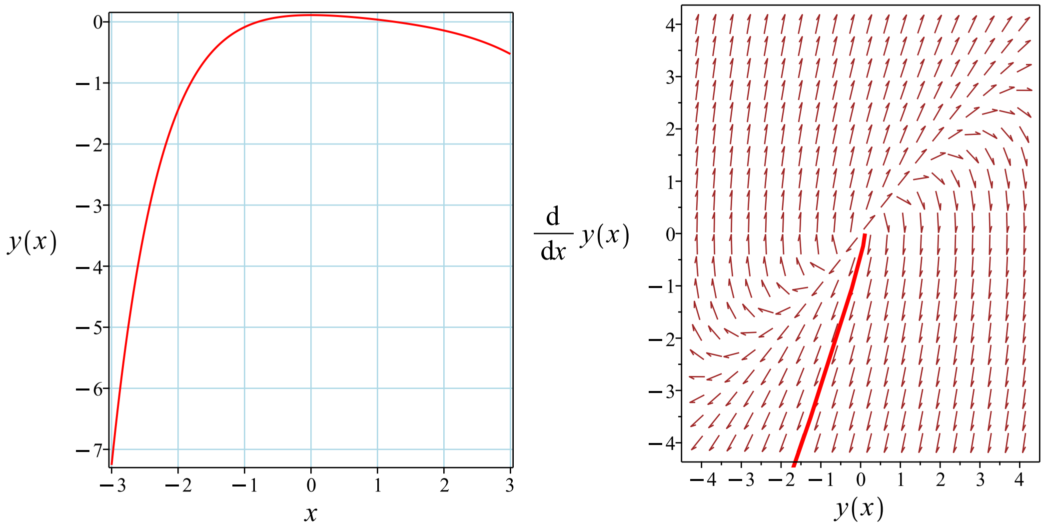
Which simplifies to

$$y = \frac{(5 + 6x)\,e^{-x}}{36} - \frac{e^{x}}{36}$$

Summary

The solution(s) found are the following

$$y = \frac{(5 + 6x)\,e^{-x}}{36} - \frac{e^{x}}{36} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{(5+6x)\,e^{-x}}{36} - \frac{e^{x}}{36}$$

Verified OK.

11.29.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 415: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x})\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(\mathrm{e}^{x}))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{x} + c_2\mathrm{e}^{2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x\,\mathrm{e}^{-x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{x}, \mathrm{e}^{2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 x\,\mathrm{e}^{-x} + A_2\mathrm{e}^{-x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-5A_1\mathrm{e}^{-x} + 6A_1 x\,\mathrm{e}^{-x} + 6A_2\mathrm{e}^{-x} = x\,\mathrm{e}^{-x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = \frac{5}{36}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{-x}}{6} + \frac{5\,\mathrm{e}^{-x}}{36}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x}\right) + \left(\frac{x\,\mathrm{e}^{-x}}{6} + \frac{5\,\mathrm{e}^{-x}}{36}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} + \frac{x\,\mathrm{e}^{-x}}{6} + \frac{5\,\mathrm{e}^{-x}}{36} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{9}$ and $x = 0$ in the above gives

$$\frac{1}{9} = c_1 + c_2 + \frac{5}{36} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^{x} + 2c_2 \mathrm{e}^{2x} + \frac{\mathrm{e}^{-x}}{36} - \frac{x\,\mathrm{e}^{-x}}{6}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1 + 2c_2 + \frac{1}{36} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{1}{36} \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{e^{x}}{36} + \frac{x\,e^{-x}}{6} + \frac{5\,e^{-x}}{36}$$

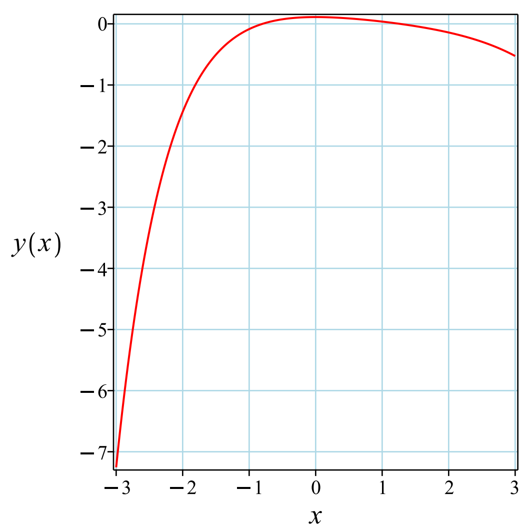
Which simplifies to

$$y = \frac{(5+6x)\,e^{-x}}{36} - \frac{e^{x}}{36}$$

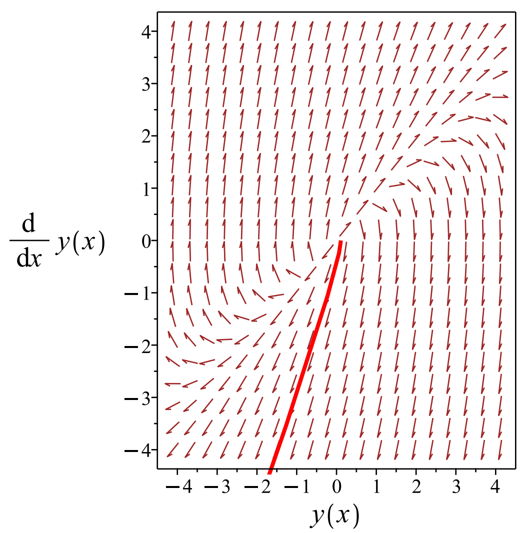
Summary

The solution(s) found are the following

$$y = \frac{(5+6x)\,e^{-x}}{36} - \frac{e^{x}}{36} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(5+6x)\,e^{-x}}{36} - \frac{e^{x}}{36}$$

Verified OK.

11.29.4 Maple step by step solution

Let's solve

$\left[y'' - 3y' + 2y = x\,\mathrm{e}^{-x}, y(0) = \frac{1}{9}, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\,\mathrm{e}^{-x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x & \mathrm{e}^{2x} \\ \mathrm{e}^x & 2\,\mathrm{e}^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{3x}$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -e^x \left(\int e^{-2x} x dx\right) + e^{2x} \left(\int x\, e^{-3x} dx\right)$$

- Compute integrals

$$y_p(x) = \frac{(5+6x)\, e^{-x}}{36}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^x + c_2 e^{2x} + \frac{(5+6x)\, e^{-x}}{36}$$

□ Check validity of solution $y = c_1 e^x + c_2 e^{2x} + \frac{(5+6x)\, e^{-x}}{36}$

- Use initial condition $y(0) = \frac{1}{9}$

$$\frac{1}{9} = c_1 + c_2 + \frac{5}{36}$$

- Compute derivative of the solution

$$y' = c_1 e^x + 2c_2 e^{2x} + \frac{e^{-x}}{6} - \frac{(5+6x)\, e^{-x}}{36}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

$$0 = c_1 + 2c_2 + \frac{1}{36}$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{1}{36}, c_2 = 0\right\}$$

- Substitute constant values into general solution and simplify

$$y = \frac{(5+6x)\, e^{-x}}{36} - \frac{e^x}{36}$$

- Solution to the IVP

$$y = \frac{(5+6x)\, e^{-x}}{36} - \frac{e^x}{36}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=x*exp(-x),y(0) = 1/9, D(y)(0) = 0],y(x), singso
```

$$y(x) = \frac{(6x+5)\,\mathrm{e}^{-x}}{36} - \frac{\mathrm{e}^{x}}{36}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 24

```
DSolve[{y''[x]-3*y'[x]+2*y[x]==x*Exp[-x],{y[0]==1/9,y'[0]==0}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to -\frac{1}{36}e^{-x}\left(-6x+e^{2x}-5\right)$$

11.30 problem 30

Internal problem ID [2169]
Internal file name [`OUTPUT/2169_Monday_February_26_2024_09_18_02_AM_51633603/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sin(x)\,e^{x}}$$

With initial conditions

$$[y(0) = 3, y'(0) = 2]$$

11.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ F &= \sin(x)\,e^{x} \end{aligned}$$

Hence the ode is

$$y'' + y = \sin(x)\,e^{x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \sin(x)\,e^{x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.30.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)\,e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\, e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,\mathrm{e}^{x}, \sin(x)\,\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)\,\mathrm{e}^{x} + A_2 \sin(x)\,\mathrm{e}^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x)\,\mathrm{e}^{x} + 2A_2 \cos(x)\,\mathrm{e}^{x} + A_1 \cos(x)\,\mathrm{e}^{x} + A_2 \sin(x)\,\mathrm{e}^{x} = \sin(x)\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{5}, A_2 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2\cos(x)\,\mathrm{e}^{x}}{5} + \frac{\sin(x)\,\mathrm{e}^{x}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{2\cos(x)\,\mathrm{e}^{x}}{5} + \frac{\sin(x)\,\mathrm{e}^{x}}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{2\cos(x)\,\mathrm{e}^{x}}{5} + \frac{\sin(x)\,\mathrm{e}^{x}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 - \frac{2}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sin(x) c_1 + c_2 \cos(x) + \frac{3 \sin(x) e^x}{5} - \frac{\cos(x) e^x}{5}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = -\frac{1}{5} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{17}{5}$$
$$c_2 = \frac{11}{5}$$

Substituting these values back in above solution results in

$$y = \frac{17 \cos(x)}{5} + \frac{11 \sin(x)}{5} - \frac{2 \cos(x) e^x}{5} + \frac{\sin(x) e^x}{5}$$

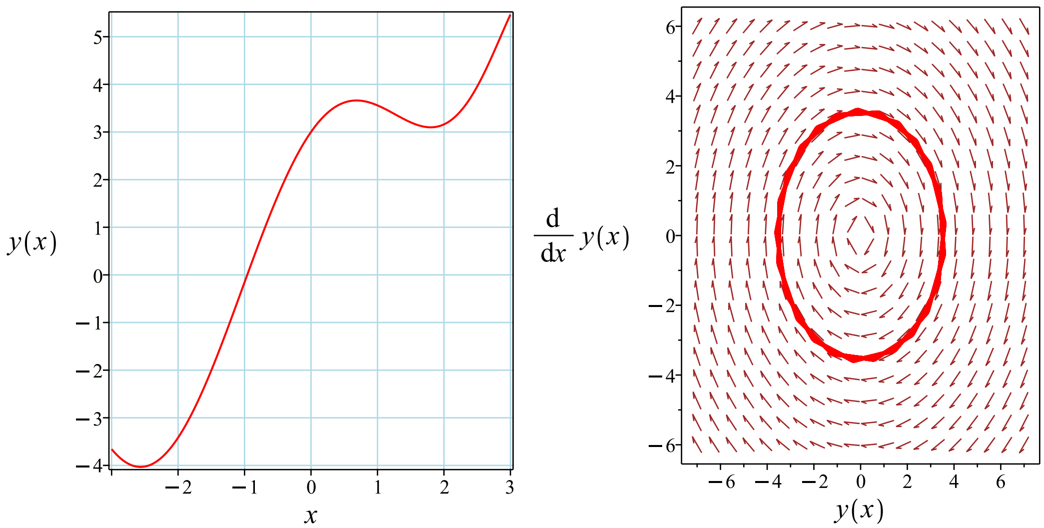
Which simplifies to

$$y = \frac{(-2 e^x + 17) \cos(x)}{5} + \frac{\sin(x) (e^x + 11)}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{(-2 e^x + 17) \cos(x)}{5} + \frac{\sin(x) (e^x + 11)}{5} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{(-2\,e^{x} + 17)\cos(x)}{5} + \frac{\sin(x)\,(e^{x} + 11)}{5}$$

Verified OK.

11.30.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 417: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\,\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,\mathrm{e}^x, \sin(x)\,\mathrm{e}^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)\,\mathrm{e}^x + A_2 \sin(x)\,\mathrm{e}^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x)\,\mathrm{e}^x + 2A_2 \cos(x)\,\mathrm{e}^x + A_1 \cos(x)\,\mathrm{e}^x + A_2 \sin(x)\,\mathrm{e}^x = \sin(x)\,\mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{5}, A_2 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2\cos(x)\,e^x}{5} + \frac{\sin(x)\,e^x}{5}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{2\cos(x)\,e^x}{5} + \frac{\sin(x)\,e^x}{5}\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{2\cos(x)\,e^x}{5} + \frac{\sin(x)\,e^x}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 - \frac{2}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sin(x)\,c_1 + c_2 \cos(x) + \frac{3\sin(x)\,e^x}{5} - \frac{\cos(x)\,e^x}{5}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = -\frac{1}{5} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}
c_1 &= \frac{17}{5} \\
c_2 &= \frac{11}{5}
\end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{17\cos(x)}{5} + \frac{11\sin(x)}{5} - \frac{2\cos(x)\,\mathrm{e}^{x}}{5} + \frac{\sin(x)\,\mathrm{e}^{x}}{5}$$

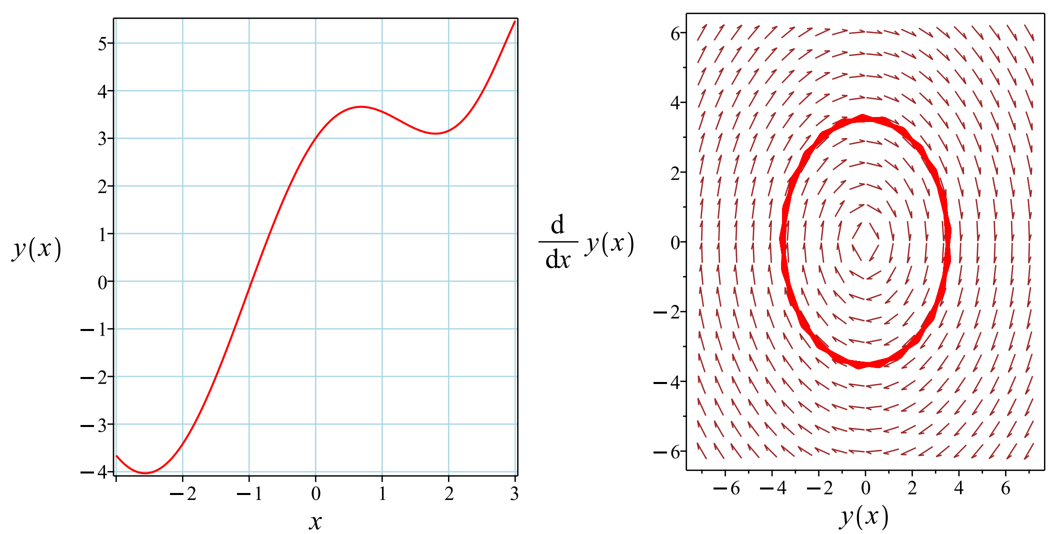
Which simplifies to

$$y = \frac{(-2\,\mathrm{e}^{x} + 17)\cos(x)}{5} + \frac{\sin(x)\,(\mathrm{e}^{x} + 11)}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{(-2\,\mathrm{e}^{x} + 17)\cos(x)}{5} + \frac{\sin(x)\,(\mathrm{e}^{x} + 11)}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{(-2\,\mathrm{e}^{x} + 17)\cos(x)}{5} + \frac{\sin(x)\,(\mathrm{e}^{x} + 11)}{5}$$

Verified OK.

11.30.4 Maple step by step solution

Let's solve

$$\left[y''+y=\sin\left(x\right)\mathrm{e}^{x}, y(0)=3, y'\Big|_{\{x=0\}}=2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+1=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm\left(\sqrt{-4}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\mathrm{I},\mathrm{I}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x)=\cos\left(x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\sin\left(x\right)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos\left(x\right)+c_2\sin\left(x\right)+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=\sin\left(x\right)\mathrm{e}^{x}\right]$$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W\left(y_1(x), y_2(x)\right)=\left[\begin{array}{cc}\cos\left(x\right) & \sin\left(x\right)\\ -\sin\left(x\right) & \cos\left(x\right)\end{array}\right]$$

  ◦ Compute Wronskian

  $W\left(y_1(x), y_2(x)\right)=1$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(x)\left(\int \sin(x)^2 \mathrm{e}^x dx\right) + \frac{\sin(x)\left(\int \mathrm{e}^x \sin(2x)dx\right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{\mathrm{e}^x(-2\cos(x)+\sin(x))}{5}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{\mathrm{e}^x(-2\cos(x)+\sin(x))}{5}$$

□ Check validity of solution $y = c_1 \cos(x) + c_2 \sin(x) + \frac{\mathrm{e}^x(-2\cos(x)+\sin(x))}{5}$

- Use initial condition $y(0) = 3$

$$3 = c_1 - \frac{2}{5}$$

- Compute derivative of the solution

$$y' = -\sin(x)\, c_1 + c_2 \cos(x) + \frac{\mathrm{e}^x(-2\cos(x)+\sin(x))}{5} + \frac{\mathrm{e}^x(\cos(x)+2\sin(x))}{5}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 2$

$$2 = -\frac{1}{5} + c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = \frac{17}{5}, c_2 = \frac{11}{5}\right\}$$

- Substitute constant values into general solution and simplify

$$y = \frac{(-2\,\mathrm{e}^x+17)\cos(x)}{5} + \frac{\sin(x)(\mathrm{e}^x+11)}{5}$$

- Solution to the IVP

$$y = \frac{(-2\,\mathrm{e}^x+17)\cos(x)}{5} + \frac{\sin(x)(\mathrm{e}^x+11)}{5}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)+y(x)=exp(x)*sin(x),y(0) = 3, D(y)(0) = 2],y(x), singsol=all)
```

$$y(x) = \frac{(-2\,e^{x}+17)\cos(x)}{5} + \frac{\sin(x)\,(e^{x}+11)}{5}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 28

```
DSolve[{y''[x]+y[x]==Exp[x]*Sin[x],{y[0]==3,y'[0]==2}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{5}((e^{x}+11)\sin(x) + (17-2e^{x})\cos(x))$$

11.31 problem 31

Internal problem ID [2170]
Internal file name [`OUTPUT/2170_Monday_February_26_2024_09_18_02_AM_24547031/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{2y'' + y' = 8\sin(2x) + \mathrm{e}^{-x}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

11.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{2} \\ q(x) &= 0 \\ F &= 4\sin(2x) + \frac{e^{-x}}{2} \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{2} = 4\sin(2x) + \frac{e^{-x}}{2}$$

The domain of $p(x) = \frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $F = 4\sin(2x) + \frac{e^{-x}}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.31.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2, B = 1, C = 0, f(x) = 8\sin(2x) + e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 \mathrm{e}^{\lambda x} + \lambda\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{1^2 - (4)\,(2)\,(0)}\\ &= -\frac{1}{4} \pm \frac{1}{4}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{4} + \frac{1}{4}$$
$$\lambda_2 = -\frac{1}{4} - \frac{1}{4}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{\left(-\frac{1}{2}\right)x}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 \mathrm{e}^{-\frac{x}{2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \sin(2x) + e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, e^{-\frac{x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-x} + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{-x} - 8A_2 \cos(2x) - 8A_3 \sin(2x) - 2A_2 \sin(2x) + 2A_3 \cos(2x) = 8 \sin(2x) + e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = -\frac{4}{17}, A_3 = -\frac{16}{17}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^{-\frac{x}{2}}\right) + \left(e^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2 \mathrm{e}^{-\frac{x}{2}} + \mathrm{e}^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 + \frac{13}{17} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_2 \mathrm{e}^{-\frac{x}{2}}}{2} - \mathrm{e}^{-x} + \frac{8\sin(2x)}{17} - \frac{32\cos(2x)}{17}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -\frac{c_2}{2} - \frac{49}{17} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

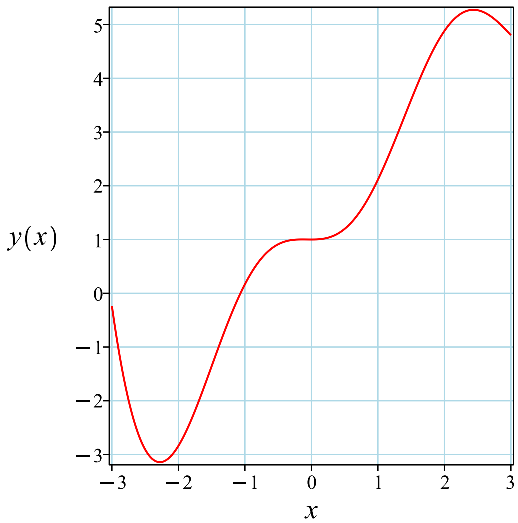
$$c_1 = 6$$
$$c_2 = -\frac{98}{17}$$

Substituting these values back in above solution results in

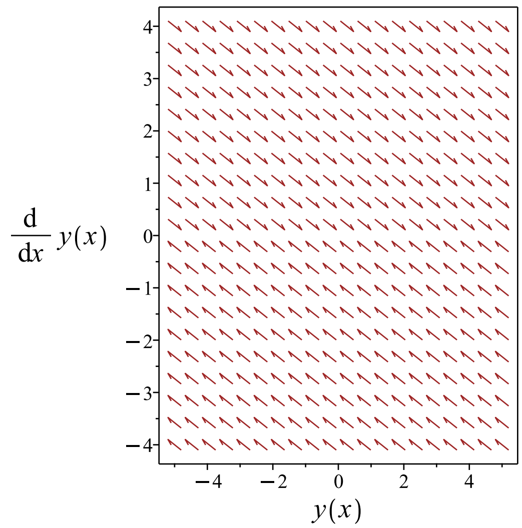
$$y = 6 - \frac{98\,\mathrm{e}^{-\frac{x}{2}}}{17} + \mathrm{e}^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$$

Summary
The solution(s) found are the following

$$y = 6 - \frac{98\,\mathrm{e}^{-\frac{x}{2}}}{17} + \mathrm{e}^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 6 - \frac{98\,\mathrm{e}^{-\frac{x}{2}}}{17} + \mathrm{e}^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$$

Verified OK.

11.31.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (2y'' + y')\,dx = \int (8\sin(2x) + \mathrm{e}^{-x})\,dx$$

$$2y' + y = -4\cos(2x) - \mathrm{e}^{-x} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{2}$$

$$q(x) = -2\cos(2x) - \frac{\mathrm{e}^{-x}}{2} + \frac{c_1}{2}$$

Hence the ode is

$$y' + \frac{y}{2} = -2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2}dx} \\ &= e^{\frac{x}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(-2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2}\right) \\ \frac{d}{dx}\left(e^{\frac{x}{2}} y\right) &= \left(e^{\frac{x}{2}}\right)\left(-2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2}\right) \\ d\left(e^{\frac{x}{2}} y\right) &= \left(\frac{(-4\,e^{x}\cos(2x) + c_1 e^{x} - 1)\,e^{-\frac{x}{2}}}{2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{\frac{x}{2}} y &= \int \frac{(-4\,e^{x}\cos(2x) + c_1 e^{x} - 1)\,e^{-\frac{x}{2}}}{2}\,dx \\ e^{\frac{x}{2}} y &= e^{-\frac{x}{2}} + c_1 e^{\frac{x}{2}} - \frac{4\,e^{\frac{x}{2}}\cos(2x)}{17} - \frac{16\,e^{\frac{x}{2}}\sin(2x)}{17} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x}{2}}$ results in

$$y = e^{-\frac{x}{2}}\left(e^{-\frac{x}{2}} + c_1 e^{\frac{x}{2}} - \frac{4\,e^{\frac{x}{2}}\cos(2x)}{17} - \frac{16\,e^{\frac{x}{2}}\sin(2x)}{17}\right) + c_2 e^{-\frac{x}{2}}$$

which simplifies to

$$y = -\frac{\left(4\,e^{x}\cos(2x) + 16\,e^{x}\sin(2x) - 17c_1 e^{x} - 17c_2 e^{\frac{x}{2}} - 17\right)e^{-x}}{17}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{\left(4\,e^{x}\cos(2x) + 16\,e^{x}\sin(2x) - 17c_1 e^{x} - 17c_2 e^{\frac{x}{2}} - 17\right)e^{-x}}{17} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 + \frac{13}{17} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(36\,\mathrm{e}^{x}\cos\left(2x\right)+8\,\mathrm{e}^{x}\sin\left(2x\right)-17c_1\mathrm{e}^{x}-\frac{17c_2\mathrm{e}^{\frac{x}{2}}}{2}\right)\mathrm{e}^{-x}}{17}+\frac{\left(4\,\mathrm{e}^{x}\cos\left(2x\right)+16\,\mathrm{e}^{x}\sin\left(2x\right)-17c_1\mathrm{e}^{x}-1\right.}{17}$$

substituting $y'=0$ and $x=0$ in the above gives

$$0 = -\frac{c_2}{2}-\frac{49}{17} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

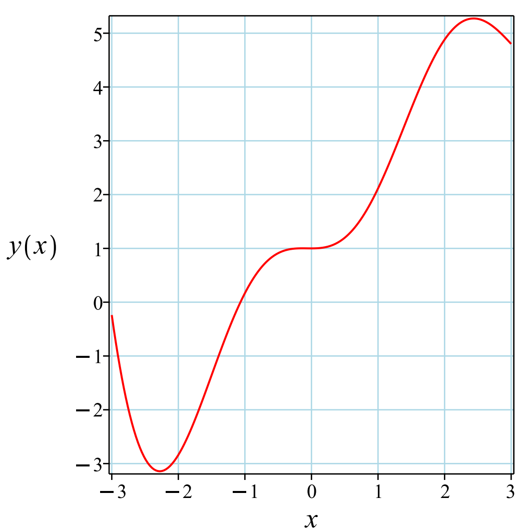
$$c_1 = 6$$
$$c_2 = -\frac{98}{17}$$

Substituting these values back in above solution results in

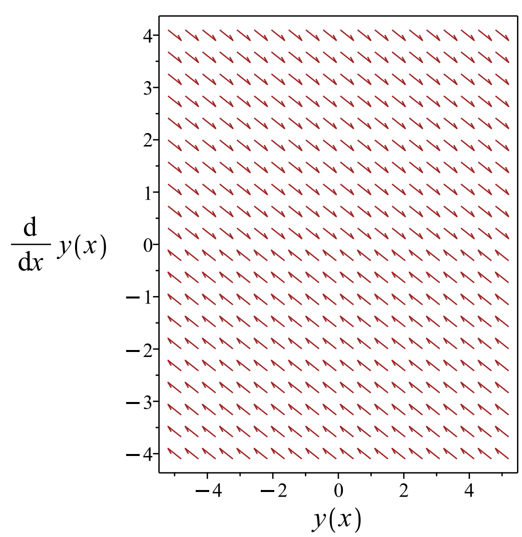
$$y = -\frac{\left(4\,\mathrm{e}^{x}\cos\left(2x\right)+16\,\mathrm{e}^{x}\sin\left(2x\right)-17+98\,\mathrm{e}^{\frac{x}{2}}-102\,\mathrm{e}^{x}\right)\mathrm{e}^{-x}}{17}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(4\,\mathrm{e}^{x}\cos\left(2x\right)+16\,\mathrm{e}^{x}\sin\left(2x\right)-17+98\,\mathrm{e}^{\frac{x}{2}}-102\,\mathrm{e}^{x}\right)\mathrm{e}^{-x}}{17} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\left(4\,\mathrm{e}^{x} \cos\left(2x\right)+16\,\mathrm{e}^{x} \sin\left(2x\right)-17+98\,\mathrm{e}^{\frac{x}{2}}-102\,\mathrm{e}^{x}\right) \mathrm{e}^{-x}}{17}$$

Verified OK.

11.31.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$2p'(x) + p(x) - 8\sin\left(2x\right) - \mathrm{e}^{-x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{2}dx}\\ &= \mathrm{e}^{\frac{x}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(4\sin\left(2x\right)+\frac{\mathrm{e}^{-x}}{2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{x}{2}} p\right) &= \left(\mathrm{e}^{\frac{x}{2}}\right)\left(4\sin\left(2x\right)+\frac{\mathrm{e}^{-x}}{2}\right)\\ \mathrm{d}\left(\mathrm{e}^{\frac{x}{2}} p\right) &= \left(4\,\mathrm{e}^{\frac{x}{2}} \sin\left(2x\right)+\frac{\mathrm{e}^{-\frac{x}{2}}}{2}\right) \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{\frac{x}{2}} p &= \int 4\,\mathrm{e}^{\frac{x}{2}} \sin\left(2x\right)+\frac{\mathrm{e}^{-\frac{x}{2}}}{2}\,\mathrm{d}x\\ \mathrm{e}^{\frac{x}{2}} p &= -\frac{32\,\mathrm{e}^{\frac{x}{2}} \cos\left(2x\right)}{17}+\frac{8\,\mathrm{e}^{\frac{x}{2}} \sin\left(2x\right)}{17}-\mathrm{e}^{-\frac{x}{2}}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{x}{2}}$ results in

$$p(x) = \mathrm{e}^{-\frac{x}{2}} \left(-\frac{32\,\mathrm{e}^{\frac{x}{2}} \cos\left(2x\right)}{17} + \frac{8\,\mathrm{e}^{\frac{x}{2}} \sin\left(2x\right)}{17} - \mathrm{e}^{-\frac{x}{2}}\right) + c_1 \mathrm{e}^{-\frac{x}{2}}$$

which simplifies to

$$p(x) = -\frac{\left(32\,\mathrm{e}^{x} \cos\left(2x\right) - 8\,\mathrm{e}^{x} \sin\left(2x\right) - 17 c_1 \mathrm{e}^{\frac{x}{2}} + 17\right) \mathrm{e}^{-x}}{17}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{49}{17} + c_1$$

$$c_1 = \frac{49}{17}$$

Substituting c_1 found above in the general solution gives

$$p(x) = -\frac{\mathrm{e}^{-x} \left(32\,\mathrm{e}^{x} \cos\left(2x\right) - 8\,\mathrm{e}^{x} \sin\left(2x\right) - 49\,\mathrm{e}^{\frac{x}{2}} + 17\right)}{17}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{\mathrm{e}^{-x} \left(32\,\mathrm{e}^{x} \cos\left(2x\right) - 8\,\mathrm{e}^{x} \sin\left(2x\right) - 49\,\mathrm{e}^{\frac{x}{2}} + 17\right)}{17}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{\mathrm{e}^{-x} \left(32\,\mathrm{e}^{x} \cos\left(2x\right) - 8\,\mathrm{e}^{x} \sin\left(2x\right) - 49\,\mathrm{e}^{\frac{x}{2}} + 17\right)}{17}\, \mathrm{d}x \\ &= \mathrm{e}^{-x} - \frac{98\,\mathrm{e}^{-\frac{x}{2}}}{17} - \frac{8 \cos\left(x\right)^{2}}{17} - \frac{32 \cos\left(x\right) \sin\left(x\right)}{17} + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{89}{17} + c_2$$

$$c_2 = \frac{106}{17}$$

Substituting c_2 found above in the general solution gives

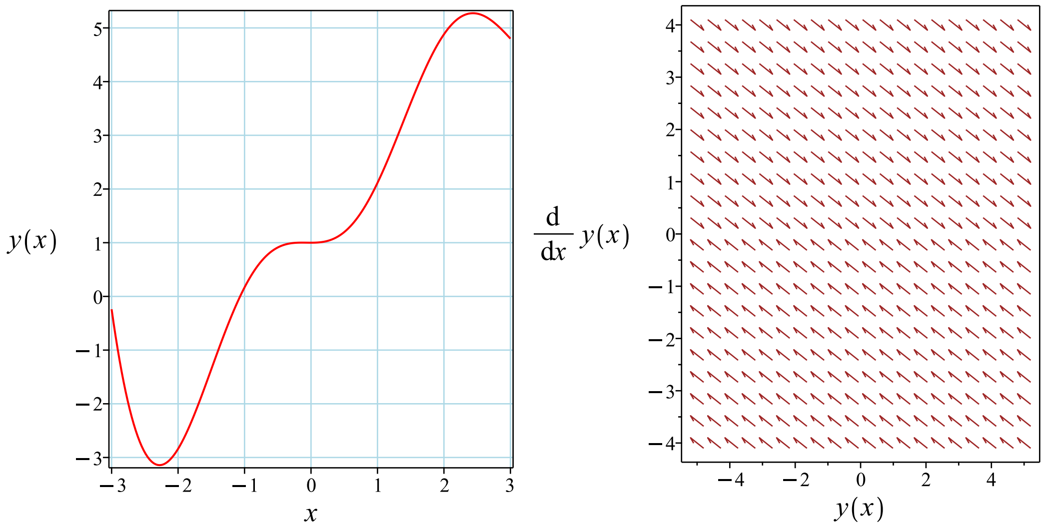
$$y = \mathrm{e}^{-x} - \frac{98\,\mathrm{e}^{-\frac{x}{2}}}{17} - \frac{8 \cos\left(x\right)^{2}}{17} - \frac{32 \cos\left(x\right) \sin\left(x\right)}{17} + \frac{106}{17}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = e^{-x} - \frac{98 e^{-\frac{x}{2}}}{17} - \frac{8 \cos(x)^2}{17} - \frac{32 \cos(x) \sin(x)}{17} + \frac{106}{17} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{-x} - \frac{98 e^{-\frac{x}{2}}}{17} - \frac{8 \cos(x)^2}{17} - \frac{32 \cos(x) \sin(x)}{17} + \frac{106}{17}$$

Verified OK.

11.31.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$2y'' + y' = 8 \sin(2x) + e^{-x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (2y'' + y')\, dx = \int (8 \sin(2x) + e^{-x})\, dx$$

$$2y' + y = -4 \cos(2x) - e^{-x} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{2} \\ q(x) &= -2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2} \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{2} = -2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{2}dx} \\ &= e^{\frac{x}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(-2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2}\right) \\ \frac{d}{dx}\left(e^{\frac{x}{2}} y\right) &= \left(e^{\frac{x}{2}}\right)\left(-2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2}\right) \\ d\left(e^{\frac{x}{2}} y\right) &= \left(\frac{(-4\,e^{x}\cos(2x) + c_1 e^{x} - 1)\,e^{-\frac{x}{2}}}{2}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{\frac{x}{2}} y &= \int \frac{(-4\,e^{x}\cos(2x) + c_1 e^{x} - 1)\,e^{-\frac{x}{2}}}{2}\, dx \\ e^{\frac{x}{2}} y &= e^{-\frac{x}{2}} + c_1 e^{\frac{x}{2}} - \frac{4\,e^{\frac{x}{2}}\cos(2x)}{17} - \frac{16\,e^{\frac{x}{2}}\sin(2x)}{17} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x}{2}}$ results in

$$y = e^{-\frac{x}{2}}\left(e^{-\frac{x}{2}} + c_1 e^{\frac{x}{2}} - \frac{4\,e^{\frac{x}{2}}\cos(2x)}{17} - \frac{16\,e^{\frac{x}{2}}\sin(2x)}{17}\right) + c_2 e^{-\frac{x}{2}}$$

which simplifies to

$$y = -\frac{\left(4\,e^{x}\cos(2x) + 16\,e^{x}\sin(2x) - 17c_1 e^{x} - 17c_2 e^{\frac{x}{2}} - 17\right)e^{-x}}{17}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{\left(4\,\mathrm{e}^{x} \cos\left(2x\right)+16\,\mathrm{e}^{x} \sin\left(2x\right)-17c_1\mathrm{e}^{x}-17c_2\mathrm{e}^{\frac{x}{2}}-17\right) \mathrm{e}^{-x}}{17} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 + \frac{13}{17} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(36\,\mathrm{e}^{x} \cos\left(2x\right)+8\,\mathrm{e}^{x} \sin\left(2x\right)-17c_1\mathrm{e}^{x}-\frac{17c_2\mathrm{e}^{\frac{x}{2}}}{2}\right) \mathrm{e}^{-x}}{17} + \frac{\left(4\,\mathrm{e}^{x} \cos\left(2x\right)+16\,\mathrm{e}^{x} \sin\left(2x\right)-17c_1\mathrm{e}^{x}-1\right.}{17}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -\frac{c_2}{2} - \frac{49}{17} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 6 \\ c_2 &= -\frac{98}{17} \end{aligned}$$

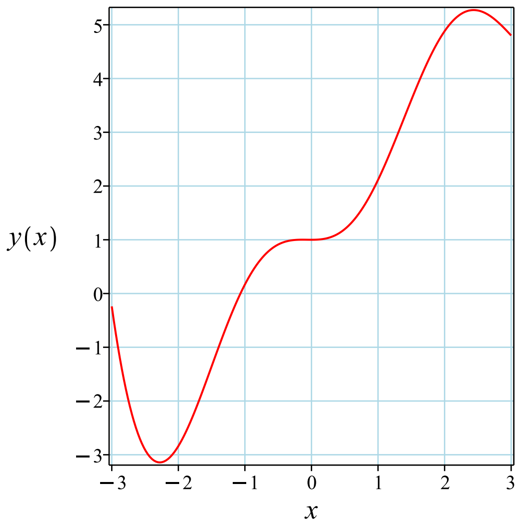
Substituting these values back in above solution results in

$$y = -\frac{\left(4\,\mathrm{e}^{x} \cos\left(2x\right)+16\,\mathrm{e}^{x} \sin\left(2x\right)-17+98\,\mathrm{e}^{\frac{x}{2}}-102\,\mathrm{e}^{x}\right) \mathrm{e}^{-x}}{17}$$

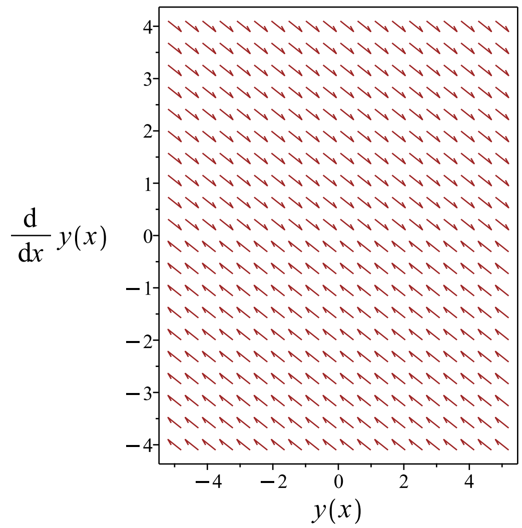
Summary

The solution(s) found are the following

$$y = -\frac{\left(4\,\mathrm{e}^{x} \cos\left(2x\right)+16\,\mathrm{e}^{x} \sin\left(2x\right)-17+98\,\mathrm{e}^{\frac{x}{2}}-102\,\mathrm{e}^{x}\right) \mathrm{e}^{-x}}{17} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\left(4\,e^{x}\cos\left(2x\right)+16\,e^{x}\sin\left(2x\right)-17+98\,e^{\frac{x}{2}}-102\,e^{x}\right)e^{-x}}{17}$$

Verified OK.

11.31.6 Solving using Kovacic algorithm

Writing the ode as

$$2y''+y'=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{16} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 16$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 419: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{2}\,dx} \\ &= z_1 e^{-\frac{x}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(2\,\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \left(2\,\mathrm{e}^{\frac{x}{2}}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
2y'' + y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-\frac{x}{2}} + 2c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
8\sin(2x) + \mathrm{e}^{-x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-x}\}, \{\cos(2x), \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{2, \mathrm{e}^{-\frac{x}{2}}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{-x} + A_2 \cos(2x) + A_3 \sin(2x)
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_1 \mathrm{e}^{-x} - 8A_2 \cos(2x) - 8A_3 \sin(2x) - 2A_2 \sin(2x) + 2A_3 \cos(2x) = 8\sin(2x) + \mathrm{e}^{-x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = -\frac{4}{17}, A_3 = -\frac{16}{17}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-\frac{x}{2}} + 2c_2\right) + \left(\mathrm{e}^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-\frac{x}{2}} + 2c_2 + \mathrm{e}^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + 2c_2 + \frac{13}{17} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{2}}}{2} - \mathrm{e}^{-x} + \frac{8\sin(2x)}{17} - \frac{32\cos(2x)}{17}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -\frac{c_1}{2} - \frac{49}{17} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{98}{17} \\ c_2 &= 3 \end{aligned}$$

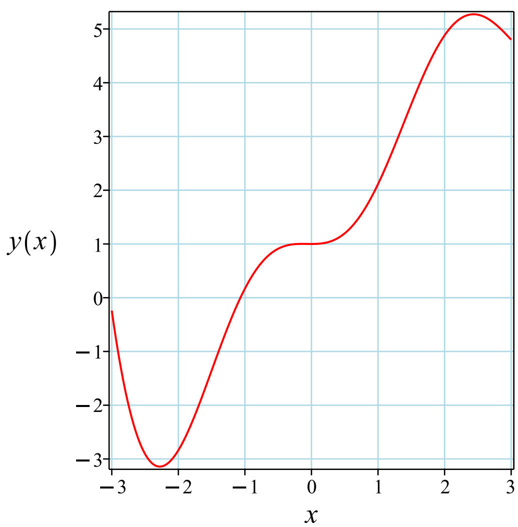
Substituting these values back in above solution results in

$$y = 6 - \frac{98\,\mathrm{e}^{-\frac{x}{2}}}{17} + \mathrm{e}^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$$

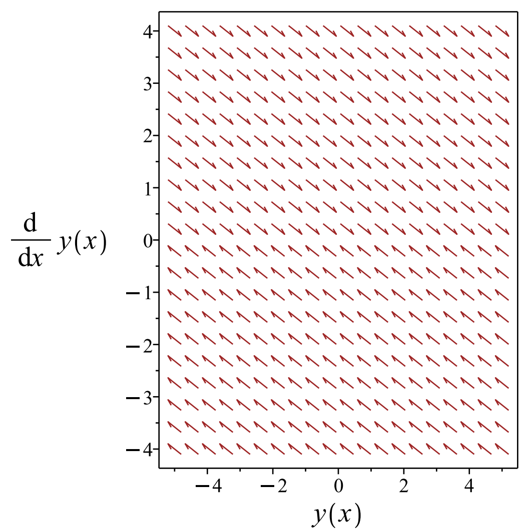
Summary

The solution(s) found are the following

$$y = 6 - \frac{98\,\mathrm{e}^{-\frac{x}{2}}}{17} + \mathrm{e}^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 6 - \frac{98\,\mathrm{e}^{-\frac{x}{2}}}{17} + \mathrm{e}^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$$

Verified OK.

11.31.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 2 \\
q(x) &= 1 \\
r(x) &= 0 \\
s(x) &= 8\sin(2x) + e^{-x}
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
2y' + y = \int 8\sin(2x) + e^{-x}\, dx
$$

We now have a first order ode to solve which is

$$
2y' + y = -4\cos(2x) - e^{-x} + c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{2} \\
q(x) &= -2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2}
\end{aligned}
$$

Hence the ode is

$$y' + \frac{y}{2} = -2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2}dx} \\ &= e^{\frac{x}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(-2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2}\right) \\ \frac{d}{dx}\left(e^{\frac{x}{2}} y\right) &= \left(e^{\frac{x}{2}}\right)\left(-2\cos(2x) - \frac{e^{-x}}{2} + \frac{c_1}{2}\right) \\ d\left(e^{\frac{x}{2}} y\right) &= \left(\frac{(-4\,e^{x}\cos(2x) + c_1 e^{x} - 1)\,e^{-\frac{x}{2}}}{2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{\frac{x}{2}} y &= \int \frac{(-4\,e^{x}\cos(2x) + c_1 e^{x} - 1)\,e^{-\frac{x}{2}}}{2}\, dx \\ e^{\frac{x}{2}} y &= e^{-\frac{x}{2}} + c_1 e^{\frac{x}{2}} - \frac{4\,e^{\frac{x}{2}}\cos(2x)}{17} - \frac{16\,e^{\frac{x}{2}}\sin(2x)}{17} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x}{2}}$ results in

$$y = e^{-\frac{x}{2}}\left(e^{-\frac{x}{2}} + c_1 e^{\frac{x}{2}} - \frac{4\,e^{\frac{x}{2}}\cos(2x)}{17} - \frac{16\,e^{\frac{x}{2}}\sin(2x)}{17}\right) + c_2 e^{-\frac{x}{2}}$$

which simplifies to

$$y = -\frac{\left(4\,e^{x}\cos(2x) + 16\,e^{x}\sin(2x) - 17 c_1 e^{x} - 17 c_2 e^{\frac{x}{2}} - 17\right) e^{-x}}{17}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{\left(4\,e^{x}\cos(2x) + 16\,e^{x}\sin(2x) - 17 c_1 e^{x} - 17 c_2 e^{\frac{x}{2}} - 17\right) e^{-x}}{17} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 + \frac{13}{17} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(36\,\mathrm{e}^{x} \cos\left(2x\right)+8\,\mathrm{e}^{x} \sin\left(2x\right)-17 c_1 \mathrm{e}^{x}-\frac{17 c_2 \mathrm{e}^{\frac{x}{2}}}{2}\right) \mathrm{e}^{-x}}{17}+\frac{\left(4\,\mathrm{e}^{x} \cos\left(2x\right)+16\,\mathrm{e}^{x} \sin\left(2x\right)-17 c_1 \mathrm{e}^{x}-1\right.}{17}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -\frac{c_2}{2}-\frac{49}{17} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

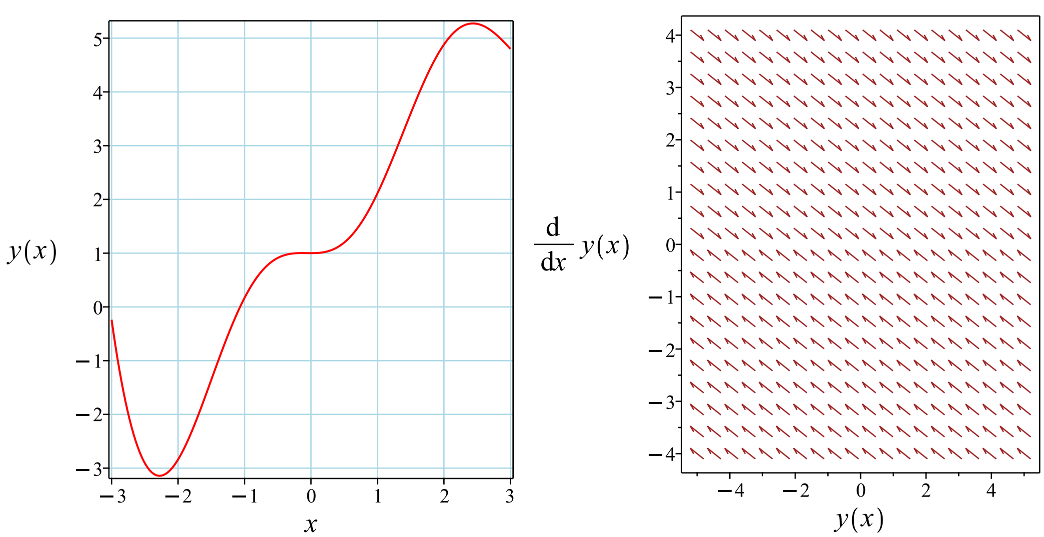
$$c_1 = 6$$
$$c_2 = -\frac{98}{17}$$

Substituting these values back in above solution results in

$$y = -\frac{\left(4\,\mathrm{e}^{x} \cos\left(2x\right)+16\,\mathrm{e}^{x} \sin\left(2x\right)-17+98\,\mathrm{e}^{\frac{x}{2}}-102\,\mathrm{e}^{x}\right) \mathrm{e}^{-x}}{17}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(4\,\mathrm{e}^{x} \cos\left(2x\right)+16\,\mathrm{e}^{x} \sin\left(2x\right)-17+98\,\mathrm{e}^{\frac{x}{2}}-102\,\mathrm{e}^{x}\right) \mathrm{e}^{-x}}{17} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{\left(4\,\mathrm{e}^{x} \cos\left(2x\right)+16\,\mathrm{e}^{x} \sin\left(2x\right)-17+98\,\mathrm{e}^{\frac{x}{2}}-102\,\mathrm{e}^{x}\right) \mathrm{e}^{-x}}{17}$$

Verified OK.

11.31.8 Maple step by step solution

Let's solve

$\left[2y''+y'=8\sin\left(2x\right)+\mathrm{e}^{-x}, y\left(0\right)=1, y'\Big|_{\{x=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'}{2}+4\sin\left(2x\right)+\frac{\mathrm{e}^{-x}}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{2}=4\sin\left(2x\right)+\frac{\mathrm{e}^{-x}}{2}$

- Characteristic polynomial of homogeneous ODE

  $r^2+\frac{1}{2}r=0$

- Factor the characteristic polynomial

  $\frac{r(2r+1)}{2}=0$

- Roots of the characteristic polynomial

  $r=\left(0,-\frac{1}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x)=1$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{-\frac{x}{2}}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1+c_2\mathrm{e}^{-\frac{x}{2}}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\sin(2x) + \frac{e^{-x}}{2}\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{-\frac{x}{2}} \\ 0 & -\frac{e^{-\frac{x}{2}}}{2} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = -\frac{e^{-\frac{x}{2}}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\left(\int \left(-8\sin(2x) - e^{-x}\right)dx\right) + e^{-\frac{x}{2}}\left(\int \left(-8\,e^{\frac{x}{2}}\sin(2x) - e^{-\frac{x}{2}}\right)dx\right)$

- Compute integrals

  $y_p(x) = e^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$

• Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{-\frac{x}{2}} + e^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$

□ Check validity of solution $y = c_1 + c_2 e^{-\frac{x}{2}} + e^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$

- Use initial condition $y(0) = 1$

  $1 = c_1 + c_2 + \frac{13}{17}$

- Compute derivative of the solution

  $y' = -\frac{c_2 e^{-\frac{x}{2}}}{2} - e^{-x} + \frac{8\sin(2x)}{17} - \frac{32\cos(2x)}{17}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = -\frac{c_2}{2} - \frac{49}{17}$

- Solve for c_1 and c_2

  $\left\{c_1 = 6, c_2 = -\frac{98}{17}\right\}$

- Substitute constant values into general solution and simplify

  $y = 6 - \frac{98\,e^{-\frac{x}{2}}}{17} + e^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$

• Solution to the IVP

  $y = 6 - \frac{98\,e^{-\frac{x}{2}}}{17} + e^{-x} - \frac{4\cos(2x)}{17} - \frac{16\sin(2x)}{17}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(1/2)*_b(_a)+4*sin(2*_a)+(1/2)*exp(-_a
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve([2*diff(y(x),x$2)+diff(y(x),x)=8*sin(2*x)+exp(-x),y(0) = 1, D(y)(0) = 0],y(x), singso
```

$$y(x) = -\frac{98\,\mathrm{e}^{-\frac{x}{2}}}{17} - \frac{16\sin\left(2x\right)}{17} + \mathrm{e}^{-x} - \frac{4\cos\left(2x\right)}{17} + 6$$

✓ Solution by Mathematica

Time used: 0.384 (sec). Leaf size: 39

```
DSolve[{2*y''[x]+y'[x]==8*Sin[2*x]+Exp[-x],{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to e^{-x} - \frac{98e^{-x/2}}{17} - \frac{16}{17}\sin(2x) - \frac{4}{17}\cos(2x) + 6$$

11.32 problem 32

Internal problem ID [2171]
Internal file name [`OUTPUT/2171_Monday_February_26_2024_09_18_05_AM_98992423/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 3 \sin (x) x}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

11.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ F &= 3 \sin (x) x \end{aligned}$$

Hence the ode is

$$y'' + y = 3\sin(x)\,x$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 3\sin(x)\,x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.32.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 3\sin(x)\,x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\sin(x)\,x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(x), \sin(x)x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x^2\cos(x), \sin(x)x, \sin(x)x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(x) + A_2 x^2\cos(x) + A_3\sin(x)x + A_4\sin(x)x^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-2A_1\sin(x) + 2A_2\cos(x) - 4A_2x\sin(x) + 2A_3\cos(x) + 4A_4\cos(x)x + 2A_4\sin(x)\\&= 3\sin(x)x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{3}{4}, A_3 = \frac{3}{4}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3x^2\cos(x)}{4} + \frac{3\sin(x)x}{4}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= (c_1\cos(x) + c_2\sin(x)) + \left(-\frac{3x^2\cos(x)}{4} + \frac{3\sin(x)x}{4}\right)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{3x^2 \cos(x)}{4} + \frac{3 \sin(x) x}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sin(x) c_1 + c_2 \cos(x) - \frac{3x \cos(x)}{4} + \frac{3 \sin(x) x^2}{4} + \frac{3 \sin(x)}{4}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

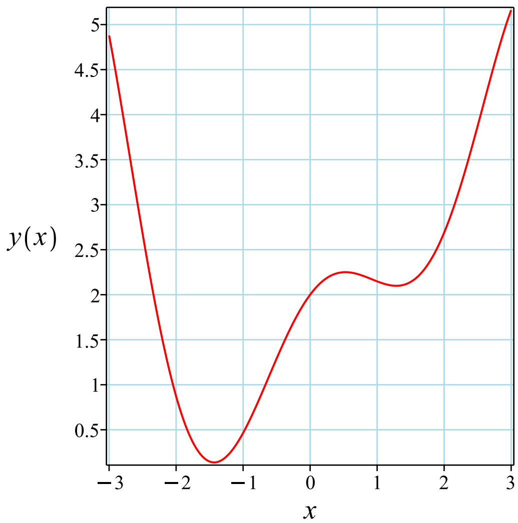
$$c_1 = 2$$
$$c_2 = 1$$

Substituting these values back in above solution results in

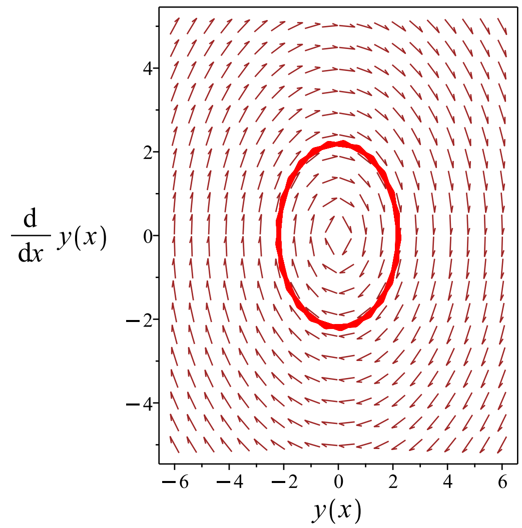
$$y = 2 \cos(x) + \sin(x) - \frac{3x^2 \cos(x)}{4} + \frac{3 \sin(x) x}{4}$$

Summary
The solution(s) found are the following

$$y = 2 \cos(x) + \sin(x) - \frac{3x^2 \cos(x)}{4} + \frac{3 \sin(x) x}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\cos(x) + \sin(x) - \frac{3x^2\cos(x)}{4} + \frac{3\sin(x)\,x}{4}$$

Verified OK.

11.32.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 421: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dx\\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx\\ &= \cos(x)(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
3 \sin(x)\, x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x \cos(x), \sin(x)\, x, \cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\cos(x), \sin(x)\}
$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x \cos(x), x^2 \cos(x), \sin(x)\, x, \sin(x)\, x^2\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 x \cos(x) + A_2 x^2 \cos(x) + A_3 \sin(x)\, x + A_4 \sin(x)\, x^2
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-2A_1 \sin(x) + 2A_2 \cos(x) - 4A_2 x \sin(x) + 2A_3 \cos(x) + 4A_4 \cos(x)\, x + 2A_4 \sin(x) \\
&= 3 \sin(x)\, x
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = 0, A_2 = -\frac{3}{4}, A_3 = \frac{3}{4}, A_4 = 0\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = -\frac{3x^2 \cos(x)}{4} + \frac{3 \sin(x)\, x}{4}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{3x^2 \cos(x)}{4} + \frac{3 \sin(x)\, x}{4}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(x) + c_2 \sin(x) - \frac{3x^2 \cos(x)}{4} + \frac{3 \sin(x)\, x}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\sin(x)\, c_1 + c_2 \cos(x) - \frac{3x \cos(x)}{4} + \frac{3 \sin(x)\, x^2}{4} + \frac{3 \sin(x)}{4}
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 1$$

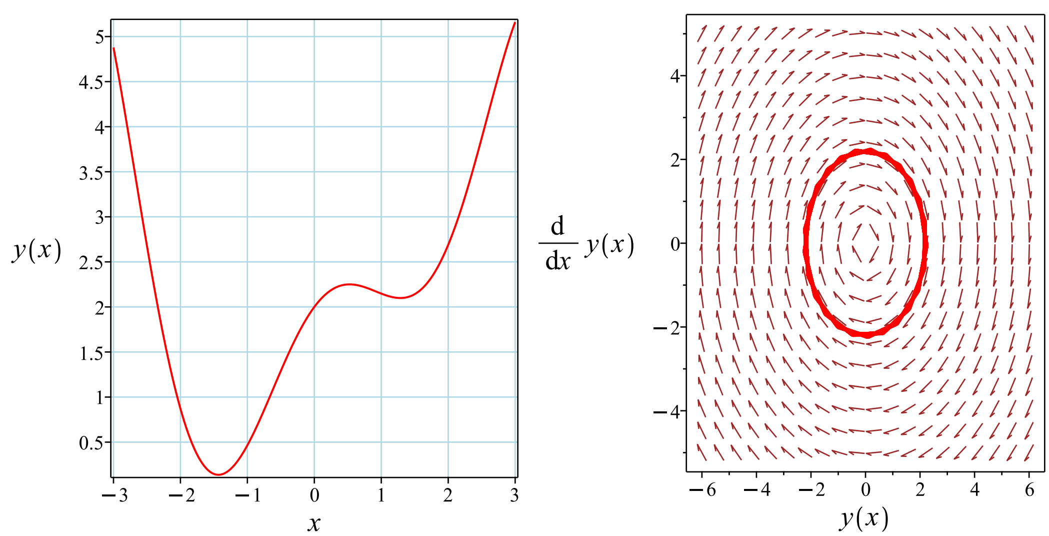
Substituting these values back in above solution results in

$$y = 2\cos(x) + \sin(x) - \frac{3x^2\cos(x)}{4} + \frac{3\sin(x)\,x}{4}$$

Summary

The solution(s) found are the following

$$y = 2\cos(x) + \sin(x) - \frac{3x^2\cos(x)}{4} + \frac{3\sin(x)\,x}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 2\cos(x) + \sin(x) - \frac{3x^2\cos(x)}{4} + \frac{3\sin(x)\,x}{4}$$

Verified OK.

11.32.4 Maple step by step solution

Let's solve

$$\left[y'' + y = 3\sin(x)\, x, y(0) = 2, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 3\sin(x)\, x\right]$$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -3\cos(x)\left(\int x\sin(x)^2\,dx\right) + \frac{3\sin(x)\left(\int x\sin(2x)dx\right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{3x(-x\cos(x)+\sin(x))}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(x) + c_2\sin(x) + \frac{3x(-x\cos(x)+\sin(x))}{4}$$

□ Check validity of solution $y = c_1\cos(x) + c_2\sin(x) + \frac{3x(-x\cos(x)+\sin(x))}{4}$

- Use initial condition $y(0) = 2$

$$2 = c_1$$

- Compute derivative of the solution

$$y' = -\sin(x)\,c_1 + c_2\cos(x) - \frac{3x\cos(x)}{4} + \frac{3\sin(x)x^2}{4} + \frac{3\sin(x)}{4}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

$$1 = c_2$$

- Solve for c_1 and c_2

$$\{c_1 = 2, c_2 = 1\}$$

- Substitute constant values into general solution and simplify

$$y = 2\cos(x) + \sin(x) - \frac{3x^2\cos(x)}{4} + \frac{3\sin(x)x}{4}$$

- Solution to the IVP

$$y = 2\cos(x) + \sin(x) - \frac{3x^2\cos(x)}{4} + \frac{3\sin(x)x}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([diff(y(x),x$2)+y(x)=3*x*sin(x),y(0) = 2, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = \sin(x) + 2\cos(x) - \frac{3\cos(x)\, x^2}{4} + \frac{3x\sin(x)}{4}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 28

```
DSolve[{y''[x]+y[x]==3*x*Sin[x],{y[0]==2,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(2 - \frac{3x^2}{4}\right)\cos(x) + \left(\frac{3x}{4} + 1\right)\sin(x)$$

11.33 problem 33

Internal problem ID [2172]
Internal file name [`OUTPUT/2172_Monday_February_26_2024_09_18_05_AM_63407399/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{2y'' + 5y' - 3y = \sin(x) - 8x}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}, y'(0) = \frac{1}{2}\right]$$

11.33.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{5}{2} \\ q(x) &= -\frac{3}{2} \\ F &= \frac{\sin(x)}{2} - 4x \end{aligned}$$

Hence the ode is

$$y'' + \frac{5y'}{2} - \frac{3y}{2} = \frac{\sin(x)}{2} - 4x$$

The domain of $p(x) = \frac{5}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{3}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \frac{\sin(x)}{2} - 4x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.33.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2, B = 5, C = -3, f(x) = \sin(x) - 8x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2y'' + 5y' - 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 5, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} + 5\lambda e^{\lambda x} - 3 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + 5\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 5, C = -3$ into the above gives

$$\lambda_{1,2} = \frac{-5}{(2)(2)} \pm \frac{1}{(2)(2)}\sqrt{5^2 - (4)(2)(-3)}$$
$$= -\frac{5}{4} \pm \frac{7}{4}$$

Hence

$$\lambda_1 = -\frac{5}{4} + \frac{7}{4}$$
$$\lambda_2 = -\frac{5}{4} - \frac{7}{4}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{-3x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{-3x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x) - 8x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-3x}, e^{\frac{x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1 + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_3 \cos(x) - 5A_4 \sin(x) + 5A_2 - 5A_3 \sin(x) + 5A_4 \cos(x) - 3A_2 x - 3A_1 = \sin(x) - 8x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{40}{9}, A_2 = \frac{8}{3}, A_3 = -\frac{1}{10}, A_4 = -\frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{8x}{3} + \frac{40}{9} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{x}{2}} + e^{-3x} c_2\right) + \left(\frac{8x}{3} + \frac{40}{9} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\frac{x}{2}} + e^{-3x} c_2 + \frac{8x}{3} + \frac{40}{9} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{2}$ and $x = 0$ in the above gives

$$\frac{1}{2} = c_1 + c_2 + \frac{391}{90} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 \mathrm{e}^{\frac{x}{2}}}{2} - 3\,\mathrm{e}^{-3x} c_2 + \frac{8}{3} + \frac{\sin(x)}{10} - \frac{\cos(x)}{10}$$

substituting $y' = \frac{1}{2}$ and $x = 0$ in the above gives

$$\frac{1}{2} = \frac{c_1}{2} - 3c_2 + \frac{77}{30} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{136}{35}$$
$$c_2 = \frac{13}{315}$$

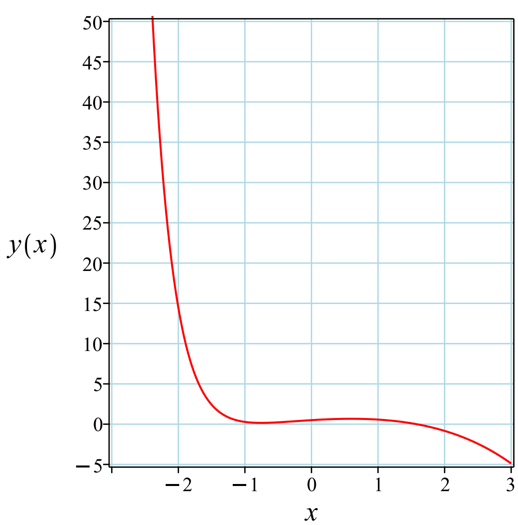
Substituting these values back in above solution results in

$$y = \frac{40}{9} - \frac{136\,\mathrm{e}^{\frac{x}{2}}}{35} + \frac{13\,\mathrm{e}^{-3x}}{315} + \frac{8x}{3} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

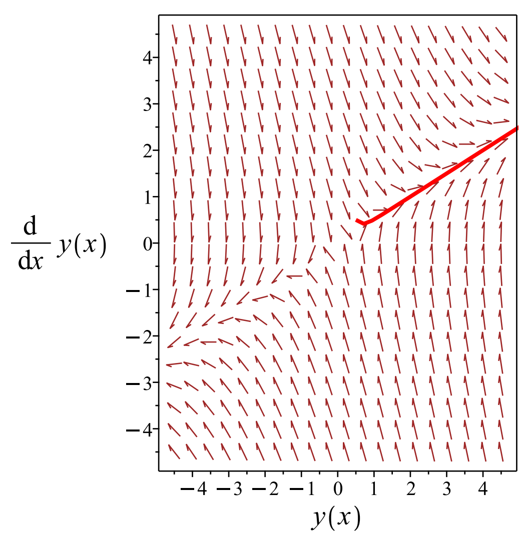
Summary

The solution(s) found are the following

$$y = \frac{40}{9} - \frac{136\,\mathrm{e}^{\frac{x}{2}}}{35} + \frac{13\,\mathrm{e}^{-3x}}{315} + \frac{8x}{3} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{40}{9} - \frac{136\,\mathrm{e}^{\frac{x}{2}}}{35} + \frac{13\,\mathrm{e}^{-3x}}{315} + \frac{8x}{3} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

Verified OK.

11.33.3 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + 5y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 5 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 49 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{49z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 423: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{49}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{7x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5}{2}\,dx} \\
&= z_1 e^{-\frac{5x}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5}{2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5x}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\,\mathrm{e}^{\frac{7x}{2}}}{7}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}\left(\frac{2\,\mathrm{e}^{\frac{7x}{2}}}{7}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$2y'' + 5y' - 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-3x} + \frac{2c_2 e^{\frac{x}{2}}}{7}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x) - 8x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{2\, e^{\frac{x}{2}}}{7}, e^{-3x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1 + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_3 \cos(x) - 5A_4 \sin(x) + 5A_2 - 5A_3 \sin(x) + 5A_4 \cos(x) - 3A_2 x - 3A_1 = \sin(x) - 8x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{40}{9}, A_2 = \frac{8}{3}, A_3 = -\frac{1}{10}, A_4 = -\frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{8x}{3} + \frac{40}{9} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + \frac{2c_2 e^{\frac{x}{2}}}{7}\right) + \left(\frac{8x}{3} + \frac{40}{9} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-3x} + \frac{2c_2 e^{\frac{x}{2}}}{7} + \frac{8x}{3} + \frac{40}{9} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{2}$ and $x = 0$ in the above gives

$$\frac{1}{2} = c_1 + \frac{2c_2}{7} + \frac{391}{90} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 e^{-3x} + \frac{c_2 e^{\frac{x}{2}}}{7} + \frac{8}{3} + \frac{\sin(x)}{10} - \frac{\cos(x)}{10}$$

substituting $y' = \frac{1}{2}$ and $x = 0$ in the above gives

$$\frac{1}{2} = -3c_1 + \frac{c_2}{7} + \frac{77}{30} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{13}{315} \\ c_2 &= -\frac{68}{5} \end{aligned}$$

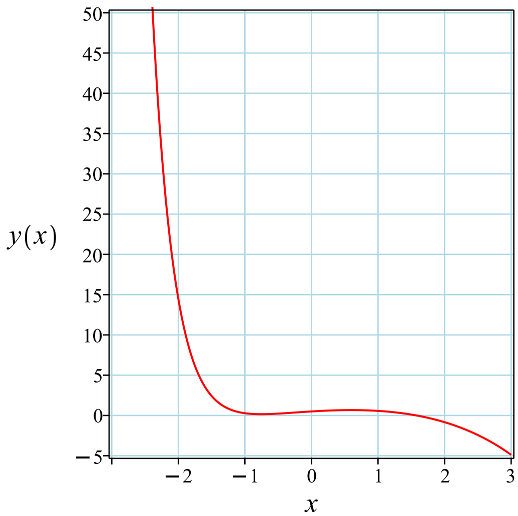
Substituting these values back in above solution results in

$$y = \frac{40}{9} - \frac{136\, e^{\frac{x}{2}}}{35} + \frac{13\, e^{-3x}}{315} + \frac{8x}{3} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

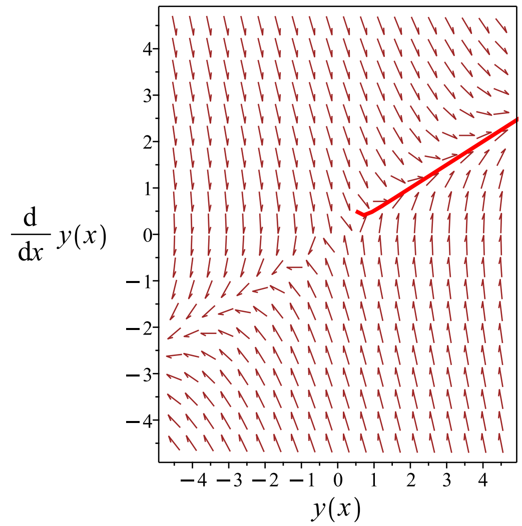
Summary

The solution(s) found are the following

$$y = \frac{40}{9} - \frac{136\,\mathrm{e}^{\frac{x}{2}}}{35} + \frac{13\,\mathrm{e}^{-3x}}{315} + \frac{8x}{3} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{40}{9} - \frac{136\,\mathrm{e}^{\frac{x}{2}}}{35} + \frac{13\,\mathrm{e}^{-3x}}{315} + \frac{8x}{3} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

Verified OK.

11.33.4 Maple step by step solution

Let's solve

$$\left[2y'' + 5y' - 3y = \sin(x) - 8x, y(0) = \frac{1}{2}, y'\Big|_{\{x=0\}} = \frac{1}{2}\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{2} + \frac{3y}{2} + \frac{\sin(x)}{2} - 4x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{5y'}{2} - \frac{3y}{2} = \frac{\sin(x)}{2} - 4x$

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{5}{2}r - \frac{3}{2} = 0$
- Factor the characteristic polynomial

  $\frac{(r+3)(2r-1)}{2} = 0$
- Roots of the characteristic polynomial

  $r = \left(-3, \frac{1}{2}\right)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{\frac{x}{2}}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{\frac{x}{2}} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{\sin(x)}{2} - 4x\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{\frac{x}{2}} \\ -3\,\mathrm{e}^{-3x} & \frac{\mathrm{e}^{\frac{x}{2}}}{2} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \frac{7\,\mathrm{e}^{-\frac{5x}{2}}}{2}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{\left(\mathrm{e}^{\frac{7x}{2}} \left(\int (\sin(x) - 8x) \mathrm{e}^{-\frac{x}{2}} dx\right) - \left(\int (\sin(x) - 8x) \mathrm{e}^{3x} dx\right)\right) \mathrm{e}^{-3x}}{7}$
  - Compute integrals

    $y_p(x) = \frac{8x}{3} + \frac{40}{9} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$
- Substitute particular solution into general solution to ODE

$$y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{\frac{x}{2}} + \frac{8x}{3} + \frac{40}{9} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$$

□ Check validity of solution $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{\frac{x}{2}} + \frac{8x}{3} + \frac{40}{9} - \frac{\cos(x)}{10} - \frac{\sin(x)}{10}$

- Use initial condition $y(0) = \frac{1}{2}$

  $$\frac{1}{2} = c_1 + c_2 + \frac{391}{90}$$

- Compute derivative of the solution

  $$y' = -3c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{\frac{x}{2}}}{2} + \frac{8}{3} + \frac{\sin(x)}{10} - \frac{\cos(x)}{10}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = \frac{1}{2}$

  $$\frac{1}{2} = -3c_1 + \frac{c_2}{2} + \frac{77}{30}$$

- Solve for c_1 and c_2

  $$\left\{c_1 = \frac{13}{315}, c_2 = -\frac{136}{35}\right\}$$

- Substitute constant values into general solution and simplify

  $$y = \frac{8\left(-\frac{51\,\mathrm{e}^{\frac{7x}{2}}}{35} + \frac{13}{840} + \left(x - \frac{3\cos(x)}{80} - \frac{3\sin(x)}{80} + \frac{5}{3}\right)\mathrm{e}^{3x}\right)\mathrm{e}^{-3x}}{3}$$

- Solution to the IVP

  $$y = \frac{8\left(-\frac{51\,\mathrm{e}^{\frac{7x}{2}}}{35} + \frac{13}{840} + \left(x - \frac{3\cos(x)}{80} - \frac{3\sin(x)}{80} + \frac{5}{3}\right)\mathrm{e}^{3x}\right)\mathrm{e}^{-3x}}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
dsolve([2*diff(y(x),x$2)+5*diff(y(x),x)-3*y(x)=sin(x)-8*x,y(0) = 1/2, D(y)(0) = 1/2],y(x), s
```

$$y(x) = \frac{8\,\mathrm{e}^{-3x}\left(-\frac{51\,\mathrm{e}^{\frac{7x}{2}}}{35}+\frac{13}{840}+\left(x-\frac{3\cos(x)}{80}-\frac{3\sin(x)}{80}+\frac{5}{3}\right)\mathrm{e}^{3x}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.208 (sec). Leaf size: 38

```
DSolve[{2*y''[x]+5*y'[x]-3*y[x]==Sin[x]-8*x,{y[0]==1/2,y'[0]==1/2}},y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{1}{630}\left(1680x+26e^{-3x}-2448e^{x/2}-63\sin(x)-63\cos(x)+2800\right)$$

11.34 problem 34

Internal problem ID [2173]
Internal file name [`OUTPUT/2173_Monday_February_26_2024_09_18_06_AM_81506055/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 19, page 86
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{8y'' - y = x\,\mathrm{e}^{-\frac{x}{2}}}$$

With initial conditions

$$[y(0) = 3, y'(0) = 5]$$

11.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{1}{8} \\ F &= \frac{x\,\mathrm{e}^{-\frac{x}{2}}}{8} \end{aligned}$$

Hence the ode is

$$y'' - \frac{y}{8} = \frac{x\,\mathrm{e}^{-\frac{x}{2}}}{8}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{1}{8}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \frac{x\,\mathrm{e}^{-\frac{x}{2}}}{8}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.34.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 8, B = 0, C = -1, f(x) = x\,\mathrm{e}^{-\frac{x}{2}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$8y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 8, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$8\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$8\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 8, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(8)} \pm \frac{1}{(2)(8)}\sqrt{0^2 - (4)(8)(-1)}\\ &= \pm\frac{\sqrt{2}}{4}\end{aligned}$$

Hence

$$\lambda_1 = +\frac{\sqrt{2}}{4}$$

$$\lambda_2 = -\frac{\sqrt{2}}{4}$$

Which simplifies to

$$\lambda_1 = \frac{\sqrt{2}}{4}$$

$$\lambda_2 = -\frac{\sqrt{2}}{4}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{\left(\frac{\sqrt{2}}{4}\right)x} + c_2 e^{\left(-\frac{\sqrt{2}}{4}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{\sqrt{2}\,x}{4}} + c_2 \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{\frac{\sqrt{2}\,x}{4}} + c_2 \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^{-\frac{x}{2}}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{x\,\mathrm{e}^{-\frac{x}{2}}, \mathrm{e}^{-\frac{x}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}, \mathrm{e}^{\frac{\sqrt{2}\,x}{4}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-\frac{x}{2}} + A_2 \mathrm{e}^{-\frac{x}{2}}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \mathrm{e}^{-\frac{x}{2}} + A_1 x\,\mathrm{e}^{-\frac{x}{2}} + A_2 \mathrm{e}^{-\frac{x}{2}} = x\,\mathrm{e}^{-\frac{x}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 8]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,\mathrm{e}^{-\frac{x}{2}} + 8\,\mathrm{e}^{-\frac{x}{2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{\frac{\sqrt{2}\,x}{4}} + c_2 \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}\right) + \left(x\,\mathrm{e}^{-\frac{x}{2}} + 8\,\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\frac{\sqrt{2}\,x}{4}} + c_2 \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}} + x\,\mathrm{e}^{-\frac{x}{2}} + 8\,\mathrm{e}^{-\frac{x}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 + c_2 + 8 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1\sqrt{2}\,e^{\frac{\sqrt{2}\,x}{4}}}{4} - \frac{c_2\sqrt{2}\,e^{-\frac{\sqrt{2}\,x}{4}}}{4} - 3\,e^{-\frac{x}{2}} - \frac{x\,e^{-\frac{x}{2}}}{2}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = -3 + \frac{\sqrt{2}\,(c_1 - c_2)}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{5}{2} + 8\sqrt{2}$$
$$c_2 = -\frac{5}{2} - 8\sqrt{2}$$

Substituting these values back in above solution results in

$$y = -\frac{5\,e^{\frac{\sqrt{2}\,x}{4}}}{2} + 8\sqrt{2}\,e^{\frac{\sqrt{2}\,x}{4}} - \frac{5\,e^{-\frac{\sqrt{2}\,x}{4}}}{2} - 8\sqrt{2}\,e^{-\frac{\sqrt{2}\,x}{4}} + x\,e^{-\frac{x}{2}} + 8\,e^{-\frac{x}{2}}$$

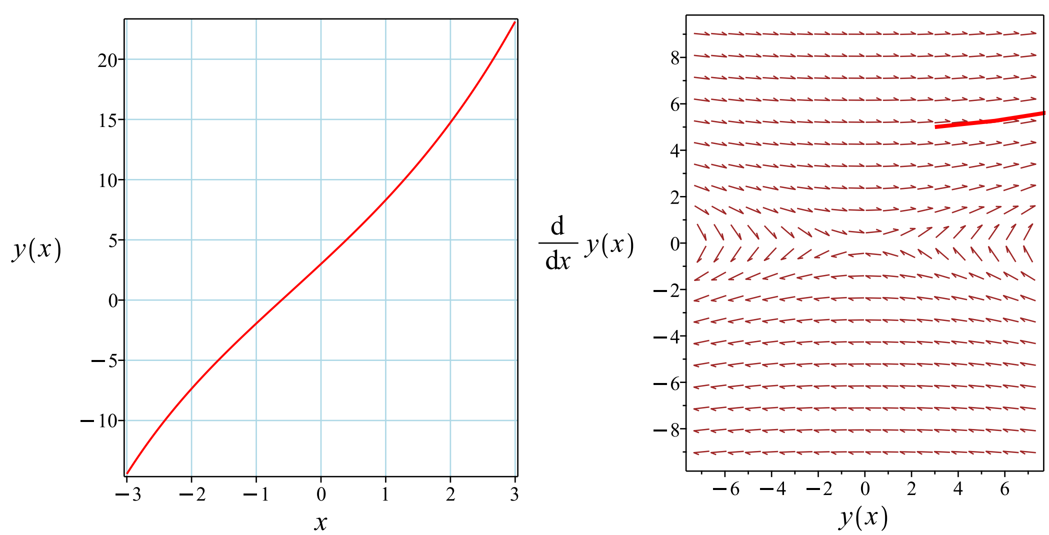
Which simplifies to

$$y = \frac{\left(-5 - 16\sqrt{2}\right) e^{-\frac{\sqrt{2}\,x}{4}}}{2} + \frac{\left(-5 + 16\sqrt{2}\right) e^{\frac{\sqrt{2}\,x}{4}}}{2} + e^{-\frac{x}{2}}(x + 8)$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-5 - 16\sqrt{2}\right) e^{-\frac{\sqrt{2}\,x}{4}}}{2} + \frac{\left(-5 + 16\sqrt{2}\right) e^{\frac{\sqrt{2}\,x}{4}}}{2} + e^{-\frac{x}{2}}(x + 8) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\left(-5-16\sqrt{2}\right) e^{-\frac{\sqrt{2}\, x}{4}}}{2} + \frac{\left(-5+16\sqrt{2}\right) e^{\frac{\sqrt{2}\, x}{4}}}{2} + e^{-\frac{x}{2}} (x+8)$$

Verified OK.

11.34.3 Solving using Kovacic algorithm

Writing the ode as

$$8y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 8 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{8} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 8 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{8} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 425: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{8}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}} \int \frac{1}{\mathrm{e}^{-\frac{\sqrt{2}\,x}{2}}}\,dx \\ &= \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}} \left(\mathrm{e}^{\frac{\sqrt{2}\,x}{2}}\sqrt{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}\right) + c_2\left(\mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}\left(\mathrm{e}^{\frac{\sqrt{2}\,x}{2}}\sqrt{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$8y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{\sqrt{2}\, x}{4}} + c_2 \sqrt{2}\, \mathrm{e}^{\frac{\sqrt{2}\, x}{4}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\, \mathrm{e}^{-\frac{x}{2}}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{x\, \mathrm{e}^{-\frac{x}{2}}, \mathrm{e}^{-\frac{x}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\sqrt{2}\, \mathrm{e}^{\frac{\sqrt{2}\, x}{4}}, \mathrm{e}^{-\frac{\sqrt{2}\, x}{4}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\, \mathrm{e}^{-\frac{x}{2}} + A_2 \mathrm{e}^{-\frac{x}{2}}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \mathrm{e}^{-\frac{x}{2}} + A_1 x\, \mathrm{e}^{-\frac{x}{2}} + A_2 \mathrm{e}^{-\frac{x}{2}} = x\, \mathrm{e}^{-\frac{x}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 8]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,\mathrm{e}^{-\frac{x}{2}} + 8\,\mathrm{e}^{-\frac{x}{2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}} + c_2 \sqrt{2}\,\mathrm{e}^{\frac{\sqrt{2}\,x}{4}}\right) + \left(x\,\mathrm{e}^{-\frac{x}{2}} + 8\,\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}} + c_2 \sqrt{2}\,\mathrm{e}^{\frac{\sqrt{2}\,x}{4}} + x\,\mathrm{e}^{-\frac{x}{2}} + 8\,\mathrm{e}^{-\frac{x}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 + \sqrt{2}\,c_2 + 8 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \sqrt{2}\,\mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}}{4} + \frac{c_2 \mathrm{e}^{\frac{\sqrt{2}\,x}{4}}}{2} - 3\,\mathrm{e}^{-\frac{x}{2}} - \frac{x\,\mathrm{e}^{-\frac{x}{2}}}{2}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = -\frac{\sqrt{2}\,c_1}{4} + \frac{c_2}{2} - 3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{5}{2} - 8\sqrt{2} \\ c_2 &= -\frac{5\sqrt{2}}{4} + 8 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{5\,\mathrm{e}^{\frac{\sqrt{2}\,x}{4}}}{2} + 8\sqrt{2}\,\mathrm{e}^{\frac{\sqrt{2}\,x}{4}} - \frac{5\,\mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}}{2} - 8\sqrt{2}\,\mathrm{e}^{-\frac{\sqrt{2}\,x}{4}} + x\,\mathrm{e}^{-\frac{x}{2}} + 8\,\mathrm{e}^{-\frac{x}{2}}$$

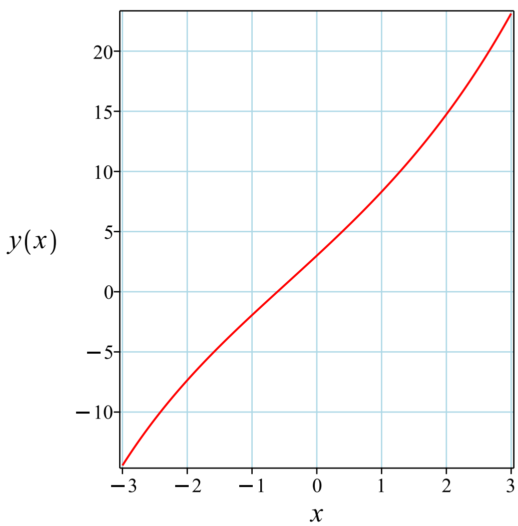
Which simplifies to

$$y = \frac{\left(-5-16\sqrt{2}\right) \mathrm{e}^{-\frac{\sqrt{2}\, x}{4}}}{2}+\frac{\left(-5+16\sqrt{2}\right) \mathrm{e}^{\frac{\sqrt{2}\, x}{4}}}{2}+\mathrm{e}^{-\frac{x}{2}} \left(x +8\right)$$

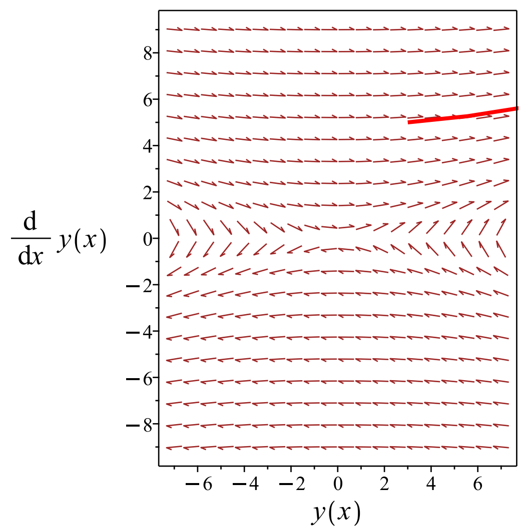
Summary

The solution(s) found are the following

$$y = \frac{\left(-5-16\sqrt{2}\right) \mathrm{e}^{-\frac{\sqrt{2}\, x}{4}}}{2}+\frac{\left(-5+16\sqrt{2}\right) \mathrm{e}^{\frac{\sqrt{2}\, x}{4}}}{2}+\mathrm{e}^{-\frac{x}{2}} \left(x +8\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\left(-5-16\sqrt{2}\right) \mathrm{e}^{-\frac{\sqrt{2}\, x}{4}}}{2}+\frac{\left(-5+16\sqrt{2}\right) \mathrm{e}^{\frac{\sqrt{2}\, x}{4}}}{2}+\mathrm{e}^{-\frac{x}{2}} \left(x +8\right)$$

Verified OK.

11.34.4 Maple step by step solution

Let's solve

$$\left[8y'' - y = x\,\mathrm{e}^{-\frac{x}{2}}, y(0) = 3, y'\Big|_{\{x=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = \frac{y}{8} + \frac{x\,\mathrm{e}^{-\frac{x}{2}}}{8}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{y}{8} = \frac{x\,\mathrm{e}^{-\frac{x}{2}}}{8}$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - \frac{1}{8} = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm \left(\sqrt{\frac{1}{2}}\right)}{2}$$

- Roots of the characteristic polynomial

$$r = \left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = \mathrm{e}^{\frac{\sqrt{2}\,x}{4}}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 \mathrm{e}^{-\frac{\sqrt{2}\,x}{4}} + c_2 \mathrm{e}^{\frac{\sqrt{2}\,x}{4}} + y_p(x)$$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{x\,\mathrm{e}^{-\frac{x}{2}}}{8}\right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x)\,,y_2(x)) = \begin{bmatrix} e^{-\frac{\sqrt{2}\,x}{4}} & e^{\frac{\sqrt{2}\,x}{4}} \\ -\frac{\sqrt{2}\,e^{-\frac{\sqrt{2}\,x}{4}}}{4} & \frac{\sqrt{2}\,e^{\frac{\sqrt{2}\,x}{4}}}{4} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x)\,,y_2(x)) = \frac{\sqrt{2}}{2}$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = \frac{\sqrt{2}\left(e^{\frac{\sqrt{2}\,x}{4}}\left(\int x\,e^{-\frac{\left(2+\sqrt{2}\right)x}{4}}dx\right)-e^{-\frac{\sqrt{2}\,x}{4}}\left(\int x\,e^{\frac{\left(-2+\sqrt{2}\right)x}{4}}dx\right)\right)}{8}$$

- Compute integrals

  $y_p(x) = e^{-\frac{x}{2}}(x+8)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-\frac{\sqrt{2}\,x}{4}} + c_2 e^{\frac{\sqrt{2}\,x}{4}} + e^{-\frac{x}{2}}(x+8)$

□ Check validity of solution $y = c_1 e^{-\frac{\sqrt{2}\,x}{4}} + c_2 e^{\frac{\sqrt{2}\,x}{4}} + e^{-\frac{x}{2}}(x+8)$

- Use initial condition $y(0) = 3$

  $3 = c_1 + c_2 + 8$

- Compute derivative of the solution

  $y' = -\frac{c_1\sqrt{2}\,e^{-\frac{\sqrt{2}\,x}{4}}}{4} + \frac{c_2\sqrt{2}\,e^{\frac{\sqrt{2}\,x}{4}}}{4} - \frac{e^{-\frac{x}{2}}(x+8)}{2} + e^{-\frac{x}{2}}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 5$

  $5 = -\frac{\sqrt{2}\,c_1}{4} + \frac{\sqrt{2}\,c_2}{4} - 3$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{5}{2} - 8\sqrt{2}, c_2 = -\frac{5}{2} + 8\sqrt{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{\left(-5-16\sqrt{2}\right)e^{-\frac{\sqrt{2}\,x}{4}}}{2} + \frac{\left(-5+16\sqrt{2}\right)e^{\frac{\sqrt{2}\,x}{4}}}{2} + e^{-\frac{x}{2}}(x+8)$

- Solution to the IVP

  $y = \frac{\left(-5-16\sqrt{2}\right)e^{-\frac{\sqrt{2}\,x}{4}}}{2} + \frac{\left(-5+16\sqrt{2}\right)e^{\frac{\sqrt{2}\,x}{4}}}{2} + e^{-\frac{x}{2}}(x+8)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.047 (sec). Leaf size: 43

```
dsolve([8*diff(y(x),x$2)-y(x)=x*exp(-x/2),y(0) = 3, D(y)(0) = 5],y(x), singsol=all)
```

$$y(x) = \frac{(-5 - 16\sqrt{2})\, e^{-\frac{\sqrt{2}\, x}{4}}}{2} + \frac{(-5 + 16\sqrt{2})\, e^{\frac{\sqrt{2}\, x}{4}}}{2} + (x+8)\, e^{-\frac{x}{2}}$$

✓ Solution by Mathematica
Time used: 0.025 (sec). Leaf size: 83

```
DSolve[{8*y''[x]-y[x]==x*Exp[-x/2],{y[0]==3,y'[0]==5}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{2} e^{-\frac{1}{4}\left(2+\sqrt{2}\right)x} \left(2e^{\frac{x}{2\sqrt{2}}}(x+8) - \left(5+16\sqrt{2}\right) e^{x/2} + \left(16\sqrt{2}-5\right) e^{\frac{1}{2}\left(1+\sqrt{2}\right)x}\right)$$

12 Exercise 20, page 90

12.1 problem 1

Internal problem ID [2174]
Internal file name [`OUTPUT/2174_Monday_February_26_2024_09_18_07_AM_40065206/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \sec(x)}$$

12.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = \mathrm{e}^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sec(x)\sin(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x) + \sin(x)\,x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\ln(\cos(x))\cos(x) + \sin(x)\,x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + \sin(x) x \tag{1}$$

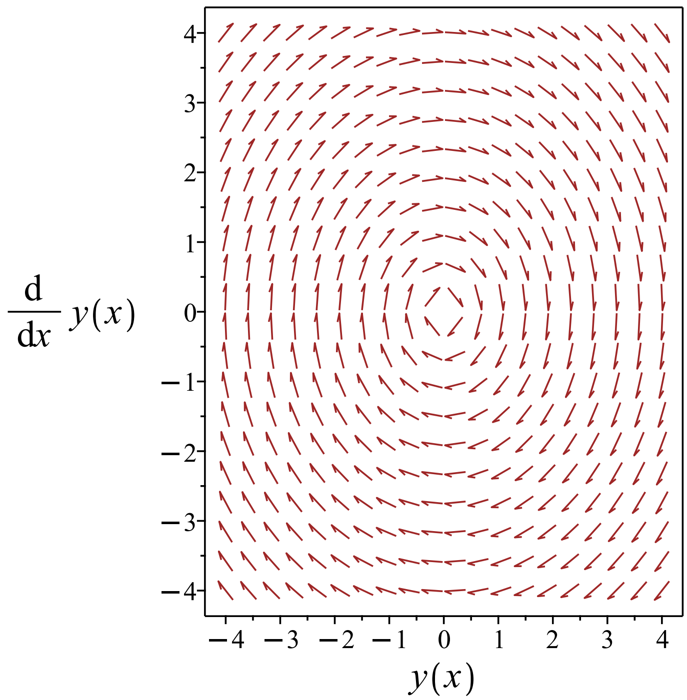


Figure 592: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + \sin(x) x$$

Verified OK.

12.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 427: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sec(x)\sin(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\, dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)}{1} dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x) + \sin(x)\, x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (\ln(\cos(x))\cos(x) + \sin(x)\, x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x))\cos(x) + \sin(x)\, x \tag{1}$$

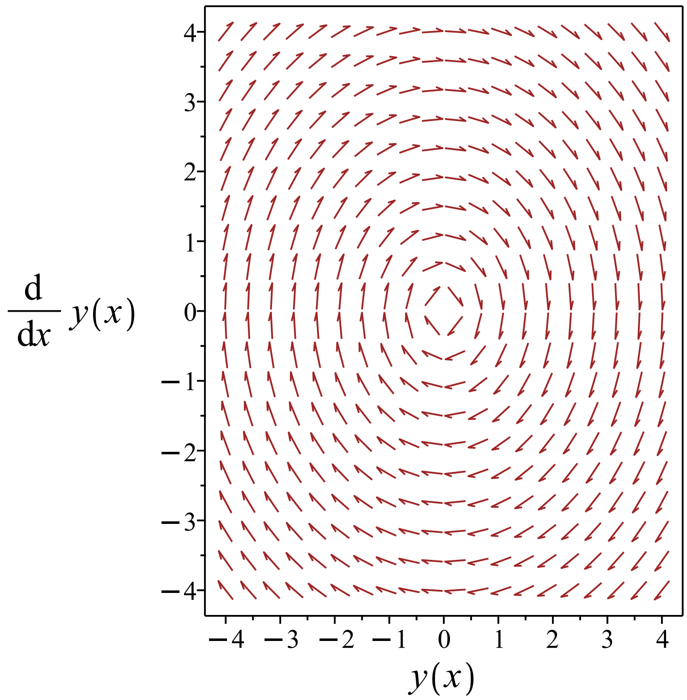


Figure 593: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + \sin(x) x$$

Verified OK.

12.1.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \tan(x)\, dx\right) + \sin(x)\left(\int 1 dx\right)$
- ○ Compute integrals

  $y_p(x) = \ln(\cos(x))\cos(x) + \sin(x)\, x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x))\cos(x) + \sin(x)\, x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+y(x)=sec(x),y(x), singsol=all)
```

$$y(x) = -\cos(x)\ln(\sec(x)) + \cos(x)\, c_1 + \sin(x)\,(c_2 + x)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 22

```
DSolve[y''[x]+y[x]==Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x + c_2)\sin(x) + \cos(x)(\log(\cos(x)) + c_1)$$

12.2 problem 2

Internal problem ID [2175]
Internal file name [`OUTPUT/2175_Monday_February_26_2024_09_18_07_AM_23402034/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 4y = e^x}$$

12.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(4)^2 - (4)\,(1)\,(4)}\\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 e^{-2x} + c_2 e^{-2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2x} + x\, e^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x} x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + x\, e^{-2x} c_2\right) + \left(\frac{e^x}{9}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_2 x + c_1) + \frac{e^x}{9}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_2 x + c_1) + \frac{e^x}{9} \tag{1}$$

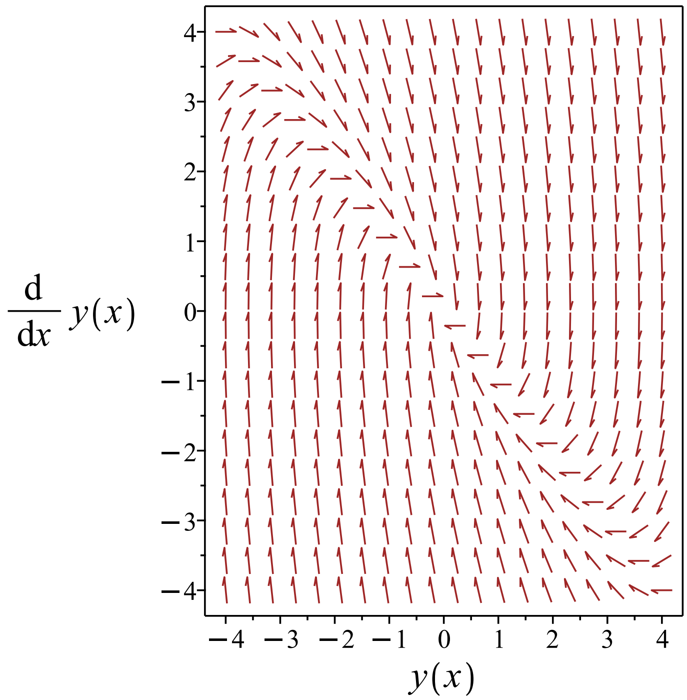


Figure 594: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) + \frac{e^x}{9}$$

Verified OK.

12.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 4\,dx} \\ &= e^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = e^{x}e^{2x}$$
$$(e^{2x}y)'' = e^{x}e^{2x}$$

Integrating once gives

$$(e^{2x}y)' = \frac{e^{3x}}{3} + c_1$$

Integrating again gives

$$(e^{2x}y) = c_1x + \frac{e^{3x}}{9} + c_2$$

Hence the solution is

$$y = \frac{c_1x + \frac{e^{3x}}{9} + c_2}{e^{2x}}$$

Or

$$y = \frac{e^{x}}{9} + c_1x\,e^{-2x} + c_2e^{-2x}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{x}}{9} + c_1x\,e^{-2x} + c_2e^{-2x} \tag{1}$$

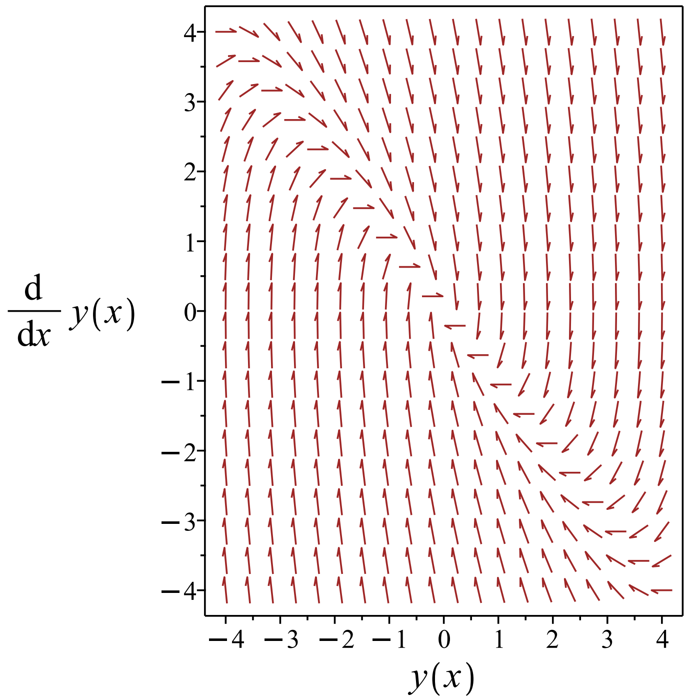


Figure 595: Slope field plot

Verification of solutions

$$y = \frac{e^x}{9} + c_1 x \, e^{-2x} + c_2 e^{-2x}$$

Verified OK.

12.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 429: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\, dx} \\ &= z_1 e^{-2x} \\ &= z_1 \left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-2x}\right) + c_2\left(e^{-2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{-2x} + x\, e^{-2x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{e^{-2x} x, e^{-2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 e^{x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
9A_1 e^{x} = e^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + x\, e^{-2x} c_2\right) + \left(\frac{e^x}{9}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_2 x + c_1) + \frac{e^x}{9}$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(c_2 x + c_1) + \frac{e^x}{9} \tag{1}$$

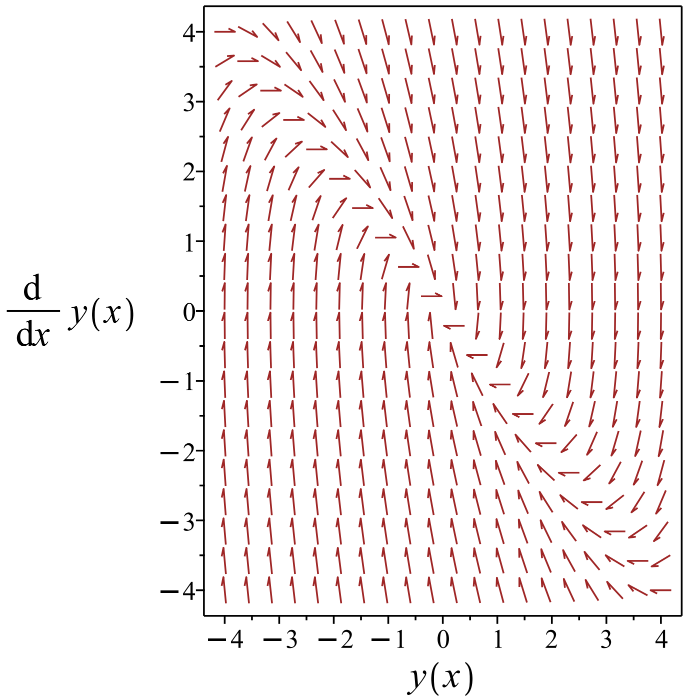


Figure 596: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2 x + c_1) + \frac{e^x}{9}$$

Verified OK.

12.2.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 2)^2 = 0$

- Root of the characteristic polynomial

$r=-2$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x)=\mathrm{e}^{-2x}x$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-2x}+x\,\mathrm{e}^{-2x}c_2+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=\mathrm{e}^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x),y_2(x))=\left[\begin{array}{cc}\mathrm{e}^{-2x} & \mathrm{e}^{-2x}x\\ -2\,\mathrm{e}^{-2x} & -2\,\mathrm{e}^{-2x}x+\mathrm{e}^{-2x}\end{array}\right]$

  - Compute Wronskian

    $W(y_1(x),y_2(x))=\mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=\mathrm{e}^{-2x}\left(-\left(\int x\,\mathrm{e}^{3x}dx\right)+\left(\int\mathrm{e}^{3x}dx\right)x\right)$

  - Compute integrals

    $y_p(x)=\frac{\mathrm{e}^{x}}{9}$

- Substitute particular solution into general solution to ODE

  $y=x\,\mathrm{e}^{-2x}c_2+c_1\mathrm{e}^{-2x}+\frac{\mathrm{e}^{x}}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = \frac{(e^{3x} + 9c_1x + 9c_2)\,e^{-2x}}{9}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 26

```
DSolve[y''[x]+4*y'[x]+4*y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x}{9} + e^{-2x}(c_2x + c_1)$$

12.3 problem 3

Internal problem ID [2176]
Internal file name [`OUTPUT/2176_Monday_February_26_2024_09_18_08_AM_97757968/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y = x^2}$$

12.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_3 x^2 + 4A_2 x + 4A_1 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = 0, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{4} - \frac{1}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{x^2}{4} - \frac{1}{8}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x^2}{4} - \frac{1}{8} \tag{1}$$

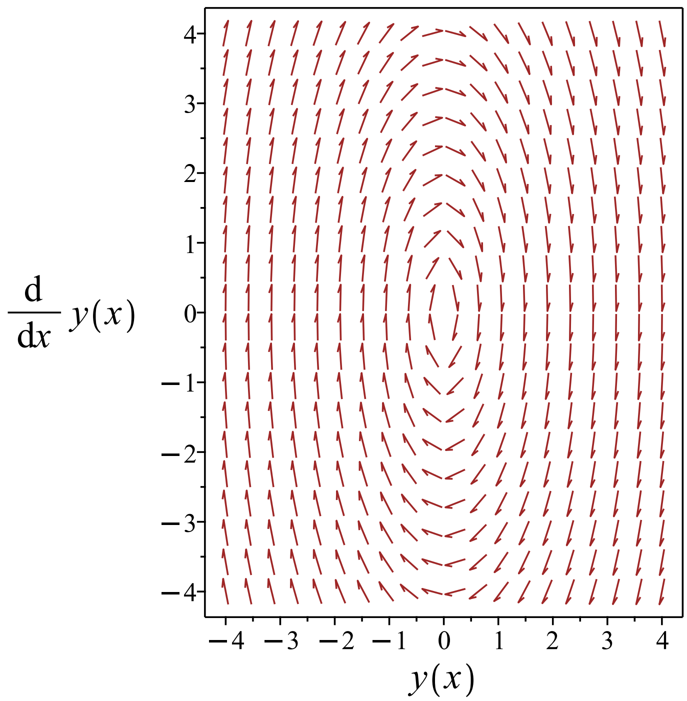


Figure 597: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x^2}{4} - \frac{1}{8}$$

Verified OK.

12.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4 z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 431: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos(2x)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_3x^2 + 4A_2x + 4A_1 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = 0, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{4} - \frac{1}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{x^2}{4} - \frac{1}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{x^2}{4} - \frac{1}{8} \tag{1}$$

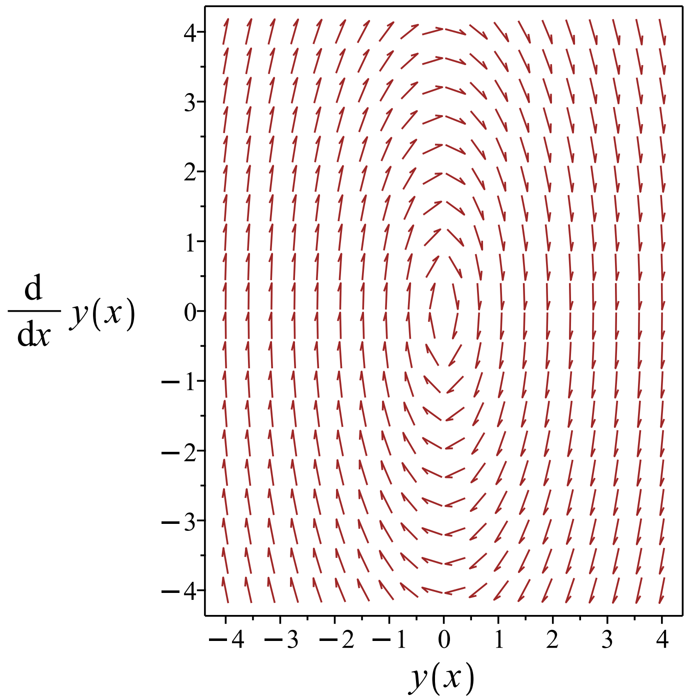


Figure 598: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{x^2}{4} - \frac{1}{8}$$

Verified OK.

12.3.3 Maple step by step solution

Let's solve

$y'' + 4y = x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x) x^2 dx\right)}{2} + \frac{\sin(2x)\left(\int x^2 \cos(2x) dx\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{x^2}{4} - \frac{1}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x^2}{4} - \frac{1}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+4*y(x)=x^2,y(x), singsol=all)
```

$$y(x) = \sin(2x) c_2 + c_1 \cos(2x) + \frac{x^2}{4} - \frac{1}{8}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 30

```
DSolve[y''[x]+4*y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{4} + c_1 \cos(2x) + c_2 \sin(2x) - \frac{1}{8}$$

12.4 problem 4

Internal problem ID [2177]
Internal file name [`OUTPUT/2177_Monday_February_26_2024_09_18_08_AM_66274892/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y' + y = e^{2x}}$$

12.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-2)^2 - (4)\,(1)\,(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 x\, e^{x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 x\, e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{x}, e^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \mathrm{e}^{2x} = \mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \mathrm{e}^{x} + c_2 x \,\mathrm{e}^{x}) + \left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x}(c_2 x + c_1) + \mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}(c_2 x + c_1) + \mathrm{e}^{2x} \tag{1}$$

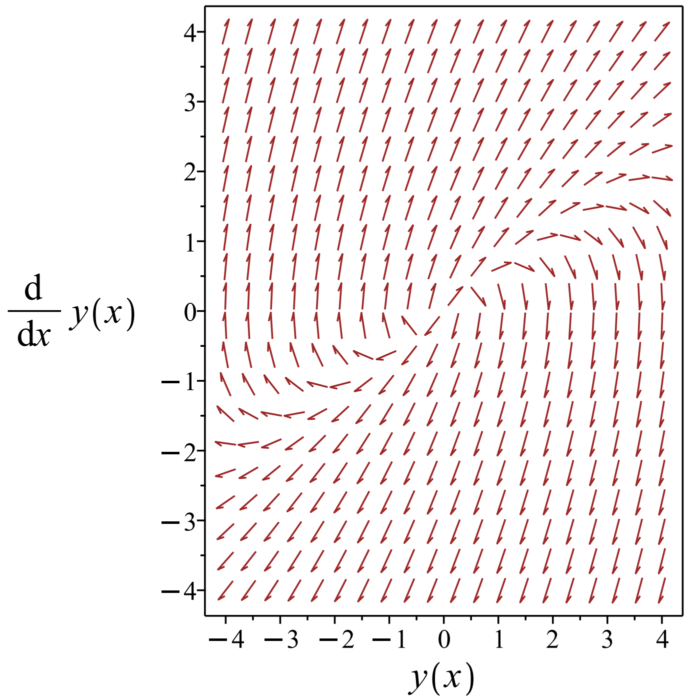


Figure 599: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + e^{2x}$$

Verified OK.

12.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = e^{-x}e^{2x}$$
$$\left(e^{-x}y\right)'' = e^{-x}e^{2x}$$

Integrating once gives

$$\left(e^{-x}y\right)' = e^{x} + c_1$$

Integrating again gives

$$\left(e^{-x}y\right) = c_1x + e^{x} + c_2$$

Hence the solution is

$$y = \frac{c_1x + e^{x} + c_2}{e^{-x}}$$

Or

$$y = c_1x\,e^{x} + c_2e^{x} + e^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1x\,e^{x} + c_2e^{x} + e^{2x} \tag{1}$$

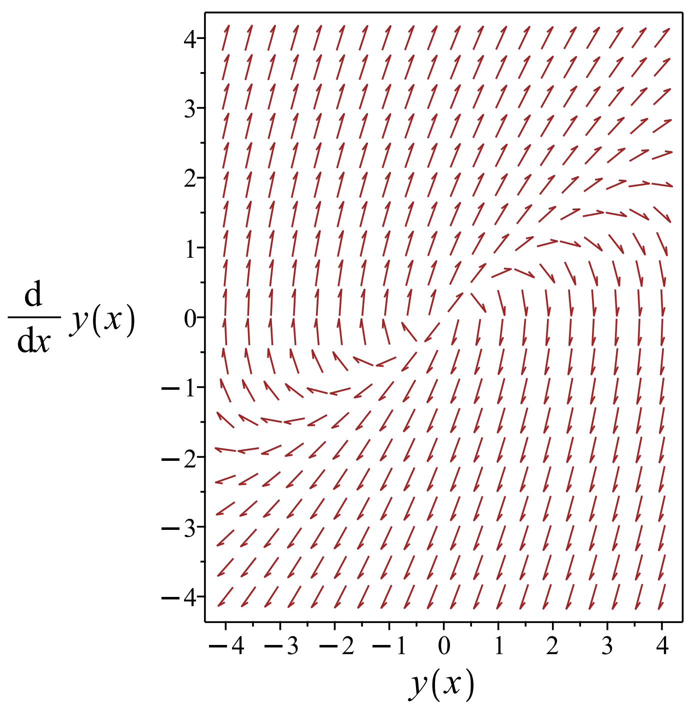


Figure 600: Slope field plot

Verification of solutions

$$y = c_1 x\, e^{x} + c_2 e^{x} + e^{2x}$$

Verified OK.

12.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^x + c_2 x\, e^x$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^x, e^x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 x\, e^x) + (e^{2x}) \end{aligned}$$

Which simplifies to

$$y = e^{x}(c_2 x + c_1) + e^{2x}$$

Summary

The solution(s) found are the following

$$y = e^{x}(c_2 x + c_1) + e^{2x} \tag{1}$$

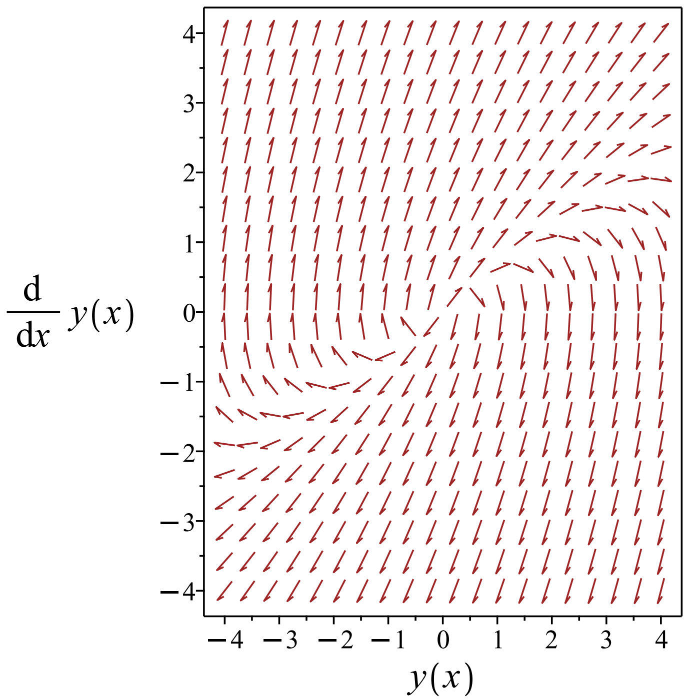


Figure 601: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + e^{2x}$$

Verified OK.

12.4.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x + c_2 x\,e^x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^{2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^x\left(-\left(\int x\,e^x dx\right) + \left(\int e^x dx\right) x\right)$

- Compute integrals

  $y_p(x) = e^{2x}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x \,e^x + c_1 e^x + e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=exp(2*x),y(x), singsol=all)
```

$$y(x) = e^{2x} + e^x(c_1 x + c_2)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 19

```
DSolve[y''[x]-2*y'[x]+y[x]==Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(e^x + c_2 x + c_1)$$

12.5 problem 5

Internal problem ID [2178]
Internal file name [`OUTPUT/2178_Monday_February_26_2024_09_18_08_AM_84295229/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = 4\sin(2x)}$$

12.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 4\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(2x) - 3A_2 \sin(2x) = 4 \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{4}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4 \sin(2x)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{4 \sin(2x)}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{4 \sin(2x)}{3} \tag{1}$$

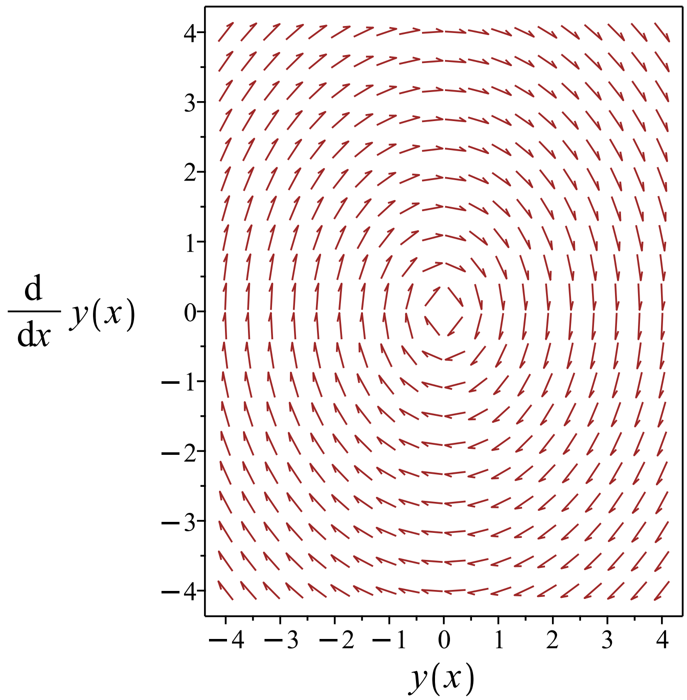


Figure 602: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{4 \sin(2x)}{3}$$

Verified OK.

12.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 435: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(2x) - 3A_2 \sin(2x) = 4 \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{4}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4 \sin(2x)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{4 \sin(2x)}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{4 \sin(2x)}{3} \tag{1}$$

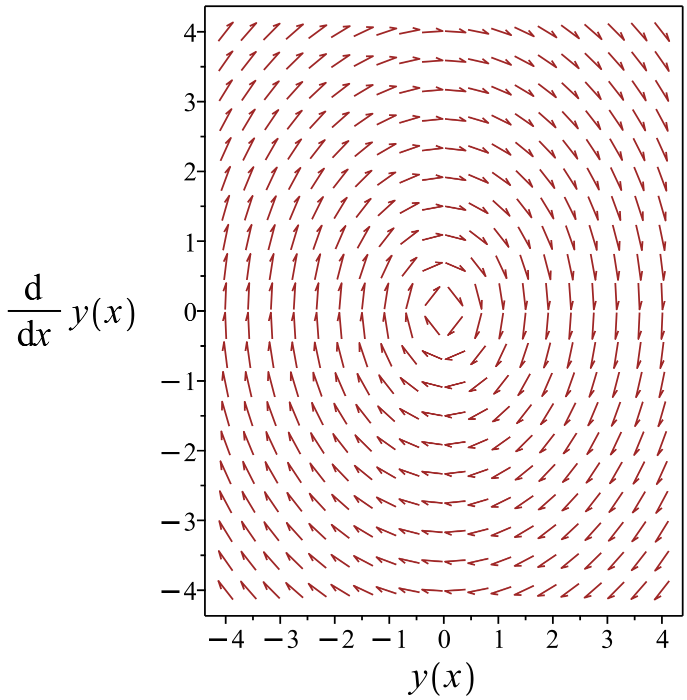


Figure 603: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{4\sin(2x)}{3}$$

Verified OK.

12.5.3 Maple step by step solution

Let's solve

$y'' + y = 4\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -8\cos(x)\left(\int \cos(x)\sin(x)^2 dx\right) + 8\sin(x)\left(\int \cos(x)^2 \sin(x) dx\right)$

  - Compute integrals

    $y_p(x) = -\frac{4\sin(2x)}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \frac{4\sin(2x)}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+y(x)=4*sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(3c_1 - 8\sin(x))\cos(x)}{3} + \sin(x)\,c_2$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 24

```
DSolve[y''[x]+y[x]==4*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{4}{3}\sin(2x) + c_1\cos(x) + c_2\sin(x)$$

12.6 problem 6

Internal problem ID [2179]
Internal file name [`OUTPUT/2179_Monday_February_26_2024_09_18_09_AM_50067208/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y = 2x - 2\sin(2x)}$$

12.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 2x - 2\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x - 2\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1, x\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x + A_1 + A_3 x\cos(2x) + A_4 x\sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_3\sin(2x) + 4A_4\cos(2x) + 4A_2 x + 4A_1 = 2x - 2\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}, A_3 = \frac{1}{2}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{2} + \frac{x\cos(2x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{x}{2} + \frac{x \cos(2x)}{2}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x}{2} + \frac{x \cos(2x)}{2} \tag{1}
$$

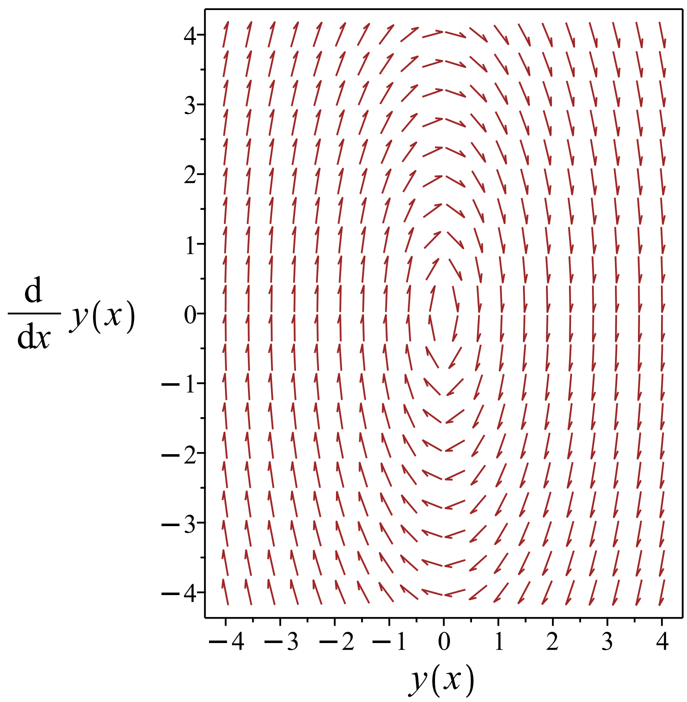


Figure 604: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x}{2} + \frac{x \cos(2x)}{2}
$$

Verified OK.

12.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(2x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\
&= \cos(2x)\left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x - 2\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1, x\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2x + A_1 + A_3x\cos(2x) + A_4x\sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_3\sin(2x) + 4A_4\cos(2x) + 4A_2x + 4A_1 = 2x - 2\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}, A_3 = \frac{1}{2}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{2} + \frac{x\cos(2x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{x}{2} + \frac{x \cos(2x)}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{x}{2} + \frac{x \cos(2x)}{2} \tag{1}
$$

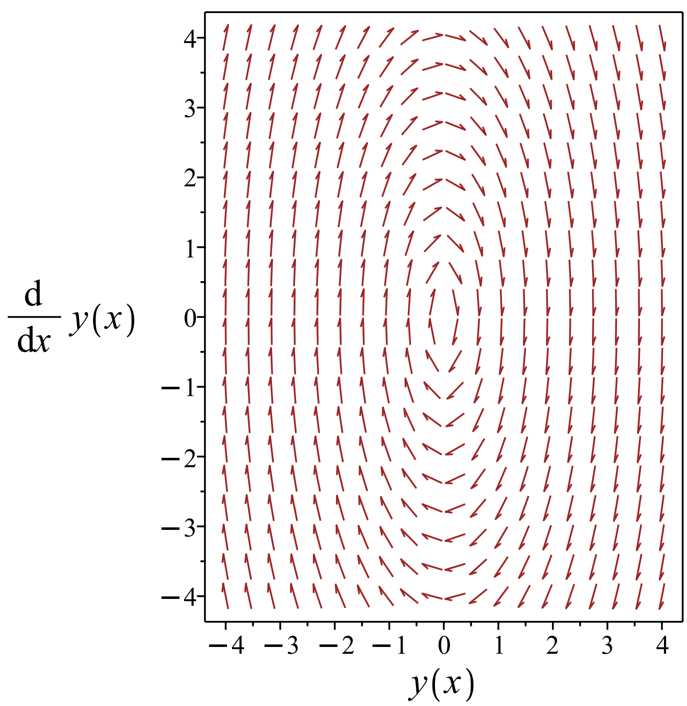


Figure 605: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{x}{2} + \frac{x \cos(2x)}{2}
$$

Verified OK.

12.6.3 Maple step by step solution

Let's solve

$y'' + 4y = 2x - 2\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2x - 2\sin(2x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \cos(2x)\left(\int \sin(2x)\left(-x+\sin(2x)\right)dx\right) - \sin(2x)\left(\int \cos(2x)\left(-x+\sin(2x)\right)dx\right)$$

- Compute integrals

$$y_p(x) = \frac{x(1+\cos(2x))}{2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(2x) + c_2\sin(2x) + \frac{x(1+\cos(2x))}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+4*y(x)=2*(x-sin(2*x)),y(x), singsol=all)
```

$$y(x) = \frac{(x+2c_1)\cos(2x)}{2} + \sin(2x)c_2 + \frac{x}{2}$$

✓ Solution by Mathematica
Time used: 0.166 (sec). Leaf size: 30

```
DSolve[y''[x]+4*y[x]==2*(x-Sin[2*x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(x + (x+2c_1)\cos(2x) + 2c_2\sin(2x))$$

12.7 problem 7

Internal problem ID [2180]
Internal file name [`OUTPUT/2180_Monday_February_26_2024_09_18_09_AM_90370423/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y = 3x + 5\,e^{x}}$$

12.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 3x + 5\,e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x + 5\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}\}, \{1, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{x} + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x} - A_2 - A_3 x = 3x + 5\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{2}, A_2 = 0, A_3 = -3\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5x\,e^{x}}{2} - 3x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{-x}\right) + \left(\frac{5x\,e^{x}}{2} - 3x\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} + \frac{5x\, e^x}{2} - 3x \tag{1}$$

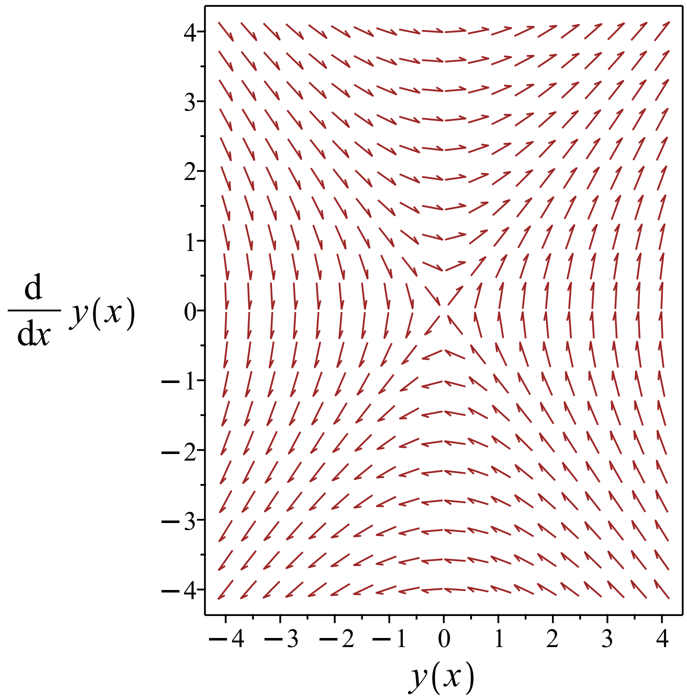


Figure 606: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} + \frac{5x\, e^x}{2} - 3x$$

Verified OK.

12.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 439: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\ &= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{-x}) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \frac{\mathrm{e}^{x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-x})\left(\frac{\mathrm{e}^{x}}{2}\right) - \left(\frac{\mathrm{e}^{x}}{2}\right)(-\mathrm{e}^{-x})$$

Which simplifies to

$$W = \mathrm{e}^{-x}\mathrm{e}^{x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^x(3x+5\,e^x)}{2}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^x(3x+5\,e^x)}{2}dx$$

Hence

$$u_1 = -\frac{5\,e^{2x}}{4} - \frac{3x\,e^x}{2} + \frac{3\,e^x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x}(3x+5\,e^x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \left(3x\,e^{-x}+5\right)dx$$

Hence

$$u_2 = 5x - 3x\,e^{-x} - 3\,e^{-x}$$

Which simplifies to

$$u_1 = -\frac{5\,e^{2x}}{4} + \frac{(-6x+6)\,e^x}{4}$$
$$u_2 = (-3x-3)\,e^{-x} + 5x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{5\,e^{2x}}{4} + \frac{(-6x+6)\,e^x}{4}\right)e^{-x} + \frac{((-3x-3)\,e^{-x}+5x)\,e^x}{2}$$

Which simplifies to

$$y_p(x) = \frac{5(2x-1)\,e^x}{4} - 3x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-x} + \frac{c_2 e^{x}}{2}\right) + \left(\frac{5(2x-1)\, e^{x}}{4} - 3x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + \frac{5(2x-1)\, e^{x}}{4} - 3x \tag{1}
$$

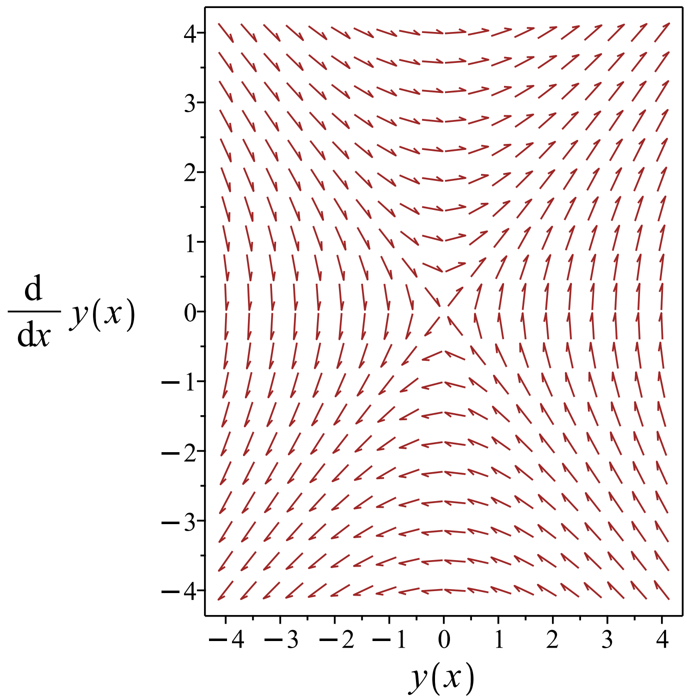


Figure 607: Slope field plot

Verification of solutions

$$
y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + \frac{5(2x-1)\, e^{x}}{4} - 3x
$$

Verified OK.

12.7.3 Maple step by step solution

Let's solve

$y'' - y = 3x + 5\,e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = 3x + 5\,e^{x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x}\left(\int e^x (3x+5\,e^x)\,dx\right)}{2} + \frac{e^x\left(\int (3x\,e^{-x}+5)\,dx\right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{5(2x-1)e^x}{4} - 3x$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x + \frac{5(2x-1)e^x}{4} - 3x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-y(x)=3*x+5*exp(x),y(x), singsol=all)
```

$$y(x) = e^{-x} c_2 + \frac{(10x + 4c_1 - 5)\,e^x}{4} - 3x$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 32

```
DSolve[y''[x]-y[x]==3*x+5*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3x + e^x\left(\frac{5x}{2} - \frac{5}{4} + c_1\right) + c_2 e^{-x}$$

12.8 problem 8

Internal problem ID [2181]
Internal file name [`OUTPUT/2181_Monday_February_26_2024_09_18_10_AM_17697769/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = e^{x} + \sin(4x)}$$

12.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = e^{x} + \sin(4x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + \sin(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x + A_2 \cos(4x) + A_3 \sin(4x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 e^x - 7A_2 \cos(4x) - 7A_3 \sin(4x) = e^x + \sin(4x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = 0, A_3 = -\frac{1}{7}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{10} - \frac{\sin(4x)}{7}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(\frac{e^x}{10} - \frac{\sin(4x)}{7}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{e^x}{10} - \frac{\sin(4x)}{7} \tag{1}$$

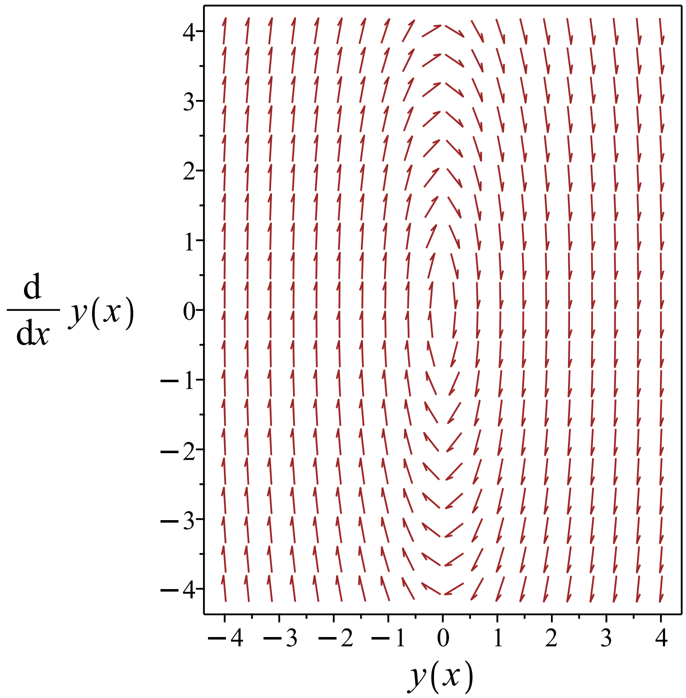


Figure 608: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{e^x}{10} - \frac{\sin(4x)}{7}$$

Verified OK.

12.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 441: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\,dx \\ &= \cos(3x)\left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + \sin(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x + A_2 \cos(4x) + A_3 \sin(4x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 e^x - 7A_2 \cos(4x) - 7A_3 \sin(4x) = e^x + \sin(4x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = 0, A_3 = -\frac{1}{7}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{10} - \frac{\sin(4x)}{7}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(\frac{e^x}{10} - \frac{\sin(4x)}{7}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{e^x}{10} - \frac{\sin(4x)}{7} \tag{1}$$

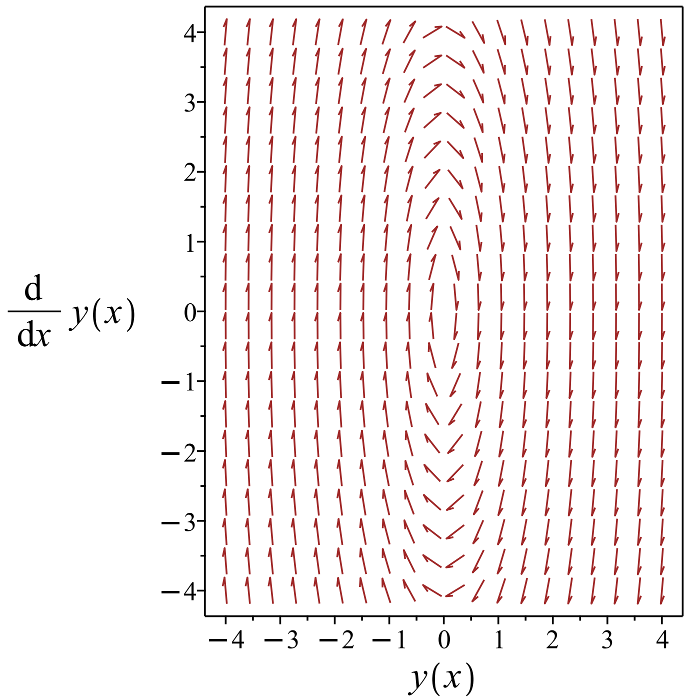


Figure 609: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{e^x}{10} - \frac{\sin(4x)}{7}$$

Verified OK.

12.8.3 Maple step by step solution

Let's solve

$y'' + 9y = e^x + \sin(4x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^x + \sin(4x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(3x)\left(\int \sin(3x)(\mathrm{e}^x + \sin(4x))dx\right)}{3} + \frac{\sin(3x)\left(\int \cos(3x)(\mathrm{e}^x + \sin(4x))dx\right)}{3}$

  - Compute integrals

    $y_p(x) = \frac{\mathrm{e}^x}{10} - \frac{\sin(4x)}{7}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\mathrm{e}^x}{10} - \frac{\sin(4x)}{7}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+9*y(x)=exp(x)+sin(4*x),y(x), singsol=all)
```

$$y(x) = \sin(3x) c_2 + \cos(3x) c_1 + \frac{e^x}{10} - \frac{\sin(4x)}{7}$$

✓ Solution by Mathematica

Time used: 0.194 (sec). Leaf size: 35

```
DSolve[y''[x]+9*y[x]==Exp[x]+Sin[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x}{10} - \frac{1}{7}\sin(4x) + c_1\cos(3x) + c_2\sin(3x)$$

12.9 problem 9

Internal problem ID [2182]
Internal file name [`OUTPUT/2182_Monday_February_26_2024_09_18_10_AM_23795280/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 3y'' - 4y' = \cos(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 3y'' - 4y' = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 - 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -4$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^x + e^{-4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = e^{-4x}$$

Now the particular solution to the given ODE is found

$$y''' + 3y'' - 4y' = \cos(2x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^x, e^{-4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_1 \sin(2x) - 16A_2 \cos(2x) - 12A_1 \cos(2x) - 12A_2 \sin(2x) = \cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{100}, A_2 = -\frac{1}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(2x)}{100} - \frac{\sin(2x)}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^x + e^{-4x} c_3\right) + \left(-\frac{3\cos(2x)}{100} - \frac{\sin(2x)}{25}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^x + e^{-4x} c_3 - \frac{3\cos(2x)}{100} - \frac{\sin(2x)}{25} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^x + e^{-4x} c_3 - \frac{3\cos(2x)}{100} - \frac{\sin(2x)}{25}$$

Verified OK.

12.9.1 Maple step by step solution

Let's solve

$y''' + 3y'' - 4y' = \cos(2x)$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = \cos(2x) - 3y_3(x) + 4y_2(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \cos(2x) - 3y_3(x) + 4y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & -3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \cos(2x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \cos(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & -3 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-4x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$$

□ Fundamental matrix

  ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix}\frac{e^{-4x}}{16} & 1 & e^{x}\\ -\frac{e^{-4x}}{4} & 0 & e^{x}\\ e^{-4x} & 0 & e^{x}\end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix}\frac{e^{-4x}}{16} & 1 & e^{x}\\ -\frac{e^{-4x}}{4} & 0 & e^{x}\\ e^{-4x} & 0 & e^{x}\end{bmatrix}\cdot\frac{1}{\begin{bmatrix}\frac{1}{16} & 1 & 1\\ -\frac{1}{4} & 0 & 1\\ 1 & 0 & 1\end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix}1 & -\frac{(-16e^{5x}+15e^{4x}+1)e^{-4x}}{20} & -\frac{(-4e^{5x}+5e^{4x}-1)e^{-4x}}{20}\\ 0 & \frac{(4e^{5x}+1)e^{-4x}}{5} & \frac{(e^{5x}-1)e^{-4x}}{5}\\ 0 & \frac{4(e^{5x}-1)e^{-4x}}{5} & \frac{(e^{5x}+4)e^{-4x}}{5}\end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{e^{-4x}\left(-(3\cos(2x)+4\sin(2x))e^{4x}+4\,e^{5x}-1\right)}{100} \\ \frac{e^{-4x}\left(-\frac{(-3\sin(2x)+4\cos(2x))e^{4x}}{2}+e^{5x}+1\right)}{25} \\ \frac{e^{-4x}\left((3\cos(2x)+4\sin(2x))e^{4x}+e^{5x}-4\right)}{25} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \begin{bmatrix} \frac{e^{-4x}\left(-(3\cos(2x)+4\sin(2x))e^{4x}+4\,e^{5x}-1\right)}{100} \\ \frac{e^{-4x}\left(-\frac{(-3\sin(2x)+4\cos(2x))e^{4x}}{2}+e^{5x}+1\right)}{25} \\ \frac{e^{-4x}\left((3\cos(2x)+4\sin(2x))e^{4x}+e^{5x}-4\right)}{25} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(\left(16c_2-\frac{12\cos(2x)}{25}-\frac{16\sin(2x)}{25}\right)e^{4x}+\left(16c_3+\frac{16}{25}\right)e^{5x}+c_1-\frac{4}{25}\right)e^{-4x}}{16}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -3*(diff(_b(_a), _a))+4*_b(_a
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)-4*diff(y(x),x)=cos(2*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-4x}\left(\left(c_3 - \frac{3\cos(2x)}{100} - \frac{\sin(2x)}{25} - \frac{3}{100}\right)\mathrm{e}^{4x} + c_1\mathrm{e}^{5x} - \frac{c_2}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.306 (sec). Leaf size: 41

```
DSolve[y'''[x]+3*y''[x]-4*y'[x]==Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{25}\sin(2x) - \frac{3}{100}\cos(2x) - \frac{1}{4}c_1e^{-4x} + c_2e^{x} + c_3$$

12.10 problem 10

Internal problem ID [2183]
Internal file name [`OUTPUT/2183_Monday_February_26_2024_09_18_10_AM_97977008/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 10.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 4y'' - 5y' = e^{3x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 4y'' - 5y' = 0$$

The characteristic equation is

$$\lambda^3 + 4\lambda^2 - 5\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -5$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^x + e^{-5x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = e^{-5x}$$

Now the particular solution to the given ODE is found

$$y''' + 4y'' - 5y' = e^{3x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^x, e^{-5x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$48 A_1 e^{3x} = e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{48}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{3x}}{48}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^x + e^{-5x} c_3\right) + \left(\frac{e^{3x}}{48}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^x + e^{-5x} c_3 + \frac{e^{3x}}{48} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^x + e^{-5x} c_3 + \frac{e^{3x}}{48}$$

Verified OK.

12.10.1 Maple step by step solution

Let's solve

$y''' + 4y'' - 5y' = e^{3x}$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = e^{3x} - 4y_3(x) + 5y_2(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = e^{3x} - 4y_3(x) + 5y_2(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & -4 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ e^{3x} \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ e^{3x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & -4 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \left[\begin{array}{c} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-5, \begin{bmatrix} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-5x} \cdot \begin{bmatrix} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$$

□ Fundamental matrix

- ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-5x}}{25} & 1 & e^{x} \\ -\frac{e^{-5x}}{5} & 0 & e^{x} \\ e^{-5x} & 0 & e^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-5x}}{25} & 1 & e^{x} \\ -\frac{e^{-5x}}{5} & 0 & e^{x} \\ e^{-5x} & 0 & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{25} & 1 & 1 \\ -\frac{1}{5} & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{(-25\,e^{6x}+24\,e^{5x}+1)e^{-5x}}{30} & -\frac{(-5\,e^{6x}+6\,e^{5x}-1)e^{-5x}}{30} \\ 0 & \frac{(5\,e^{6x}+1)e^{-5x}}{6} & \frac{(e^{6x}-1)e^{-5x}}{6} \\ 0 & \frac{5(e^{6x}-1)e^{-5x}}{6} & \frac{(e^{6x}+5)e^{-5x}}{6} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(5\,e^{8x}-20\,e^{6x}+16\,e^{5x}-1)e^{-5x}}{240} \\ -\frac{(-3\,e^{8x}+4\,e^{6x}-1)e^{-5x}}{48} \\ -\frac{(-9\,e^{8x}+4\,e^{6x}+5)e^{-5x}}{48} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \begin{bmatrix} \frac{(5\,e^{8x}-20\,e^{6x}+16\,e^{5x}-1)e^{-5x}}{240} \\ -\frac{(-3\,e^{8x}+4\,e^{6x}-1)e^{-5x}}{48} \\ -\frac{(-9\,e^{8x}+4\,e^{6x}+5)e^{-5x}}{48} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(25\,e^{8x}+1200c_3e^{6x}-100\,e^{6x}+1200c_2e^{5x}+80\,e^{5x}+48c_1-5)e^{-5x}}{1200}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -4*(diff(_b(_a), _a))+5*_b(_a
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$3)+4*diff(y(x),x$2)-5*diff(y(x),x)=exp(3*x),y(x), singsol=all)
```

$$y(x) = -\frac{e^{-5x}\left(-5\, e^{6x}c_2 - 5c_3e^{5x} + c_1 - \frac{5\, e^{8x}}{48}\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 34

```
DSolve[y'''[x]+4*y''[x]-5*y'[x]==Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{3x}}{48} - \frac{1}{5}c_1e^{-5x} + c_2e^x + c_3$$

12.11 problem 11

Internal problem ID [2184]
Internal file name [`OUTPUT/2184_Monday_February_26_2024_09_18_11_AM_37306510/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \tan(x)}$$

12.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \tan(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\tan(x)\sin(x)}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\sin(x)\, dx$$

Hence

$$u_1 = \sin(x) - \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\tan(x)\cos(x)}{1}\, dx$$

Which simplifies to

$$u_2 = \int \sin(x)\, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\sin(x) - \ln(\sec(x) + \tan(x)))\cos(x) - \cos(x)\sin(x)$$

Which simplifies to

$$y_p(x) = -\cos(x)\ln(\sec(x) + \tan(x))$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (-\cos(x) \ln(\sec(x) + \tan(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln(\sec(x) + \tan(x)) \tag{1}$$

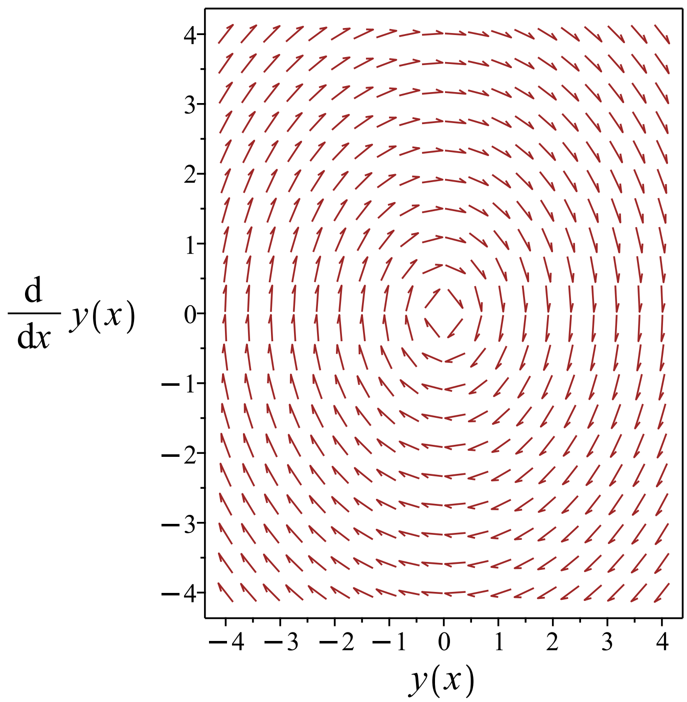


Figure 610: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln(\sec(x) + \tan(x))$$

Verified OK.

12.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 445: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\tan(x)\sin(x)}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\sin(x)\, dx$$

Hence

$$u_1 = \sin(x) - \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\tan(x)\cos(x)}{1}\, dx$$

Which simplifies to

$$u_2 = \int \sin(x)\, dx$$

Hence

$$u_2 = -\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\sin(x) - \ln(\sec(x) + \tan(x)))\cos(x) - \cos(x)\sin(x)$$

Which simplifies to

$$y_p(x) = -\cos(x)\ln(\sec(x) + \tan(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (-\cos(x)\ln(\sec(x) + \tan(x))) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln(\sec(x) + \tan(x)) \tag{1}$$

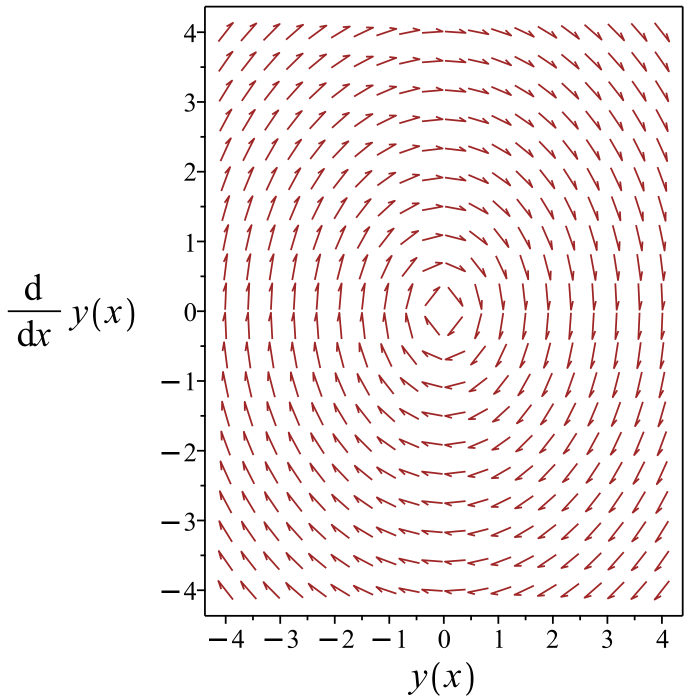


Figure 611: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln(\sec(x) + \tan(x))$$

Verified OK.

12.11.3 Maple step by step solution

Let's solve

$y'' + y = \tan(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$
- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \tan(x) \sin(x)\, dx\right) + \sin(x) \left(\int \sin(x)\, dx\right)$
  - Compute integrals

    $y_p(x) = -\cos(x) \ln(\sec(x) + \tan(x))$
- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \ln(\sec(x) + \tan(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=tan(x),y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + \cos(x) c_1 - \cos(x) \ln(\sec(x) + \tan(x))$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 23

```
DSolve[y''[x]+y[x]==Tan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x)(-\operatorname{arctanh}(\sin(x))) + c_1 \cos(x) + c_2 \sin(x)$$

12.12 problem 12

Internal problem ID [2185]
Internal file name [`OUTPUT/2185_Monday_February_26_2024_09_18_11_AM_64483771/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + a^2 y = \sec(ax)}$$

12.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = a^2, f(x) = \sec(ax)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + a^2 y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=0, C=a^2$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + a^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=0, C=a^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(a^2)} \\ &= \pm\sqrt{-a^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-a^2} \\ \lambda_2 &= -\sqrt{-a^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-a^2} \\ \lambda_2 &= -\sqrt{-a^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\sqrt{-a^2}\right)x} + c_2 e^{\left(-\sqrt{-a^2}\right)x}\end{aligned}$$

Or

$$y = c_1 e^{\sqrt{-a^2}\, x} + c_2 e^{-\sqrt{-a^2}\, x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{\sqrt{-a^2}\, x} + c_2 e^{-\sqrt{-a^2}\, x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{\sqrt{-a^2}\, x}$$
$$y_2 = \mathrm{e}^{-\sqrt{-a^2}\, x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-a^2}\, x} & \mathrm{e}^{-\sqrt{-a^2}\, x} \\ \frac{d}{dx}\left(\mathrm{e}^{\sqrt{-a^2}\, x}\right) & \frac{d}{dx}\left(\mathrm{e}^{-\sqrt{-a^2}\, x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-a^2}\, x} & \mathrm{e}^{-\sqrt{-a^2}\, x} \\ \sqrt{-a^2}\, \mathrm{e}^{\sqrt{-a^2}\, x} & -\sqrt{-a^2}\, \mathrm{e}^{-\sqrt{-a^2}\, x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{\sqrt{-a^2}\, x}\right)\left(-\sqrt{-a^2}\, \mathrm{e}^{-\sqrt{-a^2}\, x}\right) - \left(\mathrm{e}^{-\sqrt{-a^2}\, x}\right)\left(\sqrt{-a^2}\, \mathrm{e}^{\sqrt{-a^2}\, x}\right)$$

Which simplifies to

$$W = -2\, \mathrm{e}^{\sqrt{-a^2}\, x} \sqrt{-a^2}\, \mathrm{e}^{-\sqrt{-a^2}\, x}$$

Which simplifies to

$$W = -2\sqrt{-a^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-\sqrt{-a^2}\,x} \sec(ax)}{-2\sqrt{-a^2}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-\sqrt{-a^2}\,x} \sec(ax)}{2\sqrt{-a^2}}dx$$

Hence

$$u_1 = -\left(\int_0^x -\frac{e^{-\sqrt{-a^2}\,\alpha} \sec(a\alpha)}{2\sqrt{-a^2}}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\sqrt{-a^2}\,x} \sec(ax)}{-2\sqrt{-a^2}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{\sqrt{-a^2}\,x} \sec(ax)}{2\sqrt{-a^2}}dx$$

Hence

$$u_2 = \int_0^x -\frac{e^{\sqrt{-a^2}\,\alpha} \sec(a\alpha)}{2\sqrt{-a^2}}d\alpha$$

Which simplifies to

$$u_1 = \frac{\int_0^x e^{-\sqrt{-a^2}\,\alpha} \sec(a\alpha)\,d\alpha}{2\sqrt{-a^2}}$$

$$u_2 = -\frac{\int_0^x e^{\sqrt{-a^2}\,\alpha} \sec(a\alpha)\,d\alpha}{2\sqrt{-a^2}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\int_0^x \mathrm{e}^{-\sqrt{-a^2}\,\alpha} \sec(a\alpha)\, d\alpha\right) \mathrm{e}^{\sqrt{-a^2}\,x}}{2\sqrt{-a^2}} - \frac{\left(\int_0^x \mathrm{e}^{\sqrt{-a^2}\,\alpha} \sec(a\alpha)\, d\alpha\right) \mathrm{e}^{-\sqrt{-a^2}\,x}}{2\sqrt{-a^2}}$$

Which simplifies to

$$y_p(x) = \frac{\left(\int_0^x \mathrm{e}^{-\sqrt{-a^2}\,\alpha} \sec(a\alpha)\, d\alpha\right) \mathrm{e}^{\sqrt{-a^2}\,x} - \left(\int_0^x \mathrm{e}^{\sqrt{-a^2}\,\alpha} \sec(a\alpha)\, d\alpha\right) \mathrm{e}^{-\sqrt{-a^2}\,x}}{2\sqrt{-a^2}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{\sqrt{-a^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-a^2}\,x}\right) \\ &\quad + \left(\frac{\left(\int_0^x \mathrm{e}^{-\sqrt{-a^2}\,\alpha} \sec(a\alpha)\, d\alpha\right) \mathrm{e}^{\sqrt{-a^2}\,x} - \left(\int_0^x \mathrm{e}^{\sqrt{-a^2}\,\alpha} \sec(a\alpha)\, d\alpha\right) \mathrm{e}^{-\sqrt{-a^2}\,x}}{2\sqrt{-a^2}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 \mathrm{e}^{\sqrt{-a^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-a^2}\,x} \\ &\quad + \frac{\left(\int_0^x \mathrm{e}^{-\sqrt{-a^2}\,\alpha} \sec(a\alpha)\, d\alpha\right) \mathrm{e}^{\sqrt{-a^2}\,x} - \left(\int_0^x \mathrm{e}^{\sqrt{-a^2}\,\alpha} \sec(a\alpha)\, d\alpha\right) \mathrm{e}^{-\sqrt{-a^2}\,x}}{2\sqrt{-a^2}} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1 \mathrm{e}^{\sqrt{-a^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-a^2}\,x} \\ &\quad + \frac{\left(\int_0^x \mathrm{e}^{-\sqrt{-a^2}\,\alpha} \sec(a\alpha)\, d\alpha\right) \mathrm{e}^{\sqrt{-a^2}\,x} - \left(\int_0^x \mathrm{e}^{\sqrt{-a^2}\,\alpha} \sec(a\alpha)\, d\alpha\right) \mathrm{e}^{-\sqrt{-a^2}\,x}}{2\sqrt{-a^2}} \end{aligned}$$

Verified OK.

12.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + a^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (-a^2) \, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 447: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{\sqrt{-a^2}\, x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{\sqrt{-a^2}\,x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\sqrt{-a^2}\,x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \mathrm{e}^{\sqrt{-a^2}\,x} \int \frac{1}{\mathrm{e}^{2\sqrt{-a^2}\,x}}\,dx \\
&= \mathrm{e}^{\sqrt{-a^2}\,x} \left(\frac{\sqrt{-a^2}\,\mathrm{e}^{-2\sqrt{-a^2}\,x}}{2a^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\sqrt{-a^2}\,x}\right) + c_2 \left(\mathrm{e}^{\sqrt{-a^2}\,x} \left(\frac{\sqrt{-a^2}\,\mathrm{e}^{-2\sqrt{-a^2}\,x}}{2a^2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + a^2 y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{\sqrt{-a^2}\, x} + \frac{c_2 \sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x}}{2a^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{\sqrt{-a^2}\, x}$$

$$y_2 = \frac{\sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x}}{2a^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\sqrt{-a^2}\, x} & \frac{\sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x}}{2a^2} \\ \frac{d}{dx}\left(e^{\sqrt{-a^2}\, x}\right) & \frac{d}{dx}\left(\frac{\sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x}}{2a^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\sqrt{-a^2}\, x} & \frac{\sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x}}{2a^2} \\ \sqrt{-a^2}\, e^{\sqrt{-a^2}\, x} & \frac{e^{-\sqrt{-a^2}\, x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{\sqrt{-a^2}\, x}\right)\left(\frac{e^{-\sqrt{-a^2}\, x}}{2}\right) - \left(\frac{\sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x}}{2a^2}\right)\left(\sqrt{-a^2}\, e^{\sqrt{-a^2}\, x}\right)$$

Which simplifies to

$$W = e^{\sqrt{-a^2}\, x} e^{-\sqrt{-a^2}\, x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x} \sec(ax)}{2a^2}}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x} \sec(ax)}{2a^2} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\sqrt{-a^2}\, e^{-\sqrt{-a^2}\, \alpha} \sec(a\alpha)}{2a^2} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\sqrt{-a^2}\, x} \sec(ax)}{1}\, dx$$

Which simplifies to

$$u_2 = \int e^{\sqrt{-a^2}\, x} \sec(ax)\, dx$$

Hence

$$u_2 = \int_0^x e^{\sqrt{-a^2}\, \alpha} \sec(a\alpha)\, d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x \frac{\sqrt{-a^2}\,e^{-\sqrt{-a^2}\,\alpha}\sec(a\alpha)}{2a^2}d\alpha\right)e^{\sqrt{-a^2}\,x} + \frac{\left(\int_0^x e^{\sqrt{-a^2}\,\alpha}\sec(a\alpha)\,d\alpha\right)\sqrt{-a^2}\,e^{-\sqrt{-a^2}\,x}}{2a^2}$$

Which simplifies to

$$y_p(x) = -\frac{\sqrt{-a^2}\left(\left(\int_0^x e^{-\sqrt{-a^2}\,\alpha}\sec(a\alpha)\,d\alpha\right)e^{\sqrt{-a^2}\,x} - \left(\int_0^x e^{\sqrt{-a^2}\,\alpha}\sec(a\alpha)\,d\alpha\right)e^{-\sqrt{-a^2}\,x}\right)}{2a^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\sqrt{-a^2}\,x} + \frac{c_2\sqrt{-a^2}\,e^{-\sqrt{-a^2}\,x}}{2a^2}\right) \\ &\quad + \left(-\frac{\sqrt{-a^2}\left(\left(\int_0^x e^{-\sqrt{-a^2}\,\alpha}\sec(a\alpha)\,d\alpha\right)e^{\sqrt{-a^2}\,x} - \left(\int_0^x e^{\sqrt{-a^2}\,\alpha}\sec(a\alpha)\,d\alpha\right)e^{-\sqrt{-a^2}\,x}\right)}{2a^2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\sqrt{-a^2}\,x} + \frac{c_2\sqrt{-a^2}\,e^{-\sqrt{-a^2}\,x}}{2a^2} - \frac{\sqrt{-a^2}\left(\left(\int_0^x e^{-\sqrt{-a^2}\,\alpha}\sec(a\alpha)\,d\alpha\right)e^{\sqrt{-a^2}\,x} - \left(\int_0^x e^{\sqrt{-a^2}\,\alpha}\sec(a\alpha)\,d\alpha\right)e^{-\sqrt{-a^2}\,x}\right)}{2a^2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\sqrt{-a^2}\,x} + \frac{c_2\sqrt{-a^2}\,e^{-\sqrt{-a^2}\,x}}{2a^2} - \frac{\sqrt{-a^2}\left(\left(\int_0^x e^{-\sqrt{-a^2}\,\alpha}\sec(a\alpha)\,d\alpha\right)e^{\sqrt{-a^2}\,x} - \left(\int_0^x e^{\sqrt{-a^2}\,\alpha}\sec(a\alpha)\,d\alpha\right)e^{-\sqrt{-a^2}\,x}\right)}{2a^2}$$

Verified OK.

12.12.3 Maple step by step solution

Let's solve

$y'' + a^2 y = \sec(ax)$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $a^2 + r^2 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4a^2}\right)}{2}$
- Roots of the characteristic polynomial

  $r = \left(\sqrt{-a^2}, -\sqrt{-a^2}\right)$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{\sqrt{-a^2}\,x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\sqrt{-a^2}\,x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\sqrt{-a^2}\,x} + c_2 e^{-\sqrt{-a^2}\,x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(ax)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\sqrt{-a^2}\,x} & e^{-\sqrt{-a^2}\,x} \\ \sqrt{-a^2}\, e^{\sqrt{-a^2}\,x} & -\sqrt{-a^2}\, e^{-\sqrt{-a^2}\,x} \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = -2\sqrt{-a^2}$
- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{e^{\sqrt{-a^2}\,x}\left(\int e^{-\sqrt{-a^2}\,x}\sec(ax)dx\right) - e^{-\sqrt{-a^2}\,x}\left(\int e^{\sqrt{-a^2}\,x}\sec(ax)dx\right)}{2\sqrt{-a^2}}$$

- Compute integrals

$$y_p(x) = \frac{e^{\sqrt{-a^2}\,x}\left(\int e^{-\sqrt{-a^2}\,x}\sec(ax)dx\right) - e^{-\sqrt{-a^2}\,x}\left(\int e^{\sqrt{-a^2}\,x}\sec(ax)dx\right)}{2\sqrt{-a^2}}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{\sqrt{-a^2}\,x} + c_2 e^{-\sqrt{-a^2}\,x} + \frac{e^{\sqrt{-a^2}\,x}\left(\int e^{-\sqrt{-a^2}\,x}\sec(ax)dx\right) - e^{-\sqrt{-a^2}\,x}\left(\int e^{\sqrt{-a^2}\,x}\sec(ax)dx\right)}{2\sqrt{-a^2}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$2)+a^2*y(x)=sec(a*x),y(x), singsol=all)
```

$$y(x) = \sin(ax)\,c_2 + \cos(ax)\,c_1 + \frac{x\sin(ax)\,a - \ln(\sec(ax))\cos(ax)}{a^2}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 39

```
DSolve[y''[x]+a^2*y[x]==Sec[a*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\cos(ax)\left(\log(\cos(ax)) + a^2 c_1\right) + a(x + ac_2)\sin(ax)}{a^2}$$

12.13 problem 13

Internal problem ID [2186]
Internal file name [`OUTPUT/2186_Monday_February_26_2024_09_18_13_AM_92955357/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 13.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 2y'' + y' = \mathrm{e}^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 2y'' + y' = 0$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2\mathrm{e}^x + x\,\mathrm{e}^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \mathrm{e}^{x} \\
y_3 &= x\,\mathrm{e}^{x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' - 2y'' + y' = \mathrm{e}^{2x}
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{2x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{2x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1 \mathrm{e}^{2x} = \mathrm{e}^{2x}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = \frac{1}{2}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = \frac{\mathrm{e}^{2x}}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 + c_2 \mathrm{e}^{x} + x\,\mathrm{e}^{x} c_3) + \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{x}(c_3 x + c_2) + c_1 + \frac{\mathrm{e}^{2x}}{2}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{x}(c_3 x + c_2) + c_1 + \frac{\mathrm{e}^{2x}}{2} \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{x}(c_3 x + c_2) + c_1 + \frac{\mathrm{e}^{2x}}{2}
$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 2*(diff(_b(_a), _a))-_b(_a)+e
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)+diff(y(x),x)=exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{e^{2x}}{2} + (c_1(x-1) + c_2)\, e^{x} + c_3$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 30

```
DSolve[y'''[x]-2*y''[x]+y'[x]==Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2x}}{2} + e^{x}(c_2(x-1) + c_1) + c_3$$

12.14 problem 14

Internal problem ID [2187]
Internal file name [`OUTPUT/2187_Monday_February_26_2024_09_18_13_AM_74506122/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 14.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' - 2y''' + y'' = x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y''' + y'' = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + c_3 e^x + c_4 x\, e^x$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= e^x \\
y_4 &= x\, e^x
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 2y''' + y'' = x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x\, e^x, e^x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^4 + A_2 x^3 + A_1 x^2$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12x^2A_3 + 6xA_2 - 48xA_3 + 2A_1 - 12A_2 + 24A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 3, A_2 = \frac{2}{3}, A_3 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{12}x^4 + \frac{2}{3}x^3 + 3x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2x + c_1 + c_3\mathrm{e}^x + c_4x\,\mathrm{e}^x) + \left(\frac{1}{12}x^4 + \frac{2}{3}x^3 + 3x^2\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^x(c_4x + c_3) + c_2x + c_1 + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^x(c_4x + c_3) + c_2x + c_1 + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^x(c_4x + c_3) + c_2x + c_1 + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a^2-_b(_a)+2*(diff(_b(_a), _
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$3)+diff(y(x),x$2)=x^2,y(x), singsol=all)
```

$$y(x) = (c_1x - 2c_1 + c_2)\,\mathrm{e}^{x} + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 + c_3x + c_4$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 46

```
DSolve[y''''[x]-2*y'''[x]+y''[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 + c_4x + c_1e^x + c_2e^x(x-2) + c_3$$

12.15 problem 15

Internal problem ID [2188]
Internal file name [`OUTPUT/2188_Monday_February_26_2024_09_18_13_AM_42158125/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 15.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 3y'' - 4y' = \mathrm{e}^{2x} + \sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' - 4y' = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 - 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 4$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = 1$$
$$y_3 = \mathrm{e}^{4x}$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' - 4y' = \mathrm{e}^{2x} + \sin(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{2x} + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{2x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-x}, \mathrm{e}^{4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{2x} + A_2 \cos(x) + A_3 \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 \mathrm{e}^{2x} + 5A_2 \sin(x) - 5A_3 \cos(x) + 3A_2 \cos(x) + 3A_3 \sin(x) = \mathrm{e}^{2x} + \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{12}, A_2 = \frac{5}{34}, A_3 = \frac{3}{34}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{2x}}{12} + \frac{5\cos(x)}{34} + \frac{3\sin(x)}{34}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2 + e^{4x} c_3) + \left(-\frac{e^{2x}}{12} + \frac{5\cos(x)}{34} + \frac{3\sin(x)}{34}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + e^{4x} c_3 - \frac{e^{2x}}{12} + \frac{5\cos(x)}{34} + \frac{3\sin(x)}{34} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 + e^{4x} c_3 - \frac{e^{2x}}{12} + \frac{5\cos(x)}{34} + \frac{3\sin(x)}{34}$$

Verified OK.

12.15.1 Maple step by step solution

Let's solve

$y''' - 3y'' - 4y' = e^{2x} + \sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = e^{2x} + \sin(x) + 3y_3(x) + 4y_2(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = e^{2x} + \sin(x) + 3y_3(x) + 4y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ e^{2x} + \sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ e^{2x} + \sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{4x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & 1 & \frac{e^{4x}}{16} \\ -e^{-x} & 0 & \frac{e^{4x}}{4} \\ e^{-x} & 0 & e^{4x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & 1 & \frac{e^{4x}}{16} \\ -e^{-x} & 0 & \frac{e^{4x}}{4} \\ e^{-x} & 0 & e^{4x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & \frac{1}{16} \\ -1 & 0 & \frac{1}{4} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{4\,e^{-x}}{5} + \frac{3}{4} + \frac{e^{4x}}{20} & \frac{e^{-x}}{5} - \frac{1}{4} + \frac{e^{4x}}{20} \\ 0 & \frac{4\,e^{-x}}{5} + \frac{e^{4x}}{5} & -\frac{e^{-x}}{5} + \frac{e^{4x}}{5} \\ 0 & -\frac{4\,e^{-x}}{5} + \frac{4\,e^{4x}}{5} & \frac{e^{-x}}{5} + \frac{4\,e^{4x}}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \begin{bmatrix} \frac{3\sin(x)}{34} + \frac{5\cos(x)}{34} + \frac{19\,\mathrm{e}^{4x}}{680} - \frac{\mathrm{e}^{2x}}{12} - \frac{1}{8} + \frac{\mathrm{e}^{-x}}{30} \\ \frac{19\,\mathrm{e}^{4x}}{170} - \frac{\mathrm{e}^{2x}}{6} - \frac{5\sin(x)}{34} + \frac{3\cos(x)}{34} - \frac{\mathrm{e}^{-x}}{30} \\ \frac{38\,\mathrm{e}^{4x}}{85} - \frac{\mathrm{e}^{2x}}{3} - \frac{3\sin(x)}{34} - \frac{5\cos(x)}{34} + \frac{\mathrm{e}^{-x}}{30} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \begin{bmatrix} \frac{3\sin(x)}{34} + \frac{5\cos(x)}{34} + \frac{19\,\mathrm{e}^{4x}}{680} - \frac{\mathrm{e}^{2x}}{12} - \frac{1}{8} + \frac{\mathrm{e}^{-x}}{30} \\ \frac{19\,\mathrm{e}^{4x}}{170} - \frac{\mathrm{e}^{2x}}{6} - \frac{5\sin(x)}{34} + \frac{3\cos(x)}{34} - \frac{\mathrm{e}^{-x}}{30} \\ \frac{38\,\mathrm{e}^{4x}}{85} - \frac{\mathrm{e}^{2x}}{3} - \frac{3\sin(x)}{34} - \frac{5\cos(x)}{34} + \frac{\mathrm{e}^{-x}}{30} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(4080c_1+136)\mathrm{e}^{-x}}{4080} + \frac{(255c_3+114)\mathrm{e}^{4x}}{4080} + c_2 + \frac{5\cos(x)}{34} + \frac{3\sin(x)}{34} - \frac{\mathrm{e}^{2x}}{12} - \frac{1}{8}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 4*_b(_a)+3*(diff(_b(_a), _a))
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)-4*diff(y(x),x)=exp(2*x)+sin(x),y(x), singsol=all)
```

$$y(x) = \frac{c_2 e^{4x}}{4} - e^{-x} c_1 + \frac{3\sin(x)}{34} - \frac{e^{2x}}{12} + \frac{5\cos(x)}{34} + c_3$$

✓ Solution by Mathematica

Time used: 0.267 (sec). Leaf size: 49

```
DSolve[y'''[x]-3*y''[x]-4*y'[x]==Exp[2*x]+Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{2x}}{12} + \frac{3\sin(x)}{34} + \frac{5\cos(x)}{34} + c_1\left(-e^{-x}\right) + \frac{1}{4}c_2 e^{4x} + c_3$$

12.16 problem 16

Internal problem ID [2189]
Internal file name [`OUTPUT/2189_Monday_February_26_2024_09_18_14_AM_83357581/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = \frac{e^{x}}{(1-x)^{2}}}$$

12.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = \frac{e^{x}}{(x-1)^{2}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x e^x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 x e^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^x \\ y_2 &= x e^x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,e^{2x}}{(x-1)^2}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{(x-1)^2}dx$$

Hence

$$u_1 = -\ln(x-1) + \frac{1}{x-1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2x}}{(x-1)^2}}{e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{(x-1)^2} dx$$

Hence

$$u_2 = -\frac{1}{x-1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\ln(x-1) + \frac{1}{x-1}\right) e^x - \frac{x\, e^x}{x-1}$$

Which simplifies to

$$y_p(x) = -(\ln(x-1) + 1)\, e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 x\, e^x) + (-(\ln(x-1) + 1)\, e^x) \end{aligned}$$

Which simplifies to

$$y = e^x(c_2 x + c_1) - (\ln(x-1) + 1)\, e^x$$

Summary

The solution(s) found are the following

$$y = e^x(c_2 x + c_1) - (\ln(x-1) + 1)\, e^x \tag{1}$$

Verification of solutions

$$y = e^x(c_2 x + c_1) - (\ln(x-1) + 1)\, e^x$$

Verified OK.

12.16.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p \, dx} \\ &= e^{\int -2 \, dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{\mathrm{e}^{-x}\mathrm{e}^{x}}{(x-1)^2} \\ \left(\mathrm{e}^{-x}y\right)'' &= \frac{\mathrm{e}^{-x}\mathrm{e}^{x}}{(x-1)^2} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x}y\right)' = -\frac{1}{x-1} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x}y\right) = c_1 x - \ln(x-1) + c_2$$

Hence the solution is

$$y = \frac{c_1 x - \ln(x-1) + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x} - \mathrm{e}^{x} \ln(x-1)$$

Summary

The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x} - \mathrm{e}^{x} \ln(x-1) \tag{1}$$

Verification of solutions

$$y = c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x} - \mathrm{e}^{x} \ln(x-1)$$

Verified OK.

12.16.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 450: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2}{1} \, dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + c_2 x\, \mathrm{e}^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{x} \\ y_2 &= x\,\mathrm{e}^{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{x} & x\,\mathrm{e}^{x} \\ \frac{d}{dx}(\mathrm{e}^{x}) & \frac{d}{dx}(x\,\mathrm{e}^{x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{x} & x\,\mathrm{e}^{x} \\ \mathrm{e}^{x} & x\,\mathrm{e}^{x} + \mathrm{e}^{x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{x})(x\,\mathrm{e}^{x} + \mathrm{e}^{x}) - (x\,\mathrm{e}^{x})(\mathrm{e}^{x})$$

Which simplifies to

$$W = \mathrm{e}^{2x}$$

Which simplifies to

$$W = \mathrm{e}^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,e^{2x}}{(x-1)^2}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{(x-1)^2}dx$$

Hence

$$u_1 = -\ln(x-1) + \frac{1}{x-1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2x}}{(x-1)^2}}{e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{(x-1)^2}dx$$

Hence

$$u_2 = -\frac{1}{x-1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\ln(x-1) + \frac{1}{x-1}\right)e^x - \frac{x\,e^x}{x-1}$$

Which simplifies to

$$y_p(x) = -(\ln(x-1)+1)\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 x\,e^x) + (-(\ln(x-1)+1)\,e^x) \end{aligned}$$

Which simplifies to

$$y = e^{x}(c_2x + c_1) - (\ln(x-1) + 1)\, e^{x}$$

Summary
The solution(s) found are the following

$$y = e^{x}(c_2x + c_1) - (\ln(x-1) + 1)\, e^{x} \tag{1}$$

Verification of solutions

$$y = e^{x}(c_2x + c_1) - (\ln(x-1) + 1)\, e^{x}$$

Verified OK.

12.16.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = \frac{e^{x}}{(x-1)^2}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{x} + c_2 x\, e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = \frac{e^x}{(x-1)^2}\right]$$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = e^{2x}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = e^x\left(-\left(\int \frac{x}{(x-1)^2}dx\right) + \left(\int \frac{1}{(x-1)^2}dx\right)x\right)$$

- Compute integrals

  $$y_p(x) = -(\ln(x-1) + 1) e^x$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^x + c_2 x e^x - (\ln(x-1) + 1) e^x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=exp(x)/(1-x)^2,y(x), singsol=all)
```

$$y(x) = e^x(-1 + c_1 x - \ln(x-1) + c_2)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 23

```
DSolve[y''[x]-2*y'[x]+y[x]==Exp[x]/(1-x)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{x}(-\log(x-1)+c_2x-1+c_1)$$

12.17 problem 17

Internal problem ID [2190]
Internal file name [`OUTPUT/2190_Monday_February_26_2024_09_18_14_AM_51875128/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''-3y'+2y=\sin\left(\mathrm{e}^{-x}\right)}$$

12.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=1, B=-3, C=2, f(x)=\sin\left(\mathrm{e}^{-x}\right)$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-3y'+2y=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)}\\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{2x}$$
$$y_2 = e^{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{x} \\ \frac{d}{dx}\left(e^{2x}\right) & \frac{d}{dx}\left(e^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{x} \\ 2\,e^{2x} & e^{x} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(e^{x}\right) - \left(e^{x}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = -e^{x} e^{2x}$$

Which simplifies to

$$W = -e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^x \sin(e^{-x})}{-e^{3x}} dx$$

Which simplifies to

$$u_1 = -\int -\sin(e^{-x}) e^{-2x} dx$$

Hence

$$u_1 = -\frac{e^{-x} \tan\left(\frac{e^{-x}}{2}\right)^2 - e^{-x} + 2\tan\left(\frac{e^{-x}}{2}\right)}{1 + \tan\left(\frac{e^{-x}}{2}\right)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \sin(e^{-x})}{-e^{3x}} dx$$

Which simplifies to

$$u_2 = \int -\sin(e^{-x}) e^{-x} dx$$

Hence

$$u_2 = -\cos(e^{-x})$$

Which simplifies to

$$u_1 = e^{-x} \cos(e^{-x}) - \sin(e^{-x})$$
$$u_2 = -\cos(e^{-x})$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(e^{-x} \cos(e^{-x}) - \sin(e^{-x})\right) e^{2x} - \cos(e^{-x}) e^{x}$$

Which simplifies to

$$y_p(x) = -e^{2x} \sin(e^{-x})$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{2x} + c_2 e^{x}\right) + \left(-e^{2x} \sin\left(e^{-x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{2x} + c_2 e^{x} - e^{2x} \sin\left(e^{-x}\right) \tag{1}
$$

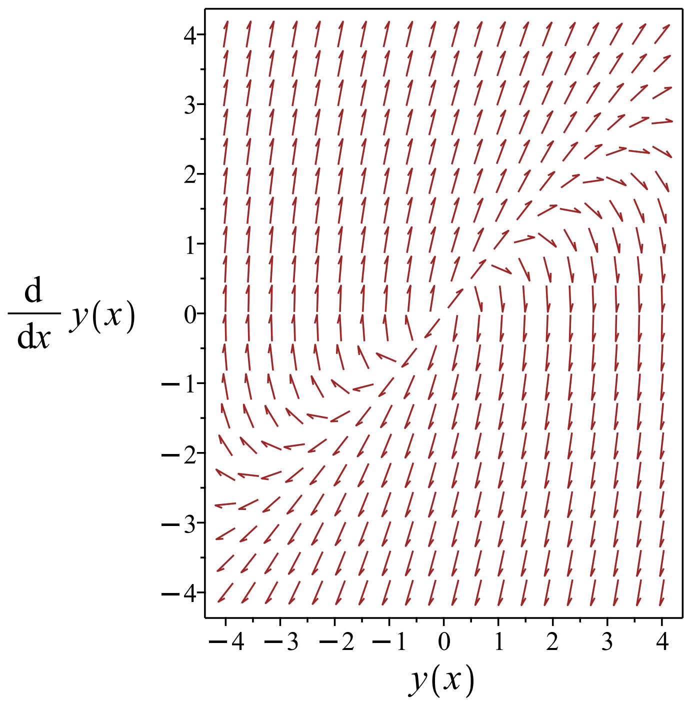


Figure 612: Slope field plot

Verification of solutions

$$
y = c_1 e^{2x} + c_2 e^{x} - e^{2x} \sin\left(e^{-x}\right)
$$

Verified OK.

12.17.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
y'' - 3y' + 2y &= 0 &(1)\\
Ay'' + By' + Cy &= 0 &(2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= -3 &(3)\\
C &= 2
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} &(5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1\\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 452: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(\mathrm{e}^{x}))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$
$$y_2 = e^{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{2x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{vmatrix}$$

Therefore

$$W = (e^x)\left(2\,e^{2x}\right) - \left(e^{2x}\right)(e^x)$$

Which simplifies to

$$W = e^x e^{2x}$$

Which simplifies to

$$W = e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{2x} \sin(e^{-x})}{e^{3x}} dx$$

Which simplifies to

$$u_1 = -\int \sin(e^{-x}) e^{-x} dx$$

Hence

$$u_1 = -\cos(e^{-x})$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{x} \sin(e^{-x})}{e^{3x}} dx$$

Which simplifies to

$$u_2 = \int \sin(e^{-x}) e^{-2x} dx$$

Hence

$$u_2 = \frac{-e^{-x} \tan\left(\frac{e^{-x}}{2}\right)^2 + e^{-x} - 2\tan\left(\frac{e^{-x}}{2}\right)}{1 + \tan\left(\frac{e^{-x}}{2}\right)^2}$$

Which simplifies to

$$u_1 = -\cos(e^{-x})$$
$$u_2 = e^{-x} \cos(e^{-x}) - \sin(e^{-x})$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (e^{-x} \cos(e^{-x}) - \sin(e^{-x})) e^{2x} - \cos(e^{-x}) e^{x}$$

Which simplifies to

$$y_p(x) = -e^{2x} \sin(e^{-x})$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^x + c_2 e^{2x}\right) + \left(-e^{2x} \sin\left(e^{-x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin\left(e^{-x}\right) \tag{1}
$$

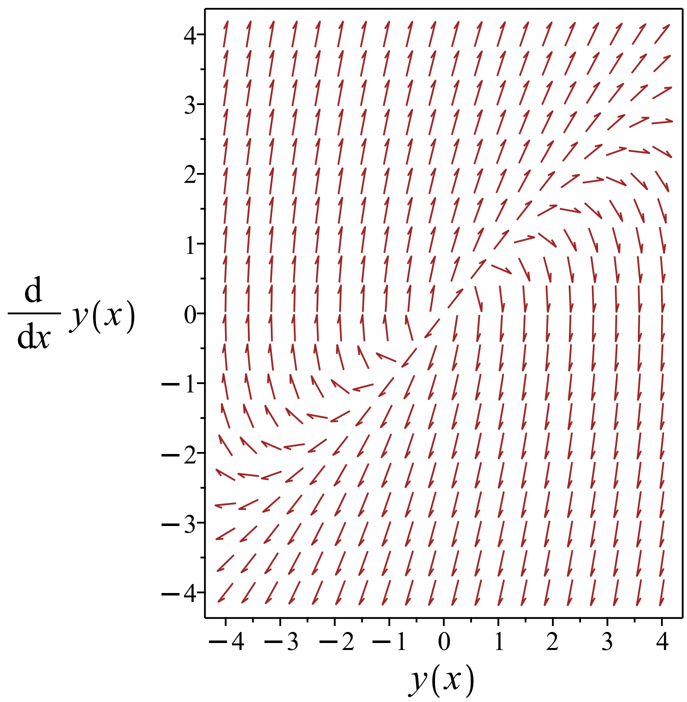


Figure 613: Slope field plot

Verification of solutions

$$
y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin\left(e^{-x}\right)
$$

Verified OK.

12.17.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = \sin(e^{-x})$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(e^{-x})\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{3x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^x\left(\int \sin(e^{-x})\,e^{-x}dx\right) + e^{2x}\left(\int \sin(e^{-x})\,e^{-2x}dx\right)$

- Compute integrals

  $y_p(x) = -e^{2x} \sin(e^{-x})$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin(e^{-x})$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=sin(exp(-x)),y(x), singsol=all)
```

$$y(x) = \left(e^x c_1 - e^x \sin\left(e^{-x}\right) + c_2\right) e^x$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 29

```
DSolve[y''[x]-3*y'[x]+2*y[x]==Sin[Exp[-x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(-e^x \sin\left(e^{-x}\right) + c_2 e^x + c_1\right)$$

12.18 problem 18

Internal problem ID [2191]
Internal file name [`OUTPUT/2191_Monday_February_26_2024_09_18_15_AM_84129772/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y = \tan(x)\sec(x)}$$

12.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \tan(x)\sec(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\
&= \pm 2i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2i \\
\lambda_2 &= -2i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(2x) \\ y_2 &= \sin(2x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}(\sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(2\cos(2x)) - (\sin(2x))(-2\sin(2x))$$

Which simplifies to

$$W = 2\cos(2x)^2 + 2\sin(2x)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(2x)\tan(x)\sec(x)}{2}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\sin(x)\,dx$$

Hence

$$u_1 = \sin(x) - \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\tan(x)\sec(x)}{2}\,dx$$

Which simplifies to

$$u_2 = \int \left(\sin(x) - \frac{\tan(x)\sec(x)}{2}\right)dx$$

Hence

$$u_2 = -\frac{\sec(x)}{2} - \cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\sin(x) - \ln(\sec(x) + \tan(x)))\cos(2x) + \left(-\frac{\sec(x)}{2} - \cos(x)\right)\sin(2x)$$

Which simplifies to

$$y_p(x) = \left(-2\cos(x)^2 + 1\right)\ln(\sec(x) + \tan(x)) - 2\sin(x)$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\left(-2\cos(x)^2 + 1\right)\ln(\sec(x) + \tan(x)) - 2\sin(x)\right)$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \left(-2\cos(x)^2 + 1\right)\ln(\sec(x) + \tan(x)) - 2\sin(x) \quad (1)$$

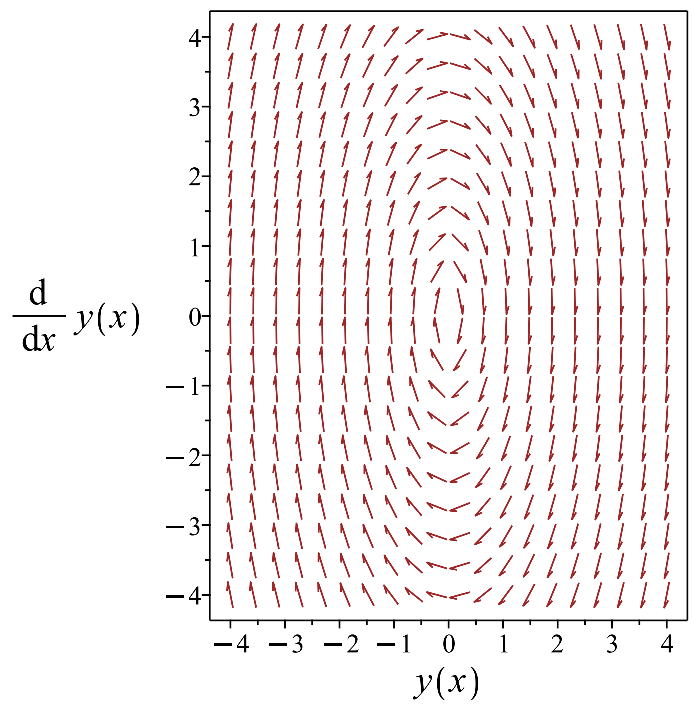


Figure 614: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \left(-2\cos(x)^2 + 1\right)\ln(\sec(x) + \tan(x)) - 2\sin(x)$$

Verified OK.

12.18.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 454: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(2x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\
&= \cos(2x) \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$
$$y_2 = \frac{\sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ -2\sin(2x) & \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(\cos(2x)) - \left(\frac{\sin(2x)}{2}\right)(-2\sin(2x))$$

Which simplifies to

$$W = \cos(2x)^2 + \sin(2x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(2x)\tan(x)\sec(x)}{2}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\sin(x)\,dx$$

Hence

$$u_1 = \sin(x) - \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\tan(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int (2\sin(x) - \tan(x)\sec(x))\,dx$$

Hence

$$u_2 = -\sec(x) - 2\cos(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\sin(x) - \ln(\sec(x) + \tan(x)))\cos(2x) + \frac{(-\sec(x) - 2\cos(x))\sin(2x)}{2}$$

Which simplifies to

$$y_p(x) = \left(-2\cos(x)^2 + 1\right)\ln(\sec(x) + \tan(x)) - 2\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\left(-2\cos(x)^2 + 1\right)\ln(\sec(x) + \tan(x)) - 2\sin(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \left(-2\cos(x)^2 + 1\right)\ln(\sec(x) + \tan(x)) - 2\sin(x) \quad (1)$$

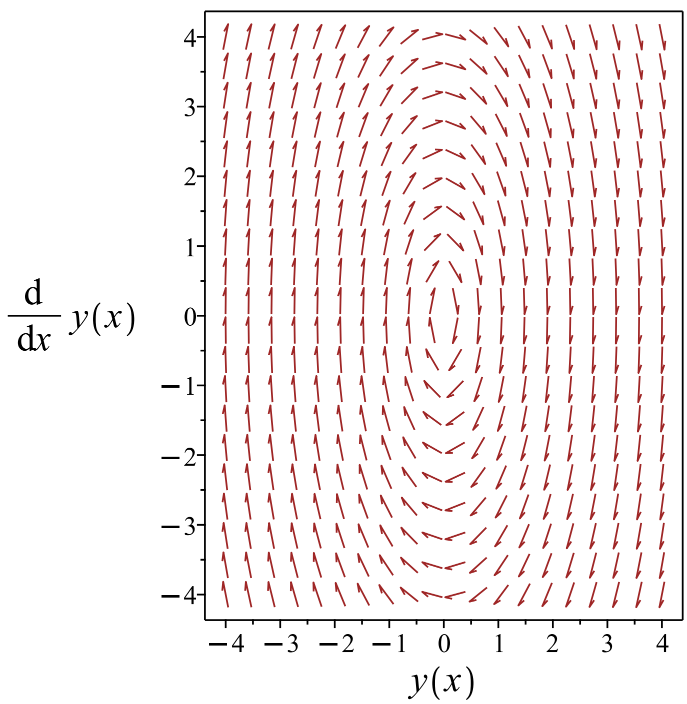


Figure 615: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \left(-2\cos(x)^2 + 1\right)\ln(\sec(x) + \tan(x)) - 2\sin(x)$$

Verified OK.

12.18.3 Maple step by step solution

Let's solve

$y'' + 4y = \tan(x)\sec(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan(x)\sec(x)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(2x)\left(\int \tan(x)\sin(x)\,dx\right) + \frac{\sin(2x)\left(\int (2\sin(x)-\tan(x)\sec(x))dx\right)}{2}$$

- Compute integrals

$$y_p(x) = \left(-2\cos(x)^2+1\right)\ln(\sec(x)+\tan(x)) - 2\sin(x)$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(2x) + c_2\sin(2x) + \left(-2\cos(x)^2+1\right)\ln(\sec(x)+\tan(x)) - 2\sin(x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$2)+4*y(x)=sec(x)*tan(x),y(x), singsol=all)
```

$$y(x) = \left(-2\cos(x)^2+1\right)\ln(\sec(x)+\tan(x)) + 2c_1\cos(x)^2 - c_1 + 2\sin(x)\cos(x)c_2 - 2\sin(x)$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 33

```
DSolve[y''[x]+4*y[x]==Sec[x]*Tan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(2x)(-\operatorname{arctanh}(\sin(x))) + c_1\cos(2x) + 2\sin(x)(-1+c_2\cos(x))$$

12.19 problem 19

Internal problem ID [2192]
Internal file name [`OUTPUT/2192_Monday_February_26_2024_09_18_16_AM_37437282/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y = \sin(2x)\,\mathrm{e}^{-x}}$$

12.19.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -2, f(x) = \sin(2x)\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-2)}\\ &= \pm\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{2}\\ \lambda_2 &= -\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{2}\\ \lambda_2 &= -\sqrt{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{\left(\sqrt{2}\right)x} + c_2 e^{\left(-\sqrt{2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{2}\,x} + c_2 \mathrm{e}^{-\sqrt{2}\,x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{\sqrt{2}\,x} + c_2 \mathrm{e}^{-\sqrt{2}\,x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x)\,\mathrm{e}^{-x}, \sin(2x)\,\mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{\sqrt{2}\,x}, \mathrm{e}^{-\sqrt{2}\,x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x)\,\mathrm{e}^{-x} + A_2 \sin(2x)\,\mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(2x)\,\mathrm{e}^{-x} + 4A_1 \sin(2x)\,\mathrm{e}^{-x} - 5A_2 \sin(2x)\,\mathrm{e}^{-x} - 4A_2 \cos(2x)\,\mathrm{e}^{-x} = \sin(2x)\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{41}, A_2 = -\frac{5}{41}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\cos(2x)\,\mathrm{e}^{-x}}{41} - \frac{5\sin(2x)\,\mathrm{e}^{-x}}{41}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{\sqrt{2}\,x} + c_2 \mathrm{e}^{-\sqrt{2}\,x}\right) + \left(\frac{4\cos(2x)\,\mathrm{e}^{-x}}{41} - \frac{5\sin(2x)\,\mathrm{e}^{-x}}{41}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\sqrt{2}\, x} + c_2 e^{-\sqrt{2}\, x} + \frac{4 \cos(2x)\, e^{-x}}{41} - \frac{5 \sin(2x)\, e^{-x}}{41} \tag{1}$$

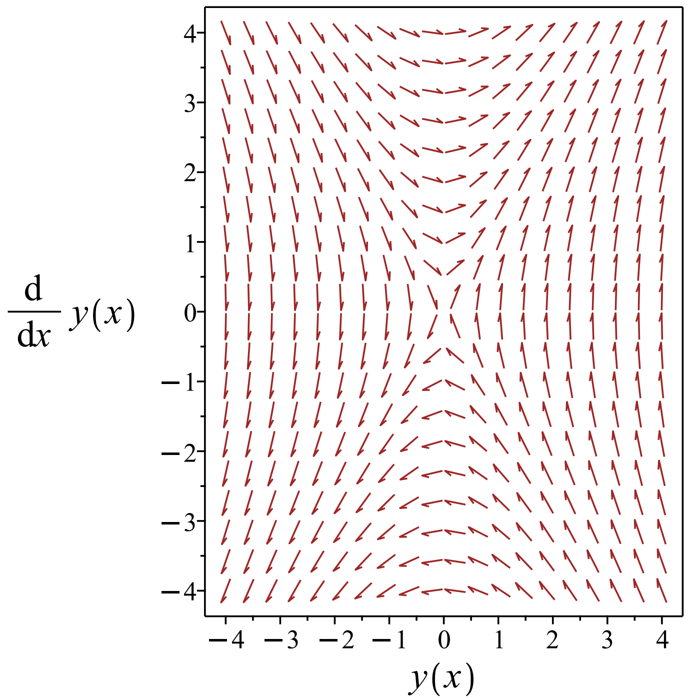


Figure 616: Slope field plot

Verification of solutions

$$y = c_1 e^{\sqrt{2}\, x} + c_2 e^{-\sqrt{2}\, x} + \frac{4 \cos(2x)\, e^{-x}}{41} - \frac{5 \sin(2x)\, e^{-x}}{41}$$

Verified OK.

12.19.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 456: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\sqrt{2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-\sqrt{2}\, x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\sqrt{2}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= e^{-\sqrt{2}\, x} \int \frac{1}{e^{-2\sqrt{2}\, x}}\, dx \\
&= e^{-\sqrt{2}\, x} \left(\frac{\sqrt{2}\, e^{2\sqrt{2}\, x}}{4} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-\sqrt{2}\, x} \right) + c_2 \left(e^{-\sqrt{2}\, x} \left(\frac{\sqrt{2}\, e^{2\sqrt{2}\, x}}{4} \right) \right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-\sqrt{2}\, x} + \frac{c_2 \sqrt{2}\, e^{\sqrt{2}\, x}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x)\,\mathrm{e}^{-x}, \sin(2x)\,\mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\,\mathrm{e}^{\sqrt{2}\,x}}{4}, \mathrm{e}^{-\sqrt{2}\,x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x)\,\mathrm{e}^{-x} + A_2 \sin(2x)\,\mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(2x)\,\mathrm{e}^{-x} + 4A_1 \sin(2x)\,\mathrm{e}^{-x} - 5A_2 \sin(2x)\,\mathrm{e}^{-x} - 4A_2 \cos(2x)\,\mathrm{e}^{-x} = \sin(2x)\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{41}, A_2 = -\frac{5}{41}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4\cos(2x)\,\mathrm{e}^{-x}}{41} - \frac{5\sin(2x)\,\mathrm{e}^{-x}}{41}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-\sqrt{2}\,x} + \frac{c_2\sqrt{2}\,\mathrm{e}^{\sqrt{2}\,x}}{4}\right) + \left(\frac{4\cos(2x)\,\mathrm{e}^{-x}}{41} - \frac{5\sin(2x)\,\mathrm{e}^{-x}}{41}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\sqrt{2}\,x} + \frac{c_2 \sqrt{2}\, e^{\sqrt{2}\,x}}{4} + \frac{4\cos(2x)\, e^{-x}}{41} - \frac{5\sin(2x)\, e^{-x}}{41} \tag{1}$$

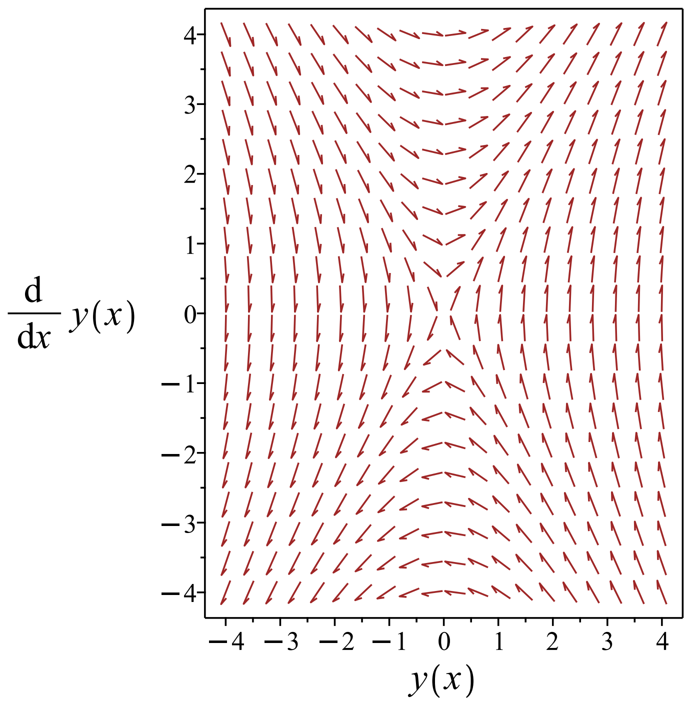


Figure 617: Slope field plot

Verification of solutions

$$y = c_1 e^{-\sqrt{2}\,x} + \frac{c_2 \sqrt{2}\, e^{\sqrt{2}\,x}}{4} + \frac{4\cos(2x)\, e^{-x}}{41} - \frac{5\sin(2x)\, e^{-x}}{41}$$

Verified OK.

12.19.3 Maple step by step solution

Let's solve

$y'' - 2y = \sin(2x)\, e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{8}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{2}, -\sqrt{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{\sqrt{2}\, x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\sqrt{2}\, x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{\sqrt{2}\, x} + c_2 \mathrm{e}^{-\sqrt{2}\, x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(2x)\, \mathrm{e}^{-x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{\sqrt{2}\, x} & \mathrm{e}^{-\sqrt{2}\, x} \\ \sqrt{2}\, \mathrm{e}^{\sqrt{2}\, x} & -\sqrt{2}\, \mathrm{e}^{-\sqrt{2}\, x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = -2\sqrt{2}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\sqrt{2}\left(\mathrm{e}^{-\sqrt{2}\, x}\left(\int \mathrm{e}^{x\left(\sqrt{2}-1\right)} \sin(2x) dx\right) - \mathrm{e}^{\sqrt{2}\, x}\left(\int \mathrm{e}^{-x\left(1+\sqrt{2}\right)} \sin(2x) dx\right)\right)}{4}$

  - Compute integrals

    $y_p(x) = \frac{\mathrm{e}^{-x}(4\cos(2x) - 5\sin(2x))}{41}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{\sqrt{2}\, x} + c_2 \mathrm{e}^{-\sqrt{2}\, x} + \frac{\mathrm{e}^{-x}(4\cos(2x) - 5\sin(2x))}{41}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)-2*y(x)=exp(-x)*sin(2*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\sqrt{2}\,x} c_1 + c_2 \mathrm{e}^{\sqrt{2}\,x} + \frac{4\,\mathrm{e}^{-x}\left(\cos\left(2x\right) - \frac{5\sin(2x)}{4}\right)}{41}$$

✓ Solution by Mathematica

Time used: 0.235 (sec). Leaf size: 57

```
DSolve[y''[x]-2*y[x]==Exp[-x]*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{5}{41} e^{-x} \sin(2x) + \frac{4}{41} e^{-x} \cos(2x) + c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}$$

12.20 problem 20

Internal problem ID [2193]
Internal file name [`OUTPUT/2193_Monday_February_26_2024_09_18_16_AM_16663807/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 9y = \sec(x)\csc(x)}$$

12.20.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = \sec(x)\csc(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(3x) \\ y_2 &= \sin(3x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}(\sin(3x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(3\cos(3x)) - (\sin(3x))(-3\sin(3x))$$

Which simplifies to

$$W = 3\cos(3x)^2 + 3\sin(3x)^2$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(3x)\sec(x)\csc(x)}{3}\,dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{\sec(x)}{3} + \frac{4\cos(x)}{3}\right) dx$$

Hence

$$u_1 = -\frac{4\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(3x)\sec(x)\csc(x)}{3}\,dx$$

Which simplifies to

$$u_2 = \int \left(\frac{4\cos(x)\cot(x)}{3} - \csc(x)\right) dx$$

Hence

$$u_2 = \frac{4\cos(x)}{3} + \frac{4\ln(\csc(x) - \cot(x))}{3} + \ln(\csc(x) + \cot(x))$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{3}\right)\cos(3x) \\ &+ \left(\frac{4\cos(x)}{3} + \frac{4\ln(\csc(x) - \cot(x))}{3} + \ln(\csc(x) + \cot(x))\right)\sin(3x) \end{aligned}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(\left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{3}\right)\cos(3x) \right. \\
&\quad \left. + \left(\frac{4\cos(x)}{3} + \frac{4\ln(\csc(x)-\cot(x))}{3} + \ln(\csc(x)+\cot(x))\right)\sin(3x)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \cos(3x) + c_2 \sin(3x) + \left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{3}\right)\cos(3x) \\
&\quad + \left(\frac{4\cos(x)}{3} + \frac{4\ln(\csc(x)-\cot(x))}{3} + \ln(\csc(x)+\cot(x))\right)\sin(3x)
\end{aligned} \tag{1}
$$

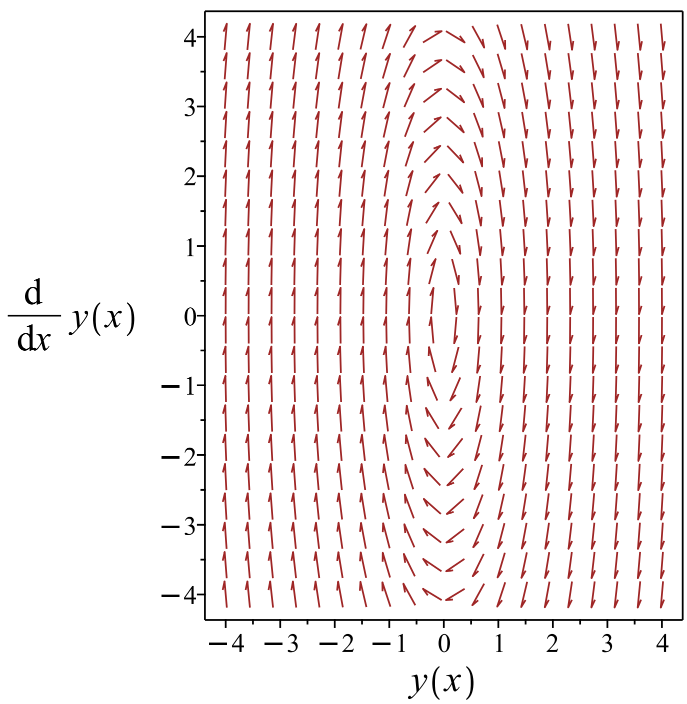


Figure 618: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{3}\right)\cos(3x) + \left(\frac{4\cos(x)}{3} + \frac{4\ln(\csc(x)-\cot(x))}{3} + \ln(\csc(x)+\cot(x))\right)\sin(3x)$$

Verified OK.

12.20.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 458: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\,dx \\ &= \cos(3x) \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3x)$$
$$y_2 = \frac{\sin(3x)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}\left(\frac{\sin(3x)}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ -3\sin(3x) & \cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(\cos(3x)) - \left(\frac{\sin(3x)}{3}\right)(-3\sin(3x))$$

Which simplifies to

$$W = \cos(3x)^2 + \sin(3x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(3x)\sec(x)\csc(x)}{3}}{1} dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{\sec(x)}{3} + \frac{4\cos(x)}{3}\right) dx$$

Hence

$$u_1 = -\frac{4\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(3x)\sec(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_2 = \int (4\cos(x)\cot(x) - 3\csc(x)) dx$$

Hence

$$u_2 = 4\cos(x) + 4\ln(\csc(x) - \cot(x)) + 3\ln(\csc(x) + \cot(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{3}\right)\cos(3x) + \frac{(4\cos(x) + 4\ln(\csc(x)-\cot(x)) + 3\ln(\csc(x)+\cot(x)))\sin(3x)}{3}$$

Which simplifies to

$$y_p(x) = \left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{3}\right)\cos(3x) + \left(\frac{4\cos(x)}{3} + \frac{4\ln(\csc(x)-\cot(x))}{3} + \ln(\csc(x)+\cot(x))\right)\sin(3x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(3x) + \frac{c_2\sin(3x)}{3}\right) + \left(\left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{3}\right)\cos(3x) \right. \\ &\quad \left. + \left(\frac{4\cos(x)}{3} + \frac{4\ln(\csc(x)-\cot(x))}{3} + \ln(\csc(x)+\cot(x))\right)\sin(3x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(3x) + \frac{c_2\sin(3x)}{3} + \left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{3}\right)\cos(3x) + \left(\frac{4\cos(x)}{3} + \frac{4\ln(\csc(x)-\cot(x))}{3} + \ln(\csc(x)+\cot(x))\right)\sin(3x) \tag{1}$$

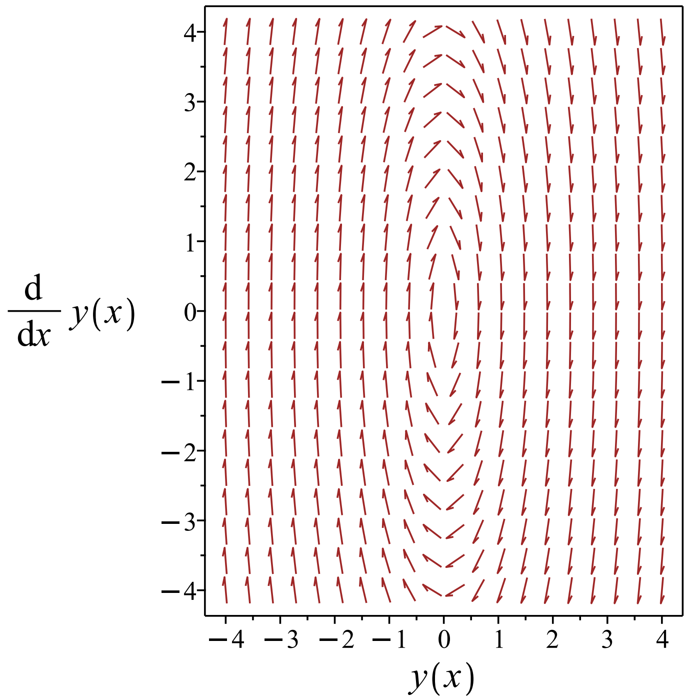


Figure 619: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{3}\right)\cos(3x) + \left(\frac{4\cos(x)}{3} + \frac{4\ln(\csc(x) - \cot(x))}{3} + \ln(\csc(x) + \cot(x))\right)\sin(3x)$$

Verified OK.

12.20.3 Maple step by step solution

Let's solve

$y'' + 9y = \sec(x)\csc(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

$r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)\csc(x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\cos(3x)\left(\int(-\sec(x)+4\cos(x))dx\right)}{3} + \frac{\sin(3x)\left(\int(4\cos(x)\cot(x)-3\csc(x))dx\right)}{3}$

- ○ Compute integrals

  $y_p(x) = \left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{3}\right)\cos(3x) + \left(\frac{4\cos(x)}{3} + \frac{4\ln(\csc(x)-\cot(x))}{3} + \ln(\csc(x)+\cot(x))\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + \left(-\frac{4\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{3}\right)\cos(3x) + \left(\frac{4\cos(x)}{3} + \frac{4\ln(\csc(x)-\cot(x))}{3} +\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)+9*y(x)=sec(x)*csc(x),y(x), singsol=all)
```

$$y(x) = \frac{\sin(x)\left(-1+4\cos(x)^2\right)\ln(\csc(x)-\cot(x))}{3} + \frac{\left(4\cos(x)^3-3\cos(x)\right)\ln(\sec(x)+\tan(x))}{3} + 4c_1\cos(x)^3 + 4\cos(x)^2\sin(x)\,c_2 + \frac{(-9c_1+8\sin(x))\cos(x)}{3} - \sin(x)\,c_2$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 65

```
DSolve[y''[x]+9*y[x]==Sec[x]*Csc[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(\cos(3x)\text{arctanh}(\sin(x)) + 4\sin(2x) + \sin(3x)\log\left(\sin\left(\frac{x}{2}\right)\right) + 3c_1\cos(3x) + 3c_2\sin(3x) - \sin(3x)\log\left(\cos\left(\frac{x}{2}\right)\right)\right)$$

12.21 problem 21

Internal problem ID [2194]
Internal file name [`OUTPUT/2194_Monday_February_26_2024_09_18_17_AM_33529992/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 9y = \csc(2x)}$$

12.21.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = \csc(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=0, C=9$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+9\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+9=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=0, C=9$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{0}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{0^2-(4)\,(1)\,(9)}\\&=\pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=+3i\\\lambda_2&=-3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=3i\\\lambda_2&=-3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha\pm i\beta$$

Where $\alpha=0$ and $\beta=3$. Therefore the final solution, when using Euler relation, can be written as

$$y=e^{\alpha x}(c_1\cos(\beta x)+c_2\sin(\beta x))$$

Which becomes

$$y=e^0(c_1\cos\left(3x\right)+c_2\sin\left(3x\right))$$

Or

$$y=c_1\cos\left(3x\right)+c_2\sin\left(3x\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3x)$$
$$y_2 = \sin(3x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}(\sin(3x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(3\cos(3x)) - (\sin(3x))(-3\sin(3x))$$

Which simplifies to

$$W = 3\cos(3x)^2 + 3\sin(3x)^2$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(3x)\csc(2x)}{3} dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{\sec(x)}{6} + \frac{2\cos(x)}{3}\right) dx$$

Hence

$$u_1 = -\frac{2\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(3x)\csc(2x)}{3} dx$$

Which simplifies to

$$u_2 = \int \left(\frac{2\cos(x)\cot(x)}{3} - \frac{\csc(x)}{2}\right) dx$$

Hence

$$u_2 = \frac{2\cos(x)}{3} + \frac{2\ln(\csc(x) - \cot(x))}{3} + \frac{\ln(\csc(x) + \cot(x))}{2}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(-\frac{2\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{6}\right)\cos(3x) \\ &+ \left(\frac{2\cos(x)}{3} + \frac{2\ln(\csc(x) - \cot(x))}{3} + \frac{\ln(\csc(x) + \cot(x))}{2}\right)\sin(3x) \end{aligned}$$

Which simplifies to

$$y_p(x) = \left(-\frac{2\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{6}\right)\cos(3x) + \frac{(4\cos(x)+4\ln(\csc(x)-\cot(x))+3\ln(\csc(x)+\cot(x)))\sin(3x)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(3x)+c_2\sin(3x)) + \left(\left(-\frac{2\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{6}\right)\cos(3x) + \frac{(4\cos(x)+4\ln(\csc(x)-\cot(x))+3\ln(\csc(x)+\cot(x)))\sin(3x)}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(3x)+c_2\sin(3x) + \left(-\frac{2\sin(x)}{3} + \frac{\ln(\sec(x)+\tan(x))}{6}\right)\cos(3x) + \frac{(4\cos(x)+4\ln(\csc(x)-\cot(x))+3\ln(\csc(x)+\cot(x)))\sin(3x)}{6} \tag{1}$$

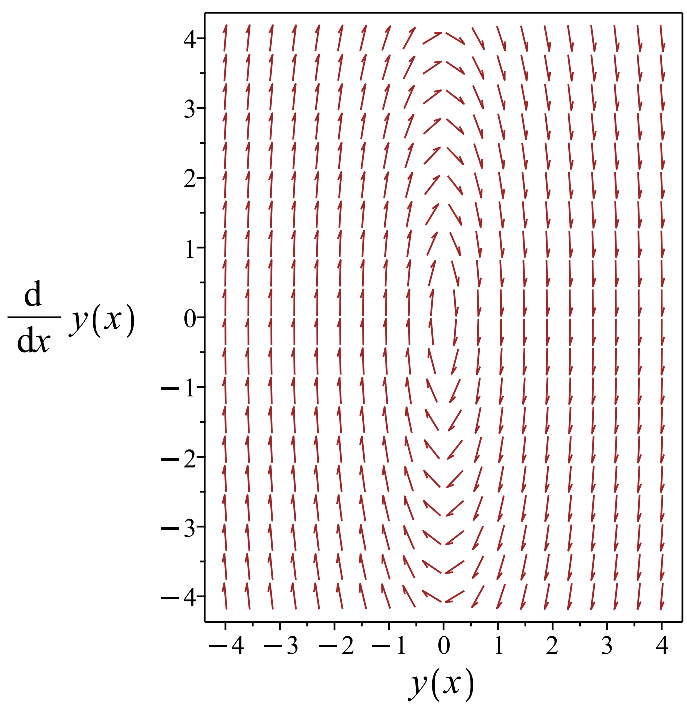


Figure 620: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \left(-\frac{2\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{6}\right)\cos(3x) + \frac{(4\cos(x) + 4\ln(\csc(x) - \cot(x)) + 3\ln(\csc(x) + \cot(x)))\sin(3x)}{6}$$

Verified OK.

12.21.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 460: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\,dx \\ &= \cos(3x)\left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3x)$$
$$y_2 = \frac{\sin(3x)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}\left(\frac{\sin(3x)}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ -3\sin(3x) & \cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(\cos(3x)) - \left(\frac{\sin(3x)}{3}\right)(-3\sin(3x))$$

Which simplifies to

$$W = \cos(3x)^2 + \sin(3x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(3x)\csc(2x)}{3}}{1} dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{\sec(x)}{6} + \frac{2\cos(x)}{3}\right) dx$$

Hence

$$u_1 = -\frac{2\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(3x)\csc(2x)}{1} dx$$

Which simplifies to

$$u_2 = \int \left(2\cos(x)\cot(x) - \frac{3\csc(x)}{2}\right) dx$$

Hence

$$u_2 = 2\cos(x) + 2\ln(\csc(x) - \cot(x)) + \frac{3\ln(\csc(x) + \cot(x))}{2}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = {} & \left(-\frac{2\sin(x)}{3}+\frac{\ln(\sec(x)+\tan(x))}{6}\right)\cos(3x) \\
& + \frac{\left(2\cos(x)+2\ln(\csc(x)-\cot(x))+\frac{3\ln(\csc(x)+\cot(x))}{2}\right)\sin(3x)}{3}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y_p(x) = {} & \left(-\frac{2\sin(x)}{3}+\frac{\ln(\sec(x)+\tan(x))}{6}\right)\cos(3x) \\
& + \frac{(4\cos(x)+4\ln(\csc(x)-\cot(x))+3\ln(\csc(x)+\cot(x)))\sin(3x)}{6}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\cos(3x)+\frac{c_2\sin(3x)}{3}\right) + \left(\left(-\frac{2\sin(x)}{3}+\frac{\ln(\sec(x)+\tan(x))}{6}\right)\cos(3x)\right. \\
&\quad \left. + \frac{(4\cos(x)+4\ln(\csc(x)-\cot(x))+3\ln(\csc(x)+\cot(x)))\sin(3x)}{6}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\cos(3x)+\frac{c_2\sin(3x)}{3}+\left(-\frac{2\sin(x)}{3}+\frac{\ln(\sec(x)+\tan(x))}{6}\right)\cos(3x) \\
& + \frac{(4\cos(x)+4\ln(\csc(x)-\cot(x))+3\ln(\csc(x)+\cot(x)))\sin(3x)}{6}
\end{aligned} \tag{1}
$$

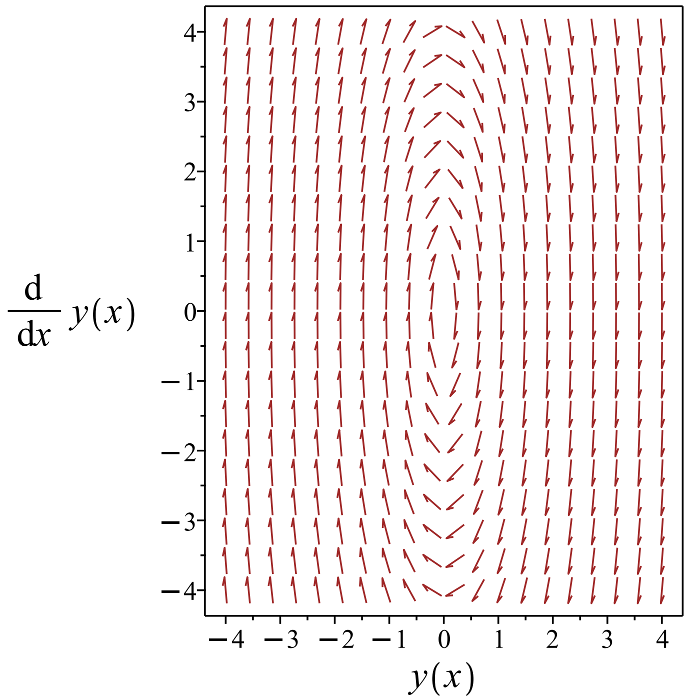


Figure 621: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \left(-\frac{2\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{6}\right)\cos(3x) + \frac{(4\cos(x) + 4\ln(\csc(x) - \cot(x)) + 3\ln(\csc(x) + \cot(x)))\sin(3x)}{6}$$

Verified OK.

12.21.3 Maple step by step solution

Let's solve

$y'' + 9y = \csc(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

$r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \csc(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(3x)\left(\int (-\sec(x) + 4\cos(x)) dx\right)}{6} + \frac{\sin(3x)\left(\int (4\cos(x)\cot(x) - 3\csc(x)) dx\right)}{6}$

  - Compute integrals

    $y_p(x) = \left(-\frac{2\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{6}\right)\cos(3x) + \frac{(4\cos(x) + 4\ln(\csc(x) - \cot(x)) + 3\ln(\csc(x) + \cot(x)))\sin(3x)}{6}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + \left(-\frac{2\sin(x)}{3} + \frac{\ln(\sec(x) + \tan(x))}{6}\right)\cos(3x) + \frac{(4\cos(x) + 4\ln(\csc(x) - \cot(x)) + 3\ln(c}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)+9*y(x)=csc(2*x),y(x), singsol=all)
```

$$y(x) = \frac{\sin(x)\left(-1+4\cos(x)^2\right)\ln(\csc(x)-\cot(x))}{6} + \frac{\left(4\cos(x)^3-3\cos(x)\right)\ln(\sec(x)+\tan(x))}{6} + 4c_1\cos(x)^3 + 4\cos(x)^2\sin(x)\,c_2 + \frac{(-9c_1+4\sin(x))\cos(x)}{3} - \sin(x)\,c_2$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 65

```
DSolve[y''[x]+9*y[x]==Csc[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}\left(\cos(3x)\operatorname{arctanh}(\sin(x)) + 4\sin(2x) + \sin(3x)\log\left(\sin\left(\frac{x}{2}\right)\right) + 6c_1\cos(3x) + 6c_2\sin(3x) - \sin(3x)\log\left(\cos\left(\frac{x}{2}\right)\right)\right)$$

12.22 problem 22

Internal problem ID [2195]
Internal file name [`OUTPUT/2195_Monday_February_26_2024_09_18_18_AM_85497231/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \tan\left(\frac{x}{3}\right)^2}$$

12.22.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \tan\left(\frac{x}{3}\right)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=0, C=1$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=0, C=1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan\left(\frac{x}{3}\right)^2}{1} dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan\left(\frac{x}{3}\right)^2 dx$$

Hence

$$u_1 = 7\left(2 + \sin\left(\frac{x}{3}\right)^2\right)\cos\left(\frac{x}{3}\right) + \frac{3\sin\left(\frac{x}{3}\right)^4}{\cos\left(\frac{x}{3}\right)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan\left(\frac{x}{3}\right)^2}{1} dx$$

Which simplifies to

$$u_2 = \int \cos(x)\tan\left(\frac{x}{3}\right)^2 dx$$

Hence

$$u_2 = 9\sin\left(\frac{x}{3}\right) - 9\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 4\sin\left(\frac{x}{3}\right)^3$$

Which simplifies to

$$u_1 = -4\cos\left(\frac{x}{3}\right)^3 + 15\cos\left(\frac{x}{3}\right) + 3\sec\left(\frac{x}{3}\right)$$
$$u_2 = 9\sin\left(\frac{x}{3}\right) - 9\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 4\sin\left(\frac{x}{3}\right)^3$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-4\cos\left(\frac{x}{3}\right)^3 + 15\cos\left(\frac{x}{3}\right) + 3\sec\left(\frac{x}{3}\right)\right)\cos(x) + \left(9\sin\left(\frac{x}{3}\right) - 9\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 4\sin\left(\frac{x}{3}\right)^3\right)\sin(x)$$

Which simplifies to

$$y_p(x) = -22 + 9\left(-4\cos\left(\frac{x}{3}\right)^2 + 1\right)\sin\left(\frac{x}{3}\right)\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) \\ &\quad + \left(-22 + 9\left(-4\cos\left(\frac{x}{3}\right)^2 + 1\right)\sin\left(\frac{x}{3}\right)\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1\cos(x) + c_2\sin(x) - 22 \\ &\quad + 9\left(-4\cos\left(\frac{x}{3}\right)^2 + 1\right)\sin\left(\frac{x}{3}\right)\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2 \end{aligned} \tag{1}$$

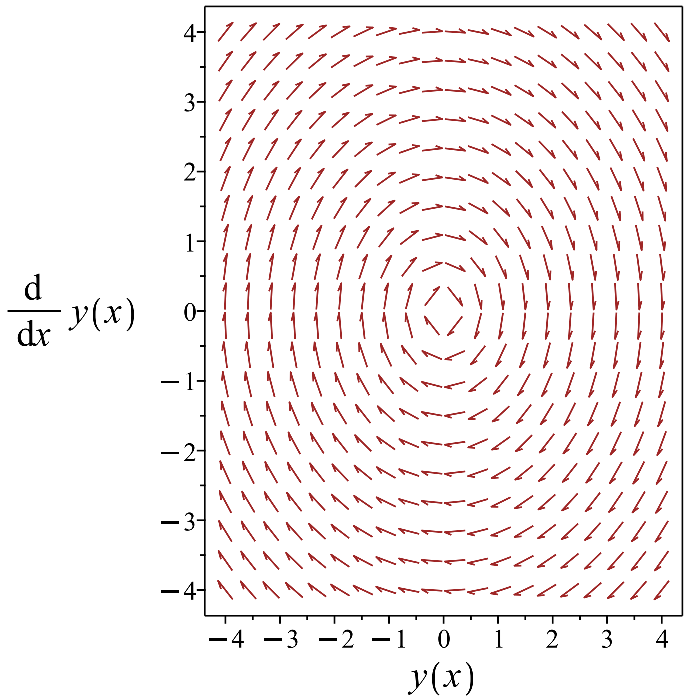


Figure 622: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - 22 + 9\left(-4\cos\left(\frac{x}{3}\right)^2 + 1\right)\sin\left(\frac{x}{3}\right)\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2$$

Verified OK.

12.22.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 462: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2} dx \\
&= \cos(x) (\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x) (\tan(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(x) \\ y_2 &= \sin(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan\left(\frac{x}{3}\right)^2}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan\left(\frac{x}{3}\right)^2 dx$$

Hence

$$u_1 = 7\left(2+\sin\left(\frac{x}{3}\right)^2\right)\cos\left(\frac{x}{3}\right) + \frac{3\sin\left(\frac{x}{3}\right)^4}{\cos\left(\frac{x}{3}\right)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan\left(\frac{x}{3}\right)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int \cos(x)\tan\left(\frac{x}{3}\right)^2 dx$$

Hence

$$u_2 = 9\sin\left(\frac{x}{3}\right) - 9\ln\left(\sec\left(\frac{x}{3}\right)+\tan\left(\frac{x}{3}\right)\right) + 4\sin\left(\frac{x}{3}\right)^3$$

Which simplifies to

$$u_1 = -4\cos\left(\frac{x}{3}\right)^3 + 15\cos\left(\frac{x}{3}\right) + 3\sec\left(\frac{x}{3}\right)$$
$$u_2 = 9\sin\left(\frac{x}{3}\right) - 9\ln\left(\sec\left(\frac{x}{3}\right)+\tan\left(\frac{x}{3}\right)\right) + 4\sin\left(\frac{x}{3}\right)^3$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(-4\cos\left(\frac{x}{3}\right)^3 + 15\cos\left(\frac{x}{3}\right) + 3\sec\left(\frac{x}{3}\right)\right)\cos(x) \\ &+ \left(9\sin\left(\frac{x}{3}\right) - 9\ln\left(\sec\left(\frac{x}{3}\right)+\tan\left(\frac{x}{3}\right)\right) + 4\sin\left(\frac{x}{3}\right)^3\right)\sin(x) \end{aligned}$$

Which simplifies to

$$y_p(x) = -22 + 9\left(-4\cos\left(\frac{x}{3}\right)^2 + 1\right)\sin\left(\frac{x}{3}\right)\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) \\ &\quad + \left(-22 + 9\left(-4\cos\left(\frac{x}{3}\right)^2 + 1\right)\sin\left(\frac{x}{3}\right)\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1\cos(x) + c_2\sin(x) - 22 \\ & + 9\left(-4\cos\left(\frac{x}{3}\right)^2 + 1\right)\sin\left(\frac{x}{3}\right)\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2 \end{aligned} \tag{1}$$

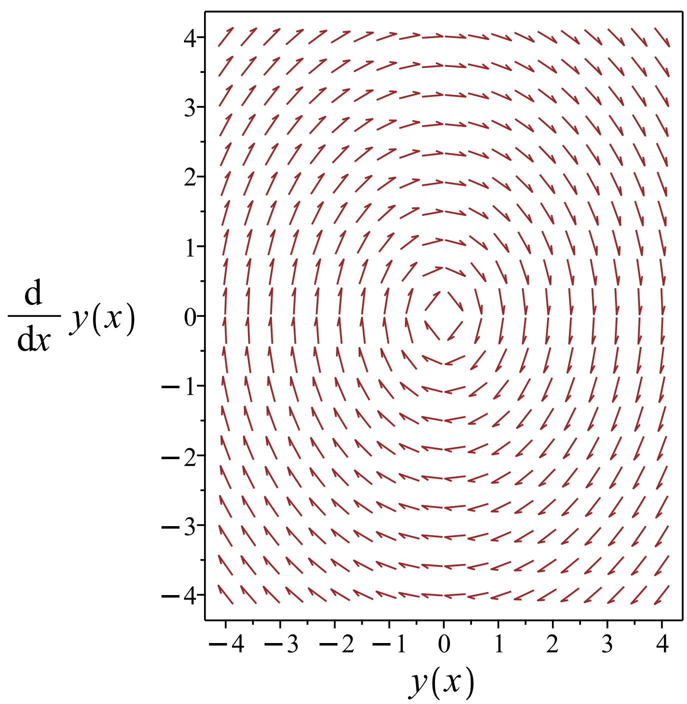


Figure 623: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - 22 + 9\left(-4\cos\left(\frac{x}{3}\right)^2 + 1\right)\sin\left(\frac{x}{3}\right)\ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2$$

Verified OK.

12.22.3 Maple step by step solution

Let's solve

$y'' + y = \tan\left(\frac{x}{3}\right)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan\left(\frac{x}{3}\right)^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = 1$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\cos(x) \left(\int \sin(x) \tan\left(\frac{x}{3}\right)^2 dx\right) + \sin(x) \left(\int \cos(x) \tan\left(\frac{x}{3}\right)^2 dx\right)$$

- Compute integrals

  $$y_p(x) = -22 + 9\left(-4\cos\left(\frac{x}{3}\right)^2 + 1\right) \sin\left(\frac{x}{3}\right) \ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 \cos(x) + c_2 \sin(x) - 22 + 9\left(-4\cos\left(\frac{x}{3}\right)^2 + 1\right) \sin\left(\frac{x}{3}\right) \ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$2)+y(x)=tan(x/3)^2,y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + \cos(x) c_1 - 22 + 9\sin\left(\frac{x}{3}\right)\left(1 - 4\cos\left(\frac{x}{3}\right)^2\right) \ln\left(\sec\left(\frac{x}{3}\right) + \tan\left(\frac{x}{3}\right)\right) + 36\cos\left(\frac{x}{3}\right)^2$$

✓ Solution by Mathematica

Time used: 0.283 (sec). Leaf size: 36

```
DSolve[y''[x]+y[x]==Tan[x/3]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -9\sin(x)\operatorname{arctanh}\left(\sin\left(\frac{x}{3}\right)\right) + 18\cos\left(\frac{2x}{3}\right) + c_1\cos(x) + c_2\sin(x) - 4$$

12.23 problem 23

Internal problem ID [2196]
Internal file name [`OUTPUT/2196_Monday_February_26_2024_09_18_18_AM_13637918/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 23.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_y]]`

$$\boxed{y''' + y' = \tan(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{-ix} + e^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \mathrm{e}^{-ix} \\
y_3 &= \mathrm{e}^{ix}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' + y' = \tan(x)
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} 1 & \mathrm{e}^{-ix} & \mathrm{e}^{ix} \\ 0 & -i\mathrm{e}^{-ix} & i\mathrm{e}^{ix} \\ 0 & -\mathrm{e}^{-ix} & -\mathrm{e}^{ix} \end{bmatrix} \\
|W| &= 2i\mathrm{e}^{-ix}\mathrm{e}^{ix}
\end{aligned}
$$

The determinant simplifies to

$$
|W| = 2i
$$

Now we determine W_i for each U_i.

$$\begin{aligned}W_1(x) &= \det\begin{bmatrix} e^{-ix} & e^{ix} \\ -ie^{-ix} & ie^{ix} \end{bmatrix} \\ &= 2i\end{aligned}$$

$$\begin{aligned}W_2(x) &= \det\begin{bmatrix} 1 & e^{ix} \\ 0 & ie^{ix} \end{bmatrix} \\ &= ie^{ix}\end{aligned}$$

$$\begin{aligned}W_3(x) &= \det\begin{bmatrix} 1 & e^{-ix} \\ 0 & -ie^{-ix} \end{bmatrix} \\ &= -ie^{-ix}\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}U_1 &= (-1)^{3-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2\int \frac{(\tan(x))(2i)}{(1)(2i)}\,dx \\ &= \int \frac{2i\tan(x)}{2i}\,dx \\ &= \int (\tan(x))\,dx \\ &= -\ln(\cos(x))\end{aligned}$$

$$\begin{aligned}U_2 &= (-1)^{3-2}\int \frac{F(x)W_2(x)}{aW(x)}\,dx \\ &= (-1)^1\int \frac{(\tan(x))(ie^{ix})}{(1)(2i)}\,dx \\ &= -\int \frac{i\tan(x)e^{ix}}{2i}\,dx \\ &= -\int \left(\frac{\tan(x)e^{ix}}{2}\right)dx \\ &= \frac{e^{ix}}{2} + \frac{i\ln(e^{ix}-i)}{2} - \frac{i\ln(e^{ix}+i)}{2}\end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(\tan(x))\left(-i \mathrm{e}^{-ix}\right)}{(1)(2i)}\, dx \\
&= \int \frac{-i \tan(x)\, \mathrm{e}^{-ix}}{2i}\, dx \\
&= \int \left(-\frac{\tan(x)\, \mathrm{e}^{-ix}}{2}\right) dx \\
&= \frac{\mathrm{e}^{-ix}}{2} + \arctan\left(\mathrm{e}^{ix}\right)
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\ln(\cos(x))\right) \\
&+ \left(\frac{\mathrm{e}^{ix}}{2} + \frac{i \ln\left(\mathrm{e}^{ix} - i\right)}{2} - \frac{i \ln\left(\mathrm{e}^{ix} + i\right)}{2}\right)\left(\mathrm{e}^{-ix}\right) \\
&+ \left(\frac{\mathrm{e}^{-ix}}{2} + \arctan\left(\mathrm{e}^{ix}\right)\right)\left(\mathrm{e}^{ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{i \mathrm{e}^{-ix} \ln\left(\mathrm{e}^{ix} - i\right)}{2} - \frac{i \mathrm{e}^{-ix} \ln\left(\mathrm{e}^{ix} + i\right)}{2} + 1 + \mathrm{e}^{ix} \arctan\left(\mathrm{e}^{ix}\right) - \ln(\cos(x))
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3\right) \\
&\quad + \left(\frac{i \mathrm{e}^{-ix} \ln\left(\mathrm{e}^{ix} - i\right)}{2} - \frac{i \mathrm{e}^{-ix} \ln\left(\mathrm{e}^{ix} + i\right)}{2} + 1 + \mathrm{e}^{ix} \arctan\left(\mathrm{e}^{ix}\right) - \ln(\cos(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3 + \frac{i \mathrm{e}^{-ix} \ln\left(\mathrm{e}^{ix} - i\right)}{2} \\
- \frac{i \mathrm{e}^{-ix} \ln\left(\mathrm{e}^{ix} + i\right)}{2} + 1 + \mathrm{e}^{ix} \arctan\left(\mathrm{e}^{ix}\right) - \ln(\cos(x))
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 + c_2 e^{-ix} + e^{ix} c_3 + \frac{i e^{-ix} \ln \left(e^{ix} - i\right)}{2} - \frac{i e^{-ix} \ln \left(e^{ix} + i\right)}{2} + 1 + e^{ix} \arctan \left(e^{ix}\right) - \ln \left(\cos \left(x\right)\right)$$

Verified OK.

12.23.1 Maple step by step solution

Let's solve

$y''' + y' = \tan \left(x\right)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \tan \left(x\right) - y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \tan \left(x\right) - y_2(x)]$

- Define vector

  $\overrightarrow{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \overrightarrow{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \tan(x) \end{bmatrix}$$

- Define the forcing function

$$\overrightarrow{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \tan(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos \left(x\right)-\mathrm{I} \sin \left(x\right)\right) \cdot \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\cos \left(x\right)+\mathrm{I} \sin \left(x\right) \\ \mathrm{I}\left(\cos \left(x\right)-\mathrm{I} \sin \left(x\right)\right) \\ \cos \left(x\right)-\mathrm{I} \sin \left(x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}\left(x\right)=\left[\begin{array}{c} -\cos \left(x\right) \\ \sin \left(x\right) \\ \cos \left(x\right) \end{array}\right], \vec{y}_{3}\left(x\right)=\left[\begin{array}{c} \sin \left(x\right) \\ \cos \left(x\right) \\ -\sin \left(x\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_{p}($

$\vec{y}\left(x\right)=c_{1} \vec{y}_{1}+c_{2} \vec{y}_{2}\left(x\right)+c_{3} \vec{y}_{3}\left(x\right)+\vec{y}_{p}\left(x\right)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\left[\begin{array}{ccc} 1 & -\cos \left(x\right) & \sin \left(x\right) \\ 0 & \sin \left(x\right) & \cos \left(x\right) \\ 0 & \cos \left(x\right) & -\sin \left(x\right) \end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$\Phi(x)=\phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} 1 & -\cos(x) & \sin(x) \\ 0 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} 1 & \sin(x) & 1-\cos(x) \\ 0 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x)=\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\overrightarrow{v}'(x)=\overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -\sin(x)\left(\int_0^x \sin(s)\tan(s)\,ds\right) + \cos(x)^2 - 2\cos(x) + \int_0^x (\tan(s) - \sin(s))\,ds + 1 \\ -\cos(x)\left(\int_0^x \sin(s)\tan(s)\,ds\right) - (\cos(x) - 1)\sin(x) \\ \sin(x)\left(\int_0^x \sin(s)\tan(s)\,ds\right) - \cos(x)^2 + \cos(x) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} -\sin(x)\left(\int_0^x \sin(s)\tan(s)\,ds\right) + \cos(x)^2 - 2\cos(x) + \int_0^ \\ -\cos(x)\left(\int_0^x \sin(s)\tan(s)\,ds\right) - (\cos(x) \\ \sin(x)\left(\int_0^x \sin(s)\tan(s)\,ds\right) - \cos(x)^2 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \int_0^x (\tan(s) - \sin(s))\,ds - \sin(x)\left(\int_0^x \sin(s)\tan(s)\,ds\right) + \cos(x)^2 + (-c_2 - 2)\cos(x) + c_3 \mathrm{si}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -_b(_a)+tan(_a), _b(_a)`   **
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 83

```
dsolve(diff(y(x),x$3)+diff(y(x),x)=tan(x),y(x), singsol=all)
```

$$y(x) = \frac{i(e^{ix} - e^{-ix}) \ln\left(\frac{ie^{ix}-1}{-e^{ix}+i}\right)}{2} + c_1 \sin(x) - c_2 \cos(x) - \ln(e^{ix} - i) - \ln(e^{ix} + i) + c_3 + \ln(e^{ix})$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 35

```
DSolve[y'''[x]+y'[x]==Tan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sin(x)\operatorname{arctanh}(\sin(x)) - \frac{1}{2}\log\left(\cos^2(x)\right) - c_2\cos(x) + c_1\sin(x) + c_3$$

12.24 problem 24

Internal problem ID [2197]
Internal file name [`OUTPUT/2197_Monday_February_26_2024_09_18_19_AM_89353891/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y'' - 4y' + y = e^{\frac{x}{2}} \ln(x)}$$

12.24.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4, B = -4, C = 1, f(x) = e^{\frac{x}{2}} \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4y'' - 4y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = -4, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 - 4\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = -4, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(4)} \pm \frac{1}{(2)\,(4)}\sqrt{(-4)^2 - (4)\,(4)\,(1)}\\ &= \frac{1}{2}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -\frac{1}{2}$. Therefore the solution is

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{\frac{x}{2}}\\ y_2 &= e^{\frac{x}{2}} x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\frac{x}{2}} & e^{\frac{x}{2}} x \\ \frac{d}{dx}\left(e^{\frac{x}{2}}\right) & \frac{d}{dx}\left(e^{\frac{x}{2}} x\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\frac{x}{2}} & e^{\frac{x}{2}} x \\ \frac{e^{\frac{x}{2}}}{2} & \frac{e^{\frac{x}{2}} x}{2} + e^{\frac{x}{2}} \end{vmatrix}$$

Therefore

$$W = \left(e^{\frac{x}{2}}\right)\left(\frac{e^{\frac{x}{2}} x}{2} + e^{\frac{x}{2}}\right) - \left(e^{\frac{x}{2}} x\right)\left(\frac{e^{\frac{x}{2}}}{2}\right)$$

Which simplifies to

$$W = e^{x}$$

Which simplifies to

$$W = e^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{x} x \ln(x)}{4\,e^{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x \ln(x)}{4} dx$$

Hence

$$u_1 = -\frac{\ln(x)\,x^2}{8} + \frac{x^2}{16}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^x \ln(x)}{4\,e^x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{4} dx$$

Hence

$$u_2 = \frac{x\ln(x)}{4} - \frac{x}{4}$$

Which simplifies to

$$u_1 = -\frac{x^2(2\ln(x) - 1)}{16}$$
$$u_2 = \frac{x(\ln(x) - 1)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2(2\ln(x) - 1)\,e^{\frac{x}{2}}}{16} + \frac{x^2(\ln(x) - 1)\,e^{\frac{x}{2}}}{4}$$

Which simplifies to

$$y_p(x) = \frac{x^2 e^{\frac{x}{2}}(2\ln(x) - 3)}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x\right) + \left(\frac{x^2 e^{\frac{x}{2}}(2\ln(x) - 3)}{16}\right) \end{aligned}$$

Which simplifies to

$$y = e^{\frac{x}{2}}(c_2 x + c_1) + \frac{x^2 e^{\frac{x}{2}}(2\ln(x) - 3)}{16}$$

Summary
The solution(s) found are the following

$$y = e^{\frac{x}{2}}(c_2x + c_1) + \frac{x^2 e^{\frac{x}{2}}(2\ln(x) - 3)}{16} \tag{1}$$

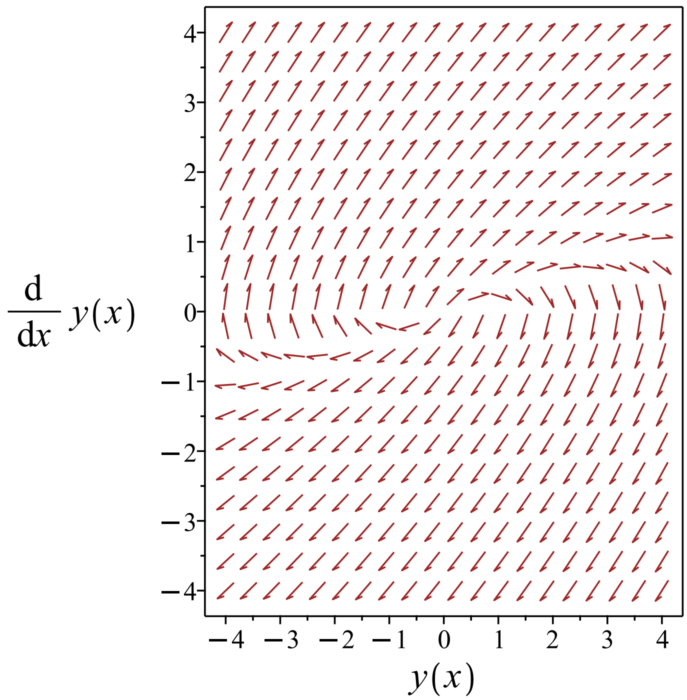


Figure 624: Slope field plot

Verification of solutions

$$y = e^{\frac{x}{2}}(c_2x + c_1) + \frac{x^2 e^{\frac{x}{2}}(2\ln(x) - 3)}{16}$$

Verified OK.

12.24.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -1$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -1\,dx} \\
&= \mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \frac{\mathrm{e}^{-\frac{x}{2}}\mathrm{e}^{\frac{x}{2}}\ln(x)}{4} \\
\left(\mathrm{e}^{-\frac{x}{2}}y\right)'' &= \frac{\mathrm{e}^{-\frac{x}{2}}\mathrm{e}^{\frac{x}{2}}\ln(x)}{4}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-\frac{x}{2}}y\right)' = \frac{x(\ln(x)-1)}{4} + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-\frac{x}{2}}y\right) = \frac{x(2x\ln(x)+16c_1-3x)}{16} + c_2
$$

Hence the solution is

$$
y = \frac{\frac{x(2x\ln(x)+16c_1-3x)}{16} + c_2}{\mathrm{e}^{-\frac{x}{2}}}
$$

Or

$$
y = \frac{x^2\mathrm{e}^{\frac{x}{2}}\ln(x)}{8} + c_1 x\,\mathrm{e}^{\frac{x}{2}} - \frac{3x^2\mathrm{e}^{\frac{x}{2}}}{16} + c_2\mathrm{e}^{\frac{x}{2}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^2\mathrm{e}^{\frac{x}{2}}\ln(x)}{8} + c_1 x\,\mathrm{e}^{\frac{x}{2}} - \frac{3x^2\mathrm{e}^{\frac{x}{2}}}{16} + c_2\mathrm{e}^{\frac{x}{2}} \tag{1}
$$

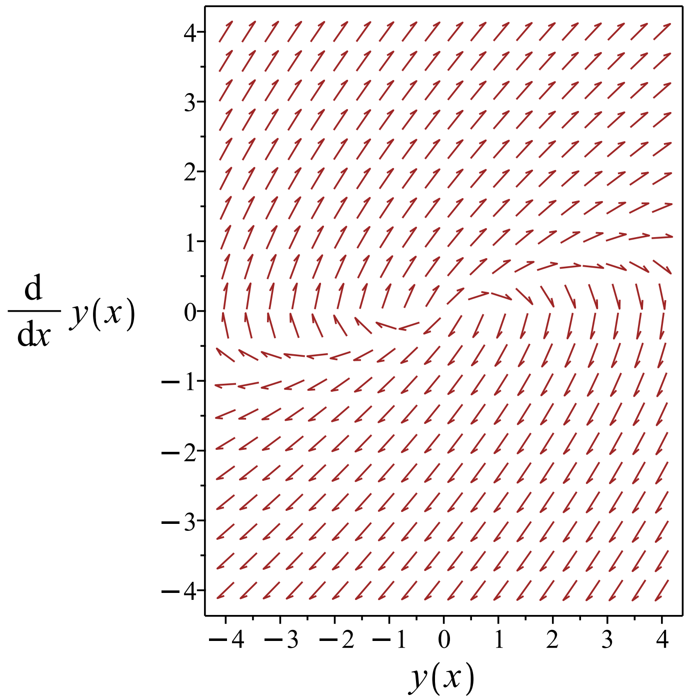


Figure 625: Slope field plot

Verification of solutions

$$y = \frac{x^2 e^{\frac{x}{2}} \ln(x)}{8} + c_1 x\, e^{\frac{x}{2}} - \frac{3x^2 e^{\frac{x}{2}}}{16} + c_2 e^{\frac{x}{2}}$$

Verified OK.

12.24.3 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - 4y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= -4 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 465: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{4}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{4}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{\frac{x}{2}}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
4y'' - 4y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{\frac{x}{2}} x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{\frac{x}{2}} \\
y_2 &= \mathrm{e}^{\frac{x}{2}} x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\frac{x}{2}} & e^{\frac{x}{2}} x \\ \frac{d}{dx}\left(e^{\frac{x}{2}}\right) & \frac{d}{dx}\left(e^{\frac{x}{2}} x\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\frac{x}{2}} & e^{\frac{x}{2}} x \\ \frac{e^{\frac{x}{2}}}{2} & \frac{e^{\frac{x}{2}} x}{2} + e^{\frac{x}{2}} \end{vmatrix}$$

Therefore

$$W = \left(e^{\frac{x}{2}}\right)\left(\frac{e^{\frac{x}{2}} x}{2} + e^{\frac{x}{2}}\right) - \left(e^{\frac{x}{2}} x\right)\left(\frac{e^{\frac{x}{2}}}{2}\right)$$

Which simplifies to

$$W = e^{x}$$

Which simplifies to

$$W = e^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{x} x \ln(x)}{4\, e^{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x \ln(x)}{4} dx$$

Hence

$$u_1 = -\frac{\ln(x)\, x^2}{8} + \frac{x^2}{16}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{x} \ln(x)}{4\, e^{x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln (x)}{4} dx$$

Hence

$$u_2 = \frac{x \ln (x)}{4} - \frac{x}{4}$$

Which simplifies to

$$u_1 = -\frac{x^2(2 \ln (x) - 1)}{16}$$
$$u_2 = \frac{x(\ln (x) - 1)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2(2 \ln (x) - 1) \, \mathrm{e}^{\frac{x}{2}}}{16} + \frac{x^2(\ln (x) - 1) \, \mathrm{e}^{\frac{x}{2}}}{4}$$

Which simplifies to

$$y_p(x) = \frac{x^2 \mathrm{e}^{\frac{x}{2}} (2 \ln (x) - 3)}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{\frac{x}{2}} x\right) + \left(\frac{x^2 \mathrm{e}^{\frac{x}{2}} (2 \ln (x) - 3)}{16}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{\frac{x}{2}} (c_2 x + c_1) + \frac{x^2 \mathrm{e}^{\frac{x}{2}} (2 \ln (x) - 3)}{16}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{x}{2}} (c_2 x + c_1) + \frac{x^2 \mathrm{e}^{\frac{x}{2}} (2 \ln (x) - 3)}{16} \tag{1}$$

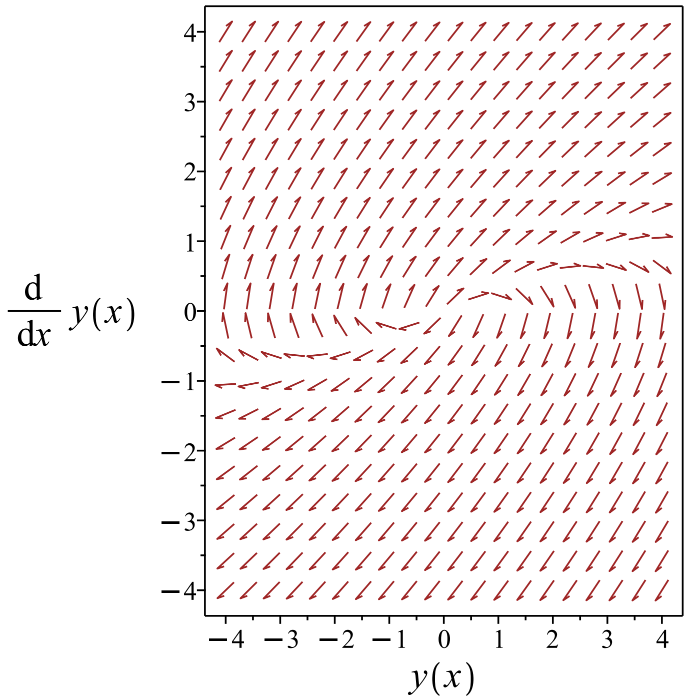


Figure 626: Slope field plot

Verification of solutions

$$y = e^{\frac{x}{2}}(c_2x + c_1) + \frac{x^2 e^{\frac{x}{2}}(2\ln(x) - 3)}{16}$$

Verified OK.

12.24.4 Maple step by step solution

Let's solve

$4y'' - 4y' + y = e^{\frac{x}{2}} \ln(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' - \frac{y}{4} + \frac{e^{\frac{x}{2}} \ln(x)}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' + \frac{y}{4} = \frac{e^{\frac{x}{2}} \ln(x)}{4}$

- Characteristic polynomial of homogeneous ODE

$r^2 - r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = \frac{1}{2}$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{\frac{x}{2}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{\frac{x}{2}} x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{e^{\frac{x}{2}} \ln(x)}{4}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\frac{x}{2}} & e^{\frac{x}{2}} x \\ \frac{e^{\frac{x}{2}}}{2} & \frac{e^{\frac{x}{2}} x}{2} + e^{\frac{x}{2}} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^x$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{\frac{x}{2}} \left(\int x \ln(x) dx - \left(\int \ln(x) dx\right) x\right)}{4}$

  ◦ Compute integrals

  $y_p(x) = \frac{x^2 e^{\frac{x}{2}} (2 \ln(x) - 3)}{16}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}} x + \frac{x^2 e^{\frac{x}{2}} (2 \ln(x) - 3)}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(4*diff(y(x),x$2)-4*diff(y(x),x)+y(x)=exp(x/2)*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{\frac{x}{2}} \left(\ln (x) x^2 - \frac{3x^2}{2} + 8c_1 x + 8c_2\right)}{8}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 38

```
DSolve[4*y''[x]-4*y'[x]+y[x]==Exp[x/2]*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{16} e^{x/2} \left(-3x^2 + 2x^2 \log(x) + 16c_2 x + 16c_1\right)$$

12.25 problem 25

Internal problem ID [2198]
Internal file name [`OUTPUT/2198_Monday_February_26_2024_09_18_20_AM_96227662/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 20, page 90
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + P(x)\, y = Q(x)}$$

12.25.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = P(x)$$
$$q(x) = Q(x)$$

Hence the ode is

$$y' + P(x)\, y = Q(x)$$

The integrating factor μ is

$$\mu = e^{\int P(x)dx}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\,(Q(x))$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int P(x)dx} y\right) = \left(\mathrm{e}^{\int P(x)dx}\right)(Q(x))$$
$$\mathrm{d}\left(\mathrm{e}^{\int P(x)dx} y\right) = \left(Q(x)\,\mathrm{e}^{\int P(x)dx}\right)\,\mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{\int P(x)dx} y = \int Q(x)\,\mathrm{e}^{\int P(x)dx}\,\mathrm{d}x$$
$$\mathrm{e}^{\int P(x)dx} y = \int Q(x)\,\mathrm{e}^{\int P(x)dx} dx + c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int P(x)dx}$ results in

$$y = \mathrm{e}^{-\left(\int P(x)dx\right)}\left(\int Q(x)\,\mathrm{e}^{\int P(x)dx} dx\right) + c_1 \mathrm{e}^{-\left(\int P(x)dx\right)}$$

which simplifies to

$$y = \mathrm{e}^{-\left(\int P(x)dx\right)}\left(\int Q(x)\,\mathrm{e}^{\int P(x)dx} dx + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\left(\int P(x)dx\right)}\left(\int Q(x)\,\mathrm{e}^{\int P(x)dx} dx + c_1\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\left(\int P(x)dx\right)}\left(\int Q(x)\,\mathrm{e}^{\int P(x)dx} dx + c_1\right)$$

Verified OK.

12.25.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -P(x)\,y + Q(x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 467: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\int -P(x)dx}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\int -P(x)dx}} dy \end{aligned}$$

12.25.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= (-P(x)\,y + Q(x))\,\mathrm{d}x \\ (P(x)\,y - Q(x))\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= P(x)\,y - Q(x) \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(P(x)\,y - Q(x)) \\ &= P(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((P(x)) - (0)) \\ &= P(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int P(x)\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\int P(x)dx} \\
&= \mathrm{e}^{\int P(x)dx}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\int P(x)dx}(P(x)\, y - Q(x)) \\
&= (P(x)\, y - Q(x))\, \mathrm{e}^{\int P(x)dx}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\int P(x)dx}(1) \\
&= \mathrm{e}^{\int P(x)dx}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left((P(x)\, y - Q(x))\, \mathrm{e}^{\int P(x)dx}\right) + \left(\mathrm{e}^{\int P(x)dx}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int \overline{M}\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int (P(x)\, y - Q(x))\, \mathrm{e}^{\int P(x)dx}\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \int^{x} (P(\_a)\, y - Q(\_a))\, \mathrm{e}^{\int P(\_a)d\_a} d\_a + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{\int^x P(\_a) d\_a} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{\int P(x) dx}$. Therefore equation (4) becomes

$$e^{\int P(x) dx} = e^{\int^x P(\_a) d\_a} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -e^{\int^x P(\_a) d\_a} + e^{\int P(x) dx}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(-e^{\int^x P(\_a) d\_a} + e^{\int P(x) dx}\right) \mathrm{d}y$$

$$f(y) = \int_0^y \left(-e^{\int^x P(\_a) d\_a} + e^{\int P(x) dx}\right) d\_a + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^x (P(\_a)\, y - Q(\_a))\, e^{\int P(\_a) d\_a} d\_a + \int_0^y \left(-e^{\int^x P(\_a) d\_a} + e^{\int P(x) dx}\right) d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x (P(\_a)\, y - Q(\_a))\, e^{\int P(\_a) d\_a} d\_a + \int_0^y \left(-e^{\int^x P(\_a) d\_a} + e^{\int P(x) dx}\right) d\_a$$

Summary

The solution(s) found are the following

$$\int^x (P(\_a)\, y - Q(\_a))\, e^{\int P(\_a) d\_a} d\_a + \int_0^y \left(-e^{\int^x P(\_a) d\_a} + e^{\int P(x) dx}\right) d\_a = c_1 \tag{1}$$

Verification of solutions

$$\int^x (P(\_a)\, y - Q(\_a))\, e^{\int P(\_a) d\_a} d\_a + \int_0^y \left(-e^{\int^x P(\_a) d\_a} + e^{\int P(x) dx}\right) d\_a = c_1$$

Verified OK.

12.25.4 Maple step by step solution

Let's solve

$y' + P(x)\, y = Q(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -P(x)\, y + Q(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + P(x)\, y = Q(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\, (y' + P(x)\, y) = \mu(x)\, Q(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\, (y' + P(x)\, y) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\, P(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\int P(x)dx}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, Q(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, Q(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)Q(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\int P(x)dx}$

  $y = \frac{\int Q(x)\mathrm{e}^{\int P(x)dx}dx + c_1}{\mathrm{e}^{\int P(x)dx}}$

- Simplify

  $y = \mathrm{e}^{-\left(\int P(x)dx\right)} \left(\int Q(x)\, \mathrm{e}^{\int P(x)dx} dx + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)+P(x)*y(x)=Q(x),y(x), singsol=all)
```

$$y(x) = \left(\int Q(x)\,\mathrm{e}^{\int P(x)dx}dx + c_1\right)\mathrm{e}^{-\left(\int P(x)dx\right)}$$

✓ Solution by Mathematica

Time used: 0.105 (sec). Leaf size: 51

```
DSolve[y'[x]+p[x]*y[x]==q[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \exp\left(\int_1^x -p(K[1])dK[1]\right)\left(\int_1^x \exp\left(-\int_1^{K[2]} -p(K[1])dK[1]\right)q(K[2])dK[2] + c_1\right)$$

13 Exercise 22, page 99

13.1 problem 1

Internal problem ID [2199]
Internal file name [`OUTPUT/2199_Monday_February_26_2024_09_18_20_AM_74108819/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 22, page 99
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 6y' + 9y = \mathrm{e}^{3x}}$$

13.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = \mathrm{e}^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-6)^2 - (4)(1)(9)}\\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 x\, e^{3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2 x\, e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{3x}, e^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{3x}\}]$$

Since $x\,\mathrm{e}^{3x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\mathrm{e}^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2\mathrm{e}^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{3x} = \mathrm{e}^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2\mathrm{e}^{3x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{3x} + c_2x\,\mathrm{e}^{3x}\right) + \left(\frac{x^2\mathrm{e}^{3x}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{3x}(c_2x + c_1) + \frac{x^2\mathrm{e}^{3x}}{2}$$

Summary
The solution(s) found are the following

$$y = e^{3x}(c_2 x + c_1) + \frac{x^2 e^{3x}}{2} \tag{1}$$

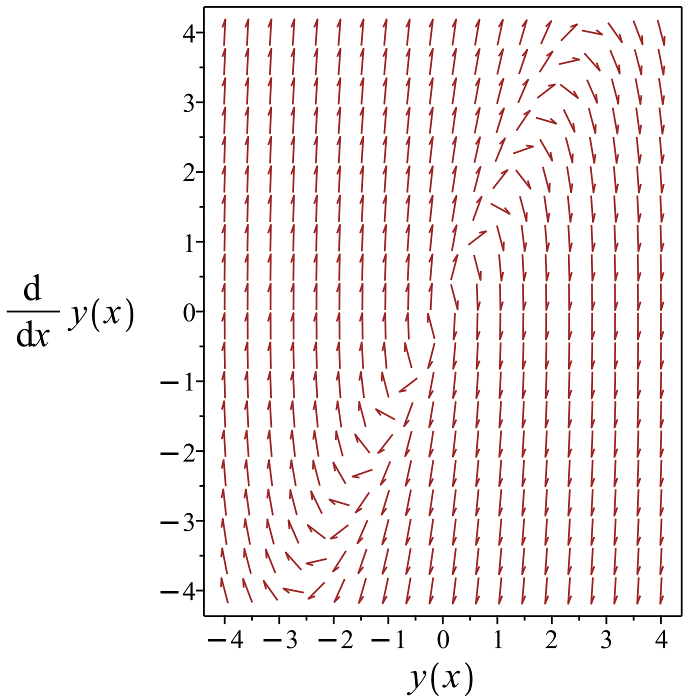


Figure 627: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2 x + c_1) + \frac{x^2 e^{3x}}{2}$$

Verified OK.

13.1.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -6\,dx} \\ &= \mathrm{e}^{-3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \mathrm{e}^{-3x}\mathrm{e}^{3x} \\ \left(\mathrm{e}^{-3x}y\right)'' &= \mathrm{e}^{-3x}\mathrm{e}^{3x} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-3x}y\right)' = x + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-3x}y\right) = \frac{x(x+2c_1)}{2} + c_2$$

Hence the solution is

$$y = \frac{\frac{x(x+2c_1)}{2} + c_2}{\mathrm{e}^{-3x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{3x} + \frac{x^2\mathrm{e}^{3x}}{2} + c_2\mathrm{e}^{3x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{3x} + \frac{x^2\mathrm{e}^{3x}}{2} + c_2\mathrm{e}^{3x} \tag{1}$$

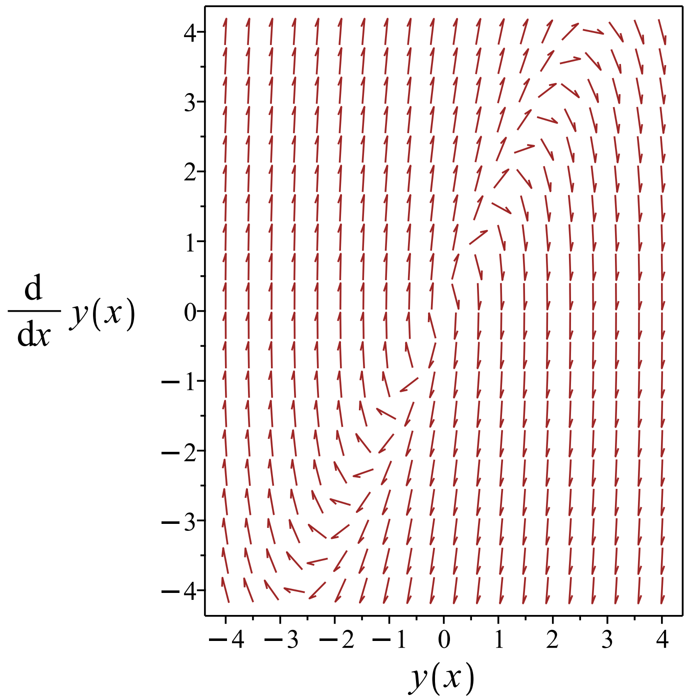


Figure 628: Slope field plot

Verification of solutions

$$y = c_1 x\,e^{3x} + \frac{x^2 e^{3x}}{2} + c_2 e^{3x}$$

Verified OK.

13.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 470: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\ &= z_1 e^{3x} \\ &= z_1\left(e^{3x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{3x}\right) + c_2 \left(\mathrm{e}^{3x}(x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{3x} + c_2 x\, \mathrm{e}^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, \mathrm{e}^{3x}, \mathrm{e}^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, \mathrm{e}^{3x}\}]$$

Since $x\, \mathrm{e}^{3x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{3x} = e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{3x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 x\, e^{3x}\right) + \left(\frac{x^2 e^{3x}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2 x + c_1) + \frac{x^2 e^{3x}}{2}$$

Summary

The solution(s) found are the following

$$y = e^{3x}(c_2 x + c_1) + \frac{x^2 e^{3x}}{2} \tag{1}$$

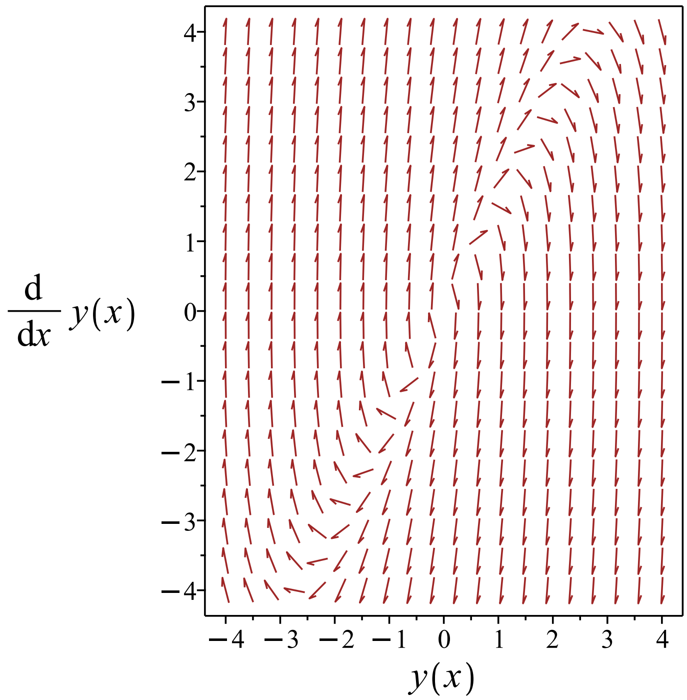


Figure 629: Slope field plot

<u>Verification of solutions</u>

$$y = e^{3x}(c_2 x + c_1) + \frac{x^2 e^{3x}}{2}$$

Verified OK.

13.1.4 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = e^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

$r = 3$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{3x} + c_2 x\, e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{3x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} e^{3x} & x\, e^{3x} \\ 3\, e^{3x} & e^{3x} + 3x\, e^{3x} \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{6x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^{3x}\left(-\left(\int x dx\right) + \left(\int 1 dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{x^2 e^{3x}}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\, e^{3x} + c_1 e^{3x} + \frac{x^2 e^{3x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=exp(3*x),y(x), singsol=all)
```

$$y(x) = e^{3x}\left(c_2 + c_1 x + \frac{1}{2}x^2\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 27

```
DSolve[y''[x]-6*y'[x]+9*y[x]==Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{3x}\left(x^2 + 2c_2 x + 2c_1\right)$$

13.2 problem 2

Internal problem ID [2200]
Internal file name [OUTPUT/2200_Monday_February_26_2024_09_18_21_AM_34700767/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 22, page 99
Problem number: 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - 3y'' + 3y' - y = e^{x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' + 3y' - y = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{x} + c_2 x\, e^{x} + x^2 e^{x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^x \\
y_2 &= x\,e^x \\
y_3 &= x^2e^x
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' - 3y'' + 3y' - y = e^x
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,e^x, x^2e^x, e^x\}
$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\,e^x\}]
$$

Since $x\,e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2e^x\}]
$$

Since x^2e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{e^x x^3\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 e^x x^3
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
6A_1 e^x = e^x
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 x\, e^x + x^2 e^x c_3\right) + \left(\frac{e^x x^3}{6}\right) \end{aligned}$$

Which simplifies to

$$y = e^x \left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^x x^3}{6}$$

Summary
The solution(s) found are the following

$$y = e^x \left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^x x^3}{6} \tag{1}$$

Verification of solutions

$$y = e^x \left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^x x^3}{6}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+3*diff(y(x),x)-y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x}\left(\frac{1}{6}x^{3} + c_1 + c_2x + c_3x^{2}\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 32

```
DSolve[y'''[x]-3*y''[x]+3*y'[x]-y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^{x}\left(x^{3} + 6c_3x^{2} + 6c_2x + 6c_1\right)$$

13.3 problem 3

Internal problem ID [2201]
Internal file name [`OUTPUT/2201_Monday_February_26_2024_09_18_21_AM_39080822/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 22, page 99
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 5y' + 6y = x^2}$$

13.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = 6, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 5\lambda\, e^{\lambda x} + 6\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-5^2 - (4)\,(1)\,(6)}\\ &= \frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{5}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{5}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= 2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(3)x} + c_2 e^{(2)x}\end{aligned}$$

Or

$$y = c_1 e^{3x} + c_2 e^{2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2 e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_3x^2 + 6A_2x - 10xA_3 + 6A_1 - 5A_2 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{19}{108}, A_2 = \frac{5}{18}, A_3 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{6}x^2 + \frac{5}{18}x + \frac{19}{108}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 e^{2x}\right) + \left(\frac{1}{6}x^2 + \frac{5}{18}x + \frac{19}{108}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} + c_2 e^{2x} + \frac{x^2}{6} + \frac{5x}{18} + \frac{19}{108} \tag{1}$$

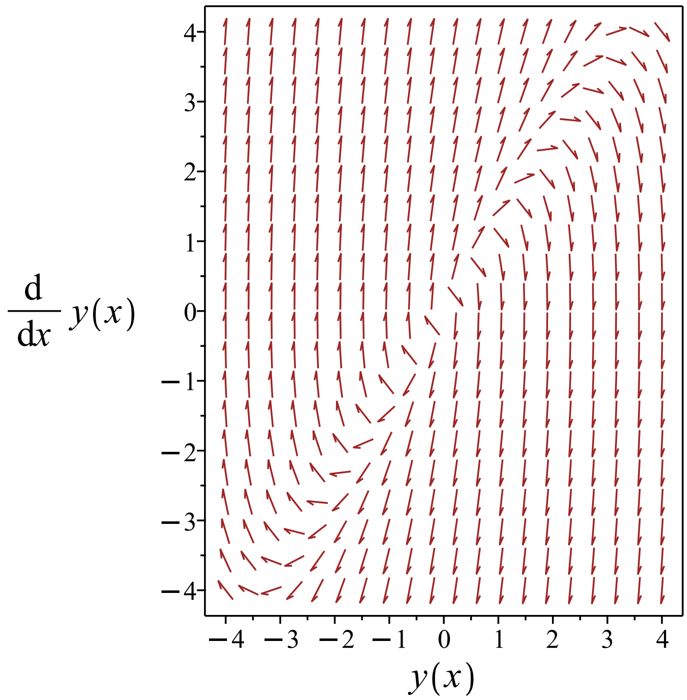


Figure 630: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{2x} + \frac{x^2}{6} + \frac{5x}{18} + \frac{19}{108}$$

Verified OK.

13.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 472: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx} \\
&= z_1 e^{\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{5x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(e^{2x}) + c_2(e^{2x}(e^x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1e^{2x} + c_2e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_3x^2 + 6A_2x - 10xA_3 + 6A_1 - 5A_2 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{19}{108}, A_2 = \frac{5}{18}, A_3 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{6}x^2 + \frac{5}{18}x + \frac{19}{108}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{3x}\right) + \left(\frac{1}{6}x^2 + \frac{5}{18}x + \frac{19}{108}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{x^2}{6} + \frac{5x}{18} + \frac{19}{108} \tag{1}$$

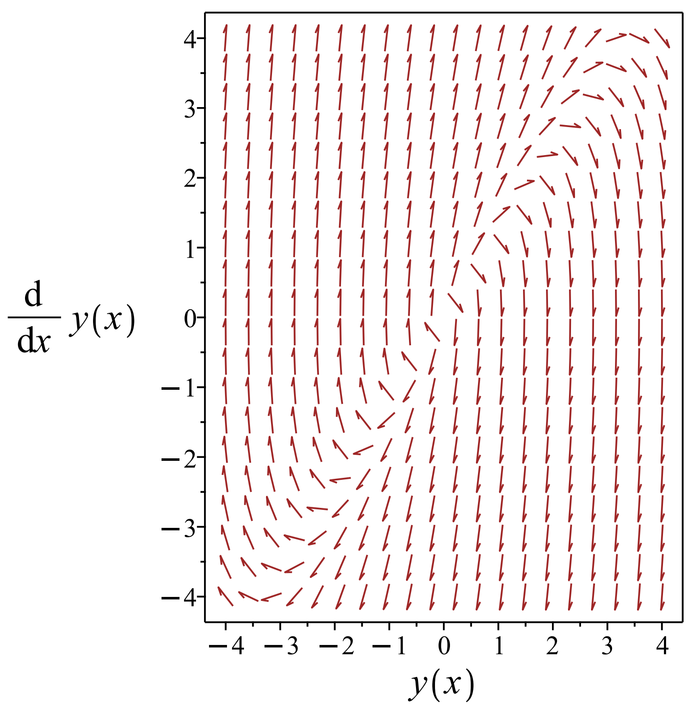


Figure 631: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{x^2}{6} + \frac{5x}{18} + \frac{19}{108}$$

Verified OK.

13.3.3 Maple step by step solution

Let's solve

$y'' - 5y' + 6y = x^2$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$
- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$
- Roots of the characteristic polynomial

  $r = (2, 3)$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + c_2 e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^2\right]$
  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{5x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{2x}\left(\int x^2 e^{-2x} dx\right) + e^{3x}\left(\int x^2 e^{-3x} dx\right)$
- Compute integrals

  $y_p(x) = \frac{1}{6}x^2 + \frac{5}{18}x + \frac{19}{108}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2x} + c_2 e^{3x} + \frac{x^2}{6} + \frac{5x}{18} + \frac{19}{108}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=x^2,y(x), singsol=all)
```

$$y(x) = c_2 e^{3x} + c_1 e^{2x} + \frac{x^2}{6} + \frac{5x}{18} + \frac{19}{108}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 37

```
DSolve[y''[x]-5*y'[x]+6*y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{6} + \frac{5x}{18} + c_1 e^{2x} + c_2 e^{3x} + \frac{19}{108}$$

14 Exercise 23, page 106

14.1 problem 1

Internal problem ID [2202]
Internal file name [`OUTPUT/2202_Monday_February_26_2024_09_18_21_AM_6999328/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 4y = 2\,\mathrm{e}^{x}}$$

14.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 2\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \mathrm{e}^{x} = 2\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2\,\mathrm{e}^{x}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{2\,\mathrm{e}^{x}}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{2\,e^x}{5} \tag{1}$$

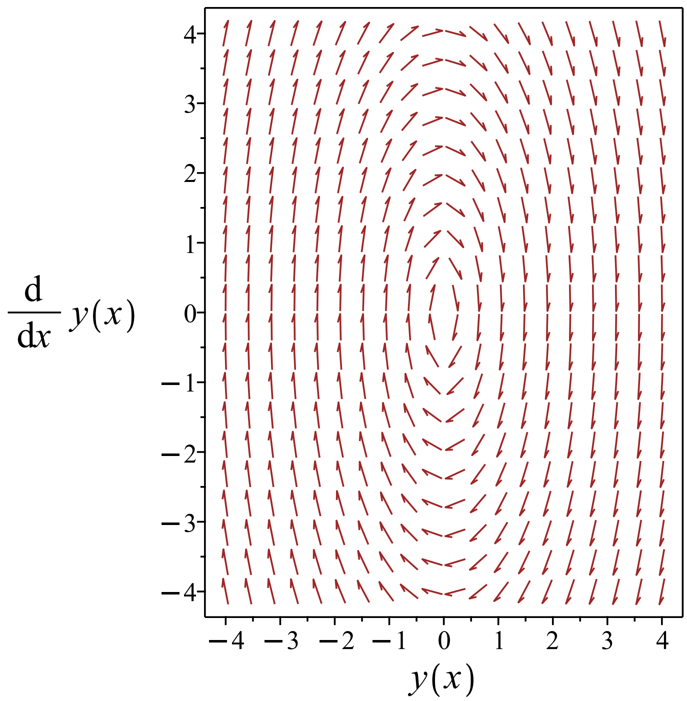


Figure 632: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{2\,e^x}{5}$$

Verified OK.

14.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 474: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(2x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^{x} = 2\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2\,e^{x}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{2\,e^{x}}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{2\,e^{x}}{5} \tag{1}$$

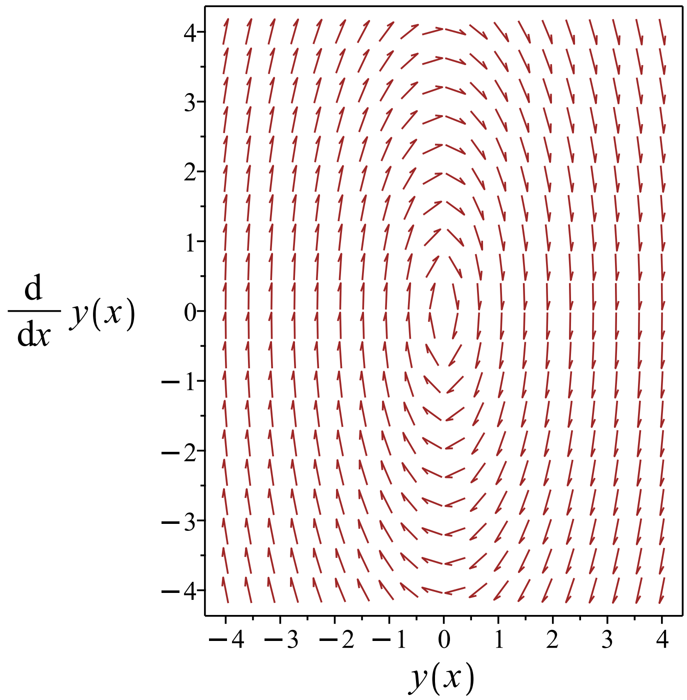


Figure 633: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{2\,e^{x}}{5}$$

Verified OK.

14.1.3 Maple step by step solution

Let's solve

$y'' + 4y = 2\,e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\,\mathrm{e}^x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(2x)\left(\int \mathrm{e}^x \sin(2x)\,dx\right) + \sin(2x)\left(\int \mathrm{e}^x \cos(2x)\,dx\right)$

  - Compute integrals

    $y_p(x) = \frac{2\,\mathrm{e}^x}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{2\,\mathrm{e}^x}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+4*y(x)=2*exp(x),y(x), singsol=all)
```

$$y(x) = \sin(2x) c_2 + c_1 \cos(2x) + \frac{2\,e^x}{5}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 27

```
DSolve[y''[x]+4*y[x]==2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2e^x}{5} + c_1 \cos(2x) + c_2 \sin(2x)$$

14.2 problem 2

Internal problem ID [2203]
Internal file name [`OUTPUT/2203_Monday_February_26_2024_09_18_22_AM_78858686/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y = 3\,\mathrm{e}^{-4x}}$$

14.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 3, f(x) = 3\,\mathrm{e}^{-4x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=0, C=3$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=0, C=3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(3)} \\ &= \pm i\sqrt{3}\end{aligned}$$

Hence

$$\lambda_1 = +i\sqrt{3}$$
$$\lambda_2 = -i\sqrt{3}$$

Which simplifies to

$$\lambda_1 = i\sqrt{3}$$
$$\lambda_2 = -i\sqrt{3}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{3}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)\right)$$

Or

$$y = c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,\mathrm{e}^{-4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{3}\, x\right), \sin\left(\sqrt{3}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-4x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$19 A_1 \mathrm{e}^{-4x} = 3\,\mathrm{e}^{-4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{19}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\,\mathrm{e}^{-4x}}{19}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)\right) + \left(\frac{3\,\mathrm{e}^{-4x}}{19}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right) + \frac{3\,\mathrm{e}^{-4x}}{19} \tag{1}$$

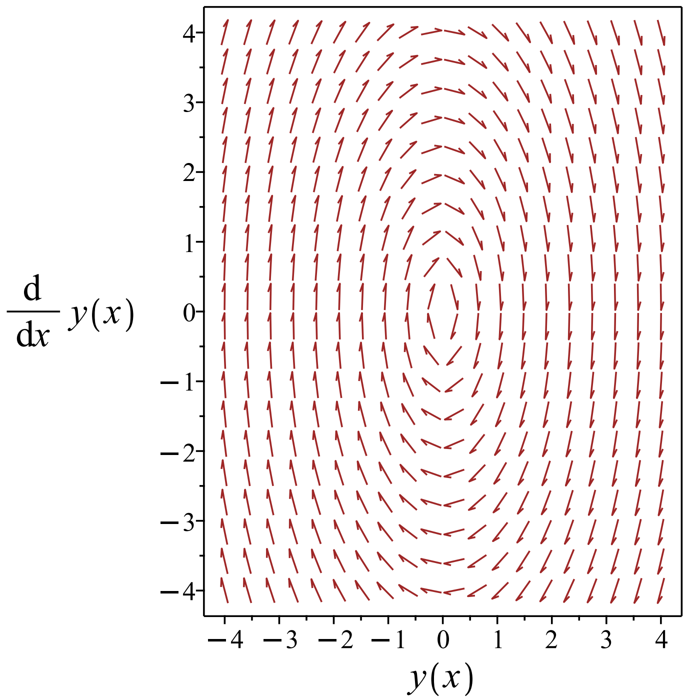


Figure 634: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right) + \frac{3\,\mathrm{e}^{-4x}}{19}$$

Verified OK.

14.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} \,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 476: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{3}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos\left(\sqrt{3}\, x\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{3}\, x\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos\left(\sqrt{3}\, x\right) \int \frac{1}{\cos\left(\sqrt{3}\, x\right)^2}\, dx \\
&= \cos\left(\sqrt{3}\, x\right) \left(\frac{\sqrt{3}\, \tan\left(\sqrt{3}\, x\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\sqrt{3}\, x\right)\right) + c_2 \left(\cos\left(\sqrt{3}\, x\right) \left(\frac{\sqrt{3}\, \tan\left(\sqrt{3}\, x\right)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(\sqrt{3}\, x\right) + \frac{\sqrt{3}\, \sin\left(\sqrt{3}\, x\right) c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,\mathrm{e}^{-4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin\left(\sqrt{3}\,x\right)\sqrt{3}}{3}, \cos\left(\sqrt{3}\,x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-4x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$19A_1 \mathrm{e}^{-4x} = 3\,\mathrm{e}^{-4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{19}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\,\mathrm{e}^{-4x}}{19}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{3}\,x\right) + \frac{\sqrt{3}\,\sin\left(\sqrt{3}\,x\right)c_2}{3}\right) + \left(\frac{3\,\mathrm{e}^{-4x}}{19}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{3}\, x\right) + \frac{\sqrt{3}\, \sin\left(\sqrt{3}\, x\right) c_2}{3} + \frac{3\, e^{-4x}}{19} \tag{1}$$

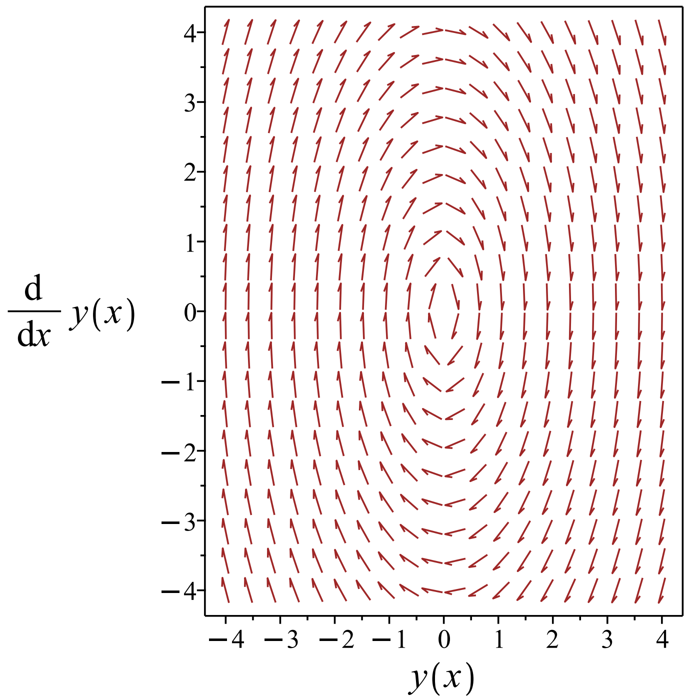


Figure 635: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{3}\, x\right) + \frac{\sqrt{3}\, \sin\left(\sqrt{3}\, x\right) c_2}{3} + \frac{3\, e^{-4x}}{19}$$

Verified OK.

14.2.3 Maple step by step solution

Let's solve

$y'' + 3y = 3\, e^{-4x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2+3=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-12})}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\mathrm{I}\sqrt{3}, \mathrm{I}\sqrt{3}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x)=\cos\left(\sqrt{3}\, x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\sin\left(\sqrt{3}\, x\right)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos\left(\sqrt{3}\, x\right)+c_2\sin\left(\sqrt{3}\, x\right)+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=3\,\mathrm{e}^{-4x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x))=\begin{bmatrix} \cos\left(\sqrt{3}\, x\right) & \sin\left(\sqrt{3}\, x\right) \\ -\sin\left(\sqrt{3}\, x\right)\sqrt{3} & \cos\left(\sqrt{3}\, x\right)\sqrt{3} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x))=\sqrt{3}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=\sqrt{3}\left(-\cos\left(\sqrt{3}\, x\right)\left(\int \sin\left(\sqrt{3}\, x\right)\mathrm{e}^{-4x}dx\right)+\sin\left(\sqrt{3}\, x\right)\left(\int \cos\left(\sqrt{3}\, x\right)\mathrm{e}^{-4x}dx\right)\right)$

  - Compute integrals

    $y_p(x)=\frac{3\,\mathrm{e}^{-4x}}{19}$

- Substitute particular solution into general solution to ODE

  $y=c_1\cos\left(\sqrt{3}\, x\right)+c_2\sin\left(\sqrt{3}\, x\right)+\frac{3\,\mathrm{e}^{-4x}}{19}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+3*y(x)=3*exp(-4*x),y(x), singsol=all)
```

$$y(x) = \sin\left(\sqrt{3}\,x\right) c_2 + \cos\left(\sqrt{3}\,x\right) c_1 + \frac{3\,\mathrm{e}^{-4x}}{19}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 37

```
DSolve[y''[x]+3*y[x]==3*Exp[-4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3e^{-4x}}{19} + c_1 \cos\left(\sqrt{3}x\right) + c_2 \sin\left(\sqrt{3}x\right)$$

14.3 problem 3

Internal problem ID [2204]
Internal file name [`OUTPUT/2204_Monday_February_26_2024_09_18_22_AM_6353427/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 4y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}}$$

14.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)} \\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 e^{-2x} + c_2 e^{-2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2x} + x\, e^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\frac{e^x}{2} + \frac{e^{-x}}{2}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x + A_2 e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^x + A_2 e^{-x} = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{18}, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{18} + \frac{e^{-x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + x\, e^{-2x} c_2\right) + \left(\frac{e^x}{18} + \frac{e^{-x}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_2 x + c_1) + \frac{e^x}{18} + \frac{e^{-x}}{2}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_2 x + c_1) + \frac{e^x}{18} + \frac{e^{-x}}{2} \tag{1}$$

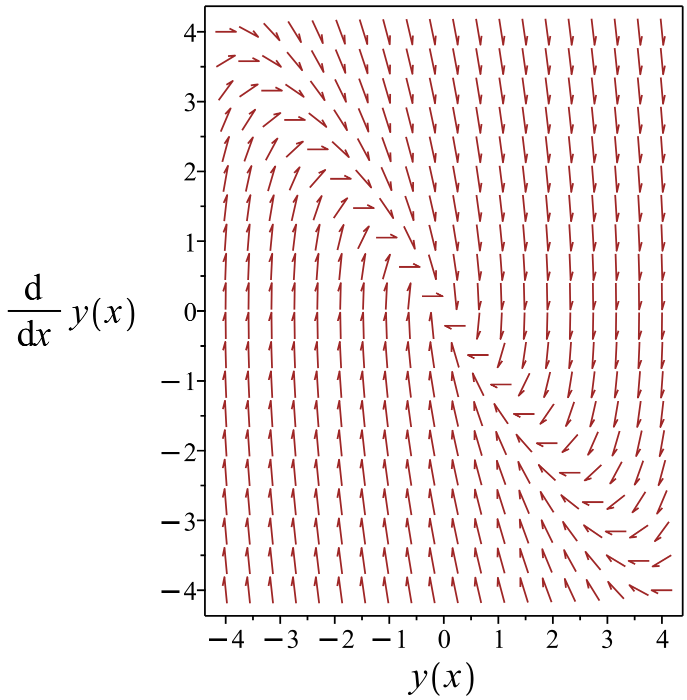


Figure 636: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) + \frac{e^x}{18} + \frac{e^{-x}}{2}$$

Verified OK.

14.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 4\, dx} \\ &= e^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= e^{2x}\left(\frac{e^{x}}{2}+\frac{e^{-x}}{2}\right) \\
(e^{2x}y)'' &= e^{2x}\left(\frac{e^{x}}{2}+\frac{e^{-x}}{2}\right)
\end{aligned}
$$

Integrating once gives

$$
(e^{2x}y)' = \frac{e^{3x}}{6}+\frac{e^{x}}{2}+c_1
$$

Integrating again gives

$$
(e^{2x}y) = c_1x+\frac{e^{x}}{2}+\frac{e^{3x}}{18}+c_2
$$

Hence the solution is

$$
y = \frac{c_1x+\frac{e^{x}}{2}+\frac{e^{3x}}{18}+c_2}{e^{2x}}
$$

Or

$$
y = \frac{e^{x}}{18}+c_1x\,e^{-2x}+c_2e^{-2x}+\frac{e^{-x}}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{e^{x}}{18}+c_1x\,e^{-2x}+c_2e^{-2x}+\frac{e^{-x}}{2} \tag{1}
$$

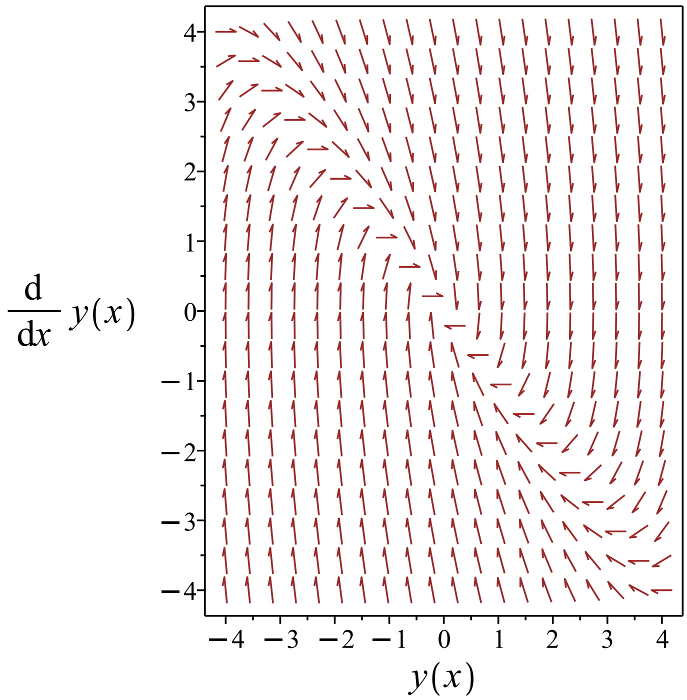


Figure 637: Slope field plot

Verification of solutions

$$y = \frac{e^{x}}{18} + c_1 x\, e^{-2x} + c_2 e^{-2x} + \frac{e^{-x}}{2}$$

Verified OK.

14.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 478: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\frac{\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{2}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{x}\}, \{\mathrm{e}^{-x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2x} x, \mathrm{e}^{-2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{x} + A_2 \mathrm{e}^{-x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
9A_1 \mathrm{e}^{x} + A_2 \mathrm{e}^{-x} = \frac{\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{2}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{18}, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{18} + \frac{e^{-x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + x\, e^{-2x} c_2\right) + \left(\frac{e^x}{18} + \frac{e^{-x}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_2 x + c_1) + \frac{e^x}{18} + \frac{e^{-x}}{2}$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(c_2 x + c_1) + \frac{e^x}{18} + \frac{e^{-x}}{2} \tag{1}$$

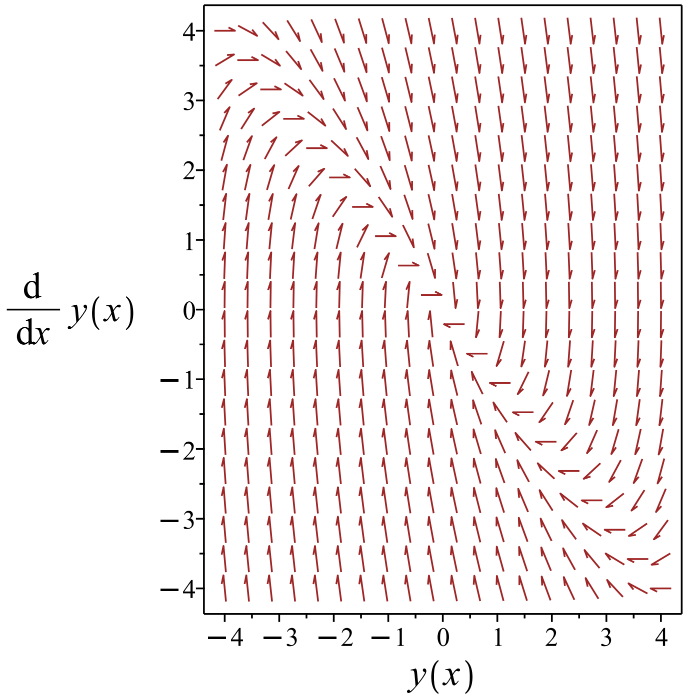


Figure 638: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2 x + c_1) + \frac{e^{x}}{18} + \frac{e^{-x}}{2}$$

Verified OK.

14.3.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 2)^2 = 0$

- Root of the characteristic polynomial

$r=-2$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x)=\mathrm{e}^{-2x}x$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-2x}+x\,\mathrm{e}^{-2x}c_2+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=\frac{\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{-x}}{2}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x))=\begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-2x}x \\ -2\,\mathrm{e}^{-2x} & -2\,\mathrm{e}^{-2x}x+\mathrm{e}^{-2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x))=\mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=-\frac{\mathrm{e}^{-2x}\left(\int x\,\mathrm{e}^{x}\left(\mathrm{e}^{2x}+1\right)dx-\left(\int \left(\mathrm{e}^{3x}+\mathrm{e}^{x}\right)dx\right)x\right)}{2}$

  - Compute integrals

    $y_p(x)=\frac{\mathrm{e}^{x}}{18}+\frac{\mathrm{e}^{-x}}{2}$

- Substitute particular solution into general solution to ODE

  $y=x\,\mathrm{e}^{-2x}c_2+c_1\mathrm{e}^{-2x}+\frac{\mathrm{e}^{x}}{18}+\frac{\mathrm{e}^{-x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=1/2*(exp(x)+exp(-x)),y(x), singsol=all)
```

$$y(x) = \frac{(e^{3x} + 18c_1x + 9\,e^{x} + 18c_2)\,e^{-2x}}{18}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 34

```
DSolve[y''[x]+4*y'[x]+4*y[x]==1/2*(Exp[x]+Exp[-x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{18}e^{-2x}\left(9e^{x} + e^{3x} + 18(c_2x + c_1)\right)$$

14.4 problem 4

Internal problem ID [2205]
Internal file name [`OUTPUT/2205_Monday_February_26_2024_09_18_23_AM_97609884/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' - 2y = \mathrm{e}^{-2x}}$$

14.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = \mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-2)} \\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-2x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-2x} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 e^{-2x} = e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{-2x} x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{-2x}\right) + \left(-\frac{e^{-2x} x}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{-2x} - \frac{e^{-2x} x}{3} \tag{1}$$

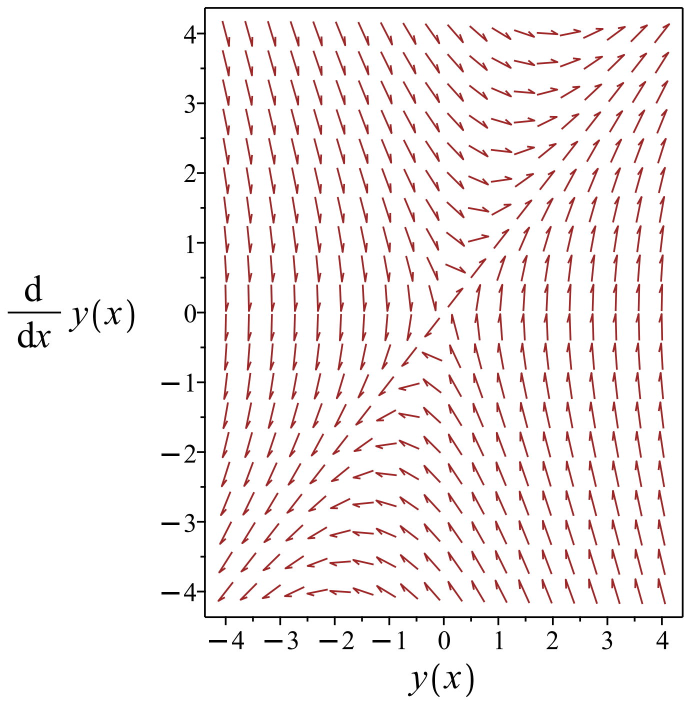


Figure 639: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-2x} - \frac{e^{-2x} x}{3}$$

Verified OK.

14.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 480: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{3x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{x}}{3}, e^{-2x}\right\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-2x} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 e^{-2x} = e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{-2x}x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{x}}{3}\right) + \left(-\frac{e^{-2x}x}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} - \frac{e^{-2x}x}{3} \tag{1}$$

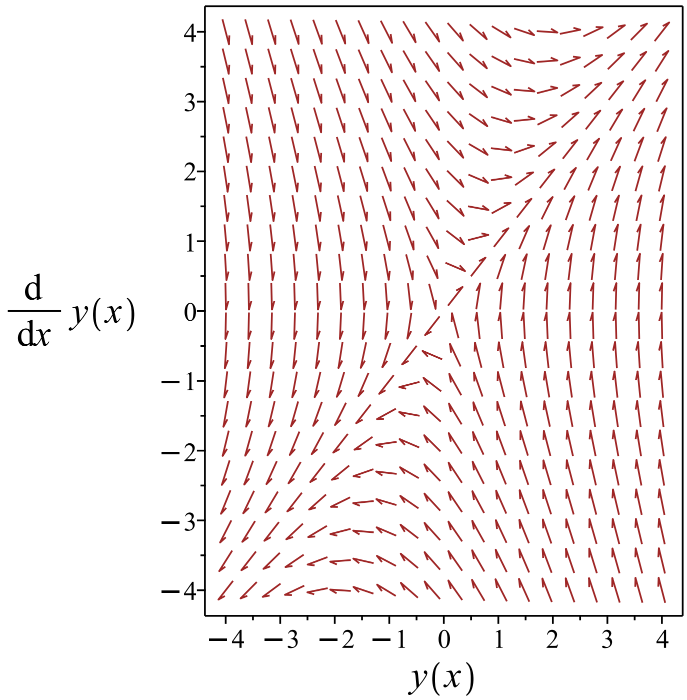


Figure 640: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^x}{3} - \frac{e^{-2x} x}{3}$$

Verified OK.

14.4.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = e^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 1) = 0$

- Roots of the characteristic polynomial

$r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \mathrm{e}^{-2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\left(-\mathrm{e}^{3x}\left(\int \mathrm{e}^{-3x}dx\right) + \int 1dx\right)\mathrm{e}^{-2x}}{3}$

  ◦ Compute integrals

  $y_p(x) = -\frac{(3x+1)\mathrm{e}^{-2x}}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} - \frac{(3x+1)\mathrm{e}^{-2x}}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=exp(-2*x),y(x), singsol=all)
```

$$y(x) = -\frac{(-3c_1 e^{3x} - 3c_2 + x)\, e^{-2x}}{3}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 32

```
DSolve[y''[x]+y'[x]-2*y[x]==Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9} e^{-2x} \left(-3x + 9c_2 e^{3x} - 1 + 9c_1\right)$$

14.5 problem 5

Internal problem ID [2206]
Internal file name [`OUTPUT/2206_Monday_February_26_2024_09_18_23_AM_49917117/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y = \sin(x)}$$

14.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 2, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{0}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)$$

Or

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{2}\, x\right), \sin\left(\sqrt{2}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(x) + A_2 \sin(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) + (\sin(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + \sin(x) \tag{1}$$

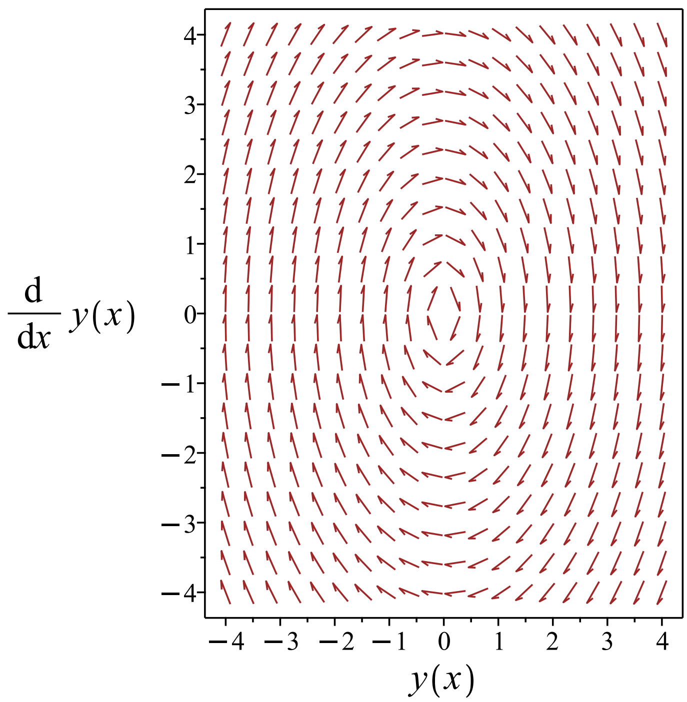


Figure 641: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + \sin(x)$$

Verified OK.

14.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 482: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\sqrt{2}\, x\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos\left(\sqrt{2}\, x\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, x\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos\left(\sqrt{2}\, x\right) \int \frac{1}{\cos\left(\sqrt{2}\, x\right)^2}\, dx \\
&= \cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\sqrt{2}\, x\right)\right) + c_2 \left(\cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(\sqrt{2}\, x\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2}, \cos\left(\sqrt{2}\,x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(x) + A_2\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\cos(x) + A_2\sin(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos\left(\sqrt{2}\,x\right) + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2}\right) + (\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos\left(\sqrt{2}\,x\right) + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} + \sin(x) \tag{1}$$

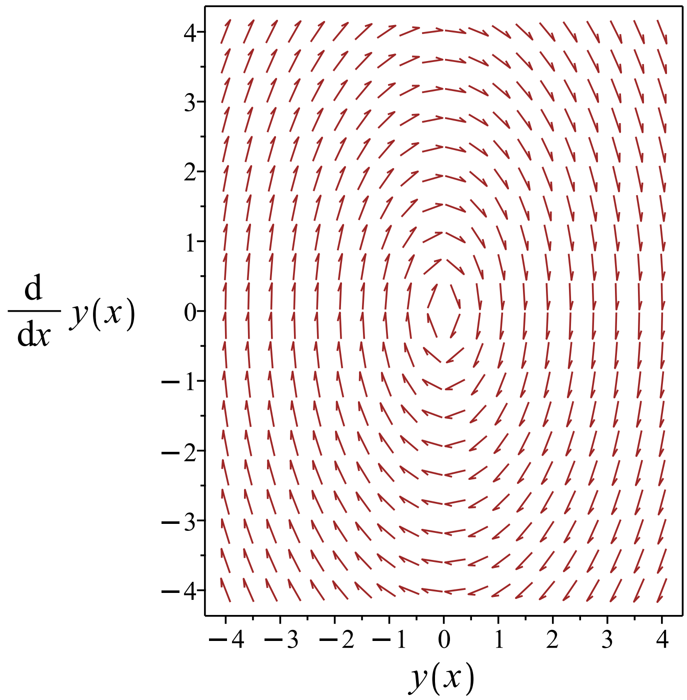


Figure 642: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + \frac{c_2\sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2} + \sin(x)$$

Verified OK.

14.5.3 Maple step by step solution

Let's solve

$y'' + 2y = \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

$r = \left(-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2}\right)$

- 1st solution of the homogeneous ODE
  $y_1(x) = \cos\left(\sqrt{2}\, x\right)$
- 2nd solution of the homogeneous ODE
  $y_2(x) = \sin\left(\sqrt{2}\, x\right)$
- General solution of the ODE
  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + y_p(x)$
- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos\left(\sqrt{2}\, x\right) & \sin\left(\sqrt{2}\, x\right) \\ -\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) & \sqrt{2}\, \cos\left(\sqrt{2}\, x\right) \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = \sqrt{2}$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = -\frac{\sqrt{2}\left(\cos\left(\sqrt{2}\, x\right)\left(\int \sin\left(\sqrt{2}\, x\right) \sin(x) dx\right) - \sin\left(\sqrt{2}\, x\right)\left(\int \cos\left(\sqrt{2}\, x\right) \sin(x) dx\right)\right)}{2}$
  - Compute integrals
    $y_p(x) = \sin(x)$
- Substitute particular solution into general solution to ODE
  $y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+2*y(x)=sin(x),y(x), singsol=all)
```

$$y(x) = c_2 \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right) c_1 + \sin\left(x\right)$$

✓ Solution by Mathematica

Time used: 0.185 (sec). Leaf size: 30

```
DSolve[y''[x]+2*y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x) + c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right)$$

14.6 problem 6

Internal problem ID [2207]
Internal file name [`OUTPUT/2207_Monday_February_26_2024_09_18_24_AM_34888450/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 4y = \frac{e^{3x}}{2} - \frac{e^{-3x}}{2}}$$

14.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = \frac{e^{3x}}{2} - \frac{e^{-3x}}{2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(4)^2 - (4)\,(1)\,(4)}\\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 e^{-2x} + c_2 e^{-2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2x} + x\, e^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\frac{e^{3x}}{2} - \frac{e^{-3x}}{2}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-3x}\}, \{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x} x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-3x} + A_2 \mathrm{e}^{3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \mathrm{e}^{-3x} + 25A_2 \mathrm{e}^{3x} = \frac{\mathrm{e}^{3x}}{2} - \frac{\mathrm{e}^{-3x}}{2}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{50}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\mathrm{e}^{-3x}}{2} + \frac{\mathrm{e}^{3x}}{50}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2\right) + \left(-\frac{\mathrm{e}^{-3x}}{2} + \frac{\mathrm{e}^{3x}}{50}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) - \frac{\mathrm{e}^{-3x}}{2} + \frac{\mathrm{e}^{3x}}{50}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) - \frac{\mathrm{e}^{-3x}}{2} + \frac{\mathrm{e}^{3x}}{50} \tag{1}$$

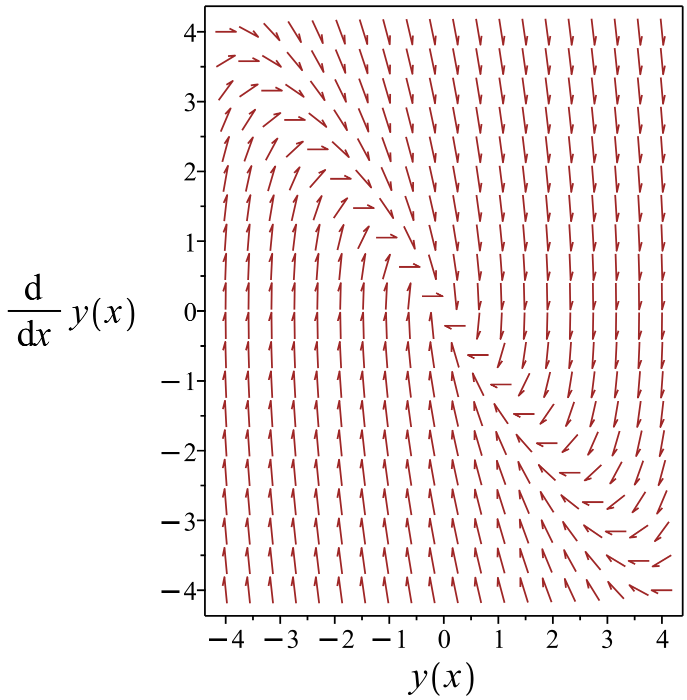


Figure 643: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) - \frac{\mathrm{e}^{-3x}}{2} + \frac{\mathrm{e}^{3x}}{50}$$

Verified OK.

14.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 4\,dx} \\ &= \mathrm{e}^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= e^{2x}\left(\frac{e^{3x}}{2}-\frac{e^{-3x}}{2}\right) \\
\left(e^{2x}y\right)'' &= e^{2x}\left(\frac{e^{3x}}{2}-\frac{e^{-3x}}{2}\right)
\end{aligned}
$$

Integrating once gives

$$
\left(e^{2x}y\right)' = \frac{e^{5x}}{10}+\frac{e^{-x}}{2}+c_1
$$

Integrating again gives

$$
\left(e^{2x}y\right) = c_1x-\frac{e^{-x}}{2}+\frac{e^{5x}}{50}+c_2
$$

Hence the solution is

$$
y = \frac{c_1x-\frac{e^{-x}}{2}+\frac{e^{5x}}{50}+c_2}{e^{2x}}
$$

Or

$$
y = \frac{e^{3x}}{50}+c_1x\,e^{-2x}+c_2e^{-2x}-\frac{e^{-3x}}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{e^{3x}}{50}+c_1x\,e^{-2x}+c_2e^{-2x}-\frac{e^{-3x}}{2} \tag{1}
$$

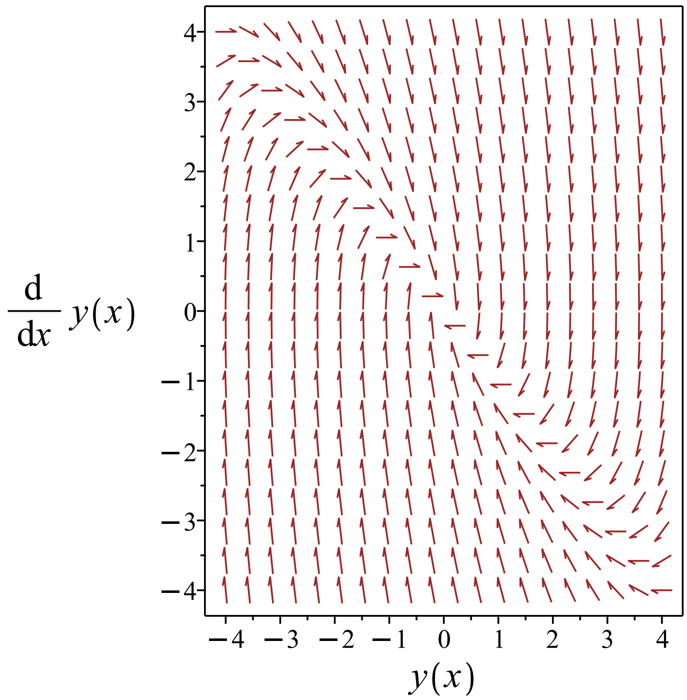


Figure 644: Slope field plot

Verification of solutions

$$y = \frac{e^{3x}}{50} + c_1 x\, e^{-2x} + c_2 e^{-2x} - \frac{e^{-3x}}{2}$$

Verified OK.

14.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 484: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1\left(\mathrm{e}^{-2x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{\mathrm{e}^{-4x}}{(y_1)^2}\,dx \\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x}(x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\frac{\mathrm{e}^{3x}}{2} - \frac{\mathrm{e}^{-3x}}{2}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-3x}\}, \{\mathrm{e}^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x} x, \mathrm{e}^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-3x} + A_2 \mathrm{e}^{3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \mathrm{e}^{-3x} + 25 A_2 \mathrm{e}^{3x} = \frac{\mathrm{e}^{3x}}{2} - \frac{\mathrm{e}^{-3x}}{2}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{50}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\mathrm{e}^{-3x}}{2} + \frac{\mathrm{e}^{3x}}{50}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2\right) + \left(-\frac{\mathrm{e}^{-3x}}{2} + \frac{\mathrm{e}^{3x}}{50}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) - \frac{\mathrm{e}^{-3x}}{2} + \frac{\mathrm{e}^{3x}}{50}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) - \frac{\mathrm{e}^{-3x}}{2} + \frac{\mathrm{e}^{3x}}{50} \tag{1}$$

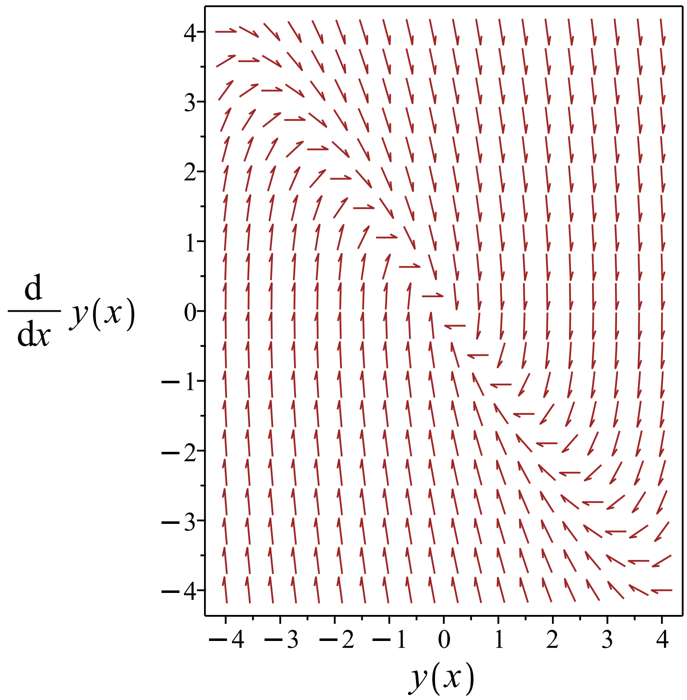


Figure 645: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) - \frac{e^{-3x}}{2} + \frac{e^{3x}}{50}$$

Verified OK.

14.6.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = \frac{e^{3x}}{2} - \frac{e^{-3x}}{2}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 2)^2 = 0$

- Root of the characteristic polynomial

$r = -2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{-2x}x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + x\, e^{-2x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{e^{3x}}{2} - \frac{e^{-3x}}{2}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-2x}x \\ -2\,e^{-2x} & -2\,e^{-2x}x + e^{-2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-2x}\left(\int x(e^{5x}-e^{-x})dx - \left(\int (e^{5x}-e^{-x})dx\right)x\right)}{2}$

  ◦ Compute integrals

  $y_p(x) = -\frac{e^{-3x}}{2} + \frac{e^{3x}}{50}$

- Substitute particular solution into general solution to ODE

  $y = x\, e^{-2x} c_2 + c_1 e^{-2x} - \frac{e^{-3x}}{2} + \frac{e^{3x}}{50}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=1/2*(exp(3*x)-exp(-3*x)),y(x), singsol=all)
```

$$y(x) = \frac{\left(-25 + e^{6x} + 50\, e^{x}(c_1 x + c_2)\right) e^{-3x}}{50}$$

✓ Solution by Mathematica

Time used: 0.217 (sec). Leaf size: 33

```
DSolve[y''[x]+4*y'[x]+4*y[x]==1/2*(Exp[3*x]-Exp[-3*x]),y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{50} e^{-3x} \left(e^{6x} + 50 e^{x}(c_2 x + c_1) - 25\right)$$

14.7 problem 7

Internal problem ID [2208]
Internal file name [`OUTPUT/2208_Monday_February_26_2024_09_18_24_AM_23921008/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' - 2y = \sin(2x)}$$

14.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = -2, f(x) = \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda \, e^{\lambda x} - 2 \, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = -2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{3^2 - (4)(1)(-2)} \\ &= -\frac{3}{2} \pm \frac{\sqrt{17}}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= -\frac{3}{2} + \frac{\sqrt{17}}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{\sqrt{17}}{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= -\frac{3}{2} + \frac{\sqrt{17}}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{\sqrt{17}}{2} \end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned} y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)x} + c_2 e^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)x} \end{aligned}$$

Or

$$y = c_1 e^{\left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)x} + c_2 e^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{\left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)x} + c_2 e^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)x}, e^{\left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \cos(2x) - 6A_2 \sin(2x) - 6A_1 \sin(2x) + 6A_2 \cos(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{12}, A_2 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(2x)}{12} - \frac{\sin(2x)}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)x} + c_2 e^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)x}\right) + \left(-\frac{\cos(2x)}{12} - \frac{\sin(2x)}{12}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 \mathrm{e}^{\frac{\left(-3+\sqrt{17}\right)x}{2}} + c_2 \mathrm{e}^{-\frac{\left(3+\sqrt{17}\right)x}{2}} - \frac{\cos\left(2x\right)}{12} - \frac{\sin\left(2x\right)}{12}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{\left(-3+\sqrt{17}\right)x}{2}} + c_2 \mathrm{e}^{-\frac{\left(3+\sqrt{17}\right)x}{2}} - \frac{\cos\left(2x\right)}{12} - \frac{\sin\left(2x\right)}{12} \tag{1}$$

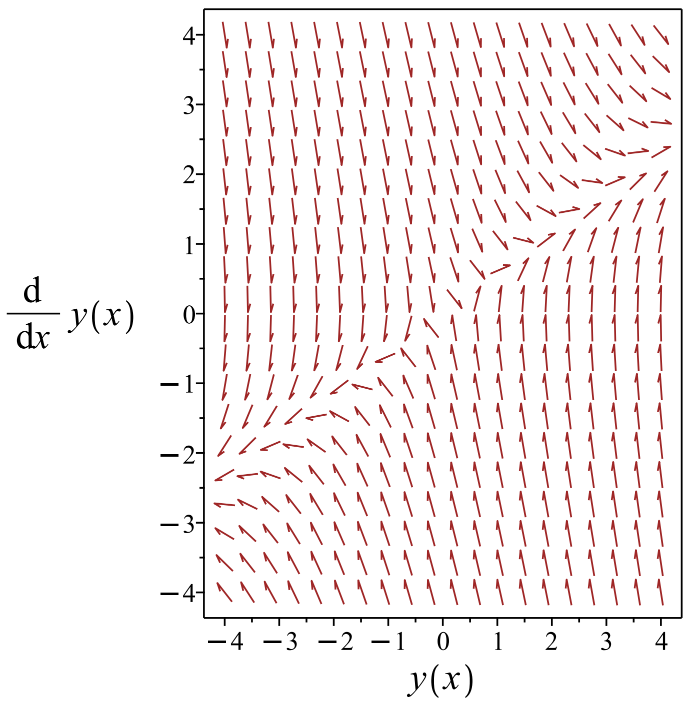


Figure 646: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{\left(-3+\sqrt{17}\right)x}{2}} + c_2 \mathrm{e}^{-\frac{\left(3+\sqrt{17}\right)x}{2}} - \frac{\cos\left(2x\right)}{12} - \frac{\sin\left(2x\right)}{12}$$

Verified OK.

14.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{17}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 17 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{17z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 486: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{17}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = e^{-\frac{x\sqrt{17}}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{\left(3+\sqrt{17}\right)x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{17}\, \mathrm{e}^{x\sqrt{17}}}{17}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{\left(3+\sqrt{17}\right)x}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{\left(3+\sqrt{17}\right)x}{2}} \left(\frac{\sqrt{17}\, \mathrm{e}^{x\sqrt{17}}}{17}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{\left(3+\sqrt{17}\right)x}{2}} + \frac{c_2 \sqrt{17}\, \mathrm{e}^{\frac{\left(-3+\sqrt{17}\right)x}{2}}}{17}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{17}\, \mathrm{e}^{\frac{\left(-3+\sqrt{17}\right)x}{2}}}{17}, \mathrm{e}^{-\frac{\left(3+\sqrt{17}\right)x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \cos(2x) - 6A_2 \sin(2x) - 6A_1 \sin(2x) + 6A_2 \cos(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{12}, A_2 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(2x)}{12} - \frac{\sin(2x)}{12}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-\frac{(3+\sqrt{17})x}{2}} + \frac{c_2\sqrt{17}\,\mathrm{e}^{\frac{(-3+\sqrt{17})x}{2}}}{17}\right) + \left(-\frac{\cos(2x)}{12} - \frac{\sin(2x)}{12}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{(3+\sqrt{17})x}{2}} + \frac{c_2\sqrt{17}\,\mathrm{e}^{\frac{(-3+\sqrt{17})x}{2}}}{17} - \frac{\cos(2x)}{12} - \frac{\sin(2x)}{12} \tag{1}
$$

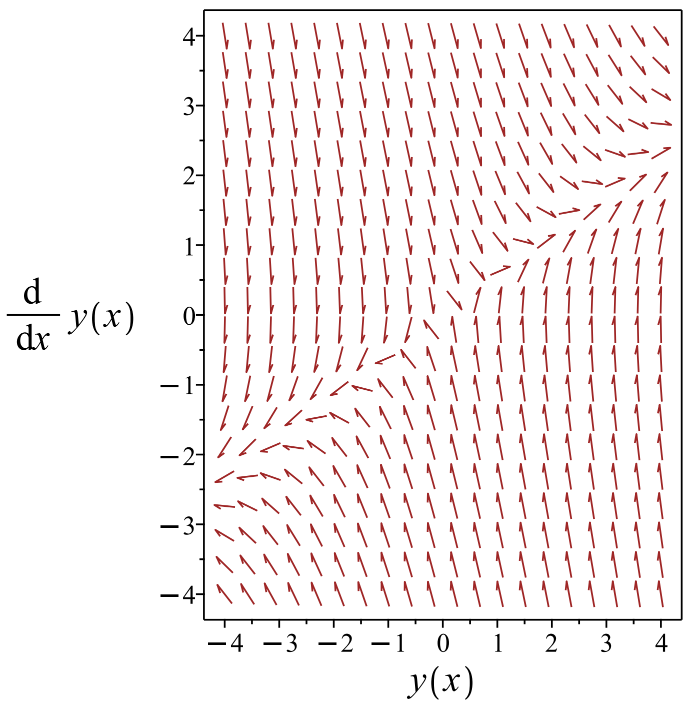


Figure 647: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{(3+\sqrt{17})x}{2}} + \frac{c_2\sqrt{17}\,\mathrm{e}^{\frac{(-3+\sqrt{17})x}{2}}}{17} - \frac{\cos(2x)}{12} - \frac{\sin(2x)}{12}
$$

Verified OK.

14.7.3 Maple step by step solution

Let's solve

$y'' + 3y' - 2y = \sin(2x)$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 3r - 2 = 0$

- Use quadratic formula to solve for r

$r = \frac{(-3)\pm\left(\sqrt{17}\right)}{2}$

- Roots of the characteristic polynomial

$r = \left(-\frac{3}{2} - \frac{\sqrt{17}}{2}, -\frac{3}{2} + \frac{\sqrt{17}}{2}\right)$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)x}$

- 2nd solution of the homogeneous ODE

$y_2(x) = e^{\left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)x}$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 e^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)x} + c_2 e^{\left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(2x)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)x} & e^{\left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)x} \\ \left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)e^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)x} & \left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)e^{\left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)x} \end{bmatrix}$

- ◦ Compute Wronskian

$W(y_1(x), y_2(x)) = \sqrt{17}\, e^{-3x}$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{\sqrt{17}\left(-\mathrm{e}^{-\frac{\left(3+\sqrt{17}\right)x}{2}}\left(\int \mathrm{e}^{\frac{\left(3+\sqrt{17}\right)x}{2}}\sin(2x)dx\right)+\mathrm{e}^{\frac{\left(-3+\sqrt{17}\right)x}{2}}\left(\int \mathrm{e}^{-\frac{\left(-3+\sqrt{17}\right)x}{2}}\sin(2x)dx\right)\right)}{17}$$

- Compute integrals

$$y_p(x) = -\frac{\cos(2x)}{12} - \frac{\sin(2x)}{12}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{\left(-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)x} + c_2\mathrm{e}^{\left(-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)x} - \frac{\cos(2x)}{12} - \frac{\sin(2x)}{12}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)-2*y(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\frac{\left(-3+\sqrt{17}\right)x}{2}}c_2 + \mathrm{e}^{-\frac{\left(3+\sqrt{17}\right)x}{2}}c_1 - \frac{\sin(2x)}{12} - \frac{\cos(2x)}{12}$$

✓ Solution by Mathematica

Time used: 0.346 (sec). Leaf size: 52

```
DSolve[y''[x]+3*y'[x]-2*y[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{12}\cos(2x) + e^{-\frac{1}{2}\left(3+\sqrt{17}\right)x}\left(c_2e^{\sqrt{17}x} + c_1\right) - \frac{1}{6}\sin(x)\cos(x)$$

14.8 problem 8

Internal problem ID [2209]
Internal file name [`OUTPUT/2209_Monday_February_26_2024_09_18_25_AM_67612745/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \sin(x)\,e^{x}}$$

14.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = \sin(x)\,e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\
&= -\frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,e^{x}, \sin(x)\,e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)\,e^{x} + A_2 \sin(x)\,e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \sin(x)\,e^{x} + 5A_2 \cos(x)\,e^{x} + 5A_1 \cos(x)\,e^{x} + 5A_2 \sin(x)\,e^{x} = \sin(x)\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{10}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,e^{x}}{10} + \frac{\sin(x)\,e^{x}}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(-\frac{\cos(x)\,e^{x}}{10} + \frac{\sin(x)\,e^{x}}{10}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{\cos(x) e^{x}}{10} + \frac{\sin(x) e^{x}}{10} \tag{1}$$

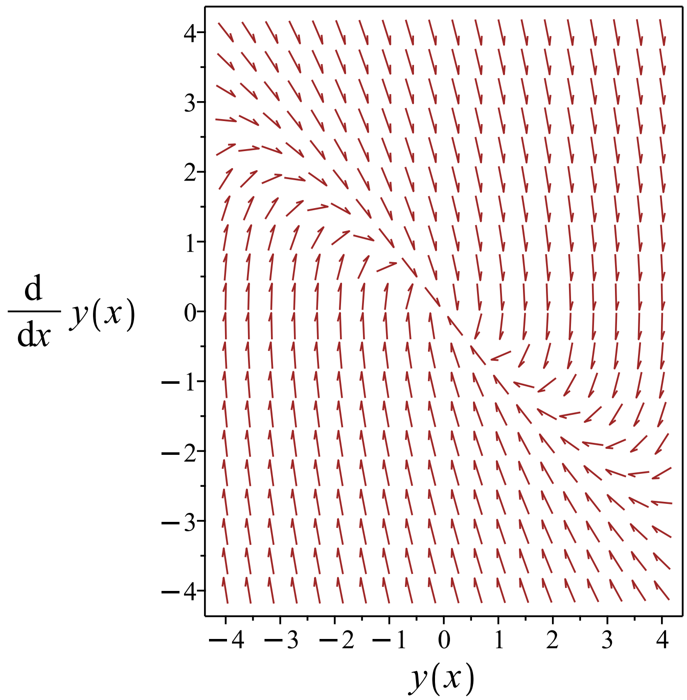


Figure 648: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{\cos(x) e^{x}}{10} + \frac{\sin(x) e^{x}}{10}$$

Verified OK.

14.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 3 \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 488: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3}{1} \, dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (\mathrm{e}^{-2x}) + c_2 (\mathrm{e}^{-2x} (\mathrm{e}^x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin (x)\, \mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,\mathrm{e}^{x}, \sin(x)\,\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)\,\mathrm{e}^{x} + A_2 \sin(x)\,\mathrm{e}^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \sin(x)\,\mathrm{e}^{x} + 5A_2 \cos(x)\,\mathrm{e}^{x} + 5A_1 \cos(x)\,\mathrm{e}^{x} + 5A_2 \sin(x)\,\mathrm{e}^{x} = \sin(x)\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{10}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,\mathrm{e}^{x}}{10} + \frac{\sin(x)\,\mathrm{e}^{x}}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}\right) + \left(-\frac{\cos(x)\,\mathrm{e}^{x}}{10} + \frac{\sin(x)\,\mathrm{e}^{x}}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} - \frac{\cos(x)\,\mathrm{e}^{x}}{10} + \frac{\sin(x)\,\mathrm{e}^{x}}{10} \tag{1}$$

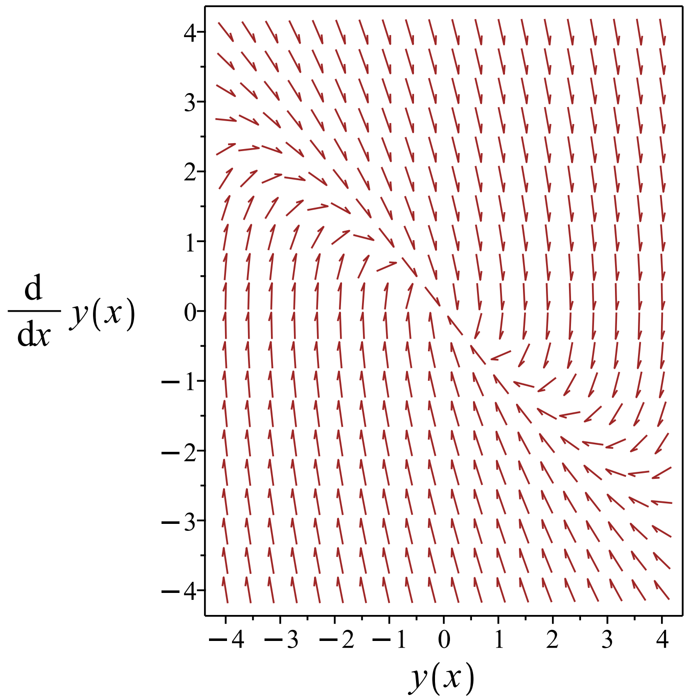


Figure 649: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} - \frac{\cos(x)\, e^{x}}{10} + \frac{\sin(x)\, e^{x}}{10}$$

Verified OK.

14.8.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = \sin(x)\, e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

$r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x) e^x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2 e^{-2x} & -e^{-x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-2x} \left(\int e^{3x} \sin(x) dx\right) + e^{-x} \left(\int e^{2x} \sin(x) dx\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{(-\cos(x) + \sin(x)) e^x}{10}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + \frac{(-\cos(x) + \sin(x)) e^x}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=exp(x)*sin(x),y(x), singsol=all)
```

$$y(x) = -\mathrm{e}^{-2x}\left(\frac{(-\sin(x)+\cos(x))\,\mathrm{e}^{3x}}{10} - c_2\mathrm{e}^{x} + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 36

```
DSolve[y''[x]+3*y'[x]+2*y[x]==Exp[x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1e^{-2x} + c_2e^{-x} + \frac{1}{10}e^{x}(\sin(x) - \cos(x))$$

14.9 problem 9

Internal problem ID [2210]
Internal file name [OUTPUT/2210_Monday_February_26_2024_09_18_25_AM_12694163/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - y = e^x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y = 0$$

The characteristic equation is

$$\lambda^3 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$

$$y_2 = e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$

$$y_3 = e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y''' - y = e^x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x, e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}, e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}\right\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\, e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^x + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3\right) + \left(\frac{x\,\mathrm{e}^x}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^x + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + \frac{x\,\mathrm{e}^x}{3} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^x + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + \frac{x\,\mathrm{e}^x}{3}$$

Verified OK.

14.9.1 Maple step by step solution

Let's solve

$y''' - y = \mathrm{e}^x$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = e^x + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = e^x + y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ e^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ e^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c}1\\1\\1\end{array}\right]\right], \left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}}\\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\\ 1\end{array}\right]\right], \left[-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}}\\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}\\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1\\1\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1\\1\\1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}}\\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}}\\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}}\\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\\ 1\end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I} \sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{-\frac{1}{2}-\frac{\mathrm{I} \sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\, x}{2}\right)-\mathrm{I} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2}-\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\, x}{2}\right) \end{array}\right], \vec{y}_{3}(x)=e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\, x}{2}\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_{p}($

$\vec{y}(x)=c_{1} \vec{y}_{1}+c_{2} \vec{y}_{2}(x)+c_{3} \vec{y}_{3}(x)+\vec{y}_{p}(x)$

☐ Fundamental matrix

  ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\left[\begin{array}{ccc} e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2}-\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) & -e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \end{array}\right]$$

  ◦ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$\Phi(x)=\phi(x) \cdot \frac{1}{\phi(0)}$

  ◦ Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\left[\begin{array}{ccc} e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2}-\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) & -e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \end{array}\right] \cdot \frac{1}{\left[\begin{array}{ccc} 1 & -\frac{1}{2} & - \\ 1 & -\frac{1}{2} & \sqrt{3} \\ 1 & 1 & 0 \end{array}\right.}$$

  ◦ Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{2\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\mathrm{e}^{x}}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}\right.}{3} \\ \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} & \frac{2\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\mathrm{e}^{x}}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}\right.}{3} \\ \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} & \frac{2\,\mathrm{e}^{-\frac{x}{2}}\cos}{3} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

    $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
  - Take the derivative of the particular solution

    $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
  - Cancel like terms

    $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
  - Integrate to solve for $\vec{v}(x)$

    $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
  - Plug $\vec{v}(x)$ into the equation for the particular solution

    $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{9} + \frac{(x-1)\mathrm{e}^{x}}{3} \\ -\frac{2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{9} + \frac{x\,\mathrm{e}^{x}}{3} \\ -\frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{9} + \frac{(x+1)\mathrm{e}^{x}}{3} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} \frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{9} + \frac{(x-1)e^{x}}{3} \\ -\frac{2\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{9} + \frac{x\,e^{x}}{3} \\ -\frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{9} + \frac{(x+1)e^{x}}{3} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\left(c_3\sqrt{3}+c_2-\frac{2}{3}\right)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\left(\left(c_2-\frac{2}{9}\right)\sqrt{3}-c_3\right)e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{x}(x+3c_1-1)}{3}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$3)-y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = c_2e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_3e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{e^{x}(x+3c_1)}{3}$$

✓ Solution by Mathematica

Time used: 0.408 (sec). Leaf size: 62

```
DSolve[y'''[x]-y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}e^{-x/2}\left(e^{3x/2}(x-1+3c_1) + 3c_2\cos\left(\frac{\sqrt{3}x}{2}\right) + 3c_3\sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

14.10 problem 10

Internal problem ID [2211]
Internal file name [`OUTPUT/2211_Monday_February_26_2024_09_18_25_AM_3871879/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 10.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 4y'' + y' - 4y = \sin(x) - e^{4x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' + y' - 4y = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + \lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 4$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{4x} + c_2 e^{-ix} + e^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{4x}$$
$$y_2 = e^{-ix}$$
$$y_3 = e^{ix}$$

Now the particular solution to the given ODE is found

$$y''' - 4y'' + y' - 4y = \sin(x) - e^{4x}$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{4x} & e^{-ix} & e^{ix} \\ 4\, e^{4x} & -i e^{-ix} & i e^{ix} \\ 16\, e^{4x} & -e^{-ix} & -e^{ix} \end{bmatrix}$$
$$|W| = 34 i e^{4x} e^{-ix} e^{ix}$$

The determinant simplifies to

$$|W| = 34i e^{4x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} e^{-ix} & e^{ix} \\ -ie^{-ix} & ie^{ix} \end{bmatrix} \\ &= 2i \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} e^{4x} & e^{ix} \\ 4\,e^{4x} & ie^{ix} \end{bmatrix} \\ &= (-4+i)\,e^{(4+i)x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} e^{4x} & e^{-ix} \\ 4\,e^{4x} & -ie^{-ix} \end{bmatrix} \\ &= (-4-i)\,e^{(4-i)x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x) W_1(x)}{a W(x)}\,dx \\ &= (-1)^2 \int \frac{(\sin(x) - e^{4x})\,(2i)}{(1)\,(34ie^{4x})}\,dx \\ &= \int \frac{2i(\sin(x) - e^{4x})}{34ie^{4x}}\,dx \\ &= \int \left(\frac{\sin(x)\,e^{-4x}}{17} - \frac{1}{17} \right) dx \\ &= -\frac{x}{17} - \frac{\cos(x)\,e^{-4x}}{289} - \frac{4\sin(x)\,e^{-4x}}{289} \\ &= -\frac{x}{17} - \frac{\cos(x)\,e^{-4x}}{289} - \frac{4\sin(x)\,e^{-4x}}{289} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(\sin(x) - \mathrm{e}^{4x})\left((-4+i)\,\mathrm{e}^{(4+i)x}\right)}{(1)\,(34i\mathrm{e}^{4x})}\,dx \\
&= -\int \frac{(-4+i)\,(\sin(x) - \mathrm{e}^{4x})\,\mathrm{e}^{(4+i)x}}{34i\mathrm{e}^{4x}}\,dx \\
&= -\int \left(\left(\frac{1}{34} + \frac{2i}{17}\right)(\sin(x) - \mathrm{e}^{4x})\,\mathrm{e}^{ix}\right)dx \\
&= \frac{x}{17} - \frac{ix}{68} + \frac{\mathrm{e}^{2ix}}{136} + \frac{i\mathrm{e}^{2ix}}{34} + \frac{4\,\mathrm{e}^{(4+i)x}}{289} + \frac{15i\mathrm{e}^{(4+i)x}}{578} \\
&= \frac{x}{17} - \frac{ix}{68} + \frac{\mathrm{e}^{2ix}}{136} + \frac{i\mathrm{e}^{2ix}}{34} + \frac{4\,\mathrm{e}^{(4+i)x}}{289} + \frac{15i\mathrm{e}^{(4+i)x}}{578}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(\sin(x) - \mathrm{e}^{4x})\left((-4-i)\,\mathrm{e}^{(4-i)x}\right)}{(1)\,(34i\mathrm{e}^{4x})}\,dx \\
&= \int \frac{(-4-i)\,(\sin(x) - \mathrm{e}^{4x})\,\mathrm{e}^{(4-i)x}}{34i\mathrm{e}^{4x}}\,dx \\
&= \int \left(\left(-\frac{1}{34} + \frac{2i}{17}\right)(\sin(x) - \mathrm{e}^{4x})\,\mathrm{e}^{-ix}\right)dx \\
&= \int \left(-\frac{1}{34} + \frac{2i}{17}\right)(\sin(x) - \mathrm{e}^{4x})\,\mathrm{e}^{-ix}dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{x}{17} - \frac{\cos(x)\,\mathrm{e}^{-4x}}{289} - \frac{4\sin(x)\,\mathrm{e}^{-4x}}{289}\right)(\mathrm{e}^{4x}) \\
&+ \left(\frac{x}{17} - \frac{ix}{68} + \frac{\mathrm{e}^{2ix}}{136} + \frac{i\mathrm{e}^{2ix}}{34} + \frac{4\,\mathrm{e}^{(4+i)x}}{289} + \frac{15i\mathrm{e}^{(4+i)x}}{578}\right)(\mathrm{e}^{-ix}) \\
&+ \left(\int \left(-\frac{1}{34} + \frac{2i}{17}\right)(\sin(x) - \mathrm{e}^{4x})\,\mathrm{e}^{-ix}dx\right)(\mathrm{e}^{ix})
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{(-17x+8)\,\mathrm{e}^{4x}}{289} + \frac{(9+68i+272x)\cos(x)}{2312} - \frac{\left(\frac{236}{17} + i + 4x\right)\sin(x)}{136}
$$

Which simplifies to

$$y_p = \frac{(-17x+8)\,\mathrm{e}^{4x}}{289} + \frac{(9+68i+272x)\cos(x)}{2312} - \frac{\left(\frac{236}{17}+i+4x\right)\sin(x)}{136}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{4x} + c_2\mathrm{e}^{-ix} + \mathrm{e}^{ix}c_3\right) \\ &\quad + \left(\frac{(-17x+8)\,\mathrm{e}^{4x}}{289} + \frac{(9+68i+272x)\cos(x)}{2312} - \frac{\left(\frac{236}{17}+i+4x\right)\sin(x)}{136}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned} y &= c_1\mathrm{e}^{4x} + c_2\mathrm{e}^{-ix} + \mathrm{e}^{ix}c_3 + \frac{(-17x+8)\,\mathrm{e}^{4x}}{289} \\ &\quad + \frac{(9+68i+272x)\cos(x)}{2312} - \frac{\left(\frac{236}{17}+i+4x\right)\sin(x)}{136} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1\mathrm{e}^{4x} + c_2\mathrm{e}^{-ix} + \mathrm{e}^{ix}c_3 + \frac{(-17x+8)\,\mathrm{e}^{4x}}{289} \\ &\quad + \frac{(9+68i+272x)\cos(x)}{2312} - \frac{\left(\frac{236}{17}+i+4x\right)\sin(x)}{136} \end{aligned}$$

Verified OK.

14.10.1 Maple step by step solution

Let's solve

$y''' - 4y'' + y' - 4y = \sin(x) - \mathrm{e}^{4x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$

  - ○ Define new variable $y_2(x)$

$y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = \sin(x) - e^{4x} + 4y_3(x) - y_2(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \sin(x) - e^{4x} + 4y_3(x) - y_2(x) + 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -1 & 4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \sin(x) - e^{4x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \sin(x) - e^{4x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -1 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[4, \left[\begin{array}{c}\frac{1}{16}\\ \frac{1}{4}\\ 1\end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c}-1\\ \mathrm{I}\\ 1\end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c}-1\\ -\mathrm{I}\\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[4, \left[\begin{array}{c}\frac{1}{16}\\ \frac{1}{4}\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{4x} \cdot \left[\begin{array}{c}\frac{1}{16}\\ \frac{1}{4}\\ 1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c}-1\\ \mathrm{I}\\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \left[\begin{array}{c}-1\\ \mathrm{I}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c}-1\\ \mathrm{I}\\ 1\end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c}-\cos(x) + \mathrm{I}\sin(x)\\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x))\\ \cos(x) - \mathrm{I}\sin(x)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{4x}}{16} & -\cos(x) & \sin(x) \\ \frac{e^{4x}}{4} & \sin(x) & \cos(x) \\ e^{4x} & \cos(x) & -\sin(x) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{4x}}{16} & -\cos(x) & \sin(x) \\ \frac{e^{4x}}{4} & \sin(x) & \cos(x) \\ e^{4x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{16} & -1 & 0 \\ \frac{1}{4} & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{4x}}{17} + \frac{16\cos(x)}{17} - \frac{4\sin(x)}{17} & \sin(x) & \frac{e^{4x}}{17} - \frac{\cos(x)}{17} - \frac{4\sin(x)}{17} \\ \frac{4e^{4x}}{17} - \frac{16\sin(x)}{17} - \frac{4\cos(x)}{17} & \cos(x) & \frac{4e^{4x}}{17} + \frac{\sin(x)}{17} - \frac{4\cos(x)}{17} \\ \frac{16e^{4x}}{17} - \frac{16\cos(x)}{17} + \frac{4\sin(x)}{17} & -\sin(x) & \frac{16e^{4x}}{17} + \frac{\cos(x)}{17} + \frac{4\sin(x)}{17} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

  - Take the derivative of the particular solution

$\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{(9-17x)\mathrm{e}^{4x}}{289}+\frac{(-9+34x)\cos(x)}{289}+\frac{(-17x-106)\sin(x)}{578} \\ \frac{(19-68x)\mathrm{e}^{4x}}{289}+\frac{(-17x-38)\cos(x)}{578}+\frac{(-68x+1)\sin(x)}{578} \\ \frac{8(1-34x)\mathrm{e}^{4x}}{289}+\frac{2(-4-17x)\cos(x)}{289}+\frac{\sin(x)(17x-30)}{578} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} \frac{(9-17x)\mathrm{e}^{4x}}{289}+\frac{(-9+34x)\cos(x)}{289}+\frac{(-17x-106)\sin(x)}{578} \\ \frac{(19-68x)\mathrm{e}^{4x}}{289}+\frac{(-17x-38)\cos(x)}{578}+\frac{(-68x+1)\sin(x)}{578} \\ \frac{8(1-34x)\mathrm{e}^{4x}}{289}+\frac{2(-4-17x)\cos(x)}{289}+\frac{\sin(x)(17x-30)}{578} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-272x+289c_1+144)\mathrm{e}^{4x}}{4624} + \frac{(-9+34x-289c_2)\cos(x)}{289} - \frac{\left(x-34c_3+\frac{106}{17}\right)\sin(x)}{34}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)+diff(y(x),x)-4*y(x)=sin(x)-exp(4*x),y(x), singsol=all
```

$$y(x) = \frac{(8-17x+289c_3)\,e^{4x}}{289} + \frac{(68x+578c_1+15)\cos(x)}{578} - \frac{\sin(x)\left(x-34c_2+\frac{8}{17}\right)}{34}$$

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 52

```
DSolve[y'''[x]-4*y''[x]+y'[x]-4*y[x]==Sin[x]-Exp[4*x],y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{289}e^{4x}(-17x+8+289c_3)+\left(\frac{2x}{17}+\frac{13}{1156}+c_1\right)\cos(x)+\left(-\frac{x}{34}-\frac{21}{289}+c_2\right)\sin(x)$$

14.11 problem 11

Internal problem ID [2212]
Internal file name [`OUTPUT/2212_Monday_February_26_2024_09_18_26_AM_92391842/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 11.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 3y'' - 4y = 4\,\mathrm{e}^{x} + 3\cos(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 3y'' - 4y = 0$$

The characteristic equation is

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= \mathrm{e}^{x} \\ y_3 &= \mathrm{e}^{2ix} \\ y_4 &= \mathrm{e}^{-2ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 3y'' - 4y = 4\,\mathrm{e}^{x} + 3\cos(2x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} & \mathrm{e}^{2ix} & \mathrm{e}^{-2ix} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & 2i\mathrm{e}^{2ix} & -2i\mathrm{e}^{-2ix} \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & -4\,\mathrm{e}^{2ix} & -4\,\mathrm{e}^{-2ix} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & -8i\mathrm{e}^{2ix} & 8i\mathrm{e}^{-2ix} \end{bmatrix}$$

$$|W| = -200i\mathrm{e}^{-x}\mathrm{e}^{x}\mathrm{e}^{2ix}\mathrm{e}^{-2ix}$$

The determinant simplifies to

$$|W| = -200i$$

Now we determine W_i for each U_i.

$$W_1(x) = \det\begin{bmatrix} e^x & e^{2ix} & e^{-2ix} \\ e^x & 2ie^{2ix} & -2ie^{-2ix} \\ e^x & -4\,e^{2ix} & -4\,e^{-2ix} \end{bmatrix}$$
$$= -20ie^x$$

$$W_2(x) = \det\begin{bmatrix} e^{-x} & e^{2ix} & e^{-2ix} \\ -e^{-x} & 2ie^{2ix} & -2ie^{-2ix} \\ e^{-x} & -4\,e^{2ix} & -4\,e^{-2ix} \end{bmatrix}$$
$$= -20ie^{-x}$$

$$W_3(x) = \det\begin{bmatrix} e^{-x} & e^x & e^{-2ix} \\ -e^{-x} & e^x & -2ie^{-2ix} \\ e^{-x} & e^x & -4\,e^{-2ix} \end{bmatrix}$$
$$= -10\,e^{-2ix}$$

$$W_4(x) = \det\begin{bmatrix} e^{-x} & e^x & e^{2ix} \\ -e^{-x} & e^x & 2ie^{2ix} \\ e^{-x} & e^x & -4\,e^{2ix} \end{bmatrix}$$
$$= -10\,e^{2ix}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3\int \frac{(4\,e^x + 3\cos(2x))\,(-20ie^x)}{(1)\,(-200i)}\,dx \\ &= -\int \frac{-20i(4\,e^x + 3\cos(2x))\,e^x}{-200i}\,dx \\ &= -\int \left(\frac{(4\,e^x + 3\cos(2x))\,e^x}{10}\right)dx \\ &= -\frac{e^{2x}}{5} - \frac{3(\cos(x) + 2\sin(x))\,e^x\cos(x)}{25} + \frac{3\,e^x}{50} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\
&= (-1)^2 \int \frac{(4\,\mathrm{e}^x + 3\cos(2x))\,(-20i\mathrm{e}^{-x})}{(1)\,(-200i)} \, dx \\
&= \int \frac{-20i(4\,\mathrm{e}^x + 3\cos(2x))\,\mathrm{e}^{-x}}{-200i} \, dx \\
&= \int \left(\frac{2}{5} + \frac{3\cos(2x)\,\mathrm{e}^{-x}}{10}\right) dx \\
&= \frac{2x}{5} - \frac{3\cos(2x)\,\mathrm{e}^{-x}}{50} + \frac{3\sin(2x)\,\mathrm{e}^{-x}}{25} \\
&= \frac{2x}{5} - \frac{3\cos(2x)\,\mathrm{e}^{-x}}{50} + \frac{3\sin(2x)\,\mathrm{e}^{-x}}{25}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)} \, dx \\
&= (-1)^1 \int \frac{(4\,\mathrm{e}^x + 3\cos(2x))\,(-10\,\mathrm{e}^{-2ix})}{(1)\,(-200i)} \, dx \\
&= -\int \frac{-10(4\,\mathrm{e}^x + 3\cos(2x))\,\mathrm{e}^{-2ix}}{-200i} \, dx \\
&= -\int \left(-\frac{i(4\,\mathrm{e}^x + 3\cos(2x))\,\mathrm{e}^{-2ix}}{20}\right) dx \\
&= -\frac{2\,\mathrm{e}^{(1-2i)x}}{25} + \frac{i\mathrm{e}^{(1-2i)x}}{25} + \frac{3ix}{40} - \frac{3\,\mathrm{e}^{-4ix}}{160}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)} \, dx \\
&= (-1)^0 \int \frac{(4\,\mathrm{e}^x + 3\cos(2x))\,(-10\,\mathrm{e}^{2ix})}{(1)\,(-200i)} \, dx \\
&= \int \frac{-10(4\,\mathrm{e}^x + 3\cos(2x))\,\mathrm{e}^{2ix}}{-200i} \, dx \\
&= \int \left(-\frac{i(4\,\mathrm{e}^x + 3\cos(2x))\,\mathrm{e}^{2ix}}{20}\right) dx \\
&= -\frac{3ix}{40} - \frac{3\,\mathrm{e}^{4ix}}{160} - \frac{2\,\mathrm{e}^{(1+2i)x}}{25} - \frac{i\mathrm{e}^{(1+2i)x}}{25}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{e^{2x}}{5}-\frac{3(\cos (x)+2\sin (x))\,e^{x}\cos (x)}{25}+\frac{3\,e^{x}}{50}\right)\left(e^{-x}\right)\\
&+\left(\frac{2x}{5}-\frac{3\cos (2x)\,e^{-x}}{50}+\frac{3\sin (2x)\,e^{-x}}{25}\right)\left(e^{x}\right)\\
&+\left(-\frac{2\,e^{(1-2i)x}}{25}+\frac{i e^{(1-2i)x}}{25}+\frac{3ix}{40}-\frac{3\,e^{-4ix}}{160}\right)\left(e^{2ix}\right)\\
&+\left(-\frac{3ix}{40}-\frac{3\,e^{4ix}}{160}-\frac{2\,e^{(1+2i)x}}{25}-\frac{i e^{(1+2i)x}}{25}\right)\left(e^{-2ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -\frac{63\cos (2x)}{400}-\frac{3x\sin (2x)}{20}+\frac{(10x-9)\,e^{x}}{25}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p\\
&= \left(c_1 e^{-x}+c_2 e^{x}+e^{2ix}c_3+e^{-2ix}c_4\right)+\left(-\frac{63\cos (2x)}{400}-\frac{3x\sin (2x)}{20}+\frac{(10x-9)\,e^{x}}{25}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 e^{-x}+c_2 e^{x}+e^{2ix}c_3+e^{-2ix}c_4-\frac{63\cos (2x)}{400}-\frac{3x\sin (2x)}{20}+\frac{(10x-9)\,e^{x}}{25} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1 e^{-x}+c_2 e^{x}+e^{2ix}c_3+e^{-2ix}c_4-\frac{63\cos (2x)}{400}-\frac{3x\sin (2x)}{20}+\frac{(10x-9)\,e^{x}}{25}
$$

Verified OK.

14.11.1 Maple step by step solution

Let's solve

$y'''' + 3y'' - 4y = 4\,e^{x} + 3\cos (2x)$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$
- Define new variable $y_2(x)$

  $y_2(x) = y'$
- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 4\,\mathrm{e}^{x} + 3\cos(2x) - 3y_3(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 4\,\mathrm{e}^{x} + 3\cos(2x) - 3y_3(x) + 4y_1(x)]$

• Define vector

$$\overrightarrow{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

• System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -3 & 0 \end{bmatrix} \cdot \overrightarrow{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4\,\mathrm{e}^{x} + 3\cos(2x) \end{bmatrix}$$

• Define the forcing function

$$\overrightarrow{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4\,\mathrm{e}^{x} + 3\cos(2x) \end{bmatrix}$$

• Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -3 & 0 \end{bmatrix}$$

- Rewrite the system as

  $$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2x) - \mathrm{I}\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\frac{\sin(2x)}{8} & -\frac{\cos(2x)}{8} \\ e^{-x} & e^{x} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ -e^{-x} & e^{x} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ e^{-x} & e^{x} & \cos(2x) & -\sin(2x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\frac{\sin(2x)}{8} & -\frac{\cos(2x)}{8} \\ e^{-x} & e^{x} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ -e^{-x} & e^{x} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ e^{-x} & e^{x} & \cos(2x) & -\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 & 0 & -\frac{1}{8} \\ 1 & 1 & -\frac{1}{4} & 0 \\ -1 & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{2e^{-x}}{5} + \frac{2e^{x}}{5} + \frac{\cos(2x)}{5} & -\frac{2e^{-x}}{5} + \frac{2e^{x}}{5} + \frac{\sin(2x)}{10} & \frac{e^{-x}}{10} + \frac{e^{x}}{10} - \frac{\cos(2x)}{5} & -\frac{e^{-x}}{10} + \frac{e^{x}}{10} - \mathrm{si} \\ -\frac{2e^{-x}}{5} + \frac{2e^{x}}{5} - \frac{2\sin(2x)}{5} & \frac{2e^{-x}}{5} + \frac{2e^{x}}{5} + \frac{\cos(2x)}{5} & -\frac{e^{-x}}{10} + \frac{e^{x}}{10} + \frac{2\sin(2x)}{5} & \frac{e^{-x}}{10} + \frac{e^{x}}{10} - \mathrm{cos} \\ \frac{2e^{-x}}{5} + \frac{2e^{x}}{5} - \frac{4\cos(2x)}{5} & -\frac{2e^{-x}}{5} + \frac{2e^{x}}{5} - \frac{2\sin(2x)}{5} & \frac{e^{-x}}{10} + \frac{e^{x}}{10} + \frac{4\cos(2x)}{5} & -\frac{e^{-x}}{10} + \frac{e^{x}}{10} + 2\mathrm{s} \\ -\frac{2e^{-x}}{5} + \frac{2e^{x}}{5} + \frac{8\sin(2x)}{5} & \frac{2e^{-x}}{5} + \frac{2e^{x}}{5} - \frac{4\cos(2x)}{5} & -\frac{e^{-x}}{10} + \frac{e^{x}}{10} - \frac{8\sin(2x)}{5} & \frac{e^{-x}}{10} + \frac{e^{x}}{10} + 4\mathrm{co} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{(-15x+8)\sin(2x)}{100} + \frac{\cos(2x)}{25} + \frac{13\,\mathrm{e}^{-x}}{50} + \frac{(4x-3)\mathrm{e}^{x}}{10} \\ \frac{(-15x+8)\cos(2x)}{50} - \frac{13\,\mathrm{e}^{-x}}{50} - \frac{23\sin(2x)}{100} + \frac{(4x+1)\mathrm{e}^{x}}{10} \\ \frac{(15x-8)\sin(2x)}{25} - \frac{19\cos(2x)}{25} + \frac{13\,\mathrm{e}^{-x}}{50} + \frac{(4x+5)\mathrm{e}^{x}}{10} \\ \frac{2(15x-8)\cos(2x)}{25} - \frac{13\,\mathrm{e}^{-x}}{50} + \frac{53\sin(2x)}{25} + \frac{(4x+9)\mathrm{e}^{x}}{10} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} \frac{(-15x+8)\sin(2x)}{100} + \frac{\cos(2x)}{25} + \frac{13\,\mathrm{e}^{-x}}{50} + \frac{(4x-3)\mathrm{e}^{x}}{10} \\ \frac{(-15x+8)\cos(2x)}{50} - \frac{13\,\mathrm{e}^{-x}}{50} - \frac{23\sin(2x)}{100} + \frac{(4x+1)\mathrm{e}^{x}}{10} \\ \frac{(15x-8)\sin(2x)}{25} - \frac{19\cos(2x)}{25} + \frac{13\,\mathrm{e}^{-x}}{50} + \frac{(4x+5)\mathrm{e}^{x}}{10} \\ \frac{2(15x-8)\cos(2x)}{25} - \frac{13\,\mathrm{e}^{-x}}{50} + \frac{53\sin(2x)}{25} + \frac{(4x+9)\mathrm{e}^{x}}{10} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-25c_3-30x+16)\sin(2x)}{200} + \frac{(-25c_4+8)\cos(2x)}{200} + \frac{(13-50c_1)\mathrm{e}^{-x}}{50} + \frac{2\left(x+\frac{5c_2}{2}-\frac{3}{4}\right)\mathrm{e}^{x}}{5}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(x),x$4)+3*diff(y(x),x$2)-4*y(x)=4*exp(x)+3*cos(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(-3 + 25c_2)\cos(2x)}{25} + \frac{(-3x + 20c_4)\sin(2x)}{20} + c_3 e^{-x} + \frac{2\,e^x\left(x + \frac{5c_1}{2} - \frac{9}{10}\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.17 (sec). Leaf size: 55

```
DSolve[y''''[x]+3*y''[x]-4*y[x]==4*Exp[x]+3*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2e^x x}{5} - \frac{9e^x}{25} - \frac{\cos(x)}{2} + c_3 e^{-x} + c_4 e^x + c_1\cos(2x) + c_2\sin(2x)$$

14.12 problem 12

Internal problem ID [2213]
Internal file name [`OUTPUT/2213_Monday_February_26_2024_09_18_27_AM_49020461/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = e^{3x}(1 + \sin(2x))}$$

14.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = e^{3x}(1 + \sin(2x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x}(1 + \sin(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}, \{e^{3x}\cos(2x), e^{3x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x} + A_2 e^{3x}\cos(2x) + A_3 e^{3x}\sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}10A_1 e^{3x} + 6A_2 e^{3x}\cos(2x) - 12A_2 e^{3x}\sin(2x) + 6A_3 e^{3x}\sin(2x) + 12A_3 e^{3x}\cos(2x) \\ = e^{3x}(1 + \sin(2x))\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = -\frac{1}{15}, A_3 = \frac{1}{30}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{3x}}{10} - \frac{e^{3x}\cos(2x)}{15} + \frac{e^{3x}\sin(2x)}{30}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{e^{3x}}{10} - \frac{e^{3x}\cos(2x)}{15} + \frac{e^{3x}\sin(2x)}{30}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{e^{3x}}{10} - \frac{e^{3x} \cos(2x)}{15} + \frac{e^{3x} \sin(2x)}{30} \tag{1}$$

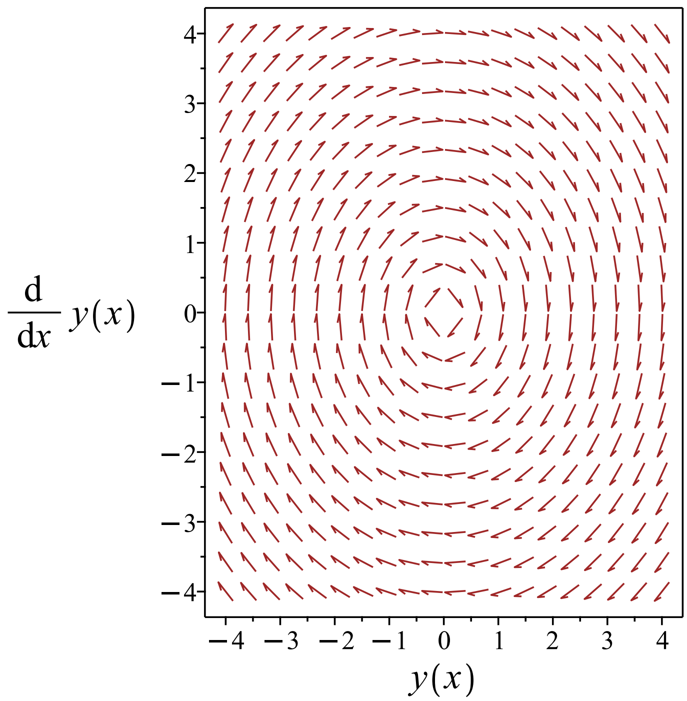


Figure 650: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{e^{3x}}{10} - \frac{e^{3x} \cos(2x)}{15} + \frac{e^{3x} \sin(2x)}{30}$$

Verified OK.

14.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 493: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x}(1 + \sin(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}, \{e^{3x}\cos(2x), e^{3x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x} + A_2 e^{3x}\cos(2x) + A_3 e^{3x}\sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 e^{3x} + 6A_2 e^{3x}\cos(2x) - 12A_2 e^{3x}\sin(2x) + 6A_3 e^{3x}\sin(2x) + 12A_3 e^{3x}\cos(2x) \\ = e^{3x}(1 + \sin(2x))$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = -\frac{1}{15}, A_3 = \frac{1}{30}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{3x}}{10} - \frac{e^{3x}\cos(2x)}{15} + \frac{e^{3x}\sin(2x)}{30}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(\frac{e^{3x}}{10} - \frac{e^{3x}\cos(2x)}{15} + \frac{e^{3x}\sin(2x)}{30}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) + \frac{e^{3x}}{10} - \frac{e^{3x}\cos(2x)}{15} + \frac{e^{3x}\sin(2x)}{30} \tag{1}$$

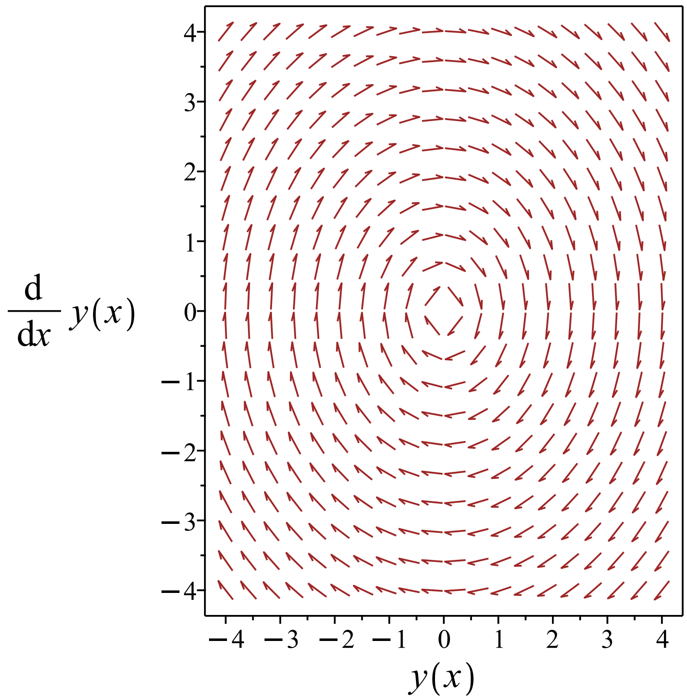


Figure 651: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{e^{3x}}{10} - \frac{e^{3x} \cos(2x)}{15} + \frac{e^{3x} \sin(2x)}{30}$$

Verified OK.

14.12.3 Maple step by step solution

Let's solve

$y'' + y = e^{3x}(1 + \sin(2x))$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r=(-\mathrm{I},\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x)=\cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x)=\sin(x)$
- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(x)+c_2\sin(x)+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=\mathrm{e}^{3x}(1+\sin(2x))\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x),y_2(x))=\left[\begin{array}{cc}\cos(x) & \sin(x)\\ -\sin(x) & \cos(x)\end{array}\right]$
- ○ Compute Wronskian

  $W(y_1(x),y_2(x))=1$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x)=-\cos(x)\left(\int\sin(x)\,\mathrm{e}^{3x}(1+\sin(2x))\,dx\right)+\sin(x)\left(\int\cos(x)\,\mathrm{e}^{3x}(1+\sin(2x))\,dx\right)$
- ○ Compute integrals

  $y_p(x)=-\frac{\mathrm{e}^{3x}(2\cos(2x)-\sin(2x)-3)}{30}$

- Substitute particular solution into general solution to ODE

  $y=c_1\cos(x)+c_2\sin(x)-\frac{\mathrm{e}^{3x}(2\cos(2x)-\sin(2x)-3)}{30}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+y(x)=exp(3*x)*(1+sin(2*x)),y(x), singsol=all)
```

$$y(x) = \frac{\left(2\cos\left(x\right)\sin\left(x\right)+5-4\cos\left(x\right)^{2}\right)\mathrm{e}^{3x}}{30}+\cos\left(x\right)c_{1}+\sin\left(x\right)c_{2}$$

✓ Solution by Mathematica

Time used: 0.288 (sec). Leaf size: 50

```
DSolve[y''[x]+y[x]==Exp[3*x]*(1+Sin[2*x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{30}\left(3e^{3x}+e^{3x}\sin(2x)-2e^{3x}\cos(2x)+30c_1\cos(x)+30c_2\sin(x)\right)$$

14.13 problem 13

Internal problem ID [2214]
Internal file name [`OUTPUT/2214_Monday_February_26_2024_09_18_27_AM_42291105/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2n^2y' + n^4y = \sin(kx)}$$

14.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2n^2, C = n^4, f(x) = \sin(kx)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2n^2y' + n^4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2n^2, C = n^4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2n^2 \lambda \,\mathrm{e}^{\lambda x} + n^4 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$n^4 + 2n^2 \lambda + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2n^2, C = n^4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2n^2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2n^2)^2 - (4)\,(1)\,(n^4)}\\ &= -n^2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = n^2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-n^2 x} + c_2 x \,\mathrm{e}^{-n^2 x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-n^2 x} + c_2 x \,\mathrm{e}^{-n^2 x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(kx)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(kx), \sin(kx)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x \,\mathrm{e}^{-n^2 x}, \mathrm{e}^{-n^2 x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(kx) + A_2 \sin(kx)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1k^2 \cos(kx) - A_2k^2 \sin(kx) + 2n^2(-A_1k \sin(kx) + A_2k \cos(kx)) \\ + n^4(A_1 \cos(kx) + A_2 \sin(kx)) = \sin(kx)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2k\,n^2}{(n^4+k^2)^2}, A_2 = \frac{n^4-k^2}{(n^4+k^2)^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2k\,n^2 \cos(kx)}{(n^4+k^2)^2} + \frac{(n^4-k^2)\sin(kx)}{(n^4+k^2)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-n^2x} + c_2x\,e^{-n^2x}\right) + \left(-\frac{2k\,n^2 \cos(kx)}{(n^4+k^2)^2} + \frac{(n^4-k^2)\sin(kx)}{(n^4+k^2)^2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-n^2x}(c_2x + c_1) - \frac{2k\,n^2 \cos(kx)}{(n^4+k^2)^2} + \frac{(n^4-k^2)\sin(kx)}{(n^4+k^2)^2}$$

Summary

The solution(s) found are the following

$$y = e^{-n^2x}(c_2x + c_1) - \frac{2k\,n^2 \cos(kx)}{(n^4+k^2)^2} + \frac{(n^4-k^2)\sin(kx)}{(n^4+k^2)^2} \tag{1}$$

Verification of solutions

$$y = e^{-n^2x}(c_2x + c_1) - \frac{2k\,n^2 \cos(kx)}{(n^4+k^2)^2} + \frac{(n^4-k^2)\sin(kx)}{(n^4+k^2)^2}$$

Verified OK.

14.13.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2n^2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 2n^2\, dx} \\ &= \mathrm{e}^{n^2 x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \mathrm{e}^{n^2 x} \sin(kx) \\ \left(\mathrm{e}^{n^2 x} y\right)'' &= \mathrm{e}^{n^2 x} \sin(kx) \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{n^2 x} y\right)' = \frac{\mathrm{e}^{n^2 x}\left(-k\cos(kx) + \sin(kx)\, n^2\right)}{n^4 + k^2} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{n^2 x} y\right) = \frac{\left((n^4 - k^2)\sin(kx) - 2k\, n^2 \cos(kx)\right) \mathrm{e}^{n^2 x} + c_1 x (n^4 + k^2)^2}{(n^4 + k^2)^2} + c_2$$

Hence the solution is

$$y = \frac{\frac{\left((n^4 - k^2)\sin(kx) - 2k\, n^2 \cos(kx)\right) \mathrm{e}^{n^2 x} + c_1 x (n^4 + k^2)^2}{(n^4 + k^2)^2} + c_2}{\mathrm{e}^{n^2 x}}$$

Or

$$\begin{aligned} y = {} & \frac{n^4 \sin(kx)}{(n^4 + k^2)^2} - \frac{2k\, n^2 \cos(kx)}{(n^4 + k^2)^2} - \frac{k^2 \sin(kx)}{(n^4 + k^2)^2} \\ & + \left(\frac{n^8 x\, \mathrm{e}^{-n^2 x}}{(n^4 + k^2)^2} + \frac{2k^2 n^4 x\, \mathrm{e}^{-n^2 x}}{(n^4 + k^2)^2} + \frac{k^4 x\, \mathrm{e}^{-n^2 x}}{(n^4 + k^2)^2}\right) c_1 + c_2 \mathrm{e}^{-n^2 x} \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= \frac{n^4 \sin(kx)}{(n^4+k^2)^2} - \frac{2k\, n^2 \cos(kx)}{(n^4+k^2)^2} - \frac{k^2 \sin(kx)}{(n^4+k^2)^2} \\
&\quad + \left(\frac{n^8 x\, \mathrm{e}^{-n^2 x}}{(n^4+k^2)^2} + \frac{2k^2 n^4 x\, \mathrm{e}^{-n^2 x}}{(n^4+k^2)^2} + \frac{k^4 x\, \mathrm{e}^{-n^2 x}}{(n^4+k^2)^2}\right) c_1 + c_2 \mathrm{e}^{-n^2 x} \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= \frac{n^4 \sin(kx)}{(n^4+k^2)^2} - \frac{2k\, n^2 \cos(kx)}{(n^4+k^2)^2} - \frac{k^2 \sin(kx)}{(n^4+k^2)^2} \\
&\quad + \left(\frac{n^8 x\, \mathrm{e}^{-n^2 x}}{(n^4+k^2)^2} + \frac{2k^2 n^4 x\, \mathrm{e}^{-n^2 x}}{(n^4+k^2)^2} + \frac{k^4 x\, \mathrm{e}^{-n^2 x}}{(n^4+k^2)^2}\right) c_1 + c_2 \mathrm{e}^{-n^2 x}
\end{align*}
$$

Verified OK.

14.13.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' + 2n^2 y' + n^4 y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 2n^2 \tag{3} \\
C &= n^4
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 495: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2n^2}{1}\,dx} \\
&= z_1 e^{-n^2 x} \\
&= z_1 \left(\mathrm{e}^{-n^2 x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-n^2 x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2n^2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2n^2 x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-n^2 x}\right) + c_2\left(\mathrm{e}^{-n^2 x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2n^2 y' + n^4 y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-n^2 x} + c_2 x\,\mathrm{e}^{-n^2 x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(kx)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(kx), \sin(kx)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{x\,\mathrm{e}^{-n^2 x}, \mathrm{e}^{-n^2 x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(kx) + A_2 \sin(kx)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-A_1 k^2 \cos(kx) - A_2 k^2 \sin(kx) + 2n^2(-A_1 k \sin(kx) + A_2 k \cos(kx)) \\
&+ n^4(A_1 \cos(kx) + A_2 \sin(kx)) = \sin(kx)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2k\,n^2}{\left(n^4+k^2\right)^2}, A_2 = \frac{n^4-k^2}{\left(n^4+k^2\right)^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2k\,n^2\cos\left(kx\right)}{\left(n^4+k^2\right)^2} + \frac{\left(n^4-k^2\right)\sin\left(kx\right)}{\left(n^4+k^2\right)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-n^2 x} + c_2 x\,\mathrm{e}^{-n^2 x}\right) + \left(-\frac{2k\,n^2\cos\left(kx\right)}{\left(n^4+k^2\right)^2} + \frac{\left(n^4-k^2\right)\sin\left(kx\right)}{\left(n^4+k^2\right)^2}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-n^2 x}(c_2 x + c_1) - \frac{2k\,n^2\cos\left(kx\right)}{\left(n^4+k^2\right)^2} + \frac{\left(n^4-k^2\right)\sin\left(kx\right)}{\left(n^4+k^2\right)^2}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-n^2 x}(c_2 x + c_1) - \frac{2k\,n^2\cos\left(kx\right)}{\left(n^4+k^2\right)^2} + \frac{\left(n^4-k^2\right)\sin\left(kx\right)}{\left(n^4+k^2\right)^2} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-n^2 x}(c_2 x + c_1) - \frac{2k\,n^2\cos\left(kx\right)}{\left(n^4+k^2\right)^2} + \frac{\left(n^4-k^2\right)\sin\left(kx\right)}{\left(n^4+k^2\right)^2}$$

Verified OK.

14.13.4 Maple step by step solution

Let's solve

$y'' + 2n^2 y' + n^4 y = \sin\left(kx\right)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $n^4 + 2n^2 r + r^2 = 0$

- Factor the characteristic polynomial

  $(n^2 + r)^2 = 0$

- Root of the characteristic polynomial

  $r = -n^2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-n^2 x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-n^2 x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-n^2 x} + c_2 x\,e^{-n^2 x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(kx)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-n^2 x} & x\,e^{-n^2 x} \\ -n^2 e^{-n^2 x} & e^{-n^2 x} - x\,n^2 e^{-n^2 x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{-2n^2 x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^{-n^2 x}\left(-\left(\int x \sin(kx)\,e^{n^2 x} dx\right) + x\left(\int e^{n^2 x} \sin(kx)\,dx\right)\right)$

  - Compute integrals

    $y_p(x) = \frac{(n^4 - k^2)\sin(kx) - 2k\,n^2 \cos(kx)}{(n^4 + k^2)^2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-n^2 x} + c_2 x\,e^{-n^2 x} + \frac{(n^4 - k^2)\sin(kx) - 2k\,n^2 \cos(kx)}{(n^4 + k^2)^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 61

```
dsolve(diff(y(x),x$2)+2*n^2*diff(y(x),x)+n^4*y(x)=sin(k*x),y(x), singsol=all)
```

$$y(x) = \frac{\left(n^4+k^2\right)^2 \left(c_1 x+c_2\right) \mathrm{e}^{-n^2 x}+\left(n^4-k^2\right) \sin \left(k x\right)-2 \cos \left(k x\right) k \, n^2}{\left(n^4+k^2\right)^2}$$

✓ Solution by Mathematica

Time used: 0.198 (sec). Leaf size: 64

```
DSolve[y''[x]+2*n^2*y'[x]+n^4*y[x]==Sin[k*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\left(n^4-k^2\right) \sin(kx)}{\left(k^2+n^4\right)^2}-\frac{2kn^2 \cos(kx)}{\left(k^2+n^4\right)^2}+\left(c_2 x+c_1\right) e^{-n^2 x}$$

14.14 problem 14

Internal problem ID [2215]
Internal file name [`OUTPUT/2215_Monday_February_26_2024_09_18_28_AM_77904488/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 5y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}}$$

14.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 5, f(x) = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(5)}\\ &= -2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\frac{e^{x}}{2}+\frac{e^{-x}}{2}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\},\{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}\cos(x), e^{-2x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x} + A_2 e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 e^{x} + 2A_2 e^{-x} = \frac{e^{x}}{2}+\frac{e^{-x}}{2}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{20}, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{x}}{20}+\frac{e^{-x}}{4}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{-2x}(c_1\cos(x)+c_2\sin(x))\right)+\left(\frac{e^{x}}{20}+\frac{e^{-x}}{4}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{e^{x}}{20} + \frac{e^{-x}}{4} \tag{1}$$

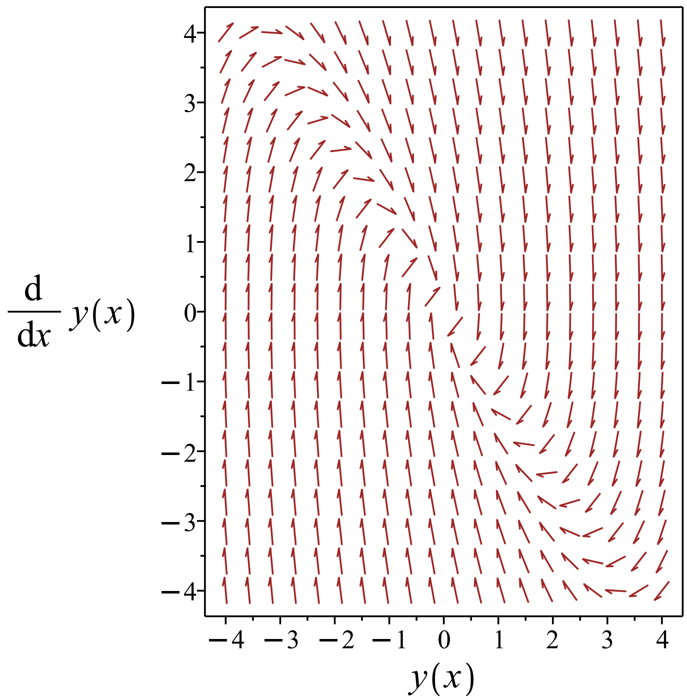


Figure 652: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{e^{x}}{20} + \frac{e^{-x}}{4}$$

Verified OK.

14.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 4 \qquad (3) \\
C &= 5
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \qquad (4)$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5) \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \qquad (6)$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \qquad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 497: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1\left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\cos(x)\right) + c_2\left(\mathrm{e}^{-2x}\cos(x)(\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\,\mathrm{e}^{-2x} c_1 + \sin(x)\,\mathrm{e}^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\frac{\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{2}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}\cos(x), e^{-2x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x} + A_2 e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 e^{x} + 2A_2 e^{-x} = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{20}, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{x}}{20} + \frac{e^{-x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(x)\, e^{-2x} c_1 + \sin(x)\, e^{-2x} c_2\right) + \left(\frac{e^{x}}{20} + \frac{e^{-x}}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{e^{x}}{20} + \frac{e^{-x}}{4}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{e^{x}}{20} + \frac{e^{-x}}{4} \tag{1}$$

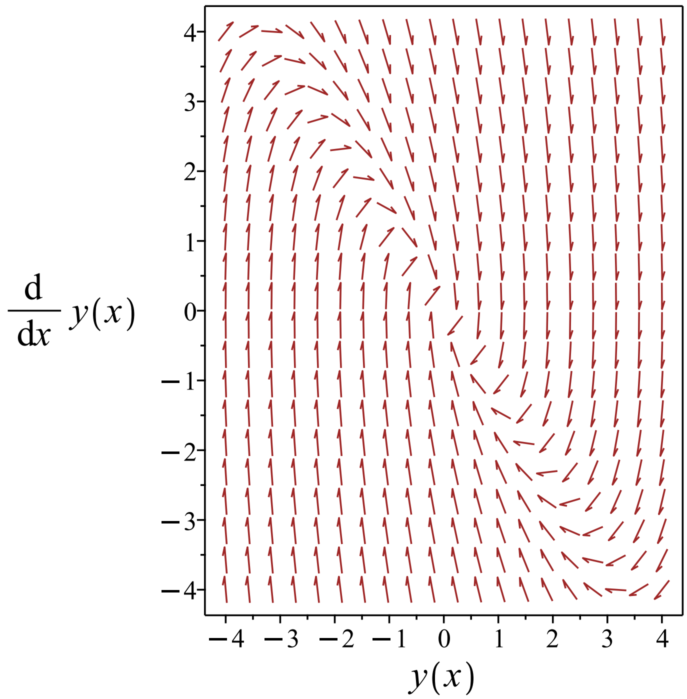


Figure 653: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{e^{x}}{20} + \frac{e^{-x}}{4}$$

Verified OK.

14.14.3 Maple step by step solution

Let's solve

$y'' + 4y' + 5y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - \mathrm{I}, -2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-2x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x)\,\mathrm{e}^{-2x} c_1 + \sin(x)\,\mathrm{e}^{-2x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{2}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x}\cos(x) & \mathrm{e}^{-2x}\sin(x) \\ -2\,\mathrm{e}^{-2x}\cos(x) - \mathrm{e}^{-2x}\sin(x) & -2\,\mathrm{e}^{-2x}\sin(x) + \mathrm{e}^{-2x}\cos(x) \end{bmatrix}$$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\mathrm{e}^{-2x}\left(-\cos(x)\left(\int (\mathrm{e}^{2x}+1)\sin(x)\mathrm{e}^{x}dx\right)+\sin(x)\left(\int (\mathrm{e}^{2x}+1)\cos(x)\mathrm{e}^{x}dx\right)\right)}{2}$

- ◦ Compute integrals

  $y_p(x) = \frac{\mathrm{e}^{x}}{20} + \frac{\mathrm{e}^{-x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = \sin(x)\,\mathrm{e}^{-2x} c_2 + \cos(x)\,\mathrm{e}^{-2x} c_1 + \frac{\mathrm{e}^{x}}{20} + \frac{\mathrm{e}^{-x}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+5*y(x)=1/2*(exp(x)+exp(-x)),y(x), singsol=all)
```

$$y(x) = \frac{(\mathrm{e}^{3x} + 20\sin(x)\, c_2 + 20\cos(x)\, c_1 + 5\,\mathrm{e}^{x})\,\mathrm{e}^{-2x}}{20}$$

✓ Solution by Mathematica

Time used: 0.203 (sec). Leaf size: 37

```
DSolve[y''[x]+4*y'[x]+5*y[x]==1/2*(Exp[x]+Exp[-x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{20}e^{-2x}\left(5e^x + e^{3x} + 20c_2\cos(x) + 20c_1\sin(x)\right)$$

14.15 problem 15

Internal problem ID [2216]
Internal file name [`OUTPUT/2216_Monday_February_26_2024_09_18_29_AM_58480841/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y' - 2y = x\,\mathrm{e}^{-x}}$$

14.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = x\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda\, e^{\lambda x} - 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)}\\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{3}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{-x} + A_2 e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-x} - 2A_1 x\,e^{-x} - 2A_2 e^{-x} = x\,e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{-x}}{2} + \frac{e^{-x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{-2x}\right) + \left(-\frac{x\,e^{-x}}{2} + \frac{e^{-x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{-2x} - \frac{x\,e^{-x}}{2} + \frac{e^{-x}}{4} \tag{1}$$

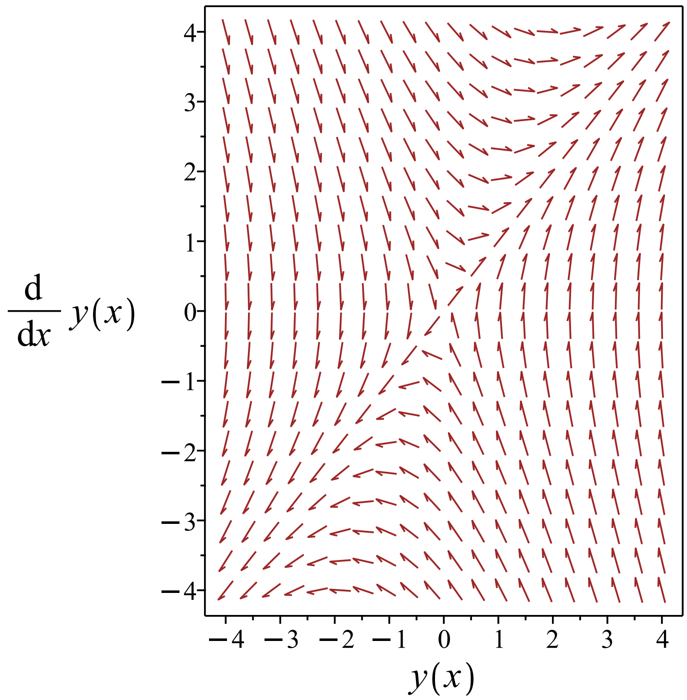


Figure 654: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-2x} - \frac{x\,e^{-x}}{2} + \frac{e^{-x}}{4}$$

Verified OK.

14.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 499: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{-2x} + \frac{c_2\mathrm{e}^{x}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x\,\mathrm{e}^{-x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{x}}{3}, \mathrm{e}^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 x\,\mathrm{e}^{-x} + A_2\mathrm{e}^{-x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \mathrm{e}^{-x} - 2A_1 x \,\mathrm{e}^{-x} - 2A_2 \mathrm{e}^{-x} = x \,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x \,\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{-x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3}\right) + \left(-\frac{x \,\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{-x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3} - \frac{x \,\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{-x}}{4} \tag{1}$$

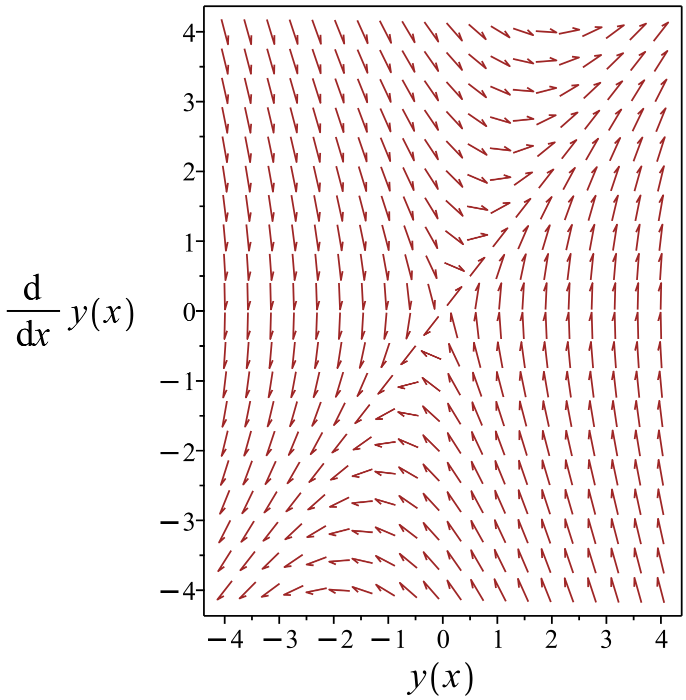


Figure 655: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} - \frac{x\, e^{-x}}{2} + \frac{e^{-x}}{4}$$

Verified OK.

14.15.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = x\, e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 1) = 0$

- Roots of the characteristic polynomial

$r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\,e^{-x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{x} \\ -2\,e^{-2x} & e^{x} \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,e^{-x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{\left(e^{3x}\left(\int e^{-2x} x dx\right) - \left(\int x\,e^{x} dx\right)\right)e^{-2x}}{3}$

  - Compute integrals

    $y_p(x) = -\frac{e^{-x}(2x-1)}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{x} - \frac{e^{-x}(2x-1)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=x*exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{(4c_1 e^{3x} + (1-2x)\, e^{x} + 4c_2)\, e^{-2x}}{4}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 34

```
DSolve[y''[x]+y'[x]-2*y[x]==x*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-x}(1-2x) + c_1 e^{-2x} + c_2 e^{x}$$

14.16 problem 16

Internal problem ID [2217]
Internal file name [`OUTPUT/2217_Monday_February_26_2024_09_18_29_AM_4771951/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = x\,e^{x}}$$

14.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = x\,e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^x + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x + 5A_1 x\,e^x + 5A_2 e^x = x\,e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = -\frac{2}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,e^x}{5} - \frac{2\,e^x}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{x\,e^x}{5} - \frac{2\,e^x}{25}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x\,e^{x}}{5} - \frac{2\,e^{x}}{25} \tag{1}$$

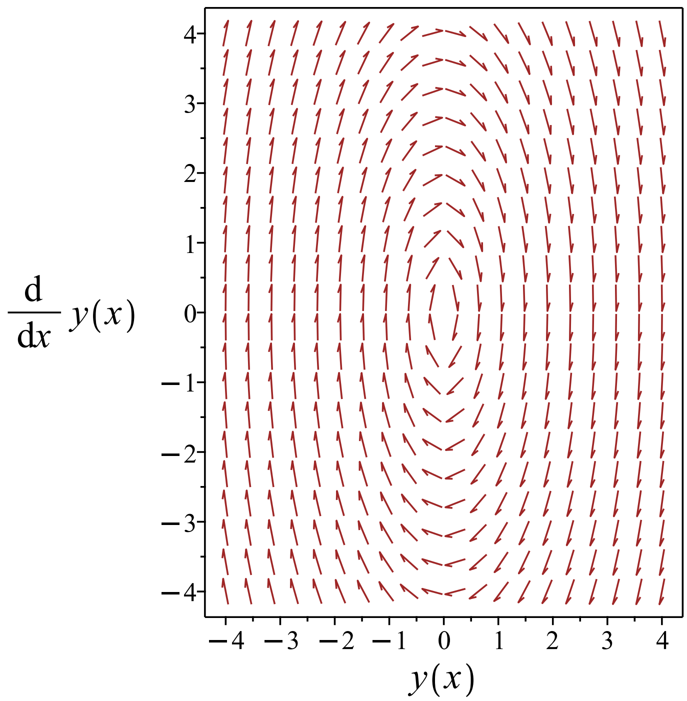


Figure 656: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x\,e^{x}}{5} - \frac{2\,e^{x}}{25}$$

Verified OK.

14.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 501: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\ &= \cos(2x)\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{x} + A_2 e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x} + 5A_1 x\,e^{x} + 5A_2 e^{x} = x\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = -\frac{2}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,e^{x}}{5} - \frac{2\,e^{x}}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{x\,e^{x}}{5} - \frac{2\,e^{x}}{25}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{x\,e^{x}}{5} - \frac{2\,e^{x}}{25} \tag{1}$$

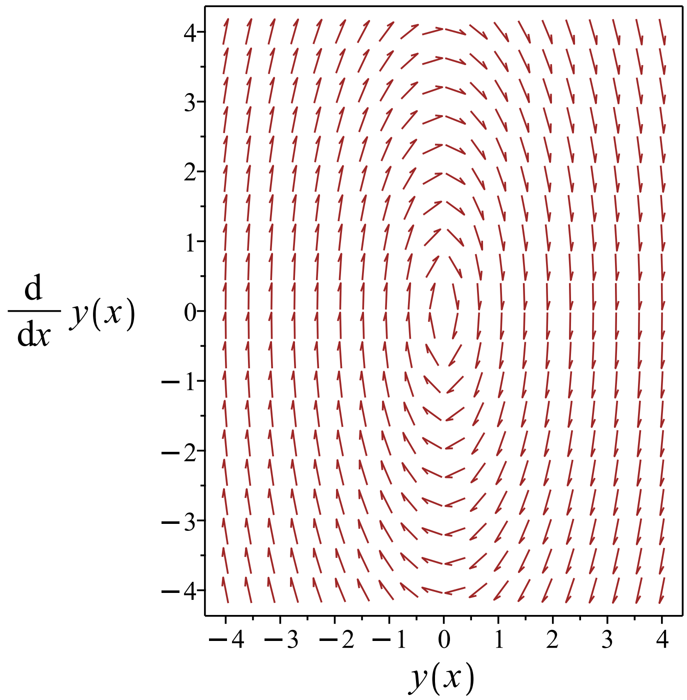


Figure 657: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{x\,e^x}{5} - \frac{2\,e^x}{25}$$

Verified OK.

14.16.3 Maple step by step solution

Let's solve

$y'' + 4y = x\,e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\,\mathrm{e}^x\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)\mathrm{e}^x x dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x)\mathrm{e}^x x dx\right)}{2}$

  ○ Compute integrals

  $y_p(x) = \frac{\mathrm{e}^x(5x-2)}{25}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\mathrm{e}^x(5x-2)}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+4*y(x)=x*exp(x),y(x), singsol=all)
```

$$y(x) = \sin(2x) c_2 + c_1 \cos(2x) + \frac{(5x-2)\,e^x}{25}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 32

```
DSolve[y''[x]+4*y[x]==x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{25} e^x (5x-2) + c_1 \cos(2x) + c_2 \sin(2x)$$

14.17 problem 17

Internal problem ID [2218]
Internal file name [`OUTPUT/2218_Monday_February_26_2024_09_18_29_AM_14033184/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y = x^2 \mathrm{e}^{-x}}$$

14.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 2, f(x) = x^2 \mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(2)}\\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)\right)$$

Or

$$y = c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{2}\, x\right), \sin\left(\sqrt{2}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-x} + A_2 x^2 \mathrm{e}^{-x} + A_3 \mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\mathrm{e}^{-x} + 3A_1 x\,\mathrm{e}^{-x} + 2A_2\mathrm{e}^{-x} - 4A_2 x\,\mathrm{e}^{-x} + 3A_2 x^2\mathrm{e}^{-x} + 3A_3\mathrm{e}^{-x} = x^2\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{9}, A_2 = \frac{1}{3}, A_3 = \frac{2}{27}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4x\,\mathrm{e}^{-x}}{9} + \frac{x^2\mathrm{e}^{-x}}{3} + \frac{2\,\mathrm{e}^{-x}}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) + \left(\frac{4x\,\mathrm{e}^{-x}}{9} + \frac{x^2\mathrm{e}^{-x}}{3} + \frac{2\,\mathrm{e}^{-x}}{27}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + \frac{4x\,\mathrm{e}^{-x}}{9} + \frac{x^2\mathrm{e}^{-x}}{3} + \frac{2\,\mathrm{e}^{-x}}{27} \tag{1}$$

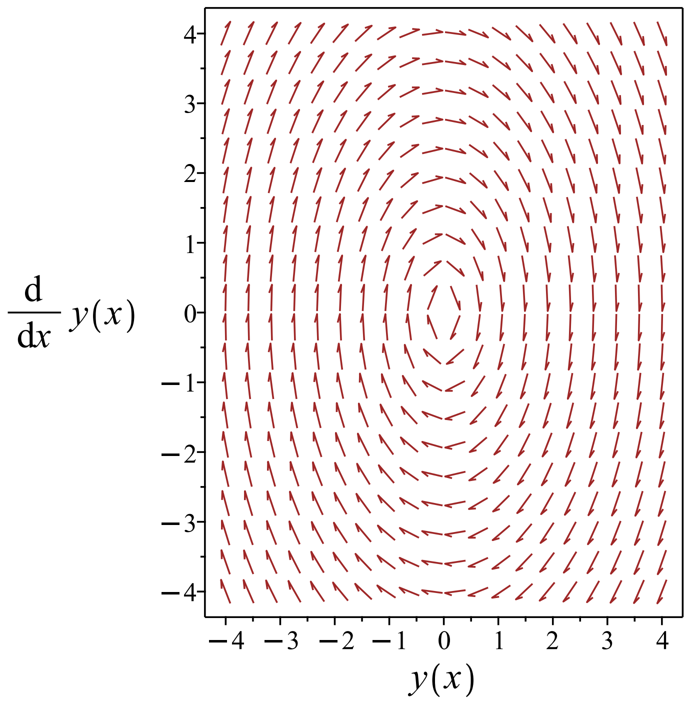


Figure 658: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + \frac{4x\,\mathrm{e}^{-x}}{9} + \frac{x^2\mathrm{e}^{-x}}{3} + \frac{2\,\mathrm{e}^{-x}}{27}$$

Verified OK.

14.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 503: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{2}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos\left(\sqrt{2}\, x\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, x\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos\left(\sqrt{2}\, x\right) \int \frac{1}{\cos\left(\sqrt{2}\, x\right)^2}\, dx \\
&= \cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\sqrt{2}\, x\right)\right) + c_2 \left(\cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(\sqrt{2}\, x\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-x}, x^2 e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2}, \cos\left(\sqrt{2}\,x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{-x} + A_2 x^2 e^{-x} + A_3 e^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{-x} + 3A_1 x\,e^{-x} + 2A_2 e^{-x} - 4A_2 x\,e^{-x} + 3A_2 x^2 e^{-x} + 3A_3 e^{-x} = x^2 e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{9}, A_2 = \frac{1}{3}, A_3 = \frac{2}{27}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{4x\,e^{-x}}{9} + \frac{x^2 e^{-x}}{3} + \frac{2\,e^{-x}}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{2}\,x\right) + \frac{c_2 \sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2}\right) + \left(\frac{4x\,e^{-x}}{9} + \frac{x^2 e^{-x}}{3} + \frac{2\,e^{-x}}{27}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2} + \frac{4x\, e^{-x}}{9} + \frac{x^2 e^{-x}}{3} + \frac{2\, e^{-x}}{27} \tag{1}$$

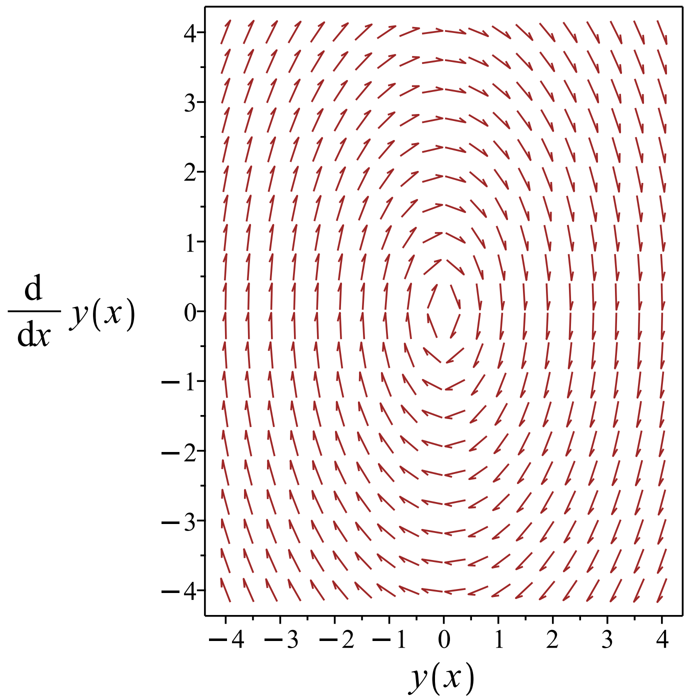


Figure 659: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2} + \frac{4x\, e^{-x}}{9} + \frac{x^2 e^{-x}}{3} + \frac{2\, e^{-x}}{27}$$

Verified OK.

14.17.3 Maple step by step solution

Let's solve

$y'' + 2y = x^2 e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos\left(\sqrt{2}\, x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin\left(\sqrt{2}\, x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 \mathrm{e}^{-x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos\left(\sqrt{2}\, x\right) & \sin\left(\sqrt{2}\, x\right) \\ -\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) & \sqrt{2}\, \cos\left(\sqrt{2}\, x\right) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \sqrt{2}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\sqrt{2}\left(\cos\left(\sqrt{2}\, x\right)\left(\int \sin\left(\sqrt{2}\, x\right) x^2 \mathrm{e}^{-x} dx\right) - \sin\left(\sqrt{2}\, x\right)\left(\int \cos\left(\sqrt{2}\, x\right) x^2 \mathrm{e}^{-x} dx\right)\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{\mathrm{e}^{-x}(9x^2 + 12x + 2)}{27}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + \frac{\mathrm{e}^{-x}(9x^2 + 12x + 2)}{27}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+2*y(x)=x^2*exp(-x),y(x), singsol=all)
```

$$y(x) = \cos\left(\sqrt{2}\,x\right)c_1 + c_2 \sin\left(\sqrt{2}\,x\right) + \frac{\left(x^2 + \frac{4}{3}x + \frac{2}{9}\right)\mathrm{e}^{-x}}{3}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 47

```
DSolve[y''[x]+2*y[x]==x^2*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{27}e^{-x}\left(9x^2 + 12x + 2\right) + c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right)$$

14.18 problem 18

Internal problem ID [2219]
Internal file name [`OUTPUT/2219_Monday_February_26_2024_09_18_30_AM_47955577/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' - 2y = x^2 - 8}$$

14.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = x^2 - 8$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \,\mathrm{e}^{\lambda x} - 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-2)} \\ &= \frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{3}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3x^2 - 2A_2x - 2xA_3 - 2A_1 - A_2 + 2A_3 = x^2 - 8$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{13}{4}, A_2 = \frac{1}{2}, A_3 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{2}x^2 + \frac{1}{2}x + \frac{13}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{2x} + c_2e^{-x}\right) + \left(-\frac{1}{2}x^2 + \frac{1}{2}x + \frac{13}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^{2x} + c_2e^{-x} - \frac{x^2}{2} + \frac{x}{2} + \frac{13}{4} \tag{1}$$

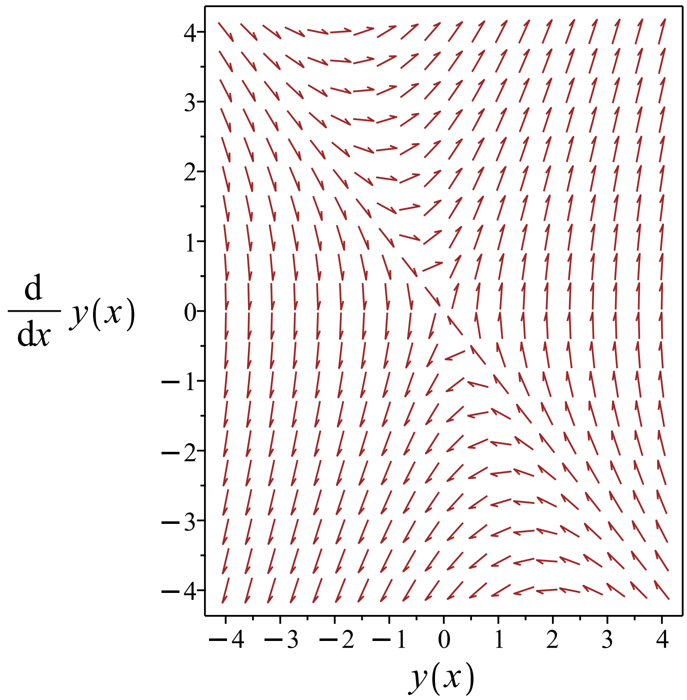


Figure 660: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-x} - \frac{x^2}{2} + \frac{x}{2} + \frac{13}{4}$$

Verified OK.

14.18.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 505: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1 \left(e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{e^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2 + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2x}}{3}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3x^2 - 2A_2x - 2xA_3 - 2A_1 - A_2 + 2A_3 = x^2 - 8$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{13}{4}, A_2 = \frac{1}{2}, A_3 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{2}x^2 + \frac{1}{2}x + \frac{13}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{2x}}{3}\right) + \left(-\frac{1}{2}x^2 + \frac{1}{2}x + \frac{13}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} - \frac{x^2}{2} + \frac{x}{2} + \frac{13}{4} \tag{1}$$

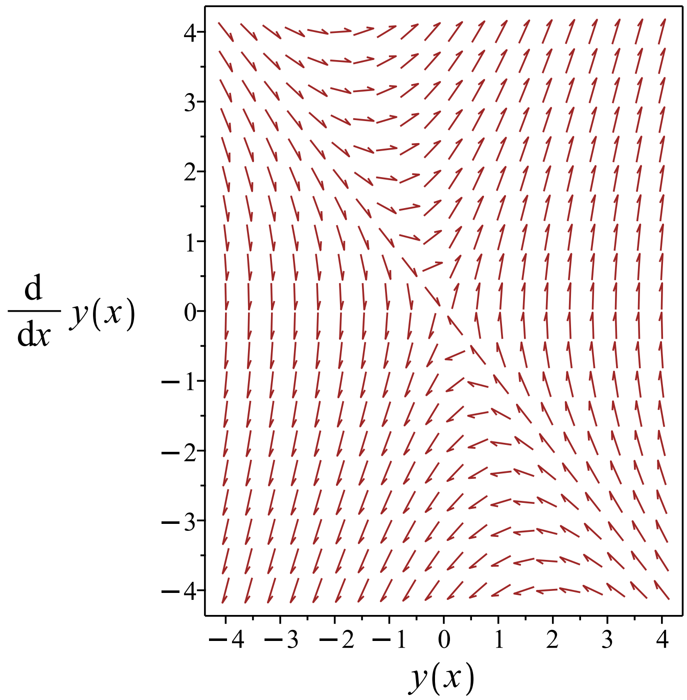


Figure 661: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} - \frac{x^2}{2} + \frac{x}{2} + \frac{13}{4}$$

Verified OK.

14.18.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = x^2 - 8$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

$r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 - 8\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3e^{x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{e^{-x}\left(\int (x^2-8)e^{x} dx\right)}{3} + \frac{e^{2x}\left(\int e^{-2x}(x^2-8) dx\right)}{3}$

  - Compute integrals

    $y_p(x) = -\frac{1}{2}x^2 + \frac{1}{2}x + \frac{13}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{2x} - \frac{x^2}{2} + \frac{x}{2} + \frac{13}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-2*y(x)=x^2-8,y(x), singsol=all)
```

$$y(x) = c_2 e^{2x} + e^{-x} c_1 - \frac{x^2}{2} + \frac{x}{2} + \frac{13}{4}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 36

```
DSolve[y''[x]-y'[x]-2*y[x]==x^2-8,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-2x^2 + 2x + 13\right) + c_1 e^{-x} + c_2 e^{2x}$$

14.19 problem 19

Internal problem ID [2220]
Internal file name [`OUTPUT/2220_Monday_February_26_2024_09_18_30_AM_30198627/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 19.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - y = x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y = 0$$

The characteristic equation is

$$\lambda^3 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$

$$y_2 = e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$

$$y_3 = e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y''' - y = x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x, e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}, e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3 x^2 - A_2 x - A_1 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3\right) + \left(-x^2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 - x^2 \tag{1}$$

Verification of solutions

$$y = c_1 e^x + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 - x^2$$

Verified OK.

14.19.1 Maple step by step solution

Let's solve

$y''' - y = x^2$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = x^2 + y_1(x)$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 + y_1(x)]$$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ x^2 \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ x^2 \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right], \left[-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = e^{x} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{I\sqrt{3}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\, x}{2}\right) - I\sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) \cdot \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1\end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \left[\begin{array}{c}\frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-I\sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\, x}{2}\right)-I\sin\left(\frac{\sqrt{3}\, x}{2}\right)}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\, x}{2}\right)-I\sin\left(\frac{\sqrt{3}\, x}{2}\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right], \vec{y}_3(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($
  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \left[\begin{array}{ccc} \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t
  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{ccc} \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ \mathrm{e}^{x} & \mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right] \cdot \frac{1}{\left[\begin{array}{ccc} 1 & -\frac{1}{2} & -\frac{\sqrt{\ }}{2} \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 1 & 0 \end{array}\right.}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{ccc} \frac{2\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\mathrm{e}^{x}}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}\right.}{3} \\ \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} & \frac{2\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\mathrm{e}^{x}}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}\right.}{3} \\ \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} & \frac{\mathrm{e}^{x}}{3} - \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3} & \frac{2\,\mathrm{e}^{-\frac{x}{2}}\cos}{3} \end{array}\right.$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{2\,\mathrm{e}^{x}}{3}-x^{2}-\frac{2 \sin \left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}-\frac{2\,\mathrm{e}^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}\, x}{2}\right)}{3} \\ \frac{2\,\mathrm{e}^{x}}{3}+\frac{2 \sin \left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}-2x-\frac{2\,\mathrm{e}^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}\, x}{2}\right)}{3} \\ \frac{2\,\mathrm{e}^{x}}{3}-2+\frac{4\,\mathrm{e}^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}\, x}{2}\right)}{3} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \left[\begin{array}{c} \frac{2\,\mathrm{e}^{x}}{3}-x^{2}-\frac{2 \sin \left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}-\frac{2\,\mathrm{e}^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}\, x}{2}\right)}{3} \\ \frac{2\,\mathrm{e}^{x}}{3}+\frac{2 \sin \left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}-2x-\frac{2\,\mathrm{e}^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}\, x}{2}\right)}{3} \\ \frac{2\,\mathrm{e}^{x}}{3}-2+\frac{4\,\mathrm{e}^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}\, x}{2}\right)}{3} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\left(c_3\sqrt{3}+c_2+\frac{4}{3}\right)\mathrm{e}^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}\, x}{2}\right)}{2}-\frac{\left(\left(c_2+\frac{4}{3}\right)\sqrt{3}-c_3\right)\mathrm{e}^{-\frac{x}{2}} \sin \left(\frac{\sqrt{3}\, x}{2}\right)}{2}+\frac{(6c_1+4)\mathrm{e}^{x}}{6}-x^{2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$3)-y(x)=x^2,y(x), singsol=all)
```

$$y(x) = -x^2 + e^x c_1 + c_2 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_3 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 59

```
DSolve[y'''[x]-y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 + c_1 e^x + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

14.20 problem 20

Internal problem ID [2221]
Internal file name [`OUTPUT/2221_Monday_February_26_2024_09_18_31_AM_34446081/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 20.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 4y'' - 5y' = x^2 \mathrm{e}^{-x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 4y'' - 5y' = 0$$

The characteristic equation is

$$\lambda^3 + 4\lambda^2 - 5\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -5$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^x + e^{-5x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = e^{-5x}$$

Now the particular solution to the given ODE is found

$$y''' + 4y'' - 5y' = x^2 e^{-x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{-x}, x^2 e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^x, e^{-5x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\, e^{-x} + A_2 x^2 e^{-x} + A_3 e^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-10A_1 e^{-x} + 8A_1 x\, e^{-x} + 2A_2 e^{-x} - 20A_2 x\, e^{-x} + 8A_2 x^2 e^{-x} + 8A_3 e^{-x} = x^2 e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{16}, A_2 = \frac{1}{8}, A_3 = \frac{23}{64}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5x\,e^{-x}}{16} + \frac{x^2 e^{-x}}{8} + \frac{23\,e^{-x}}{64}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^x + e^{-5x} c_3\right) + \left(\frac{5x\,e^{-x}}{16} + \frac{x^2 e^{-x}}{8} + \frac{23\,e^{-x}}{64}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 + c_2 e^x + e^{-5x} c_3 + \frac{5x\,e^{-x}}{16} + \frac{x^2 e^{-x}}{8} + \frac{23\,e^{-x}}{64} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 + c_2 e^x + e^{-5x} c_3 + \frac{5x\,e^{-x}}{16} + \frac{x^2 e^{-x}}{8} + \frac{23\,e^{-x}}{64}$$

Verified OK.

14.20.1 Maple step by step solution

Let's solve

$y''' + 4y'' - 5y' = x^2 e^{-x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = x^2 \mathrm{e}^{-x} - 4y_3(x) + 5y_2(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 \mathrm{e}^{-x} - 4y_3(x) + 5y_2(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & -4 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ x^2 \mathrm{e}^{-x} \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ x^2 \mathrm{e}^{-x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & -4 \end{array}\right]$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-5, \left[\begin{array}{c} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-5, \left[\begin{array}{c} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-5x} \cdot \left[\begin{array}{c} \frac{1}{25} \\ -\frac{1}{5} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + \overrightarrow{y}_p(x)$$

□ Fundamental matrix

- ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-5x}}{25} & 1 & e^{x} \\ -\frac{e^{-5x}}{5} & 0 & e^{x} \\ e^{-5x} & 0 & e^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-5x}}{25} & 1 & e^{x} \\ -\frac{e^{-5x}}{5} & 0 & e^{x} \\ e^{-5x} & 0 & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{25} & 1 & 1 \\ -\frac{1}{5} & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{(-25\,e^{6x}+24\,e^{5x}+1)e^{-5x}}{30} & -\frac{(-5\,e^{6x}+6\,e^{5x}-1)e^{-5x}}{30} \\ 0 & \frac{(5\,e^{6x}+1)e^{-5x}}{6} & \frac{(e^{6x}-1)e^{-5x}}{6} \\ 0 & \frac{5(e^{6x}-1)e^{-5x}}{6} & \frac{(e^{6x}+5)e^{-5x}}{6} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \begin{bmatrix} \frac{(120x^2+300x+345)e^{-5x}e^{4x}}{960} + \frac{(-384\,e^{5x}+40\,e^{6x}-1)e^{-5x}}{960} \\ \frac{(-24x^2-12x-9)e^{-5x}e^{4x}}{192} + \frac{(8\,e^{6x}+1)e^{-5x}}{192} \\ \frac{(24x^2-36x-3)e^{-5x}e^{4x}}{192} + \frac{(8\,e^{6x}-5)e^{-5x}}{192} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \begin{bmatrix} \frac{(120x^2+300x+345)e^{-5x}e^{4x}}{960} + \frac{(-384\,e^{5x}+40\,e^{6x}-1)e^{-5x}}{960} \\ \frac{(-24x^2-12x-9)e^{-5x}e^{4x}}{192} + \frac{(8\,e^{6x}+1)e^{-5x}}{192} \\ \frac{(24x^2-36x-3)e^{-5x}e^{4x}}{192} + \frac{(8\,e^{6x}-5)e^{-5x}}{192} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{e^{-5x}\left((8x^2+20x+23)e^{4x}+64c_2e^{5x}+64c_3e^{6x}+\frac{64c_1}{25}-\frac{128\,e^{5x}}{5}+\frac{8\,e^{6x}}{3}-\frac{1}{15}\right)}{64}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = exp(-_a)*_a^2+5*_b(_a)-4*(dif
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$3)+4*diff(y(x),x$2)-5*diff(y(x),x)=x^2*exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(x^2 + \frac{5}{2}x + \frac{23}{8}\right) e^{4x} + 8\, e^{6x} c_2 + 8 c_3 e^{5x} - \frac{8 c_1}{5}\right) e^{-5x}}{8}$$

✓ Solution by Mathematica

Time used: 0.192 (sec). Leaf size: 44

```
DSolve[y'''[x]+4*y''[x]-5*y'[x]==x^2*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{64} e^{-x}\left(8x^2 + 20x + 23\right) - \frac{1}{5} c_1 e^{-5x} + c_2 e^x + c_3$$

14.21 problem 21

Internal problem ID [2222]
Internal file name [OUTPUT/2222_Monday_February_26_2024_09_18_31_AM_86073294/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 21.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' - 2y''' + y'' = x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y''' + y'' = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + c_3 \mathrm{e}^{x} + c_4 x \,\mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = \mathrm{e}^{x}$$
$$y_4 = x \,\mathrm{e}^{x}$$

Now the particular solution to the given ODE is found

$$y'''' - 2y''' + y'' = x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^4 + A_2 x^3 + A_1 x^2$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12x^2A_3 + 6xA_2 - 48xA_3 + 2A_1 - 12A_2 + 24A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 3, A_2 = \frac{2}{3}, A_3 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{12}x^4 + \frac{2}{3}x^3 + 3x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2x + c_1 + c_3\mathrm{e}^x + c_4x\,\mathrm{e}^x) + \left(\frac{1}{12}x^4 + \frac{2}{3}x^3 + 3x^2\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^x(c_4x + c_3) + c_2x + c_1 + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x(c_4x + c_3) + c_2x + c_1 + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^x(c_4x + c_3) + c_2x + c_1 + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$3)+diff(y(x),x$2)=x^2,y(x), singsol=all)
```

$$y(x) = (c_1 x - 2c_1 + c_2) e^x + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 + c_3 x + c_4$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 46

```
DSolve[y''''[x]-2*y'''[x]+y''[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 + c_4 x + c_1 e^x + c_2 e^x (x-2) + c_3$$

14.22 problem 22

Internal problem ID [2223]
Internal file name [`OUTPUT/2223_Monday_February_26_2024_09_18_31_AM_16990448/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 22.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - y' = e^{x} \left(\sin \left(x \right) - x^{2}\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y' = 0$$

The characteristic equation is

$$\lambda^{3} - \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 e^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= 1 \\
y_3 &= \mathrm{e}^{x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' - y' = \mathrm{e}^{x} \left(\sin \left(x \right)-x^{2}\right)
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{x} \left(\sin \left(x \right)-x^{2}\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos \left(x \right) \mathrm{e}^{x}, \sin \left(x \right) \mathrm{e}^{x}\}, \{x \,\mathrm{e}^{x}, x^{2} \mathrm{e}^{x}, \mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, \mathrm{e}^{x}, \mathrm{e}^{-x}\}
$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{\cos \left(x \right) \mathrm{e}^{x}, \sin \left(x \right) \mathrm{e}^{x}\}, \{x \,\mathrm{e}^{x}, x^{2} \mathrm{e}^{x}, \mathrm{e}^{x} x^{3}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_{1} \cos \left(x \right) \mathrm{e}^{x}+A_{2} \sin \left(x \right) \mathrm{e}^{x}+A_{3} x \,\mathrm{e}^{x}+A_{4} x^{2} \mathrm{e}^{x}+A_{5} \mathrm{e}^{x} x^{3}
$$

The unknowns $\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-3 A_{1} \cos \left(x \right) \mathrm{e}^{x}-A_{1} \sin \left(x \right) \mathrm{e}^{x}-3 A_{2} \sin \left(x \right) \mathrm{e}^{x}+A_{2} \cos \left(x \right) \mathrm{e}^{x}+2 A_{3} \mathrm{e}^{x} \\
&+6 A_{4} \mathrm{e}^{x}+4 A_{4} x \,\mathrm{e}^{x}+6 A_{5} \mathrm{e}^{x} x^{2}+18 A_{5} \mathrm{e}^{x} x +6 A_{5} \mathrm{e}^{x} = \mathrm{e}^{x} \left(\sin \left(x \right)-x^{2}\right)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{10}, A_2 = -\frac{3}{10}, A_3 = -\frac{7}{4}, A_4 = \frac{3}{4}, A_5 = -\frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,\mathrm{e}^x}{10} - \frac{3\sin(x)\,\mathrm{e}^x}{10} - \frac{7x\,\mathrm{e}^x}{4} + \frac{3x^2\mathrm{e}^x}{4} - \frac{\mathrm{e}^x x^3}{6}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(c_1\mathrm{e}^{-x} + c_2 + c_3\mathrm{e}^x\right) + \left(-\frac{\cos(x)\,\mathrm{e}^x}{10} - \frac{3\sin(x)\,\mathrm{e}^x}{10} - \frac{7x\,\mathrm{e}^x}{4} + \frac{3x^2\mathrm{e}^x}{4} - \frac{\mathrm{e}^x x^3}{6}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-x} + c_2 + c_3\mathrm{e}^x - \frac{\cos(x)\,\mathrm{e}^x}{10} - \frac{3\sin(x)\,\mathrm{e}^x}{10} - \frac{7x\,\mathrm{e}^x}{4} + \frac{3x^2\mathrm{e}^x}{4} - \frac{\mathrm{e}^x x^3}{6} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-x} + c_2 + c_3\mathrm{e}^x - \frac{\cos(x)\,\mathrm{e}^x}{10} - \frac{3\sin(x)\,\mathrm{e}^x}{10} - \frac{7x\,\mathrm{e}^x}{4} + \frac{3x^2\mathrm{e}^x}{4} - \frac{\mathrm{e}^x x^3}{6}$$

Verified OK.

14.22.1 Maple step by step solution

Let's solve

$y''' - y' = \mathrm{e}^x(\sin(x) - x^2)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -x^2 \mathrm{e}^x + \sin(x)\,\mathrm{e}^x + y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -x^2 \mathrm{e}^x + \sin(x)\,\mathrm{e}^x + y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -x^2 \mathrm{e}^x + \sin(x)\,\mathrm{e}^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -x^2 \mathrm{e}^x + \sin(x)\,\mathrm{e}^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right], \left[0, \left[\begin{array}{c}1 \\ 0 \\ 0\end{array}\right]\right], \left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c}1 \\ 0 \\ 0\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \left[\begin{array}{c}1 \\ 0 \\ 0\end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

- ☐ Fundamental matrix
  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $$\phi(x) = \begin{bmatrix} \mathrm{e}^{-x} & 1 & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & 0 & \mathrm{e}^{x} \\ \mathrm{e}^{-x} & 0 & \mathrm{e}^{x} \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

    $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

    $$\Phi(x) = \begin{bmatrix} \mathrm{e}^{-x} & 1 & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & 0 & \mathrm{e}^{x} \\ \mathrm{e}^{-x} & 0 & \mathrm{e}^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

    $$\Phi(x) = \begin{bmatrix} 1 & -\frac{\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x}}{2} & \frac{\mathrm{e}^{-x}}{2} - 1 + \frac{\mathrm{e}^{x}}{2} \\ 0 & \frac{\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{2} & -\frac{\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x}}{2} \\ 0 & -\frac{\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x}}{2} & \frac{\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{-x}}{2} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

  - Take the derivative of the particular solution

    $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - Cancel like terms

$\Phi(x)\cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)}\cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{9\,\mathrm{e}^{-x}}{40}-\frac{5}{2}+\frac{\left(-20x^3+90x^2-210x-12\cos(x)-36\sin(x)+285\right)\mathrm{e}^{x}}{120} \\ -\frac{9\,\mathrm{e}^{-x}}{40}+\frac{\left(-20x^3+30x^2-30x-48\cos(x)-24\sin(x)+75\right)\mathrm{e}^{x}}{120} \\ \frac{9\,\mathrm{e}^{-x}}{40}+\frac{\left(-20x^3-30x^2+30x-72\cos(x)+24\sin(x)+45\right)\mathrm{e}^{x}}{120} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \left[\begin{array}{c} \frac{9\,\mathrm{e}^{-x}}{40}-\frac{5}{2}+\frac{\left(-20x^3+90x^2-210x-12\cos(x)-36\sin(x)+285\right)\mathrm{e}^{x}}{120} \\ -\frac{9\,\mathrm{e}^{-x}}{40}+\frac{\left(-20x^3+30x^2-30x-48\cos(x)-24\sin(x)+75\right)\mathrm{e}^{x}}{120} \\ \frac{9\,\mathrm{e}^{-x}}{40}+\frac{\left(-20x^3-30x^2+30x-72\cos(x)+24\sin(x)+45\right)\mathrm{e}^{x}}{120} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{5}{2}+\frac{(9+40c_1)\mathrm{e}^{-x}}{40}+\frac{\left(-20x^3+90x^2-210x+120c_3-12\cos(x)-36\sin(x)+285\right)\mathrm{e}^{x}}{120}+c_2$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -_a^2*exp(_a)+exp(_a)*sin(_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$3)-diff(y(x),x)=exp(x)*(sin(x)-x^2),y(x), singsol=all)
```

$$y(x) = -\mathrm{e}^{-x} c_1 + \frac{\left(-20x^3 + 90x^2 - 210x + 120c_2 - 12\cos\left(x\right) - 36\sin\left(x\right) + 225\right)\mathrm{e}^{x}}{120} + c_3$$

✓ Solution by Mathematica

Time used: 1.016 (sec). Leaf size: 63

```
DSolve[y'''[x]-y'[x]==Exp[x]*(Sin[x]-x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{24}e^x\left(-4x^3 + 18x^2 - 42x + 45\right) - \frac{3}{10}e^x\sin(x) - \frac{1}{10}e^x\cos(x) + c_1e^x - c_2e^{-x} + c_3$$

14.23 problem 23

Internal problem ID [2224]
Internal file name [`OUTPUT/2224_Monday_February_26_2024_09_18_32_AM_49298837/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 23.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 4y'' = e^{2x}(x - 3)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 4$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + e^{4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= e^{4x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$y''' - 4y'' = e^{2x}(x - 3)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}(x - 3)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, e^{4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{2x} + A_2 e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{2x} - 8A_1 x\,e^{2x} - 8A_2 e^{2x} = e^{2x}(x - 3)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = \frac{7}{16}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{2x}}{8} + \frac{7\,e^{2x}}{16}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_2 x + c_1 + e^{4x} c_3) + \left(-\frac{x\, e^{2x}}{8} + \frac{7\, e^{2x}}{16}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 x + c_1 + e^{4x} c_3 - \frac{x\, e^{2x}}{8} + \frac{7\, e^{2x}}{16} \tag{1}
$$

Verification of solutions

$$
y = c_2 x + c_1 + e^{4x} c_3 - \frac{x\, e^{2x}}{8} + \frac{7\, e^{2x}}{16}
$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = exp(2*_a)*_a-3*exp(2*_a)+4*_b(_a), _b(_
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)=exp(2*x)*(x-3),y(x), singsol=all)
```

$$
y(x) = \frac{(-2x + 7)\, e^{2x}}{16} + c_2 x + \frac{e^{4x} c_1}{16} + c_3
$$

✓ Solution by Mathematica

Time used: 0.268 (sec). Leaf size: 34

```
DSolve[y'''[x]-4*y''[x]==Exp[2*x]*(x-3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{16}e^{2x}\left(-2x + c_1e^{2x} + 7\right) + c_3x + c_2$$

14.24 problem 24

Internal problem ID [2225]
Internal file name [`OUTPUT/2225_Monday_February_26_2024_09_18_32_AM_17887916/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 24.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' - 6y''' + 9y'' = \sin(3x) + x\,\mathrm{e}^{x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 6y''' + 9y'' = 0$$

The characteristic equation is

$$\lambda^4 - 6\lambda^3 + 9\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 3$$
$$\lambda_4 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_4 x\,\mathrm{e}^{3x} + c_3\mathrm{e}^{3x} + c_2 x + c_1$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= x \\ y_3 &= \mathrm{e}^{3x} \\ y_4 &= x\,\mathrm{e}^{3x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 6y''' + 9y'' = \sin(3x) + x\,\mathrm{e}^{x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(3x) + x\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x\,\mathrm{e}^{3x}, \mathrm{e}^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{x} + A_2\mathrm{e}^{x} + A_3\cos(3x) + A_4\sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\mathrm{e}^{x} + 4A_1 x\,\mathrm{e}^{x} + 4A_2\mathrm{e}^{x} - 162A_3\sin(3x) + 162A_4\cos(3x) = \sin(3x) + x\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}, A_2 = -\frac{1}{4}, A_3 = -\frac{1}{162}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,e^x}{4} - \frac{e^x}{4} - \frac{\cos(3x)}{162}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_4 x\,e^{3x} + c_3 e^{3x} + c_2 x + c_1\right) + \left(\frac{x\,e^x}{4} - \frac{e^x}{4} - \frac{\cos(3x)}{162}\right) \end{aligned}$$

Which simplifies to

$$y = (c_4 x + c_3)\,e^{3x} + c_2 x + c_1 + \frac{x\,e^x}{4} - \frac{e^x}{4} - \frac{\cos(3x)}{162}$$

Summary

The solution(s) found are the following

$$y = (c_4 x + c_3)\,e^{3x} + c_2 x + c_1 + \frac{x\,e^x}{4} - \frac{e^x}{4} - \frac{\cos(3x)}{162} \tag{1}$$

Verification of solutions

$$y = (c_4 x + c_3)\,e^{3x} + c_2 x + c_1 + \frac{x\,e^x}{4} - \frac{e^x}{4} - \frac{\cos(3x)}{162}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a*exp(_a)+sin(3*_a)-9*_b(_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$4)-6*diff(y(x),x$3)+9*diff(y(x),x$2)=sin(3*x)+x*exp(x),y(x), singsol=all)
```

$$y(x) = \frac{(3c_1x - 2c_1 + 3c_2)\,e^{3x}}{27} - \frac{\cos(3x)}{162} + \frac{(x-1)\,e^{x}}{4} + c_3x + c_4$$

✓ Solution by Mathematica

Time used: 1.069 (sec). Leaf size: 52

```
DSolve[y''''[x]-6*y'''[x]+9*y''[x]==Sin[3*x]+x*Exp[x],y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{4}e^{x}(x-1) - \frac{1}{162}\cos(3x) + \frac{1}{27}e^{3x}(c_2(3x-2) + 3c_1) + c_4x + c_3$$

14.25 problem 25

Internal problem ID [2226]
Internal file name [`OUTPUT/2226_Monday_February_26_2024_09_18_32_AM_83729563/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 25.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' - 6y'' + 11y' - 6y = x^2 \mathrm{e}^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 11y' - 6y = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$

$$y_2 = e^{2x}$$

$$y_3 = e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 11y' - 6y = x^2 e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{2x}, x^2 e^{2x}, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}, e^{3x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}, x^2 e^{2x}, e^{2x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{2x} + A_2 x^2 e^{2x} + A_3 e^{2x} x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_2 x\,e^{2x} - 3A_3 e^{2x} x^2 - A_1 e^{2x} + 6A_3 e^{2x} = x^2 e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -2, A_2 = 0, A_3 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x\,\mathrm{e}^{2x} - \frac{\mathrm{e}^{2x}x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{2x} + c_3\mathrm{e}^{3x}\right) + \left(-2x\,\mathrm{e}^{2x} - \frac{\mathrm{e}^{2x}x^3}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{x} + c_2\mathrm{e}^{2x} + c_3\mathrm{e}^{3x} - 2x\,\mathrm{e}^{2x} - \frac{\mathrm{e}^{2x}x^3}{3} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{x} + c_2\mathrm{e}^{2x} + c_3\mathrm{e}^{3x} - 2x\,\mathrm{e}^{2x} - \frac{\mathrm{e}^{2x}x^3}{3}$$

Verified OK.

14.25.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 11y' - 6y = x^2\mathrm{e}^{2x}$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = x^2 e^{2x} + 6y_3(x) - 11y_2(x) + 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 e^{2x} + 6y_3(x) - 11y_2(x) + 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 e^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 e^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3+\vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(x)=\begin{bmatrix} e^{x} & \frac{e^{2x}}{4} & \frac{e^{3x}}{9} \\ e^{x} & \frac{e^{2x}}{2} & \frac{e^{3x}}{3} \\ e^{x} & e^{2x} & e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x)=\begin{bmatrix} e^{x} & \frac{e^{2x}}{4} & \frac{e^{3x}}{9} \\ e^{x} & \frac{e^{2x}}{2} & \frac{e^{3x}}{3} \\ e^{x} & e^{2x} & e^{3x} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{9} \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x)=\begin{bmatrix} 3e^{x}-3e^{2x}+e^{3x} & -\frac{5e^{x}}{2}+4e^{2x}-\frac{3e^{3x}}{2} & \frac{e^{x}}{2}-e^{2x}+\frac{e^{3x}}{2} \\ 3e^{x}-6e^{2x}+3e^{3x} & -\frac{5e^{x}}{2}+8e^{2x}-\frac{9e^{3x}}{2} & \frac{e^{x}}{2}-2e^{2x}+\frac{3e^{3x}}{2} \\ 3e^{x}-12e^{2x}+9e^{3x} & -\frac{5e^{x}}{2}+16e^{2x}-\frac{27e^{3x}}{2} & \frac{e^{x}}{2}-4e^{2x}+\frac{9e^{3x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{(-x^3-6x)\mathrm{e}^{2x}}{3} - \mathrm{e}^{x} + \mathrm{e}^{3x} \\ \frac{(-2x^3-3x^2-12x-6)\mathrm{e}^{2x}}{3} - \mathrm{e}^{x} + 3\,\mathrm{e}^{3x} \\ \frac{2(-2x^3-6x^2-15x-12)\mathrm{e}^{2x}}{3} - \mathrm{e}^{x} + 9\,\mathrm{e}^{3x} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \left[\begin{array}{c} \frac{(-x^3-6x)\mathrm{e}^{2x}}{3} - \mathrm{e}^{x} + \mathrm{e}^{3x} \\ \frac{(-2x^3-3x^2-12x-6)\mathrm{e}^{2x}}{3} - \mathrm{e}^{x} + 3\,\mathrm{e}^{3x} \\ \frac{2(-2x^3-6x^2-15x-12)\mathrm{e}^{2x}}{3} - \mathrm{e}^{x} + 9\,\mathrm{e}^{3x} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = \frac{(-4x^3+3c_2-24x)\mathrm{e}^{2x}}{12} + \frac{(c_3+9)\mathrm{e}^{3x}}{9} + \mathrm{e}^{x}(c_1-1)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+11*diff(y(x),x)-6*y(x)=x^2*exp(2*x),y(x), singsol=all
```

$$y(x)=\frac{(-x^3+3c_2-6x)\,\mathrm{e}^{2x}}{3}+\mathrm{e}^{x}c_1+c_3\mathrm{e}^{3x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 37

```
DSolve[y'''[x]-6*y''[x]+11*y'[x]-6*y[x]==x^2*Exp[2*x],y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x)\to e^x\left(e^x\left(-\frac{x^3}{3}-2x+c_2\right)+c_3e^{2x}+c_1\right)$$

14.26 problem 26

Internal problem ID [2227]
Internal file name [OUTPUT/2227_Monday_February_26_2024_09_18_33_AM_24631714/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 26.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 2y' = x^2 + \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y' = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i\sqrt{2}$$
$$\lambda_3 = -i\sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{-i\sqrt{2}\,x} c_2 + e^{i\sqrt{2}\,x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{-i\sqrt{2}\,x}$$
$$y_3 = e^{i\sqrt{2}\,x}$$

Now the particular solution to the given ODE is found

$$y''' + 2y' = x^2 + \cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, e^{i\sqrt{2}\,x}, e^{-i\sqrt{2}\,x}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x), \sin(x)\}, \{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 x + A_4 x^2 + A_5 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) + 6A_5 + 2A_3 + 4A_4 x + 6A_5 x^2 = x^2 + \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 1, A_3 = -\frac{1}{2}, A_4 = 0, A_5 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(x) - \frac{x}{2} + \frac{x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + e^{-i\sqrt{2}\,x}c_2 + e^{i\sqrt{2}\,x}c_3\right) + \left(\sin(x) - \frac{x}{2} + \frac{x^3}{6}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 + e^{-i\sqrt{2}\,x}c_2 + e^{i\sqrt{2}\,x}c_3 + \sin(x) - \frac{x}{2} + \frac{x^3}{6} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 + e^{-i\sqrt{2}\,x}c_2 + e^{i\sqrt{2}\,x}c_3 + \sin(x) - \frac{x}{2} + \frac{x^3}{6}$$

Verified OK.

14.26.1 Maple step by step solution

Let's solve

$y''' + 2y' = x^2 + \cos(x)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = x^2 + \cos(x) - 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 + \cos(x) - 2y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 + \cos(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 + \cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-I\sqrt{2}, \begin{bmatrix} -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[I\sqrt{2}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I\sqrt{2}, \begin{bmatrix} -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-I\sqrt{2}\,x} \cdot \begin{bmatrix} -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right)\right) \cdot \begin{bmatrix} -\frac{1}{2} \\ \frac{I}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\cos\left(\sqrt{2}\,x\right)}{2} + \frac{I\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{I}{2}\left(\cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2} \\ \cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x)=\begin{bmatrix} -\frac{\cos\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} \\ \cos\left(\sqrt{2}\,x\right) \end{bmatrix}, \vec{y}_3(x)=\begin{bmatrix} \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ -\sin\left(\sqrt{2}\,x\right) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+\vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} 1 & -\frac{\cos\left(\sqrt{2}\,x\right)}{2} & \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ 0 & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} & \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ 0 & \cos\left(\sqrt{2}\,x\right) & -\sin\left(\sqrt{2}\,x\right) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} 1 & -\frac{\cos\left(\sqrt{2}\,x\right)}{2} & \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ 0 & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} & \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ 0 & \cos\left(\sqrt{2}\,x\right) & -\sin\left(\sqrt{2}\,x\right) \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} 1 & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} & \frac{1}{2}-\frac{\cos\left(\sqrt{2}\,x\right)}{2} \\ 0 & \cos\left(\sqrt{2}\,x\right) & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} \\ 0 & -\sqrt{2}\,\sin\left(\sqrt{2}\,x\right) & \cos\left(\sqrt{2}\,x\right) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x) = \begin{bmatrix} \frac{x^3}{6} - \frac{x}{2} - \frac{\sqrt{2}\sin\left(\sqrt{2}\,x\right)}{4} + \sin(x) \\ -\frac{1}{2} + \frac{x^2}{2} + \cos(x) - \frac{\cos\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\sin\left(\sqrt{2}\,x\right)}{2} - \sin(x) + x \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \begin{bmatrix} \frac{x^3}{6} - \frac{x}{2} - \frac{\sqrt{2}\sin\left(\sqrt{2}\,x\right)}{4} + \sin(x) \\ -\frac{1}{2} + \frac{x^2}{2} + \cos(x) - \frac{\cos\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\sin\left(\sqrt{2}\,x\right)}{2} - \sin(x) + x \end{bmatrix}$$
- First component of the vector is the solution to the ODE

  $y = \frac{\left(2c_3 - \sqrt{2}\right)\sin\left(\sqrt{2}\,x\right)}{4} + \frac{x^3}{6} - \frac{c_2\cos\left(\sqrt{2}\,x\right)}{2} - \frac{x}{2} + c_1 + \sin(x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a^2+cos(_a)-2*_b(_a), _b(_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x)=x^2+cos(x),y(x), singsol=all)
```

$$y(x) = \frac{x^3}{6} - \frac{c_2\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} + \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)c_1}{2} + \sin\left(x\right) - \frac{x}{2} + c_3$$

✓ Solution by Mathematica

Time used: 0.417 (sec). Leaf size: 55

```
DSolve[y'''[x]+2*y'[x]==x^2+Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{6} - \frac{x}{2} + \sin(x) - \frac{c_2\cos\left(\sqrt{2}x\right)}{\sqrt{2}} + \frac{c_1\sin\left(\sqrt{2}x\right)}{\sqrt{2}} + c_3$$

14.27 problem 27

Internal problem ID [2228]
Internal file name [`OUTPUT/2228_Monday_February_26_2024_09_18_33_AM_59470393/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 27.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _linear, _nonhomogeneous]]`

$$\boxed{y'''' + 3y'' - y' + 2y = \sin(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 3y'' - y' + 2y = 0$$

The characteristic equation is

$$\lambda^4 + 3\lambda^2 - \lambda + 2 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= \text{RootOf}\left(_Z^4 + 3_Z^2 - _Z + 2, \text{index} = 1\right) \\
\lambda_2 &= \text{RootOf}\left(_Z^4 + 3_Z^2 - _Z + 2, \text{index} = 2\right) \\
\lambda_3 &= \text{RootOf}\left(_Z^4 + 3_Z^2 - _Z + 2, \text{index} = 3\right) \\
\lambda_4 &= \text{RootOf}\left(_Z^4 + 3_Z^2 - _Z + 2, \text{index} = 4\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=1)x}c_1+e^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=3)x}c_2+e^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=4)}$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= e^{\mathrm{RootOf}(_Z^4+3_Z^2-_Z+2,\mathrm{index}=1)x} \\
y_2 &= e^{\mathrm{RootOf}(_Z^4+3_Z^2-_Z+2,\mathrm{index}=3)x} \\
y_3 &= e^{\mathrm{RootOf}(_Z^4+3_Z^2-_Z+2,\mathrm{index}=4)x} \\
y_4 &= e^{\mathrm{RootOf}(_Z^4+3_Z^2-_Z+2,\mathrm{index}=2)x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 3y'' - y' + 2y = \sin(2x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=1)x}, e^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=2)x}, e^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=3)x}, e^{\mathrm{RootOf}(\_Z^4}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 \cos(2x) + 6A_2 \sin(2x) + 2A_1 \sin(2x) - 2A_2 \cos(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{20}, A_2 = \frac{3}{20}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=1)x} c_1 + \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=3)x} c_2 \right. \\ &\quad \left. + \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=4)x} c_3 + \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=2)x} c_4\right) \\ &\quad + \left(\frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=1)x} c_1 + \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=3)x} c_2 \\ &\quad + \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=4)x} c_3 \\ &\quad + \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=2)x} c_4 + \frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=1)x} c_1 + \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=3)x} c_2 \\ &\quad + \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=4)x} c_3 \\ &\quad + \mathrm{e}^{\mathrm{RootOf}(\_Z^4+3\_Z^2-\_Z+2,\mathrm{index}=2)x} c_4 + \frac{\cos(2x)}{20} + \frac{3\sin(2x)}{20} \end{aligned}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 6182

```
dsolve(diff(y(x),x$4)+3*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=sin(2*x),y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 1.487 (sec). Leaf size: 1124

```
DSolve[y''''[x]+3*y''[x]-y'[x]+2*y[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x\text{Root}[\#1^4+3\#1^2-\#1+2\&,1]}c_1 + e^{x\text{Root}[\#1^4+3\#1^2-\#1+2\&,2]}c_2 + e^{x\text{Root}[\#1^4+3\#1^2-\#1+2\&,3]}c_3 + e^{x\text{Root}[\#1^4+3\#1^2-\#1+2\&,4]}c_4 - \frac{(\text{Root}[\#1^4+3\#1^2-\#1+2\&,1] - \text{Root}[\#1^4+3\#1^2-\#1+2\&,3])\,(\text{Root}[\#1^4+3\#1^2-\#1+}{} + \frac{(\text{Root}[\#1^4+3\#1^2-\#1+2\&,1] - \text{Root}[\#1^4+3\#1^2-\#1+2\&,2])\,(\text{Root}[\#1^4+3\#1^2-\#1+}{} - \frac{(\text{Root}[\#1^4+3\#1^2-\#1+2\&,1] - \text{Root}[\#1^4+3\#1^2-\#1+2\&,2])\,(\text{Root}[\#1^4+3\#1^2-\#1+}{} - \frac{e^{(\text{Root}[\#1^4+3\#1^2-\#1+2\&,2]+\text{Root}[\#1^4+3\#1^2-\#1+2\&,3]+\text{Root}[\#1^4+3\#1^2-\#1+2\&,4])x+\text{Root}[\#1^4+3\#1^2-\#1+2\&,1]x}(\text{Roo}}{}$$

14.28 problem 30

Internal problem ID [2229]
Internal file name [`OUTPUT/2229_Monday_February_26_2024_09_18_33_AM_72525973/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 30.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_y]]`

$$\boxed{y'''' + 2y'' + y' = x^3 - \frac{\cos(2x)}{2}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y'' + y' = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$

$$\lambda_2 = -\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} + \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}$$

$$\lambda_3 = \frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_4 = \frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + \mathrm{e}^{\left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$

$$y_2 = \mathrm{e}^{\left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)x}$$

$$y_3 = \mathrm{e}^{\left(\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{12} - \frac{2}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} - \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)}{2}\right)x}$$

$$y_4 = \mathrm{e}^{\left(-\frac{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}{6} + \frac{4}{\left(108+12\sqrt{177}\right)^{\frac{1}{3}}}\right)x}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y'' + y' = x^3 - \frac{\cos(2x)}{2}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 - \frac{\cos(2x)}{2}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, e^{\left(-\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{6}+\frac{4}{(108+12\sqrt{177})^{\frac{1}{3}}}\right)x}, e^{\left(\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{12}-\frac{2}{(108+12\sqrt{177})^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(-\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{6}-\frac{4}{(108+12\sqrt{177})^{\frac{1}{3}}}\right)}{2}\right)x}, e^{\left(\frac{(108}{}\right.}\right.$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(2x), \sin(2x)\}, \{x, x^2, x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x) + A_3 x + A_4 x^2 + A_5 x^3 + A_6 x^4$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&8A_1 \cos(2x) + 8A_2 \sin(2x) + 24A_6 + 4A_4 + 12A_5 x + 24A_6 x^2 - 2A_1 \sin(2x)\\&+ 2A_2 \cos(2x) + A_3 + 2A_4 x + 3A_5 x^2 + 4A_6 x^3 = x^3 - \frac{\cos(2x)}{2}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{17}, A_2 = -\frac{1}{68}, A_3 = -54, A_4 = 12, A_5 = -2, A_6 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(2x)}{17} - \frac{\sin(2x)}{68} - 54x + 12x^2 - 2x^3 + \frac{x^4}{4}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + e^{\left(\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{12} - \frac{2}{(108+12\sqrt{177})^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(-\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{6} - \frac{4}{(108+12\sqrt{177})^{\frac{1}{3}}}\right)}{2}\right)x} c_2 \right. \\
&\quad + e^{\left(\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{12} - \frac{2}{(108+12\sqrt{177})^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{6} - \frac{4}{(108+12\sqrt{177})^{\frac{1}{3}}}\right)}{2}\right)x} c_3 \\
&\quad \left. + e^{\left(-\frac{(108+12\sqrt{177})^{\frac{1}{3}}}{6} + \frac{4}{(108+12\sqrt{177})^{\frac{1}{3}}}\right)x} c_4 \right) \\
&\quad + \left(-\frac{\cos(2x)}{17} - \frac{\sin(2x)}{68} - 54x + 12x^2 - 2x^3 + \frac{x^4}{4}\right)
\end{aligned}$$

Which simplifies to

$$y = c_1 + e^{\frac{\left(i(108+12\sqrt{3}\sqrt{59})^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}+(108+12\sqrt{3}\sqrt{59})^{\frac{2}{3}}-24\right)x}{12(108+12\sqrt{3}\sqrt{59})^{\frac{1}{3}}}} c_2 + e^{-\frac{x\left((i\sqrt{3}-1)(108+12\sqrt{3}\sqrt{59})^{\frac{2}{3}}+24i\sqrt{3}+24\right)}{12(108+12\sqrt{3}\sqrt{59})^{\frac{1}{3}}}} c_3 + e^{-\frac{\left((108+12\sqrt{3}\sqrt{59})^{\frac{2}{3}}-24\right)x}{6(108+12\sqrt{3}\sqrt{59})^{\frac{1}{3}}}} c_4 - \frac{\cos(2x)}{17} - \frac{\sin(2x)}{68} - 54x + 12x^2 - 2x^3 + \frac{x^4}{4}$$

Summary
The solution(s) found are the following

$$y = c_1 + e^{\frac{\left(i\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}+\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}-24\right)x}{12\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{1}{3}}}} c_2 + e^{-\frac{x\left(\left(i\sqrt{3}-1\right)\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}+24i\sqrt{3}+24\right)}{12\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{1}{3}}}} c_3 + e^{-\frac{\left(\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}-24\right)x}{6\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{1}{3}}}} c_4 - \frac{\cos(2x)}{17} - \frac{\sin(2x)}{68} - 54x + 12x^2 - 2x^3 + \frac{x^4}{4} \tag{1}$$

Verification of solutions

$$y = c_1 + e^{\frac{\left(i\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}\sqrt{3}+24i\sqrt{3}+\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}-24\right)x}{12\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{1}{3}}}} c_2 + e^{-\frac{x\left(\left(i\sqrt{3}-1\right)\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}+24i\sqrt{3}+24\right)}{12\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{1}{3}}}} c_3 + e^{-\frac{\left(\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{2}{3}}-24\right)x}{6\left(108+12\sqrt{3}\,\sqrt{59}\right)^{\frac{1}{3}}}} c_4 - \frac{\cos(2x)}{17} - \frac{\sin(2x)}{68} - 54x + 12x^2 - 2x^3 + \frac{x^4}{4}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = _a^3-(1/2)*cos(2*_a
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.531 (sec). Leaf size: 1302

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$2)+diff(y(x),x)=x^3-1/2*cos(2*x),y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 5.87 (sec). Leaf size: 1293

```
DSolve[y''''[x]+2*y''[x]+y'[x]==x^3-1/2*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{x\text{Root}[\#1^3+2\#1+1\&,1]}c_1}{\text{Root}\left[\#1^3+2\#1+1\&,1\right]}+c_4$$

$$+\frac{ix(\text{Root}[\#1^3+2\#1+1\&,1]-\text{Root}[\#1^3+2\#1+1\&,2])\left(4+\text{Root}[\#1^3+2\#1+1\&,3]^2\right)(\text{Roo}}{4\sqrt{59}\left(-1+4\text{Root}\left[\#\right.\right.}$$

$$+\frac{ix(\text{Root}[\#1^3+2\#1+1\&,2]-\text{Root}[\#1^3+2\#1+1\&,3])\left(\text{Root}[\#1^3+2\#1+1\&,3]^3x^3+3\text{Root}\right.}{}$$

$$-\frac{i(\text{Root}[\#1^3+2\#1+1\&,1]-\text{Root}[\#1^3+2\#1+1\&,2])\text{Root}[\#1^3+2\#1+1\&,3]^5\sin(2x)}{4\sqrt{59}\left(-1+4\text{Root}\left[\#1^3+2\#1+1\&,3\right]^2\right)}$$

$$-\frac{i(\text{Root}[\#1^3+2\#1+1\&,2]-\text{Root}[\#1^3+2\#1+1\&,3])(\text{Root}[\#1^3+2\#1+1\&,2]+\text{Root}\left[\right.}{4\sqrt{59}\left(-2i+\text{Root}\left[\#1^3+2\#1+1\&,2\right]+\text{Root}\left[\#1^3+2\#1+1\&,3\right]\right)\left(2i+\text{Root}\left[\#1^3+2\#1+1\right.\right.}$$

$$-\frac{i\text{Root}[\#1^3+2\#1+1\&,2]^5(\text{Root}[\#1^3+2\#1+1\&,1]-\text{Root}[\#1^3+2\#1+1\&,3])\sin(2x)}{4\sqrt{59}\left(1-4\text{Root}\left[\#1^3+2\#1+1\&,2\right]^2\right)}$$

$$-\frac{i\cos(2x)\left(\text{Root}[\#1^3+2\#1+1\&,1]-\text{Root}[\#1^3+2\#1+1\&,2]\right)\text{Root}[\#1^3+2\#1+1\&,3]^4}{2\sqrt{59}\left(-1+4\text{Root}\left[\#1^3+2\#1+1\&,3\right]^2\right)}$$

$$+\frac{i\cos(2x)\left(\text{Root}[\#1^3+2\#1+1\&,2]-\text{Root}[\#1^3+2\#1+1\&,3]\right)}{2\sqrt{59}\left(-2i+\text{Root}\left[\#1^3+2\#1+1\&,2\right]+\text{Root}\left[\#1^3+2\#1+1\&,3\right]\right)\left(2i+\text{Root}\left[\#1^3+2\#1+1\right.\right.}$$

$$+\frac{e^{x\text{Root}[\#1^3+2\#1+1\&,3]}c_3}{\text{Root}\left[\#1^3+2\#1+1\&,3\right]}$$

$$+\frac{ix\left(4+\text{Root}[\#1^3+2\#1+1\&,2]^2\right)\left(\text{Root}[\#1^3+2\#1+1\&,2]^3x^3+4\text{Root}[\#1^3+2\#1+1\&,2]^2\right.}{4\sqrt{59}\left(1-4\text{Root}\left[\#\right.\right.}$$

$$-\frac{i\cos(2x)\text{Root}[\#1^3+2\#1+1\&,2]^4(\text{Root}[\#1^3+2\#1+1\&,1]-\text{Root}[\#1^3+2\#1+1\&,3])}{2\sqrt{59}\left(1-4\text{Root}\left[\#1^3+2\#1+1\&,2\right]^2\right)}$$

$$+\frac{e^{x\text{Root}[\#1^3+2\#1+1\&,2]}c_2}{\text{Root}\left[\#1^3+2\#1+1\&,2\right]}$$

14.29 problem 31

Internal problem ID [2230]
Internal file name [`OUTPUT/2230_Monday_February_26_2024_09_18_34_AM_27483324/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 31.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 4y'' + 5y' = \mathrm{e}^{-2x} \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 4y'' + 5y' = 0$$

The characteristic equation is

$$\lambda^3 + 4\lambda^2 + 5\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -2 + i$$
$$\lambda_3 = -2 - i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{(-2+i)x} c_2 + e^{(-2-i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= 1 \\
y_2 &= e^{(-2+i)x} \\
y_3 &= e^{(-2-i)x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + 4y'' + 5y' = e^{-2x} \cos(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$\begin{aligned}
W &= \begin{bmatrix} 1 & e^{(-2+i)x} & e^{(-2-i)x} \\ 0 & (-2+i)\, e^{(-2+i)x} & (-2-i)\, e^{(-2-i)x} \\ 0 & (3-4i)\, e^{(-2+i)x} & (3+4i)\, e^{(-2-i)x} \end{bmatrix} \\
|W| &= -10i e^{(-2+i)x} e^{(-2-i)x}
\end{aligned}$$

The determinant simplifies to

$$|W| = -10i\mathrm{e}^{-4x}$$

Now we determine W_i for each U_i.

$$\begin{aligned}W_1(x) &= \det\begin{bmatrix}\mathrm{e}^{(-2+i)x} & \mathrm{e}^{(-2-i)x}\\ (-2+i)\,\mathrm{e}^{(-2+i)x} & (-2-i)\,\mathrm{e}^{(-2-i)x}\end{bmatrix}\\ &= -2i\mathrm{e}^{-4x}\end{aligned}$$

$$\begin{aligned}W_2(x) &= \det\begin{bmatrix}1 & \mathrm{e}^{(-2-i)x}\\ 0 & (-2-i)\,\mathrm{e}^{(-2-i)x}\end{bmatrix}\\ &= (-2-i)\,\mathrm{e}^{(-2-i)x}\end{aligned}$$

$$\begin{aligned}W_3(x) &= \det\begin{bmatrix}1 & \mathrm{e}^{(-2+i)x}\\ 0 & (-2+i)\,\mathrm{e}^{(-2+i)x}\end{bmatrix}\\ &= (-2+i)\,\mathrm{e}^{(-2+i)x}\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}U_1 &= (-1)^{3-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx\\ &= (-1)^2\int \frac{\left(\mathrm{e}^{-2x}\cos(x)\right)\left(-2i\mathrm{e}^{-4x}\right)}{(1)\left(-10i\mathrm{e}^{-4x}\right)}\,dx\\ &= \int \frac{-2i\mathrm{e}^{-2x}\cos(x)\,\mathrm{e}^{-4x}}{-10i\mathrm{e}^{-4x}}\,dx\\ &= \int \left(\frac{\mathrm{e}^{-2x}\cos(x)}{5}\right)dx\\ &= -\frac{2\,\mathrm{e}^{-2x}\cos(x)}{25} + \frac{\mathrm{e}^{-2x}\sin(x)}{25}\end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{\left(e^{-2x}\cos(x)\right)\left((-2-i)\,e^{(-2-i)x}\right)}{(1)\left(-10i e^{-4x}\right)}\,dx \\
&= -\int \frac{(-2-i)\,e^{-2x}\cos(x)\,e^{(-2-i)x}}{-10i e^{-4x}}\,dx \\
&= -\int \left(\left(\frac{1}{10}-\frac{i}{5}\right)\cos(x)\,e^{-ix}\right) dx \\
&= -\left(\int \left(\frac{1}{10}-\frac{i}{5}\right)\cos(x)\,e^{-ix}dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{\left(e^{-2x}\cos(x)\right)\left((-2+i)\,e^{(-2+i)x}\right)}{(1)\left(-10i e^{-4x}\right)}\,dx \\
&= \int \frac{(-2+i)\,e^{-2x}\cos(x)\,e^{(-2+i)x}}{-10i e^{-4x}}\,dx \\
&= \int \left(\left(-\frac{1}{10}-\frac{i}{5}\right)\cos(x)\,e^{ix}\right) dx \\
&= -\frac{x}{20}-\frac{ix}{10}-\frac{e^{2ix}}{20}+\frac{i e^{2ix}}{40} \\
&= -\frac{x}{20}-\frac{ix}{10}-\frac{e^{2ix}}{20}+\frac{i e^{2ix}}{40}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{2\,e^{-2x}\cos(x)}{25}+\frac{e^{-2x}\sin(x)}{25}\right) \\
&+ \left(-\left(\int \left(\frac{1}{10}-\frac{i}{5}\right)\cos(x)\,e^{-ix}dx\right)\right)\left(e^{(-2+i)x}\right) \\
&+ \left(-\frac{x}{20}-\frac{ix}{10}-\frac{e^{2ix}}{20}+\frac{i e^{2ix}}{40}\right)\left(e^{(-2-i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{e^{-2x}\left(\left(-\frac{26}{5}+i-4x\right)\cos(x)+2\left(-\frac{7}{10}+i-4x\right)\sin(x)\right)}{40}
$$

Which simplifies to

$$y_p = \frac{e^{-2x}\left(\left(-\frac{26}{5}+i-4x\right)\cos(x)+2\left(-\frac{7}{10}+i-4x\right)\sin(x)\right)}{40}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + e^{(-2+i)x}c_2 + e^{(-2-i)x}c_3\right) \\ &\quad + \left(\frac{e^{-2x}\left(\left(-\frac{26}{5}+i-4x\right)\cos(x)+2\left(-\frac{7}{10}+i-4x\right)\sin(x)\right)}{40}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 + e^{(-2+i)x}c_2 + e^{(-2-i)x}c_3 \\ &\quad + \frac{e^{-2x}\left(\left(-\frac{26}{5}+i-4x\right)\cos(x)+2\left(-\frac{7}{10}+i-4x\right)\sin(x)\right)}{40} \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 + e^{(-2+i)x}c_2 + e^{(-2-i)x}c_3 + \frac{e^{-2x}\left(\left(-\frac{26}{5}+i-4x\right)\cos(x)+2\left(-\frac{7}{10}+i-4x\right)\sin(x)\right)}{40}$$

Verified OK.

14.29.1 Maple step by step solution

Let's solve

$y''' + 4y'' + 5y' = e^{-2x}\cos(x)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

$y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = e^{-2x}\cos(x) - 4y_3(x) - 5y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = e^{-2x}\cos(x) - 4y_3(x) - 5y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ e^{-2x}\cos(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ e^{-2x}\cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -4 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-2 - I, \begin{bmatrix} \frac{3}{25} - \frac{4\,I}{25} \\ -\frac{2}{5} + \frac{I}{5} \\ 1 \end{bmatrix}\right], \left[-2 + I, \begin{bmatrix} \frac{3}{25} + \frac{4\,I}{25} \\ -\frac{2}{5} - \frac{I}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2 - \mathrm{I}, \begin{bmatrix} \frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ -\frac{2}{5} + \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-2-\mathrm{I})x} \cdot \begin{bmatrix} \frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ -\frac{2}{5} + \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-2x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} \frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ -\frac{2}{5} + \frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-2x} \cdot \begin{bmatrix} \left(\frac{3}{25} - \frac{4\,\mathrm{I}}{25}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \left(-\frac{2}{5} + \frac{\mathrm{I}}{5}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{3\cos(x)}{25} - \frac{4\sin(x)}{25} \\ -\frac{2\cos(x)}{5} + \frac{\sin(x)}{5} \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{3\sin(x)}{25} - \frac{4\cos(x)}{25} \\ \frac{2\sin(x)}{5} + \frac{\cos(x)}{5} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & \mathrm{e}^{-2x}\left(\frac{3\cos(x)}{25} - \frac{4\sin(x)}{25}\right) & \mathrm{e}^{-2x}\left(-\frac{3\sin(x)}{25} - \frac{4\cos(x)}{25}\right) \\ 0 & \mathrm{e}^{-2x}\left(-\frac{2\cos(x)}{5} + \frac{\sin(x)}{5}\right) & \mathrm{e}^{-2x}\left(\frac{2\sin(x)}{5} + \frac{\cos(x)}{5}\right) \\ 0 & \mathrm{e}^{-2x}\cos(x) & -\mathrm{e}^{-2x}\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & \mathrm{e}^{-2x}\left(\frac{3\cos(x)}{25} - \frac{4\sin(x)}{25}\right) & \mathrm{e}^{-2x}\left(-\frac{3\sin(x)}{25} - \frac{4\cos(x)}{25}\right) \\ 0 & \mathrm{e}^{-2x}\left(-\frac{2\cos(x)}{5} + \frac{\sin(x)}{5}\right) & \mathrm{e}^{-2x}\left(\frac{2\sin(x)}{5} + \frac{\cos(x)}{5}\right) \\ 0 & \mathrm{e}^{-2x}\cos(x) & -\mathrm{e}^{-2x}\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{3}{25} & -\frac{4}{25} \\ 0 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & \frac{4}{5} + \frac{\mathrm{e}^{-2x}(-3\sin(x)-4\cos(x))}{5} & \frac{1}{5} + \frac{(-\cos(x)-2\sin(x))\mathrm{e}^{-2x}}{5} \\ 0 & \mathrm{e}^{-2x}(\cos(x) + 2\sin(x)) & \mathrm{e}^{-2x}\sin(x) \\ 0 & -5\,\mathrm{e}^{-2x}\sin(x) & \mathrm{e}^{-2x}(-2\sin(x) + \cos(x)) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot \vec{v}(x)+\Phi(x)\cdot \vec{v}'(x)=A\cdot \Phi(x)\cdot \vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot \Phi(x)\cdot \vec{v}(x)+\Phi(x)\cdot \vec{v}'(x)=A\cdot \Phi(x)\cdot \vec{v}(x)+\vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot \vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot \left(\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c}\frac{2}{25}+\frac{((-5x-4)\cos(x)+(-10x-3)\sin(x))\mathrm{e}^{-2x}}{50}\\ \frac{\mathrm{e}^{-2x}\sin(x)x}{2}\\ \frac{\mathrm{e}^{-2x}(\sin(x)+x\cos(x)-2\sin(x)x)}{2}\end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+\left[\begin{array}{c}\frac{2}{25}+\frac{((-5x-4)\cos(x)+(-10x-3)\sin(x))\mathrm{e}^{-2x}}{50}\\ \frac{\mathrm{e}^{-2x}\sin(x)x}{2}\\ \frac{\mathrm{e}^{-2x}(\sin(x)+x\cos(x)-2\sin(x)x)}{2}\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=\frac{\left((-5x+6c_2-8c_3-4)\cos(x)-10\left(x+\frac{4c_2}{5}+\frac{3c_3}{5}+\frac{3}{10}\right)\sin(x)\right)\mathrm{e}^{-2x}}{50}+c_1+\frac{2}{25}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = exp(-2*_a)*cos(_a)-5*_b(_a)-4
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$3)+4*diff(y(x),x$2)+5*diff(y(x),x)=exp(-2*x)*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(-5x-20c_1-10c_2-4\right)\cos\left(x\right)-10\sin\left(x\right)\left(x-c_1+2c_2+\frac{3}{10}\right)\right)\mathrm{e}^{-2x}}{50}+c_3$$

✓ Solution by Mathematica

Time used: 0.337 (sec). Leaf size: 52

```
DSolve[y'''[x]+4*y''[x]+5*y'[x]==Exp[-2*x]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\rightarrow \frac{1}{50}e^{-2x}(2(-5x+1-10c_1+5c_2)\sin(x)-(5x+14+10c_1+20c_2)\cos(x))+c_3$$

14.30 problem 32

Internal problem ID [2231]
Internal file name [`OUTPUT/2231_Monday_February_26_2024_09_18_34_AM_74806069/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 32.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + y'' - 2y' = e^{-2x} \cos(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' - 2y' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{-2x} + c_3 e^x$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \mathrm{e}^{-2x} \\
y_3 &= \mathrm{e}^{x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' + y'' - 2y' = \mathrm{e}^{-2x} \cos(2x)
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{-2x} \cos(2x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-2x} \cos(2x), \mathrm{e}^{-2x} \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, \mathrm{e}^{x}, \mathrm{e}^{-2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{-2x} \cos(2x) + A_2 \mathrm{e}^{-2x} \sin(2x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-4A_1 \mathrm{e}^{-2x} \sin(2x) + 20A_1 \mathrm{e}^{-2x} \cos(2x) + 4A_2 \mathrm{e}^{-2x} \cos(2x) + 20A_2 \mathrm{e}^{-2x} \sin(2x) \\
&= \mathrm{e}^{-2x} \cos(2x)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = \frac{5}{104}, A_2 = \frac{1}{104}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5\,\mathrm{e}^{-2x}\cos(2x)}{104} + \frac{\mathrm{e}^{-2x}\sin(2x)}{104}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2\mathrm{e}^{-2x} + c_3\mathrm{e}^{x}\right) + \left(\frac{5\,\mathrm{e}^{-2x}\cos(2x)}{104} + \frac{\mathrm{e}^{-2x}\sin(2x)}{104}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2\mathrm{e}^{-2x} + c_3\mathrm{e}^{x} + \frac{5\,\mathrm{e}^{-2x}\cos(2x)}{104} + \frac{\mathrm{e}^{-2x}\sin(2x)}{104} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2\mathrm{e}^{-2x} + c_3\mathrm{e}^{x} + \frac{5\,\mathrm{e}^{-2x}\cos(2x)}{104} + \frac{\mathrm{e}^{-2x}\sin(2x)}{104}$$

Verified OK.

14.30.1 Maple step by step solution

Let's solve

$y''' + y'' - 2y' = \mathrm{e}^{-2x}\cos(2x)$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = \mathrm{e}^{-2x} \cos(2x) - y_3(x) + 2y_2(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \mathrm{e}^{-2x} \cos(2x) - y_3(x) + 2y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-2x} \cos(2x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-2x} \cos(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$$

□ Fundamental matrix

  ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 1 & e^{x} \\ -\frac{e^{-2x}}{2} & 0 & e^{x} \\ e^{-2x} & 0 & e^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 1 & e^{x} \\ -\frac{e^{-2x}}{2} & 0 & e^{x} \\ e^{-2x} & 0 & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & 1 \\ -\frac{1}{2} & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} 1 & \frac{(4\,e^{3x}-3\,e^{2x}-1)e^{-2x}}{6} & \frac{(2\,e^{3x}-3\,e^{2x}+1)e^{-2x}}{6} \\ 0 & \frac{(2\,e^{3x}+1)e^{-2x}}{3} & \frac{(e^{3x}-1)e^{-2x}}{3} \\ 0 & \frac{2(e^{3x}-1)e^{-2x}}{3} & \frac{(e^{3x}+2)e^{-2x}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{e^{-2x}\left(8\,e^{3x}-13\,e^{2x}+5\cos(2x)+\sin(2x)\right)}{104} \\ -\frac{\left(-2\,e^{3x}+2\cos(2x)+3\sin(2x)\right)e^{-2x}}{26} \\ -\frac{\left(-e^{3x}+\cos(2x)-5\sin(2x)\right)e^{-2x}}{13} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \left[\begin{array}{c} \frac{e^{-2x}\left(8\,e^{3x}-13\,e^{2x}+5\cos(2x)+\sin(2x)\right)}{104} \\ -\frac{\left(-2\,e^{3x}+2\cos(2x)+3\sin(2x)\right)e^{-2x}}{26} \\ -\frac{\left(-e^{3x}+\cos(2x)-5\sin(2x)\right)e^{-2x}}{13} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(104c_3e^{3x}+8\,e^{3x}+104c_2e^{2x}-13\,e^{2x}+5\cos(2x)+\sin(2x)+26c_1\right)e^{-2x}}{104}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = exp(-2*_a)*cos(2*_a)+2*_b(_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-2*diff(y(x),x)=exp(-2*x)*cos(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{e^{-2x}\left(-2c_2 e^{3x} - 2c_3 e^{2x} + c_1 - \frac{5\cos(2x)}{52} - \frac{\sin(2x)}{52}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.872 (sec). Leaf size: 42

```
DSolve[y'''[x]+y''[x]-2*y'[x]==Exp[-2*x]*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{104} e^{-2x}\left(\sin(2x) + 5\cos(2x) - 52\left(c_1 - 2c_2 e^{3x}\right)\right) + c_3$$

14.31 problem 33

Internal problem ID [2232]
Internal file name [`OUTPUT/2232_Monday_February_26_2024_09_18_35_AM_92822207/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 33.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 2y' = \sin(x)\, x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y' = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i\sqrt{2}$$
$$\lambda_3 = -i\sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{-i\sqrt{2}\,x}c_2 + e^{i\sqrt{2}\,x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{-i\sqrt{2}\,x}$$
$$y_3 = e^{i\sqrt{2}\,x}$$

Now the particular solution to the given ODE is found

$$y''' + 2y' = \sin(x)\,x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\,x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(x), x^2\cos(x), \sin(x)\,x, \sin(x)\,x^2, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, e^{i\sqrt{2}\,x}, e^{-i\sqrt{2}\,x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1x\cos(x) + A_2x^2\cos(x) + A_3\sin(x)\,x + A_4\sin(x)\,x^2 + A_5\cos(x) + A_6\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-A_1\cos(x) - A_1x\sin(x) - 6A_2\sin(x) - 2A_2x\cos(x)\\ &- A_2x^2\sin(x) + A_3\cos(x)\,x - A_3\sin(x) + A_4\cos(x)\,x^2\\ &- 2A_4\sin(x)\,x + 6A_4\cos(x) - A_5\sin(x) + A_6\cos(x) = \sin(x)\,x^2\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1, A_3 = -2, A_4 = 0, A_5 = 8, A_6 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 \cos(x) - 2\sin(x)\, x + 8\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \mathrm{e}^{-i\sqrt{2}\,x} c_2 + \mathrm{e}^{i\sqrt{2}\,x} c_3\right) + \left(-x^2 \cos(x) - 2\sin(x)\, x + 8\cos(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \mathrm{e}^{-i\sqrt{2}\,x} c_2 + \mathrm{e}^{i\sqrt{2}\,x} c_3 - x^2 \cos(x) - 2\sin(x)\, x + 8\cos(x) \tag{1}$$

Verification of solutions

$$y = c_1 + \mathrm{e}^{-i\sqrt{2}\,x} c_2 + \mathrm{e}^{i\sqrt{2}\,x} c_3 - x^2 \cos(x) - 2\sin(x)\, x + 8\cos(x)$$

Verified OK.

14.31.1 Maple step by step solution

Let's solve

$y''' + 2y' = \sin(x)\, x^2$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = \sin(x)\, x^2 - 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \sin(x)\, x^2 - 2y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \sin(x)\, x^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \sin(x)\, x^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{2}, \left[\begin{array}{c} -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}\sqrt{2}\,x} \cdot \left[\begin{array}{c} -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(\sqrt{2}\,x\right) - \mathrm{I}\sin\left(\sqrt{2}\,x\right)\right) \cdot \left[\begin{array}{c} -\frac{1}{2} \\ \frac{\mathrm{I}}{2}\sqrt{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{\cos\left(\sqrt{2}\,x\right)}{2} + \frac{\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{\mathrm{I}}{2}\left(\cos\left(\sqrt{2}\,x\right) - \mathrm{I}\sin\left(\sqrt{2}\,x\right)\right)\sqrt{2} \\ \cos\left(\sqrt{2}\,x\right) - \mathrm{I}\sin\left(\sqrt{2}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\frac{\cos\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} \\ \cos\left(\sqrt{2}\,x\right) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ -\sin\left(\sqrt{2}\,x\right) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & -\frac{\cos\left(\sqrt{2}\,x\right)}{2} & \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ 0 & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} & \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ 0 & \cos\left(\sqrt{2}\,x\right) & -\sin\left(\sqrt{2}\,x\right) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{\cos\left(\sqrt{2}\,x\right)}{2} & \frac{\sin\left(\sqrt{2}\,x\right)}{2} \\ 0 & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} & \frac{\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} \\ 0 & \cos\left(\sqrt{2}\,x\right) & -\sin\left(\sqrt{2}\,x\right) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} & \frac{1}{2} - \frac{\cos\left(\sqrt{2}\,x\right)}{2} \\ 0 & \cos\left(\sqrt{2}\,x\right) & \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} \\ 0 & -\sqrt{2}\,\sin\left(\sqrt{2}\,x\right) & \cos\left(\sqrt{2}\,x\right) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}_p'(x) = \Phi'(x)\cdot\vec{v}(x) + \Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x) + \Phi(x)\cdot\vec{v}'(x) = A\cdot\Phi(x)\cdot\vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\vec{v}(x) + \Phi(x)\cdot\vec{v}'(x) = A\cdot\Phi(x)\cdot\vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)}\cdot\vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -1 - x^2\cos(x) - 2\sin(x)x + 8\cos(x) - 7\cos(\sqrt{2}\,x) \\ \frac{\left(14\sin\left(\sqrt{2}\,x\right)+\sqrt{2}\,(-4x\cos(x)+\sin(x)x^2-10\sin(x))\right)\sqrt{2}}{2\left(1+\sqrt{2}\right)^3\left(\sqrt{2}-1\right)^3} \\ x^2\cos(x) + 6\sin(x)x - 14\cos(x) + 14\cos(\sqrt{2}\,x) \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} -1 - x^2\cos(x) - 2\sin(x)x + 8\cos(x) - 7\cos(\sqrt{2}\,x) \\ \frac{\left(14\sin\left(\sqrt{2}\,x\right)+\sqrt{2}\,(-4x\cos(x)+\sin(x)x^2-10\sin(x))\right)\sqrt{2}}{2\left(1+\sqrt{2}\right)^3\left(\sqrt{2}-1\right)^3} \\ x^2\cos(x) + 6\sin(x)x - 14\cos(x) + 14\cos(\sqrt{2}\,x) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -1 - x^2\cos(x) - 2\sin(x)x + 8\cos(x) - 7\cos(\sqrt{2}\,x) + \frac{c_3\sin\left(\sqrt{2}\,x\right)}{2} - \frac{c_2\cos\left(\sqrt{2}\,x\right)}{2} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = sin(_a)*_a^2-2*_b(_a), _b(_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x)=x^2*sin(x),y(x), singsol=all)
```

$$y(x) = -\frac{c_2\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)}{2} + \frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)c_1}{2} - \cos\left(x\right)x^2 + 8\cos\left(x\right) - 2x\sin\left(x\right) + c_3$$

✓ Solution by Mathematica

Time used: 0.164 (sec). Leaf size: 55

```
DSolve[y'''[x]+2*y'[x]==x^2*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\left(x^2-8\right)\cos(x) - 2x\sin(x) - \frac{c_2\cos\left(\sqrt{2}x\right)}{\sqrt{2}} + \frac{c_1\sin\left(\sqrt{2}x\right)}{\sqrt{2}} + c_3$$

14.32 problem 34

Internal problem ID [2233]
Internal file name [`OUTPUT/2233_Monday_February_26_2024_09_18_35_AM_76711494/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 23, page 106
Problem number: 34.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - y = x^2 \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y = 0$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= \mathrm{e}^{x} \\ y_3 &= \mathrm{e}^{-ix} \\ y_4 &= \mathrm{e}^{ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - y = x^2 \cos(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} & \mathrm{e}^{-ix} & \mathrm{e}^{ix} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & -i\mathrm{e}^{-ix} & i\mathrm{e}^{ix} \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & -\mathrm{e}^{-ix} & -\mathrm{e}^{ix} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & i\mathrm{e}^{-ix} & -i\mathrm{e}^{ix} \end{bmatrix}$$

$$|W| = 16i \mathrm{e}^{-x} \mathrm{e}^{x} \mathrm{e}^{-ix} \mathrm{e}^{ix}$$

The determinant simplifies to

$$|W| = 16i$$

Now we determine W_i for each U_i.

$$W_1(x) = \det\begin{bmatrix} e^x & e^{-ix} & e^{ix} \\ e^x & -ie^{-ix} & ie^{ix} \\ e^x & -e^{-ix} & -e^{ix} \end{bmatrix}$$
$$= 4ie^x$$

$$W_2(x) = \det\begin{bmatrix} e^{-x} & e^{-ix} & e^{ix} \\ -e^{-x} & -ie^{-ix} & ie^{ix} \\ e^{-x} & -e^{-ix} & -e^{ix} \end{bmatrix}$$
$$= 4ie^{-x}$$

$$W_3(x) = \det\begin{bmatrix} e^{-x} & e^x & e^{ix} \\ -e^{-x} & e^x & ie^{ix} \\ e^{-x} & e^x & -e^{ix} \end{bmatrix}$$
$$= -4\,e^{ix}$$

$$W_4(x) = \det\begin{bmatrix} e^{-x} & e^x & e^{-ix} \\ -e^{-x} & e^x & -ie^{-ix} \\ e^{-x} & e^x & -e^{-ix} \end{bmatrix}$$
$$= -4\,e^{-ix}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3\int \frac{(x^2\cos(x))\,(4ie^x)}{(1)\,(16i)}\,dx \\ &= -\int \frac{4ix^2\cos(x)\,e^x}{16i}\,dx \\ &= -\int \left(\frac{\cos(x)\,e^x x^2}{4}\right)dx \\ &= -\frac{\left(\frac{x^2}{2}-\frac{1}{2}\right)e^x\cos(x)}{4} + \frac{\left(x-\frac{1}{2}x^2-\frac{1}{2}\right)e^x\sin(x)}{4} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(x^2 \cos(x)\right)\left(4i\mathrm{e}^{-x}\right)}{(1)\,(16i)}\,dx \\
&= \int \frac{4ix^2 \cos(x)\,\mathrm{e}^{-x}}{16i}\,dx \\
&= \int \left(\frac{x^2 \cos(x)\,\mathrm{e}^{-x}}{4}\right) dx \\
&= \frac{\left(-\frac{x^2}{2} + \frac{1}{2}\right)\mathrm{e}^{-x}\cos(x)}{4} - \frac{\left(-\frac{1}{2}x^2 - x - \frac{1}{2}\right)\mathrm{e}^{-x}\sin(x)}{4}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{\left(x^2 \cos(x)\right)\left(-4\,\mathrm{e}^{ix}\right)}{(1)\,(16i)}\,dx \\
&= -\int \frac{-4x^2 \cos(x)\,\mathrm{e}^{ix}}{16i}\,dx \\
&= -\int \left(\frac{ix^2 \cos(x)\,\mathrm{e}^{ix}}{4}\right) dx \\
&= -\frac{ix^3}{24} - \frac{\left(2x^2 + 2ix - 1\right)\mathrm{e}^{2ix}}{32}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{\left(x^2 \cos(x)\right)\left(-4\,\mathrm{e}^{-ix}\right)}{(1)\,(16i)}\,dx \\
&= \int \frac{-4x^2 \cos(x)\,\mathrm{e}^{-ix}}{16i}\,dx \\
&= \int \left(\frac{ix^2 \cos(x)\,\mathrm{e}^{-ix}}{4}\right) dx \\
&= \int \frac{ix^2 \cos(x)\,\mathrm{e}^{-ix}}{4} dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$y_p = \left(-\frac{\left(\frac{x^2}{2}-\frac{1}{2}\right)\mathrm{e}^{x}\cos\left(x\right)}{4}+\frac{\left(x-\frac{1}{2}x^2-\frac{1}{2}\right)\mathrm{e}^{x}\sin\left(x\right)}{4}\right)\left(\mathrm{e}^{-x}\right)$$
$$+\left(\frac{\left(-\frac{x^2}{2}+\frac{1}{2}\right)\mathrm{e}^{-x}\cos\left(x\right)}{4}-\frac{\left(-\frac{1}{2}x^2-x-\frac{1}{2}\right)\mathrm{e}^{-x}\sin\left(x\right)}{4}\right)\left(\mathrm{e}^{x}\right)$$
$$+\left(-\frac{ix^3}{24}-\frac{\left(2x^2+2ix-1\right)\mathrm{e}^{2ix}}{32}\right)\left(\mathrm{e}^{-ix}\right)$$
$$+\left(\int\frac{ix^2\cos\left(x\right)\mathrm{e}^{-ix}}{4}dx\right)\left(\mathrm{e}^{ix}\right)$$

Therefore the particular solution is

$$y_p = \frac{\sin\left(x\right)\left(-8x^3+60x+3i\right)}{96}+\frac{\left(-36x^2+33\right)\cos\left(x\right)}{96}$$

Which simplifies to

$$y_p = \frac{\sin\left(x\right)\left(-8x^3+60x+3i\right)}{96}+\frac{\left(-36x^2+33\right)\cos\left(x\right)}{96}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{x}+\mathrm{e}^{-ix}c_3+\mathrm{e}^{ix}c_4\right)+\left(\frac{\sin\left(x\right)\left(-8x^3+60x+3i\right)}{96}+\frac{\left(-36x^2+33\right)\cos\left(x\right)}{96}\right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{x}+\mathrm{e}^{-ix}c_3+\mathrm{e}^{ix}c_4+\frac{\sin\left(x\right)\left(-8x^3+60x+3i\right)}{96}+\frac{\left(-36x^2+33\right)\cos\left(x\right)}{96} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{x}+\mathrm{e}^{-ix}c_3+\mathrm{e}^{ix}c_4+\frac{\sin\left(x\right)\left(-8x^3+60x+3i\right)}{96}+\frac{\left(-36x^2+33\right)\cos\left(x\right)}{96}$$

Verified OK.

14.32.1 Maple step by step solution

Let's solve

$y'''' - y = x^2 \cos(x)$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = x^2 \cos(x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = x^2 \cos(x) + y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^2 \cos(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^2 \cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \overrightarrow{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3(x) + c_4\overrightarrow{y}_4(x) + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\mathrm{e}^{-x} & \mathrm{e}^{x} & -\sin(x) & -\cos(x) \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & -\cos(x) & \sin(x) \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & \sin(x) & \cos(x) \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\mathrm{e}^{-x} & \mathrm{e}^{x} & -\sin(x) & -\cos(x) \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & -\cos(x) & \sin(x) \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & \sin(x) & \cos(x) \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} \\ \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

    $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
  - Take the derivative of the particular solution

    $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
  - Cancel like terms

    $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
  - Integrate to solve for $\vec{v}(x)$

    $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
  - Plug $\vec{v}(x)$ into the equation for the particular solution

    $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -\frac{e^{-x}}{8} + \frac{(-3x^2+2)\cos(x)}{8} + \frac{\sin(x)(-2x^3+15x)}{24} - \frac{e^{x}}{8} \\ \frac{e^{-x}}{8} + \frac{(-2x^3-3x)\cos(x)}{24} + \frac{(x^2+3)\sin(x)}{8} - \frac{e^{x}}{8} \\ -\frac{e^{-x}}{8} + \frac{(-x^2+2)\cos(x)}{8} + \frac{(2x^3+9x)\sin(x)}{24} - \frac{e^{x}}{8} \\ \frac{e^{-x}}{8} + \frac{(2x^3+3x)\cos(x)}{24} + \frac{(3x^2+1)\sin(x)}{8} - \frac{e^{x}}{8} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} -\frac{e^{-x}}{8} + \frac{(-3x^2+2)\cos(x)}{8} + \frac{\sin(x)(-2x^3+15x)}{24} - \frac{e^x}{8} \\ \frac{e^{-x}}{8} + \frac{(-2x^3-3x)\cos(x)}{24} + \frac{(x^2+3)\sin(x)}{8} - \frac{e^x}{8} \\ -\frac{e^{-x}}{8} + \frac{(-x^2+2)\cos(x)}{8} + \frac{(2x^3+9x)\sin(x)}{24} - \frac{e^x}{8} \\ \frac{e^{-x}}{8} + \frac{(2x^3+3x)\cos(x)}{24} + \frac{(3x^2+1)\sin(x)}{8} - \frac{e^x}{8} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-8c_1-1)e^{-x}}{8} + \frac{(-3x^2-8c_4+2)\cos(x)}{8} + \frac{(-2x^3-24c_3+15x)\sin(x)}{24} + \frac{e^x(8c_2-1)}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
dsolve(diff(y(x),x$4)-y(x)=x^2*cos(x),y(x), singsol=all)
```

$$y(x) = c_4e^{-x} + \frac{(-3x^2 + 8c_1 + 2)\cos(x)}{8} + \frac{(-2x^3 + 24c_3 + 15x)\sin(x)}{24} + c_2e^x$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 54

```
DSolve[y''''[x]-y[x]==x^2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-\frac{x^3}{12} + \frac{5x}{8} + c_4\right)\sin(x) + \left(-\frac{3x^2}{8} + \frac{5}{16} + c_2\right)\cos(x) + c_1e^x + c_3e^{-x}$$

15 Exercise 24, page 109

15.1 problem 1

Internal problem ID [2234]
Internal file name [`OUTPUT/2234_Monday_February_26_2024_09_18_36_AM_49956565/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y = \sin(x) x}$$

15.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sin(x) x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\, x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(x), \sin(x)\, x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x \cos(x) + A_2 \sin(x)\, x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-2A_1 \sin(x) + 3A_1 x \cos(x) + 3A_2 \sin(x)\, x + 2A_2 \cos(x) + 3A_3 \cos(x) + 3A_4 \sin(x) \\
&= \sin(x)\, x
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{3}, A_3 = -\frac{2}{9}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)\, x}{3} - \frac{2\cos(x)}{9}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{\sin(x)\, x}{3} - \frac{2\cos(x)}{9}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(x)\, x}{3} - \frac{2\cos(x)}{9} \tag{1}$$

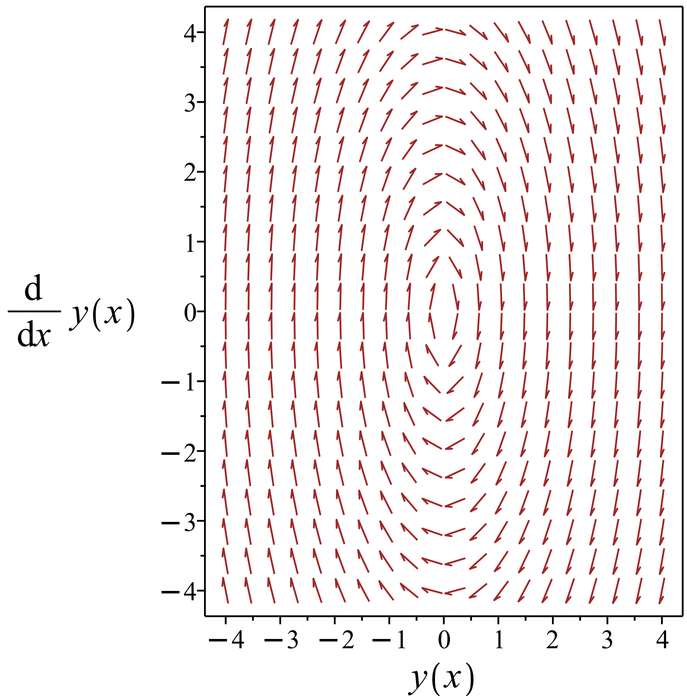


Figure 662: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(x)\, x}{3} - \frac{2\cos(x)}{9}$$

Verified OK.

15.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 516: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\
&= \cos(2x) \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\,x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(x),\sin(x)\,x,\cos(x),\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2},\cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1x\cos(x)+A_2\sin(x)\,x+A_3\cos(x)+A_4\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sin(x)+3A_1x\cos(x)+3A_2\sin(x)\,x+2A_2\cos(x)+3A_3\cos(x)+3A_4\sin(x) = \sin(x)\,x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=0, A_2=\frac{1}{3}, A_3=-\frac{2}{9}, A_4=0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)\,x}{3}-\frac{2\cos(x)}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x)+\frac{c_2\sin(2x)}{2}\right)+\left(\frac{\sin(x)\,x}{3}-\frac{2\cos(x)}{9}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\sin(x)\, x}{3} - \frac{2\cos(x)}{9} \tag{1}$$

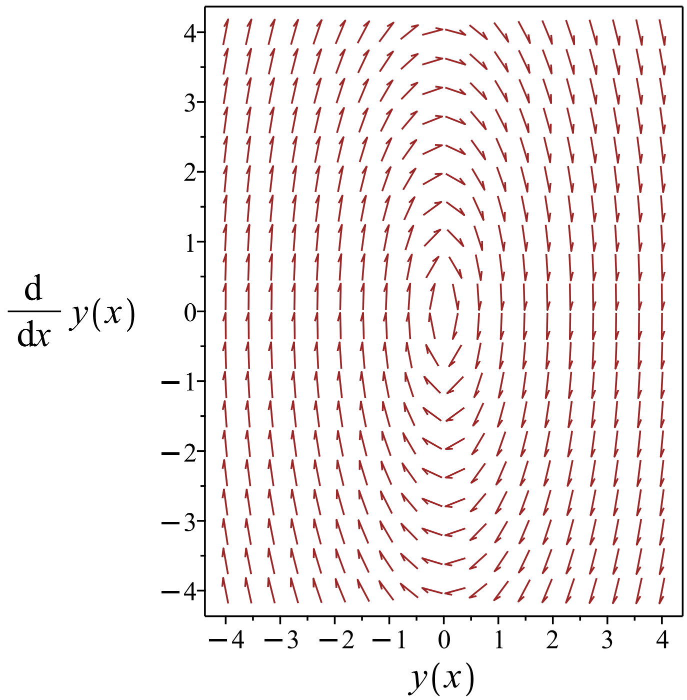


Figure 663: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\sin(x)\, x}{3} - \frac{2\cos(x)}{9}$$

Verified OK.

15.1.3 Maple step by step solution

Let's solve

$y'' + 4y = \sin(x)\, x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\, x\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)\sin(x) x dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x)\sin(x) x dx\right)}{2}$

- ○ Compute integrals

  $y_p(x) = \frac{\sin(x) x}{3} - \frac{2\cos(x)}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(x) x}{3} - \frac{2\cos(x)}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+4*y(x)=x*sin(x),y(x), singsol=all)
```

$$y(x) = \sin(2x) c_2 + c_1 \cos(2x) - \frac{2\cos(x)}{9} + \frac{x\sin(x)}{3}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 33

```
DSolve[y''[x]+4*y[x]==x*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}x\sin(x) + c_1\cos(2x) + \cos(x)\left(-\frac{2}{9} + 2c_2\sin(x)\right)$$

15.2 problem 2

Internal problem ID [2235]
Internal file name [`OUTPUT/2235_Monday_February_26_2024_09_18_36_AM_34766132/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = x^2 \cos(x)}$$

15.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x^2 \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(x), x^2 \cos(x), \sin(x) x, \sin(x) x^2, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(x), x^2 \cos(x), x^3 \cos(x), x^3 \sin(x), \sin(x) x, \sin(x) x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(x) + A_2 x^2 \cos(x) + A_3 x^3 \cos(x) + A_4 x^3 \sin(x) + A_5 \sin(x) x + A_6 \sin(x) x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-2A_1 \sin(x) + 2A_2 \cos(x) - 4A_2 x \sin(x) + 6A_3 x \cos(x) - 6A_3 x^2 \sin(x) \\ &+ 6A_4 x \sin(x) + 6A_4 x^2 \cos(x) + 2A_5 \cos(x) + 4A_6 \cos(x) x + 2A_6 \sin(x) = x^2 \cos(x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}, A_3 = 0, A_4 = \frac{1}{6}, A_5 = -\frac{1}{4}, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 \cos(x)}{4} + \frac{x^3 \sin(x)}{6} - \frac{\sin(x) x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{x^2 \cos(x)}{4} + \frac{x^3 \sin(x)}{6} - \frac{\sin(x) x}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{x^2 \cos(x)}{4} + \frac{x^3 \sin(x)}{6} - \frac{\sin(x) x}{4} \tag{1}$$

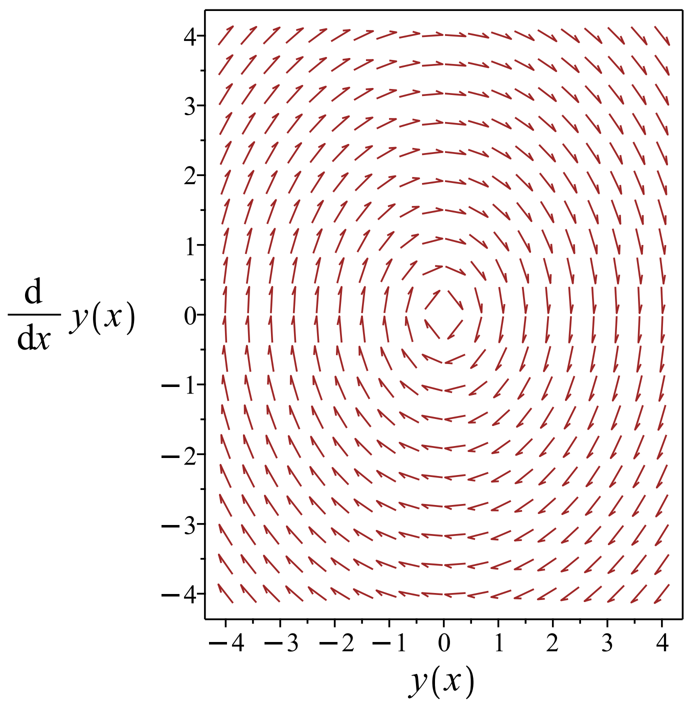


Figure 664: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{x^2 \cos(x)}{4} + \frac{x^3 \sin(x)}{6} - \frac{\sin(x) x}{4}$$

Verified OK.

15.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 518: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \cos (x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos (x), x^2 \cos (x), \sin (x) x, \sin (x) x^2, \cos (x), \sin (x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos (x), \sin (x)\}$$

Since $\cos (x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos (x), x^2 \cos (x), x^3 \cos (x), x^3 \sin (x), \sin (x) x, \sin (x) x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos (x) + A_2 x^2 \cos (x) + A_3 x^3 \cos (x) + A_4 x^3 \sin (x) + A_5 \sin (x) x + A_6 \sin (x) x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &-2A_1 \sin (x) + 2A_2 \cos (x) - 4A_2 x \sin (x) + 6A_3 x \cos (x) - 6A_3 x^2 \sin (x) \\ &+ 6A_4 x \sin (x) + 6A_4 x^2 \cos (x) + 2A_5 \cos (x) + 4A_6 \cos (x) x + 2A_6 \sin (x) = x^2 \cos (x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}, A_3 = 0, A_4 = \frac{1}{6}, A_5 = -\frac{1}{4}, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 \cos (x)}{4} + \frac{x^3 \sin (x)}{6} - \frac{\sin (x) x}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{x^2 \cos(x)}{4} + \frac{x^3 \sin(x)}{6} - \frac{\sin(x)\, x}{4}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{x^2 \cos(x)}{4} + \frac{x^3 \sin(x)}{6} - \frac{\sin(x)\, x}{4} \tag{1}
$$

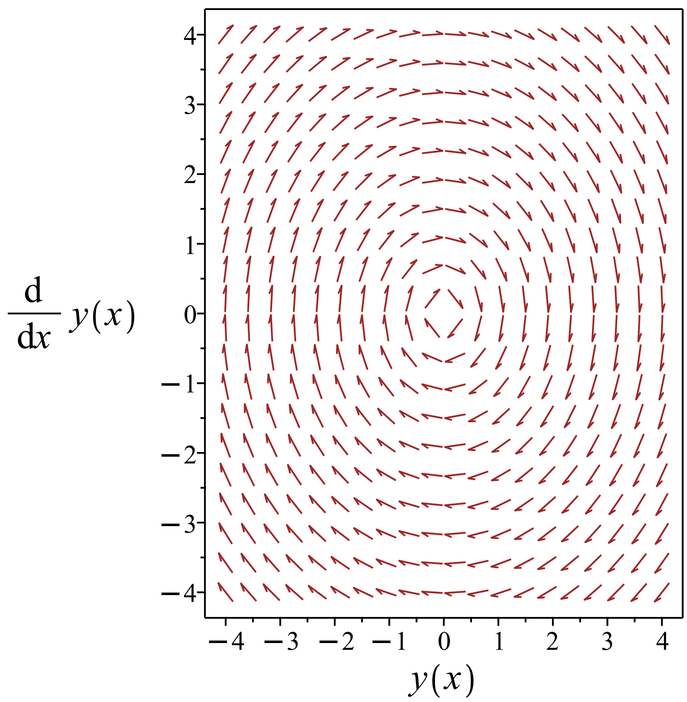


Figure 665: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{x^2 \cos(x)}{4} + \frac{x^3 \sin(x)}{6} - \frac{\sin(x)\, x}{4}
$$

Verified OK.

15.2.3 Maple step by step solution

Let's solve

$y'' + y = x^2 \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 \cos(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(x) \left(\int \sin(2x) x^2 dx\right)}{2} + \sin(x) \left(\int x^2 \cos(x)^2 dx\right)$$

- Compute integrals

$$y_p(x) = \frac{\cos(x) (2x^2 - 1)}{8} + \frac{(2x^3 - 3x) \sin(x)}{12}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{\cos(x) (2x^2 - 1)}{8} + \frac{(2x^3 - 3x) \sin(x)}{12}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+y(x)=x^2*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{(2x^3 + 12c_2 - 3x) \sin(x)}{12} + \frac{\cos(x) (x^2 + 4c_1)}{4}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 41

```
DSolve[y''[x]+y[x]==x^2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{12}(2x^3 - 3x + 12c_2) \sin(x) + \left(\frac{x^2}{4} - \frac{1}{8} + c_1\right) \cos(x)$$

15.3 problem 3

Internal problem ID [2236]
Internal file name [`OUTPUT/2236_Monday_February_26_2024_09_18_36_AM_9070686/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - y = x^2 \cos(x)}$$

15.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = x^2 \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(x), x^2\cos(x), \sin(x)x, \sin(x)x^2, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\cos(x) + A_2 x^2\cos(x) + A_3\sin(x)x + A_4\sin(x)x^2 + A_5\cos(x) + A_6\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-2A_1\sin(x) - 2A_1x\cos(x) + 2A_2\cos(x) - 4A_2x\sin(x)\\&- 2A_2x^2\cos(x) - 2A_3\sin(x)x + 2A_3\cos(x) - 2A_4\sin(x)x^2\\&+ 4A_4\cos(x)x + 2A_4\sin(x) - 2A_5\cos(x) - 2A_6\sin(x) = x^2\cos(x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{2}, A_3 = 1, A_4 = 0, A_5 = \frac{1}{2}, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x^2\cos(x)}{2} + \sin(x)x + \frac{\cos(x)}{2}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(c_1e^x + c_2e^{-x}\right) + \left(-\frac{x^2\cos(x)}{2} + \sin(x)x + \frac{\cos(x)}{2}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} - \frac{x^2 \cos(x)}{2} + \sin(x) x + \frac{\cos(x)}{2} \tag{1}$$

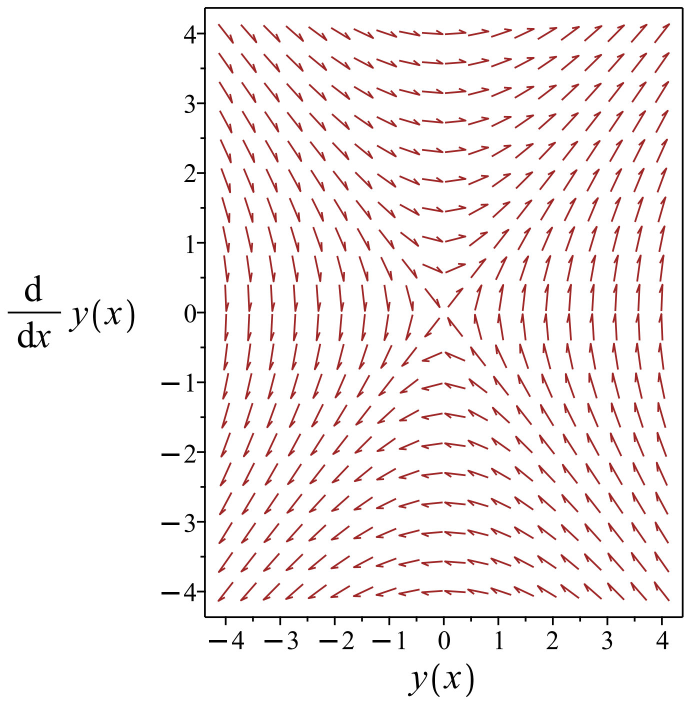


Figure 666: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - \frac{x^2 \cos(x)}{2} + \sin(x) x + \frac{\cos(x)}{2}$$

Verified OK.

15.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1\\ B &= 0\\ C &= -1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= 1\\ t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 520: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\
&= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{x}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(x), x^2 \cos(x), \sin(x) x, \sin(x) x^2, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^x}{2}, e^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x \cos(x) + A_2 x^2 \cos(x) + A_3 \sin(x) x + A_4 \sin(x) x^2 + A_5 \cos(x) + A_6 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-2A_1 \sin(x) - 2A_1 x \cos(x) + 2A_2 \cos(x) - 4A_2 x \sin(x) \\
&- 2A_2 x^2 \cos(x) - 2A_3 \sin(x) x + 2A_3 \cos(x) - 2A_4 \sin(x) x^2 \\
&+ 4A_4 \cos(x) x + 2A_4 \sin(x) - 2A_5 \cos(x) - 2A_6 \sin(x) = x^2 \cos(x)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{2}, A_3 = 1, A_4 = 0, A_5 = \frac{1}{2}, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x^2 \cos(x)}{2} + \sin(x) x + \frac{\cos(x)}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-x} + \frac{c_2 e^x}{2}\right) + \left(-\frac{x^2 \cos(x)}{2} + \sin(x) x + \frac{\cos(x)}{2}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} - \frac{x^2 \cos(x)}{2} + \sin(x) x + \frac{\cos(x)}{2} \tag{1}$$

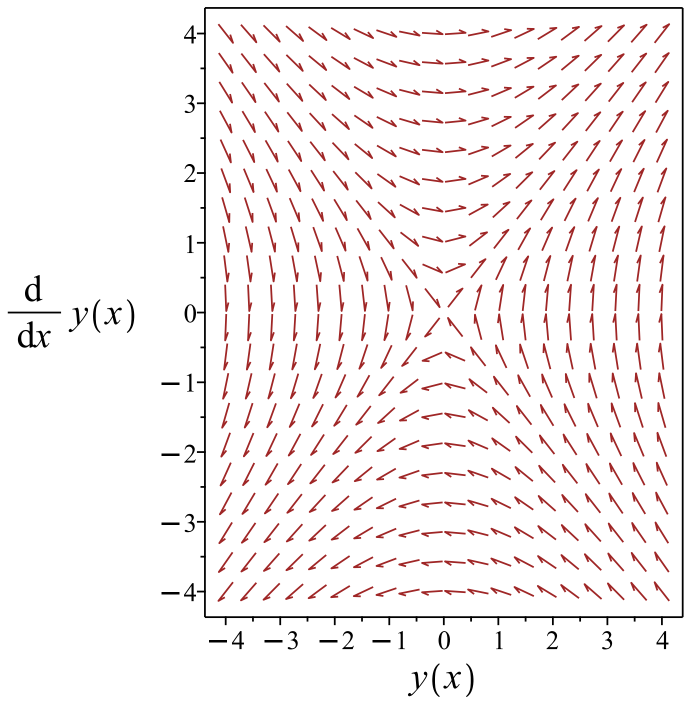


Figure 667: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} - \frac{x^2 \cos(x)}{2} + \sin(x) x + \frac{\cos(x)}{2}$$

Verified OK.

15.3.3 Maple step by step solution

Let's solve

$y'' - y = x^2 \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^2\cos(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{e^{-x}\left(\int \cos(x)e^{x}x^2 dx\right)}{2} + \frac{e^{x}\left(\int x^2\cos(x)e^{-x}dx\right)}{2}$

  - Compute integrals

    $y_p(x) = -\frac{x^2\cos(x)}{2} + \sin(x)\,x + \frac{\cos(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{x} - \frac{x^2\cos(x)}{2} + \sin(x)\,x + \frac{\cos(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)-y(x)=x^2*cos(x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} c_2 + \mathrm{e}^{x} c_1 - \frac{\cos(x)\, x^2}{2} + \frac{\cos(x)}{2} + x \sin(x)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 35

```
DSolve[y''[x]-y[x]==x^2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}(x^2 - 1)\cos(x) + x\sin(x) + c_1 e^x + c_2 e^{-x}$$

15.4 problem 4

Internal problem ID [2237]
Internal file name [`OUTPUT/2237_Monday_February_26_2024_09_18_37_AM_58340241/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 4.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 4y' = e^{x} + \sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 4y' = 0$$

The characteristic equation is

$$\lambda^3 + 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2i$$
$$\lambda_3 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{2ix}c_2 + e^{-2ix}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{2ix}$$
$$y_3 = e^{-2ix}$$

Now the particular solution to the given ODE is found

$$y''' + 4y' = e^x + \sin(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2ix}, e^{2ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x + A_2 \cos(x) + A_3 \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^x - 3A_2 \sin(x) + 3A_3 \cos(x) = e^x + \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = -\frac{1}{3}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{5} - \frac{\cos(x)}{3}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(c_1 + e^{2ix}c_2 + e^{-2ix}c_3\right) + \left(\frac{e^x}{5} - \frac{\cos(x)}{3}\right)$$

Summary
The solution(s) found are the following

$$y = c_1 + e^{2ix}c_2 + e^{-2ix}c_3 + \frac{e^x}{5} - \frac{\cos(x)}{3} \tag{1}$$

Verification of solutions

$$y = c_1 + e^{2ix}c_2 + e^{-2ix}c_3 + \frac{e^x}{5} - \frac{\cos(x)}{3}$$

Verified OK.

15.4.1 Maple step by step solution

Let's solve

$y''' + 4y' = e^x + \sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = e^x + \sin(x) - 4y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = e^x + \sin(x) - 4y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ e^x + \sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ e^x + \sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_2(x) = \begin{bmatrix} -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \overrightarrow{y}_3(x) = \begin{bmatrix} \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$$

☐ Fundamental matrix

  ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ 0 & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ 0 & \cos(2x) & -\sin(2x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} 1 & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ 0 & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ 0 & \cos(2x) & -\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} 1 & \frac{\sin(2x)}{2} & \frac{1}{4} - \frac{\cos(2x)}{4} \\ 0 & \cos(2x) & \frac{\sin(2x)}{2} \\ 0 & -2\sin(2x) & \cos(2x) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \begin{bmatrix} \frac{4\cos(x)^2}{15} + \frac{(-3\sin(x)-5)\cos(x)}{15} + \frac{e^x}{5} - \frac{2}{15} \\ \frac{(-8\cos(x)+5)\sin(x)}{15} - \frac{2\cos(x)^2}{5} + \frac{e^x}{5} + \frac{1}{5} \\ \frac{4\cos(x)\sin(x)}{5} - \frac{16\cos(x)^2}{15} + \frac{e^x}{5} + \frac{\cos(x)}{3} + \frac{8}{15} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \begin{bmatrix} \frac{4\cos(x)^2}{15} + \frac{(-3\sin(x)-5)\cos(x)}{15} + \frac{e^x}{5} - \frac{2}{15} \\ \frac{(-8\cos(x)+5)\sin(x)}{15} - \frac{2\cos(x)^2}{5} + \frac{e^x}{5} + \frac{1}{5} \\ \frac{4\cos(x)\sin(x)}{5} - \frac{16\cos(x)^2}{15} + \frac{e^x}{5} + \frac{\cos(x)}{3} + \frac{8}{15} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-30c_2+16)\cos(x)^2}{60} + \frac{(-20+(30c_3-12)\sin(x))\cos(x)}{60} + c_1 + \frac{c_2}{4} + \frac{e^x}{5} - \frac{2}{15}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -4*_b(_a)+exp(_a)+sin(_a), _b
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$3)+4*diff(y(x),x)=exp(x)+sin(x),y(x), singsol=all)
```

$$y(x) = \frac{\sin(2x)\, c_1}{2} - \frac{c_2\cos(2x)}{2} + \frac{e^x}{5} - \frac{\cos(x)}{3} + c_3$$

✓ Solution by Mathematica

Time used: 0.241 (sec). Leaf size: 37

```
DSolve[y'''[x]+4*y'[x]==Exp[x]+Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x}{5} - \frac{1}{2}c_2\cos(2x) + \cos(x)\left(-\frac{1}{3} + c_1\sin(x)\right) + c_3$$

15.5 problem 5

Internal problem ID [2238]
Internal file name [`OUTPUT/2238_Monday_February_26_2024_09_18_37_AM_94999196/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 5.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_y]]`

$$\boxed{y^{(5)} + y'''' = x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y^{(5)} + y'''' = 0$$

The characteristic equation is

$$\lambda^5 + \lambda^4 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$
$$\lambda_5 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 x + c_4 x^2 + c_5 x^3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = 1$$
$$y_3 = x$$
$$y_4 = x^2$$
$$y_5 = x^3$$

Now the particular solution to the given ODE is found

$$y^{(5)} + y'''' = x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2, x^3, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3, x^4\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3, x^4, x^5\}]$$

Since x^3 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^4, x^5, x^6\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3x^6 + A_2x^5 + A_1x^4$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$360x^2A_3 + 120xA_2 + 720xA_3 + 24A_1 + 120A_2 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{12}, A_2 = -\frac{1}{60}, A_3 = \frac{1}{360}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{360}x^6 - \frac{1}{60}x^5 + \frac{1}{12}x^4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^{-x} + c_2 + c_3x + c_4x^2 + c_5x^3) + \left(\frac{1}{360}x^6 - \frac{1}{60}x^5 + \frac{1}{12}x^4\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1e^{-x} + c_2 + c_3x + c_4x^2 + c_5x^3 + \frac{x^6}{360} - \frac{x^5}{60} + \frac{x^4}{12} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1e^{-x} + c_2 + c_3x + c_4x^2 + c_5x^3 + \frac{x^6}{360} - \frac{x^5}{60} + \frac{x^4}{12}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^2-_b(_a), _b(_a)`   *** Sublevel 2 *
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$5)+diff(y(x),x$4)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^6}{360} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{c_2 x^3}{6} + \frac{c_3 x^2}{2} + \mathrm{e}^{-x} c_1 + c_4 x + c_5$$

✓ Solution by Mathematica

Time used: 0.134 (sec). Leaf size: 53

```
DSolve[y'''''[x]+y''''[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^6}{360} - \frac{x^5}{60} + \frac{x^4}{12} + c_5 x^3 + c_4 x^2 + c_3 x + c_1 e^{-x} + c_2$$

15.6 problem 6

Internal problem ID [2239]
Internal file name [`OUTPUT/2239_Monday_February_26_2024_09_18_37_AM_42527952/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{2y'' + 3y' - 2y = x^2 e^x}$$

15.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2, B = 3, C = -2, f(x) = x^2 e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2y'' + 3y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 3, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} - 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + 3\lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 3, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{3^2 - (4)\,(2)\,(-2)} \\ &= -\frac{3}{4} \pm \frac{5}{4}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{4} + \frac{5}{4} \\ \lambda_2 &= -\frac{3}{4} - \frac{5}{4}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{1}{2} \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{\frac{x}{2}} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^x, x^2 e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-2x}, e^{\frac{x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^x + A_2 x^2 e^x + A_3 e^x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1 e^x + 3A_1 x\,e^x + 4A_2 e^x + 14A_2 x\,e^x + 3A_2 x^2 e^x + 3A_3 e^x = x^2 e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{14}{9}, A_2 = \frac{1}{3}, A_3 = \frac{86}{27}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{14x\,e^x}{9} + \frac{x^2 e^x}{3} + \frac{86\,e^x}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{x}{2}} + c_2 e^{-2x}\right) + \left(-\frac{14x\,e^x}{9} + \frac{x^2 e^x}{3} + \frac{86\,e^x}{27}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{-2x} - \frac{14x\,e^x}{9} + \frac{x^2 e^x}{3} + \frac{86\,e^x}{27} \tag{1}$$

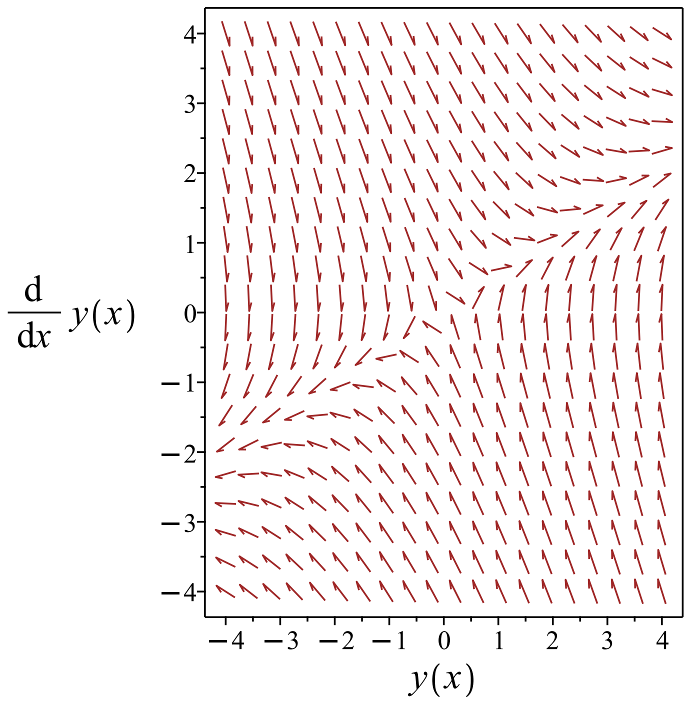


Figure 668: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{-2x} - \frac{14x\,e^x}{9} + \frac{x^2 e^x}{3} + \frac{86\,e^x}{27}$$

Verified OK.

15.6.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + 3y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 2 \\
B &= 3 \\
C &= -2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{25}{16} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 25 \\
t &= 16
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{25z(x)}{16} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 523: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{2}\,dx} \\ &= z_1 e^{-\frac{3x}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{3x}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{2\,\mathrm{e}^{\frac{5x}{2}}}{5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{2\,\mathrm{e}^{\frac{5x}{2}}}{5}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$2y'' + 3y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{2c_2 \mathrm{e}^{\frac{x}{2}}}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \,\mathrm{e}^x, x^2 \mathrm{e}^x, \mathrm{e}^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{2 \,\mathrm{e}^{\frac{x}{2}}}{5}, \mathrm{e}^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x \,\mathrm{e}^x + A_2 x^2 \mathrm{e}^x + A_3 \mathrm{e}^x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1 \mathrm{e}^x + 3A_1 x \,\mathrm{e}^x + 4A_2 \mathrm{e}^x + 14A_2 x \,\mathrm{e}^x + 3A_2 x^2 \mathrm{e}^x + 3A_3 \mathrm{e}^x = x^2 \mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{14}{9}, A_2 = \frac{1}{3}, A_3 = \frac{86}{27}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{14x \,\mathrm{e}^x}{9} + \frac{x^2 \mathrm{e}^x}{3} + \frac{86 \,\mathrm{e}^x}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + \frac{2c_2 \mathrm{e}^{\frac{x}{2}}}{5}\right) + \left(-\frac{14x \,\mathrm{e}^x}{9} + \frac{x^2 \mathrm{e}^x}{3} + \frac{86 \,\mathrm{e}^x}{27}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{2c_2 e^{\frac{x}{2}}}{5} - \frac{14x\,e^{x}}{9} + \frac{x^2 e^{x}}{3} + \frac{86\,e^{x}}{27} \tag{1}$$

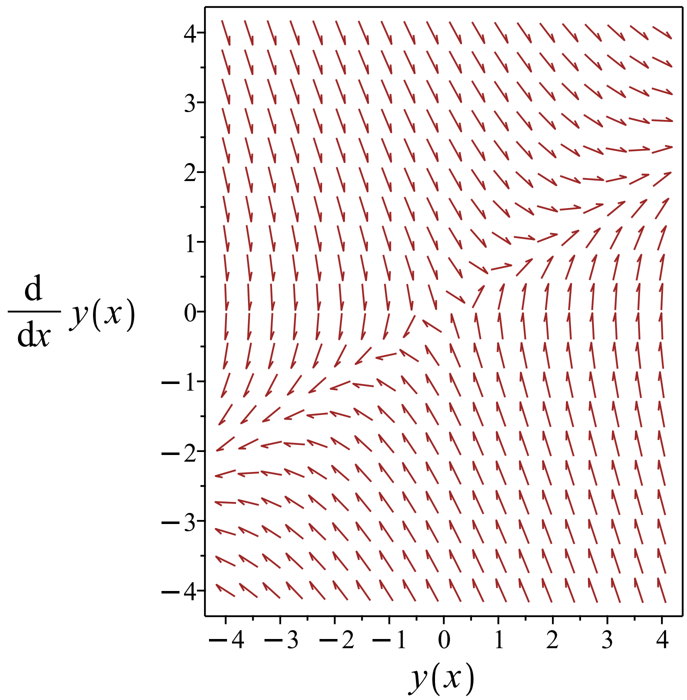


Figure 669: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{2c_2 e^{\frac{x}{2}}}{5} - \frac{14x\,e^{x}}{9} + \frac{x^2 e^{x}}{3} + \frac{86\,e^{x}}{27}$$

Verified OK.

15.6.3 Maple step by step solution

Let's solve

$2y'' + 3y' - 2y = x^2 e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{3y'}{2} + y + \frac{x^2 e^x}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2} - y = \frac{x^2 e^x}{2}$
- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{3}{2}r - 1 = 0$
- Factor the characteristic polynomial

  $\frac{(r+2)(2r-1)}{2} = 0$
- Roots of the characteristic polynomial

  $r = \left(-2, \frac{1}{2}\right)$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{\frac{x}{2}}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{\frac{x}{2}} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{x^2 e^x}{2}\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{\frac{x}{2}} \\ -2e^{-2x} & \frac{e^{\frac{x}{2}}}{2} \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{5e^{-\frac{3x}{2}}}{2}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\left(e^{\frac{5x}{2}}\left(\int x^2 e^{\frac{x}{2}}dx\right) - \left(\int x^2 e^{3x}dx\right)\right)e^{-2x}}{5}$
- Compute integrals

$$y_p(x) = \frac{e^x (9x^2-42x+86)}{27}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-2x} + c_2 e^{\frac{x}{2}} + \frac{e^x (9x^2-42x+86)}{27}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(2*diff(y(x),x$2)+3*diff(y(x),x)-2*y(x)=x^2*exp(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{-2x} \left(3c_2 e^{\frac{5x}{2}} + \left(x^2 - \frac{14}{3}x + \frac{86}{9}\right) e^{3x} + 3c_1\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 41

```
DSolve[2*y''[x]+3*y'[x]-2*y[x]==x^2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{27} e^x \left(9x^2 - 42x + 86\right) + c_1 e^{x/2} + c_2 e^{-2x}$$

15.7 problem 8

Internal problem ID [2240]
Internal file name [`OUTPUT/2240_Monday_February_26_2024_09_18_38_AM_1305094/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 8.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + y' = \sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{-ix} + e^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \mathrm{e}^{-ix} \\
y_3 &= \mathrm{e}^{ix}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' + y' = \sin(x)
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} 1 & \mathrm{e}^{-ix} & \mathrm{e}^{ix} \\ 0 & -i\mathrm{e}^{-ix} & i\mathrm{e}^{ix} \\ 0 & -\mathrm{e}^{-ix} & -\mathrm{e}^{ix} \end{bmatrix} \\
|W| &= 2i\mathrm{e}^{-ix}\mathrm{e}^{ix}
\end{aligned}
$$

The determinant simplifies to

$$
|W| = 2i
$$

Now we determine W_i for each U_i.

$$
\begin{aligned}
W_1(x) &= \det \begin{bmatrix} e^{-ix} & e^{ix} \\ -ie^{-ix} & ie^{ix} \end{bmatrix} \\
&= 2i
\end{aligned}
$$

$$
\begin{aligned}
W_2(x) &= \det \begin{bmatrix} 1 & e^{ix} \\ 0 & ie^{ix} \end{bmatrix} \\
&= ie^{ix}
\end{aligned}
$$

$$
\begin{aligned}
W_3(x) &= \det \begin{bmatrix} 1 & e^{-ix} \\ 0 & -ie^{-ix} \end{bmatrix} \\
&= -ie^{-ix}
\end{aligned}
$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(\sin(x))\,(2i)}{(1)\,(2i)}\,dx \\
&= \int \frac{2i\sin(x)}{2i}\,dx \\
&= \int (\sin(x))\,dx \\
&= -\cos(x)
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(\sin(x))\,(ie^{ix})}{(1)\,(2i)}\,dx \\
&= -\int \frac{i\sin(x)\,e^{ix}}{2i}\,dx \\
&= -\int \left(\frac{e^{ix}\sin(x)}{2}\right)dx \\
&= -\left(\int \frac{e^{ix}\sin(x)}{2}\,dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(\sin(x)) \left(-i \mathrm{e}^{-ix}\right)}{(1)\,(2i)}\, dx \\
&= \int \frac{-i \sin(x)\, \mathrm{e}^{-ix}}{2i}\, dx \\
&= \int \left(-\frac{\mathrm{e}^{-ix} \sin(x)}{2}\right) dx \\
&= \int -\frac{\mathrm{e}^{-ix} \sin(x)}{2} dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = {} & (-\cos(x)) \\
& + \left(-\left(\int \frac{\mathrm{e}^{ix} \sin(x)}{2} dx\right)\right) \left(\mathrm{e}^{-ix}\right) \\
& + \left(\int -\frac{\mathrm{e}^{-ix} \sin(x)}{2} dx\right) \left(\mathrm{e}^{ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -\cos(x) - \frac{\left(\int \mathrm{e}^{ix} \sin(x)\, dx\right) \mathrm{e}^{-ix}}{2} - \frac{\left(\int \mathrm{e}^{-ix} \sin(x)\, dx\right) \mathrm{e}^{ix}}{2}
$$

Which simplifies to

$$
y_p = -\left(\int \sin(x)^2\, dx\right) \sin(x) - \frac{\cos(x) \left(\int \sin(2x)\, dx + 2\right)}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3\right) + \left(-\left(\int \sin(x)^2\, dx\right) \sin(x) - \frac{\cos(x) \left(\int \sin(2x)\, dx + 2\right)}{2}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 + c_2 \mathrm{e}^{-ix} + \mathrm{e}^{ix} c_3 - \left(\int \sin(x)^2\, dx\right) \sin(x) - \frac{\cos(x) \left(\int \sin(2x)\, dx + 2\right)}{2} \tag{1}
$$

Verification of solutions

$$y = c_1 + c_2 e^{-ix} + e^{ix} c_3 - \left(\int \sin(x)^2 dx\right) \sin(x) - \frac{\cos(x) \left(\int \sin(2x) dx + 2\right)}{2}$$

Verified OK.

15.7.1 Maple step by step solution

Let's solve

$y''' + y' = \sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \sin(x) - y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \sin(x) - y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ \sin \left(x \right) \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} -1 \\ -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c} -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & -\cos(x) & \sin(x) \\ 0 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix}1 & -\cos(x) & \sin(x)\\ 0 & \sin(x) & \cos(x)\\ 0 & \cos(x) & -\sin(x)\end{bmatrix}\cdot\frac{1}{\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix}1 & \sin(x) & 1-\cos(x)\\ 0 & \cos(x) & \sin(x)\\ 0 & -\sin(x) & \cos(x)\end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x)=\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\Phi(x)\cdot\overrightarrow{v}'(x)=A\cdot\Phi(x)\cdot\overrightarrow{v}(x)+\overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\overrightarrow{v}'(x)=\overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot\overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot\overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} 1 - \frac{\sin(x)x}{2} - \cos(x) \\ -\frac{x\cos(x)}{2} + \frac{\sin(x)}{2} \\ \frac{\sin(x)x}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} 1 - \frac{\sin(x)x}{2} - \cos(x) \\ -\frac{x\cos(x)}{2} + \frac{\sin(x)}{2} \\ \frac{\sin(x)x}{2} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = 1 + (-c_2 - 1)\cos(x) + \frac{(2c_3 - x)\sin(x)}{2} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -_b(_a)+sin(_a), _b(_a)`   **
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$3)+diff(y(x),x)=sin(x),y(x), singsol=all)
```

$$y(x) = (-1 - c_2)\cos(x) + \frac{(2c_1 - x)\sin(x)}{2} + c_3$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 31

```
DSolve[y'''[x]+y'[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}(1 + 2c_2)\cos(x) + \left(-\frac{x}{2} + c_1\right)\sin(x) + c_3$$

15.8 problem 9

Internal problem ID [2241]
Internal file name [OUTPUT/2241_Monday_February_26_2024_09_18_38_AM_60975173/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - y' = \sin(x) x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y' = 0$$

The characteristic equation is

$$\lambda^3 - \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 e^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= 1 \\
y_3 &= \mathrm{e}^{x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' - y' = \sin(x)\, x
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(x)\, x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\cos(x), \sin(x)\, x, \cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, \mathrm{e}^{x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 x\cos(x) + A_2\sin(x)\, x + A_3\cos(x) + A_4\sin(x)
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-4A_1\cos(x) + 2A_1 x\sin(x) - 2A_2\cos(x)\, x - 4A_2\sin(x) + 2A_3\sin(x) - 2A_4\cos(x) \\
&= \sin(x)\, x
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = \frac{1}{2}, A_2 = 0, A_3 = 0, A_4 = -1\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x \cos(x)}{2} - \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 + c_3 e^{x}\right) + \left(\frac{x \cos(x)}{2} - \sin(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + c_3 e^{x} + \frac{x \cos(x)}{2} - \sin(x) \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 + c_3 e^{x} + \frac{x \cos(x)}{2} - \sin(x)$$

Verified OK.

15.8.1 Maple step by step solution

Let's solve

$y''' - y' = \sin(x)\, x$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = \sin(x)\, x + y_2(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \sin(x)\, x + y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \sin(x)\, x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \sin(x)\, x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$$

□ Fundamental matrix

 ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & 1 & e^{x} \\ -e^{-x} & 0 & e^{x} \\ e^{-x} & 0 & e^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & 1 & e^{x} \\ -e^{-x} & 0 & e^{x} \\ e^{-x} & 0 & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{e^{-x}}{2} - 1 + \frac{e^{x}}{2} \\ 0 & \frac{e^{x}}{2} + \frac{e^{-x}}{2} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} \\ 0 & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{e^{x}}{2} + \frac{e^{-x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x) = \begin{bmatrix} \frac{x\cos(x)}{2} - \sin(x) + \frac{e^x}{4} - \frac{e^{-x}}{4} \\ \frac{e^x}{4} + \frac{e^{-x}}{4} - \frac{\sin(x)x}{2} - \frac{\cos(x)}{2} \\ -\frac{x\cos(x)}{2} + \frac{e^x}{4} - \frac{e^{-x}}{4} \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \begin{bmatrix} \frac{x\cos(x)}{2} - \sin(x) + \frac{e^x}{4} - \frac{e^{-x}}{4} \\ \frac{e^x}{4} + \frac{e^{-x}}{4} - \frac{\sin(x)x}{2} - \frac{\cos(x)}{2} \\ -\frac{x\cos(x)}{2} + \frac{e^x}{4} - \frac{e^{-x}}{4} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = c_1 e^{-x} + c_3 e^x + \frac{x\cos(x)}{2} - \sin(x) + \frac{e^x}{4} - \frac{e^{-x}}{4} + c_2$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a*sin(_a)+_b(_a), _b(_a)`
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$3)-diff(y(x),x)=x*sin(x),y(x), singsol=all)
```

$$y(x) = -\sin(x) + \frac{x\cos(x)}{2} - e^{-x}c_2 + e^{x}c_1 + c_3$$

✓ Solution by Mathematica

Time used: 0.142 (sec). Leaf size: 34

```
DSolve[y'''[x]-y'[x]==x*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sin(x) + \frac{1}{2}x\cos(x) + c_1e^x - c_2e^{-x} + c_3$$

15.9 problem 10

Internal problem ID [2242]
Internal file name [`OUTPUT/2242_Monday_February_26_2024_09_18_39_AM_47990902/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 10.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 2y'' = x\cos(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y'' = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -2$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + e^{-2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= e^{-2x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + 2y'' = x \cos(2x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(2x), x \sin(2x), \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x) + A_3 \cos(2x) + A_4 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-12A_1 \cos(2x) + 8A_1 x \sin(2x) - 12A_2 \sin(2x) - 8A_2 x \cos(2x) \\
&+ 8A_3 \sin(2x) - 8A_4 \cos(2x) - 8A_1 \sin(2x) - 8A_1 x \cos(2x) \\
&+ 8A_2 \cos(2x) - 8A_2 x \sin(2x) - 8A_3 \cos(2x) - 8A_4 \sin(2x) = x \cos(2x)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{16}, A_2 = -\frac{1}{16}, A_3 = -\frac{1}{16}, A_4 = \frac{3}{32}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(2x)}{16} - \frac{x\sin(2x)}{16} - \frac{\cos(2x)}{16} + \frac{3\sin(2x)}{32}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x + c_1 + \mathrm{e}^{-2x} c_3\right) + \left(-\frac{x\cos(2x)}{16} - \frac{x\sin(2x)}{16} - \frac{\cos(2x)}{16} + \frac{3\sin(2x)}{32}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 + \mathrm{e}^{-2x} c_3 - \frac{x\cos(2x)}{16} - \frac{x\sin(2x)}{16} - \frac{\cos(2x)}{16} + \frac{3\sin(2x)}{32} \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + \mathrm{e}^{-2x} c_3 - \frac{x\cos(2x)}{16} - \frac{x\sin(2x)}{16} - \frac{\cos(2x)}{16} + \frac{3\sin(2x)}{32}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a*cos(2*_a)-2*_b(_a), _b(_a)`   *** Su
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)=x*cos(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(-2x-2)\cos(2x)}{32} + \frac{(3-2x)\sin(2x)}{32} + c_2 x + \frac{e^{-2x} c_1}{4} + c_3$$

✓ Solution by Mathematica

Time used: 1.447 (sec). Leaf size: 48

```
DSolve[y'''[x]+2*y''[x]==x*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 x + \frac{1}{16}\left(-x\sin(2x) - (x+1)\cos(2x) + 4c_1 e^{-2x} + 3\sin(x)\cos(x)\right) + c_2$$

15.10 problem 11

Internal problem ID [2243]
Internal file name [`OUTPUT/2243_Monday_February_26_2024_09_18_39_AM_10265833/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 3y' + 2y = x^2 \cos(x)}$$

15.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = x^2 \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=3, C=2$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+3\lambda\,\mathrm{e}^{\lambda x}+2\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+3\lambda+2=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=3, C=2$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-3}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{3^2-(4)\,(1)\,(2)}\\&=-\frac{3}{2}\pm\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{3}{2}+\frac{1}{2}\\\lambda_2&=-\frac{3}{2}-\frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=-1\\\lambda_2&=-2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(-1)x}+c_2e^{(-2)x}\end{aligned}$$

Or

$$y=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(x), x^2 \cos(x), \sin(x) x, \sin(x) x^2, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x \cos(x) + A_2 x^2 \cos(x) + A_3 \sin(x) x + A_4 \sin(x) x^2 + A_5 \cos(x) + A_6 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&6A_2 x \cos(x) + 6A_4 \sin(x) x + 4A_4 \cos(x) x - 3A_5 \sin(x) + 3A_6 \cos(x) \\
&- 2A_1 \sin(x) + 3A_1 \cos(x) - 3A_1 x \sin(x) - 3A_2 x^2 \sin(x) \\
&+ 3A_3 \cos(x) x + 3A_3 \sin(x) + 3A_4 \cos(x) x^2 + 2A_2 \cos(x) \\
&- 4A_2 x \sin(x) + 2A_3 \cos(x) + 2A_4 \sin(x) + A_1 x \cos(x) + A_2 x^2 \cos(x) \\
&+ A_3 \sin(x) x + A_4 \sin(x) x^2 + A_5 \cos(x) + A_6 \sin(x) = x^2 \cos(x)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{6}{25}, A_2 = \frac{1}{10}, A_3 = -\frac{17}{25}, A_4 = \frac{3}{10}, A_5 = -\frac{133}{250}, A_6 = \frac{81}{250}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{6x \cos(x)}{25} + \frac{x^2 \cos(x)}{10} - \frac{17 \sin(x) x}{25} + \frac{3 \sin(x) x^2}{10} - \frac{133 \cos(x)}{250} + \frac{81 \sin(x)}{250}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-x} + c_2 e^{-2x}\right) \\
&\quad + \left(\frac{6x \cos(x)}{25} + \frac{x^2 \cos(x)}{10} - \frac{17 \sin(x) x}{25} + \frac{3 \sin(x) x^2}{10} - \frac{133 \cos(x)}{250} + \frac{81 \sin(x)}{250}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{6x\cos(x)}{25} + \frac{x^2\cos(x)}{10} - \frac{17\sin(x)x}{25} + \frac{3\sin(x)x^2}{10} - \frac{133\cos(x)}{250} + \frac{81\sin(x)}{250} \tag{1}$$

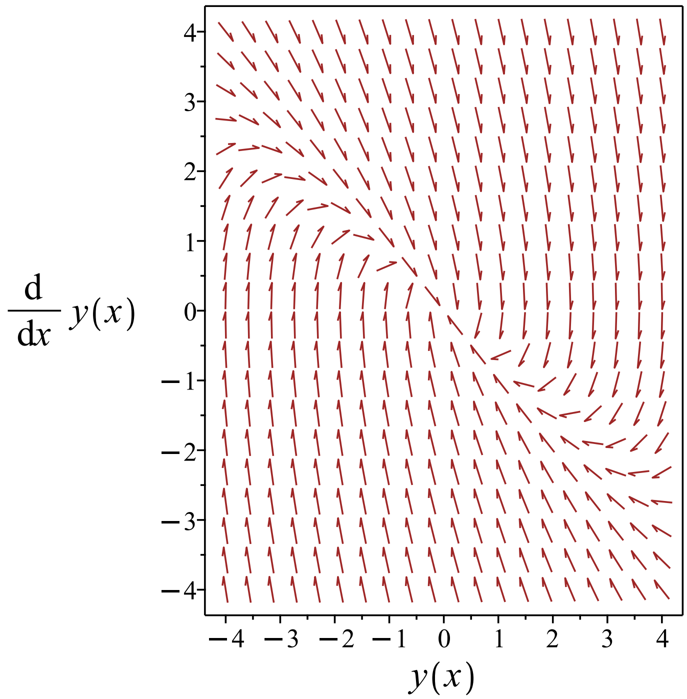


Figure 670: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{6x\cos(x)}{25} + \frac{x^2\cos(x)}{10} - \frac{17\sin(x)x}{25} + \frac{3\sin(x)x^2}{10} - \frac{133\cos(x)}{250} + \frac{81\sin(x)}{250}$$

Verified OK.

15.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 527: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\
&= y_1 (\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} (\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(x), x^2 \cos(x), \sin(x) x, \sin(x) x^2, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x \cos(x) + A_2 x^2 \cos(x) + A_3 \sin(x) x + A_4 \sin(x) x^2 + A_5 \cos(x) + A_6 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-3A_5 \sin(x) + 3A_6 \cos(x) - 2A_1 \sin(x) + 2A_3 \cos(x) + 2A_4 \sin(x)\\&+ 3A_1 \cos(x) - 3A_1 x \sin(x) - 3A_2 x^2 \sin(x) + 3A_3 \cos(x) x\\&+ 3A_3 \sin(x) + 3A_4 \cos(x) x^2 + 2A_2 \cos(x) - 4A_2 x \sin(x)\\&+ A_5 \cos(x) + A_6 \sin(x) + A_1 x \cos(x) + A_2 x^2 \cos(x) + A_3 \sin(x) x\\&+ A_4 \sin(x) x^2 + 6A_2 x \cos(x) + 6A_4 \sin(x) x + 4A_4 \cos(x) x = x^2 \cos(x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{6}{25}, A_2 = \frac{1}{10}, A_3 = -\frac{17}{25}, A_4 = \frac{3}{10}, A_5 = -\frac{133}{250}, A_6 = \frac{81}{250}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{6x \cos(x)}{25} + \frac{x^2 \cos(x)}{10} - \frac{17 \sin(x) x}{25} + \frac{3 \sin(x) x^2}{10} - \frac{133 \cos(x)}{250} + \frac{81 \sin(x)}{250}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(c_1 e^{-2x} + c_2 e^{-x}\right)\\&\quad + \left(\frac{6x \cos(x)}{25} + \frac{x^2 \cos(x)}{10} - \frac{17 \sin(x) x}{25} + \frac{3 \sin(x) x^2}{10} - \frac{133 \cos(x)}{250} + \frac{81 \sin(x)}{250}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-x} + \frac{6x \cos(x)}{25} + \frac{x^2 \cos(x)}{10} - \frac{17 \sin(x) x}{25} + \frac{3 \sin(x) x^2}{10} - \frac{133 \cos(x)}{250} + \frac{81 \sin(x)}{250} \tag{1}$$

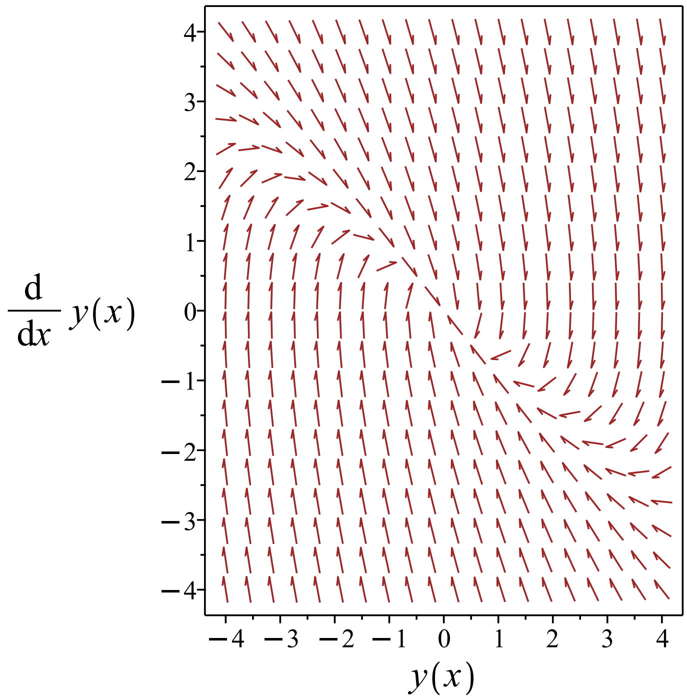


Figure 671: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} + \frac{6x \cos(x)}{25} + \frac{x^2 \cos(x)}{10} - \frac{17 \sin(x) x}{25} + \frac{3 \sin(x) x^2}{10} - \frac{133 \cos(x)}{250} + \frac{81 \sin(x)}{250}$$

Verified OK.

15.10.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = x^2 \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 \cos(x)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-2x}\left(\int e^{2x} \cos(x)\, x^2 dx\right) + e^{-x}\left(\int \cos(x)\, e^x x^2 dx\right)$

- Compute integrals

  $y_p(x) = \frac{(25x^2+60x-133)\cos(x)}{250} + \frac{\sin(x)(75x^2-170x+81)}{250}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + \frac{(25x^2+60x-133)\cos(x)}{250} + \frac{\sin(x)(75x^2-170x+81)}{250}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=x^2*cos(x),y(x), singsol=all)
```

$$y(x) = -e^{-2x}c_1 + e^{-x}c_2 + \frac{(25x^2 + 60x - 133)\cos(x)}{250} + \frac{\sin(x)(75x^2 - 170x + 81)}{250}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 53

```
DSolve[y''[x]+3*y'[x]+2*y[x]==x^2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{250}\left(\left(75x^2 - 170x + 81\right)\sin(x) + \left(25x^2 + 60x - 133\right)\cos(x)\right) + c_1 e^{-2x} + c_2 e^{-x}$$

15.11 problem 12

Internal problem ID [2244]
Internal file name [`OUTPUT/2244_Monday_February_26_2024_09_18_39_AM_45048317/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 4y' + 3y = \sin(x)\, x^2}$$

15.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 3, f(x) = \sin(x)\, x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-4^2 - (4)(1)(3)}\\ &= 2 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = 2 + 1$$
$$\lambda_2 = 2 - 1$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 e^{3x} + c_2 e^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2 e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\,x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(x)\,,x^2\cos(x)\,,\sin(x)\,x,\sin(x)\,x^2,\cos(x)\,,\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1x\cos(x) + A_2x^2\cos(x) + A_3\sin(x)\,x + A_4\sin(x)\,x^2 + A_5\cos(x) + A_6\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&4A_5\sin(x) - 4A_6\cos(x) - 2A_1\sin(x) + 2A_3\cos(x) + 2A_4\sin(x)\\
&- 4A_1\cos(x) + 4A_1x\sin(x) + 4A_2x^2\sin(x) - 4A_3\cos(x)\,x - 4A_3\sin(x)\\
&- 4A_4\cos(x)\,x^2 + 2A_2\cos(x) - 4A_2x\sin(x) + 2A_1x\cos(x)\\
&+ 2A_2x^2\cos(x) + 2A_3\sin(x)\,x + 2A_4\sin(x)\,x^2 + 2A_5\cos(x)\\
&+ 2A_6\sin(x) + 4A_4\cos(x)\,x - 8A_2x\cos(x) - 8A_4\sin(x)\,x = \sin(x)\,x^2
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{11}{25}, A_2 = \frac{1}{5}, A_3 = -\frac{2}{25}, A_4 = \frac{1}{10}, A_5 = \frac{28}{125}, A_6 = -\frac{67}{250}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{11x\cos(x)}{25} + \frac{x^2\cos(x)}{5} - \frac{2\sin(x)\,x}{25} + \frac{\sin(x)\,x^2}{10} + \frac{28\cos(x)}{125} - \frac{67\sin(x)}{250}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p\\
&= \left(c_1e^{3x} + c_2e^x\right)\\
&\quad + \left(\frac{11x\cos(x)}{25} + \frac{x^2\cos(x)}{5} - \frac{2\sin(x)\,x}{25} + \frac{\sin(x)\,x^2}{10} + \frac{28\cos(x)}{125} - \frac{67\sin(x)}{250}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} + c_2 e^{x} + \frac{11x \cos(x)}{25} + \frac{x^2 \cos(x)}{5} - \frac{2 \sin(x) x}{25} + \frac{\sin(x) x^2}{10} + \frac{28 \cos(x)}{125} - \frac{67 \sin(x)}{250} \tag{1}$$

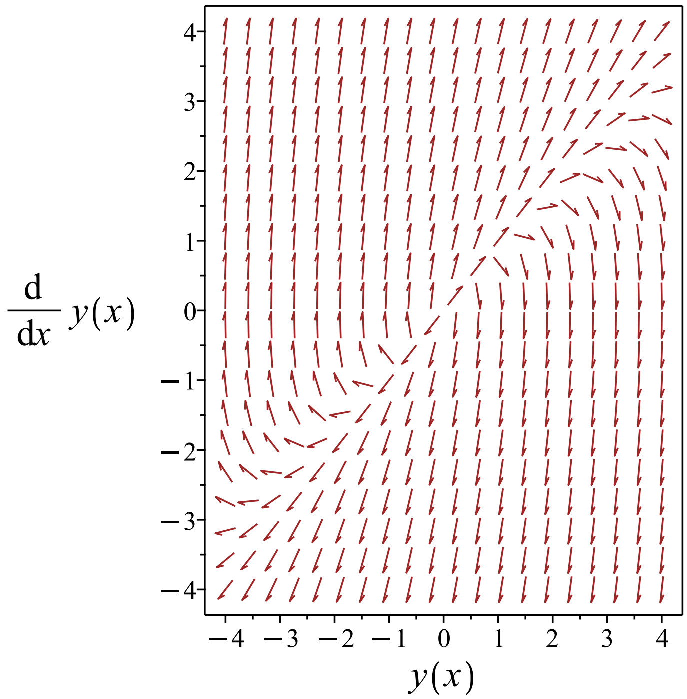


Figure 672: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{x} + \frac{11x \cos(x)}{25} + \frac{x^2 \cos(x)}{5} - \frac{2 \sin(x) x}{25} + \frac{\sin(x) x^2}{10} + \frac{28 \cos(x)}{125} - \frac{67 \sin(x)}{250}$$

Verified OK.

15.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 529: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} \, dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2 \left(\mathrm{e}^x \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + \frac{c_2 \mathrm{e}^{3x}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\, x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(x), x^2\cos(x), \sin(x)\, x, \sin(x)\, x^2, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{3x}}{2}, e^{x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\cos(x) + A_2 x^2\cos(x) + A_3\sin(x)\, x + A_4\sin(x)\, x^2 + A_5\cos(x) + A_6\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-8A_2 x\cos(x) - 8A_4\sin(x)\, x + 4A_4\cos(x)\, x + 4A_5\sin(x) - 4A_6\cos(x) \\
&- 2A_1\sin(x) + 2A_3\cos(x) + 2A_4\sin(x) - 4A_1\cos(x) + 4A_1 x\sin(x) \\
&+ 4A_2 x^2\sin(x) - 4A_3\cos(x)\, x - 4A_3\sin(x) - 4A_4\cos(x)\, x^2 \\
&+ 2A_2\cos(x) - 4A_2 x\sin(x) + 2A_5\cos(x) + 2A_6\sin(x) + 2A_1 x\cos(x) \\
&+ 2A_2 x^2\cos(x) + 2A_3\sin(x)\, x + 2A_4\sin(x)\, x^2 = \sin(x)\, x^2
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{11}{25}, A_2 = \frac{1}{5}, A_3 = -\frac{2}{25}, A_4 = \frac{1}{10}, A_5 = \frac{28}{125}, A_6 = -\frac{67}{250}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{11x\cos(x)}{25} + \frac{x^2\cos(x)}{5} - \frac{2\sin(x)\, x}{25} + \frac{\sin(x)\, x^2}{10} + \frac{28\cos(x)}{125} - \frac{67\sin(x)}{250}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{x} + \frac{c_2 \mathrm{e}^{3x}}{2}\right) \\
&\quad + \left(\frac{11x\cos(x)}{25} + \frac{x^2\cos(x)}{5} - \frac{2\sin(x)\,x}{25} + \frac{\sin(x)\,x^2}{10} + \frac{28\cos(x)}{125} - \frac{67\sin(x)}{250}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \mathrm{e}^{x} + \frac{c_2 \mathrm{e}^{3x}}{2} + \frac{11x\cos(x)}{25} + \frac{x^2\cos(x)}{5} \\
- \frac{2\sin(x)\,x}{25} + \frac{\sin(x)\,x^2}{10} + \frac{28\cos(x)}{125} - \frac{67\sin(x)}{250}
\end{aligned}
\tag{1}
$$

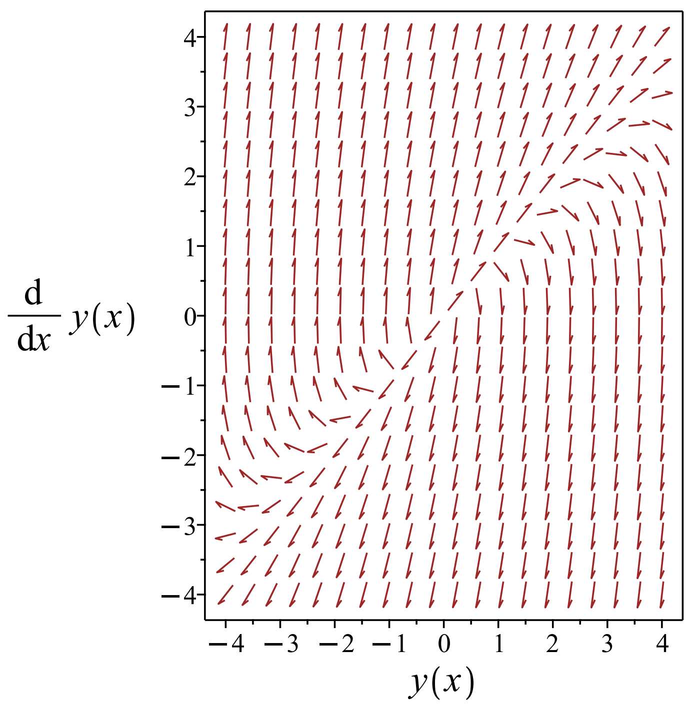


Figure 673: Slope field plot

Verification of solutions

$$y = c_1 e^x + \frac{c_2 e^{3x}}{2} + \frac{11x \cos(x)}{25} + \frac{x^2 \cos(x)}{5} - \frac{2\sin(x)\, x}{25} + \frac{\sin(x)\, x^2}{10} + \frac{28\cos(x)}{125} - \frac{67\sin(x)}{250}$$

Verified OK.

15.11.3 Maple step by step solution

Let's solve

$y'' - 4y' + 3y = \sin(x)\, x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (1, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x + c_2 e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\, x^2\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{3x} \\ e^x & 3\,e^{3x} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = 2\,e^{4x}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{e^x\left(\int x^2 \sin(x) e^{-x} dx\right)}{2} + \frac{e^{3x}\left(\int x^2 e^{-3x} \sin(x) dx\right)}{2}$$

- Compute integrals

  $$y_p(x) = \frac{(25x^2+55x+28)\cos(x)}{125} + \frac{\left(x^2-\frac{4}{5}x-\frac{67}{25}\right)\sin(x)}{10}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^x + c_2 e^{3x} + \frac{(25x^2+55x+28)\cos(x)}{125} + \frac{\left(x^2-\frac{4}{5}x-\frac{67}{25}\right)\sin(x)}{10}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=x^2*sin(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{3x} + \frac{(25x^2 + 55x + 28)\cos(x)}{125} + \frac{(25x^2 - 20x - 67)\sin(x)}{250} + e^x c_1$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 52

```
DSolve[y''[x]-4*y'[x]+3*y[x]==x^2*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{250}\left(\left(25x^2-20x-67\right)\sin(x)+2\left(25x^2+55x+28\right)\cos(x)\right)+c_1e^x+c_2e^{3x}$$

15.12 problem 13

Internal problem ID [2245]
Internal file name [`OUTPUT/2245_Monday_February_26_2024_09_18_40_AM_83842828/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - y = x \sin(2x)}$$

15.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = x \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(2x), x\sin(2x), \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\cos(2x) + A_2 x\sin(2x) + A_3\cos(2x) + A_4\sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-4A_1\sin(2x) - 5A_1x\cos(2x) + 4A_2\cos(2x)\\&- 5A_2x\sin(2x) - 5A_3\cos(2x) - 5A_4\sin(2x) = x\sin(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{5}, A_3 = -\frac{4}{25}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\sin(2x)}{5} - \frac{4\cos(2x)}{25}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(c_1 e^{x} + c_2 e^{-x}\right) + \left(-\frac{x\sin(2x)}{5} - \frac{4\cos(2x)}{25}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} - \frac{x \sin(2x)}{5} - \frac{4 \cos(2x)}{25} \tag{1}$$

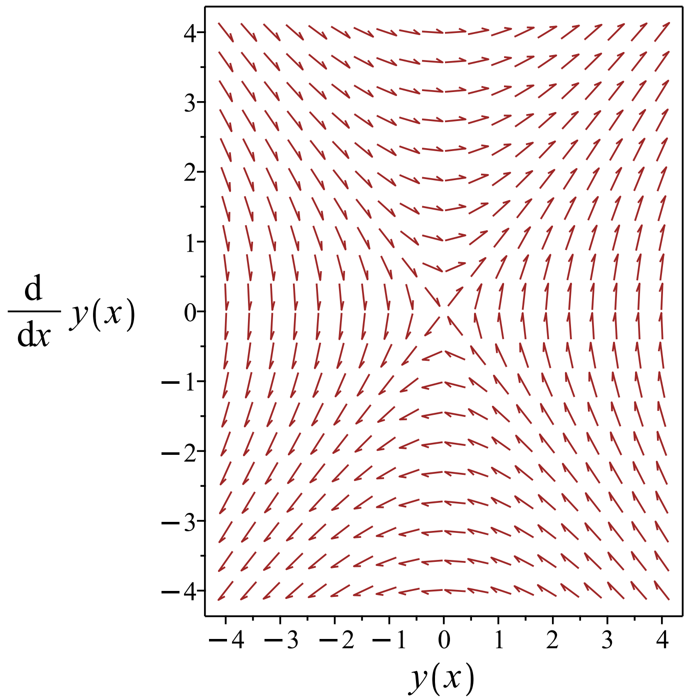


Figure 674: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - \frac{x \sin(2x)}{5} - \frac{4 \cos(2x)}{25}$$

Verified OK.

15.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 531: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} dx \\
&= e^{-x} \int \frac{1}{e^{-2x}} dx \\
&= e^{-x} \left(\frac{e^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(2x), x \sin(2x), \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^x}{2}, e^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x) + A_3 \cos(2x) + A_4 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-4A_1 \sin(2x) - 5A_1 x \cos(2x) + 4A_2 \cos(2x)\\ &\quad - 5A_2 x \sin(2x) - 5A_3 \cos(2x) - 5A_4 \sin(2x) = x \sin(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{5}, A_3 = -\frac{4}{25}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x \sin(2x)}{5} - \frac{4 \cos(2x)}{25}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2}\right) + \left(-\frac{x \sin(2x)}{5} - \frac{4 \cos(2x)}{25}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - \frac{x \sin(2x)}{5} - \frac{4\cos(2x)}{25} \tag{1}$$

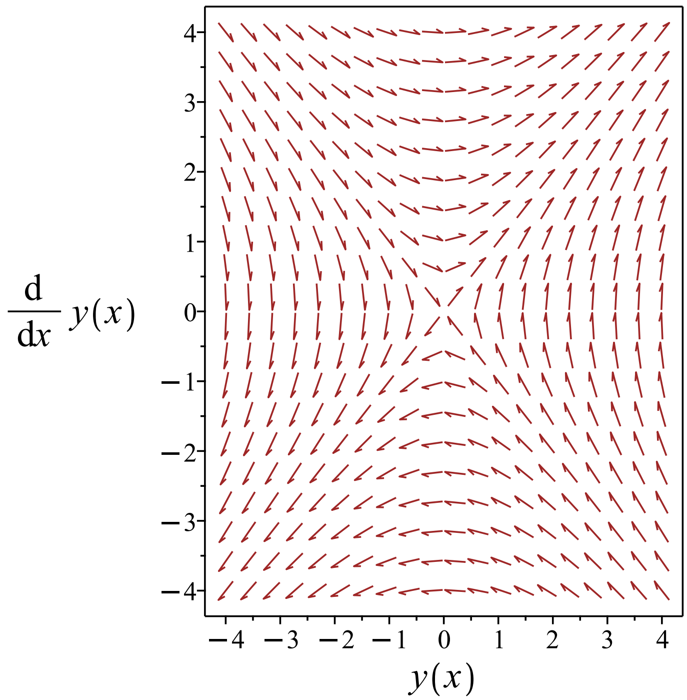


Figure 675: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - \frac{x \sin(2x)}{5} - \frac{4\cos(2x)}{25}$$

Verified OK.

15.12.3 Maple step by step solution

Let's solve

$y'' - y = x \sin(2x)$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x \sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\mathrm{e}^{-x}\left(\int \sin(2x)\mathrm{e}^{x} x dx\right)}{2} + \frac{\mathrm{e}^{x}\left(\int \sin(2x)\mathrm{e}^{-x} x dx\right)}{2}$

  - Compute integrals

    $y_p(x) = -\frac{x \sin(2x)}{5} - \frac{4\cos(2x)}{25}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - \frac{x \sin(2x)}{5} - \frac{4\cos(2x)}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-y(x)=x*sin(2*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} c_2 + \mathrm{e}^{x} c_1 - \frac{4\cos(2x)}{25} - \frac{\sin(2x)\, x}{5}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 37

```
DSolve[y''[x]-y[x]==x*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{5} x \sin(2x) - \frac{4}{25} \cos(2x) + c_1 e^x + c_2 e^{-x}$$

15.13 problem 14

Internal problem ID [2246]
Internal file name [`OUTPUT/2246_Monday_February_26_2024_09_18_41_AM_97191660/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 2y' = x^3 \sin(2x)}$$

15.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 0, f(x) = x^3 \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+2y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=2, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}+2\lambda\, e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+2\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=2, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-2}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{2^2-(4)(1)(0)}\\&=-1\pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-1+1\\\lambda_2&=-1-1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(2x), x \sin(2x), x^2 \cos(2x), x^3 \cos(2x), x^3 \sin(2x), \sin(2x) x^2, \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$\begin{aligned} y_p = {} & A_1 x \cos(2x) + A_2 x \sin(2x) + A_3 x^2 \cos(2x) + A_4 x^3 \cos(2x) \\ & + A_5 x^3 \sin(2x) + A_6 \sin(2x) x^2 + A_7 \cos(2x) + A_8 \sin(2x) \end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} & 4A_3 x \cos(2x) + 6A_4 x^2 \cos(2x) + 6A_5 x^2 \sin(2x) + 4A_6 \sin(2x) x \\ & + 6A_4 x \cos(2x) + 6A_5 x \sin(2x) + 8A_6 \cos(2x) x + 2A_3 \cos(2x) \\ & - 8A_3 x \sin(2x) - 12A_4 x^2 \sin(2x) + 12A_5 x^2 \cos(2x) + 2A_6 \sin(2x) \\ & + 2A_1 \cos(2x) - 4A_1 x \sin(2x) + 2A_2 \sin(2x) + 4A_2 x \cos(2x) \\ & - 4A_3 x^2 \sin(2x) - 4A_4 x^3 \sin(2x) + 4A_5 x^3 \cos(2x) + 4A_6 \cos(2x) x^2 \\ & - 4A_7 \cos(2x) - 4A_8 \sin(2x) - 4A_1 x \cos(2x) - 4A_2 x \sin(2x) \\ & - 4A_3 x^2 \cos(2x) - 4A_4 x^3 \cos(2x) - 4A_5 x^3 \sin(2x) - 4A_6 \sin(2x) x^2 \\ & - 4A_1 \sin(2x) + 4A_2 \cos(2x) - 4A_7 \sin(2x) + 4A_8 \cos(2x) = x^3 \sin(2x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{15}{32}, A_2 = \frac{3}{32}, A_3 = -\frac{3}{16}, A_4 = -\frac{1}{8}, A_5 = -\frac{1}{8}, A_6 = \frac{3}{8}, A_7 = 0, A_8 = -\frac{15}{64}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$\begin{aligned} y_p = &\frac{15x\cos(2x)}{32} + \frac{3x\sin(2x)}{32} - \frac{3x^2\cos(2x)}{16} - \frac{x^3\cos(2x)}{8} \\ &- \frac{x^3\sin(2x)}{8} + \frac{3\sin(2x)\,x^2}{8} - \frac{15\sin(2x)}{64} \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^{-2x}\right) + \left(\frac{15x\cos(2x)}{32} + \frac{3x\sin(2x)}{32} - \frac{3x^2\cos(2x)}{16} - \frac{x^3\cos(2x)}{8} \right. \\ &\qquad \left. - \frac{x^3\sin(2x)}{8} + \frac{3\sin(2x)\,x^2}{8} - \frac{15\sin(2x)}{64}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 + c_2 e^{-2x} &+ \frac{15x\cos(2x)}{32} + \frac{3x\sin(2x)}{32} - \frac{3x^2\cos(2x)}{16} \\ &- \frac{x^3\cos(2x)}{8} - \frac{x^3\sin(2x)}{8} + \frac{3\sin(2x)\,x^2}{8} - \frac{15\sin(2x)}{64} \end{aligned} \tag{1}$$

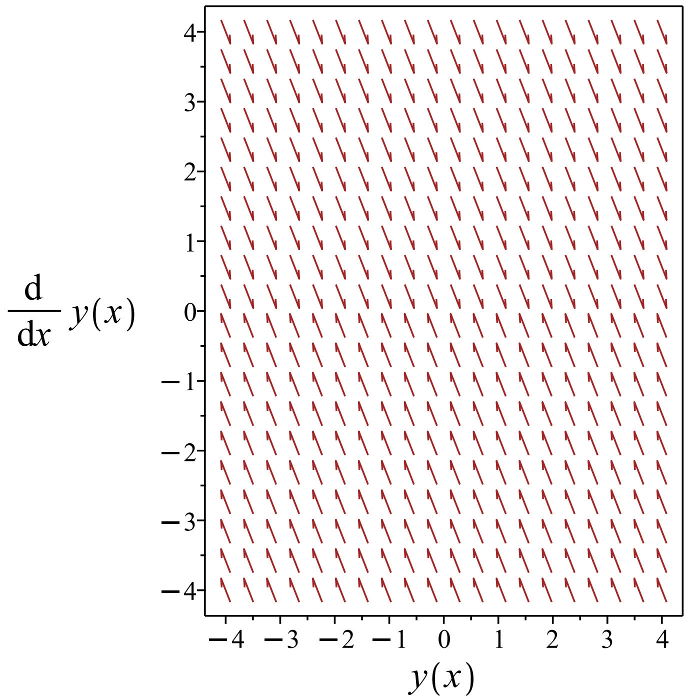


Figure 676: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-2x} + \frac{15x\cos(2x)}{32} + \frac{3x\sin(2x)}{32} - \frac{3x^2\cos(2x)}{16} - \frac{x^3\cos(2x)}{8} - \frac{x^3\sin(2x)}{8} + \frac{3\sin(2x)\,x^2}{8} - \frac{15\sin(2x)}{64}$$

Verified OK.

15.13.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\,dx = \int x^3\sin(2x)\,dx$$

$$2y + y' = -\frac{x^3\cos(2x)}{2} + \frac{3\sin(2x)\,x^2}{4} - \frac{3\sin(2x)}{8} + \frac{3x\cos(2x)}{4} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 2\\
q(x) &= \frac{(-4x^3+6x)\cos(2x)}{8}+\frac{3\sin(2x)\,x^2}{4}+c_1-\frac{3\sin(2x)}{8}
\end{aligned}
$$

Hence the ode is

$$
2y+y' = \frac{(-4x^3+6x)\cos(2x)}{8}+\frac{3\sin(2x)\,x^2}{4}+c_1-\frac{3\sin(2x)}{8}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 2dx}\\
&= e^{2x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{(-4x^3+6x)\cos(2x)}{8}+\frac{3\sin(2x)\,x^2}{4}+c_1-\frac{3\sin(2x)}{8}\right)\\
\frac{d}{dx}(e^{2x}y) &= (e^{2x})\left(\frac{(-4x^3+6x)\cos(2x)}{8}+\frac{3\sin(2x)\,x^2}{4}+c_1-\frac{3\sin(2x)}{8}\right)\\
d(e^{2x}y) &= \left(-\frac{(4x^3\cos(2x)-6\sin(2x)\,x^2-6x\cos(2x)+3\sin(2x)-8c_1)\,e^{2x}}{8}\right)dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{2x}y &= \int -\frac{(4x^3\cos(2x)-6\sin(2x)\,x^2-6x\cos(2x)+3\sin(2x)-8c_1)\,e^{2x}}{8}\,dx\\
e^{2x}y &= -\frac{(4x^3-3x+\frac{3}{2})\,e^{2x}\cos(2x)}{32}+\frac{(-4x^3+6x^2-3x)\,e^{2x}\sin(2x)}{32}+\frac{3(-2x^2+2x-\frac{1}{2})\,e^{2x}\cos(2x)}{32}+
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$
y = e^{-2x}\left(-\frac{(4x^3-3x+\frac{3}{2})\,e^{2x}\cos(2x)}{32}+\frac{(-4x^3+6x^2-3x)\,e^{2x}\sin(2x)}{32}+\frac{3(-2x^2+2x-\frac{1}{2})\,e^{2x}\cos(2x}{32}\right.
$$

which simplifies to

$$
y = \frac{(-8x^3+24x^2+6x-15)\sin(2x)}{64}+\frac{(-4x^3-6x^2+15x)\cos(2x)}{32}+c_2e^{-2x}+\frac{c_1}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(-8x^3+24x^2+6x-15)\sin(2x)}{64}+\frac{(-4x^3-6x^2+15x)\cos(2x)}{32}+c_2e^{-2x}+\frac{c_1}{2} \tag{1}
$$

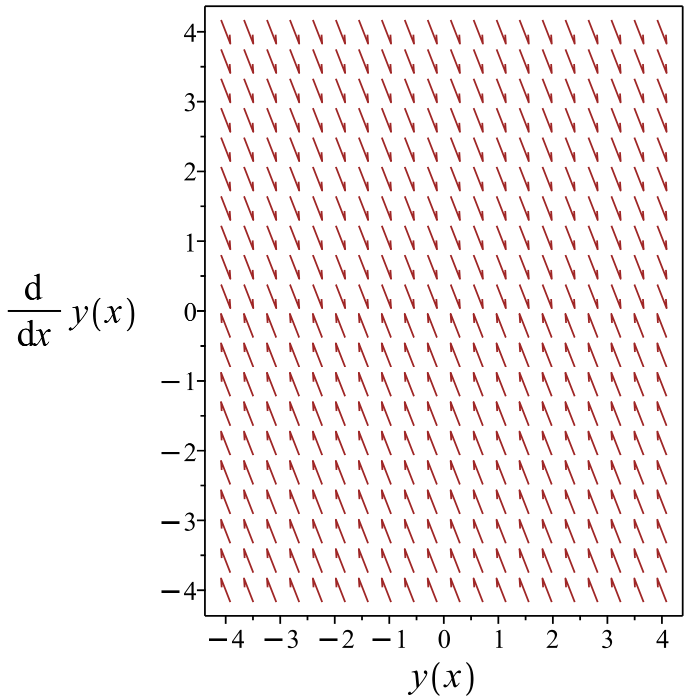


Figure 677: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{(-8x^3 + 24x^2 + 6x - 15)\sin(2x)}{64} + \frac{(-4x^3 - 6x^2 + 15x)\cos(2x)}{32} + c_2 e^{-2x} + \frac{c_1}{2}$$

Verified OK.

15.13.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 2p(x) - x^3 \sin(2x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= 2\\ q(x) &= x^3 \sin(2x)\end{aligned}$$

Hence the ode is

$$p'(x) + 2p(x) = x^3 \sin(2x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2dx}\\ &= e^{2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu)\left(x^3 \sin(2x)\right)\\ \frac{d}{dx}\left(e^{2x} p\right) &= \left(e^{2x}\right)\left(x^3 \sin(2x)\right)\\ d\left(e^{2x} p\right) &= \left(x^3 \sin(2x) e^{2x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2x} p &= \int x^3 \sin(2x) e^{2x} dx\\ e^{2x} p &= \frac{(-4x^3 + 6x^2 - 3x) e^{2x} \cos(2x)}{16} + \frac{\left(4x^3 - 3x + \frac{3}{2}\right) e^{2x} \sin(2x)}{16} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$p(x) = e^{-2x}\left(\frac{(-4x^3 + 6x^2 - 3x) e^{2x} \cos(2x)}{16} + \frac{\left(4x^3 - 3x + \frac{3}{2}\right) e^{2x} \sin(2x)}{16}\right) + c_1 e^{-2x}$$

which simplifies to

$$p(x) = \frac{(-4x^3 + 6x^2 - 3x) \cos(2x)}{16} + \frac{(8x^3 - 6x + 3) \sin(2x)}{32} + c_1 e^{-2x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{(-4x^3 + 6x^2 - 3x) \cos(2x)}{16} + \frac{(8x^3 - 6x + 3) \sin(2x)}{32} + c_1 e^{-2x}$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{x^{3} \sin (2x)}{4}-\frac{x^{3} \cos (2x)}{4}+\frac{3x^{2} \cos (2x)}{8}-\frac{3x \sin (2x)}{16}-\frac{3x \cos (2x)}{16}+\frac{3 \sin (2x)}{32}+c_{1} \mathrm{e}^{-2x} \,\mathrm{d}x \\
&= \frac{3 \sin (2x)\, x^{2}}{8}-\frac{15 \sin (2x)}{64}+\frac{15x \cos (2x)}{32}-\frac{x^{3} \cos (2x)}{8}-\frac{c_{1} \mathrm{e}^{-2x}}{2}+\frac{3x \sin (2x)}{32}-\frac{x^{3} \sin (2x)}{8}-3x^{2}\, \mathrm{c}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{3 \sin (2x)\, x^{2}}{8}-\frac{15 \sin (2x)}{64}+\frac{15x \cos (2x)}{32}-\frac{x^{3} \cos (2x)}{8} \\
& -\frac{c_{1} \mathrm{e}^{-2x}}{2}+\frac{3x \sin (2x)}{32}-\frac{x^{3} \sin (2x)}{8}-\frac{3x^{2} \cos (2x)}{16}+c_{2}
\end{aligned} \tag{1}
$$

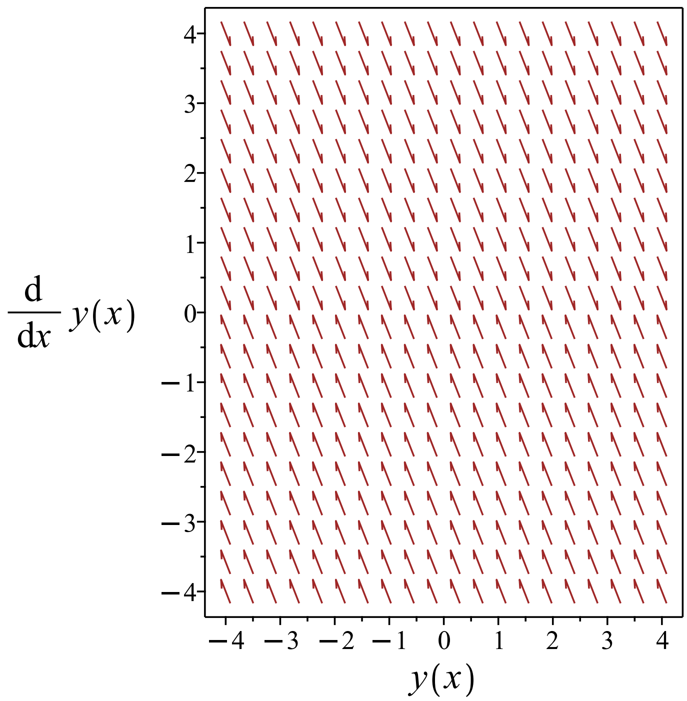


Figure 678: Slope field plot

<u>Verification of solutions</u>

$$
\begin{aligned}
y = {} & \frac{3 \sin (2x)\, x^{2}}{8}-\frac{15 \sin (2x)}{64}+\frac{15x \cos (2x)}{32}-\frac{x^{3} \cos (2x)}{8} \\
& -\frac{c_{1} \mathrm{e}^{-2x}}{2}+\frac{3x \sin (2x)}{32}-\frac{x^{3} \sin (2x)}{8}-\frac{3x^{2} \cos (2x)}{16}+c_{2}
\end{aligned}
$$

Verified OK.

15.13.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 2y' = x^3 \sin(2x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\,dx = \int x^3 \sin(2x)\,dx$$

$$2y + y' = -\frac{x^3 \cos(2x)}{2} + \frac{3 \sin(2x)\,x^2}{4} - \frac{3 \sin(2x)}{8} + \frac{3x \cos(2x)}{4} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = \frac{(-4x^3 + 6x)\cos(2x)}{8} + \frac{3 \sin(2x)\,x^2}{4} + c_1 - \frac{3 \sin(2x)}{8}$$

Hence the ode is

$$2y + y' = \frac{(-4x^3 + 6x)\cos(2x)}{8} + \frac{3 \sin(2x)\,x^2}{4} + c_1 - \frac{3 \sin(2x)}{8}$$

The integrating factor μ is

$$\mu = e^{\int 2dx}$$
$$= e^{2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{(-4x^3 + 6x)\cos(2x)}{8} + \frac{3 \sin(2x)\,x^2}{4} + c_1 - \frac{3 \sin(2x)}{8}\right)$$
$$\frac{d}{dx}(e^{2x} y) = (e^{2x})\left(\frac{(-4x^3 + 6x)\cos(2x)}{8} + \frac{3 \sin(2x)\,x^2}{4} + c_1 - \frac{3 \sin(2x)}{8}\right)$$
$$d(e^{2x} y) = \left(-\frac{(4x^3 \cos(2x) - 6 \sin(2x)\,x^2 - 6x \cos(2x) + 3 \sin(2x) - 8c_1)\,e^{2x}}{8}\right) dx$$

Integrating gives

$$e^{2x}y = \int -\frac{(4x^3\cos(2x) - 6\sin(2x)x^2 - 6x\cos(2x) + 3\sin(2x) - 8c_1)e^{2x}}{8}\,dx$$

$$e^{2x}y = -\frac{\left(4x^3 - 3x + \frac{3}{2}\right)e^{2x}\cos(2x)}{32} + \frac{(-4x^3 + 6x^2 - 3x)e^{2x}\sin(2x)}{32} + \frac{3\left(-2x^2 + 2x - \frac{1}{2}\right)e^{2x}\cos(2x)}{32} +$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = e^{-2x}\left(-\frac{\left(4x^3 - 3x + \frac{3}{2}\right)e^{2x}\cos(2x)}{32} + \frac{(-4x^3 + 6x^2 - 3x)e^{2x}\sin(2x)}{32} + \frac{3\left(-2x^2 + 2x - \frac{1}{2}\right)e^{2x}\cos(2x}{32}\right.$$

which simplifies to

$$y = \frac{(-8x^3 + 24x^2 + 6x - 15)\sin(2x)}{64} + \frac{(-4x^3 - 6x^2 + 15x)\cos(2x)}{32} + c_2e^{-2x} + \frac{c_1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(-8x^3 + 24x^2 + 6x - 15)\sin(2x)}{64} + \frac{(-4x^3 - 6x^2 + 15x)\cos(2x)}{32} + c_2e^{-2x} + \frac{c_1}{2} \tag{1}$$

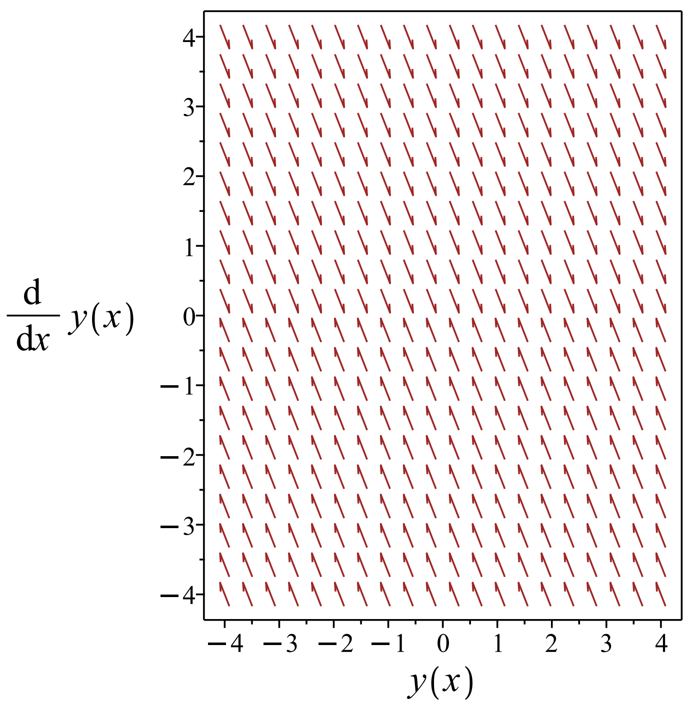


Figure 679: Slope field plot

Verification of solutions

$$y = \frac{(-8x^3 + 24x^2 + 6x - 15)\sin(2x)}{64} + \frac{(-4x^3 - 6x^2 + 15x)\cos(2x)}{32} + c_2 \mathrm{e}^{-2x} + \frac{c_1}{2}$$

Verified OK.

15.13.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 533: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+2y'=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(2x), x\sin(2x), x^2\cos(2x), x^3\cos(2x), x^3\sin(2x), \sin(2x)x^2, \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{1}{2}, \mathrm{e}^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$\begin{aligned}y_p = {} & A_1x\cos(2x) + A_2x\sin(2x) + A_3x^2\cos(2x) + A_4x^3\cos(2x) \\ & + A_5x^3\sin(2x) + A_6\sin(2x)x^2 + A_7\cos(2x) + A_8\sin(2x)\end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-4A_7\sin(2x) + 4A_8\cos(2x) - 4A_1\sin(2x) + 4A_2\cos(2x) + 2A_6\sin(2x) \\ &+ 2A_1\cos(2x) - 4A_1x\sin(2x) + 2A_2\sin(2x) + 4A_2x\cos(2x) \\ &- 4A_3x^2\sin(2x) - 4A_4x^3\sin(2x) + 4A_5x^3\cos(2x) + 4A_6\cos(2x)x^2 \\ &+ 2A_3\cos(2x) - 8A_3x\sin(2x) - 12A_4x^2\sin(2x) + 12A_5x^2\cos(2x) \\ &- 4A_7\cos(2x) - 4A_8\sin(2x) - 4A_1x\cos(2x) - 4A_2x\sin(2x) \\ &- 4A_3x^2\cos(2x) - 4A_4x^3\cos(2x) - 4A_5x^3\sin(2x) - 4A_6\sin(2x)x^2 \\ &+ 6A_4x\cos(2x) + 6A_5x\sin(2x) + 8A_6\cos(2x)x + 4A_3x\cos(2x) \\ &+ 6A_4x^2\cos(2x) + 6A_5x^2\sin(2x) + 4A_6\sin(2x)x = x^3\sin(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{15}{32}, A_2 = \frac{3}{32}, A_3 = -\frac{3}{16}, A_4 = -\frac{1}{8}, A_5 = -\frac{1}{8}, A_6 = \frac{3}{8}, A_7 = 0, A_8 = -\frac{15}{64}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$\begin{aligned} y_p = {} & \frac{15x\cos(2x)}{32} + \frac{3x\sin(2x)}{32} - \frac{3x^2\cos(2x)}{16} - \frac{x^3\cos(2x)}{8} \\ & - \frac{x^3\sin(2x)}{8} + \frac{3\sin(2x)\,x^2}{8} - \frac{15\sin(2x)}{64} \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2}{2}\right) + \left(\frac{15x\cos(2x)}{32} + \frac{3x\sin(2x)}{32} - \frac{3x^2\cos(2x)}{16} - \frac{x^3\cos(2x)}{8} \right. \\ & \qquad \left. - \frac{x^3\sin(2x)}{8} + \frac{3\sin(2x)\,x^2}{8} - \frac{15\sin(2x)}{64}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1 e^{-2x} + \frac{c_2}{2} + \frac{15x\cos(2x)}{32} + \frac{3x\sin(2x)}{32} - \frac{3x^2\cos(2x)}{16} \\ & - \frac{x^3\cos(2x)}{8} - \frac{x^3\sin(2x)}{8} + \frac{3\sin(2x)\,x^2}{8} - \frac{15\sin(2x)}{64} \end{aligned} \tag{1}$$

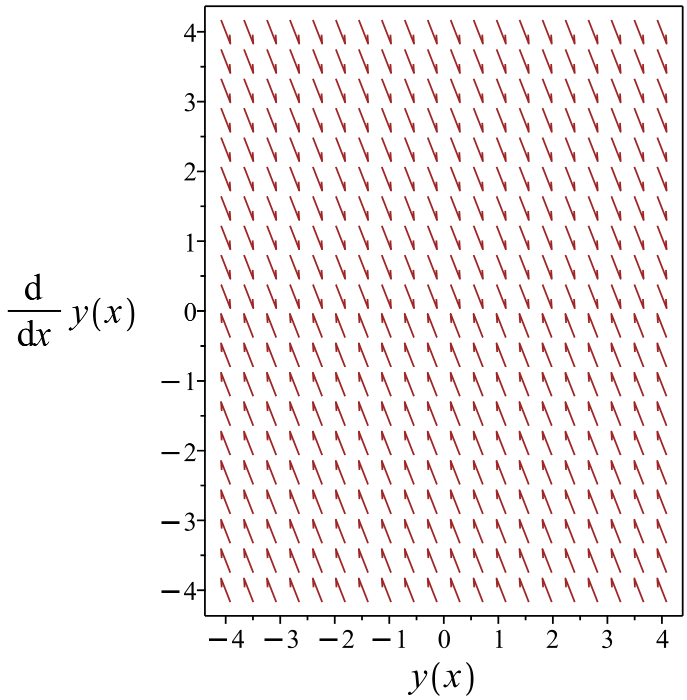


Figure 680: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2}{2} + \frac{15x\cos(2x)}{32} + \frac{3x\sin(2x)}{32} - \frac{3x^2\cos(2x)}{16} - \frac{x^3\cos(2x)}{8} - \frac{x^3\sin(2x)}{8} + \frac{3\sin(2x)\,x^2}{8} - \frac{15\sin(2x)}{64}$$

Verified OK.

15.13.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= 1\\ q(x) &= 2\\ r(x) &= 0\\ s(x) &= x^3 \sin(2x)\end{aligned}$$

Hence

$$\begin{aligned}p''(x) &= 0\\ q'(x) &= 0\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$2y + y' = \int x^3 \sin(2x)\, dx$$

We now have a first order ode to solve which is

$$2y + y' = -\frac{x^3 \cos(2x)}{2} + \frac{3 \sin(2x)\, x^2}{4} - \frac{3 \sin(2x)}{8} + \frac{3x \cos(2x)}{4} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 2\\ q(x) &= \frac{(-4x^3 + 6x)\cos(2x)}{8} + \frac{3 \sin(2x)\, x^2}{4} + c_1 - \frac{3 \sin(2x)}{8}\end{aligned}$$

Hence the ode is

$$2y + y' = \frac{(-4x^3 + 6x)\cos(2x)}{8} + \frac{3\sin(2x)\,x^2}{4} + c_1 - \frac{3\sin(2x)}{8}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 2dx}\\ &= \mathrm{e}^{2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(-4x^3 + 6x)\cos(2x)}{8} + \frac{3\sin(2x)\,x^2}{4} + c_1 - \frac{3\sin(2x)}{8}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}y) &= (\mathrm{e}^{2x})\left(\frac{(-4x^3 + 6x)\cos(2x)}{8} + \frac{3\sin(2x)\,x^2}{4} + c_1 - \frac{3\sin(2x)}{8}\right)\\ \mathrm{d}(\mathrm{e}^{2x}y) &= \left(-\frac{(4x^3\cos(2x) - 6\sin(2x)\,x^2 - 6x\cos(2x) + 3\sin(2x) - 8c_1)\,\mathrm{e}^{2x}}{8}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{2x}y &= \int -\frac{(4x^3\cos(2x) - 6\sin(2x)\,x^2 - 6x\cos(2x) + 3\sin(2x) - 8c_1)\,\mathrm{e}^{2x}}{8}\,\mathrm{d}x\\ \mathrm{e}^{2x}y &= -\frac{\left(4x^3 - 3x + \frac{3}{2}\right)\mathrm{e}^{2x}\cos(2x)}{32} + \frac{(-4x^3 + 6x^2 - 3x)\,\mathrm{e}^{2x}\sin(2x)}{32} + \frac{3\left(-2x^2 + 2x - \frac{1}{2}\right)\mathrm{e}^{2x}\cos(2x)}{32} +\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$y = \mathrm{e}^{-2x}\left(-\frac{\left(4x^3 - 3x + \frac{3}{2}\right)\mathrm{e}^{2x}\cos(2x)}{32} + \frac{(-4x^3 + 6x^2 - 3x)\,\mathrm{e}^{2x}\sin(2x)}{32} + \frac{3\left(-2x^2 + 2x - \frac{1}{2}\right)\mathrm{e}^{2x}\cos(2x}{32}\right.$$

which simplifies to

$$y = \frac{(-8x^3 + 24x^2 + 6x - 15)\sin(2x)}{64} + \frac{(-4x^3 - 6x^2 + 15x)\cos(2x)}{32} + c_2\mathrm{e}^{-2x} + \frac{c_1}{2}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{(-8x^3 + 24x^2 + 6x - 15)\sin(2x)}{64} + \frac{(-4x^3 - 6x^2 + 15x)\cos(2x)}{32} + c_2\mathrm{e}^{-2x} + \frac{c_1}{2} \quad (1)$$

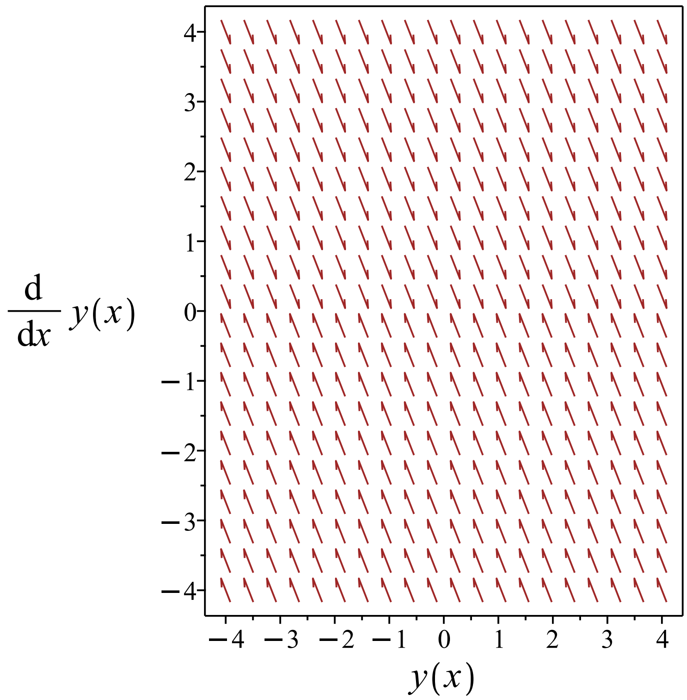


Figure 681: Slope field plot

Verification of solutions

$$y = \frac{(-8x^3 + 24x^2 + 6x - 15)\sin(2x)}{64} + \frac{(-4x^3 - 6x^2 + 15x)\cos(2x)}{32} + c_2 e^{-2x} + \frac{c_1}{2}$$

Verified OK.

15.13.7 Maple step by step solution

Let's solve

$y'' + 2y' = x^3 \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r = 0$

- Factor the characteristic polynomial

  $r(r + 2) = 0$

- Roots of the characteristic polynomial

$r = (-2, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^3 \sin(2x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & 1 \\ -2\,\mathrm{e}^{-2x} & 0 \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-2x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-2x}\left(\int x^3 \sin(2x)\mathrm{e}^{2x}dx\right)}{2} + \frac{\left(\int x^3 \sin(2x)dx\right)}{2}$

- ○ Compute integrals

  $y_p(x) = \frac{(-8x^3+24x^2+6x-15)\sin(2x)}{64} - \frac{\left(x^2+\frac{3}{2}x-\frac{15}{4}\right)x\cos(2x)}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 + \frac{(-8x^3+24x^2+6x-15)\sin(2x)}{64} - \frac{\left(x^2+\frac{3}{2}x-\frac{15}{4}\right)x\cos(2x)}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^3*sin(2*_a)-2*_b(_a), _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)=x^3*sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(-8x^3 + 24x^2 + 6x - 15)\sin(2x)}{64} + \frac{(-4x^3 - 6x^2 + 15x)\cos(2x)}{32} - \frac{e^{-2x}c_1}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.435 (sec). Leaf size: 61

```
DSolve[y''[x]+2*y'[x]==x^3*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{32}x(4x^2+6x-15)\cos(2x)+\frac{1}{64}(-8x^3+24x^2+6x-15)\sin(2x)-\frac{1}{2}c_1e^{-2x}+c_2$$

15.14 problem 15

Internal problem ID [2247]
Internal file name [`OUTPUT/2247_Monday_February_26_2024_09_18_43_AM_30563614/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - y' = e^{2x} \sin(x) x}$$

15.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = 0, f(x) = e^{2x} \sin(x) x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)}\\ &= \frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(0)x}\end{aligned}$$

Or

$$y = c_1 e^x + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x} \sin(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x} \cos(x), e^{2x} \sin(x), e^{2x} \cos(x) x, e^{2x} \sin(x) x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} \cos(x) + A_2 e^{2x} \sin(x) + A_3 e^{2x} \cos(x) x + A_4 e^{2x} \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} A_1 e^{2x} \cos(x) - 3A_1 e^{2x} \sin(x) + A_2 e^{2x} \sin(x) + 3A_2 e^{2x} \cos(x) + A_3 e^{2x} \cos(x) x \\ - 3A_3 e^{2x} \sin(x) x + 3A_3 e^{2x} \cos(x) + A_4 e^{2x} \sin(x) x + 3A_4 e^{2x} \cos(x) x \\ + 3A_4 e^{2x} \sin(x) - 2A_3 e^{2x} \sin(x) + 2A_4 e^{2x} \cos(x) = e^{2x} \sin(x) x \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{17}{50}, A_2 = \frac{3}{25}, A_3 = -\frac{3}{10}, A_4 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{17 e^{2x} \cos(x)}{50} + \frac{3 e^{2x} \sin(x)}{25} - \frac{3 e^{2x} \cos(x) x}{10} + \frac{e^{2x} \sin(x) x}{10}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= (c_1 e^x + c_2) + \left(\frac{17\, e^{2x} \cos(x)}{50} + \frac{3\, e^{2x} \sin(x)}{25} - \frac{3\, e^{2x} \cos(x)\, x}{10} + \frac{e^{2x} \sin(x)\, x}{10}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 + \frac{17\, e^{2x} \cos(x)}{50} + \frac{3\, e^{2x} \sin(x)}{25} - \frac{3\, e^{2x} \cos(x)\, x}{10} + \frac{e^{2x} \sin(x)\, x}{10} \tag{1}$$

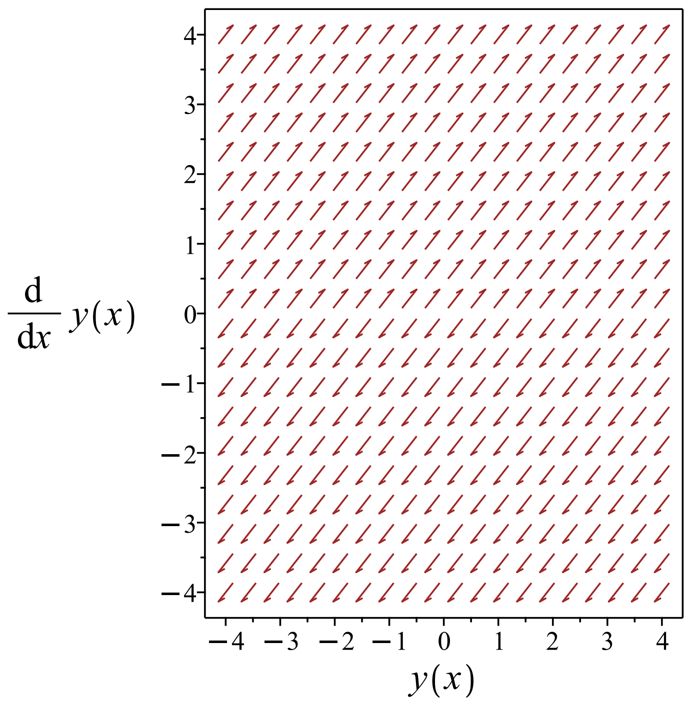


Figure 682: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 + \frac{17\, e^{2x} \cos(x)}{50} + \frac{3\, e^{2x} \sin(x)}{25} - \frac{3\, e^{2x} \cos(x)\, x}{10} + \frac{e^{2x} \sin(x)\, x}{10}$$

Verified OK.

15.14.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - y')\, dx = \int e^{2x} \sin(x)\, x dx$$

$$y' - y = \left(-\frac{x}{5} + \frac{4}{25}\right) e^{2x} \cos(x) + \left(\frac{2x}{5} - \frac{3}{25}\right) e^{2x} \sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= \frac{((4-5x)\cos(x) + \sin(x)(10x-3))\, e^{2x}}{25} + c_1 \end{aligned}$$

Hence the ode is

$$y' - y = \frac{((4-5x)\cos(x) + \sin(x)(10x-3))\, e^{2x}}{25} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1) dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{((4-5x)\cos(x) + \sin(x)(10x-3))\, e^{2x}}{25} + c_1\right) \\ \frac{d}{dx}\left(e^{-x} y\right) &= \left(e^{-x}\right)\left(\frac{((4-5x)\cos(x) + \sin(x)(10x-3))\, e^{2x}}{25} + c_1\right) \\ d\left(e^{-x} y\right) &= \left(c_1 e^{-x} - \frac{\left(\left(-\frac{4}{5} + x\right)\cos(x) + \sin(x)\left(\frac{3}{5} - 2x\right)\right) e^{x}}{5}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-x} y &= \int c_1 e^{-x} - \frac{\left(\left(-\frac{4}{5} + x\right)\cos(x) + \sin(x)\left(\frac{3}{5} - 2x\right)\right) e^{x}}{5}\, dx \\ e^{-x} y &= -c_1 e^{-x} - \frac{e^{x}\cos(x)\, x}{10} + \frac{\left(-\frac{x}{2} + \frac{1}{2}\right) e^{x} \sin(x)}{5} + \frac{2\left(-\frac{x}{2} + \frac{1}{2}\right) e^{x} \cos(x)}{5} + \frac{e^{x}\sin(x)\, x}{5} + \frac{7\cos(x)\, e^{x}}{50} + \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$y = \mathrm{e}^{x}\left(-c_1\mathrm{e}^{-x} - \frac{\mathrm{e}^{x}\cos\left(x\right)x}{10} + \frac{\left(-\frac{x}{2}+\frac{1}{2}\right)\mathrm{e}^{x}\sin\left(x\right)}{5} + \frac{2\left(-\frac{x}{2}+\frac{1}{2}\right)\mathrm{e}^{x}\cos\left(x\right)}{5} + \frac{\mathrm{e}^{x}\sin\left(x\right)x}{5} + \frac{7\cos\left(x\right)\mathrm{e}^{x}}{50}\right.$$

which simplifies to

$$y = \frac{\left(\left(-15x+17\right)\cos\left(x\right)+\sin\left(x\right)\left(5x+6\right)\right)\mathrm{e}^{2x}}{50} + c_2\mathrm{e}^{x} - c_1$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\left(-15x+17\right)\cos\left(x\right)+\sin\left(x\right)\left(5x+6\right)\right)\mathrm{e}^{2x}}{50} + c_2\mathrm{e}^{x} - c_1 \tag{1}$$

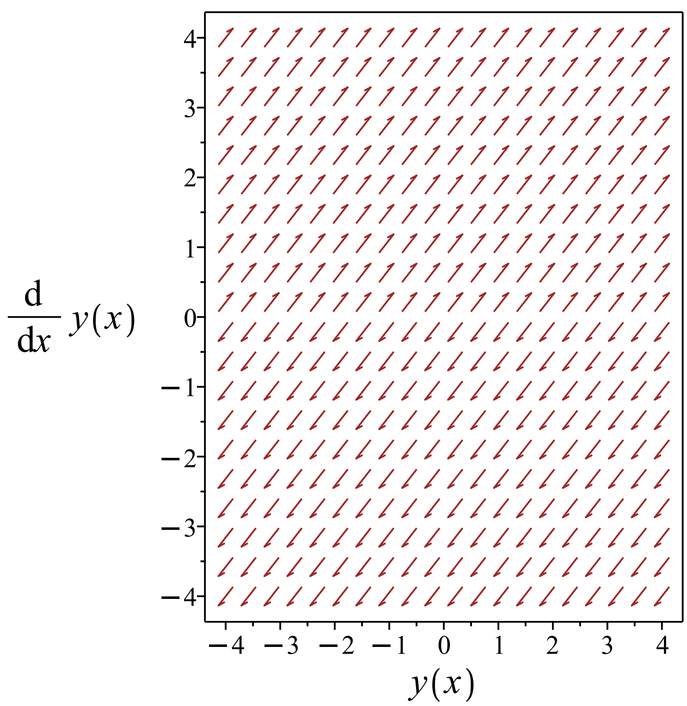


Figure 683: Slope field plot

Verification of solutions

$$y = \frac{\left(\left(-15x+17\right)\cos\left(x\right)+\sin\left(x\right)\left(5x+6\right)\right)\mathrm{e}^{2x}}{50} + c_2\mathrm{e}^{x} - c_1$$

Verified OK.

15.14.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$-p(x) + p'(x) - e^{2x} \sin(x) x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= e^{2x} \sin(x) x \end{aligned}$$

Hence the ode is

$$-p(x) + p'(x) = e^{2x} \sin(x) x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1)dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu) \left(e^{2x} \sin(x) x\right) \\ \frac{d}{dx}\left(e^{-x} p\right) &= \left(e^{-x}\right) \left(e^{2x} \sin(x) x\right) \\ d\left(e^{-x} p\right) &= \left(e^{x} \sin(x) x\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-x} p &= \int e^{x} \sin(x) x \, dx \\ e^{-x} p &= \left(-\frac{x}{2} + \frac{1}{2}\right) e^{x} \cos(x) + \frac{e^{x} \sin(x) x}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$p(x) = \mathrm{e}^{x}\left(\left(-\frac{x}{2}+\frac{1}{2}\right)\mathrm{e}^{x}\cos(x)+\frac{\mathrm{e}^{x}\sin(x)\,x}{2}\right)+c_1\mathrm{e}^{x}$$

which simplifies to

$$p(x) = \frac{\mathrm{e}^{2x}((1-x)\cos(x)+\sin(x)\,x)}{2}+c_1\mathrm{e}^{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{\mathrm{e}^{2x}((1-x)\cos(x)+\sin(x)\,x)}{2}+c_1\mathrm{e}^{x}$$

Integrating both sides gives

$$\begin{aligned}y &= \int -\frac{\mathrm{e}^{2x}\cos(x)\,x}{2}+\frac{\mathrm{e}^{2x}\sin(x)\,x}{2}+\frac{\mathrm{e}^{2x}\cos(x)}{2}+c_1\mathrm{e}^{x}\,\mathrm{d}x\\ &= \frac{\mathrm{e}^{2x}\cos(x)}{5}+\frac{\mathrm{e}^{2x}\sin(x)}{10}+\frac{\left(-\frac{x}{5}+\frac{4}{25}\right)\mathrm{e}^{2x}\cos(x)}{2}+\frac{\left(\frac{2x}{5}-\frac{3}{25}\right)\mathrm{e}^{2x}\sin(x)}{2}+c_1\mathrm{e}^{x}-\frac{\left(\frac{2x}{5}-\frac{3}{25}\right)\mathrm{e}^{2x}\cos(x)}{2}\end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned}y = {}&\frac{\mathrm{e}^{2x}\cos(x)}{5}+\frac{\mathrm{e}^{2x}\sin(x)}{10}+\frac{\left(-\frac{x}{5}+\frac{4}{25}\right)\mathrm{e}^{2x}\cos(x)}{2}+\frac{\left(\frac{2x}{5}-\frac{3}{25}\right)\mathrm{e}^{2x}\sin(x)}{2}\\ &+c_1\mathrm{e}^{x}-\frac{\left(\frac{2x}{5}-\frac{3}{25}\right)\mathrm{e}^{2x}\cos(x)}{2}+\frac{\left(-\frac{x}{5}+\frac{4}{25}\right)\mathrm{e}^{2x}\sin(x)}{2}+c_2\end{aligned} \tag{1}$$

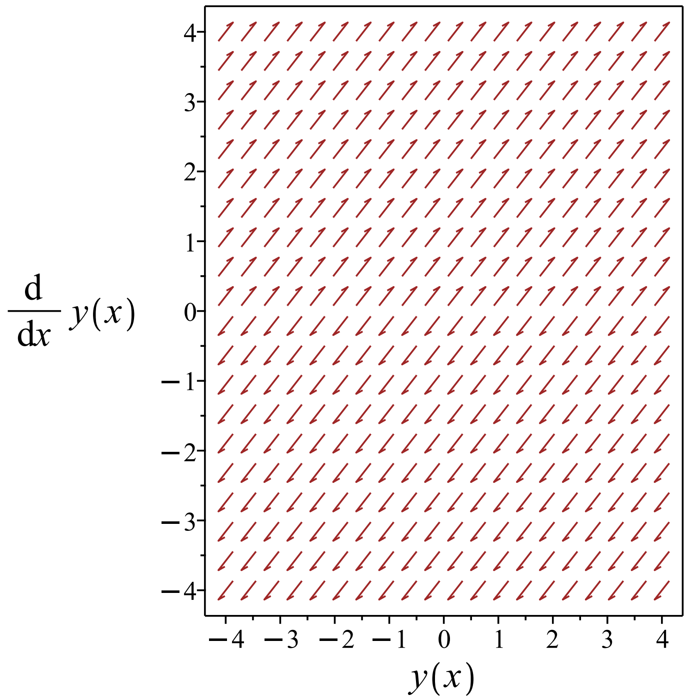


Figure 684: Slope field plot

Verification of solutions

$$y = \frac{e^{2x}\cos(x)}{5} + \frac{e^{2x}\sin(x)}{10} + \frac{\left(-\frac{x}{5}+\frac{4}{25}\right)e^{2x}\cos(x)}{2} + \frac{\left(\frac{2x}{5}-\frac{3}{25}\right)e^{2x}\sin(x)}{2} + c_1 e^{x} - \frac{\left(\frac{2x}{5}-\frac{3}{25}\right)e^{2x}\cos(x)}{2} + \frac{\left(-\frac{x}{5}+\frac{4}{25}\right)e^{2x}\sin(x)}{2} + c_2$$

Verified OK.

15.14.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - y' = e^{2x}\sin(x)\,x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - y')\,dx = \int e^{2x}\sin(x)\,x dx$$

$$y' - y = \left(-\frac{x}{5}+\frac{4}{25}\right)e^{2x}\cos(x) + \left(\frac{2x}{5}-\frac{3}{25}\right)e^{2x}\sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -1\\ q(x) &= \frac{((4-5x)\cos(x) + \sin(x)(10x-3))\,\mathrm{e}^{2x}}{25} + c_1\end{aligned}$$

Hence the ode is

$$y' - y = \frac{((4-5x)\cos(x) + \sin(x)(10x-3))\,\mathrm{e}^{2x}}{25} + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int(-1)dx}\\ &= \mathrm{e}^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{((4-5x)\cos(x) + \sin(x)(10x-3))\,\mathrm{e}^{2x}}{25} + c_1\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x}y) &= (\mathrm{e}^{-x})\left(\frac{((4-5x)\cos(x) + \sin(x)(10x-3))\,\mathrm{e}^{2x}}{25} + c_1\right)\\ \mathrm{d}(\mathrm{e}^{-x}y) &= \left(c_1\mathrm{e}^{-x} - \frac{\left(\left(-\frac{4}{5}+x\right)\cos(x) + \sin(x)\left(\frac{3}{5}-2x\right)\right)\mathrm{e}^{x}}{5}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-x}y &= \int c_1\mathrm{e}^{-x} - \frac{\left(\left(-\frac{4}{5}+x\right)\cos(x) + \sin(x)\left(\frac{3}{5}-2x\right)\right)\mathrm{e}^{x}}{5}\,\mathrm{d}x\\ \mathrm{e}^{-x}y &= -c_1\mathrm{e}^{-x} - \frac{\mathrm{e}^{x}\cos(x)\,x}{10} + \frac{\left(-\frac{x}{2}+\frac{1}{2}\right)\mathrm{e}^{x}\sin(x)}{5} + \frac{2\left(-\frac{x}{2}+\frac{1}{2}\right)\mathrm{e}^{x}\cos(x)}{5} + \frac{\mathrm{e}^{x}\sin(x)\,x}{5} + \frac{7\cos(x)\,\mathrm{e}^{x}}{50} +\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$y = \mathrm{e}^{x}\left(-c_1\mathrm{e}^{-x} - \frac{\mathrm{e}^{x}\cos(x)\,x}{10} + \frac{\left(-\frac{x}{2}+\frac{1}{2}\right)\mathrm{e}^{x}\sin(x)}{5} + \frac{2\left(-\frac{x}{2}+\frac{1}{2}\right)\mathrm{e}^{x}\cos(x)}{5} + \frac{\mathrm{e}^{x}\sin(x)\,x}{5} + \frac{7\cos(x)\,\mathrm{e}^{x}}{50}\right.$$

which simplifies to

$$y = \frac{((-15x+17)\cos(x) + \sin(x)(5x+6))\,\mathrm{e}^{2x}}{50} + c_2\mathrm{e}^{x} - c_1$$

Summary

The solution(s) found are the following

$$y = \frac{((-15x+17)\cos(x)+\sin(x)(5x+6))\,e^{2x}}{50}+c_2e^{x}-c_1 \tag{1}$$

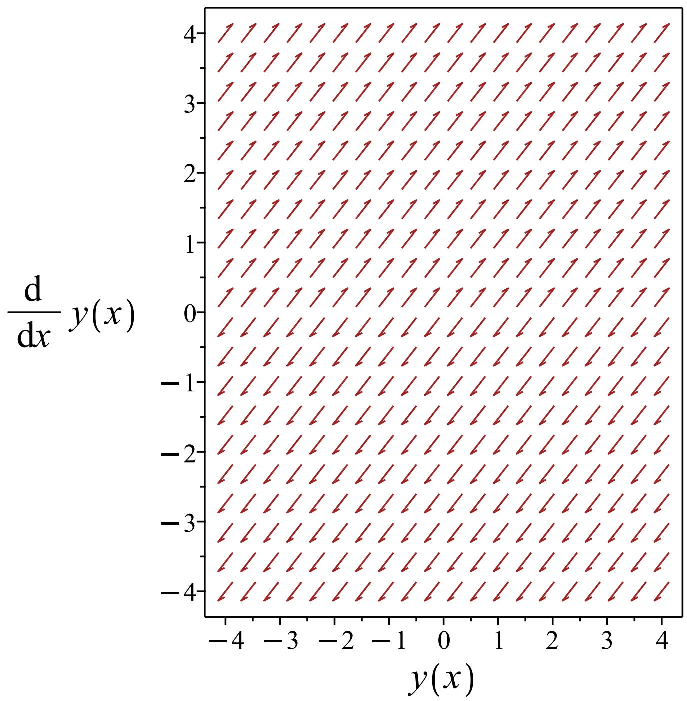


Figure 685: Slope field plot

Verification of solutions

$$y = \frac{((-15x+17)\cos(x)+\sin(x)(5x+6))\,e^{2x}}{50}+c_2e^{x}-c_1$$

Verified OK.

15.14.5 Solving using Kovacic algorithm

Writing the ode as

$$y''-y'=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 535: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^x}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(\mathrm{e}^x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + c_2\mathrm{e}^x$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{2x} \sin(x)\, x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\cos(x), e^{2x}\sin(x), e^{2x}\cos(x)x, e^{2x}\sin(x)x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}\cos(x) + A_2 e^{2x}\sin(x) + A_3 e^{2x}\cos(x)x + A_4 e^{2x}\sin(x)x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&A_1 e^{2x}\cos(x) - 3A_1 e^{2x}\sin(x) + A_2 e^{2x}\sin(x) + 3A_2 e^{2x}\cos(x) + A_3 e^{2x}\cos(x)x \\ &- 3A_3 e^{2x}\sin(x)x + 3A_3 e^{2x}\cos(x) + A_4 e^{2x}\sin(x)x + 3A_4 e^{2x}\cos(x)x \\ &+ 3A_4 e^{2x}\sin(x) - 2A_3 e^{2x}\sin(x) + 2A_4 e^{2x}\cos(x) = e^{2x}\sin(x)x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{17}{50}, A_2 = \frac{3}{25}, A_3 = -\frac{3}{10}, A_4 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{17\,e^{2x}\cos(x)}{50} + \frac{3\,e^{2x}\sin(x)}{25} - \frac{3\,e^{2x}\cos(x)x}{10} + \frac{e^{2x}\sin(x)x}{10}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 + c_2 e^{x}) + \left(\frac{17\,e^{2x}\cos(x)}{50} + \frac{3\,e^{2x}\sin(x)}{25} - \frac{3\,e^{2x}\cos(x)x}{10} + \frac{e^{2x}\sin(x)x}{10}\right)\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 + c_2 e^{x} + \frac{17\,e^{2x}\cos(x)}{50} + \frac{3\,e^{2x}\sin(x)}{25} - \frac{3\,e^{2x}\cos(x)x}{10} + \frac{e^{2x}\sin(x)x}{10} \tag{1}$$

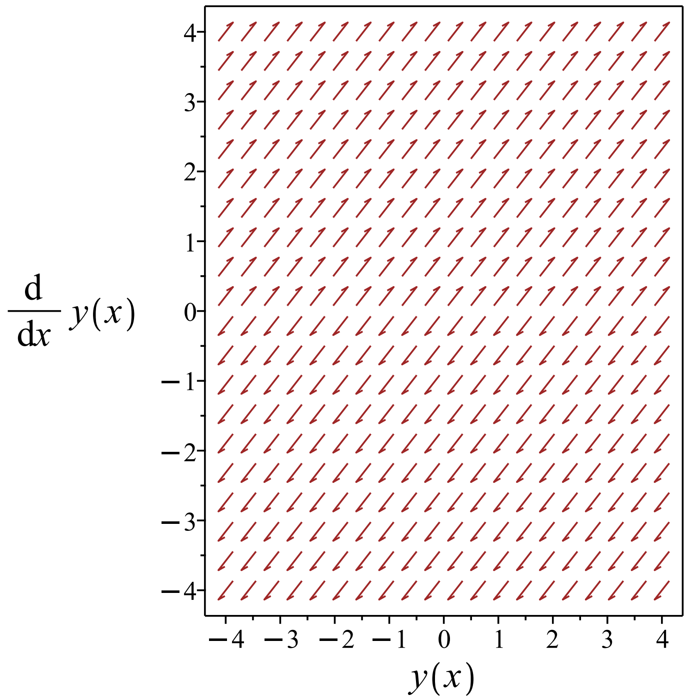


Figure 686: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^x + \frac{17\,e^{2x}\cos(x)}{50} + \frac{3\,e^{2x}\sin(x)}{25} - \frac{3\,e^{2x}\cos(x)\,x}{10} + \frac{e^{2x}\sin(x)\,x}{10}$$

Verified OK.

15.14.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= e^{2x}\sin(x)\,x \end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' - y = \int \mathrm{e}^{2x} \sin(x)\, x\, dx$$

We now have a first order ode to solve which is

$$y' - y = \left(-\frac{x}{5} + \frac{4}{25}\right) \mathrm{e}^{2x} \cos(x) + \left(\frac{2x}{5} - \frac{3}{25}\right) \mathrm{e}^{2x} \sin(x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = \frac{((4 - 5x)\cos(x) + \sin(x)(10x - 3))\, \mathrm{e}^{2x}}{25} + c_1$$

Hence the ode is

$$y' - y = \frac{((4 - 5x)\cos(x) + \sin(x)(10x - 3))\, \mathrm{e}^{2x}}{25} + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int (-1) dx}$$
$$= \mathrm{e}^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{((4-5x)\cos(x)+\sin(x)(10x-3))\,e^{2x}}{25}+c_1\right)$$

$$\frac{d}{dx}\left(e^{-x}y\right) = \left(e^{-x}\right)\left(\frac{((4-5x)\cos(x)+\sin(x)(10x-3))\,e^{2x}}{25}+c_1\right)$$

$$d\left(e^{-x}y\right) = \left(c_1e^{-x} - \frac{\left(\left(-\frac{4}{5}+x\right)\cos(x)+\sin(x)\left(\frac{3}{5}-2x\right)\right)e^{x}}{5}\right)dx$$

Integrating gives

$$e^{-x}y = \int c_1e^{-x} - \frac{\left(\left(-\frac{4}{5}+x\right)\cos(x)+\sin(x)\left(\frac{3}{5}-2x\right)\right)e^{x}}{5}\,dx$$

$$e^{-x}y = -c_1e^{-x} - \frac{e^{x}\cos(x)\,x}{10} + \frac{\left(-\frac{x}{2}+\frac{1}{2}\right)e^{x}\sin(x)}{5} + \frac{2\left(-\frac{x}{2}+\frac{1}{2}\right)e^{x}\cos(x)}{5} + \frac{e^{x}\sin(x)\,x}{5} + \frac{7\cos(x)\,e^{x}}{50} +$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = e^{x}\left(-c_1e^{-x} - \frac{e^{x}\cos(x)\,x}{10} + \frac{\left(-\frac{x}{2}+\frac{1}{2}\right)e^{x}\sin(x)}{5} + \frac{2\left(-\frac{x}{2}+\frac{1}{2}\right)e^{x}\cos(x)}{5} + \frac{e^{x}\sin(x)\,x}{5} + \frac{7\cos(x)\,e^{x}}{50} +\right.$$

which simplifies to

$$y = \frac{((-15x+17)\cos(x)+\sin(x)(5x+6))\,e^{2x}}{50} + c_2e^{x} - c_1$$

Summary
The solution(s) found are the following

$$y = \frac{((-15x+17)\cos(x)+\sin(x)(5x+6))\,e^{2x}}{50} + c_2e^{x} - c_1 \tag{1}$$

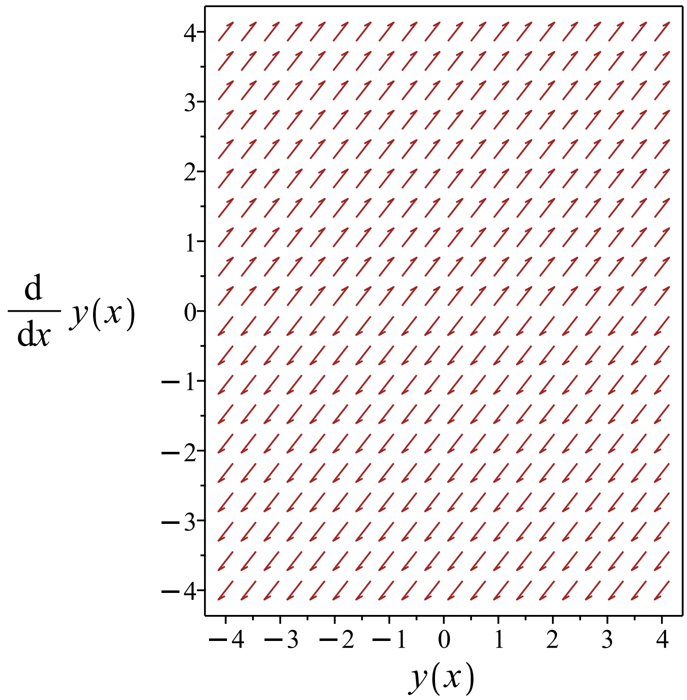


Figure 687: Slope field plot

Verification of solutions

$$y = \frac{((-15x+17)\cos(x) + \sin(x)(5x+6))\, e^{2x}}{50} + c_2 e^{x} - c_1$$

Verified OK.

15.14.7 Maple step by step solution

Let's solve

$-y' + y'' = e^{2x} \sin(x)\, x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r = 0$

- Factor the characteristic polynomial

  $r(r-1) = 0$

- Roots of the characteristic polynomial

$r = (0, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{2x} \sin(x) x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^x \\ 0 & e^x \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^x$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\left(\int e^{2x} \sin(x) x dx\right) + e^x \left(\int e^x \sin(x) x dx\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{((-15x+17)\cos(x)+\sin(x)(5x+6))e^{2x}}{50}$

- Substitute particular solution into general solution to ODE

  $y = \frac{((-15x+17)\cos(x)+\sin(x)(5x+6))e^{2x}}{50} + c_2 e^x + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = sin(_a)*exp(2*_a)*_a+_b(_a), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)-diff(y(x),x)=x*exp(2*x)*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{((-15x+17)\cos(x) + \sin(x)(5x+6))\,e^{2x}}{50} + e^{x}c_1 + c_2$$

✓ Solution by Mathematica

Time used: 0.486 (sec). Leaf size: 46

```
DSolve[y''[x]-y'[x]==x*Exp[2*x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{50}e^{2x}(5x+6)\sin(x) - \frac{1}{50}e^{2x}(15x-17)\cos(x) + c_1e^{x} + c_2$$

15.15 problem 16

Internal problem ID [2248]
Internal file name [`OUTPUT/2248_Monday_February_26_2024_09_18_45_AM_25209105/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 4y = \mathrm{e}^{2x} \cos(x)\, x}$$

15.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = \mathrm{e}^{2x} \cos(x)\, x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x} \cos(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x} \cos(x), e^{2x} \sin(x), e^{2x} \cos(x) x, e^{2x} \sin(x) x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} \cos(x) + A_2 e^{2x} \sin(x) + A_3 e^{2x} \cos(x) x + A_4 e^{2x} \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-A_1 e^{2x} \cos(x) - 4A_1 e^{2x} \sin(x) - A_2 e^{2x} \sin(x) + 4A_2 e^{2x} \cos(x) - A_3 e^{2x} \cos(x) x \\ &- 4A_3 e^{2x} \sin(x) x + 4A_3 e^{2x} \cos(x) - 2A_3 e^{2x} \sin(x) - A_4 e^{2x} \sin(x) x \\ &+ 4A_4 e^{2x} \cos(x) x + 4A_4 e^{2x} \sin(x) + 2A_4 e^{2x} \cos(x) = e^{2x} \cos(x) x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{76}{289}, A_2 = \frac{2}{289}, A_3 = -\frac{1}{17}, A_4 = \frac{4}{17}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{76 e^{2x} \cos(x)}{289} + \frac{2 e^{2x} \sin(x)}{289} - \frac{e^{2x} \cos(x) x}{17} + \frac{4 e^{2x} \sin(x) x}{17}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 e^{2x} + c_2 e^{-2x}) + \left(\frac{76 e^{2x} \cos(x)}{289} + \frac{2 e^{2x} \sin(x)}{289} - \frac{e^{2x} \cos(x) x}{17} + \frac{4 e^{2x} \sin(x) x}{17}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{76\, e^{2x} \cos(x)}{289} + \frac{2\, e^{2x} \sin(x)}{289} - \frac{e^{2x} \cos(x)\, x}{17} + \frac{4\, e^{2x} \sin(x)\, x}{17} \tag{1}$$

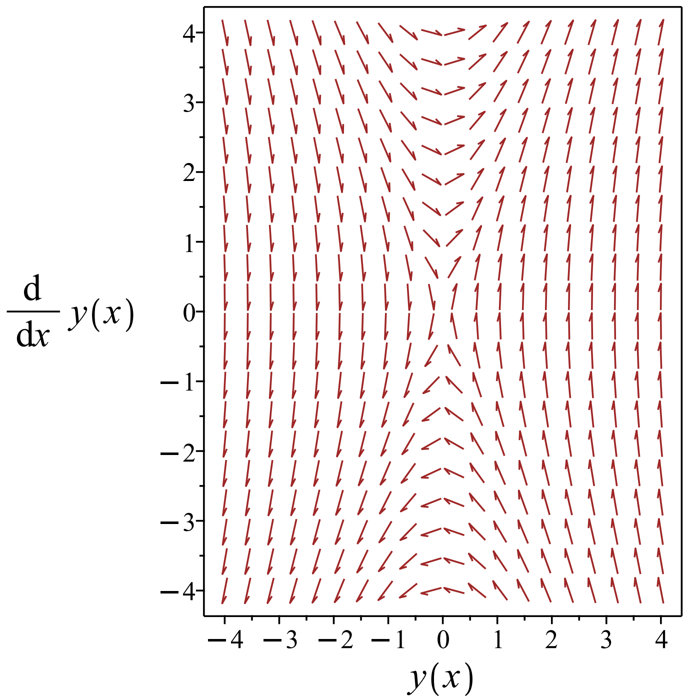


Figure 688: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{76\, e^{2x} \cos(x)}{289} + \frac{2\, e^{2x} \sin(x)}{289} - \frac{e^{2x} \cos(x)\, x}{17} + \frac{4\, e^{2x} \sin(x)\, x}{17}$$

Verified OK.

15.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 537: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-2x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-2x}
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-2x} \int \frac{1}{\mathrm{e}^{-4x}}\, dx \\ &= \mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^{2x} \cos(x) x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{2x} \cos(x), e^{2x} \sin(x), e^{2x} \cos(x) x, e^{2x} \sin(x) x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{e^{2x}}{4}, e^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 e^{2x} \cos(x) + A_2 e^{2x} \sin(x) + A_3 e^{2x} \cos(x) x + A_4 e^{2x} \sin(x) x
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-A_1 e^{2x} \cos(x) - 4A_1 e^{2x} \sin(x) - A_2 e^{2x} \sin(x) + 4A_2 e^{2x} \cos(x) - A_3 e^{2x} \cos(x) x \\
&- 4A_3 e^{2x} \sin(x) x + 4A_3 e^{2x} \cos(x) - 2A_3 e^{2x} \sin(x) - A_4 e^{2x} \sin(x) x \\
&+ 4A_4 e^{2x} \cos(x) x + 4A_4 e^{2x} \sin(x) + 2A_4 e^{2x} \cos(x) = e^{2x} \cos(x) x
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = \frac{76}{289}, A_2 = \frac{2}{289}, A_3 = -\frac{1}{17}, A_4 = \frac{4}{17}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = \frac{76\, e^{2x} \cos(x)}{289} + \frac{2\, e^{2x} \sin(x)}{289} - \frac{e^{2x} \cos(x) x}{17} + \frac{4\, e^{2x} \sin(x) x}{17}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}\right) + \left(\frac{76\, e^{2x} \cos(x)}{289} + \frac{2\, e^{2x} \sin(x)}{289} - \frac{e^{2x} \cos(x) x}{17} + \frac{4\, e^{2x} \sin(x) x}{17}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} + \frac{76\, e^{2x} \cos(x)}{289} + \frac{2\, e^{2x} \sin(x)}{289} - \frac{e^{2x} \cos(x)\, x}{17} + \frac{4\, e^{2x} \sin(x)\, x}{17} \tag{1}$$

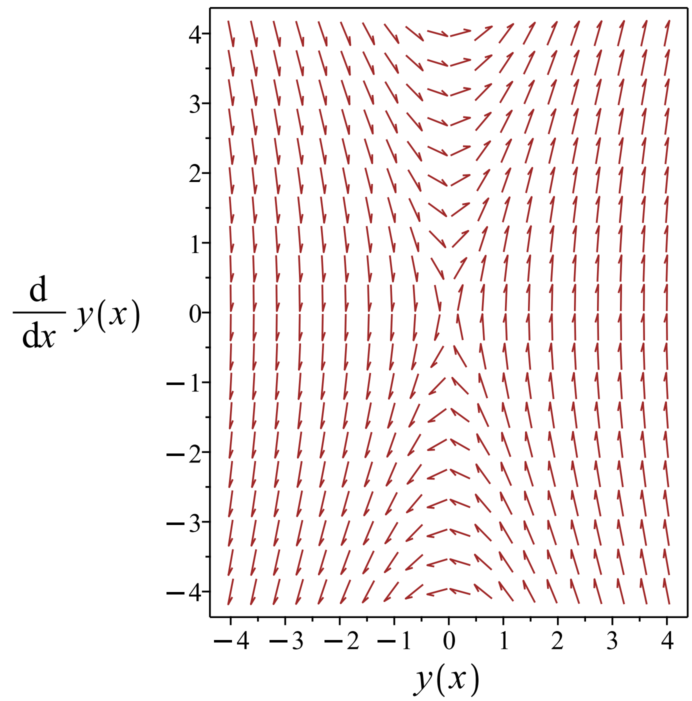


Figure 689: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} + \frac{76\, e^{2x} \cos(x)}{289} + \frac{2\, e^{2x} \sin(x)}{289} - \frac{e^{2x} \cos(x)\, x}{17} + \frac{4\, e^{2x} \sin(x)\, x}{17}$$

Verified OK.

15.15.3 Maple step by step solution

Let's solve

$y'' - 4y = e^{2x} \cos(x)\, x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2-4=0$

- Factor the characteristic polynomial

  $(r-2)(r+2)=0$

- Roots of the characteristic polynomial

  $r=(-2,2)$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{2x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{2x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=\mathrm{e}^{2x}\cos(x)\,x\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x),y_2(x))=\begin{bmatrix}\mathrm{e}^{-2x} & \mathrm{e}^{2x}\\ -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x}\end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(x),y_2(x))=4$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x)=-\frac{\mathrm{e}^{-2x}\left(\int\cos(x)\mathrm{e}^{4x}xdx\right)}{4}+\frac{\mathrm{e}^{2x}\left(\int x\cos(x)dx\right)}{4}$

- ○ Compute integrals

  $y_p(x)=\frac{((-17x+76)\cos(x)+(68x+2)\sin(x))\mathrm{e}^{2x}}{289}$

- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{2x}+\frac{((-17x+76)\cos(x)+(68x+2)\sin(x))\mathrm{e}^{2x}}{289}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)-4*y(x)=x*exp(2*x)*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{((-17x+76)\cos(x)+(68x+2)\sin(x)+289c_2)\,\mathrm{e}^{2x}}{289}+\mathrm{e}^{-2x}c_1$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 49

```
DSolve[y''[x]-4*y[x]==x*Exp[2*x]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1e^{2x}+c_2e^{-2x}+\frac{1}{289}e^{2x}(2(34x+1)\sin(x)+(76-17x)\cos(x))$$

15.16 problem 17

Internal problem ID [2249]
Internal file name [`OUTPUT/2249_Monday_February_26_2024_09_18_46_AM_99744366/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 24, page 109
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 2y' = x^2 \sin(x) e^{-x}}$$

15.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 0, f(x) = x^2 \sin(x) e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+2y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=2, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+2\lambda\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+2\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=2, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-2}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{2^2-(4)\,(1)\,(0)}\\&=-1\pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-1+1\\\lambda_2&=-1-1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \sin(x) e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) e^{-x}, \sin(x) e^{-x}, x\cos(x) e^{-x}, x^2 \cos(x) e^{-x}, x^2 \sin(x) e^{-x}, \sin(x) x e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$\begin{aligned} y_p = {} & A_1 \cos(x) e^{-x} + A_2 \sin(x) e^{-x} + A_3 x \cos(x) e^{-x} \\ & + A_4 x^2 \cos(x) e^{-x} + A_5 x^2 \sin(x) e^{-x} + A_6 \sin(x) x e^{-x} \end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} & 2A_4 \cos(x) e^{-x} - 4A_4 x \sin(x) e^{-x} + 2A_5 \sin(x) e^{-x} + 4A_5 x \cos(x) e^{-x} \\ & + 2A_6 \cos(x) e^{-x} - 2A_4 x^2 \cos(x) e^{-x} - 2A_5 x^2 \sin(x) e^{-x} \\ & - 2A_6 \sin(x) x e^{-x} - 2A_3 x \cos(x) e^{-x} - 2A_3 \sin(x) e^{-x} \\ & - 2A_1 \cos(x) e^{-x} - 2A_2 \sin(x) e^{-x} = x^2 \sin(x) e^{-x} \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}, A_3 = -1, A_4 = 0, A_5 = -\frac{1}{2}, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)\,e^{-x}}{2} - x\cos(x)\,e^{-x} - \frac{x^2\sin(x)\,e^{-x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^{-2x}\right) + \left(\frac{\sin(x)\,e^{-x}}{2} - x\cos(x)\,e^{-x} - \frac{x^2\sin(x)\,e^{-x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-2x} + \frac{\sin(x)\,e^{-x}}{2} - x\cos(x)\,e^{-x} - \frac{x^2\sin(x)\,e^{-x}}{2} \tag{1}$$

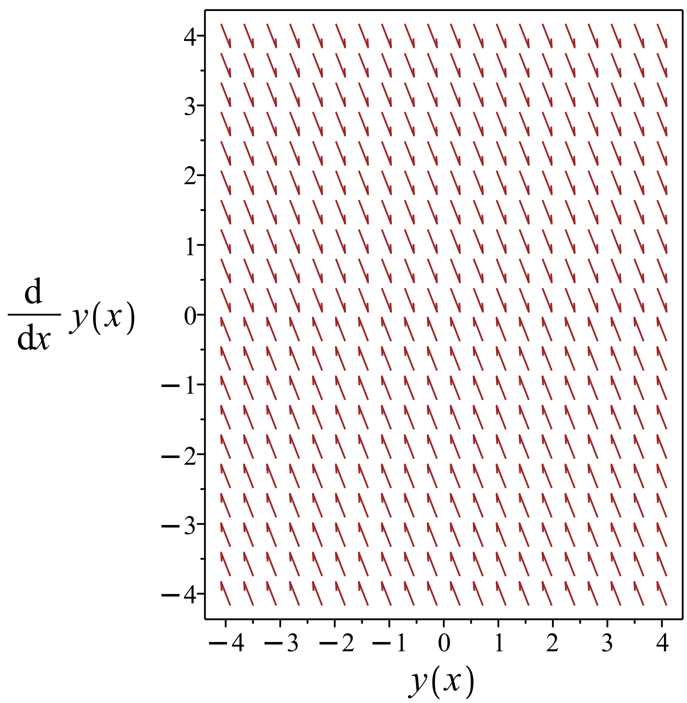


Figure 690: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-2x} + \frac{\sin(x)\,e^{-x}}{2} - x\cos(x)\,e^{-x} - \frac{x^2\sin(x)\,e^{-x}}{2}$$

Verified OK.

15.16.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\, dx = \int x^2 \sin(x)\, e^{-x} dx$$

$$2y + y' = \left(-\frac{1}{2}x^2 - x - \frac{1}{2}\right) e^{-x} \cos(x) + \left(-\frac{x^2}{2} + \frac{1}{2}\right) e^{-x} \sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= \frac{\left(-(x+1)^2 \cos(x) + (1-x^2)\sin(x)\right) e^{-x}}{2} + c_1 \end{aligned}$$

Hence the ode is

$$2y + y' = \frac{\left(-(x+1)^2 \cos(x) + (1-x^2)\sin(x)\right) e^{-x}}{2} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 2dx} \\ &= e^{2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\left(-(x+1)^2 \cos(x) + (1-x^2)\sin(x)\right) e^{-x}}{2} + c_1\right) \\ \frac{d}{dx}\left(e^{2x} y\right) &= \left(e^{2x}\right)\left(\frac{\left(-(x+1)^2 \cos(x) + (1-x^2)\sin(x)\right) e^{-x}}{2} + c_1\right) \\ d\left(e^{2x} y\right) &= \left(-\frac{e^{x}\left((x+1)^2 \cos(x) + \sin(x)\, x^2 - 2c_1 e^{x} - \sin(x)\right)}{2}\right) dx \end{aligned}$$

Integrating gives

$$e^{2x} y = \int -\frac{e^{x}\left((x+1)^2 \cos(x) + \sin(x)\, x^2 - 2c_1 e^{x} - \sin(x)\right)}{2}\, dx$$

$$e^{2x} y = -\frac{\left(\frac{x^2}{2} - \frac{1}{2}\right) e^{x} \cos(x)}{2} + \frac{\left(x - \frac{1}{2}x^2 - \frac{1}{2}\right) e^{x} \sin(x)}{2} - \frac{\left(x - \frac{1}{2}x^2 - \frac{1}{2}\right) e^{x} \cos(x)}{2} - \frac{\left(\frac{x^2}{2} - \frac{1}{2}\right) e^{x} \sin(x)}{2}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$y = \mathrm{e}^{-2x}\left(-\frac{\left(\frac{x^2}{2}-\frac{1}{2}\right)\mathrm{e}^{x}\cos(x)}{2}+\frac{\left(x-\frac{1}{2}x^2-\frac{1}{2}\right)\mathrm{e}^{x}\sin(x)}{2}-\frac{\left(x-\frac{1}{2}x^2-\frac{1}{2}\right)\mathrm{e}^{x}\cos(x)}{2}-\frac{\left(\frac{x^2}{2}-\frac{1}{2}\right)\mathrm{e}^{x}\sin}{2}\right.$$

which simplifies to

$$y = -\frac{\left(-c_1\mathrm{e}^{2x}+\mathrm{e}^{x}\left(2x\cos(x)+\sin(x)\,x^2-\sin(x)\right)-2c_2\right)\mathrm{e}^{-2x}}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\left(-c_1\mathrm{e}^{2x}+\mathrm{e}^{x}\left(2x\cos(x)+\sin(x)\,x^2-\sin(x)\right)-2c_2\right)\mathrm{e}^{-2x}}{2} \tag{1}$$

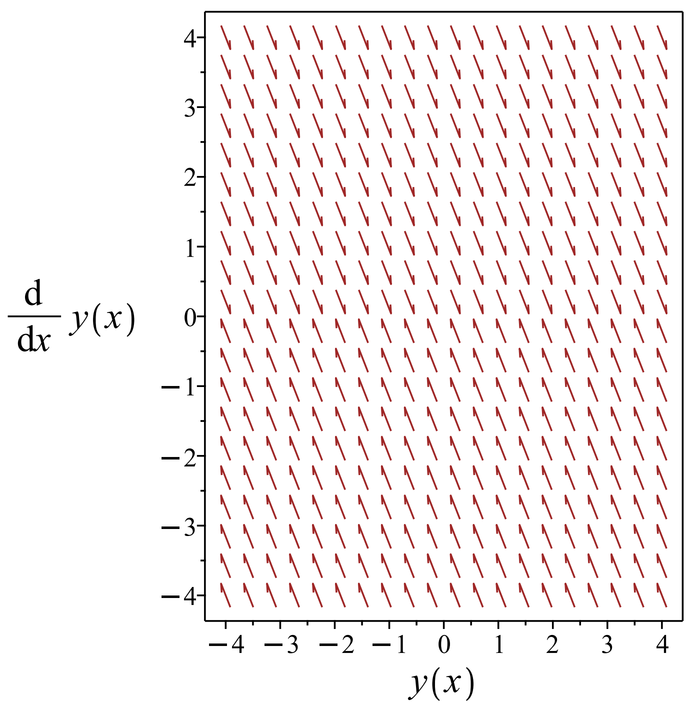


Figure 691: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\left(-c_1\mathrm{e}^{2x}+\mathrm{e}^{x}\left(2x\cos(x)+\sin(x)\,x^2-\sin(x)\right)-2c_2\right)\mathrm{e}^{-2x}}{2}$$

Verified OK.

15.16.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 2p(x) - x^2 \sin(x) e^{-x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= x^2 \sin(x) e^{-x} \end{aligned}$$

Hence the ode is

$$p'(x) + 2p(x) = x^2 \sin(x) e^{-x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 2dx} \\ &= e^{2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(x^2 \sin(x) e^{-x}\right) \\ \frac{d}{dx}\left(e^{2x} p\right) &= \left(e^{2x}\right)\left(x^2 \sin(x) e^{-x}\right) \\ d\left(e^{2x} p\right) &= \left(e^{x} \sin(x) x^2\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{2x} p &= \int e^{x} \sin(x) x^2 \, dx \\ e^{2x} p &= \left(x - \frac{1}{2}x^2 - \frac{1}{2}\right) e^{x} \cos(x) + \left(\frac{x^2}{2} - \frac{1}{2}\right) e^{x} \sin(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$p(x) = \mathrm{e}^{-2x}\left(\left(x - \frac{1}{2}x^2 - \frac{1}{2}\right)\mathrm{e}^{x}\cos(x) + \left(\frac{x^2}{2} - \frac{1}{2}\right)\mathrm{e}^{x}\sin(x)\right) + c_1\mathrm{e}^{-2x}$$

which simplifies to

$$p(x) = -\frac{(((x-1)\cos(x) - (x+1)\sin(x))(x-1)\,\mathrm{e}^{x} - 2c_1)\,\mathrm{e}^{-2x}}{2}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{(((x-1)\cos(x) - (x+1)\sin(x))(x-1)\,\mathrm{e}^{x} - 2c_1)\,\mathrm{e}^{-2x}}{2}$$

Integrating both sides gives

$$y = \int -\frac{\mathrm{e}^{-2x}(\cos(x)\,\mathrm{e}^{x}x^2 - \mathrm{e}^{x}\sin(x)\,x^2 - 2\,\mathrm{e}^{x}\cos(x)\,x + \cos(x)\,\mathrm{e}^{x} + \sin(x)\,\mathrm{e}^{x} - 2c_1)}{2}\,\mathrm{d}x$$

$$= -\frac{\left(-\frac{x^2}{2} + \frac{1}{2}\right)\mathrm{e}^{-x}\cos(x)}{2} + \frac{\left(-\frac{1}{2}x^2 - x - \frac{1}{2}\right)\mathrm{e}^{-x}\sin(x)}{2} + \frac{\left(-\frac{1}{2}x^2 - x - \frac{1}{2}\right)\mathrm{e}^{-x}\cos(x)}{2} + \frac{\left(-\frac{x^2}{2} + \frac{1}{2}\right)\mathrm{e}}{2}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &-\frac{\left(-\frac{x^2}{2} + \frac{1}{2}\right)\mathrm{e}^{-x}\cos(x)}{2} + \frac{\left(-\frac{1}{2}x^2 - x - \frac{1}{2}\right)\mathrm{e}^{-x}\sin(x)}{2} \\ &+ \frac{\left(-\frac{1}{2}x^2 - x - \frac{1}{2}\right)\mathrm{e}^{-x}\cos(x)}{2} + \frac{\left(-\frac{x^2}{2} + \frac{1}{2}\right)\mathrm{e}^{-x}\sin(x)}{2} \\ &- \frac{x\cos(x)\,\mathrm{e}^{-x}}{2} - \left(-\frac{x}{2} - \frac{1}{2}\right)\mathrm{e}^{-x}\sin(x) + \frac{\cos(x)\,\mathrm{e}^{-x}}{2} - \frac{c_1\mathrm{e}^{-2x}}{2} + c_2 \end{aligned} \tag{1}$$

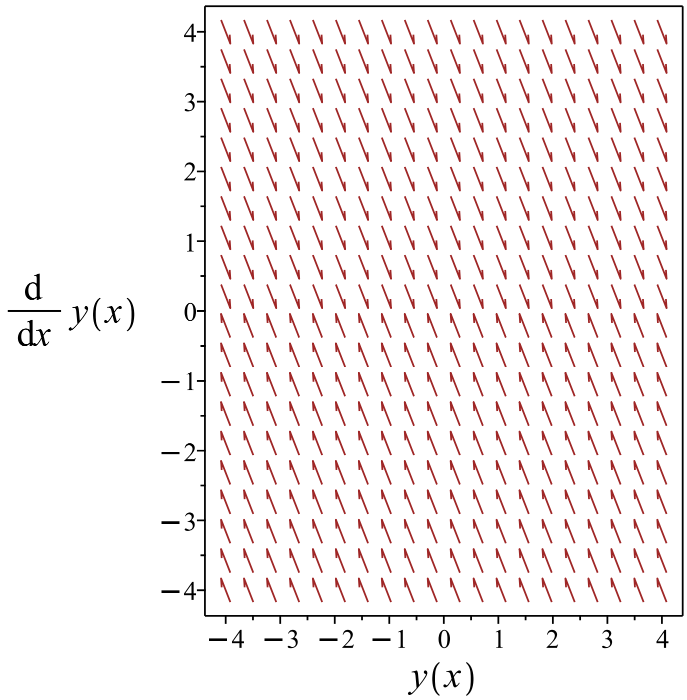


Figure 692: Slope field plot

Verification of solutions

$$y = -\frac{\left(-\frac{x^2}{2}+\frac{1}{2}\right)\mathrm{e}^{-x}\cos(x)}{2} + \frac{\left(-\frac{1}{2}x^2 - x - \frac{1}{2}\right)\mathrm{e}^{-x}\sin(x)}{2} + \frac{\left(-\frac{1}{2}x^2 - x - \frac{1}{2}\right)\mathrm{e}^{-x}\cos(x)}{2} + \frac{\left(-\frac{x^2}{2}+\frac{1}{2}\right)\mathrm{e}^{-x}\sin(x)}{2} - \frac{x\cos(x)\,\mathrm{e}^{-x}}{2} - \left(-\frac{x}{2}-\frac{1}{2}\right)\mathrm{e}^{-x}\sin(x) + \frac{\cos(x)\,\mathrm{e}^{-x}}{2} - \frac{c_1\mathrm{e}^{-2x}}{2} + c_2$$

Verified OK.

15.16.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 2y' = x^2\sin(x)\,\mathrm{e}^{-x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y''+2y'\right)dx = \int x^2\sin\left(x\right)\mathrm{e}^{-x}dx$$

$$2y+y' = \left(-\frac{1}{2}x^2-x-\frac{1}{2}\right)\mathrm{e}^{-x}\cos\left(x\right)+\left(-\frac{x^2}{2}+\frac{1}{2}\right)\mathrm{e}^{-x}\sin\left(x\right)+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= 2 \\
q(x) &= \frac{\left(-(x+1)^2\cos\left(x\right)+\left(1-x^2\right)\sin\left(x\right)\right)\mathrm{e}^{-x}}{2}+c_1
\end{aligned}$$

Hence the ode is

$$2y+y' = \frac{\left(-(x+1)^2\cos\left(x\right)+\left(1-x^2\right)\sin\left(x\right)\right)\mathrm{e}^{-x}}{2}+c_1$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int 2dx} \\
&= \mathrm{e}^{2x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\left(-(x+1)^2\cos\left(x\right)+\left(1-x^2\right)\sin\left(x\right)\right)\mathrm{e}^{-x}}{2}+c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x}y\right) &= \left(\mathrm{e}^{2x}\right)\left(\frac{\left(-(x+1)^2\cos\left(x\right)+\left(1-x^2\right)\sin\left(x\right)\right)\mathrm{e}^{-x}}{2}+c_1\right) \\
\mathrm{d}\left(\mathrm{e}^{2x}y\right) &= \left(-\frac{\mathrm{e}^{x}\left((x+1)^2\cos\left(x\right)+\sin\left(x\right)x^2-2c_1\mathrm{e}^{x}-\sin\left(x\right)\right)}{2}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\mathrm{e}^{2x}y &= \int -\frac{\mathrm{e}^{x}\left((x+1)^2\cos\left(x\right)+\sin\left(x\right)x^2-2c_1\mathrm{e}^{x}-\sin\left(x\right)\right)}{2}\,\mathrm{d}x \\
\mathrm{e}^{2x}y &= -\frac{\left(\frac{x^2}{2}-\frac{1}{2}\right)\mathrm{e}^{x}\cos\left(x\right)}{2}+\frac{\left(x-\frac{1}{2}x^2-\frac{1}{2}\right)\mathrm{e}^{x}\sin\left(x\right)}{2}-\frac{\left(x-\frac{1}{2}x^2-\frac{1}{2}\right)\mathrm{e}^{x}\cos\left(x\right)}{2}-\frac{\left(\frac{x^2}{2}-\frac{1}{2}\right)\mathrm{e}^{x}\sin\left(x\right)}{2}
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$y = \mathrm{e}^{-2x}\left(-\frac{\left(\frac{x^2}{2}-\frac{1}{2}\right)\mathrm{e}^{x}\cos\left(x\right)}{2}+\frac{\left(x-\frac{1}{2}x^2-\frac{1}{2}\right)\mathrm{e}^{x}\sin\left(x\right)}{2}-\frac{\left(x-\frac{1}{2}x^2-\frac{1}{2}\right)\mathrm{e}^{x}\cos\left(x\right)}{2}-\frac{\left(\frac{x^2}{2}-\frac{1}{2}\right)\mathrm{e}^{x}\sin}{2}\right.$$

which simplifies to

$$y = -\frac{\left(-c_1\mathrm{e}^{2x}+\mathrm{e}^{x}\left(2x\cos\left(x\right)+\sin\left(x\right)x^2-\sin\left(x\right)\right)-2c_2\right)\mathrm{e}^{-2x}}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(-c_1\mathrm{e}^{2x}+\mathrm{e}^{x}\left(2x\cos\left(x\right)+\sin\left(x\right)x^2-\sin\left(x\right)\right)-2c_2\right)\mathrm{e}^{-2x}}{2} \tag{1}$$

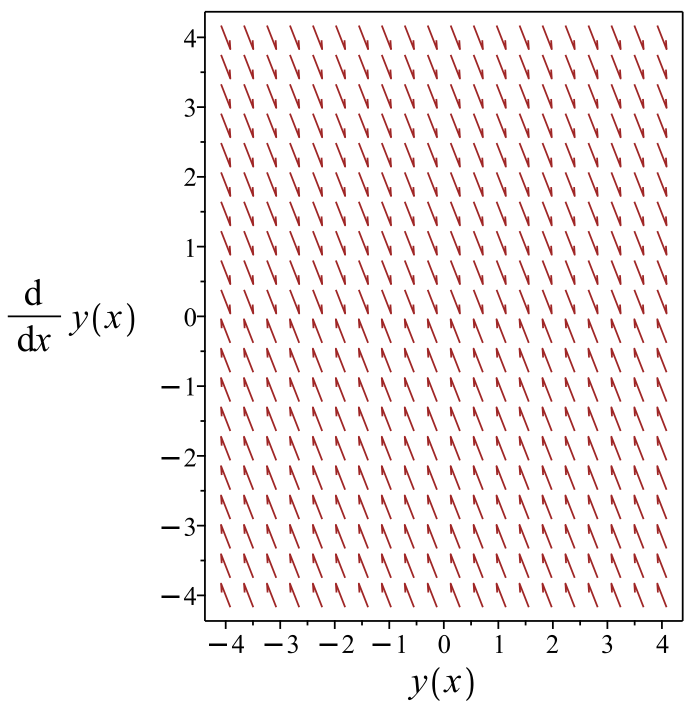


Figure 693: Slope field plot

Verification of solutions

$$y = -\frac{\left(-c_1\mathrm{e}^{2x}+\mathrm{e}^{x}\left(2x\cos\left(x\right)+\sin\left(x\right)x^2-\sin\left(x\right)\right)-2c_2\right)\mathrm{e}^{-2x}}{2}$$

Verified OK.

15.16.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 539: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2}{1} \, dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \sin(x) e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) e^{-x}, \sin(x) e^{-x}, x \cos(x) e^{-x}, x^2 \cos(x) e^{-x}, x^2 \sin(x) e^{-x}, \sin(x) x e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{1}{2}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$\begin{aligned}y_p = &A_1 \cos(x) e^{-x} + A_2 \sin(x) e^{-x} + A_3 x \cos(x) e^{-x} \\ &+ A_4 x^2 \cos(x) e^{-x} + A_5 x^2 \sin(x) e^{-x} + A_6 \sin(x) x e^{-x}\end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-2A_4 x^2 \cos(x) e^{-x} - 2A_5 x^2 \sin(x) e^{-x} - 2A_6 \sin(x) x e^{-x} \\ &- 2A_3 x \cos(x) e^{-x} + 2A_4 \cos(x) e^{-x} - 4A_4 x \sin(x) e^{-x} \\ &+ 2A_5 \sin(x) e^{-x} + 4A_5 x \cos(x) e^{-x} + 2A_6 \cos(x) e^{-x} \\ &- 2A_3 \sin(x) e^{-x} - 2A_1 \cos(x) e^{-x} - 2A_2 \sin(x) e^{-x} = x^2 \sin(x) e^{-x}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}, A_3 = -1, A_4 = 0, A_5 = -\frac{1}{2}, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x) e^{-x}}{2} - x \cos(x) e^{-x} - \frac{x^2 \sin(x) e^{-x}}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} + \frac{c_2}{2}\right) + \left(\frac{\sin(x)\,\mathrm{e}^{-x}}{2} - x\cos(x)\,\mathrm{e}^{-x} - \frac{x^2 \sin(x)\,\mathrm{e}^{-x}}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2} + \frac{\sin(x)\,\mathrm{e}^{-x}}{2} - x\cos(x)\,\mathrm{e}^{-x} - \frac{x^2 \sin(x)\,\mathrm{e}^{-x}}{2} \tag{1}
$$

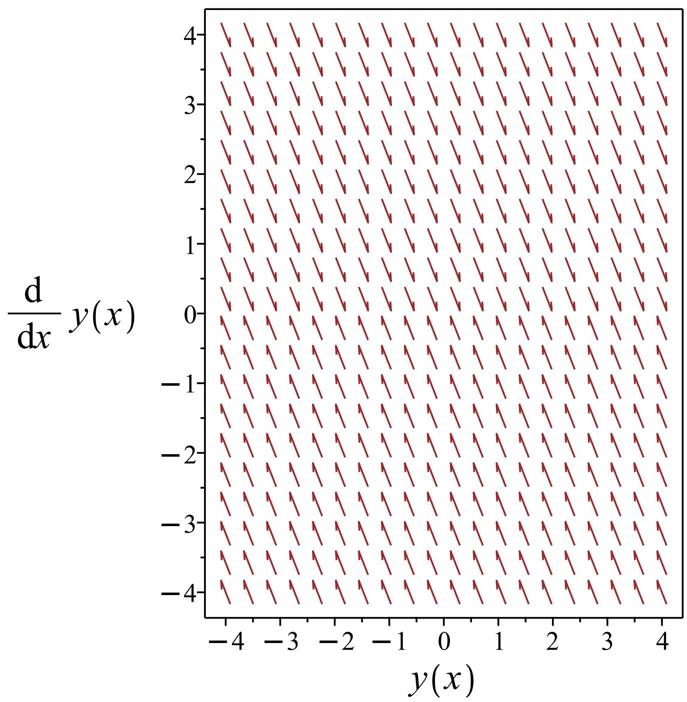


Figure 694: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2} + \frac{\sin(x)\,\mathrm{e}^{-x}}{2} - x\cos(x)\,\mathrm{e}^{-x} - \frac{x^2 \sin(x)\,\mathrm{e}^{-x}}{2}
$$

Verified OK.

15.16.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=1\\q(x)&=2\\r(x)&=0\\s(x)&=x^{2}\sin\left(x\right){\mathrm e}^{-x}\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=0\\q'(x)&=0\end{aligned}$$

Therefore (1) becomes

$$0-\left(0\right)+\left(0\right)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$2y+y'=\int x^{2}\sin\left(x\right){\mathrm e}^{-x}\,dx$$

We now have a first order ode to solve which is

$$2y+y'=\left(-\frac{1}{2}x^{2}-x-\frac{1}{2}\right){\mathrm e}^{-x}\cos\left(x\right)+\left(-\frac{x^{2}}{2}+\frac{1}{2}\right){\mathrm e}^{-x}\sin\left(x\right)+c_{1}$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 2\\ q(x) &= \frac{\left(-(x+1)^2 \cos(x) + (1-x^2)\sin(x)\right) e^{-x}}{2} + c_1\end{aligned}$$

Hence the ode is

$$2y + y' = \frac{\left(-(x+1)^2 \cos(x) + (1-x^2)\sin(x)\right) e^{-x}}{2} + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2dx}\\ &= e^{2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\left(-(x+1)^2 \cos(x) + (1-x^2)\sin(x)\right) e^{-x}}{2} + c_1\right)\\ \frac{d}{dx}\left(e^{2x} y\right) &= \left(e^{2x}\right)\left(\frac{\left(-(x+1)^2 \cos(x) + (1-x^2)\sin(x)\right) e^{-x}}{2} + c_1\right)\\ d\left(e^{2x} y\right) &= \left(-\frac{e^{x}\left((x+1)^2 \cos(x) + \sin(x)\, x^2 - 2c_1 e^{x} - \sin(x)\right)}{2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2x} y &= \int -\frac{e^{x}\left((x+1)^2 \cos(x) + \sin(x)\, x^2 - 2c_1 e^{x} - \sin(x)\right)}{2}\, dx\\ e^{2x} y &= -\frac{\left(\frac{x^2}{2} - \frac{1}{2}\right) e^{x} \cos(x)}{2} + \frac{\left(x - \frac{1}{2}x^2 - \frac{1}{2}\right) e^{x} \sin(x)}{2} - \frac{\left(x - \frac{1}{2}x^2 - \frac{1}{2}\right) e^{x} \cos(x)}{2} - \frac{\left(\frac{x^2}{2} - \frac{1}{2}\right) e^{x} \sin(x)}{2}\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = e^{-2x}\left(-\frac{\left(\frac{x^2}{2} - \frac{1}{2}\right) e^{x} \cos(x)}{2} + \frac{\left(x - \frac{1}{2}x^2 - \frac{1}{2}\right) e^{x} \sin(x)}{2} - \frac{\left(x - \frac{1}{2}x^2 - \frac{1}{2}\right) e^{x} \cos(x)}{2} - \frac{\left(\frac{x^2}{2} - \frac{1}{2}\right) e^{x} \sin}{2}\right.$$

which simplifies to

$$y = -\frac{\left(-c_1 e^{2x} + e^x\left(2x\cos(x) + \sin(x)\, x^2 - \sin(x)\right) - 2c_2\right) e^{-2x}}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(-c_1 e^{2x} + e^x\left(2x\cos(x) + \sin(x)\, x^2 - \sin(x)\right) - 2c_2\right) e^{-2x}}{2} \tag{1}$$

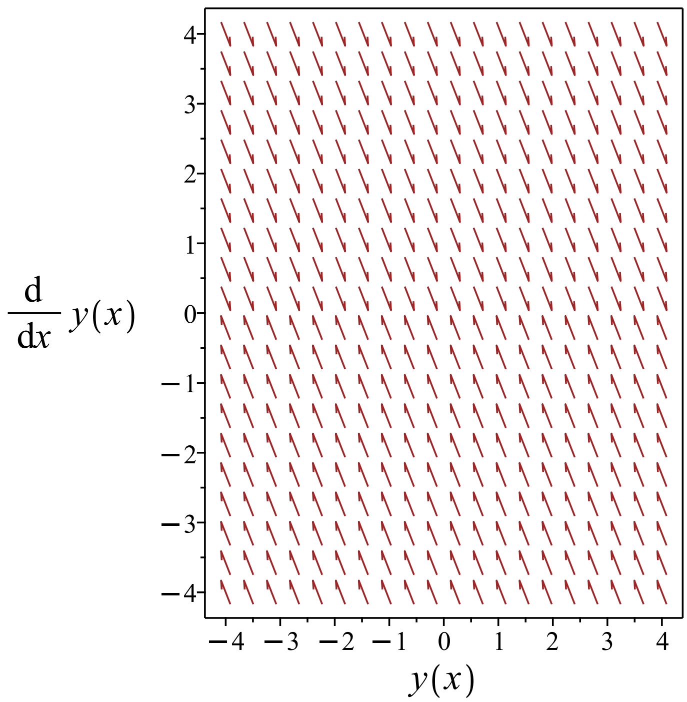


Figure 695: Slope field plot

Verification of solutions

$$y = -\frac{\left(-c_1 e^{2x} + e^x\left(2x\cos(x) + \sin(x)\, x^2 - \sin(x)\right) - 2c_2\right) e^{-2x}}{2}$$

Verified OK.

15.16.7 Maple step by step solution

Let's solve

$y'' + 2y' = x^2 \sin(x) e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r = 0$

- Factor the characteristic polynomial

  $r(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 \sin(x) e^{-x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & 1 \\ -2e^{-2x} & 0 \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2e^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\mathrm{e}^{-2x} \left(\int \mathrm{e}^{x} \sin(x) x^2 dx\right)}{2} + \frac{\left(\int x^2 \sin(x) \mathrm{e}^{-x} dx\right)}{2}$$

- Compute integrals

$$y_p(x) = -\frac{\mathrm{e}^{-x} \left(2x \cos(x) + \sin(x) x^2 - \sin(x)\right)}{2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \mathrm{e}^{-2x} + c_2 - \frac{\mathrm{e}^{-x} \left(2x \cos(x) + \sin(x) x^2 - \sin(x)\right)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = exp(-_a)*sin(_a)*_a^2-2*_b(_a), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)=x^2*exp(-x)*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{-x}\left(-\sin\left(x\right) x^2 - 2x \cos\left(x\right) + \sin\left(x\right)\right)}{2} - \frac{\mathrm{e}^{-2x} c_1}{2} + c_2$$

✓ Solution by Mathematica

Time used: 1.508 (sec). Leaf size: 39

```
DSolve[y''[x]+2*y'[x]==x^2*Exp[-x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{2} e^{-2x} \left(e^x \left(x^2 - 1\right) \sin(x) + 2 e^x x \cos(x) + c_1\right)$$

16 Exercise 25, page 112

16.1 problem 1

Internal problem ID [2250]
Internal file name [`OUTPUT/2250_Monday_February_26_2024_09_18_48_AM_64896791/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - 4y'x + y = 0}$$

16.1.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 1 = 0$$

Or

$$r^2 - 5r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{5}{2} - \frac{\sqrt{21}}{2}$$
$$r_2 = \frac{5}{2} + \frac{\sqrt{21}}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1 x^{\frac{5}{2} - \frac{\sqrt{21}}{2}} + c_2 x^{\frac{5}{2} + \frac{\sqrt{21}}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{5}{2} - \frac{\sqrt{21}}{2}} + c_2 x^{\frac{5}{2} + \frac{\sqrt{21}}{2}} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{5}{2} - \frac{\sqrt{21}}{2}} + c_2 x^{\frac{5}{2} + \frac{\sqrt{21}}{2}}$$

Verified OK.

16.1.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 4y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)}dx \\
&= \int e^{4\ln(x)}\,dx \\
&= \int x^4 dx \\
&= \frac{x^5}{5}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{x^8} \\
&= \frac{1}{x^{10}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{x^{10}} = 0$$

But in terms of τ

$$\frac{1}{x^{10}} = \frac{1}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 1 = 0$$

Or

$$25r^2 - 25r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{\sqrt{21}}{10}$$
$$r_2 = \frac{1}{2} + \frac{\sqrt{21}}{10}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{2}-\frac{\sqrt{21}}{10}} + c_2\tau^{\frac{1}{2}+\frac{\sqrt{21}}{10}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{5}\,\sqrt{x^5}\left(c_1 5^{\frac{\sqrt{21}}{10}} \left(x^5\right)^{-\frac{\sqrt{21}}{10}} + c_2 5^{-\frac{\sqrt{21}}{10}} \left(x^5\right)^{\frac{\sqrt{21}}{10}}\right)}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{5}\,\sqrt{x^5}\left(c_1 5^{\frac{\sqrt{21}}{10}} \left(x^5\right)^{-\frac{\sqrt{21}}{10}} + c_2 5^{-\frac{\sqrt{21}}{10}} \left(x^5\right)^{\frac{\sqrt{21}}{10}}\right)}{5} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{5}\,\sqrt{x^5}\left(c_1 5^{\frac{\sqrt{21}}{10}} \left(x^5\right)^{-\frac{\sqrt{21}}{10}} + c_2 5^{-\frac{\sqrt{21}}{10}} \left(x^5\right)^{\frac{\sqrt{21}}{10}}\right)}{5}$$

Verified OK.

16.1.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 4y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\, x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -5c
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 5c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5c\tau}{2}}\left(c_1 \cosh\left(\frac{c\sqrt{21}\,\tau}{2}\right) + i c_2 \sinh\left(\frac{c\sqrt{21}\,\tau}{2}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\sqrt{21}\, \ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\sqrt{21}\, \ln(x)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\sqrt{21}\, \ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\sqrt{21}\, \ln(x)}{2}\right)\right) \tag{1}$$

Verification of solutions

$$y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\sqrt{21}\, \ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\sqrt{21}\, \ln(x)}{2}\right)\right)$$

Verified OK.

16.1.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' - 4y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{5}{2} + \frac{\sqrt{21}}{2} \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{5+\sqrt{21}}{x} - \frac{4}{x}\right) v'(x) = 0$$

$$v''(x) + \frac{(1+\sqrt{21})\, v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+\sqrt{21})\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(-1-\sqrt{21})\, u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-\sqrt{21}}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1-\sqrt{21}}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-1-\sqrt{21}}{x}\, dx \\ \ln(u) &= \left(-1-\sqrt{21}\right) \ln(x) + c_1 \\ u &= \mathrm{e}^{\left(-1-\sqrt{21}\right) \ln(x)+c_1} \\ &= c_1 \mathrm{e}^{\left(-1-\sqrt{21}\right) \ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-\sqrt{21}}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{\sqrt{21}\, c_1 x^{-\sqrt{21}}}{21} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{\sqrt{21}\, c_1 x^{-\sqrt{21}}}{21} + c_2\right) x^{\frac{5}{2}+\frac{\sqrt{21}}{2}} \\
&= -\frac{x^{\frac{5}{2}-\frac{\sqrt{21}}{2}} \left(\sqrt{21}\, c_1 - 21 c_2 x^{\sqrt{21}}\right)}{21}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{\sqrt{21}\, c_1 x^{-\sqrt{21}}}{21} + c_2\right) x^{\frac{5}{2}+\frac{\sqrt{21}}{2}} \tag{1}$$

Verification of solutions

$$y = \left(-\frac{\sqrt{21}\, c_1 x^{-\sqrt{21}}}{21} + c_2\right) x^{\frac{5}{2}+\frac{\sqrt{21}}{2}}$$

Verified OK.

16.1.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4y'x + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 541: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 5$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{21}}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{21}}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 5$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{21}}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{21}}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{\sqrt{21}}{2}$	$\frac{1}{2} - \frac{\sqrt{21}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + \frac{\sqrt{21}}{2}$	$\frac{1}{2} - \frac{\sqrt{21}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2} - \frac{\sqrt{21}}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \frac{\sqrt{21}}{2} - \left(\frac{1}{2} - \frac{\sqrt{21}}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{\sqrt{21}}{2}}{x} + (-)\,(0) \\
&= \frac{\frac{1}{2} - \frac{\sqrt{21}}{2}}{x} \\
&= -\frac{\sqrt{21} - 1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{21}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{21}}{2}}{x^2} \right) + \left(\frac{\frac{1}{2} - \frac{\sqrt{21}}{2}}{x} \right)^2 - \left(\frac{5}{x^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{\frac{1}{2} - \frac{\sqrt{21}}{2}}{x} dx} \\
&= x^{\frac{1}{2} - \frac{\sqrt{21}}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1 \left(x^2\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{5}{2}-\frac{\sqrt{21}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^{\sqrt{21}}\sqrt{21}}{21}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{5}{2}-\frac{\sqrt{21}}{2}}\right) + c_2 \left(x^{\frac{5}{2}-\frac{\sqrt{21}}{2}} \left(\frac{x^{\sqrt{21}}\sqrt{21}}{21}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{5}{2}-\frac{\sqrt{21}}{2}} + \frac{c_2\sqrt{21}\, x^{\frac{5}{2}+\frac{\sqrt{21}}{2}}}{21} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{5}{2}-\frac{\sqrt{21}}{2}} + \frac{c_2\sqrt{21}\, x^{\frac{5}{2}+\frac{\sqrt{21}}{2}}}{21}$$

Verified OK.

16.1.6 Maple step by step solution

Let's solve

$x^2y'' - 4y'x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{x} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - 4y'x + y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)-4\frac{d}{dt}y(t)+y(t)=0$$

- Simplify

$$\frac{d^2}{dt^2}y(t)-5\frac{d}{dt}y(t)+y(t)=0$$

- Characteristic polynomial of ODE

$$r^2-5r+1=0$$

- Use quadratic formula to solve for r

$$r=\frac{5\pm\left(\sqrt{21}\right)}{2}$$

- Roots of the characteristic polynomial

$$r=\left(\frac{5}{2}-\frac{\sqrt{21}}{2},\frac{5}{2}+\frac{\sqrt{21}}{2}\right)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^{\left(\frac{5}{2}-\frac{\sqrt{21}}{2}\right)t}$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{\left(\frac{5}{2}+\frac{\sqrt{21}}{2}\right)t}$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^{\left(\frac{5}{2}-\frac{\sqrt{21}}{2}\right)t}+c_2\mathrm{e}^{\left(\frac{5}{2}+\frac{\sqrt{21}}{2}\right)t}$$

- Change variables back using $t=\ln(x)$

$$y=c_1\mathrm{e}^{\left(\frac{5}{2}-\frac{\sqrt{21}}{2}\right)\ln(x)}+c_2\mathrm{e}^{\left(\frac{5}{2}+\frac{\sqrt{21}}{2}\right)\ln(x)}$$

- Simplify

$$y=x^{\frac{5}{2}}\left(x^{\frac{\sqrt{21}}{2}}c_2+x^{-\frac{\sqrt{21}}{2}}c_1\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = x^{\frac{5}{2}}\left(x^{\frac{\sqrt{21}}{2}}c_1 + x^{-\frac{\sqrt{21}}{2}}c_2\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 34

```
DSolve[x^2*y''[x]-4*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{\frac{5}{2}-\frac{\sqrt{21}}{2}}\left(c_2 x^{\sqrt{21}} + c_1\right)$$

16.2 problem 2

Internal problem ID [2251]
Internal file name [`OUTPUT/2251_Monday_February_26_2024_09_18_49_AM_69219871/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + y'x + 16y = 0}$$

16.2.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + 16x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + 16x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 16 = 0$$

Or

$$r^2 + 16 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -4i \\ r_2 &= 4i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 0$ and $\beta = -4$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta} \\ &= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^{\alpha}\left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 0, \beta = -4$, the above becomes

$$y = x^0\left(c_1 e^{-4i\ln(x)} + c_2 e^{4i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos(4\ln(x)) + c_2 \sin(4\ln(x))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(4\ln(x)) + c_2 \sin(4\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(4\ln(x)) + c_2 \sin(4\ln(x))$$

Verified OK.

16.2.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + y'x + 16y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{16}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\ &= \int e^{-\ln(x)}\, dx \\ &= \int \frac{1}{x}dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{16}{x^2}}{\frac{1}{x^2}} \\ &= 16 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + 16y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 16$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + 16\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 16$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(16)} \\ &= \pm 4i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +4i \\ \lambda_2 &= -4i \end{aligned}$$

Which simplifies to

$$\lambda_1 = 4i$$
$$\lambda_2 = -4i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha \tau}(c_1 \cos(\beta \tau) + c_2 \sin(\beta \tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(4\tau) + c_2 \sin(4\tau))$$

Or

$$y(\tau) = c_1 \cos(4\tau) + c_2 \sin(4\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(4 \ln(x)) + c_2 \sin(4 \ln(x))$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(4 \ln(x)) + c_2 \sin(4 \ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(4 \ln(x)) + c_2 \sin(4 \ln(x))$$

Verified OK.

16.2.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + y'x + 16y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{16}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{4\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{4}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{4}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{4\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{4\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 4\sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{4\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(4\ln(x)) + c_2 \sin(4\ln(x))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(4\ln(x)) + c_2 \sin(4\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(4\ln(x)) + c_2 \sin(4\ln(x))$$

Verified OK.

16.2.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + y'x + 16y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x} \\ q(x) &= \frac{16}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{16}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 4i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{8i}{x} + \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+8i)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+8i)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(-1-8i)\, u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-8i}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-8i}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{-1-8i}{x}\,dx \\
\ln(u) &= (-1-8i)\ln(x) + c_1 \\
u &= e^{(-1-8i)\ln(x)+c_1} \\
&= c_1 e^{(-1-8i)\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-8i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{i c_1 x^{-8i}}{8} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{i c_1 x^{-8i}}{8} + c_2\right) x^{4i} \\
&= x^{4i} c_2 + \frac{i x^{-4i} c_1}{8}
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(\frac{i c_1 x^{-8i}}{8} + c_2\right) x^{4i} \tag{1}$$

<u>Verification of solutions</u>

$$y = \left(\frac{i c_1 x^{-8i}}{8} + c_2\right) x^{4i}$$

Verified OK.

16.2.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y'x + 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-65}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -65 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{65}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 543: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{65}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{65}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 4i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 4i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{65}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{65}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 4i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 4i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{65}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 4i$	$\frac{1}{2} - 4i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + 4i$	$\frac{1}{2} - 4i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - 4i$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - 4i - \left(\frac{1}{2} - 4i\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - 4i}{x} + (-)\,(0) \\
&= \frac{\frac{1}{2} - 4i}{x} \\
&= \frac{\frac{1}{2} - 4i}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - 4i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 4i}{x^2}\right) + \left(\frac{\frac{1}{2} - 4i}{x}\right)^2 - \left(-\frac{65}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{\frac{1}{2} - 4i}{x} dx} \\
&= x^{\frac{1}{2} - 4i}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-4i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i x^{8i}}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-4i}\right) + c_2 \left(x^{-4i}\left(-\frac{i x^{8i}}{8}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x^{-4i} c_1 - \frac{i c_2 x^{4i}}{8} \tag{1}
$$

Verification of solutions

$$
y = x^{-4i} c_1 - \frac{i c_2 x^{4i}}{8}
$$

Verified OK.

16.2.6 Maple step by step solution

Let's solve

$x^2 y'' + y'x + 16y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{16y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{16y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + y'x + 16y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) + 16y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 16y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 16 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$
- Roots of the characteristic polynomial

  $r = (-4\,\mathrm{I}, 4\,\mathrm{I})$
- 1st solution of the ODE

  $y_1(t) = \cos(4t)$
- 2nd solution of the ODE

  $y_2(t) = \sin(4t)$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \cos(4t) + c_2 \sin(4t)$
- Change variables back using $t = \ln(x)$

  $y = c_1 \cos(4\ln(x)) + c_2 \sin(4\ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(4\ln(x)) + c_2 \cos(4\ln(x))$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]+x*y'[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(4 \log(x)) + c_2 \sin(4 \log(x))$$

16.3 problem 3

Internal problem ID [2252]
Internal file name [`OUTPUT/2252_Monday_February_26_2024_09_18_50_AM_52693940/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{4x^2y'' - 16y'x + 25y = 0}$$

16.3.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$4x^2(r(r-1))x^{r-2} - 16xrx^{r-1} + 25x^r = 0$$

Simplifying gives

$$4r(r-1)\,x^r - 16r\,x^r + 25x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$4r(r-1) - 16r + 25 = 0$$

Or

$$4r^2 - 20r + 25 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{5}{2}$$
$$r_2 = \frac{5}{2}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = x^{\frac{5}{2}} c_1 + c_2 x^{\frac{5}{2}} \ln(x)$$

Summary

The solution(s) found are the following

$$y = x^{\frac{5}{2}} c_1 + c_2 x^{\frac{5}{2}} \ln(x) \tag{1}$$

Verification of solutions

$$y = x^{\frac{5}{2}} c_1 + c_2 x^{\frac{5}{2}} \ln(x)$$

Verified OK.

16.3.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$4x^2 y'' - 16y'x + 25y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{25}{4x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\ &= \int e^{-\left(\int -\frac{4}{x}dx\right)}dx \\ &= \int e^{4\ln(x)}\,dx \\ &= \int x^4 dx \\ &= \frac{x^5}{5}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{25}{4x^2}}{x^8} \\ &= \frac{25}{4x^{10}}\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{25y(\tau)}{4x^{10}} = 0$$

But in terms of τ

$$\frac{25}{4x^{10}} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{5}\,\sqrt{x^5}\,(c_1 + c_2\ln(x^5) - c_2\ln(5))}{5}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{5}\,\sqrt{x^5}\,(c_1 + c_2 \ln(x^5) - c_2 \ln(5))}{5} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{5}\,\sqrt{x^5}\,(c_1 + c_2 \ln(x^5) - c_2 \ln(5))}{5}$$

Verified OK.

16.3.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$4x^2y'' - 16y'x + 25y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{25}{4x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{5\sqrt{\frac{1}{x^2}}}{2c} \\ \tau'' &= -\frac{5}{2c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{5}{2c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{5\sqrt{\frac{1}{x^2}}}{2c}}{\left(\frac{5\sqrt{\frac{1}{x^2}}}{2c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \frac{5\sqrt{\frac{1}{x^2}}}{2}dx}{c} \\
&= \frac{5\sqrt{\frac{1}{x^2}}\,x\ln(x)}{2c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}} c_1
$$

Summary
The solution(s) found are the following

$$
y = x^{\frac{5}{2}} c_1 \qquad (1)
$$

Verification of solutions

$$
y = x^{\frac{5}{2}} c_1
$$

Verified OK.

16.3.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$4x^2y'' - 16y'x + 25y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{25}{4x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{25}{4x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{5}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= e^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= (c_1 \ln(x) + c_2)\,x^{\frac{5}{2}} \\ &= (c_1 \ln(x) + c_2)\,x^{\frac{5}{2}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \ln(x) + c_2)\,x^{\frac{5}{2}} \tag{1}$$

Verification of solutions

$$y = (c_1 \ln(x) + c_2)\,x^{\frac{5}{2}}$$

Verified OK.

16.3.5 Solving using Kovacic algorithm

Writing the ode as

$$4x^2y'' - 16y'x + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -16x \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 545: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-}=\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-16x}{4x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1\left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{\frac{5}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\frac{5}{2}}\right) + c_2\left(x^{\frac{5}{2}}(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x^{\frac{5}{2}} c_1 + c_2 x^{\frac{5}{2}} \ln(x) \tag{1}
$$

Verification of solutions

$$
y = x^{\frac{5}{2}} c_1 + c_2 x^{\frac{5}{2}} \ln(x)
$$

Verified OK.

16.3.6 Maple step by step solution

Let's solve

$4x^2y'' - 16y'x + 25y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{x} - \frac{25y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{25y}{4x^2} = 0$

- Multiply by denominators of the ODE

  $4x^2y'' - 16y'x + 25y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 16\frac{d}{dt}y(t) + 25y(t) = 0$

- Simplify

  $4\frac{d^2}{dt^2}y(t) - 20\frac{d}{dt}y(t) + 25y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = 5\frac{d}{dt}y(t) - \frac{25y(t)}{4}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + \frac{25y(t)}{4} = 0$
- Characteristic polynomial of ODE

  $r^2 - 5r + \frac{25}{4} = 0$
- Factor the characteristic polynomial

  $\frac{(2r-5)^2}{4} = 0$
- Root of the characteristic polynomial

  $r = \frac{5}{2}$
- 1st solution of the ODE

  $y_1(t) = e^{\frac{5t}{2}}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{\frac{5t}{2}}$
- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$
- Substitute in solutions

  $y(t) = c_1e^{\frac{5t}{2}} + c_2t\,e^{\frac{5t}{2}}$
- Change variables back using $t = \ln(x)$

  $y = x^{\frac{5}{2}}c_1 + c_2x^{\frac{5}{2}}\ln(x)$
- Simplify

  $y = (c_1 + c_2\ln(x))\,x^{\frac{5}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(4*x^2*diff(y(x),x$2)-16*x*diff(y(x),x)+25*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_2 \ln(x) + c_1) x^{\frac{5}{2}}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 25

```
DSolve[4*x^2*y''[x]-16*x*y'[x]+25*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} x^{5/2} (5c_2 \log(x) + 2c_1)$$

16.4 problem 4

Internal problem ID [2253]
Internal file name [`OUTPUT/2253_Monday_February_26_2024_09_18_51_AM_9694866/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 5y'x + 10y = 0}$$

16.4.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 5xrx^{r-1} + 10x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 5r\,x^r + 10x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 5r + 10 = 0$$

Or

$$r^2 + 4r + 10 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 - i\sqrt{6} \\ r_2 &= -2 + i\sqrt{6} \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = -2$ and $\beta = -\sqrt{6}$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^{\alpha}\left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = -2, \beta = -\sqrt{6}$, the above becomes

$$y = x^{-2}\left(c_1 e^{-i\sqrt{6}\,\ln(x)} + c_2 e^{i\sqrt{6}\,\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = \frac{1}{x^2}\left(c_1 \cos\left(\sqrt{6}\,\ln(x)\right) + c_2 \sin\left(\sqrt{6}\,\ln(x)\right)\right)$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \cos\left(\sqrt{6}\,\ln(x)\right) + c_2 \sin\left(\sqrt{6}\,\ln(x)\right)}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos\left(\sqrt{6}\,\ln(x)\right) + c_2 \sin\left(\sqrt{6}\,\ln(x)\right)}{x^2}$$

Verified OK.

16.4.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + 5y'x + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{10}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{5}{x}dx\right)}dx \\ &= \int e^{-5\ln(x)}\,dx \\ &= \int \frac{1}{x^5}dx \\ &= -\frac{1}{4x^4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{10}{x^2}}{\frac{1}{x^{10}}} \\
&= 10x^8
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 10x^8 y(\tau) &= 0
\end{aligned}$$

But in terms of τ

$$10x^8 = \frac{5}{8\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{8\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$8\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 5y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$8\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0$$

Simplifying gives

$$8r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$8r(r-1) + 0 + 5 = 0$$

Or

$$8r^2 - 8r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{i\sqrt{6}}{4}$$
$$r_2 = \frac{1}{2} + \frac{i\sqrt{6}}{4}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{\sqrt{6}}{4}$. Hence the solution becomes

$$\begin{aligned}
y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\
&= \tau^{\alpha}\left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\
&= \tau^{\alpha}\left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\
&= \tau^{\alpha}\left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right)
\end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{\sqrt{6}}{4}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-\frac{i\sqrt{6}\,\ln(\tau)}{4}} + c_2 e^{\frac{i\sqrt{6}\,\ln(\tau)}{4}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{\sqrt{6}\,\ln(\tau)}{4}\right) + c_2 \sin\left(\frac{\sqrt{6}\,\ln(\tau)}{4}\right)\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{-\frac{1}{x^4}}\left(c_1 \cos\left(\frac{\sqrt{6}\left(-2\ln(2)+\ln\left(-\frac{1}{x^4}\right)\right)}{4}\right) + c_2 \sin\left(\frac{\sqrt{6}\left(-2\ln(2)+\ln\left(-\frac{1}{x^4}\right)\right)}{4}\right)\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-\frac{1}{x^4}}\left(c_1 \cos\left(\frac{\sqrt{6}\left(-2\ln(2)+\ln\left(-\frac{1}{x^4}\right)\right)}{4}\right) + c_2 \sin\left(\frac{\sqrt{6}\left(-2\ln(2)+\ln\left(-\frac{1}{x^4}\right)\right)}{4}\right)\right)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{-\frac{1}{x^4}}\left(c_1 \cos\left(\frac{\sqrt{6}\left(-2\ln(2)+\ln\left(-\frac{1}{x^4}\right)\right)}{4}\right) + c_2 \sin\left(\frac{\sqrt{6}\left(-2\ln(2)+\ln\left(-\frac{1}{x^4}\right)\right)}{4}\right)\right)}{2}$$

Verified OK.

16.4.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + 5y'x + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{10}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{\sqrt{10}}{c\sqrt{\frac{1}{x^2}}\,x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \\
&= \frac{-\frac{\sqrt{10}}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{5}{x}\frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= \frac{2c\sqrt{10}}{5}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{2c\sqrt{10}\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = e^{-\frac{\sqrt{10}\,c\tau}{5}}\left(c_1 \cos\left(\frac{c\sqrt{15}\,\tau}{5}\right) + c_2 \sin\left(\frac{c\sqrt{15}\,\tau}{5}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{10}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1 \cos\left(\sqrt{6}\,\ln(x)\right) + c_2 \sin\left(\sqrt{6}\,\ln(x)\right)}{x^2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos\left(\sqrt{6}\,\ln(x)\right) + c_2 \sin\left(\sqrt{6}\,\ln(x)\right)}{x^2} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 \cos\left(\sqrt{6}\, \ln(x)\right) + c_2 \sin\left(\sqrt{6}\, \ln(x)\right)}{x^2}$$

Verified OK.

16.4.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' + 5y'x + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{5}{x}$$
$$q(x) = \frac{10}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{5n}{x^2} + \frac{10}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -2 + i\sqrt{6} \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{-4 + 2i\sqrt{6}}{x} + \frac{5}{x}\right) v'(x) = 0$$
$$v''(x) + \frac{(2i\sqrt{6} + 1)\, v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(2i\sqrt{6} + 1)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{(-2i\sqrt{6} - 1)\, u}{x} \end{aligned}$$

Where $f(x) = \frac{-2i\sqrt{6}-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-2i\sqrt{6} - 1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-2i\sqrt{6} - 1}{x}\, dx \\ \ln(u) &= \left(-2i\sqrt{6} - 1\right) \ln(x) + c_1 \\ u &= \mathrm{e}^{\left(-2i\sqrt{6}-1\right) \ln(x) + c_1} \\ &= c_1 \mathrm{e}^{\left(-2i\sqrt{6}-1\right) \ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-2i\sqrt{6}}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= \frac{i\sqrt{6}\, c_1 x^{-2i\sqrt{6}}}{12} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(\frac{i\sqrt{6}\, c_1 x^{-2i\sqrt{6}}}{12} + c_2\right) x^{-2+i\sqrt{6}} \\
&= \frac{\frac{i x^{-i\sqrt{6}} \sqrt{6}\, c_1}{12} + x^{i\sqrt{6}} c_2}{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(\frac{i\sqrt{6}\, c_1 x^{-2i\sqrt{6}}}{12} + c_2\right) x^{-2+i\sqrt{6}} \tag{1}
$$

Verification of solutions

$$
y = \left(\frac{i\sqrt{6}\, c_1 x^{-2i\sqrt{6}}}{12} + c_2\right) x^{-2+i\sqrt{6}}
$$

Verified OK.

16.4.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
x^2 y'' + 5y'x + 10y &= 0 && (1) \\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= 5x \\
C &= 10
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-25}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -25$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{25}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 547: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{25}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{25}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i\sqrt{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i\sqrt{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{25}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{25}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i\sqrt{6} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i\sqrt{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{25}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i\sqrt{6}$	$\frac{1}{2} - i\sqrt{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + i\sqrt{6}$	$\frac{1}{2} - i\sqrt{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - i\sqrt{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - i\sqrt{6} - \left(\frac{1}{2} - i\sqrt{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - i\sqrt{6}}{x} + (-)(0) \\ &= \frac{\frac{1}{2} - i\sqrt{6}}{x} \\ &= \frac{-2i\sqrt{6} + 1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - i\sqrt{6}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - i\sqrt{6}}{x^2}\right) + \left(\frac{\frac{1}{2} - i\sqrt{6}}{x}\right)^2 - \left(-\frac{25}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{\frac{1}{2} - i\sqrt{6}}{x} dx} \\
&= x^{\frac{1}{2} - i\sqrt{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{5x}{x^2} \, dx} \\
&= z_1 e^{-\frac{5 \ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{5}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{-2 - i\sqrt{6}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{5x}{x^2}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{-5\ln(x)}}{(y_1)^2}\,dx$$

$$= y_1 \left(-\frac{i x^{2i\sqrt{6}}\sqrt{6}}{12}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(x^{-2-i\sqrt{6}}\right) + c_2 \left(x^{-2-i\sqrt{6}} \left(-\frac{i x^{2i\sqrt{6}}\sqrt{6}}{12}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{-2-i\sqrt{6}} - \frac{i c_2 \sqrt{6}\, x^{-2+i\sqrt{6}}}{12} \tag{1}$$

Verification of solutions

$$y = c_1 x^{-2-i\sqrt{6}} - \frac{i c_2 \sqrt{6}\, x^{-2+i\sqrt{6}}}{12}$$

Verified OK.

16.4.6 Maple step by step solution

Let's solve

$x^2 y'' + 5y'x + 10y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{x} - \frac{10y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{5y'}{x} + \frac{10y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + 5y'x + 10y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 5\frac{d}{dt}y(t) + 10y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 10y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 4r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-24})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-2 - \mathrm{I}\sqrt{6}, -2 + \mathrm{I}\sqrt{6}\right)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-2t}\cos\left(\sqrt{6}\,t\right)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-2t}\sin\left(\sqrt{6}\,t\right)$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-2t} \cos\left(\sqrt{6}\, t\right) + c_2 \mathrm{e}^{-2t} \sin\left(\sqrt{6}\, t\right)$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1 \cos\left(\sqrt{6}\, \ln(x)\right)}{x^2} + \frac{c_2 \sin\left(\sqrt{6}\, \ln(x)\right)}{x^2}$

- Simplify

  $y = \frac{c_1 \cos\left(\sqrt{6}\, \ln(x)\right)}{x^2} + \frac{c_2 \sin\left(\sqrt{6}\, \ln(x)\right)}{x^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+10*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\sqrt{6}\, \ln(x)\right) + c_2 \cos\left(\sqrt{6}\, \ln(x)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 34

```
DSolve[x^2*y''[x]+5*x*y'[x]+10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 \cos\left(\sqrt{6} \log(x)\right) + c_1 \sin\left(\sqrt{6} \log(x)\right)}{x^2}$$

16.5 problem 5

Internal problem ID [2254]
Internal file name [`OUTPUT/2254_Monday_February_26_2024_09_18_52_AM_93778860/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - 3y'x - 18y = \ln(x)}$$

16.5.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2x^2, B = -3x, C = -18, f(x) = \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$2x^2y'' - 3y'x - 18y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$2x^2(r(r-1))x^{r-2} - 3xrx^{r-1} - 18x^r = 0$$

Simplifying gives

$$2r(r-1)\,x^r - 3r\,x^r - 18x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$2r(r-1) - 3r - 18 = 0$$

Or

$$2r^2 - 5r - 18 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2$$
$$r_2 = \frac{9}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2x^{\frac{9}{2}}$$

Next, we find the particular solution to the ODE

$$2x^2y'' - 3y'x - 18y = \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$

$$y_2 = x^{\frac{9}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^{\frac{9}{2}} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(x^{\frac{9}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^{\frac{9}{2}} \\ -\frac{2}{x^3} & \frac{9x^{\frac{7}{2}}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(\frac{9x^{\frac{7}{2}}}{2}\right) - \left(x^{\frac{9}{2}}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{13x^{\frac{3}{2}}}{2}$$

Which simplifies to

$$W = \frac{13x^{\frac{3}{2}}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{9}{2}} \ln(x)}{13x^{\frac{7}{2}}} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{x \ln(x)}{13} dx$$

Hence

$$u_1 = -\frac{\ln(x)\, x^2}{26} + \frac{x^2}{52}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x)}{x^2}}{13x^{\frac{7}{2}}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{13x^{\frac{11}{2}}} dx$$

Hence

$$u_2 = -\frac{2\ln(x)}{117x^{\frac{9}{2}}} - \frac{4}{1053x^{\frac{9}{2}}}$$

Which simplifies to

$$u_1 = -\frac{x^2(2\ln(x) - 1)}{52}$$

$$u_2 = \frac{-\frac{2\ln(x)}{117} - \frac{4}{1053}}{x^{\frac{9}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{18} + \frac{5}{324}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -\frac{\ln(x)}{18} + \frac{5}{324} + \frac{c_1}{x^2} + c_2 x^{\frac{9}{2}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(x)}{18} + \frac{5}{324} + \frac{c_1}{x^2} + c_2 x^{\frac{9}{2}} \tag{1}$$

Verification of solutions

$$y = -\frac{\ln(x)}{18} + \frac{5}{324} + \frac{c_1}{x^2} + c_2 x^{\frac{9}{2}}$$

Verified OK.

16.5.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2x^2y'' - 3y'x - 18y = 0$$

In normal form the ode

$$2x^2y'' - 3y'x - 18y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{2x} \\ q(x) &= -\frac{9}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int -\frac{3}{2x} dx\right)} dx \\
&= \int e^{\frac{3\ln(x)}{2}} dx \\
&= \int x^{\frac{3}{2}} dx \\
&= \frac{2x^{\frac{5}{2}}}{5}
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{9}{x^2}}{x^3} \\
&= -\frac{9}{x^5}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{9y(\tau)}{x^5} &= 0
\end{aligned}
$$

But in terms of τ

$$
-\frac{9}{x^5} = -\frac{36}{25\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) - \frac{36y(\tau)}{25\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 36y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 36\tau^r = 0
$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r - 36\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 - 36 = 0$$

Or

$$25r^2 - 25r - 36 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{4}{5}$$
$$r_2 = \frac{9}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau^{\frac{4}{5}}} + c_2 \tau^{\frac{9}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(4c_2 2^{\frac{3}{5}} x^5 \left(x^{\frac{5}{2}}\right)^{\frac{3}{5}} + 25c_1 5^{\frac{3}{5}}\right) 10^{\frac{1}{5}}}{50 \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\left(4c_2 2^{\frac{3}{5}} x^5 \left(x^{\frac{5}{2}}\right)^{\frac{3}{5}} + 25c_1 5^{\frac{3}{5}}\right) 10^{\frac{1}{5}}}{50 \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}$$

$$y_2 = \frac{x^5}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}} & \frac{x^5}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}} \\ \frac{d}{dx}\left(\frac{1}{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}\right) & \frac{d}{dx}\left(\frac{x^5}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}} & \frac{x^5}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}} \\ -\frac{2x^{\frac{3}{2}}}{\left(x^{\frac{5}{2}}\right)^{\frac{9}{5}}} & -\frac{x^{\frac{13}{2}}}{2\left(x^{\frac{5}{2}}\right)^{\frac{6}{5}}} + \frac{5x^4}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}\right)\left(-\frac{x^{\frac{13}{2}}}{2\left(x^{\frac{5}{2}}\right)^{\frac{6}{5}}} + \frac{5x^4}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}\right) - \left(\frac{x^5}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}\right)\left(-\frac{2x^{\frac{3}{2}}}{\left(x^{\frac{5}{2}}\right)^{\frac{9}{5}}}\right)$$

Which simplifies to

$$W = \frac{13x^{\frac{3}{2}}}{2}$$

Which simplifies to

$$W = \frac{13x^{\frac{3}{2}}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^5 \ln(x)}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}}{13x^{\frac{7}{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{\frac{3}{2}} \ln(x)}{13 \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}} dx$$

Hence

$$u_1 = -\frac{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}} (2\ln(x) - 1)}{52}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x)}{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}}{13x^{\frac{7}{2}}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{13 \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}} x^{\frac{7}{2}}} dx$$

Hence

$$u_2 = \int_0^x \frac{\ln(\alpha)}{13 \left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \alpha^{\frac{7}{2}}} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{26} + \frac{1}{52} + \frac{\left(\int_0^x \frac{\ln(\alpha)}{13\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \alpha^{\frac{7}{2}}} d\alpha\right) x^5}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}$$

Which simplifies to

$$y_p(x) = -\frac{\ln(x)}{26} + \frac{1}{52} + \frac{\left(\int_0^x \frac{\ln(\alpha)}{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \alpha^{\frac{7}{2}}} d\alpha\right) x^5}{13\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\left(4c_2 2^{\frac{3}{5}} x^5 \left(x^{\frac{5}{2}}\right)^{\frac{3}{5}} + 25c_1 5^{\frac{3}{5}}\right) 10^{\frac{1}{5}}}{50\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}\right) + \left(-\frac{\ln(x)}{26} + \frac{1}{52} + \frac{\left(\int_0^x \frac{\ln(\alpha)}{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \alpha^{\frac{7}{2}}} d\alpha\right) x^5}{13\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(4c_2 2^{\frac{3}{5}} x^5 \left(x^{\frac{5}{2}}\right)^{\frac{3}{5}} + 25c_1 5^{\frac{3}{5}}\right) 10^{\frac{1}{5}}}{50\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}} - \frac{\ln(x)}{26} + \frac{1}{52} + \frac{\left(\int_0^x \frac{\ln(\alpha)}{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \alpha^{\frac{7}{2}}} d\alpha\right) x^5}{13\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}} \tag{1}$$

Verification of solutions

$$y = \frac{\left(4c_2 2^{\frac{3}{5}} x^5 \left(x^{\frac{5}{2}}\right)^{\frac{3}{5}} + 25c_1 5^{\frac{3}{5}}\right) 10^{\frac{1}{5}}}{50\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}} - \frac{\ln(x)}{26} + \frac{1}{52} + \frac{\left(\int_0^x \frac{\ln(\alpha)}{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \alpha^{\frac{7}{2}}} d\alpha\right) x^5}{13\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}$$

Verified OK.

16.5.3 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2x^2, B = -3x, C = -18, f(x) = \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$2x^2y'' - 3y'x - 18y = 0$$

In normal form the ode

$$2x^2y'' - 3y'x - 18y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{2x} \\ q(x) &= -\frac{9}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{2x^2} - \frac{9}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{9}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{15v'(x)}{2x} &= 0 \\ v''(x) + \frac{15v'(x)}{2x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{15u(x)}{2x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{15u}{2x} \end{aligned}$$

Where $f(x) = -\frac{15}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{15}{2x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{15}{2x}\, dx \\ \ln(u) &= -\frac{15\ln(x)}{2} + c_1 \\ u &= \mathrm{e}^{-\frac{15\ln(x)}{2} + c_1} \\ &= \frac{c_1}{x^{\frac{15}{2}}} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{2c_1}{13x^{\frac{13}{2}}} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{2c_1}{13x^{\frac{13}{2}}} + c_2\right) x^{\frac{9}{2}} \\
&= \frac{13c_2 x^{\frac{13}{2}} - 2c_1}{13x^2}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
2x^2 y'' - 3y'x - 18y = \ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^2} \\
y_2 &= x^{\frac{9}{2}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \frac{1}{x^2} & x^{\frac{9}{2}} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(x^{\frac{9}{2}}\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^{\frac{9}{2}} \\ -\frac{2}{x^3} & \frac{9x^{\frac{7}{2}}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(\frac{9x^{\frac{7}{2}}}{2}\right) - \left(x^{\frac{9}{2}}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{13x^{\frac{3}{2}}}{2}$$

Which simplifies to

$$W = \frac{13x^{\frac{3}{2}}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{9}{2}} \ln (x)}{13x^{\frac{7}{2}}} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{x \ln (x)}{13} dx$$

Hence

$$u_1 = -\frac{\ln (x) \, x^2}{26} + \frac{x^2}{52}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x)}{x^2}}{13x^{\frac{7}{2}}} \, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln (x)}{13x^{\frac{11}{2}}} dx$$

Hence

$$u_2 = -\frac{2 \ln (x)}{117x^{\frac{9}{2}}} - \frac{4}{1053x^{\frac{9}{2}}}$$

Which simplifies to

$$u_1 = -\frac{x^2(2\ln(x) - 1)}{52}$$

$$u_2 = \frac{-\frac{2\ln(x)}{117} - \frac{4}{1053}}{x^{\frac{9}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{18} + \frac{5}{324}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{2c_1}{13x^{\frac{13}{2}}} + c_2\right)x^{\frac{9}{2}}\right) + \left(-\frac{\ln(x)}{18} + \frac{5}{324}\right) \\ &= -\frac{\ln(x)}{18} + \frac{5}{324} + \left(-\frac{2c_1}{13x^{\frac{13}{2}}} + c_2\right)x^{\frac{9}{2}} \end{aligned}$$

Which simplifies to

$$y = -\frac{\ln(x)}{18} + \frac{5}{324} + \left(-\frac{2c_1}{13x^{\frac{13}{2}}} + c_2\right)x^{\frac{9}{2}}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(x)}{18} + \frac{5}{324} + \left(-\frac{2c_1}{13x^{\frac{13}{2}}} + c_2\right)x^{\frac{9}{2}} \tag{1}$$

Verification of solutions

$$y = -\frac{\ln(x)}{18} + \frac{5}{324} + \left(-\frac{2c_1}{13x^{\frac{13}{2}}} + c_2\right)x^{\frac{9}{2}}$$

Verified OK.

16.5.4 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
2x^2y'' - 3y'x - 18y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 2x^2 \\
B &= -3x \tag{3} \\
C &= -18
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{165}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 165 \\
t &= 16x^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{165}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 549: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{165}{16x^2}
$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{165}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{15}{4} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{11}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{165}{16x^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{165}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{15}{4} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{11}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{165}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{15}{4}$	$-\frac{11}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{15}{4}$	$-\frac{11}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{11}{4}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{11}{4} - \left(-\frac{11}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{11}{4x} + (-)\,(0) \\ &= -\frac{11}{4x} \\ &= -\frac{11}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{11}{4x}\right)(0) + \left(\left(\frac{11}{4x^2}\right) + \left(-\frac{11}{4x}\right)^2 - \left(\frac{165}{16x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{11}{4x} dx} \\ &= \frac{1}{x^{\frac{11}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{2x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{3}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2x^{\frac{13}{2}}}{13}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2}\right) + c_2 \left(\frac{1}{x^2}\left(\frac{2x^{\frac{13}{2}}}{13}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$2x^2y'' - 3y'x - 18y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x^2} + \frac{2c_2x^{\frac{9}{2}}}{13}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$

$$y_2 = \frac{2x^{\frac{9}{2}}}{13}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{2x^{\frac{9}{2}}}{13} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{2x^{\frac{9}{2}}}{13}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{2x^{\frac{9}{2}}}{13} \\ -\frac{2}{x^3} & \frac{9x^{\frac{7}{2}}}{13} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(\frac{9x^{\frac{7}{2}}}{13}\right) - \left(\frac{2x^{\frac{9}{2}}}{13}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = x^{\frac{3}{2}}$$

Which simplifies to

$$W = x^{\frac{3}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2x^{\frac{9}{2}}\ln(x)}{13}}{2x^{\frac{7}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x\ln(x)}{13}dx$$

Hence

$$u_1 = -\frac{\ln(x)\,x^2}{26} + \frac{x^2}{52}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x)}{x^2}}{2x^{\frac{7}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{2x^{\frac{11}{2}}}dx$$

Hence

$$u_2 = -\frac{\ln(x)}{9x^{\frac{9}{2}}} - \frac{2}{81x^{\frac{9}{2}}}$$

Which simplifies to

$$u_1 = -\frac{x^2(2\ln(x) - 1)}{52}$$

$$u_2 = \frac{-9\ln(x) - 2}{81x^{\frac{9}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{18} + \frac{5}{324}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^2} + \frac{2c_2x^{\frac{9}{2}}}{13}\right) + \left(-\frac{\ln(x)}{18} + \frac{5}{324}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2} + \frac{2c_2x^{\frac{9}{2}}}{13} - \frac{\ln(x)}{18} + \frac{5}{324} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^2} + \frac{2c_2x^{\frac{9}{2}}}{13} - \frac{\ln(x)}{18} + \frac{5}{324}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve(2*x^2*diff(y(x),x$2)-3*x*diff(y(x),x)-18*y(x)=ln(x),y(x), singsol=all)
```

$$y(x) = \frac{c_2}{x^2} + x^{\frac{9}{2}} c_1 - \frac{\ln(x)}{18} + \frac{5}{324}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 29

```
DSolve[2*x^2*y''[x]-3*x*y'[x]-18*y[x]==Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x^{9/2} + \frac{c_1}{x^2} - \frac{\log(x)}{18} + \frac{5}{324}$$

16.6 problem 6

Internal problem ID [2255]
Internal file name [`OUTPUT/2255_Monday_February_26_2024_09_18_53_AM_22386589/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - 3y'x + 2y = \ln(x^2)}$$

16.6.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2x^2, B = -3x, C = 2, f(x) = \ln(x^2)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$2x^2y'' - 3y'x + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$2x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$2r(r-1)\,x^r - 3r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$2r(r-1) - 3r + 2 = 0$$

Or

$$2r^2 - 5r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= \frac{1}{2} \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1 x^2 + \sqrt{x}\, c_2$$

Next, we find the particular solution to the ODE

$$2x^2 y'' - 3y'x + 2y = \ln\left(x^2\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= \sqrt{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \sqrt{x} \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\sqrt{x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \sqrt{x} \\ 2x & \frac{1}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = (x^2)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x})(2x)$$

Which simplifies to

$$W = -\frac{3x^{\frac{3}{2}}}{2}$$

Which simplifies to

$$W = -\frac{3x^{\frac{3}{2}}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sqrt{x}\,\ln(x^2)}{-3x^{\frac{7}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\ln(x^2)}{3x^3}\,dx$$

Hence

$$u_1 = -\frac{\ln(x^2)}{6x^2} - \frac{1}{6x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2 \ln(x^2)}{-3x^{\frac{7}{2}}}dx$$

Which simplifies to

$$u_2 = \int -\frac{\ln(x^2)}{3x^{\frac{3}{2}}}dx$$

Hence

$$u_2 = \frac{2\ln(x^2)}{3\sqrt{x}} + \frac{8}{3\sqrt{x}}$$

Which simplifies to

$$u_1 = \frac{-1 - \ln(x^2)}{6x^2}$$

$$u_2 = \frac{\frac{2\ln(x^2)}{3} + \frac{8}{3}}{\sqrt{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{5}{2} + \frac{\ln(x^2)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{5}{2} + \frac{\ln(x^2)}{2} + c_1x^2 + \sqrt{x}\, c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{5}{2} + \frac{\ln(x^2)}{2} + c_1x^2 + \sqrt{x}\, c_2 \tag{1}$$

Verification of solutions

$$y = \frac{5}{2} + \frac{\ln(x^2)}{2} + c_1x^2 + \sqrt{x}\, c_2$$

Verified OK.

16.6.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2x^2y'' - 3y'x + 2y = 0$$

In normal form the ode

$$2x^2y'' - 3y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{2x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int -\frac{3}{2x}dx\right)} dx \\
&= \int e^{\frac{3\ln(x)}{2}} dx \\
&= \int x^{\frac{3}{2}} dx \\
&= \frac{2x^{\frac{5}{2}}}{5}
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{x^3} \\
&= \frac{1}{x^5}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{x^5} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{1}{x^5} = \frac{4}{25\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{25\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 4y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 4\tau^r = 0
$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 4\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 4 = 0$$

Or

$$25r^2 - 25r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{5}$$
$$r_2 = \frac{4}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{5}} + c_2 \tau^{\frac{4}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 2^{\frac{1}{5}} 5^{\frac{4}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{5} + \frac{c_2 2^{\frac{4}{5}} 5^{\frac{1}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{5}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 2^{\frac{1}{5}} 5^{\frac{4}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{5} + \frac{c_2 2^{\frac{4}{5}} 5^{\frac{1}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}$$

$$y_2 = \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}} & \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}} \\ \frac{d}{dx}\left(\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}\right) & \frac{d}{dx}\left(\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}} & \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}} \\ \frac{x^{\frac{3}{2}}}{2\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}} & \frac{2x^{\frac{3}{2}}}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}\right)\left(\frac{2x^{\frac{3}{2}}}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}\right) - \left(\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}\right)\left(\frac{x^{\frac{3}{2}}}{2\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}\right)$$

Which simplifies to

$$W = \frac{3x^{\frac{3}{2}}}{2}$$

Which simplifies to

$$W = \frac{3x^{\frac{3}{2}}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln\left(x^2\right)}{3x^{\frac{7}{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln\left(x^2\right)}{3x^{\frac{7}{2}}} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln\left(\alpha^2\right)}{3\alpha^{\frac{7}{2}}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln\left(x^2\right)}{3x^{\frac{7}{2}}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln\left(x^2\right)}{3x^{\frac{7}{2}}} dx$$

Hence

$$u_2 = \int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln\left(\alpha^2\right)}{3\alpha^{\frac{7}{2}}} d\alpha$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right)}{3}$$

$$u_2 = \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{3} + \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 2^{\frac{1}{5}} 5^{\frac{4}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{5} + \frac{c_2 2^{\frac{4}{5}} 5^{\frac{1}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{5}\right) \\ &\quad + \left(-\frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{3} + \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{3}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned} y = {} & \frac{c_1 2^{\frac{1}{5}} 5^{\frac{4}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{5} + \frac{c_2 2^{\frac{4}{5}} 5^{\frac{1}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{5} \\ & - \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{3} + \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{3} \end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 2^{\frac{1}{5}} 5^{\frac{4}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{5} + \frac{c_2 2^{\frac{4}{5}} 5^{\frac{1}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{5} - \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right) \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{3} + \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right) \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{3}$$

Verified OK.

16.6.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2x^2, B = -3x, C = 2, f(x) = \ln(x^2)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$2x^2y'' - 3y'x + 2y = 0$$

In normal form the ode

$$2x^2y'' - 3y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{2x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{2x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{5c}{2}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{5c\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5c\tau}{4}}\left(c_1\cosh\left(\frac{3c\tau}{4}\right) + ic_2\sinh\left(\frac{3c\tau}{4}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{4}}\left(c_1\cosh\left(\frac{3\ln(x)}{4}\right) + ic_2\sinh\left(\frac{3\ln(x)}{4}\right)\right)
$$

Now the particular solution to this ODE is found

$$
2x^2y'' - 3y'x + 2y = \ln\left(x^2\right)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}} \\
y_2 &= \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}} & \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}} \\ \frac{d}{dx}\left(\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}\right) & \frac{d}{dx}\left(\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}} & \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}} \\ \frac{x^{\frac{3}{2}}}{2\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}} & \frac{2x^{\frac{3}{2}}}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}\right)\left(\frac{2x^{\frac{3}{2}}}{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}\right) - \left(\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}\right)\left(\frac{x^{\frac{3}{2}}}{2\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}\right)$$

Which simplifies to

$$W = \frac{3x^{\frac{3}{2}}}{2}$$

Which simplifies to

$$W = \frac{3x^{\frac{3}{2}}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln\left(x^2\right)}{3x^{\frac{7}{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln\left(x^2\right)}{3x^{\frac{7}{2}}} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln\left(\alpha^2\right)}{3\alpha^{\frac{7}{2}}} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln\left(x^2\right)}{3x^{\frac{7}{2}}} dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln\left(x^2\right)}{3x^{\frac{7}{2}}} dx$$

Hence

$$u_2 = \int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln\left(\alpha^2\right)}{3\alpha^{\frac{7}{2}}} d\alpha$$

Which simplifies to

$$u_1 = -\frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right)}{3}$$

$$u_2 = \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right) \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{3} + \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln(\alpha^2)}{\alpha^{\frac{7}{2}}} d\alpha\right) \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(x^{\frac{5}{4}}\left(c_1 \cosh\left(\frac{3\ln(x)}{4}\right) + i c_2 \sinh\left(\frac{3\ln(x)}{4}\right)\right)\right) \\
&\quad + \left(-\frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}}\ln(\alpha^2)}{\alpha^{\frac{7}{2}}}d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{3} + \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}}\ln(\alpha^2)}{\alpha^{\frac{7}{2}}}d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{3}\right) \\
&= -\frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}}\ln(\alpha^2)}{\alpha^{\frac{7}{2}}}d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{3} + \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}}\ln(\alpha^2)}{\alpha^{\frac{7}{2}}}d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{3} \\
&\quad + x^{\frac{5}{4}}\left(c_1 \cosh\left(\frac{3\ln(x)}{4}\right) + i c_2 \sinh\left(\frac{3\ln(x)}{4}\right)\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y &= i x^{\frac{5}{4}} \sinh\left(\frac{3\ln(x)}{4}\right) c_2 + x^{\frac{5}{4}} \cosh\left(\frac{3\ln(x)}{4}\right) c_1 \\
&\quad + \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}}\ln(\alpha^2)}{\alpha^{\frac{7}{2}}}d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{3} - \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}}\ln(\alpha^2)}{\alpha^{\frac{7}{2}}}d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= i x^{\frac{5}{4}} \sinh\left(\frac{3\ln(x)}{4}\right) c_2 + x^{\frac{5}{4}} \cosh\left(\frac{3\ln(x)}{4}\right) c_1 \\
&\quad + \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}}\ln(\alpha^2)}{\alpha^{\frac{7}{2}}}d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{3} - \frac{\left(\int_0^x \frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}}\ln(\alpha^2)}{\alpha^{\frac{7}{2}}}d\alpha\right)\left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{3}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = i x^{\frac{5}{4}} \sinh\left(\frac{3 \ln\left(x\right)}{4}\right) c_{2}+x^{\frac{5}{4}} \cosh\left(\frac{3 \ln\left(x\right)}{4}\right) c_{1} \\
+\frac{\left(\int_{0}^{x}\frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{1}{5}} \ln\left(\alpha^{2}\right)}{\alpha^{\frac{7}{2}}}d\alpha\right) \left(x^{\frac{5}{2}}\right)^{\frac{4}{5}}}{3}-\frac{\left(\int_{0}^{x}\frac{\left(\alpha^{\frac{5}{2}}\right)^{\frac{4}{5}} \ln\left(\alpha^{2}\right)}{\alpha^{\frac{7}{2}}}d\alpha\right) \left(x^{\frac{5}{2}}\right)^{\frac{1}{5}}}{3}
$$

Verified OK.

16.6.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
A y''(x) + B y'(x) + C y(x) = f(x)
$$

Where $A = 2x^{2}, B = -3x, C = 2, f(x) = \ln\left(x^{2}\right)$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$
2x^{2}y''-3y'x+2y = 0
$$

In normal form the ode

$$
2x^{2}y''-3y'x+2y = 0 \tag{1}
$$

Becomes

$$
y''+p(x) y'+q(x) y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{3}{2x} \\
q(x) &= \frac{1}{x^{2}}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x) x^{n}$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x)+\left(\frac{2n}{x}+p\right) v'(x)+\left(\frac{n(n-1)}{x^{2}}+\frac{np}{x}+q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{2x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{5v'(x)}{2x} = 0$$
$$v''(x) + \frac{5v'(x)}{2x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{2x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{5u}{2x} \end{aligned}$$

Where $f(x) = -\frac{5}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{5}{2x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{5}{2x}\,dx \\ \ln(u) &= -\frac{5\ln(x)}{2} + c_1 \\ u &= e^{-\frac{5\ln(x)}{2} + c_1} \\ &= \frac{c_1}{x^{\frac{5}{2}}} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{2c_1}{3x^{\frac{3}{2}}} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right) x^2 \\
&= c_2 x^2 - \frac{2\sqrt{x}\, c_1}{3}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
2x^2y'' - 3y'x + 2y = \ln\left(x^2\right)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= \sqrt{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \sqrt{x} \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\sqrt{x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \sqrt{x} \\ 2x & \frac{1}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = (x^2)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x})\,(2x)$$

Which simplifies to

$$W = -\frac{3x^{\frac{3}{2}}}{2}$$

Which simplifies to

$$W = -\frac{3x^{\frac{3}{2}}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sqrt{x}\,\ln(x^2)}{-3x^{\frac{7}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\ln(x^2)}{3x^3}dx$$

Hence

$$u_1 = -\frac{\ln(x^2)}{6x^2} - \frac{1}{6x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2\ln(x^2)}{-3x^{\frac{7}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\ln(x^2)}{3x^{\frac{3}{2}}} dx$$

Hence

$$u_2 = \frac{2\ln(x^2)}{3\sqrt{x}} + \frac{8}{3\sqrt{x}}$$

Which simplifies to

$$u_1 = \frac{-1 - \ln(x^2)}{6x^2}$$

$$u_2 = \frac{\frac{2\ln(x^2)}{3} + \frac{8}{3}}{\sqrt{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{5}{2} + \frac{\ln(x^2)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x^2\right) + \left(\frac{5}{2} + \frac{\ln(x^2)}{2}\right) \\ &= \frac{5}{2} + \frac{\ln(x^2)}{2} + \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x^2 \end{aligned}$$

Which simplifies to

$$y = c_2x^2 - \frac{2\sqrt{x}\,c_1}{3} + \frac{\ln(x^2)}{2} + \frac{5}{2}$$

Summary

The solution(s) found are the following

$$y = c_2x^2 - \frac{2\sqrt{x}\,c_1}{3} + \frac{\ln(x^2)}{2} + \frac{5}{2} \tag{1}$$

Verification of solutions

$$y = c_2x^2 - \frac{2\sqrt{x}\,c_1}{3} + \frac{\ln(x^2)}{2} + \frac{5}{2}$$

Verified OK.

16.6.5 Solving using Kovacic algorithm

Writing the ode as

$$2x^2y'' - 3y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -3x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 550: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{5}{16x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5}{16x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{5}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4x} + (-)\,(0) \\
&= -\frac{1}{4x} \\
&= -\frac{1}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{4x}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x}\right)^2 - \left(\frac{5}{16x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{4x} dx} \\
&= \frac{1}{x^{\frac{1}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{2x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{4}} \\
&= z_1\left(x^{\frac{3}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{2x^{\frac{3}{2}}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(\frac{2x^{\frac{3}{2}}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$2x^2y'' - 3y'x + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \sqrt{x}\, c_1 + \frac{2c_2x^2}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x}$$
$$y_2 = \frac{2x^2}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x} & \frac{2x^2}{3} \\ \frac{d}{dx}\left(\sqrt{x}\right) & \frac{d}{dx}\left(\frac{2x^2}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x} & \frac{2x^2}{3} \\ \frac{1}{2\sqrt{x}} & \frac{4x}{3} \end{vmatrix}$$

Therefore

$$W = (\sqrt{x})\left(\frac{4x}{3}\right) - \left(\frac{2x^2}{3}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = x^{\frac{3}{2}}$$

Which simplifies to

$$W = x^{\frac{3}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2x^2 \ln(x^2)}{3}}{2x^{\frac{7}{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x^2)}{3x^{\frac{3}{2}}} dx$$

Hence

$$u_1 = \frac{2\ln(x^2)}{3\sqrt{x}} + \frac{8}{3\sqrt{x}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x}\, \ln(x^2)}{2x^{\frac{7}{2}}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x^2)}{2x^3} dx$$

Hence

$$u_2 = -\frac{\ln(x^2)}{4x^2} - \frac{1}{4x^2}$$

Which simplifies to

$$u_1 = \frac{\frac{2\ln(x^2)}{3} + \frac{8}{3}}{\sqrt{x}}$$

$$u_2 = \frac{-1 - \ln(x^2)}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{5}{2} + \frac{\ln(x^2)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\sqrt{x}\, c_1 + \frac{2c_2x^2}{3}\right) + \left(\frac{5}{2} + \frac{\ln(x^2)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x}\, c_1 + \frac{2c_2x^2}{3} + \frac{5}{2} + \frac{\ln(x^2)}{2} \tag{1}$$

Verification of solutions

$$y = \sqrt{x}\, c_1 + \frac{2c_2x^2}{3} + \frac{5}{2} + \frac{\ln(x^2)}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(2*x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+2*y(x)=ln(x^2),y(x), singsol=all)
```

$$y(x) = \frac{5}{2} + \frac{\ln(x^2)}{2} + \frac{2c_1x^2}{3} + \sqrt{x}\, c_2$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 30

```
DSolve[2*x^2*y''[x]-3*x*y'[x]+2*y[x]==Log[x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\log\left(x^2\right) + 5\right) + c_2x^2 + c_1\sqrt{x}$$

16.7 problem 7

Internal problem ID [2256]
Internal file name [`OUTPUT/2256_Monday_February_26_2024_09_18_55_AM_69991001/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3y'x + 4y = x^3}$$

16.7.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3y'x + 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x^2 + c_2 \ln(x)\,x^2$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3y'x + 4y = x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= \ln(x)\,x^2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2x\ln(x)) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5 \ln(x)}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \ln(x)\, dx$$

Hence

$$u_1 = -x\ln(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^5}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-x \ln(x) + x)\, x^2 + x^3 \ln(x)$$

Which simplifies to

$$y_p(x) = x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^2(c_2 \ln(x) + c_1 + x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^2(c_2 \ln(x) + c_1 + x) \tag{1}$$

Verification of solutions

$$y = x^2(c_2 \ln(x) + c_1 + x)$$

Verified OK.

16.7.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\ &= \int e^{-\left(\int -\frac{3}{x}dx\right)}dx \\ &= \int e^{3\ln(x)}\,dx \\ &= \int x^3 dx \\ &= \frac{x^4}{4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{4}{x^2}}{x^6} \\ &= \frac{4}{x^8} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(-2c_2 \ln(2) + c_2 \ln(x^4) + c_1)\sqrt{x^4}}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{(-2c_2 \ln(2) + c_2 \ln(x^4) + c_1)\sqrt{x^4}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x^4}$$

$$y_2 = -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{2x^3}{\sqrt{x^4}} & -\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(-\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}}\right) - \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)x^3}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int \left(\ln(x) - \frac{\ln(2)}{2}\right) dx$$

Hence

$$u_1 = \frac{x(\ln(2) - 2\ln(x) + 2)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4}\,x^3}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{1}{2} dx$$

Hence

$$u_2 = \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x(\ln(2) - 2\ln(x) + 2)\sqrt{x^4}}{2} + \frac{x\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(-2c_2\ln(2) + c_2\ln(x^4) + c_1)\sqrt{x^4}}{2}\right) + (x^3) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(-2c_2 \ln(2) + c_2 \ln(x^4) + c_1\right)\sqrt{x^4}}{2} + x^3 \tag{1}$$

Verification of solutions

$$y = \frac{\left(-2c_2 \ln(2) + c_2 \ln(x^4) + c_1\right)\sqrt{x^4}}{2} + x^3$$

Verified OK. `{0 < x}`

16.7.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau} c_1$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 x^2$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3y'x + 4y = x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \sqrt{x^4} \\ y_2 &= -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{2x^3}{\sqrt{x^4}} & -\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(-\frac{2\ln(2)\,x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)\,x^3}{\sqrt{x^4}}\right) - \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)x^3}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int \left(\ln(x) - \frac{\ln(2)}{2}\right) dx$$

Hence

$$u_1 = \frac{x(\ln(2) - 2\ln(x) + 2)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4}\,x^3}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{1}{2}dx$$

Hence

$$u_2 = \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x(\ln (2) - 2 \ln (x) + 2) \sqrt{x^4}}{2} + \frac{x\left(-\ln (2) \sqrt{x^4} + \frac{\ln(x^4) \sqrt{x^4}}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2) + (x^3) \\ &= c_1 x^2 + x^3 \end{aligned}$$

Which simplifies to

$$y = x^2(x + c_1)$$

Summary

The solution(s) found are the following

$$y = x^2(x + c_1) \tag{1}$$

Verification of solutions

$$y = x^2(x + c_1)$$

Verified OK. `{0 < x}`

16.7.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= e^{-\ln(x) + c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= (c_1 \ln(x) + c_2)\, x^2 \\
&= (c_1 \ln(x) + c_2)\, x^2
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 3y'x + 4y = x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= \ln(x)\, x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)\,(x + 2x\ln(x)) - (\ln(x)\, x^2)\,(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5 \ln(x)}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \ln(x)\, dx$$

Hence

$$u_1 = -x(\ln(x) - 1)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^5}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^3(\ln(x) - 1) + x^3 \ln(x)$$

Which simplifies to

$$y_p(x) = x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((c_1 \ln(x) + c_2) x^2\right) + \left(x^3\right) \\ &= x^3 + (c_1 \ln(x) + c_2) x^2 \end{aligned}$$

Which simplifies to

$$y = x^2(c_1 \ln(x) + c_2 + x)$$

Summary
The solution(s) found are the following

$$y = x^2(c_1 \ln(x) + c_2 + x) \tag{1}$$

Verification of solutions

$$y = x^2(c_1 \ln(x) + c_2 + x)$$

Verified OK. `{0 < x}`

16.7.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 3y'x + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 551: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3x}{x^2} \, dx} \\ &= z_1 e^{\frac{3 \ln(x)}{2}} \\ &= z_1 \left(x^{\frac{3}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
x^2 y'' - 3y'x + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x^2 + c_2 \ln(x)\, x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= \ln(x)\, x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2x\ln(x)) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5 \ln(x)}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \ln(x)\, dx$$

Hence

$$u_1 = -x(\ln(x) - 1)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^5}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^3(\ln(x) - 1) + x^3 \ln(x)$$

Which simplifies to

$$y_p(x) = x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 + c_2 \ln(x)\, x^2\right) + \left(x^3\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_1 + c_2 \ln(x)) + x^3$$

Summary
The solution(s) found are the following

$$y = x^2(c_1 + c_2 \ln(x)) + x^3 \tag{1}$$

Verification of solutions

$$y = x^2(c_1 + c_2 \ln(x)) + x^3$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=x^3,y(x), singsol=all)
```

$$y(x) = x^2(c_2 + \ln(x)\, c_1 + x)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 19

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(x + 2c_2 \log(x) + c_1)$$

16.8 problem 8

Internal problem ID [2257]
Internal file name [`OUTPUT/2257_Monday_February_26_2024_09_18_57_AM_92912275/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + 3y'x + y = 1 - x}$$

16.8.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 3x, C = 1, f(x) = 1 - x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 3y'x + y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 3xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\, x^r + 3r\, x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 3r + 1 = 0$$

Or

$$r^2 + 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = -1$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$$

Next, we find the particular solution to the ODE

$$x^2y'' + 3y'x + y = 1 - x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{\ln(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & \frac{1}{x^2} - \frac{\ln(x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{x^2} - \frac{\ln(x)}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x)(1-x)}{x}}{\frac{1}{x}} dx$$

Which simplifies to

$$u_1 = -\int \ln(x)(1-x)\,dx$$

Hence

$$u_1 = \frac{\ln(x)\,x^2}{2} - \frac{x^2}{4} - x\ln(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1-x}{x}}{\frac{1}{x}} dx$$

Which simplifies to

$$u_2 = \int (1-x)\,dx$$

Hence

$$u_2 = x - \frac{1}{2}x^2$$

Which simplifies to

$$u_1 = \frac{(2x^2 - 4x)\ln(x)}{4} - \frac{x^2}{4} + x$$

$$u_2 = x - \frac{1}{2}x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\frac{(2x^2-4x)\ln(x)}{4} - \frac{x^2}{4} + x}{x} + \frac{\left(x - \frac{1}{2}x^2\right)\ln(x)}{x}$$

Which simplifies to

$$y_p(x) = -\frac{x}{4} + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -\frac{x}{4} + 1 + \frac{c_1}{x} + \frac{c_2 \ln(x)}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{4} + 1 + \frac{c_1}{x} + \frac{c_2 \ln(x)}{x} \tag{1}$$

Verification of solutions

$$y = -\frac{x}{4} + 1 + \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$$

Verified OK.

16.8.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 3y'x + y = 0$$

In normal form the ode

$$x^2y'' + 3y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{3}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x) dx\right)} dx \\
&= \int e^{-\left(\int \frac{3}{x} dx\right)} dx \\
&= \int e^{-3\ln(x)}\, dx \\
&= \int \frac{1}{x^3} dx \\
&= -\frac{1}{2x^2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{\frac{1}{x^6}} \\
&= x^4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + x^4 y(\tau) &= 0
\end{aligned}
$$

But in terms of τ

$$
x^4 = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$
4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
4r(r-1) + 0 + 1 = 0
$$

Or

$$
4r^2 - 4r + 1 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since the roots are equal, then the general solution is

$$
y(\tau) = c_1 y_1 + c_2 y_2
$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{x^2}\right)\right)}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{x^2}\right)\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\frac{1}{x^2}}$$

$$y_2 = -\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{x^2}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3} & -\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}}{x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{x^2}}\right)\left(-\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}}{x}\right) - \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right)(1-x)}{\frac{\sqrt{2}}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\frac{1}{x^2}}\,\left(\ln(2) - \ln\left(-\frac{1}{x^2}\right)\right)(x-1)\,x}{2} dx$$

Hence

$$u_1 = -\frac{\sqrt{-\frac{1}{x^2}}\,x\left(x - \frac{1}{2}x^2\right)\ln\left(-\frac{1}{x^2}\right)}{2} - \frac{\sqrt{-\frac{1}{x^2}}\,x^3\ln(2)}{4} + \frac{\sqrt{-\frac{1}{x^2}}\,x^2\ln(2)}{2} + \frac{\sqrt{-\frac{1}{x^2}}\,x^3}{4} - \sqrt{-\frac{1}{x^2}}\,x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-\frac{1}{x^2}}\,(1-x)}{\frac{\sqrt{2}}{x}} dx$$

Which simplifies to

$$u_2 = \int -\frac{\sqrt{-\frac{1}{x^2}}\,(x-1)\,x\sqrt{2}}{2} dx$$

Hence

$$u_2 = -\frac{x^2(-2+x)\sqrt{-\frac{1}{x^2}}\,\sqrt{2}}{4}$$

Which simplifies to

$$u_1 = -\frac{\left((2-x)\ln\left(-\frac{1}{x^2}\right) + (-2+x)\ln(2) - x + 4\right)x^2\sqrt{-\frac{1}{x^2}}}{4}$$

$$u_2 = -\frac{x^2(-2+x)\sqrt{-\frac{1}{x^2}}\,\sqrt{2}}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(2-x)\ln\left(-\frac{1}{x^2}\right)}{4} + \frac{(-2+x)\ln(2)}{4} - \frac{x}{4} + 1 - \frac{x^2(-2+x)\sqrt{-\frac{1}{x^2}}\,\sqrt{2}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}\right)}{4}$$

Which simplifies to

$$y_p(x) = -\frac{x}{4} + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^2}\right)\right)}{2}\right) + \left(-\frac{x}{4} + 1\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^2}\right)\right)}{2} - \frac{x}{4} + 1 \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^2}\right)\right)}{2} - \frac{x}{4} + 1$$

Verified OK.

16.8.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 3x, C = 1, f(x) = 1 - x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 3y'x + y = 0$$

In normal form the ode

$$x^2y'' + 3y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\, x^3} + \frac{3}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= 2c \end{aligned}$$

Therefore ode (3) now becomes

$$y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2} y(\tau) + 2c\left(\frac{d}{d\tau} y(\tau)\right) + c^2 y(\tau) = 0 \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-c\tau} c_1$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\ &= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1}{x}$$

Now the particular solution to this ODE is found

$$x^2 y'' + 3y'x + y = 1 - x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\frac{1}{x^2}}$$
$$y_2 = -\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\ln\left(-\frac{1}{x^2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{x^2}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{x^2}} & -\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3} & -\frac{\sqrt{2}\,\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\,\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}}{x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{x^2}}\right)\left(-\frac{\sqrt{2}\,\ln(2)}{2\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{\sqrt{2}\,\ln\left(-\frac{1}{x^2}\right)}{2\sqrt{-\frac{1}{x^2}}\,x^3} - \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}}{x}\right) - \left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{x^2}}\,x^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}\right)(1-x)}{\frac{\sqrt{2}}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-\frac{1}{x^2}}\,\left(\ln\left(2\right) - \ln\left(-\frac{1}{x^2}\right)\right)(x-1)\,x}{2}dx$$

Hence

$$u_1 = -\frac{\sqrt{-\frac{1}{x^2}}\,x\left(x - \frac{1}{2}x^2\right)\ln\left(-\frac{1}{x^2}\right)}{2} - \frac{\sqrt{-\frac{1}{x^2}}\,x^3\ln\left(2\right)}{4} + \frac{\sqrt{-\frac{1}{x^2}}\,x^2\ln\left(2\right)}{2} + \frac{\sqrt{-\frac{1}{x^2}}\,x^3}{4} - \sqrt{-\frac{1}{x^2}}\,x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-\frac{1}{x^2}}\,(1-x)}{\frac{\sqrt{2}}{x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\sqrt{-\frac{1}{x^2}}\,(x-1)\,x\sqrt{2}}{2}dx$$

Hence

$$u_2 = -\frac{x^2(-2+x)\,\sqrt{-\frac{1}{x^2}}\,\sqrt{2}}{4}$$

Which simplifies to

$$u_1 = -\frac{\left((2-x)\ln\left(-\frac{1}{x^2}\right) + (-2+x)\ln\left(2\right) - x + 4\right)x^2\sqrt{-\frac{1}{x^2}}}{4}$$

$$u_2 = -\frac{x^2(-2+x)\,\sqrt{-\frac{1}{x^2}}\,\sqrt{2}}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(2-x)\ln\left(-\frac{1}{x^2}\right)}{4} + \frac{(-2+x)\ln(2)}{4} - \frac{x}{4} + 1 - \frac{x^2(-2+x)\sqrt{-\frac{1}{x^2}}\,\sqrt{2}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{x^2}}\,\ln\left(-\frac{1}{x^2}\right)}{2}\right)}{4}$$

Which simplifies to

$$y_p(x) = -\frac{x}{4} + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x}\right) + \left(-\frac{x}{4} + 1\right) \\ &= -\frac{x}{4} + 1 + \frac{c_1}{x} \end{aligned}$$

Which simplifies to

$$y = -\frac{x}{4} + 1 + \frac{c_1}{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{4} + 1 + \frac{c_1}{x} \tag{1}$$

Verification of solutions

$$y = -\frac{x}{4} + 1 + \frac{c_1}{x}$$

Verified OK.

16.8.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 3x, C = 1, f(x) = 1 - x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 3y'x + y = 0$$

In normal form the ode

$$x^2y'' + 3y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{3n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \frac{c_1 \ln(x) + c_2}{x} \\
&= \frac{c_1 \ln(x) + c_2}{x}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + 3y'x + y = 1 - x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= \frac{\ln(x)}{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & \frac{1}{x^2} - \frac{\ln(x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{x^2} - \frac{\ln(x)}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x)(1-x)}{x}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \ln(x)\,(1-x)\,dx$$

Hence

$$u_1 = \frac{\ln(x)\,x^2}{2} - \frac{x^2}{4} - x\ln(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1-x}{x}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_2 = \int (1-x)\,dx$$

Hence

$$u_2 = x - \frac{1}{2}x^2$$

Which simplifies to

$$u_1 = \frac{(2x^2 - 4x)\ln(x)}{4} - \frac{x^2}{4} + x$$

$$u_2 = x - \frac{1}{2}x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\frac{(2x^2-4x)\ln(x)}{4} - \frac{x^2}{4} + x}{x} + \frac{\left(x - \frac{1}{2}x^2\right)\ln(x)}{x}$$

Which simplifies to

$$y_p(x) = -\frac{x}{4} + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \ln(x) + c_2}{x}\right) + \left(-\frac{x}{4} + 1\right) \\ &= -\frac{x}{4} + 1 + \frac{c_1 \ln(x) + c_2}{x} \end{aligned}$$

Which simplifies to

$$y = -\frac{x}{4} + 1 + \frac{c_1 \ln(x) + c_2}{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{x}{4} + 1 + \frac{c_1 \ln(x) + c_2}{x} \tag{1}$$

Verification of solutions

$$y = -\frac{x}{4} + 1 + \frac{c_1 \ln(x) + c_2}{x}$$

Verified OK.

16.8.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (x^2y'' + 3y'x + y)\, dx &= \int (1 - x)\, dx \\
y'x^2 + yx &= x - \frac{1}{2}x^2 + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{-x^2 + 2c_1 + 2x}{2x^2}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{y}{x} = \frac{-x^2 + 2c_1 + 2x}{2x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{-x^2 + 2c_1 + 2x}{2x^2}\right) \\
\frac{d}{dx}(yx) &= (x)\left(\frac{-x^2 + 2c_1 + 2x}{2x^2}\right) \\
d(yx) &= \left(\frac{-x^2 + 2c_1 + 2x}{2x}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
yx &= \int \frac{-x^2 + 2c_1 + 2x}{2x}\, dx \\
yx &= x - \frac{x^2}{4} + c_1 \ln(x) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{x - \frac{x^2}{4} + c_1 \ln(x)}{x} + \frac{c_2}{x}$$

which simplifies to

$$y = \frac{c_1 \ln(x) - \frac{x^2}{4} + c_2 + x}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) - \frac{x^2}{4} + c_2 + x}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) - \frac{x^2}{4} + c_2 + x}{x}$$

Verified OK.

16.8.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2 y'' + 3y'x + y = 1 - x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2 y'' + 3y'x + y)\, dx = \int (1 - x)\, dx$$
$$y'x^2 + yx = x - \frac{1}{2}x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{-x^2 + 2c_1 + 2x}{2x^2}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{-x^2 + 2c_1 + 2x}{2x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x} dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{-x^2 + 2c_1 + 2x}{2x^2}\right) \\ \frac{d}{dx}(yx) &= (x)\left(\frac{-x^2 + 2c_1 + 2x}{2x^2}\right) \\ d(yx) &= \left(\frac{-x^2 + 2c_1 + 2x}{2x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}yx &= \int \frac{-x^2 + 2c_1 + 2x}{2x} dx \\ yx &= x - \frac{x^2}{4} + c_1 \ln(x) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{x - \frac{x^2}{4} + c_1 \ln(x)}{x} + \frac{c_2}{x}$$

which simplifies to

$$y = \frac{c_1 \ln(x) - \frac{x^2}{4} + c_2 + x}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) - \frac{x^2}{4} + c_2 + x}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) - \frac{x^2}{4} + c_2 + x}{x}$$

Verified OK.

16.8.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 3y'x + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 552: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}(\ln(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + 3y'x + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{\ln(x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{\ln(x)}{x} \\ -\frac{1}{x^2} & \frac{1}{x^2} - \frac{\ln(x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{x^2} - \frac{\ln(x)}{x^2}\right) - \left(\frac{\ln(x)}{x}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Which simplifies to

$$W = \frac{1}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x)(1-x)}{x}}{\frac{1}{x}} dx$$

Which simplifies to

$$u_1 = -\int \ln(x)(1-x)\, dx$$

Hence

$$u_1 = \frac{\ln(x)\, x^2}{2} - \frac{x^2}{4} - x\ln(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1-x}{x}}{\frac{1}{x}} dx$$

Which simplifies to

$$u_2 = \int (1-x)\, dx$$

Hence

$$u_2 = x - \frac{1}{2}x^2$$

Which simplifies to

$$u_1 = \frac{(2x^2 - 4x)\ln(x)}{4} - \frac{x^2}{4} + x$$

$$u_2 = x - \frac{1}{2}x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\frac{(2x^2-4x)\ln(x)}{4} - \frac{x^2}{4} + x}{x} + \frac{\left(x - \frac{1}{2}x^2\right)\ln(x)}{x}$$

Which simplifies to

$$y_p(x) = -\frac{x}{4} + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 \ln(x)}{x}\right) + \left(-\frac{x}{4} + 1\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_1 + c_2 \ln(x)}{x} - \frac{x}{4} + 1$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 + c_2 \ln(x)}{x} - \frac{x}{4} + 1 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 + c_2 \ln(x)}{x} - \frac{x}{4} + 1$$

Verified OK.

16.8.8 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= x^2 \\
q(x) &= 3x \\
r(x) &= 1 \\
s(x) &= 1-x
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 3
\end{aligned}$$

Therefore (1) becomes

$$2-(3)+(1)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y'x^2 + yx = \int 1 - x\, dx$$

We now have a first order ode to solve which is

$$y'x^2 + yx = -\frac{1}{2}x^2 + c_1 + x$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{-x^2 + 2c_1 + 2x}{2x^2}
\end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{-x^2 + 2c_1 + 2x}{2x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx}\\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{-x^2 + 2c_1 + 2x}{2x^2}\right)\\ \frac{d}{dx}(yx) &= (x)\left(\frac{-x^2 + 2c_1 + 2x}{2x^2}\right)\\ d(yx) &= \left(\frac{-x^2 + 2c_1 + 2x}{2x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}yx &= \int \frac{-x^2 + 2c_1 + 2x}{2x} dx\\ yx &= x - \frac{x^2}{4} + c_1 \ln(x) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{x - \frac{x^2}{4} + c_1 \ln(x)}{x} + \frac{c_2}{x}$$

which simplifies to

$$y = \frac{c_1 \ln(x) - \frac{x^2}{4} + c_2 + x}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) - \frac{x^2}{4} + c_2 + x}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) - \frac{x^2}{4} + c_2 + x}{x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+y(x)=1-x,y(x), singsol=all)
```

$$y(x) = \frac{c_2}{x} - \frac{x}{4} + 1 + \frac{\ln(x) c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]+3*x*y'[x]+y[x]==1-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{4} + \frac{c_1}{x} + \frac{c_2 \log(x)}{x} + 1$$

16.9 problem 9

Internal problem ID [2258]
Internal file name [`OUTPUT/2258_Monday_February_26_2024_09_18_59_AM_52935214/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^3y''' + 2x^2y'' - y'x + y = \frac{1}{x}}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + 2x^2y'' - y'x + y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' + 2x^2y'' - y'x + y = \frac{1}{x}$$

gives

$$-x\lambda\, x^{\lambda-1} + 2x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + x^\lambda = 0$$

Which simplifies to

$$-\lambda\, x^\lambda + 2\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-\lambda + 2\lambda(\lambda-1) + \lambda(\lambda-1)\,(\lambda-2) + 1 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda+1)\,(\lambda-1)^2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
1	2	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1\cos(\beta\ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + c_2x + \ln(x)\, c_3x$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= x \\
y_3 &= x \ln(x)
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x^3 y''' + 2x^2 y'' - y'x + y = \frac{1}{x}
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} \frac{1}{x} & x & x \ln(x) \\ -\frac{1}{x^2} & 1 & \ln(x) + 1 \\ \frac{2}{x^3} & 0 & \frac{1}{x} \end{bmatrix} \\
|W| &= \frac{4}{x^2}
\end{aligned}
$$

The determinant simplifies to

$$|W| = \frac{4}{x^2}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x & x\ln(x) \\ 1 & \ln(x)+1 \end{bmatrix} \\ &= x \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} \frac{1}{x} & x\ln(x) \\ -\frac{1}{x^2} & \ln(x)+1 \end{bmatrix} \\ &= \frac{2\ln(x)+1}{x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{bmatrix} \\ &= \frac{2}{x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{\left(\frac{1}{x}\right)(x)}{(x^3)\left(\frac{4}{x^2}\right)}\,dx \\ &= \int \frac{1}{4x}\,dx \\ &= \int \left(\frac{1}{4x}\right)\,dx \\ &= \frac{\ln(x)}{4} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{\left(\frac{1}{x}\right)\left(\frac{2\ln(x)+1}{x}\right)}{(x^3)\left(\frac{4}{x^2}\right)}\, dx \\
&= -\int \frac{\frac{2\ln(x)+1}{x^2}}{4x}\, dx \\
&= -\int \left(\frac{2\ln(x)+1}{4x^3}\right) dx \\
&= \frac{\ln(x)}{4x^2} + \frac{1}{4x^2}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{\left(\frac{1}{x}\right)\left(\frac{2}{x}\right)}{(x^3)\left(\frac{4}{x^2}\right)}\, dx \\
&= \int \frac{\frac{2}{x^2}}{4x}\, dx \\
&= \int \left(\frac{1}{2x^3}\right) dx \\
&= -\frac{1}{4x^2}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = {} & \left(\frac{\ln(x)}{4}\right)\left(\frac{1}{x}\right) \\
& + \left(\frac{\ln(x)}{4x^2} + \frac{1}{4x^2}\right)(x) \\
& + \left(-\frac{1}{4x^2}\right)(x \ln(x))
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\ln(x) + 1}{4x}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x} + c_2 x + \ln(x)\, c_3 x\right) + \left(\frac{\ln(x) + 1}{4x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + c_2 x + \ln(x)\, c_3 x + \frac{\ln(x) + 1}{4x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + c_2 x + \ln(x)\, c_3 x + \frac{\ln(x) + 1}{4x}
$$

Verified OK.

16.9.1 Maple step by step solution

Let's solve

$x^3 y''' + 2x^2 y'' - y'x + y = \frac{1}{x}$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = (c__1-_a*_b(_a)+(diff(_b(_a),
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   <- high order exact linear fully integrable successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(x^3*diff(y(x),x$3)+2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=1/x,y(x), singsol=all)
```

$$y(x) = \frac{4c_2x^2\ln(x) + 4c_3x^2 + \ln(x) + c_1 + 1}{4x}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 33

```
DSolve[x^3*y'''[x]+2*x^2*y''[x]-x*y'[x]+y[x]==1/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\log(x) + 1}{4x} + \frac{c_1}{x} + c_2x + c_3x\log(x)$$

16.10 problem 10

Internal problem ID [2259]
Internal file name [`OUTPUT/2259_Monday_February_26_2024_09_18_59_AM_68061762/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2y'x + 2y = 4x + \sin(\ln(x))}$$

16.10.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = 2, f(x) = 4x + \sin(\ln(x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2y'x + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 2xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 2r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 2r + 2 = 0$$

Or

$$r^2 - 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^2 + c_1x$$

Next, we find the particular solution to the ODE

$$x^2y'' - 2y'x + 2y = 4x + \sin(\ln(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2(4x + \sin(\ln(x)))}{x^4}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{4x + \sin(\ln(x))}{x^2} dx$$

Hence

$$u_1 = -4\ln(x) - \frac{-\frac{1}{2} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{2} - \tan\left(\frac{\ln(x)}{2}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(4x + \sin(\ln(x)))}{x^4}\, dx$$

Which simplifies to

$$u_2 = \int \frac{4x + \sin(\ln(x))}{x^3} dx$$

Hence

$$u_2 = -\frac{4}{x} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2}$$

Which simplifies to

$$u_1 = \frac{-8x\ln(x) + \sin(\ln(x)) + \cos(\ln(x))}{2x}$$

$$u_2 = \frac{-\frac{\cos(\ln(x))}{5} - \frac{2\sin(\ln(x))}{5} - 4x}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x + c_2x^2 + c_1x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x + c_2x^2 + c_1x \tag{1}$$

Verification of solutions

$$y = \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x + c_2x^2 + c_1x$$

Verified OK.

16.10.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{2}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{2}{x}\,dx} \\ &= \frac{1}{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{4x + \sin(\ln(x))}{x^3} \\ \left(\frac{y}{x}\right)'' &= \frac{4x + \sin(\ln(x))}{x^3} \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x}\right)' = \frac{-\frac{\cos(\ln(x))}{5} - \frac{2\sin(\ln(x))}{5} - 4x}{x^2} + c_1$$

Integrating again gives

$$\left(\frac{y}{x}\right) = \frac{\frac{\sin(\ln(x))}{10} + c_1x^2 - 4x\ln(x) + \frac{3\cos(\ln(x))}{10}}{x} + c_2$$

Hence the solution is

$$y = \frac{\frac{\frac{\sin(\ln(x))}{10} + c_1x^2 - 4x\ln(x) + \frac{3\cos(\ln(x))}{10}}{x} + c_2}{\frac{1}{x}}$$

Or

$$y = c_1x^2 + c_2x - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} + \frac{\sin(\ln(x))}{10}$$

Summary

The solution(s) found are the following

$$y = c_1x^2 + c_2x - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} + \frac{\sin(\ln(x))}{10} \tag{1}$$

Verification of solutions

$$y = c_1x^2 + c_2x - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} + \frac{\sin(\ln(x))}{10}$$

Verified OK.

16.10.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2y'x + 2y = 0$$

In normal form the ode

$$x^2y'' - 2y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x)dx\right)}dx \\ &= \int e^{-\left(\int -\frac{2}{x}dx\right)}dx \\ &= \int e^{2\ln(x)}\,dx \\ &= \int x^2dx \\ &= \frac{x^3}{3} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{x^2}}{x^4} \\
&= \frac{2}{x^6}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{x^6} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{2}{x^6} = \frac{2}{9\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0
$$

Simplifying gives

$$
9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
9r(r-1) + 0 + 2 = 0
$$

Or

$$
9r^2 - 9r + 2 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^3)^{\frac{1}{3}}$$
$$y_2 = (x^3)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{d}{dx}\left((x^3)^{\frac{1}{3}}\right) & \frac{d}{dx}\left((x^3)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{x^2}{(x^3)^{\frac{2}{3}}} & \frac{2x^2}{(x^3)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^3)^{\frac{1}{3}}\right)\left(\frac{2x^2}{(x^3)^{\frac{1}{3}}}\right) - \left((x^3)^{\frac{2}{3}}\right)\left(\frac{x^2}{(x^3)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}}\,(4x + \sin(\ln(x)))}{x^4}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}}\,(4x + \sin(\ln(x)))}{x^4}dx$$

Hence

$$u_1 = \mathit{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}}\,(4x + \sin(\ln(x)))}{x^4}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^3\right)^{\frac{1}{3}} \left(4x + \sin\left(\ln\left(x\right)\right)\right)}{x^4} dx$$

Hence

$$u_2 = \mathit{undefined}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined} \left(x^3\right)^{\frac{1}{3}} + \text{undefined} \left(x^3\right)^{\frac{2}{3}}$$

Which simplifies to

$$y_p(x) = \left(x^3\right)^{\frac{1}{3}} \left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 3^{\frac{2}{3}} \left(x^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(x^3\right)^{\frac{2}{3}}}{3}\right) + \left(\left(x^3\right)^{\frac{1}{3}} \left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} \left(x^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(x^3\right)^{\frac{2}{3}}}{3} + \left(x^3\right)^{\frac{1}{3}} \left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{2}{3}} \left(x^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(x^3\right)^{\frac{2}{3}}}{3} + \left(x^3\right)^{\frac{1}{3}} \left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined}$$

Verified OK.

16.10.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = 2, f(x) = 4x + \sin(\ln(x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2y'x + 2y = 0$$

In normal form the ode

$$x^2y'' - 2y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2}{x^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \qquad (6) \\
\tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{2}{x}\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{3c\sqrt{2}}{2}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{2}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{2}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 2y'x + 2y = 4x + \sin(\ln(x))
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^3\right)^{\frac{1}{3}}$$
$$y_2 = \left(x^3\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^3\right)^{\frac{1}{3}} & \left(x^3\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(x^3\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(x^3\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^3\right)^{\frac{1}{3}} & \left(x^3\right)^{\frac{2}{3}} \\ \frac{x^2}{\left(x^3\right)^{\frac{2}{3}}} & \frac{2x^2}{\left(x^3\right)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^3\right)^{\frac{1}{3}}\right)\left(\frac{2x^2}{\left(x^3\right)^{\frac{1}{3}}}\right) - \left(\left(x^3\right)^{\frac{2}{3}}\right)\left(\frac{x^2}{\left(x^3\right)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}} (4x + \sin(\ln(x)))}{x^4} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}} (4x + \sin(\ln(x)))}{x^4} dx$$

Hence

$$u_1 = \mathit{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} (4x + \sin(\ln(x)))}{x^4} \, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} (4x + \sin(\ln(x)))}{x^4} dx$$

Hence

$$u_2 = \mathit{undefined}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined}\,(x^3)^{\frac{1}{3}} + \text{undefined}\,(x^3)^{\frac{2}{3}}$$

Which simplifies to

$$y_p(x) = (x^3)^{\frac{1}{3}} \left((x^3)^{\frac{1}{3}} + 1\right) \text{undefined}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)\right) + \left(\left(x^3\right)^{\frac{1}{3}}\left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined}\right) \\
&= \left(x^3\right)^{\frac{1}{3}}\left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined} + x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = \left(x^3\right)^{\frac{1}{3}}\left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined} + x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = \left(x^3\right)^{\frac{1}{3}}\left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined} + x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \tag{1}
$$

Verification of solutions

$$
y = \left(x^3\right)^{\frac{1}{3}}\left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined} + x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Verified OK.

16.10.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 2y'x + 2y = 0
$$

In normal form the given ode is written as

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2}{x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\
&= \frac{2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= \frac{2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{2}{x}}{2}} \\
&= x
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^3 v''(x) = 4x + \sin(\ln(x))
$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$
v''(x) = \frac{4x + \sin(\ln(x))}{x^3}
$$

Integrating once gives

$$v'(x) = -\frac{4}{x} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x^2} + c_1$$

Integrating again gives

$$v(x) = \frac{\frac{3}{10} - \frac{3\tan\left(\frac{\ln(x)}{2}\right)^2}{10} + \frac{\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x} - 4\ln(x) + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(\frac{\frac{3}{10} - \frac{3\tan\left(\frac{\ln(x)}{2}\right)^2}{10} + \frac{\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x} - 4\ln(x) + c_1 x + c_2\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = \left(\frac{\frac{3}{10} - \frac{3\tan\left(\frac{\ln(x)}{2}\right)^2}{10} + \frac{\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x} - 4\ln(x) + c_1 x + c_2\right)x$$

Therefore the homogeneous solution y_h is

$$y_h = \left(\frac{\frac{3}{10} - \frac{3\tan\left(\frac{\ln(x)}{2}\right)^2}{10} + \frac{\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x} - 4\ln(x) + c_1 x + c_2\right)x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^3)^{\frac{1}{3}}$$

$$y_2 = (x^3)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{d}{dx}\left((x^3)^{\frac{1}{3}}\right) & \frac{d}{dx}\left((x^3)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{x^2}{(x^3)^{\frac{2}{3}}} & \frac{2x^2}{(x^3)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^3)^{\frac{1}{3}}\right)\left(\frac{2x^2}{(x^3)^{\frac{1}{3}}}\right) - \left((x^3)^{\frac{2}{3}}\right)\left(\frac{x^2}{(x^3)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^3\right)^{\frac{2}{3}} \left(4x + \sin\left(\ln\left(x\right)\right)\right)}{x^4} dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^3\right)^{\frac{2}{3}} \left(4x + \sin\left(\ln\left(x\right)\right)\right)}{x^4} dx$$

Hence

$$u_1 = undefined$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^3\right)^{\frac{1}{3}} \left(4x + \sin\left(\ln\left(x\right)\right)\right)}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^3\right)^{\frac{1}{3}} \left(4x + \sin\left(\ln\left(x\right)\right)\right)}{x^4} dx$$

Hence

$$u_2 = undefined$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined} \left(x^3\right)^{\frac{1}{3}} + \text{undefined} \left(x^3\right)^{\frac{2}{3}}$$

Which simplifies to

$$y_p(x) = \left(x^3\right)^{\frac{1}{3}} \left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{\frac{3}{10} - \frac{3\tan\left(\frac{\ln(x)}{2}\right)^2}{10} + \frac{\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x} - 4\ln\left(x\right) + c_1 x + c_2\right) x\right) \\ &\quad + \left(\left(x^3\right)^{\frac{1}{3}} \left(\left(x^3\right)^{\frac{1}{3}} + 1\right) \text{undefined}\right) \end{aligned}$$

Which simplifies to

$$y = c_1x^2 + c_2x - 4x \ln(x) + \frac{3\cos(\ln(x))}{10} + \frac{\sin(\ln(x))}{10} + (x^3)^{\frac{1}{3}} \left((x^3)^{\frac{1}{3}} + 1\right) \text{undefined}$$

Summary

The solution(s) found are the following

$$y = c_1x^2 + c_2x - 4x \ln(x) + \frac{3\cos(\ln(x))}{10} + \frac{\sin(\ln(x))}{10} + (x^3)^{\frac{1}{3}} \left((x^3)^{\frac{1}{3}} + 1\right) \text{undefined} \tag{1}$$

Verification of solutions

$$y = c_1x^2 + c_2x - 4x \ln(x) + \frac{3\cos(\ln(x))}{10} + \frac{\sin(\ln(x))}{10} + (x^3)^{\frac{1}{3}} \left((x^3)^{\frac{1}{3}} + 1\right) \text{undefined}$$

Verified OK.

16.10.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = 2, f(x) = 4x + \sin(\ln(x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2y'x + 2y = 0$$

In normal form the ode

$$x^2y'' - 2y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} - \frac{2n}{x^2} + \frac{2}{x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = 2 \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \frac{2v'(x)}{x} &= 0 \\
v''(x) + \frac{2v'(x)}{x} &= 0
\end{aligned} \tag{7}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{2u(x)}{x} = 0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(-\frac{c_1}{x} + c_2\right)x^2 \\ &= x(c_2 x - c_1) \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' - 2y'x + 2y = 4x + \sin(\ln(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2(4x + \sin(\ln(x)))}{x^4}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{4x + \sin(\ln(x))}{x^2} dx$$

Hence

$$u_1 = -4\ln(x) - \frac{-\frac{1}{2} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{2} - \tan\left(\frac{\ln(x)}{2}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(4x + \sin(\ln(x)))}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{4x + \sin(\ln(x))}{x^3} dx$$

Hence

$$u_2 = -\frac{4}{x} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2}$$

Which simplifies to

$$u_1 = \frac{-8x\ln(x) + \sin(\ln(x)) + \cos(\ln(x))}{2x}$$
$$u_2 = \frac{-\frac{\cos(\ln(x))}{5} - \frac{2\sin(\ln(x))}{5} - 4x}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{c_1}{x} + c_2\right) x^2\right) + \left(\frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x\right) \\
&= \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x + \left(-\frac{c_1}{x} + c_2\right) x^2
\end{aligned}
$$

Which simplifies to

$$
y = c_2x^2 - 4x\ln(x) - c_1x + \frac{\sin(\ln(x))}{10} + \frac{3\cos(\ln(x))}{10} - 4x
$$

Summary

The solution(s) found are the following

$$
y = c_2x^2 - 4x\ln(x) - c_1x + \frac{\sin(\ln(x))}{10} + \frac{3\cos(\ln(x))}{10} - 4x \tag{1}
$$

Verification of solutions

$$
y = c_2x^2 - 4x\ln(x) - c_1x + \frac{\sin(\ln(x))}{10} + \frac{3\cos(\ln(x))}{10} - 4x
$$

Verified OK.

16.10.7 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$\begin{aligned}A(v''B+2v'B'+B''v)+B(B'v+v'B)+CBv&=0\\ ABv''+\left(2AB'+B^2\right)v'+\left(AB''+BB'+CB\right)v&=0\end{aligned}\tag{1}$$

If the term $AB''+BB'+CB$ is zero, then this method works and can be used to solve

$$ABv''+\left(2AB'+B^2\right)v'=0$$

By Using $u=v'$ which reduces the order of the above ode to one. The new ode is

$$ABu'+\left(2AB'+B^2\right)u=0$$

The above ode is first order ode which is solved for u. Now a new ode $v'=u$ is solved for v as first order ode. Then the final solution is obtain from $y=Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$\begin{aligned}A&=x^2\\ B&=-2x\\ C&=2\\ F&=4x+\sin\left(\ln\left(x\right)\right)\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}AB''+BB'+CB&=\left(x^2\right)(0)+(-2x)(-2)+(2)(-2x)\\ &=0\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x^3v''+(0)\,v'=0$$

Now by applying $v'=u$ the above becomes

$$-2x^3u'(x)=0$$

Which is now solved for u. Integrating both sides gives

$$\begin{aligned}u(x)&=\int 0\,\mathrm{d}x\\ &=c_1\end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 \, \mathrm{d}x \\
&= c_1 x + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-2x)\left(c_1 x + c_2\right) \\
&= -2x(c_1 x + c_2)
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2(4x + \sin(\ln(x)))}{x^4} dx$$

Which simplifies to

$$u_1 = -\int \frac{4x + \sin(\ln(x))}{x^2} dx$$

Hence

$$u_1 = -4\ln(x) - \frac{-\frac{1}{2} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{2} - \tan\left(\frac{\ln(x)}{2}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(4x + \sin(\ln(x)))}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{4x + \sin(\ln(x))}{x^3} dx$$

Hence

$$u_2 = -\frac{4}{x} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x^2}$$

Which simplifies to

$$u_1 = \frac{-8x\ln(x) + \sin(\ln(x)) + \cos(\ln(x))}{2x}$$
$$u_2 = \frac{-\frac{\cos(\ln(x))}{5} - \frac{2\sin(\ln(x))}{5} - 4x}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (-2x(c_1x + c_2)) + \left(\frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x\right) \\ &= -2c_1x^2 - 4x\ln(x) - 2c_2x + \frac{\sin(\ln(x))}{10} + \frac{3\cos(\ln(x))}{10} - 4x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -2c_1x^2 - 4x\ln(x) - 2c_2x + \frac{\sin(\ln(x))}{10} + \frac{3\cos(\ln(x))}{10} - 4x \tag{1}$$

Verification of solutions

$$y = -2c_1x^2 - 4x\ln(x) - 2c_2x + \frac{\sin(\ln(x))}{10} + \frac{3\cos(\ln(x))}{10} - 4x$$

Verified OK.

16.10.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 554: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2} \, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - 2y'x + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x^2 + c_1 x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2(4x + \sin(\ln(x)))}{x^4} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{4x + \sin(\ln(x))}{x^2} dx$$

Hence

$$u_1 = -4\ln(x) - \frac{-\frac{1}{2} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{2} - \tan\left(\frac{\ln(x)}{2}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(4x + \sin(\ln(x)))}{x^4} \, dx$$

Which simplifies to

$$u_2 = \int \frac{4x + \sin(\ln(x))}{x^3} dx$$

Hence

$$u_2 = -\frac{4}{x} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2}$$

Which simplifies to

$$u_1 = \frac{-8x\ln(x) + \sin(\ln(x)) + \cos(\ln(x))}{2x}$$
$$u_2 = \frac{-\frac{\cos(\ln(x))}{5} - \frac{2\sin(\ln(x))}{5} - 4x}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_2x^2 + c_1x) + \left(\frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x\right)
\end{aligned}
$$

Which simplifies to

$$
y = x(c_2x + c_1) + \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x
$$

Summary

The solution(s) found are the following

$$
y = x(c_2x + c_1) + \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x \tag{1}
$$

Verification of solutions

$$
y = x(c_2x + c_1) + \frac{\sin(\ln(x))}{10} - 4x\ln(x) + \frac{3\cos(\ln(x))}{10} - 4x
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=4*x+sin(ln(x)),y(x), singsol=all)
```

$$y(x) = \frac{3\cos(\ln(x))}{10} + \frac{\sin(\ln(x))}{10} - 4x\ln(x) + c_2x^2 + (-4 + c_1)x$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+2*y[x]==4*x+Sin[Log[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10}(\sin(\log(x)) + 3\cos(\log(x)) + 10x(-4\log(x) + c_2x - 4 + c_1))$$

16.11 problem 11

Internal problem ID [2260]
Internal file name [`OUTPUT/2260_Monday_February_26_2024_09_19_06_AM_6641865/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - y'x + 2y = \ln(x)\,x^2}$$

16.11.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 2, f(x) = \ln(x)\,x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - y'x + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 2 = 0$$

Or

$$r^2 - 2r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 1 - i \\ r_2 &= 1 + i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 1$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta} \\ &= x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^\alpha \left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha \left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 1, \beta = -1$, the above becomes

$$y = x^1 \left(c_1 e^{-i\ln(x)} + c_2 e^{i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = x(c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)))$$

Next, we find the particular solution to the ODE

$$x^2y'' - y'x + 2y = \ln(x)\, x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(\ln(x))\, x \\ y_2 &= -\sin(\ln(x))\, x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(\ln(x))\, x & -\sin(\ln(x))\, x \\ \frac{d}{dx}(\cos(\ln(x))\, x) & \frac{d}{dx}(-\sin(\ln(x))\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(\ln(x))\, x & -\sin(\ln(x))\, x \\ -\sin(\ln(x)) + \cos(\ln(x)) & -\sin(\ln(x)) - \cos(\ln(x)) \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & (\cos(\ln(x))\, x)\,(-\sin(\ln(x)) - \cos(\ln(x))) \\ & - (-\sin(\ln(x))\, x)\,(-\sin(\ln(x)) + \cos(\ln(x))) \end{aligned}$$

Which simplifies to

$$W = -\sin(\ln(x))^2 x - \cos(\ln(x))^2 x$$

Which simplifies to

$$W = -x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\sin(\ln(x))\, x^3 \ln(x)}{-x^3} dx$$

Which simplifies to

$$u_1 = -\int \sin(\ln(x)) \ln(x)\, dx$$

Hence

$$u_1 = -\left(-\frac{\ln(x)}{2} + \frac{1}{2}\right) x \cos(\ln(x)) - \frac{\sin(\ln(x)) \ln(x)\, x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(\ln(x))\, x^3 \ln(x)}{-x^3} dx$$

Which simplifies to

$$u_2 = \int -\cos(\ln(x)) \ln(x)\, dx$$

Hence

$$u_2 = -\frac{\ln(x)\cos(\ln(x))\, x}{2} + \left(-\frac{\ln(x)}{2} + \frac{1}{2}\right) x \sin(\ln(x))$$

Which simplifies to

$$u_1 = \left(-\frac{(\sin(\ln(x)) - \cos(\ln(x))) \ln(x)}{2} - \frac{\cos(\ln(x))}{2}\right) x$$

$$u_2 = -\frac{((\ln(x) - 1)\sin(\ln(x)) + \cos(\ln(x)) \ln(x))\, x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{(\sin(\ln(x)) - \cos(\ln(x)))\ln(x)}{2} - \frac{\cos(\ln(x))}{2}\right)x^2\cos(\ln(x)) + \frac{((\ln(x)-1)\sin(\ln(x)) + \cos(\ln(x))\ln(x))\,x^2\sin(\ln(x))}{2}$$

Which simplifies to

$$y_p(x) = \frac{(\ln(x)-1)\,x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \sin(\ln(x))\,c_2 x + \cos(\ln(x))\,c_1 x - \frac{x^2}{2} + \frac{\ln(x)\,x^2}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sin(\ln(x))\,c_2 x + \cos(\ln(x))\,c_1 x - \frac{x^2}{2} + \frac{\ln(x)\,x^2}{2} \tag{1}$$

Verification of solutions

$$y = \sin(\ln(x))\,c_2 x + \cos(\ln(x))\,c_1 x - \frac{x^2}{2} + \frac{\ln(x)\,x^2}{2}$$

Verified OK.

16.11.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - y'x + 2y = 0$$

In normal form the ode

$$x^2y'' - y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= \frac{2}{x^2}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(x)dx\right)}dx\\ &= \int e^{-\left(\int -\frac{1}{x}dx\right)}dx\\ &= \int e^{\ln(x)}\,dx\\ &= \int x dx\\ &= \frac{x^2}{2}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2}\\ &= \frac{\frac{2}{x^2}}{x^2}\\ &= \frac{2}{x^4}\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{x^4} = 0$$

But in terms of τ

$$\frac{2}{x^4} = \frac{1}{2\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{2\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$2\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$2\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$2r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$2r(r-1) + 0 + 1 = 0$$

Or

$$2r^2 - 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{i}{2}$$
$$r_2 = \frac{1}{2} + \frac{i}{2}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$. Hence the solution becomes

$$\begin{aligned}
y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\
&= \tau^{\alpha}\left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\
&= \tau^{\alpha}\left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\
&= \tau^{\alpha}\left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right)
\end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-\frac{i \ln(\tau)}{2}} + c_2 e^{\frac{i \ln(\tau)}{2}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{\ln(\tau)}{2}\right) + c_2 \sin\left(\frac{\ln(\tau)}{2}\right)\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\, x\left(c_1 \cos\left(\ln(x) - \frac{\ln(2)}{2}\right) + c_2 \sin\left(\ln(x) - \frac{\ln(2)}{2}\right)\right)}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\, x\left(c_1 \cos\left(\ln(x) - \frac{\ln(2)}{2}\right) + c_2 \sin\left(\ln(x) - \frac{\ln(2)}{2}\right)\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{\sqrt{2}\, x \cos(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, x \sin(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} \\
y_2 &= \frac{\sqrt{2}\, x \sin(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\, x \cos(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix}
\frac{\sqrt{2}\, x \cos(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, x \sin(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} & \frac{\sqrt{2}\, x \sin(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\, x \cos(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} \\
\frac{d}{dx}\left(\frac{\sqrt{2}\, x \cos(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, x \sin(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2}\right) & \frac{d}{dx}\left(\frac{\sqrt{2}\, x \sin(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\, x \cos(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2}\right)
\end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix}
\frac{\sqrt{2}\, x \cos(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, x \sin(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} \\
\frac{\sqrt{2}\, \cos(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\, \sin(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, \sin(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, \cos(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} \quad \frac{\sqrt{2}\, \sin(\ln(x)) \cos(}{2}
\end{vmatrix}
$$

Therefore

$$W = \left(\frac{\sqrt{2}\, x \cos\left(\ln\left(x\right)\right) \cos\left(\frac{\ln\left(2\right)}{2}\right)}{2} + \frac{\sqrt{2}\, x \sin\left(\ln\left(x\right)\right) \sin\left(\frac{\ln\left(2\right)}{2}\right)}{2}\right) \left(\frac{\sqrt{2}\, \sin\left(\ln\left(x\right)\right) \cos\left(\frac{\ln\left(2\right)}{2}\right)}{2} + \frac{\sqrt{2}\, \cos\left(\ln\left(x\right)\right) \cos\left(\frac{\ln\left(2\right)}{2}\right)}{2} - \frac{\sqrt{2}\, \cos\left(\ln\left(x\right)\right) \sin\left(\frac{\ln\left(2\right)}{2}\right)}{2} + \frac{\sqrt{2}\, \sin\left(\ln\left(x\right)\right) \sin\left(\frac{\ln\left(2\right)}{2}\right)}{2}\right) - \left(\frac{\sqrt{2}\, x \sin\left(\ln\left(x\right)\right) \cos\left(\frac{\ln\left(2\right)}{2}\right)}{2} - \frac{\sqrt{2}\, x \cos\left(\ln\left(x\right)\right) \sin\left(\frac{\ln\left(2\right)}{2}\right)}{2}\right) \left(\frac{\sqrt{2}\, \cos\left(\ln\left(x\right)\right) \cos\left(\frac{\ln\left(2\right)}{2}\right)}{2} - \frac{\sqrt{2}\, \sin\left(\ln\left(x\right)\right) \cos\left(\frac{\ln\left(2\right)}{2}\right)}{2} + \frac{\sqrt{2}\, \sin\left(\ln\left(x\right)\right) \sin\left(\frac{\ln\left(2\right)}{2}\right)}{2} + \frac{\sqrt{2}\, \cos\left(\ln\left(x\right)\right) \sin\left(\frac{\ln\left(2\right)}{2}\right)}{2}\right)$$

Which simplifies to

$$W = \frac{x \cos\left(\ln\left(x\right)\right)^{2} \cos\left(\frac{\ln\left(2\right)}{2}\right)^{2}}{2} + \frac{x \sin\left(\ln\left(x\right)\right)^{2} \sin\left(\frac{\ln\left(2\right)}{2}\right)^{2}}{2} + \frac{x \sin\left(\ln\left(x\right)\right)^{2} \cos\left(\frac{\ln\left(2\right)}{2}\right)^{2}}{2} + \frac{x \cos\left(\ln\left(x\right)\right)^{2} \sin\left(\frac{\ln\left(2\right)}{2}\right)^{2}}{2}$$

Which simplifies to

$$W = \frac{x}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{\sqrt{2}\, x \sin\left(\ln\left(x\right)\right) \cos\left(\frac{\ln\left(2\right)}{2}\right)}{2} - \frac{\sqrt{2}\, x \cos\left(\ln\left(x\right)\right) \sin\left(\frac{\ln\left(2\right)}{2}\right)}{2}\right) \ln\left(x\right) x^{2}}{\frac{x^{3}}{2}} dx$$

Which simplifies to

$$u_1 = -\int \sqrt{2}\left(\sin\left(\ln\left(x\right)\right)\cos\left(\frac{\ln\left(2\right)}{2}\right) - \cos\left(\ln\left(x\right)\right)\sin\left(\frac{\ln\left(2\right)}{2}\right)\right)\ln\left(x\right)dx$$

Hence

$$u_1 = -\frac{\left(-\frac{\sqrt{2}\,\cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\,\sin\left(\frac{\ln(2)}{2}\right)}{2}\right)x\ln\left(x\right) + \left(\sqrt{2}\,\cos\left(\frac{\ln(2)}{2}\right) - \sqrt{2}\,\sin\left(\frac{\ln(2)}{2}\right)\right)x\ln\left(x\right)\tan\left(\frac{\ln(x)}{2}\right) + \left(\cdots\right.}{1\cdots}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(\frac{\sqrt{2}\,x\cos(\ln(x))\cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\,x\sin(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)}{2}\right)\ln\left(x\right)x^2}{\frac{x^3}{2}}dx$$

Which simplifies to

$$u_2 = \int \sqrt{2}\left(\cos\left(\ln\left(x\right)\right)\cos\left(\frac{\ln\left(2\right)}{2}\right) + \sin\left(\ln\left(x\right)\right)\sin\left(\frac{\ln\left(2\right)}{2}\right)\right)\ln\left(x\right)dx$$

Hence

$$u_2 = \frac{\left(\frac{\sqrt{2}\,\cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\,\sin\left(\frac{\ln(2)}{2}\right)}{2}\right)x\ln\left(x\right) + \left(\sqrt{2}\,\cos\left(\frac{\ln(2)}{2}\right) + \sqrt{2}\,\sin\left(\frac{\ln(2)}{2}\right)\right)x\ln\left(x\right)\tan\left(\frac{\ln(x)}{2}\right) + \left(-\frac{\sqrt{2}\,c\cdots}{}\right.}{1+\cdots}$$

Which simplifies to

$$u_1 = \frac{\left(\left(\cos\left(\ln\left(x\right)\right)\left(\ln\left(x\right)-1\right) - \sin\left(\ln\left(x\right)\right)\ln\left(x\right)\right)\cos\left(\frac{\ln(2)}{2}\right) + \left(\left(\ln\left(x\right)-1\right)\sin\left(\ln\left(x\right)\right) + \cos\left(\ln\left(x\right)\right)\ln\left(x\right)\right)\cdots}{2}$$

$$u_2 = \frac{\left(\left(\left(\ln\left(x\right)-1\right)\sin\left(\ln\left(x\right)\right) + \cos\left(\ln\left(x\right)\right)\ln\left(x\right)\right)\cos\left(\frac{\ln(2)}{2}\right) - \sin\left(\frac{\ln(2)}{2}\right)\left(\cos\left(\ln\left(x\right)\right)\left(\ln\left(x\right)-1\right) - \sin\left(\ln\cdots\right.\right.}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\left(\cos\left(\ln\left(x\right)\right)\left(\ln\left(x\right)-1\right)-\sin\left(\ln\left(x\right)\right)\ln\left(x\right)\right)\cos\left(\frac{\ln(2)}{2}\right)+\left(\left(\ln\left(x\right)-1\right)\sin\left(\ln\left(x\right)\right)+\cos\left(\ln\left(x\right)\right)\ln\left(x\right)\right)\right.}{2} + \frac{\left(\left(\left(\ln\left(x\right)-1\right)\sin\left(\ln\left(x\right)\right)+\cos\left(\ln\left(x\right)\right)\ln\left(x\right)\right)\cos\left(\frac{\ln(2)}{2}\right)-\sin\left(\frac{\ln(2)}{2}\right)\left(\cos\left(\ln\left(x\right)\right)\left(\ln\left(x\right)-1\right)-\sin\left(\right.\right.\right.}{2}$$

Which simplifies to

$$y_p(x) = \frac{\left(\ln\left(x\right)-1\right)x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{2}\,x\left(c_1\cos\left(\ln\left(x\right)-\frac{\ln(2)}{2}\right)+c_2\sin\left(\ln\left(x\right)-\frac{\ln(2)}{2}\right)\right)}{2}\right)+\left(\frac{\left(\ln\left(x\right)-1\right)x^2}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\,x\left(c_1\cos\left(\ln\left(x\right)-\frac{\ln(2)}{2}\right)+c_2\sin\left(\ln\left(x\right)-\frac{\ln(2)}{2}\right)\right)}{2}+\frac{\left(\ln\left(x\right)-1\right)x^2}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\,x\left(c_1\cos\left(\ln\left(x\right)-\frac{\ln(2)}{2}\right)+c_2\sin\left(\ln\left(x\right)-\frac{\ln(2)}{2}\right)\right)}{2}+\frac{\left(\ln\left(x\right)-1\right)x^2}{2}$$

Verified OK.

16.11.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 2, f(x) = \ln(x)\, x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - y'x + 2y = 0$$

In normal form the ode

$$x^2y'' - y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -c\sqrt{2}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned}
\tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{\sqrt{2}\,c\tau}{2}}\left(c_1\cos\left(\frac{\sqrt{2}\,c\tau}{2}\right) + c_2\sin\left(\frac{\sqrt{2}\,c\tau}{2}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{2}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{2}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x(c_1\cos(\ln(x)) + c_2\sin(\ln(x)))
$$

Now the particular solution to this ODE is found

$$
x^2y'' - y'x + 2y = \ln(x)\,x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\sqrt{2}\, x \cos\left(\ln\left(x\right)\right) \cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, x \sin\left(\ln\left(x\right)\right) \sin\left(\frac{\ln(2)}{2}\right)}{2}$$

$$y_2 = \frac{\sqrt{2}\, x \sin\left(\ln\left(x\right)\right) \cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\, x \cos\left(\ln\left(x\right)\right) \sin\left(\frac{\ln(2)}{2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\sqrt{2}\, x \cos(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, x \sin(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} & \frac{\sqrt{2}\, x \sin(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\, x \cos(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} \\ \frac{d}{dx}\left(\frac{\sqrt{2}\, x \cos(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, x \sin(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2}\right) & \frac{d}{dx}\left(\frac{\sqrt{2}\, x \sin(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\, x \cos(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\sqrt{2}\, x \cos(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, x \sin(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} \\ \frac{\sqrt{2}\, \cos(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\, \sin(\ln(x)) \cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, \sin(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\, \cos(\ln(x)) \sin\left(\frac{\ln(2)}{2}\right)}{2} \quad \frac{\sqrt{2}\, \sin(\ln(x)) \cos\left(\right.}{2}$$

Therefore

$$W = \left(\frac{\sqrt{2}\,x\cos(\ln(x))\cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\,x\sin(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)}{2}\right)\left(\frac{\sqrt{2}\,\sin(\ln(x))\cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\,\cos(\ln(x))\cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\,\cos(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\,\sin(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)}{2}\right) - \left(\frac{\sqrt{2}\,x\sin(\ln(x))\cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\,x\cos(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)}{2}\right)\left(\frac{\sqrt{2}\,\cos(\ln(x))\cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\,\sin(\ln(x))\cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\,\sin(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\,\cos(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)}{2}\right)$$

Which simplifies to

$$W = \frac{x\cos(\ln(x))^2\cos\left(\frac{\ln(2)}{2}\right)^2}{2} + \frac{x\sin(\ln(x))^2\sin\left(\frac{\ln(2)}{2}\right)^2}{2} + \frac{x\sin(\ln(x))^2\cos\left(\frac{\ln(2)}{2}\right)^2}{2} + \frac{x\cos(\ln(x))^2\sin\left(\frac{\ln(2)}{2}\right)^2}{2}$$

Which simplifies to

$$W = \frac{x}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{\sqrt{2}\,x\sin(\ln(x))\cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\,x\cos(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)}{2}\right)\ln(x)\,x^2}{\frac{x^3}{2}}\,dx$$

Which simplifies to

$$u_1 = -\int \sqrt{2}\left(\sin(\ln(x))\cos\left(\frac{\ln(2)}{2}\right) - \cos(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)\right)\ln(x)\,dx$$

Hence

$$u_1 = -\frac{\left(-\frac{\sqrt{2}\,\cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\,\sin\left(\frac{\ln(2)}{2}\right)}{2}\right) x\ln(x) + \left(\sqrt{2}\,\cos\left(\frac{\ln(2)}{2}\right) - \sqrt{2}\,\sin\left(\frac{\ln(2)}{2}\right)\right) x\ln(x)\tan\left(\frac{\ln(x)}{2}\right) + \left(\sqrt{\ }\right.}{1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(\frac{\sqrt{2}\,x\cos(\ln(x))\cos\left(\frac{\ln(2)}{2}\right)}{2} + \frac{\sqrt{2}\,x\sin(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)}{2}\right)\ln(x)\,x^2}{\frac{x^3}{2}}\,dx$$

Which simplifies to

$$u_2 = \int \sqrt{2}\left(\cos(\ln(x))\cos\left(\frac{\ln(2)}{2}\right) + \sin(\ln(x))\sin\left(\frac{\ln(2)}{2}\right)\right)\ln(x)\,dx$$

Hence

$$u_2 = \frac{\left(\frac{\sqrt{2}\,\cos\left(\frac{\ln(2)}{2}\right)}{2} - \frac{\sqrt{2}\,\sin\left(\frac{\ln(2)}{2}\right)}{2}\right) x\ln(x) + \left(\sqrt{2}\,\cos\left(\frac{\ln(2)}{2}\right) + \sqrt{2}\,\sin\left(\frac{\ln(2)}{2}\right)\right) x\ln(x)\tan\left(\frac{\ln(x)}{2}\right) + \left(-\frac{\sqrt{2}\,c}{}\right.}{1+}$$

Which simplifies to

$$u_1 = \frac{\left((\cos(\ln(x))(\ln(x)-1) - \sin(\ln(x))\ln(x))\cos\left(\frac{\ln(2)}{2}\right) + ((\ln(x)-1)\sin(\ln(x)) + \cos(\ln(x))\ln(x))\right.}{2}$$

$$u_2 = \frac{\left(((\ln(x)-1)\sin(\ln(x)) + \cos(\ln(x))\ln(x))\cos\left(\frac{\ln(2)}{2}\right) - \sin\left(\frac{\ln(2)}{2}\right)(\cos(\ln(x))(\ln(x)-1) - \sin(\ln\right.}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\left(\cos(\ln(x))(\ln(x)-1) - \sin(\ln(x))\ln(x)\right)\cos\left(\frac{\ln(2)}{2}\right) + \left((\ln(x)-1)\sin(\ln(x)) + \cos(\ln(x))\ln(x)\right)\right.}{2} + \frac{\left(\left((\ln(x)-1)\sin(\ln(x)) + \cos(\ln(x))\ln(x)\right)\cos\left(\frac{\ln(2)}{2}\right) - \sin\left(\frac{\ln(2)}{2}\right)\left(\cos(\ln(x))(\ln(x)-1) - \sin(\right.\right.}{2}$$

Which simplifies to

$$y_p(x) = \frac{(\ln(x)-1)\,x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (x(c_1\cos(\ln(x)) + c_2\sin(\ln(x)))) + \left(\frac{(\ln(x)-1)\,x^2}{2}\right) \\ &= \frac{(\ln(x)-1)\,x^2}{2} + x(c_1\cos(\ln(x)) + c_2\sin(\ln(x))) \end{aligned}$$

Which simplifies to

$$y = \sin(\ln(x))\,c_2x + \cos(\ln(x))\,c_1x - \frac{x^2}{2} + \frac{\ln(x)\,x^2}{2}$$

Summary

The solution(s) found are the following

$$y = \sin(\ln(x))\,c_2x + \cos(\ln(x))\,c_1x - \frac{x^2}{2} + \frac{\ln(x)\,x^2}{2} \tag{1}$$

Verification of solutions

$$y = \sin(\ln(x))\,c_2x + \cos(\ln(x))\,c_1x - \frac{x^2}{2} + \frac{\ln(x)\,x^2}{2}$$

Verified OK.

16.11.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 2, f(x) = \ln(x)\,x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - y'x + 2y = 0$$

In normal form the ode

$$x^2y'' - y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 + i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2+2i}{x} - \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+2i)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+2i)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(-1-2i)\, u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-2i}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1-2i}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-1-2i}{x}\, dx \\ \ln(u) &= (-1-2i)\ln(x) + c_1 \\ u &= \mathrm{e}^{(-1-2i)\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{(-1-2i)\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-2i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{i c_1 x^{-2i}}{2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^{1+i} \\
&= x^{1+i} c_2 + \frac{i x^{1-i} c_1}{2}
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' - y' x + 2y = \ln(x)\, x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x\, x^{-i} \\
y_2 &= x\, x^{i}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\,x^{-i} & x\,x^{i} \\ \frac{d}{dx}(x\,x^{-i}) & \frac{d}{dx}(x\,x^{i}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\,x^{-i} & x\,x^{i} \\ x^{-i} - ix^{-i} & x^{i} + ix^{i} \end{vmatrix}$$

Therefore

$$W = (x\,x^{-i})\,(x^{i} + ix^{i}) - (x\,x^{i})\,(x^{-i} - ix^{-i})$$

Which simplifies to

$$W = 2ix$$

Which simplifies to

$$W = 2ix$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3 x^i \ln(x)}{2ix^3}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{ix^i \ln(x)}{2}dx$$

Hence

$$u_1 = \frac{x^{1+i}(-1 + (1+i)\ln(x))}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3 x^{-i} \ln(x)}{2ix^3}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{ix^{-i} \ln(x)}{2}dx$$

Hence

$$u_2 = -\frac{(1+(-1+i)\ln(x))\, x^{1-i}}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^{1+i}(-1+(1+i)\ln(x))\, x\, x^{-i}}{4} - \frac{(1+(-1+i)\ln(x))\, x^{1-i} x\, x^{i}}{4}$$

Which simplifies to

$$y_p(x) = \frac{(\ln(x)-1)\, x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^{1+i}\right) + \left(\frac{(\ln(x)-1)\, x^2}{2}\right) \\ &= \frac{(\ln(x)-1)\, x^2}{2} + \left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^{1+i} \end{aligned}$$

Which simplifies to

$$y = \frac{i x^{1-i} c_1}{2} + x^{1+i} c_2 + \frac{(\ln(x)-1)\, x^2}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{i x^{1-i} c_1}{2} + x^{1+i} c_2 + \frac{(\ln(x)-1)\, x^2}{2} \tag{1}$$

Verification of solutions

$$y = \frac{i x^{1-i} c_1}{2} + x^{1+i} c_2 + \frac{(\ln(x)-1)\, x^2}{2}$$

Verified OK.

16.11.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 555: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{5}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{5}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{5}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - i$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{\frac{1}{2} - i}{x} + (-)(0) \\ &= \frac{\frac{1}{2} - i}{x} \\ &= \frac{\frac{1}{2} - i}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{\frac{1}{2} - i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{x^2}\right) + \left(\frac{\frac{1}{2} - i}{x}\right)^2 - \left(-\frac{5}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{\frac{1}{2} - i}{x} dx} \\ &= x^{\frac{1}{2} - i} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{1-i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i x^{2i}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{1-i}\right) + c_2\left(x^{1-i}\left(-\frac{i x^{2i}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - y'x + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = x^{1-i} c_1 - \frac{i c_2 x^{1+i}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{1-i}$$
$$y_2 = -\frac{i x^{1+i}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{1-i} & -\frac{i x^{1+i}}{2} \\ \frac{d}{dx}\left(x^{1-i}\right) & \frac{d}{dx}\left(-\frac{i x^{1+i}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{1-i} & -\frac{i x^{1+i}}{2} \\ \frac{(1-i) x^{1-i}}{x} & \frac{\left(\frac{1}{2}-\frac{i}{2}\right) x^{1+i}}{x} \end{vmatrix}$$

Therefore

$$W = \left(x^{1-i}\right)\left(\frac{\left(\frac{1}{2}-\frac{i}{2}\right)x^{1+i}}{x}\right) - \left(-\frac{i x^{1+i}}{2}\right)\left(\frac{(1-i)\,x^{1-i}}{x}\right)$$

Which simplifies to

$$W = \frac{x^{1-i}x^{1+i}}{x}$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i x^{1+i}\ln(x)x^2}{2}}{x^3}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{i x^{i}\ln(x)}{2}dx$$

Hence

$$u_1 = \frac{x^{1+i}(-1+(1+i)\ln(x))}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{1-i}\ln(x)\,x^2}{x^3}\,dx$$

Which simplifies to

$$u_2 = \int x^{-i}\ln(x)\,dx$$

Hence

$$u_2 = \frac{(-i+(1+i)\ln(x))\,x^{1-i}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^{1+i}(-1+(1+i)\ln(x))\,x^{1-i}}{4} - \frac{i(-i+(1+i)\ln(x))\,x^{1-i}x^{1+i}}{4}$$

Which simplifies to

$$y_p(x) = \frac{(\ln (x) - 1) x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{1-i} c_1 - \frac{i c_2 x^{1+i}}{2}\right) + \left(\frac{(\ln (x) - 1) x^2}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{1-i} c_1 - \frac{i c_2 x^{1+i}}{2} + \frac{(\ln (x) - 1) x^2}{2} \tag{1}$$

Verification of solutions

$$y = x^{1-i} c_1 - \frac{i c_2 x^{1+i}}{2} + \frac{(\ln (x) - 1) x^2}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=x^2*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{(2\cos(\ln(x))\, c_1 + 2\sin(\ln(x))\, c_2 + x(-1 + \ln(x)))\, x}{2}$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 31

```
DSolve[x^2*y''[x]-x*y'[x]+2*y[x]==x^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}x(x(\log(x) - 1) + 2c_2\cos(\log(x)) + 2c_1\sin(\log(x)))$$

16.12 problem 12

Internal problem ID [2261]
Internal file name [`OUTPUT/2261_Monday_February_26_2024_09_19_09_AM_18785997/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + 4y'x + 3y = \ln(x)(x-1)}$$

16.12.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 3, f(x) = \ln(x)(x-1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4y'x + 3y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 4xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 4r\,x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 4r + 3 = 0$$

Or

$$r^2 + 3r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -\frac{i\sqrt{3}}{2} - \frac{3}{2} \\ r_2 &= \frac{i\sqrt{3}}{2} - \frac{3}{2} \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = -\frac{3}{2}$ and $\beta = -\frac{\sqrt{3}}{2}$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^\alpha \left(c_1 e^{\ln(x^{i\beta})} + c_2 e^{\ln(x^{-i\beta})}\right) \\ &= x^\alpha \left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = -\frac{3}{2}, \beta = -\frac{\sqrt{3}}{2}$, the above becomes

$$y = x^{-\frac{3}{2}} \left(c_1 e^{-\frac{i\sqrt{3}\,\ln(x)}{2}} + c_2 e^{\frac{i\sqrt{3}\,\ln(x)}{2}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = \frac{1}{x^{\frac{3}{2}}} \left(c_1 \cos \left(\frac{\sqrt{3}\, \ln(x)}{2} \right) + c_2 \sin \left(\frac{\sqrt{3}\, \ln(x)}{2} \right) \right)$$

Next, we find the particular solution to the ODE

$$x^2 y'' + 4y'x + 3y = \ln(x)\,(x - 1)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

$$y_2 = -\frac{\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}} & -\frac{\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \\ \frac{d}{dx}\left(\frac{\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}}\right) & \frac{d}{dx}\left(-\frac{\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}} & -\frac{\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \\ -\frac{3\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{2x^{\frac{5}{2}}} - \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{2x^{\frac{5}{2}}} & \frac{3\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{2x^{\frac{5}{2}}} - \frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{2x^{\frac{5}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}}\right)\left(\frac{3\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{2x^{\frac{5}{2}}} - \frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{2x^{\frac{5}{2}}}\right) - \left(-\frac{\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}}\right)\left(-\frac{3\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{2x^{\frac{5}{2}}} - \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{2x^{\frac{5}{2}}}\right)$$

Which simplifies to

$$W = -\frac{\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)^2 + \sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)^2\right)}{2x^4}$$

Which simplifies to

$$W = -\frac{\sqrt{3}}{2x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\ln(x)(x-1)}{x^{\frac{3}{2}}}}{-\frac{\sqrt{3}}{2x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2\sqrt{x}\,\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\ln(x)\,(x-1)\,\sqrt{3}}{3}dx$$

Hence

$$u_1 = \frac{\left(\left(\left(x-\frac{7}{3}\right)\ln(x) - \frac{5x}{7} + \frac{7}{3}\right)\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right) - 5\left(\left(x-\frac{7}{5}\right)\ln(x) - \frac{11x}{35} + \frac{7}{15}\right)\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\right)x^{\frac{3}{2}}\sqrt{3}}{21}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\ln(x)(x-1)}{x^{\frac{3}{2}}}}{-\frac{\sqrt{3}}{2x^2}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{2\sqrt{x}\,\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\ln\left(x\right)\left(x-1\right)\sqrt{3}}{3}dx$$

Hence

$$u_2 = -\frac{\left(\left((5x-7)\ln\left(x\right)-\frac{11x}{7}+\frac{7}{3}\right)\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)+\left(\left(x-\frac{7}{3}\right)\ln\left(x\right)-\frac{5x}{7}+\frac{7}{3}\right)\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\right)x^{\frac{3}{2}}\sqrt{3}}{21}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\left(\left(x-\frac{7}{3}\right)\ln\left(x\right)-\frac{5x}{7}+\frac{7}{3}\right)\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)-5\left(\left(x-\frac{7}{5}\right)\ln\left(x\right)-\frac{11x}{35}+\frac{7}{15}\right)\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\right)\sqrt{3}\,\cos\left(\frac{\sqrt{3}\right.}{21} + \frac{\left(\left((5x-7)\ln\left(x\right)-\frac{11x}{7}+\frac{7}{3}\right)\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)+\left(\left(x-\frac{7}{3}\right)\ln\left(x\right)-\frac{5x}{7}+\frac{7}{3}\right)\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\right)\sqrt{3}\,\sin\left(\frac{\sqrt{3}\right.}{21}$$

Which simplifies to

$$y_p(x) = \frac{1}{3} + \frac{(3x-7)\ln\left(x\right)}{21} - \frac{5x}{49}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{1}{3} + \frac{(3x-7)\ln\left(x\right)}{21} - \frac{5x}{49} + \frac{c_1\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3} + \frac{(3x-7)\ln\left(x\right)}{21} - \frac{5x}{49} + \frac{c_1\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y=\frac{1}{3}+\frac{(3x-7)\ln(x)}{21}-\frac{5x}{49}+\frac{c_1\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

Verified OK.

16.12.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''+4y'x+3y=0$$

In normal form the ode

$$x^2y''+4y'x+3y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{4}{x}$$
$$q(x)=\frac{3}{x^2}$$

Applying change of variables $\tau=g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau)=\frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{4}{x}dx\right)} dx \\
&= \int e^{-4\ln(x)}\, dx \\
&= \int \frac{1}{x^4} dx \\
&= -\frac{1}{3x^3} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{3}{x^2}}{\frac{1}{x^8}} \\
&= 3x^6 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 3x^6 y(\tau) &= 0
\end{aligned}$$

But in terms of τ

$$3x^6 = \frac{1}{3\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{3\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$3\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$3\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$3r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$3r(r-1) + 0 + 1 = 0$$

Or

$$3r^2 - 3r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{i\sqrt{3}}{6}$$
$$r_2 = \frac{1}{2} + \frac{i\sqrt{3}}{6}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{\sqrt{3}}{6}$. Hence the solution becomes

$$\begin{aligned} y(\tau) &= c_1\tau^{r_1} + c_2\tau^{r_2} \\ &= c_1\tau^{\alpha+i\beta} + c_2\tau^{\alpha-i\beta} \\ &= \tau^\alpha\left(c_1\tau^{i\beta} + c_2\tau^{-i\beta}\right) \\ &= \tau^\alpha\left(c_1e^{\ln(\tau^{i\beta})} + c_2e^{\ln(\tau^{-i\beta})}\right) \\ &= \tau^\alpha\left(c_1e^{i(\beta\ln\tau)} + c_2e^{-i(\beta\ln\tau)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{\sqrt{3}}{6}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}}\left(c_1e^{-\frac{i\sqrt{3}\,\ln(\tau)}{6}} + c_2e^{\frac{i\sqrt{3}\,\ln(\tau)}{6}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{\sqrt{3}\,\ln(\tau)}{6}\right) + c_2 \sin\left(\frac{\sqrt{3}\,\ln(\tau)}{6}\right)\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(c_1 \cos\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right) + c_2 \sin\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right)\right)\sqrt{3}\,\sqrt{-\frac{1}{x^3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\left(c_1 \cos\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right) + c_2 \sin\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right)\right)\sqrt{3}\,\sqrt{-\frac{1}{x^3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3}$$

$$y_2 = -\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} & -\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}}{\ldots}\right.}{\ldots} \\ \frac{d}{dx}\left(\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3}\right) & \frac{d}{dx}\left(-\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\ldots\right.}{\ldots}\right.\end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{\ldots} \\ \frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} + \frac{\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2x} + \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} - \frac{\sqrt{\ldots}}{\ldots} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3}\right)\left(-\frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} - \frac{\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2x} + \frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} - \frac{\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{2x}\right) - \left(-\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{3}\right)\left(\frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} + \frac{\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2x} + \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} - \frac{\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2x}\right)$$

Which simplifies to

$$W = \frac{\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)^2\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)^2 + \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)^2\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)^2 + \cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)^2\sin\left(\frac{\sqrt{3}\,\ln(-}{6}\right.}{6x^4}$$

Which simplifies to

$$W = \frac{\sqrt{3}}{6x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{3}\right)\ln(x)\,(x-1)}{\frac{\sqrt{3}}{6x^2}}dx$$

Which simplifies to

$$u_1 = -\int -2\sqrt{-\frac{1}{x^3}}\left(-\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) + \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\ln(x)\,(x-1)\,x^2dx$$

Hence

$$u_1 = -\left(\int_0^x -2\sqrt{-\frac{1}{\alpha^3}}\left(-\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) + \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\ln(\alpha)\,(\alpha-1)\,\alpha^2d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3}\right)\ln(x)\,(x-1)}{\frac{\sqrt{3}}{6x^2}}dx$$

Which simplifies to

$$u_2 = \int 2\sqrt{-\frac{1}{x^3}}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) + \sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\ln(x)(x-1)\,x^2 dx$$

Hence

$$u_2 = \int_0^x 2\sqrt{-\frac{1}{\alpha^3}}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) + \sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\ln(\alpha)(\alpha-1)\,\alpha^2 d\alpha$$

Which simplifies to

$$u_1 = -2\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) - \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\sqrt{-\frac{1}{\alpha^3}}\,\ln(\alpha)(\alpha-1)\,\alpha^2 d\alpha\right)$$

$$u_2 = 2\left(\int_0^x \sqrt{-\frac{1}{\alpha^3}}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) + \sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\ln(\alpha)(\alpha-1)\,\alpha^2 d\alpha\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) - \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\sqrt{-\frac{1}{\alpha^3}}\,\ln(\alpha)\,(\alpha - 1)\,\alpha^2 d\alpha\right)\left(\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3}\right) + 2\left(\int_0^x \sqrt{-\frac{1}{\alpha^3}}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) + \sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\ln(\alpha)\,(\alpha - 1)\,\alpha^2 d\alpha\right)\left(-\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{3}\right)$$

Which simplifies to

$$y_p(x) = -\frac{2\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) - \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right.\right.\right.\right.}{}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(\frac{\left(c_1 \cos\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right) + c_2 \sin\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right)\right)\sqrt{3}\sqrt{-\frac{1}{x^3}}}{3} \right)$$

$$+ \left(-\frac{2\sqrt{3}\sqrt{-\frac{1}{x^3}}\left(\cos\left(\frac{\sqrt{3}\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\ln(3)}{6}\right)\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\ln(3)}{6}\right) - \cos\left(\frac{\sqrt{3}\ln(\right.\right.\right.\right.}{}\right.$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_1 \cos\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right) + c_2 \sin\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right)\right)\sqrt{3}\sqrt{-\frac{1}{x^3}}}{3} \quad (1)$$

$$- \frac{2\sqrt{3}\sqrt{-\frac{1}{x^3}}\left(\cos\left(\frac{\sqrt{3}\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\ln(3)}{6}\right)\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\ln(3)}{6}\right) - \cos\left(\frac{\sqrt{3}\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right.\right.\right.\right.}{}$$

Verification of solutions

$$y = \frac{\left(c_1 \cos\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right) + c_2 \sin\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right)\right)\sqrt{3}\sqrt{-\frac{1}{x^3}}}{3}$$

$$- \frac{2\sqrt{3}\sqrt{-\frac{1}{x^3}}\left(\cos\left(\frac{\sqrt{3}\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\ln(3)}{6}\right)\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\ln(3)}{6}\right) - \cos\left(\frac{\sqrt{3}\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right.\right.\right.\right.}{}$$

Verified OK.

16.12.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 3, f(x) = \ln(x)(x-1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4y'x + 3y = 0$$

In normal form the ode

$$x^2y'' + 4y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\, x^3} + \frac{4}{x}\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= c\sqrt{3} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c\sqrt{3}\left(\frac{d}{d\tau} y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned}
\tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{\sqrt{3}\,c\tau}{2}} \left(c_1 \cos\left(\frac{c\tau}{2}\right) + c_2 \sin\left(\frac{c\tau}{2}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{3}\sqrt{\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{3}\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1 \cos\left(\frac{\sqrt{3}\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)}{x^{\frac{3}{2}}}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + 4y'x + 3y = \ln(x)(x - 1)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3}$$

$$y_2 = -\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} & -\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\cdots\right.}{} \\ \frac{d}{dx}\left(\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3}\right) & \frac{d}{dx}\left(-\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\cdots\right.}{}\right. \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} \cdots \\ \frac{\sqrt{3}\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} + \frac{\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2x} + \frac{\sqrt{3}\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} - \cdots \end{vmatrix}$$

Therefore

$$W = \left(\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3}\right)\left(-\frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} - \frac{\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2x} + \frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} - \frac{\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{2x}\right) - \left(-\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{3}\right)\left(\frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} + \frac{\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2x} + \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2\sqrt{-\frac{1}{x^3}}\,x^4} - \frac{\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{2x}\right)$$

Which simplifies to

$$W = \frac{\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)^2\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)^2 + \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)^2\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)^2 + \cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)^2\sin\left(\frac{\sqrt{3}\,\ln\left(-\right.}{6}\right.}{6x^4}$$

Which simplifies to

$$W = \frac{\sqrt{3}}{6x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{3}\right)\ln(x)(x-1)}{\frac{\sqrt{3}}{6x^2}}dx$$

Which simplifies to

$$u_1 = -\int -2\sqrt{-\frac{1}{x^3}}\left(-\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) + \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\ln(x)(x-1)\,x^2 dx$$

Hence

$$u_1 = -\left(\int_0^x -2\sqrt{-\frac{1}{\alpha^3}}\left(-\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) + \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\ln(\alpha)(\alpha-1)\,\alpha^2 d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3}\right)\ln(x)(x-1)}{\frac{\sqrt{3}}{6x^2}}dx$$

Which simplifies to

$$u_2 = \int 2\sqrt{-\frac{1}{x^3}}\, \left(\cos\left(\frac{\sqrt{3}\, \ln\left(-\frac{1}{x^3}\right)}{6}\right) \cos\left(\frac{\sqrt{3}\, \ln\left(3\right)}{6}\right) + \sin\left(\frac{\sqrt{3}\, \ln\left(-\frac{1}{x^3}\right)}{6}\right) \sin\left(\frac{\sqrt{3}\, \ln\left(3\right)}{6}\right)\right) \ln\left(x\right) \left(x-1\right) x^2 dx$$

Hence

$$u_2 = \int_0^x 2\sqrt{-\frac{1}{\alpha^3}}\, \left(\cos\left(\frac{\sqrt{3}\, \ln\left(-\frac{1}{\alpha^3}\right)}{6}\right) \cos\left(\frac{\sqrt{3}\, \ln\left(3\right)}{6}\right) + \sin\left(\frac{\sqrt{3}\, \ln\left(-\frac{1}{\alpha^3}\right)}{6}\right) \sin\left(\frac{\sqrt{3}\, \ln\left(3\right)}{6}\right)\right) \ln\left(\alpha\right) \left(\alpha-1\right) \alpha^2 d\alpha$$

Which simplifies to

$$u_1 = -2\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\, \ln\left(-\frac{1}{\alpha^3}\right)}{6}\right) \cos\left(\frac{\sqrt{3}\, \ln\left(3\right)}{6}\right) - \cos\left(\frac{\sqrt{3}\, \ln\left(-\frac{1}{\alpha^3}\right)}{6}\right) \sin\left(\frac{\sqrt{3}\, \ln\left(3\right)}{6}\right)\right) \sqrt{-\frac{1}{\alpha^3}}\, \ln\left(\alpha\right) \left(\alpha-1\right) \alpha^2 d\alpha\right)$$

$$u_2 = 2\left(\int_0^x \sqrt{-\frac{1}{\alpha^3}}\, \left(\cos\left(\frac{\sqrt{3}\, \ln\left(-\frac{1}{\alpha^3}\right)}{6}\right) \cos\left(\frac{\sqrt{3}\, \ln\left(3\right)}{6}\right) + \sin\left(\frac{\sqrt{3}\, \ln\left(-\frac{1}{\alpha^3}\right)}{6}\right) \sin\left(\frac{\sqrt{3}\, \ln\left(3\right)}{6}\right)\right) \ln\left(\alpha\right) \left(\alpha-1\right) \alpha^2 d\alpha\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) - \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\sqrt{-\frac{1}{\alpha^3}}\,\ln(\alpha)\,(\alpha - 1)\,\alpha^2 d\alpha\right)\left(\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3}\right) + 2\left(\int_0^x \sqrt{-\frac{1}{\alpha^3}}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) + \sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\right)\ln(\alpha)\,(\alpha - 1)\,\alpha^2 d\alpha\right)\left(-\frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\,\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)}{3}\right)$$

Which simplifies to

$$y_p(x) = -\frac{2\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) - \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right.\right.\right.\right.}{}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(\frac{c_1 \cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}}\right)$$

$$+ \left(-\frac{2\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) - \cos\left(\frac{\sqrt{3}\,\ln\left(\right.}{}\right.\right.\right.\right.}{}\right.$$

$$= -\frac{2\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\left(\cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{x^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right)\left(\int_0^x \left(\sin\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right)}{6}\right)\cos\left(\frac{\sqrt{3}\,\ln(3)}{6}\right) - \cos\left(\frac{\sqrt{3}\,\ln\left(-\frac{1}{\alpha^3}\right.}{6}\right.\right.\right.\right.}{}$$

$$+ \frac{c_1 \cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

Which simplifies to

$$y = -\frac{2x^{\frac{3}{2}}\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\left(\int_0^x -\sqrt{-\frac{1}{\alpha^3}}\,\ln(\alpha)(\alpha-1)\,\alpha^2 \sin\left(\frac{\sqrt{3}\left(-\ln\left(-\frac{1}{\alpha^3}\right)+\ln(3)\right)}{6}\right)d\alpha\right)\cos\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right.}{}$$

Summary

The solution(s) found are the following

$$y = -\frac{2x^{\frac{3}{2}}\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\left(\int_0^x -\sqrt{-\frac{1}{\alpha^3}}\,\ln(\alpha)(\alpha-1)\,\alpha^2 \sin\left(\frac{\sqrt{3}\left(-\ln\left(-\frac{1}{\alpha^3}\right)+\ln(3)\right)}{6}\right)d\alpha\right)\cos\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right.}{} \tag{1}$$

Verification of solutions

$$y = -\frac{2x^{\frac{3}{2}}\sqrt{3}\,\sqrt{-\frac{1}{x^3}}\left(\int_0^x -\sqrt{-\frac{1}{\alpha^3}}\,\ln(\alpha)(\alpha-1)\,\alpha^2 \sin\left(\frac{\sqrt{3}\left(-\ln\left(-\frac{1}{\alpha^3}\right)+\ln(3)\right)}{6}\right)d\alpha\right)\cos\left(\frac{\sqrt{3}\left(-\ln(3)+\ln\left(-\frac{1}{x^3}\right)\right)}{6}\right.}{}$$

Verified OK.

16.12.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 3, f(x) = \ln(x)(x-1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4y'x + 3y = 0$$

In normal form the ode

$$x^2y'' + 4y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{4n}{x^2} + \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{i\sqrt{3}}{2} - \frac{3}{2} \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{i\sqrt{3}-3}{x} + \frac{4}{x}\right) v'(x) = 0$$
$$v''(x) + \frac{\left(1+i\sqrt{3}\right) v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{\left(1+i\sqrt{3}\right) u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\left(-i\sqrt{3}-1\right) u}{x} \end{aligned}$$

Where $f(x) = \frac{-i\sqrt{3}-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-i\sqrt{3}-1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-i\sqrt{3}-1}{x}\, dx \\ \ln(u) &= \left(-i\sqrt{3}-1\right)\ln(x) + c_1 \\ u &= \mathrm{e}^{\left(-i\sqrt{3}-1\right)\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{\left(-i\sqrt{3}-1\right)\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-i\sqrt{3}}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= \frac{i\sqrt{3}\, c_1 x^{-i\sqrt{3}}}{3} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(\frac{i\sqrt{3}\, c_1 x^{-i\sqrt{3}}}{3} + c_2\right) x^{\frac{i\sqrt{3}}{2} - \frac{3}{2}} \\ &= \frac{x^{-\frac{i\sqrt{3}}{2} - \frac{3}{2}} \left(i\sqrt{3}\, c_1 + 3 c_2 x^{i\sqrt{3}}\right)}{3} \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' + 4y' x + 3y = \ln(x)\,(x - 1)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{x^{\frac{i\sqrt{3}}{2}}}{x^{\frac{3}{2}}}$$

$$y_2 = \frac{x^{\frac{i\sqrt{3}}{2}} x^{-i\sqrt{3}}}{x^{\frac{3}{2}}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{x^{\frac{i\sqrt{3}}{2}}}{x^{\frac{3}{2}}} & \frac{x^{\frac{i\sqrt{3}}{2}} x^{-i\sqrt{3}}}{x^{\frac{3}{2}}} \\ \frac{d}{dx}\left(\frac{x^{\frac{i\sqrt{3}}{2}}}{x^{\frac{3}{2}}}\right) & \frac{d}{dx}\left(\frac{x^{\frac{i\sqrt{3}}{2}} x^{-i\sqrt{3}}}{x^{\frac{3}{2}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{x^{\frac{i\sqrt{3}}{2}}}{x^{\frac{3}{2}}} & \frac{x^{\frac{i\sqrt{3}}{2}} x^{-i\sqrt{3}}}{x^{\frac{3}{2}}} \\ \frac{i x^{\frac{i\sqrt{3}}{2}} \sqrt{3}}{2x^{\frac{5}{2}}} - \frac{3x^{\frac{i\sqrt{3}}{2}}}{2x^{\frac{5}{2}}} & -\frac{i x^{\frac{i\sqrt{3}}{2}} \sqrt{3}\, x^{-i\sqrt{3}}}{2x^{\frac{5}{2}}} - \frac{3x^{\frac{i\sqrt{3}}{2}} x^{-i\sqrt{3}}}{2x^{\frac{5}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{x^{\frac{i\sqrt{3}}{2}}}{x^{\frac{3}{2}}}\right)\left(-\frac{i x^{\frac{i\sqrt{3}}{2}} \sqrt{3}\, x^{-i\sqrt{3}}}{2x^{\frac{5}{2}}} - \frac{3x^{\frac{i\sqrt{3}}{2}} x^{-i\sqrt{3}}}{2x^{\frac{5}{2}}}\right) - \left(\frac{x^{\frac{i\sqrt{3}}{2}} x^{-i\sqrt{3}}}{x^{\frac{3}{2}}}\right)\left(\frac{i x^{\frac{i\sqrt{3}}{2}} \sqrt{3}}{2x^{\frac{5}{2}}} - \frac{3x^{\frac{i\sqrt{3}}{2}}}{2x^{\frac{5}{2}}}\right)$$

Which simplifies to

$$W = -\frac{i x^{i\sqrt{3}} \sqrt{3}\, x^{-i\sqrt{3}}}{x^4}$$

Which simplifies to

$$W = -\frac{i\sqrt{3}}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^{\frac{i\sqrt{3}}{2}} x^{-i\sqrt{3}} \ln(x)(x-1)}{x^{\frac{3}{2}}}}{-\frac{i\sqrt{3}}{x^2}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{i \ln(x) \sqrt{3}\, x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} (x-1)}{3} dx$$

Hence

$$u_1 = -\frac{5x^{\frac{3}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{7}{5} + i\left(\left(x-\frac{7}{5}\right)\ln(x) - \frac{11x}{35} + \frac{7}{15}\right)\sqrt{3} + \frac{(7-3x)\ln(x)}{5} + \frac{3x}{7}\right)}{42}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^{\frac{i\sqrt{3}}{2}} \ln(x)(x-1)}{x^{\frac{3}{2}}}}{-\frac{i\sqrt{3}}{x^2}} dx$$

Which simplifies to

$$u_2 = \int \frac{i \ln(x) \sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} (x-1)}{3} dx$$

Hence

$$u_2 = \frac{5x^{\frac{3}{2}+\frac{i\sqrt{3}}{2}} \left(\frac{7}{5} + i\left(\left(x-\frac{7}{5}\right)\ln(x) - \frac{11x}{35} + \frac{7}{15}\right)\sqrt{3} + \frac{(3x-7)\ln(x)}{5} - \frac{3x}{7}\right)}{42}$$

Which simplifies to

$$u_1 = \frac{5\left(\frac{7}{5} - i\left(\left(x-\frac{7}{5}\right)\ln(x) - \frac{11x}{35} + \frac{7}{15}\right)\sqrt{3} - \frac{(7-3x)\ln(x)}{5} - \frac{3x}{7}\right) x^{\frac{3}{2}} x^{-\frac{i\sqrt{3}}{2}}}{42}$$

$$u_2 = \frac{5x^{\frac{3}{2}+\frac{i\sqrt{3}}{2}} \left(\frac{7}{5} + i\left(\left(x-\frac{7}{5}\right)\ln(x) - \frac{11x}{35} + \frac{7}{15}\right)\sqrt{3} + \frac{(3x-7)\ln(x)}{5} - \frac{3x}{7}\right)}{42}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{5\left(\frac{7}{5} - i\left(\left(x-\frac{7}{5}\right)\ln(x) - \frac{11x}{35} + \frac{7}{15}\right)\sqrt{3} - \frac{(7-3x)\ln(x)}{5} - \frac{3x}{7}\right) x^{-\frac{i\sqrt{3}}{2}} x^{\frac{i\sqrt{3}}{2}}}{42} + \frac{5x^{\frac{3}{2}+\frac{i\sqrt{3}}{2}} \left(\frac{7}{5} + i\left(\left(x-\frac{7}{5}\right)\ln(x) - \frac{11x}{35} + \frac{7}{15}\right)\sqrt{3} + \frac{(3x-7)\ln(x)}{5} - \frac{3x}{7}\right) x^{\frac{i\sqrt{3}}{2}} x^{-i\sqrt{3}}}{42x^{\frac{3}{2}}}$$

Which simplifies to

$$y_p(x) = \frac{1}{3} + \frac{(3x-7)\ln(x)}{21} - \frac{5x}{49}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{i\sqrt{3}\,c_1 x^{-i\sqrt{3}}}{3} + c_2\right) x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}\right) + \left(\frac{1}{3} + \frac{(3x-7)\ln(x)}{21} - \frac{5x}{49}\right) \\ &= \frac{1}{3} + \frac{(3x-7)\ln(x)}{21} - \frac{5x}{49} + \left(\frac{i\sqrt{3}\,c_1 x^{-i\sqrt{3}}}{3} + c_2\right) x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}} \end{aligned}$$

Which simplifies to

$$y = \frac{1}{3} + \frac{(3x-7)\ln(x)}{21} - \frac{5x}{49} + \left(\frac{i\sqrt{3}\,c_1 x^{-i\sqrt{3}}}{3} + c_2\right) x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{3} + \frac{(3x-7)\ln(x)}{21} - \frac{5x}{49} + \left(\frac{i\sqrt{3}\,c_1 x^{-i\sqrt{3}}}{3} + c_2\right) x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}} \tag{1}$$

Verification of solutions

$$y = \frac{1}{3} + \frac{(3x-7)\ln(x)}{21} - \frac{5x}{49} + \left(\frac{i\sqrt{3}\,c_1 x^{-i\sqrt{3}}}{3} + c_2\right) x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}$$

Verified OK.

16.12.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 4y'x + 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= 4x \\
C &= 3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 556: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b=-1$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}+\frac{i\sqrt{3}}{2} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=-\frac{1}{x^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-1$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}+\frac{i\sqrt{3}}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{1}{2}-\frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{1}{2}-\frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \frac{i\sqrt{3}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} + (-)\,(0) \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} \\
&= \frac{1 - i\sqrt{3}}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)^2 - \left(-\frac{1}{x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} dx} \\
&= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1\left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}}\right) + c_2\left(x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}}\left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''+4y'x+3y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}} - \frac{i c_2 \sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}}$$

$$y_2 = -\frac{i\sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}} & -\frac{i\sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{3} \\ \frac{d}{dx}\left(x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}}\right) & \frac{d}{dx}\left(-\frac{i\sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}} & -\frac{i\sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{3} \\ \frac{x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}}\left(-\frac{i\sqrt{3}}{2}-\frac{3}{2}\right)}{x} & -\frac{i\sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}\left(\frac{i\sqrt{3}}{2}-\frac{3}{2}\right)}{3x} \end{vmatrix}$$

Therefore

$$\begin{aligned} W &= \left(x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}}\right)\left(-\frac{i\sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}\left(\frac{i\sqrt{3}}{2}-\frac{3}{2}\right)}{3x}\right) \\ &\quad - \left(-\frac{i\sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{3}\right)\left(\frac{x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}}\left(-\frac{i\sqrt{3}}{2}-\frac{3}{2}\right)}{x}\right) \end{aligned}$$

Which simplifies to

$$W = \frac{x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{x}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i\sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}} \ln(x)(x-1)}{3}}{\frac{1}{x^2}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{i \ln(x)\sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}(x-1)}{3} dx$$

Hence

$$u_1 = -\frac{5\left(-\frac{7}{5} - i\left(\left(x-\frac{7}{5}\right)\ln(x) - \frac{11x}{35} + \frac{7}{15}\right)\sqrt{3} - \frac{(3x-7)\ln(x)}{5} + \frac{3x}{7}\right) x^{\frac{3}{2}} x^{\frac{i\sqrt{3}}{2}}}{42}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}} \ln\left(x\right) \left(x-1\right)}{\frac{1}{x^2}} \, dx$$

Which simplifies to

$$u_2 = \int \ln\left(x\right) x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(x-1\right) dx$$

Hence

$$u_2 = \frac{x^{\frac{3}{2}-\frac{i\sqrt{3}}{2}} \left(\frac{7}{3}+i\left(\left(x-\frac{7}{3}\right)\ln\left(x\right)-\frac{5x}{7}+\frac{7}{3}\right)\sqrt{3}+\left(5x-7\right)\ln\left(x\right)-\frac{11x}{7}\right)}{14}$$

Which simplifies to

$$u_1 = \frac{5x^{\frac{3}{2}+\frac{i\sqrt{3}}{2}} \left(\frac{7}{5}+i\left(\left(x-\frac{7}{5}\right)\ln\left(x\right)-\frac{11x}{35}+\frac{7}{15}\right)\sqrt{3}+\frac{\left(3x-7\right)\ln\left(x\right)}{5}-\frac{3x}{7}\right)}{42}$$

$$u_2 = \frac{x^{\frac{3}{2}-\frac{i\sqrt{3}}{2}} \left(\frac{7}{3}+i\left(\left(x-\frac{7}{3}\right)\ln\left(x\right)-\frac{5x}{7}+\frac{7}{3}\right)\sqrt{3}+\left(5x-7\right)\ln\left(x\right)-\frac{11x}{7}\right)}{14}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{5x^{\frac{3}{2}+\frac{i\sqrt{3}}{2}} \left(\frac{7}{5}+i\left(\left(x-\frac{7}{5}\right)\ln\left(x\right)-\frac{11x}{35}+\frac{7}{15}\right)\sqrt{3}+\frac{\left(3x-7\right)\ln\left(x\right)}{5}-\frac{3x}{7}\right) x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{42} - \frac{i x^{\frac{3}{2}-\frac{i\sqrt{3}}{2}} \left(\frac{7}{3}+i\left(\left(x-\frac{7}{3}\right)\ln\left(x\right)-\frac{5x}{7}+\frac{7}{3}\right)\sqrt{3}+\left(5x-7\right)\ln\left(x\right)-\frac{11x}{7}\right)\sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{42}$$

Which simplifies to

$$y_p(x) = \frac{1}{3}+\frac{\left(3x-7\right)\ln\left(x\right)}{21}-\frac{5x}{49}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}} - \frac{i c_2 \sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{3}\right) + \left(\frac{1}{3}+\frac{\left(3x-7\right)\ln\left(x\right)}{21}-\frac{5x}{49}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}} - \frac{i c_2 \sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{3} + \frac{1}{3} + \frac{(3x-7)\ln(x)}{21} - \frac{5x}{49} \tag{1}$$

Verification of solutions

$$y = c_1 x^{-\frac{i\sqrt{3}}{2}-\frac{3}{2}} - \frac{i c_2 \sqrt{3}\, x^{\frac{i\sqrt{3}}{2}-\frac{3}{2}}}{3} + \frac{1}{3} + \frac{(3x-7)\ln(x)}{21} - \frac{5x}{49}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 44

```
dsolve(x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+3*y(x)=(x-1)*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right) c_2}{x^{\frac{3}{2}}} + \frac{\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right) c_1}{x^{\frac{3}{2}}} + \frac{1}{3} + \frac{(3x-7)\ln(x)}{21} - \frac{5x}{49}$$

✓ Solution by Mathematica

Time used: 0.516 (sec). Leaf size: 67

```
DSolve[x^2*y''[x]+4*x*y'[x]+3*y[x]==(x-1)*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 \cos\left(\frac{1}{2}\sqrt{3}\log(x)\right)}{x^{3/2}} + \frac{c_1 \sin\left(\frac{1}{2}\sqrt{3}\log(x)\right)}{x^{3/2}} - \frac{5x}{49} + \frac{1}{7}x\log(x) - \frac{\log(x)}{3} + \frac{1}{3}$$

16.13 problem 13

Internal problem ID [2262]
Internal file name [`OUTPUT/2262_Monday_February_26_2024_10_05_02_AM_96052526/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 13.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{4x^3y''' + 8x^2y'' - y'x + y = x + \ln(x)}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$4x^3y''' + 8x^2y'' - y'x + y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$4x^3y''' + 8x^2y'' - y'x + y = x + \ln(x)$$

gives

$$-x\lambda\, x^{\lambda-1}+8x^{2}\lambda(\lambda-1)\, x^{\lambda-2}+4x^{3}\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}+x^{\lambda}=0$$

Which simplifies to

$$-\lambda\, x^{\lambda}+8\lambda(\lambda-1)\, x^{\lambda}+4\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda}+x^{\lambda}=0$$

And since $x^{\lambda}\neq 0$ then dividing through by x^{λ}, the above becomes

$$-\lambda+8\lambda(\lambda-1)+4\lambda(\lambda-1)\,(\lambda-2)+1=0$$

Simplifying gives the characteristic equation as

$$4\lambda^{3}-4\lambda^{2}-\lambda+1=0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{1}{2}$$
$$\lambda_3 = \frac{1}{2}$$

This table summarises the result

root	multiplicity	type of root
1	1	real root
$-\frac{1}{2}$	1	real root
$\frac{1}{2}$	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^{\lambda}$ basis solution. Each real root of multiplicty two, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda}\ln(x)$ and $c_3 x^{\lambda}\ln(x)^{2}$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\, x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^{2} x^{\alpha}(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1 x+\frac{c_2}{\sqrt{x}}+\sqrt{x}\, c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{1}{\sqrt{x}} \\
y_3 &= \sqrt{x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
4x^3y''' + 8x^2y'' - y'x + y = x + \ln(x)
$$

Let the particular solution be

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)}\,dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} x & \frac{1}{\sqrt{x}} & \sqrt{x} \\ 1 & -\frac{1}{2x^{\frac{3}{2}}} & \frac{1}{2\sqrt{x}} \\ 0 & \frac{3}{4x^{\frac{5}{2}}} & -\frac{1}{4x^{\frac{3}{2}}} \end{bmatrix} \\
|W| &= \frac{3}{4x^2}
\end{aligned}
$$

The determinant simplifies to

$$|W| = \frac{3}{4x^2}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} \frac{1}{\sqrt{x}} & \sqrt{x} \\ -\frac{1}{2x^{\frac{3}{2}}} & \frac{1}{2\sqrt{x}} \end{bmatrix} \\ &= \frac{1}{x} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} x & \sqrt{x} \\ 1 & \frac{1}{2\sqrt{x}} \end{bmatrix} \\ &= -\frac{\sqrt{x}}{2} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} x & \frac{1}{\sqrt{x}} \\ 1 & -\frac{1}{2x^{\frac{3}{2}}} \end{bmatrix} \\ &= -\frac{3}{2\sqrt{x}} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^2 \int \frac{(x + \ln(x))\left(\frac{1}{x}\right)}{(4x^3)\left(\frac{3}{4x^2}\right)}\, dx \\ &= \int \frac{\frac{x+\ln(x)}{x}}{3x}\, dx \\ &= \int \left(\frac{x + \ln(x)}{3x^2}\right) dx \\ &= \frac{\ln(x)}{3} - \frac{\ln(x)}{3x} - \frac{1}{3x} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^{1} \int \frac{(x+\ln(x))\left(-\frac{\sqrt{x}}{2}\right)}{(4x^3)\left(\frac{3}{4x^2}\right)}\, dx \\
&= -\int \frac{-\frac{(x+\ln(x))\sqrt{x}}{2}}{3x}\, dx \\
&= -\int \left(-\frac{x+\ln(x)}{6\sqrt{x}}\right) dx \\
&= \frac{x^{\frac{3}{2}}}{9} + \frac{\ln(x)\sqrt{x}}{3} - \frac{2\sqrt{x}}{3}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^{0} \int \frac{(x+\ln(x))\left(-\frac{3}{2\sqrt{x}}\right)}{(4x^3)\left(\frac{3}{4x^2}\right)}\, dx \\
&= \int \frac{-\frac{3(x+\ln(x))}{2\sqrt{x}}}{3x}\, dx \\
&= \int \left(-\frac{x+\ln(x)}{2x^{\frac{3}{2}}}\right) dx \\
&= -\sqrt{x} + \frac{\ln(x)}{\sqrt{x}} + \frac{2}{\sqrt{x}}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = {} & \left(\frac{\ln(x)}{3} - \frac{\ln(x)}{3x} - \frac{1}{3x}\right)(x) \\
& + \left(\frac{x^{\frac{3}{2}}}{9} + \frac{\ln(x)\sqrt{x}}{3} - \frac{2\sqrt{x}}{3}\right)\left(\frac{1}{\sqrt{x}}\right) \\
& + \left(-\sqrt{x} + \frac{\ln(x)}{\sqrt{x}} + \frac{2}{\sqrt{x}}\right)(\sqrt{x})
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{x\ln(x)}{3} + \ln(x) + 1 - \frac{8x}{9}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x + \frac{c_2}{\sqrt{x}} + \sqrt{x}\, c_3\right) + \left(\frac{x \ln(x)}{3} + \ln(x) + 1 - \frac{8x}{9}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + \frac{c_2}{\sqrt{x}} + \sqrt{x}\, c_3 + \frac{x \ln(x)}{3} + \ln(x) + 1 - \frac{8x}{9} \tag{1}
$$

Verification of solutions

$$
y = c_1 x + \frac{c_2}{\sqrt{x}} + \sqrt{x}\, c_3 + \frac{x \ln(x)}{3} + \ln(x) + 1 - \frac{8x}{9}
$$

Verified OK.

16.13.1 Maple step by step solution

Let's solve

$4x^3 y''' + 8x^2 y'' - y'x + y = x + \ln(x)$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(4*x^3*diff(y(x),x$3)+8*x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=x+ln(x),y(x), singsol=a
```

$$y(x) = \frac{x \ln(x)}{3} + \ln(x) + 1 - \frac{8x}{9} + c_1 x + \frac{c_2}{\sqrt{x}} + c_3\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 40

```
DSolve[4*x^3*y'''[x]+8*x^2*y''[x]-x*y'[x]+y[x]==x+Log[x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{3}(x+3)\log(x) + \frac{c_1}{\sqrt{x}} + c_2\sqrt{x} + \left(-\frac{8}{9} + c_3\right)x + 1$$

16.14 problem 14

Internal problem ID [2263]
Internal file name [`OUTPUT/2263_Monday_February_26_2024_10_05_03_AM_85331185/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 14.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{3x^3y''' + 4x^2y'' - 10y'x + 10y = \frac{4}{x^2}}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$3x^3y''' + 4x^2y'' - 10y'x + 10y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$3x^3y''' + 4x^2y'' - 10y'x + 10y = \frac{4}{x^2}$$

gives

$$-10x\lambda\, x^{\lambda-1} + 4x^2\lambda(\lambda-1)\, x^{\lambda-2} + 3x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 10x^\lambda = 0$$

Which simplifies to

$$-10\lambda\, x^\lambda + 4\lambda(\lambda-1)\, x^\lambda + 3\lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 10x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-10\lambda + 4\lambda(\lambda-1) + 3\lambda(\lambda-1)\,(\lambda-2) + 10 = 0$$

Simplifying gives the characteristic equation as

$$3\lambda^3 - 5\lambda^2 - 8\lambda + 10 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = \frac{1}{3} - \frac{\sqrt{31}}{3}$$
$$\lambda_3 = \frac{1}{3} + \frac{\sqrt{31}}{3}$$

This table summarises the result

root	multiplicity	type of root
1	1	real root
$\frac{1}{3} - \frac{\sqrt{31}}{3}$	1	real root
$\frac{1}{3} + \frac{\sqrt{31}}{3}$	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$
y = c_1 x + c_2 x^{\frac{1}{3}-\frac{\sqrt{31}}{3}} + c_3 x^{\frac{1}{3}+\frac{\sqrt{31}}{3}}
$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x^{\frac{1}{3}-\frac{\sqrt{31}}{3}} \\
y_3 &= x^{\frac{1}{3}+\frac{\sqrt{31}}{3}}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
3x^3 y''' + 4x^2 y'' - 10y'x + 10y = \frac{4}{x^2}
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
W = \begin{bmatrix} x & x^{\frac{1}{3}-\frac{\sqrt{31}}{3}} & x^{\frac{1}{3}+\frac{\sqrt{31}}{3}} \\ 1 & -\frac{x^{-\frac{2}{3}-\frac{\sqrt{31}}{3}}\left(-1+\sqrt{31}\right)}{3} & \frac{x^{-\frac{2}{3}+\frac{\sqrt{31}}{3}}\left(1+\sqrt{31}\right)}{3} \\ 0 & \frac{x^{-\frac{5}{3}-\frac{\sqrt{31}}{3}}\left(-1+\sqrt{31}\right)\left(2+\sqrt{31}\right)}{9} & \frac{x^{-\frac{5}{3}+\frac{\sqrt{31}}{3}}\left(-2+\sqrt{31}\right)\left(1+\sqrt{31}\right)}{9} \end{bmatrix}
$$

$$
|W| = -\frac{10x\, x^{-\frac{2}{3}-\frac{\sqrt{31}}{3}} x^{-\frac{5}{3}+\frac{\sqrt{31}}{3}} \sqrt{31}}{9} + \frac{20x\, x^{-\frac{2}{3}-\frac{\sqrt{31}}{3}} x^{-\frac{5}{3}+\frac{\sqrt{31}}{3}}}{9} - \frac{10x\, x^{-\frac{5}{3}-\frac{\sqrt{31}}{3}} x^{-\frac{2}{3}+\frac{\sqrt{31}}{3}} \sqrt{31}}{9} - \frac{20x\, x^{-\frac{5}{3}-\frac{\sqrt{31}}{3}} x^{-}}{9}
$$

The determinant simplifies to

$$|W| = -\frac{2\sqrt{31}}{x^{\frac{4}{3}}}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x^{\frac{1}{3}-\frac{\sqrt{31}}{3}} & x^{\frac{1}{3}+\frac{\sqrt{31}}{3}} \\ -\frac{x^{-\frac{2}{3}-\frac{\sqrt{31}}{3}}\left(-1+\sqrt{31}\right)}{3} & \frac{x^{-\frac{2}{3}+\frac{\sqrt{31}}{3}}\left(1+\sqrt{31}\right)}{3} \end{bmatrix} \\ &= \frac{2\sqrt{31}}{3x^{\frac{1}{3}}} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} x & x^{\frac{1}{3}+\frac{\sqrt{31}}{3}} \\ 1 & \frac{x^{-\frac{2}{3}+\frac{\sqrt{31}}{3}}\left(1+\sqrt{31}\right)}{3} \end{bmatrix} \\ &= \frac{\left(-2+\sqrt{31}\right) x^{\frac{1}{3}+\frac{\sqrt{31}}{3}}}{3} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} x & x^{\frac{1}{3}-\frac{\sqrt{31}}{3}} \\ 1 & -\frac{x^{-\frac{2}{3}-\frac{\sqrt{31}}{3}}\left(-1+\sqrt{31}\right)}{3} \end{bmatrix} \\ &= -\frac{\left(2+\sqrt{31}\right) x^{\frac{1}{3}-\frac{\sqrt{31}}{3}}}{3} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^2 \int \frac{\left(\frac{4}{x^2}\right)\left(\frac{2\sqrt{31}}{3x^{\frac{1}{3}}}\right)}{(3x^3)\left(-\frac{2\sqrt{31}}{x^{\frac{4}{3}}}\right)}\, dx \\ &= \int \frac{\frac{8\sqrt{31}}{3x^{\frac{7}{3}}}}{-6x^{\frac{5}{3}}\sqrt{31}}\, dx \\ &= \int \left(-\frac{4}{9x^4}\right) dx \\ &= \frac{4}{27x^3} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{\left(\frac{4}{x^2}\right)\left(\frac{\left(-2+\sqrt{31}\right) x^{\frac{1}{3}+\frac{\sqrt{31}}{3}}}{3}\right)}{\left(3x^3\right)\left(-\frac{2\sqrt{31}}{x^{\frac{4}{3}}}\right)}\, dx \\
&= -\int \frac{\frac{4\left(-2+\sqrt{31}\right) x^{\frac{1}{3}+\frac{\sqrt{31}}{3}}}{3x^2}}{-6x^{\frac{5}{3}}\sqrt{31}}\, dx \\
&= -\int \left(-\frac{2\sqrt{31}\, x^{-\frac{10}{3}+\frac{\sqrt{31}}{3}}\left(-2+\sqrt{31}\right)}{279}\right) dx \\
&= -\frac{x^{-\frac{7}{3}+\frac{\sqrt{31}}{3}}\sqrt{31}\left(-2+\sqrt{31}\right)\left(7+\sqrt{31}\right)}{837}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{\left(\frac{4}{x^2}\right)\left(-\frac{\left(2+\sqrt{31}\right) x^{\frac{1}{3}-\frac{\sqrt{31}}{3}}}{3}\right)}{\left(3x^3\right)\left(-\frac{2\sqrt{31}}{x^{\frac{4}{3}}}\right)}\, dx \\
&= \int \frac{-\frac{4\left(2+\sqrt{31}\right) x^{\frac{1}{3}-\frac{\sqrt{31}}{3}}}{3x^2}}{-6x^{\frac{5}{3}}\sqrt{31}}\, dx \\
&= \int \left(\frac{2\sqrt{31}\, x^{-\frac{10}{3}-\frac{\sqrt{31}}{3}}\left(2+\sqrt{31}\right)}{279}\right) dx \\
&= \frac{x^{-\frac{7}{3}-\frac{\sqrt{31}}{3}}\sqrt{31}\left(-7+\sqrt{31}\right)\left(2+\sqrt{31}\right)}{837}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{4}{27x^3}\right)(x) \\
&+ \left(-\frac{x^{-\frac{7}{3}+\frac{\sqrt{31}}{3}}\sqrt{31}\left(-2+\sqrt{31}\right)\left(7+\sqrt{31}\right)}{837}\right)\left(x^{\frac{1}{3}-\frac{\sqrt{31}}{3}}\right) \\
&+ \left(\frac{x^{-\frac{7}{3}-\frac{\sqrt{31}}{3}}\sqrt{31}\left(-7+\sqrt{31}\right)\left(2+\sqrt{31}\right)}{837}\right)\left(x^{\frac{1}{3}+\frac{\sqrt{31}}{3}}\right)
\end{aligned}
$$

Therefore the particular solution is

$$y_p = -\frac{2}{9x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x + c_2 x^{\frac{1}{3}-\frac{\sqrt{31}}{3}} + c_3 x^{\frac{1}{3}+\frac{\sqrt{31}}{3}}\right) + \left(-\frac{2}{9x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 x^{\frac{1}{3}-\frac{\sqrt{31}}{3}} + c_3 x^{\frac{1}{3}+\frac{\sqrt{31}}{3}} - \frac{2}{9x^2} \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2 x^{\frac{1}{3}-\frac{\sqrt{31}}{3}} + c_3 x^{\frac{1}{3}+\frac{\sqrt{31}}{3}} - \frac{2}{9x^2}$$

Verified OK.

16.14.1 Maple step by step solution

Let's solve

$3x^3 y''' + 4x^2 y'' - 10y'x + 10y = \frac{4}{x^2}$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
dsolve(3*x^3*diff(y(x),x$3)+4*x^2*diff(y(x),x$2)-10*x*diff(y(x),x)+10*y(x)=4/x^2,y(x), sings
```

$$y(x) = \frac{9c_3x^{\frac{7}{3}+\frac{\sqrt{31}}{3}} + 9c_2x^{\frac{7}{3}-\frac{\sqrt{31}}{3}} + 9c_1x^3 - 2}{9x^2}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 51

```
DSolve[3*x^3*y'''[x]+4*x^2*y''[x]-10*x*y'[x]+10*y[x]==4/x^2,y[x],x,IncludeSingularSolutions
```

$$y(x) \to c_2x^{\frac{1}{3}\left(1+\sqrt{31}\right)} + c_1x^{\frac{1}{3}-\frac{\sqrt{31}}{3}} - \frac{2}{9x^2} + c_3x$$

16.15 problem 15

Internal problem ID [2264]
Internal file name [`OUTPUT/2264_Monday_February_26_2024_10_05_03_AM_61469210/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 15.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_high_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^4y'''' + 7x^3y''' + 9x^2y'' - 6xy' - 6y = \cos(\ln(x))}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^4y'''' + 7x^3y''' + 9x^2y'' - 6xy' - 6y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}
\end{aligned}$$

Substituting these back into

$$x^4y'''' + 7x^3y''' + 9x^2y'' - 6xy' - 6y = \cos(\ln(x))$$

gives

$$
\begin{aligned}
&-6x\lambda\, x^{\lambda-1} + 9x^2\lambda(\lambda-1)\, x^{\lambda-2} + 7x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
&+ x^4\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4} - 6x^\lambda = 0
\end{aligned}
$$

Which simplifies to

$$
-6\lambda\, x^\lambda + 9\lambda(\lambda-1)\, x^\lambda + 7\lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^\lambda - 6x^\lambda = 0
$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$
-6\lambda + 9\lambda(\lambda-1) + 7\lambda(\lambda-1)\,(\lambda-2) + \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3) - 6 = 0
$$

Simplifying gives the characteristic equation as

$$
\lambda^4 + \lambda^3 - \lambda^2 - 7\lambda - 6 = 0
$$

Solving the above gives the following roots

$$
\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -1 \\
\lambda_3 &= -1 - i\sqrt{2} \\
\lambda_4 &= -1 + i\sqrt{2}
\end{aligned}
$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
2	1	real root
$-1 \pm \sqrt{2}i$	1	complex conjugate root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$
y = \frac{c_1}{x} + c_2 x^2 + \frac{c_3 \cos\left(\sqrt{2}\, \ln\left(x \right)\right) + c_4 \sin\left(\sqrt{2}\, \ln\left(x \right)\right)}{x}
$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{align*}
y_1 &= \frac{1}{x} \\
y_2 &= x^2 \\
y_3 &= \frac{\cos\left(\sqrt{2}\, \ln\left(x \right)\right)}{x} \\
y_4 &= \frac{\sin\left(\sqrt{2}\, \ln\left(x \right)\right)}{x}
\end{align*}
$$

Now the particular solution to the given ODE is found

$$
x^4 y'''' + 7x^3 y''' + 9x^2 y'' - 6x y' - 6y = \cos\left(\ln\left(x \right)\right)
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \frac{1}{x} & x^2 & \frac{\cos\left(\sqrt{2}\,\ln(x)\right)}{x} & \frac{\sin\left(\sqrt{2}\,\ln(x)\right)}{x} \\ -\frac{1}{x^2} & 2x & \frac{-\cos\left(\sqrt{2}\,\ln(x)\right)-\sqrt{2}\,\sin\left(\sqrt{2}\,\ln(x)\right)}{x^2} & \frac{-\sin\left(\sqrt{2}\,\ln(x)\right)+\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)}{x^2} \\ \frac{2}{x^3} & 2 & \frac{3\sqrt{2}\,\sin\left(\sqrt{2}\,\ln(x)\right)}{x^3} & -\frac{3\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)}{x^3} \\ -\frac{6}{x^4} & 0 & \frac{-9\sqrt{2}\,\sin\left(\sqrt{2}\,\ln(x)\right)+6\cos\left(\sqrt{2}\,\ln(x)\right)}{x^4} & \frac{9\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)+6\sin\left(\sqrt{2}\,\ln(x)\right)}{x^4} \end{bmatrix}$$

$$|W| = \frac{66\sqrt{2}\left(\cos\left(\sqrt{2}\,\ln\left(x\right)\right)^2+\sin\left(\sqrt{2}\,\ln\left(x\right)\right)^2\right)}{x^7}$$

The determinant simplifies to

$$|W| = \frac{66\sqrt{2}}{x^7}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det\begin{bmatrix} x^2 & \frac{\cos\left(\sqrt{2}\,\ln(x)\right)}{x} & \frac{\sin\left(\sqrt{2}\,\ln(x)\right)}{x} \\ 2x & \frac{-\cos\left(\sqrt{2}\,\ln(x)\right)-\sqrt{2}\,\sin\left(\sqrt{2}\,\ln(x)\right)}{x^2} & \frac{-\sin\left(\sqrt{2}\,\ln(x)\right)+\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)}{x^2} \\ 2 & \frac{3\sqrt{2}\,\sin\left(\sqrt{2}\,\ln(x)\right)}{x^3} & -\frac{3\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)}{x^3} \end{bmatrix}$$
$$= \frac{11\sqrt{2}}{x^3}$$

$$W_2(x) = \det\begin{bmatrix} \frac{1}{x} & \frac{\cos\left(\sqrt{2}\,\ln(x)\right)}{x} & \frac{\sin\left(\sqrt{2}\,\ln(x)\right)}{x} \\ -\frac{1}{x^2} & \frac{-\cos\left(\sqrt{2}\,\ln(x)\right)-\sqrt{2}\,\sin\left(\sqrt{2}\,\ln(x)\right)}{x^2} & \frac{-\sin\left(\sqrt{2}\,\ln(x)\right)+\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)}{x^2} \\ \frac{2}{x^3} & \frac{3\sqrt{2}\,\sin\left(\sqrt{2}\,\ln(x)\right)}{x^3} & -\frac{3\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)}{x^3} \end{bmatrix}$$
$$= \frac{2\sqrt{2}}{x^6}$$

$$W_3(x) = \det \begin{bmatrix} \frac{1}{x} & x^2 & \frac{\sin\left(\sqrt{2}\ln(x)\right)}{x} \\ -\frac{1}{x^2} & 2x & \frac{-\sin\left(\sqrt{2}\ln(x)\right)+\sqrt{2}\cos\left(\sqrt{2}\ln(x)\right)}{x^2} \\ \frac{2}{x^3} & 2 & -\frac{3\sqrt{2}\cos\left(\sqrt{2}\ln(x)\right)}{x^3} \end{bmatrix}$$

$$= \frac{-9\sqrt{2}\cos\left(\sqrt{2}\ln(x)\right) - 6\sin\left(\sqrt{2}\ln(x)\right)}{x^3}$$

$$W_4(x) = \det \begin{bmatrix} \frac{1}{x} & x^2 & \frac{\cos\left(\sqrt{2}\ln(x)\right)}{x} \\ -\frac{1}{x^2} & 2x & \frac{-\cos\left(\sqrt{2}\ln(x)\right)-\sqrt{2}\sin\left(\sqrt{2}\ln(x)\right)}{x^2} \\ \frac{2}{x^3} & 2 & \frac{3\sqrt{2}\sin\left(\sqrt{2}\ln(x)\right)}{x^3} \end{bmatrix}$$

$$= \frac{9\sqrt{2}\sin\left(\sqrt{2}\ln(x)\right) - 6\cos\left(\sqrt{2}\ln(x)\right)}{x^3}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3 \int \frac{(\cos(\ln(x)))\left(\frac{11\sqrt{2}}{x^3}\right)}{(x^4)\left(\frac{66\sqrt{2}}{x^7}\right)}\,dx \\ &= -\int \frac{\frac{11\cos(\ln(x))\sqrt{2}}{x^3}}{\frac{66\sqrt{2}}{x^3}}\,dx \\ &= -\int \left(\frac{\cos(\ln(x))}{6}\right)dx \\ &= -\frac{\cos(\ln(x))\,x}{12} - \frac{\sin(\ln(x))\,x}{12} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(\cos(\ln(x)))\left(\frac{2\sqrt{2}}{x^6}\right)}{(x^4)\left(\frac{66\sqrt{2}}{x^7}\right)}\,dx \\
&= \int \frac{\frac{2\cos(\ln(x))\sqrt{2}}{x^6}}{\frac{66\sqrt{2}}{x^3}}\,dx \\
&= \int \left(\frac{\cos(\ln(x))}{33x^3}\right)\,dx \\
&= \frac{-\frac{2}{165} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)^2}{165} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)}{165}}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)x^2}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(\cos(\ln(x)))\left(\frac{-9\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)-6\sin\left(\sqrt{2}\,\ln(x)\right)}{x^3}\right)}{(x^4)\left(\frac{66\sqrt{2}}{x^7}\right)}\,dx \\
&= -\int \frac{\frac{\cos(\ln(x))\left(-9\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)-6\sin\left(\sqrt{2}\,\ln(x)\right)\right)}{x^3}}{\frac{66\sqrt{2}}{x^3}}\,dx \\
&= -\int \left(\frac{\cos(\ln(x))\left(-9\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)-6\sin\left(\sqrt{2}\,\ln(x)\right)\right)\sqrt{2}}{132}\right)\,dx \\
&= -\left(\int \frac{\cos(\ln(x))\left(-9\sqrt{2}\,\cos\left(\sqrt{2}\,\ln(x)\right)-6\sin\left(\sqrt{2}\,\ln(x)\right)\right)\sqrt{2}}{132}dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(\cos(\ln(x))) \left(\frac{9\sqrt{2}\, \sin\left(\sqrt{2}\, \ln(x)\right) - 6\cos\left(\sqrt{2}\, \ln(x)\right)}{x^3}\right)}{(x^4)\left(\frac{66\sqrt{2}}{x^7}\right)}\, dx \\
&= \int \frac{\frac{\cos(\ln(x))\left(9\sqrt{2}\, \sin\left(\sqrt{2}\, \ln(x)\right) - 6\cos\left(\sqrt{2}\, \ln(x)\right)\right)}{x^3}}{\frac{66\sqrt{2}}{x^3}}\, dx \\
&= \int \left(\frac{\cos(\ln(x))\left(3\sqrt{2}\, \sin\left(\sqrt{2}\, \ln(x)\right) - 2\cos\left(\sqrt{2}\, \ln(x)\right)\right)\sqrt{2}}{44}\right) dx \\
&= \int \frac{\cos(\ln(x))\left(3\sqrt{2}\, \sin\left(\sqrt{2}\, \ln(x)\right) - 2\cos\left(\sqrt{2}\, \ln(x)\right)\right)\sqrt{2}}{44} dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{\cos(\ln(x))\, x}{12} - \frac{\sin(\ln(x))\, x}{12}\right)\left(\frac{1}{x}\right) \\
&+ \left(\frac{-\frac{2}{165} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)^2}{165} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)}{165}}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x^2}\right)(x^2) \\
&+ \left(-\left(\int \frac{\cos(\ln(x))\left(-9\sqrt{2}\, \cos\left(\sqrt{2}\, \ln(x)\right) - 6\sin\left(\sqrt{2}\, \ln(x)\right)\right)\sqrt{2}}{132} dx\right)\right)\left(\frac{\cos\left(\sqrt{2}\, \ln(x)\right)}{x}\right) \\
&+ \left(\int \frac{\cos(\ln(x))\left(3\sqrt{2}\, \sin\left(\sqrt{2}\, \ln(x)\right) - 2\cos\left(\sqrt{2}\, \ln(x)\right)\right)\sqrt{2}}{44} dx\right)\left(\frac{\sin\left(\sqrt{2}\, \ln(x)\right)}{x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{5\sqrt{2}\left(\int \cos(\ln(x))\left(3\sqrt{2}\, \cos\left(\sqrt{2}\, \ln(x)\right) + 2\sin\left(\sqrt{2}\, \ln(x)\right)\right) dx\right)\cos\left(\sqrt{2}\, \ln(x)\right) + 5\sqrt{2}\left(\int \cos(\ln(x)\right.}{220}
$$

Therefore the general solution is

$$
y = y_h + y_p
$$

$$= \left(\frac{c_1}{x} + c_2 x^2 + \frac{c_3 \cos\left(\sqrt{2}\, \ln(x)\right) + c_4 \sin\left(\sqrt{2}\, \ln(x)\right)}{x}\right)$$

$$+ \left(\frac{5\sqrt{2}\left(\int \cos(\ln(x)) \left(3\sqrt{2}\, \cos\left(\sqrt{2}\, \ln(x)\right) + 2\sin\left(\sqrt{2}\, \ln(x)\right)\right) dx\right) \cos\left(\sqrt{2}\, \ln(x)\right) + 5\sqrt{2}\left(\int \cos(\ln\right.}{2}\right.$$

Which simplifies to

$$y = \frac{c_2 x^3 + c_4 \sin\left(\sqrt{2}\, \ln(x)\right) + c_3 \cos\left(\sqrt{2}\, \ln(x)\right) + c_1}{x}$$
$$+ \frac{5\sqrt{2}\left(\int \cos(\ln(x)) \left(3\sqrt{2}\, \cos\left(\sqrt{2}\, \ln(x)\right) + 2\sin\left(\sqrt{2}\, \ln(x)\right)\right) dx\right) \cos\left(\sqrt{2}\, \ln(x)\right) + 5\sqrt{2}\left(\int \cos(\ln(x\right.}{220x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 x^3 + c_4 \sin\left(\sqrt{2}\, \ln(x)\right) + c_3 \cos\left(\sqrt{2}\, \ln(x)\right) + c_1}{x} \tag{1}$$
$$+ \frac{5\sqrt{2}\left(\int \cos(\ln(x)) \left(3\sqrt{2}\, \cos\left(\sqrt{2}\, \ln(x)\right) + 2\sin\left(\sqrt{2}\, \ln(x)\right)\right) dx\right) \cos\left(\sqrt{2}\, \ln(x)\right) + 5\sqrt{2}\left(\int \cos(\ln(x\right.}{220x}$$

Verification of solutions

$$y = \frac{c_2 x^3 + c_4 \sin\left(\sqrt{2}\, \ln(x)\right) + c_3 \cos\left(\sqrt{2}\, \ln(x)\right) + c_1}{x}$$
$$+ \frac{5\sqrt{2}\left(\int \cos(\ln(x)) \left(3\sqrt{2}\, \cos\left(\sqrt{2}\, \ln(x)\right) + 2\sin\left(\sqrt{2}\, \ln(x)\right)\right) dx\right) \cos\left(\sqrt{2}\, \ln(x)\right) + 5\sqrt{2}\left(\int \cos(\ln(x\right.}{220x}$$

Verified OK.

16.15.1 Maple step by step solution

Let's solve

$x^4 y'''' + 7x^3 y''' + 9x^2 y'' - 6xy' - 6y = \cos(\ln(x))$

- Highest derivative means the order of the ODE is 4

  y''''

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = (c__1+6*_a*_b(_a)-3
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   -> Calling odsolve with the ODE`, diff(diff(_g(_f), _f), _f) = c__2-3*_g(_f)/_f^2-3*(diff
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying high order exact linear fully integrable
      trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
      trying a double symmetry of the form [xi=0, eta=F(x)]
      -> Try solving first the homogeneous part of the ODE
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         <- LODE of Euler type successful
      <- solving first the homogeneous part of the ODE successful
   <- high order exact_linear_nonhomogeneous successful
<- high order exact_linear_nonhomogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 289

```
dsolve(x^4*diff(y(x),x$4)+7*x^3*diff(y(x),x$3)+9*x^2*diff(y(x),x$2)-6*x*diff(y(x),x)-6*y(x)=
```

$$y(x) = \frac{\left(\left(\left(-66-132i+(-33+99i)\sqrt{2}\right)x^{1-i}+\left(-66+132i+(33+99i)\sqrt{2}\right)x^{1+i}-360i\sqrt{2}\,c_2x^3+240c_2x^3\right.\right.}{}$$

✓ Solution by Mathematica

Time used: 0.318 (sec). Leaf size: 62

```
DSolve[x^4*y''''[x]+7*x^3*y'''[x]+9*x^2*y''[x]-6*x*y'[x]-6*y[x]==Cos[Log[x]],y[x],x,IncludeS
```

$$y(x) \to c_4x^2 + \frac{c_3}{x} - \frac{1}{10}\sin(\log(x)) - \frac{1}{20}\cos(\log(x)) + \frac{c_2\cos\left(\sqrt{2}\log(x)\right)}{x} + \frac{c_1\sin\left(\sqrt{2}\log(x)\right)}{x}$$

16.16 problem 16

Internal problem ID [2265]
Internal file name [OUTPUT/2265_Monday_February_26_2024_10_05_05_AM_85293657/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 25, page 112
Problem number: 16.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^3y''' - 2x^2y'' - xy' + 4y = \sin(\ln(x))}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' - 2x^2y'' - xy' + 4y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' - 2x^2y'' - xy' + 4y = \sin(\ln(x))$$

gives

$$-x\lambda\, x^{\lambda-1} - 2x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 4x^\lambda = 0$$

Which simplifies to

$$-\lambda\, x^\lambda - 2\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 4x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$-\lambda - 2\lambda(\lambda-1) + \lambda(\lambda-1)\,(\lambda-2) + 4 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 5\lambda^2 + 3\lambda + 4 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 4$$
$$\lambda_2 = \frac{\sqrt{5}}{2} + \frac{1}{2}$$
$$\lambda_3 = -\frac{\sqrt{5}}{2} + \frac{1}{2}$$

This table summarises the result

root	multiplicity	type of root
4	1	real root
$-\frac{\sqrt{5}}{2} + \frac{1}{2}$	1	real root
$\frac{\sqrt{5}}{2} + \frac{1}{2}$	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1 x^4 + c_2 x^{-\frac{\sqrt{5}}{2}+\frac{1}{2}} + c_3 x^{\frac{\sqrt{5}}{2}+\frac{1}{2}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= x^4 \\
y_2 &= x^{-\frac{\sqrt{5}}{2}+\frac{1}{2}} \\
y_3 &= x^{\frac{\sqrt{5}}{2}+\frac{1}{2}}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x^3y''' - 2x^2y'' - xy' + 4y = \sin(\ln(x))
$$

Let the particular solution be

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i}\int \frac{F(x)W_i(x)}{aW(x)}\,dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
W = \begin{bmatrix} x^4 & x^{-\frac{\sqrt{5}}{2}+\frac{1}{2}} & x^{\frac{\sqrt{5}}{2}+\frac{1}{2}} \\ 4x^3 & -\frac{x^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\sqrt{5}-1\right)}{2} & \frac{x^{-\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(\sqrt{5}+1\right)}{2} \\ 12x^2 & x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}} & x^{-\frac{3}{2}+\frac{\sqrt{5}}{2}} \end{bmatrix}
$$

$$
|W| = -\frac{x^4x^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}x^{-\frac{3}{2}+\frac{\sqrt{5}}{2}}\sqrt{5}}{2} + \frac{x^4x^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}x^{-\frac{3}{2}+\frac{\sqrt{5}}{2}}}{2} - \frac{x^4x^{-\frac{1}{2}+\frac{\sqrt{5}}{2}}x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}}\sqrt{5}}{2} - \frac{x^4x^{-\frac{1}{2}+\frac{\sqrt{5}}{2}}x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}}}{2} + 4x^{\frac{\sqrt{5}}{2}+}
$$

The determinant simplifies to

$$|W| = 11x^2\sqrt{5}$$

Now we determine W_i for each U_i.

$$\begin{aligned}
W_1(x) &= \det \begin{bmatrix} x^{-\frac{\sqrt{5}}{2}+\frac{1}{2}} & x^{\frac{\sqrt{5}}{2}+\frac{1}{2}} \\ -\frac{x^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\sqrt{5}-1\right)}{2} & \frac{x^{-\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(\sqrt{5}+1\right)}{2} \end{bmatrix} \\
&= \sqrt{5}
\end{aligned}$$

$$\begin{aligned}
W_2(x) &= \det \begin{bmatrix} x^4 & x^{\frac{\sqrt{5}}{2}+\frac{1}{2}} \\ 4x^3 & \frac{x^{-\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(\sqrt{5}+1\right)}{2} \end{bmatrix} \\
&= \frac{\left(\sqrt{5}-7\right) x^{\frac{7}{2}+\frac{\sqrt{5}}{2}}}{2}
\end{aligned}$$

$$\begin{aligned}
W_3(x) &= \det \begin{bmatrix} x^4 & x^{-\frac{\sqrt{5}}{2}+\frac{1}{2}} \\ 4x^3 & -\frac{x^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\sqrt{5}-1\right)}{2} \end{bmatrix} \\
&= -\frac{\left(7+\sqrt{5}\right) x^{\frac{7}{2}-\frac{\sqrt{5}}{2}}}{2}
\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{\left(\sin\left(\ln\left(x\right)\right)\right)\left(\sqrt{5}\right)}{\left(x^3\right)\left(11x^2\sqrt{5}\right)}\, dx \\
&= \int \frac{\sin\left(\ln\left(x\right)\right)\sqrt{5}}{11x^5\sqrt{5}}\, dx \\
&= \int \left(\frac{\sin\left(\ln\left(x\right)\right)}{11x^5}\right) dx \\
&= \frac{\left(-\frac{1}{374}+\frac{2i}{187}\right) x^{i}}{x^4} + \frac{\left(-\frac{1}{374}-\frac{2i}{187}\right) x^{-i}}{x^4}
\end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{(\sin(\ln(x))) \left(\frac{\left(\sqrt{5}-7\right) x^{\frac{7}{2}+\frac{\sqrt{5}}{2}}}{2}\right)}{(x^3)\left(11 x^2 \sqrt{5}\right)}\, dx \\
&= -\int \frac{\frac{\sin(\ln(x))\left(\sqrt{5}-7\right) x^{\frac{7}{2}+\frac{\sqrt{5}}{2}}}{2}}{11 x^5 \sqrt{5}}\, dx \\
&= -\int \left(\frac{x^{-\frac{3}{2}+\frac{\sqrt{5}}{2}} \sqrt{5}\, \sin(\ln(x)) \left(\sqrt{5}-7\right)}{110}\right) dx \\
&= -\frac{i\sqrt{5}\left(\sqrt{5}-7\right)\left(i\sqrt{5}+5i-2\sqrt{5}\right) x\, x^{-\frac{3}{2}+\frac{\sqrt{5}}{2}} x^{i}}{2200} - \frac{i\sqrt{5}\left(\sqrt{5}-7\right)\left(i\sqrt{5}+5i+2\sqrt{5}\right) x\, x^{-\frac{3}{2}+\frac{\sqrt{5}}{2}} x^{-i}}{2200} \\
&= -\frac{i\sqrt{5}\left(\sqrt{5}-7\right)\left(i\sqrt{5}+5i-2\sqrt{5}\right) x\, x^{-\frac{3}{2}+\frac{\sqrt{5}}{2}} x^{i}}{2200} - \frac{i\sqrt{5}\left(\sqrt{5}-7\right)\left(i\sqrt{5}+5i+2\sqrt{5}\right) x\, x^{-\frac{3}{2}+\frac{\sqrt{5}}{2}} x^{-i}}{2200}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(\sin(\ln(x))) \left(-\frac{\left(7+\sqrt{5}\right) x^{\frac{7}{2}-\frac{\sqrt{5}}{2}}}{2}\right)}{(x^3)\left(11 x^2 \sqrt{5}\right)}\, dx \\
&= \int \frac{-\frac{\sin(\ln(x))\left(7+\sqrt{5}\right) x^{\frac{7}{2}-\frac{\sqrt{5}}{2}}}{2}}{11 x^5 \sqrt{5}}\, dx \\
&= \int \left(-\frac{x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}} \sqrt{5}\, \sin(\ln(x)) \left(7+\sqrt{5}\right)}{110}\right) dx \\
&= \frac{i\sqrt{5}\left(7+\sqrt{5}\right)\left(i\sqrt{5}-5i-2\sqrt{5}\right) x\, x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}} x^{i}}{2200} + \frac{i\sqrt{5}\left(7+\sqrt{5}\right)\left(i\sqrt{5}-5i+2\sqrt{5}\right) x\, x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}} x^{-i}}{2200} \\
&= \frac{i\sqrt{5}\left(7+\sqrt{5}\right)\left(i\sqrt{5}-5i-2\sqrt{5}\right) x\, x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}} x^{i}}{2200} + \frac{i\sqrt{5}\left(7+\sqrt{5}\right)\left(i\sqrt{5}-5i+2\sqrt{5}\right) x\, x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}} x^{-i}}{2200}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$y_p = \left(\frac{\left(-\frac{1}{374}+\frac{2i}{187}\right)x^{i}}{x^{4}}+\frac{\left(-\frac{1}{374}-\frac{2i}{187}\right)x^{-i}}{x^{4}}\right)\left(x^{4}\right)$$
$$+\left(-\frac{i\sqrt{5}\left(\sqrt{5}-7\right)\left(i\sqrt{5}+5i-2\sqrt{5}\right)x\,x^{-\frac{3}{2}+\frac{\sqrt{5}}{2}}x^{i}}{2200}-\frac{i\sqrt{5}\left(\sqrt{5}-7\right)\left(i\sqrt{5}+5i+2\sqrt{5}\right)x\,x^{-\frac{3}{2}+\frac{\sqrt{5}}{2}}x^{-i}}{2200}\right)$$
$$+\left(\frac{i\sqrt{5}\left(7+\sqrt{5}\right)\left(i\sqrt{5}-5i-2\sqrt{5}\right)x\,x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}}x^{i}}{2200}+\frac{i\sqrt{5}\left(7+\sqrt{5}\right)\left(i\sqrt{5}-5i+2\sqrt{5}\right)x\,x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}}x^{-i}}{2200}\right)\left(x\right.$$

Therefore the particular solution is

$$y_p = -\frac{9\left(\left(\frac{2}{9}+i\right)x^{2i}+\frac{2}{9}-i\right)x^{-i}}{170}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^4 + c_2 x^{-\frac{\sqrt{5}}{2}+\frac{1}{2}} + c_3 x^{\frac{\sqrt{5}}{2}+\frac{1}{2}}\right) + \left(-\frac{9\left(\left(\frac{2}{9}+i\right)x^{2i}+\frac{2}{9}-i\right)x^{-i}}{170}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^4 + c_2 x^{-\frac{\sqrt{5}}{2}+\frac{1}{2}} + c_3 x^{\frac{\sqrt{5}}{2}+\frac{1}{2}} - \frac{9\left(\left(\frac{2}{9}+i\right)x^{2i}+\frac{2}{9}-i\right)x^{-i}}{170} \tag{1}$$

Verification of solutions

$$y = c_1 x^4 + c_2 x^{-\frac{\sqrt{5}}{2}+\frac{1}{2}} + c_3 x^{\frac{\sqrt{5}}{2}+\frac{1}{2}} - \frac{9\left(\left(\frac{2}{9}+i\right)x^{2i}+\frac{2}{9}-i\right)x^{-i}}{170}$$

Verified OK.

16.16.1 Maple step by step solution

Let's solve

$x^3y''' - 2x^2y'' - xy' + 4y = \sin(\ln(x))$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 48

```
dsolve(x^3*diff(y(x),x$3)-2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+4*y(x)=sin(ln(x)),y(x), singso
```

$$y(x) = c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + c_3 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}} + \left(-\frac{1}{85}+\frac{9i}{170}\right) x^{-i} + \left(-\frac{1}{85}-\frac{9i}{170}\right) x^{i} + c_1 x^4$$

✓ Solution by Mathematica

Time used: 0.212 (sec). Leaf size: 60

```
DSolve[x^3*y'''[x]-2*x^2*y''[x]-x*y'[x]+4*y[x]==Sin[Log[x]],y[x],x,IncludeSingularSolutions
```

$$y(x) \to c_2 x^{\frac{1}{2}\left(1+\sqrt{5}\right)} + c_1 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + c_3 x^4 + \frac{9}{85}\sin(\log(x)) - \frac{2}{85}\cos(\log(x))$$

17 Exercise 26, page 115

17.1 problem 1

Internal problem ID [2266]
Internal file name [`OUTPUT/2266_Monday_February_26_2024_10_05_06_AM_82600808/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 26, page 115
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = x(t) + \cos(t)$$
$$y'(t) = -y(t) + 4t$$

17.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \cos(t) \\ 4t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{t} c_1 \\ e^{-t} c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t) \, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \int \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} \cos(t) \\ 4t \end{bmatrix} dt \\ &= \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -\frac{e^{-t}(\cos(t) - \sin(t))}{2} \\ 4(t-1)\, e^{t} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ 4t - 4 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} e^{t} c_1 - \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ e^{-t} c_2 + 4t - 4 \end{bmatrix} \end{aligned}$$

17.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \cos(t) \\ 4t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(-1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 0 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{t} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ 0 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 0 & \mathrm{e}^{t} \\ \mathrm{e}^{-t} & 0 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} 0 & \mathrm{e}^{t} \\ \mathrm{e}^{-t} & 0 \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 0 & \mathrm{e}^{t} \\ \mathrm{e}^{-t} & 0 \end{bmatrix} \int \begin{bmatrix} 0 & \mathrm{e}^{t} \\ \mathrm{e}^{-t} & 0 \end{bmatrix} \begin{bmatrix} \cos(t) \\ 4t \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & \mathrm{e}^{t} \\ \mathrm{e}^{-t} & 0 \end{bmatrix} \int \begin{bmatrix} 4\,\mathrm{e}^{t} t \\ \mathrm{e}^{-t} \cos(t) \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & \mathrm{e}^{t} \\ \mathrm{e}^{-t} & 0 \end{bmatrix} \begin{bmatrix} 4(t-1)\,\mathrm{e}^{t} \\ -\frac{\mathrm{e}^{-t}(\cos(t)-\sin(t))}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ 4t - 4 \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ c_1 \mathrm{e}^{-t} \end{bmatrix} + \begin{bmatrix} c_2 \mathrm{e}^{t} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ 4t - 4 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 \mathrm{e}^{t} - \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ c_1 \mathrm{e}^{-t} + 4t - 4 \end{bmatrix}$$

17.1.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + \cos(t), y'(t) = -y(t) + 4t]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \cos(t) \\ 4t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \cos(t) \\ 4t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \cos(t) \\ 4t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $\left[\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$
- Consider eigenpair

  $\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$
- Solution to homogeneous system from eigenpair

  $\overrightarrow{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- Consider eigenpair

  $\left[1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$
- Solution to homogeneous system from eigenpair

  $\overrightarrow{x}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

□ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t) = \begin{bmatrix} 0 & \mathrm{e}^{t} \\ \mathrm{e}^{-t} & 0 \end{bmatrix}$
- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} 0 & e^{t} \\ e^{-t} & 0 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{t}}{2} - \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ 4\,e^{-t} + 4t - 4 \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \left[\begin{array}{c} \frac{e^t}{2} - \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ 4\,e^{-t} + 4t - 4 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} c_2 e^t + \frac{e^t}{2} - \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ (c_1 + 4)\,e^{-t} + 4t - 4 \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = c_2 e^t + \frac{e^t}{2} - \frac{\cos(t)}{2} + \frac{\sin(t)}{2}, y(t) = (c_1 + 4)\,e^{-t} + 4t - 4\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve([diff(x(t),t)-x(t)=cos(t),diff(y(t),t)+y(t)=4*t],singsol=all)
```

$$x(t) = -\frac{\cos(t)}{2} + \frac{\sin(t)}{2} + c_1 e^t$$
$$y(t) = 4t - 4 + c_2 e^{-t}$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 39

```
DSolve[{x'[t]-x[t]==Cos[t],y'[t]+y[t]==4*t},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2}(\sin(t) - \cos(t) + 2c_1 e^t)$$
$$y(t) \to 4t + c_2 e^{-t} - 4$$

17.2 problem 2

Internal problem ID [2267]
Internal file name [`OUTPUT/2267_Tuesday_February_27_2024_08_23_51_AM_67136864/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 26, page 115
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 3t^2 - 5x(t) \\
y'(t) &= -y(t) + \mathrm{e}^{3t}
\end{aligned}
$$

17.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 3t^2 \\ \mathrm{e}^{3t} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-5t}c_1 \\ e^{-t}c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{t} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{-t} \end{bmatrix} \int \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 3t^2 \\ e^{3t} \end{bmatrix} dt \\ &= \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{3(25t^2-10t+2)e^{5t}}{125} \\ \frac{e^{4t}}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{5}t^2 - \frac{6}{25}t + \frac{6}{125} \\ \frac{e^{3t}}{4} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} e^{-5t}c_1 + \frac{3t^2}{5} - \frac{6t}{25} + \frac{6}{125} \\ e^{-t}c_2 + \frac{e^{3t}}{4} \end{bmatrix} \end{aligned}$$

17.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 3t^2 \\ e^{3t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -5-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-5-\lambda)(-1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} - (-5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 4 & 0 \end{array}\right]$$

Since the current pivot $A(1, 2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 0 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-5t} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-5t} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-5t} \\ 0 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 0 & e^{-5t} \\ e^{-t} & 0 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} 0 & e^{t} \\ e^{5t} & 0 \end{bmatrix}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} 0 & e^{-5t} \\ e^{-t} & 0 \end{bmatrix} \int \begin{bmatrix} 0 & e^{t} \\ e^{5t} & 0 \end{bmatrix} \begin{bmatrix} 3t^2 \\ e^{3t} \end{bmatrix} dt \\ &= \begin{bmatrix} 0 & e^{-5t} \\ e^{-t} & 0 \end{bmatrix} \int \begin{bmatrix} e^{4t} \\ 3\,e^{5t} t^2 \end{bmatrix} dt \\ &= \begin{bmatrix} 0 & e^{-5t} \\ e^{-t} & 0 \end{bmatrix} \begin{bmatrix} \frac{e^{4t}}{4} \\ \frac{3(25t^2 - 10t + 2)\,e^{5t}}{125} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{5}t^2 - \frac{6}{25}t + \frac{6}{125} \\ \frac{e^{3t}}{4} \end{bmatrix} \end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ c_1 e^{-t} \end{bmatrix} + \begin{bmatrix} c_2 e^{-5t} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{5}t^2 - \frac{6}{25}t + \frac{6}{125} \\ \frac{e^{3t}}{4} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 e^{-5t} + \frac{3t^2}{5} - \frac{6t}{25} + \frac{6}{125} \\ c_1 e^{-t} + \frac{e^{3t}}{4} \end{bmatrix}$$

17.2.3 Maple step by step solution

Let's solve

$$\left[x'(t) = 3t^2 - 5x(t), y'(t) = -y(t) + (e^t)^3\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 3t^2 \\ (e^t)^3 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 3t^2 \\ (e^t)^3 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 3t^2 \\ (e^t)^3 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $\left[\left[-5, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]\right]$
- Consider eigenpair

  $\left[-5, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$
- Solution to homogeneous system from eigenpair

  $\vec{x}_1 = e^{-5t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Consider eigenpair

  $\left[-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$
- Solution to homogeneous system from eigenpair

  $\vec{x}_2 = e^{-t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \vec{x}_p(t)$

□ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t) = \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{-t} \end{bmatrix}$
- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{-t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{3t^2}{5} - \frac{6\,e^{-5t}}{125} - \frac{6t}{25} + \frac{6}{125} \\ \frac{e^{3t}}{4} - \frac{e^{-t}}{4} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \left[\begin{array}{c} \frac{3t^2}{5} - \frac{6\,\mathrm{e}^{-5t}}{125} - \frac{6t}{25} + \frac{6}{125} \\ \frac{\mathrm{e}^{3t}}{4} - \frac{\mathrm{e}^{-t}}{4} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} c_1\mathrm{e}^{-5t} + \frac{3t^2}{5} - \frac{6\,\mathrm{e}^{-5t}}{125} - \frac{6t}{25} + \frac{6}{125} \\ c_2\mathrm{e}^{-t} + \frac{\mathrm{e}^{3t}}{4} - \frac{\mathrm{e}^{-t}}{4} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = c_1\mathrm{e}^{-5t} + \frac{3t^2}{5} - \frac{6\,\mathrm{e}^{-5t}}{125} - \frac{6t}{25} + \frac{6}{125}, y(t) = c_2\mathrm{e}^{-t} + \frac{\mathrm{e}^{3t}}{4} - \frac{\mathrm{e}^{-t}}{4}\right\}$$

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 37

```
dsolve([diff(x(t),t)+5*x(t)=3*t^2,diff(y(t),t)+y(t)=exp(3*t)],singsol=all)
```

$$x(t) = \frac{3t^2}{5} - \frac{6t}{25} + \frac{6}{125} + c_2\mathrm{e}^{-5t}$$
$$y(t) = \frac{\mathrm{e}^{3t}}{4} + \mathrm{e}^{-t}c_1$$

✓ Solution by Mathematica

Time used: 0.216 (sec). Leaf size: 50

```
DSolve[{x'[t]+5*x[t]==3*t^2,y'[t]+y[t]==Exp[3*t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{3t^2}{5} - \frac{6t}{25} + c_1e^{-5t} + \frac{6}{125}$$
$$y(t) \to \frac{e^{3t}}{4} + c_2e^{-t}$$

17.3 problem 3

Internal problem ID [2268]
Internal file name [`OUTPUT/2268_Tuesday_February_27_2024_08_23_52_AM_5854370/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 26, page 115
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) + 3t \\
y'(t) &= x(t) - \frac{3t}{2} - \frac{y(t)}{2} + \cos(t)^2 - \frac{1}{2}
\end{aligned}
$$

17.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 3t \\ -\frac{3t}{2} + \cos(t)^2 - \frac{1}{2} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-2t} & 0 \\ \frac{2\,e^{-\frac{t}{2}}}{3} - \frac{2\,e^{-2t}}{3} & e^{-\frac{t}{2}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-2t} & 0 \\ \frac{2\,e^{-\frac{t}{2}}}{3} - \frac{2\,e^{-2t}}{3} & e^{-\frac{t}{2}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t} c_1 \\ \left(\frac{2\,e^{-\frac{t}{2}}}{3} - \frac{2\,e^{-2t}}{3}\right) c_1 + e^{-\frac{t}{2}} c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t} c_1 \\ \frac{(2c_1+3c_2)e^{-\frac{t}{2}}}{3} - \frac{2\,e^{-2t} c_1}{3} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} e^{2t} & 0 \\ -\frac{2\left(e^{\frac{3t}{2}}-1\right)e^{\frac{t}{2}}}{3} & e^{\frac{t}{2}} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{-2t} & 0 \\ \frac{2\,e^{-\frac{t}{2}}}{3} - \frac{2\,e^{-2t}}{3} & e^{-\frac{t}{2}} \end{bmatrix} \int \begin{bmatrix} e^{2t} & 0 \\ -\frac{2\left(e^{\frac{3t}{2}}-1\right)e^{\frac{t}{2}}}{3} & e^{\frac{t}{2}} \end{bmatrix} \begin{bmatrix} 3t \\ -\frac{3t}{2} + \cos(t)^2 - \frac{1}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{-2t} & 0 \\ \frac{2\,e^{-\frac{t}{2}}}{3} - \frac{2\,e^{-2t}}{3} & e^{-\frac{t}{2}} \end{bmatrix} \begin{bmatrix} \frac{3(2t-1)e^{2t}}{4} \\ \frac{(-34+17t+\cos(2t)+4\sin(2t))e^{\frac{t}{2}}}{17} + \frac{(-2t+1)e^{2t}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3t}{2} - \frac{3}{4} \\ \frac{\cos(2t)}{17} + \frac{4\sin(2t)}{17} - \frac{3}{2} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \mathrm{e}^{-2t} c_1 + \frac{3t}{2} - \frac{3}{4} \\ \frac{(2c_1+3c_2)\mathrm{e}^{-\frac{t}{2}}}{3} - \frac{2\,\mathrm{e}^{-2t} c_1}{3} + \frac{\cos(2t)}{17} + \frac{4\sin(2t)}{17} - \frac{3}{2} \end{bmatrix}
\end{aligned}
$$

17.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 3t \\ -\frac{3t}{2} + \cos(t)^2 - \frac{1}{2} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -2 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -2-\lambda & 0 \\ 1 & -\frac{1}{2}-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-2-\lambda)(-\frac{1}{2}-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
$-\frac{1}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & \frac{3}{2} & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} - \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{2} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{3}{2} & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{3} \implies \left[\begin{array}{cc|c} -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{3}{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-\frac{1}{2}$	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-\frac{t}{2}} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-\frac{t}{2}}
\end{aligned}
$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-2t} \\
&= \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} e^{-2t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{-\frac{t}{2}} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{3\,\mathrm{e}^{-2t}}{2} \\ \mathrm{e}^{-2t} \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} 0 & -\frac{3\,\mathrm{e}^{-2t}}{2} \\ \mathrm{e}^{-\frac{t}{2}} & \mathrm{e}^{-2t} \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{2\,e^{\frac{t}{2}}}{3} & e^{\frac{t}{2}} \\ -\frac{2\,e^{2t}}{3} & 0 \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 0 & -\frac{3\,e^{-2t}}{2} \\ e^{-\frac{t}{2}} & e^{-2t} \end{bmatrix} \int \begin{bmatrix} \frac{2\,e^{\frac{t}{2}}}{3} & e^{\frac{t}{2}} \\ -\frac{2\,e^{2t}}{3} & 0 \end{bmatrix} \begin{bmatrix} 3t \\ -\frac{3t}{2} + \cos(t)^2 - \frac{1}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & -\frac{3\,e^{-2t}}{2} \\ e^{-\frac{t}{2}} & e^{-2t} \end{bmatrix} \int \begin{bmatrix} \frac{e^{\frac{t}{2}}\left(2\cos(t)^2+t-1\right)}{2} \\ -2\,e^{2t}t \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & -\frac{3\,e^{-2t}}{2} \\ e^{-\frac{t}{2}} & e^{-2t} \end{bmatrix} \begin{bmatrix} \frac{(-34+17t+\cos(2t)+4\sin(2t))e^{\frac{t}{2}}}{17} \\ -\frac{(2t-1)e^{2t}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3t}{2} - \frac{3}{4} \\ \frac{\cos(2t)}{17} + \frac{4\sin(2t)}{17} - \frac{3}{2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} 0 \\ c_1 e^{-\frac{t}{2}} \end{bmatrix} + \begin{bmatrix} -\frac{3c_2 e^{-2t}}{2} \\ c_2 e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{3t}{2} - \frac{3}{4} \\ \frac{\cos(2t)}{17} + \frac{4\sin(2t)}{17} - \frac{3}{2} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{3c_2 e^{-2t}}{2} + \frac{3t}{2} - \frac{3}{4} \\ c_1 e^{-\frac{t}{2}} + c_2 e^{-2t} + \frac{\cos(2t)}{17} + \frac{4\sin(2t)}{17} - \frac{3}{2} \end{bmatrix}$$

17.3.3 Maple step by step solution

Let's solve

$\left[x'(t) = -2x(t) + 3t, y'(t) = x(t) - \frac{3t}{2} - \frac{y(t)}{2} + \cos(t)^2 - \frac{1}{2}\right]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -2 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 3t \\ -\frac{3t}{2} + \cos(t)^2 - \frac{1}{2} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -2 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 3t \\ -\frac{3t}{2} + \cos(t)^2 - \frac{1}{2} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 3t \\ -\frac{3t}{2} + \cos(t)^2 - \frac{1}{2} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{1}{2}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{-\frac{t}{2}} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \vec{x}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -\frac{3\,e^{-2t}}{2} & 0 \\ e^{-2t} & e^{-\frac{t}{2}} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{3\,e^{-2t}}{2} & 0 \\ e^{-2t} & e^{-\frac{t}{2}} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{3}{2} & 0 \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 0 \\ \frac{2\,e^{-\frac{t}{2}}}{3} - \frac{2\,e^{-2t}}{3} & e^{-\frac{t}{2}} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$

- Cancel like terms

  $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot\vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t)=\left[\begin{array}{c}\frac{3\,\mathrm{e}^{-2t}}{4}+\frac{3t}{2}-\frac{3}{4}\\ \frac{33\,\mathrm{e}^{-\frac{t}{2}}}{17}-\frac{\mathrm{e}^{-2t}}{2}-\frac{3}{2}+\frac{\cos(2t)}{17}+\frac{4\sin(2t)}{17}\end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{x}(t)=c_1\vec{x}_1+c_2\vec{x}_2+\left[\begin{array}{c}\frac{3\,\mathrm{e}^{-2t}}{4}+\frac{3t}{2}-\frac{3}{4}\\ \frac{33\,\mathrm{e}^{-\frac{t}{2}}}{17}-\frac{\mathrm{e}^{-2t}}{2}-\frac{3}{2}+\frac{\cos(2t)}{17}+\frac{4\sin(2t)}{17}\end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c}x(t)\\ y(t)\end{array}\right]=\left[\begin{array}{c}\frac{(-6c_1+3)\mathrm{e}^{-2t}}{4}+\frac{3t}{2}-\frac{3}{4}\\ c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{-\frac{t}{2}}+\frac{33\,\mathrm{e}^{-\frac{t}{2}}}{17}-\frac{\mathrm{e}^{-2t}}{2}-\frac{3}{2}+\frac{\cos(2t)}{17}+\frac{4\sin(2t)}{17}\end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t)=\frac{(-6c_1+3)\mathrm{e}^{-2t}}{4}+\frac{3t}{2}-\frac{3}{4},y(t)=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{-\frac{t}{2}}+\frac{33\,\mathrm{e}^{-\frac{t}{2}}}{17}-\frac{\mathrm{e}^{-2t}}{2}-\frac{3}{2}+\frac{\cos(2t)}{17}+\frac{4\sin(2t)}{17}\right\}$$

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 46

```
dsolve([diff(x(t),t)+2*x(t)=3*t,diff(x(t),t)+2*diff(y(t),t)+y(t)=cos(2*t)],singsol=all)
```

$$x(t) = \frac{3t}{2} - \frac{3}{4} + c_2 e^{-2t}$$
$$y(t) = \frac{\cos(2t)}{17} + \frac{4\sin(2t)}{17} - \frac{2c_2 e^{-2t}}{3} - \frac{3}{2} + c_1 e^{-\frac{t}{2}}$$

✓ Solution by Mathematica

Time used: 0.348 (sec). Leaf size: 74

```
DSolve[{x'[t]+2*x[t]==3*t,x'[t]+2*y'[t]+y[t]==Cos[2*t]},{x[t],y[t]},t,IncludeSingularSolutio
```

$$x(t) \to \frac{3t}{2} + c_1 e^{-2t} - \frac{3}{4}$$
$$y(t) \to \frac{4}{17}\sin(2t) + \frac{1}{17}\cos(2t) + \frac{1}{6}\left(-4c_1 e^{-2t} + (4c_1 + 6c_2)e^{-t/2} - 9\right)$$

17.4 problem 4

Internal problem ID [2269]
Internal file name [`OUTPUT/2269_Tuesday_February_27_2024_08_23_52_AM_96573020/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 26, page 115
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -y(t) + x(t) + 2\sin(t) \\
y'(t) &= 4y(t) - 4x(t) - 2\sin(t)
\end{aligned}
$$

17.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2\sin(t) \\ -2\sin(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{4}{5} + \frac{e^{5t}}{5} & -\frac{e^{5t}}{5} + \frac{1}{5} \\ -\frac{4\,e^{5t}}{5} + \frac{4}{5} & \frac{1}{5} + \frac{4\,e^{5t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{4}{5} + \frac{e^{5t}}{5} & -\frac{e^{5t}}{5} + \frac{1}{5} \\ -\frac{4\,e^{5t}}{5} + \frac{4}{5} & \frac{1}{5} + \frac{4\,e^{5t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{4}{5} + \frac{e^{5t}}{5}\right) c_1 + \left(-\frac{e^{5t}}{5} + \frac{1}{5}\right) c_2 \\ \left(-\frac{4\,e^{5t}}{5} + \frac{4}{5}\right) c_1 + \left(\frac{1}{5} + \frac{4\,e^{5t}}{5}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{5t}(-c_2+c_1)}{5} + \frac{4c_1}{5} + \frac{c_2}{5} \\ \frac{(-4c_1+4c_2)e^{5t}}{5} + \frac{4c_1}{5} + \frac{c_2}{5} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{e^{-5t}}{5} + \frac{4}{5} & \frac{1}{5} - \frac{e^{-5t}}{5} \\ \frac{4}{5} - \frac{4\,e^{-5t}}{5} & \frac{4\,e^{-5t}}{5} + \frac{1}{5} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{4}{5} + \frac{e^{5t}}{5} & -\frac{e^{5t}}{5} + \frac{1}{5} \\ -\frac{4\,e^{5t}}{5} + \frac{4}{5} & \frac{1}{5} + \frac{4\,e^{5t}}{5} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-5t}}{5} + \frac{4}{5} & \frac{1}{5} - \frac{e^{-5t}}{5} \\ \frac{4}{5} - \frac{4\,e^{-5t}}{5} & \frac{4\,e^{-5t}}{5} + \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2\sin(t) \\ -2\sin(t) \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{4}{5} + \frac{e^{5t}}{5} & -\frac{e^{5t}}{5} + \frac{1}{5} \\ -\frac{4\,e^{5t}}{5} + \frac{4}{5} & \frac{1}{5} + \frac{4\,e^{5t}}{5} \end{bmatrix} \begin{bmatrix} \frac{2(-\cos(t)-5\sin(t))e^{-5t}}{65} - \frac{6\cos(t)}{5} \\ \frac{8(\cos(t)+5\sin(t))e^{-5t}}{65} - \frac{6\cos(t)}{5} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{16\cos(t)}{13} - \frac{2\sin(t)}{13} \\ -\frac{14\cos(t)}{13} + \frac{8\sin(t)}{13} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} \frac{e^{5t}(-c_2+c_1)}{5} + \frac{4c_1}{5} + \frac{c_2}{5} - \frac{16\cos(t)}{13} - \frac{2\sin(t)}{13} \\ \frac{4(c_2-c_1)e^{5t}}{5} + \frac{4c_1}{5} + \frac{c_2}{5} - \frac{14\cos(t)}{13} + \frac{8\sin(t)}{13} \end{bmatrix}$$

17.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2\sin(t) \\ -2\sin(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ -4 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 5\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -4 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 4R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -1 & 0 \\ -4 & -1 & 0 \end{array} \right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -4 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	1	1	No	$\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{5t}$$
$$= \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} e^{5t}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{0}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{5t}}{4} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{e^{5t}}{4} & 1 \\ e^{5t} & 1 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{4 e^{-5t}}{5} & \frac{4 e^{-5t}}{5} \\ \frac{4}{5} & \frac{1}{5} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{e^{5t}}{4} & 1 \\ e^{5t} & 1 \end{bmatrix} \int \begin{bmatrix} -\frac{4\,e^{-5t}}{5} & \frac{4\,e^{-5t}}{5} \\ \frac{4}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2\sin(t) \\ -2\sin(t) \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{e^{5t}}{4} & 1 \\ e^{5t} & 1 \end{bmatrix} \int \begin{bmatrix} -\frac{16\sin(t)e^{-5t}}{5} \\ \frac{6\sin(t)}{5} \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{e^{5t}}{4} & 1 \\ e^{5t} & 1 \end{bmatrix} \begin{bmatrix} \frac{8(\cos(t)+5\sin(t))e^{-5t}}{65} \\ -\frac{6\cos(t)}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{16\cos(t)}{13} - \frac{2\sin(t)}{13} \\ -\frac{14\cos(t)}{13} + \frac{8\sin(t)}{13} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{5t}}{4} \\ c_1 e^{5t} \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} -\frac{16\cos(t)}{13} - \frac{2\sin(t)}{13} \\ -\frac{14\cos(t)}{13} + \frac{8\sin(t)}{13} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{5t}}{4} + c_2 - \frac{16\cos(t)}{13} - \frac{2\sin(t)}{13} \\ c_1 e^{5t} + c_2 - \frac{14\cos(t)}{13} + \frac{8\sin(t)}{13} \end{bmatrix}$$

17.4.3 Maple step by step solution

Let's solve

$[x'(t) = -y(t) + x(t) + 2\sin(t), y'(t) = 4y(t) - 4x(t) - 2\sin(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2\sin(t) \\ -2\sin(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2\sin(t) \\ -2\sin(t) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 2\sin(t) \\ -2\sin(t) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{5t} \cdot \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \begin{bmatrix} 1 & -\frac{e^{5t}}{4} \\ 1 & e^{5t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \begin{bmatrix} 1 & -\frac{e^{5t}}{4} \\ 1 & e^{5t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{4} \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} \frac{4}{5} + \frac{e^{5t}}{5} & -\frac{e^{5t}}{5} + \frac{1}{5} \\ -\frac{4\,e^{5t}}{5} + \frac{4}{5} & \frac{1}{5} + \frac{4\,e^{5t}}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

  $\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \left[\begin{array}{c} \frac{6}{5} - \frac{16\cos(t)}{13} + \frac{2\,e^{5t}}{65} - \frac{2\sin(t)}{13} \\ \frac{6}{5} - \frac{14\cos(t)}{13} - \frac{8\,e^{5t}}{65} + \frac{8\sin(t)}{13} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \left[\begin{array}{c} \frac{6}{5} - \frac{16\cos(t)}{13} + \frac{2\,e^{5t}}{65} - \frac{2\sin(t)}{13} \\ \frac{6}{5} - \frac{14\cos(t)}{13} - \frac{8\,e^{5t}}{65} + \frac{8\sin(t)}{13} \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{c_2 e^{5t}}{4} + \frac{6}{5} - \frac{16\cos(t)}{13} + \frac{2\,e^{5t}}{65} - \frac{2\sin(t)}{13} + c_1 \\ c_2 e^{5t} + \frac{6}{5} - \frac{14\cos(t)}{13} - \frac{8\,e^{5t}}{65} + \frac{8\sin(t)}{13} + c_1 \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t) = -\frac{c_2 e^{5t}}{4} + \frac{6}{5} - \frac{16\cos(t)}{13} + \frac{2\,e^{5t}}{65} - \frac{2\sin(t)}{13} + c_1, y(t) = c_2 e^{5t} + \frac{6}{5} - \frac{14\cos(t)}{13} - \frac{8\,e^{5t}}{65} + \frac{8\sin(t)}{13} + c_1\right\}$$

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 42

```
dsolve([diff(x(t),t)-x(t)+y(t)=2*sin(t),diff(x(t),t)+diff(y(t),t)=3*y(t)-3*x(t)],singsol=all
```

$$x(t) = -\frac{e^{5t}c_1}{20} - \frac{16\cos(t)}{13} - \frac{2\sin(t)}{13} + c_2$$
$$y(t) = \frac{e^{5t}c_1}{5} + \frac{8\sin(t)}{13} - \frac{14\cos(t)}{13} + c_2$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 80

```
DSolve[{x'[t]-x[t]+y[t]==2*Sin[t],x'[t]+y'[t]==3*y[t]-3*x[t]},{x[t],y[t]},t,IncludeSingularS
```

$$x(t) \to \frac{1}{65}\left(-10\sin(t) - 80\cos(t) + 13c_1\left(e^{5t} + 4\right) - 13c_2\left(e^{5t} - 1\right)\right)$$
$$y(t) \to \frac{1}{65}\left(40\sin(t) - 70\cos(t) - 52c_1\left(e^{5t} - 1\right) + 13c_2\left(4e^{5t} + 1\right)\right)$$

17.5 problem 5

Internal problem ID [2270]
Internal file name [`OUTPUT/2270_Tuesday_February_27_2024_08_23_53_AM_33243587/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 26, page 115
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -\frac{3x(t)}{2} + \frac{y(t)}{2} + \frac{e^t}{2}$$
$$y'(t) = \frac{5x(t)}{3} - \frac{y(t)}{3} - \frac{2t}{3}$$

17.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{5}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \frac{e^t}{2} \\ -\frac{2t}{3} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(3\,e^{\frac{13t}{6}}+10\right)e^{-2t}}{13} & \frac{3\left(e^{\frac{13t}{6}}-1\right)e^{-2t}}{13} \\ \frac{10\left(e^{\frac{13t}{6}}-1\right)e^{-2t}}{13} & \frac{\left(10\,e^{\frac{13t}{6}}+3\right)e^{-2t}}{13} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{\left(3\,e^{\frac{13t}{6}}+10\right)e^{-2t}}{13} & \frac{3\left(e^{\frac{13t}{6}}-1\right)e^{-2t}}{13} \\ \frac{10\left(e^{\frac{13t}{6}}-1\right)e^{-2t}}{13} & \frac{\left(10\,e^{\frac{13t}{6}}+3\right)e^{-2t}}{13} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\left(3\,e^{\frac{13t}{6}}+10\right)e^{-2t}c_1}{13} + \frac{3\left(e^{\frac{13t}{6}}-1\right)e^{-2t}c_2}{13} \\ \frac{10\left(e^{\frac{13t}{6}}-1\right)e^{-2t}c_1}{13} + \frac{\left(10\,e^{\frac{13t}{6}}+3\right)e^{-2t}c_2}{13} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3\left((c_1+c_2)e^{\frac{13t}{6}}+\frac{10c_1}{3}-c_2\right)e^{-2t}}{13} \\ \frac{10\,e^{-2t}\left((c_1+c_2)e^{\frac{13t}{6}}-c_1+\frac{3c_2}{10}\right)}{13} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} \frac{10\,e^{2t}}{13} + \frac{3\,e^{-\frac{t}{6}}}{13} & -\frac{3\,e^{2t}}{13} + \frac{3\,e^{-\frac{t}{6}}}{13} \\ -\frac{10\,e^{2t}}{13} + \frac{10\,e^{-\frac{t}{6}}}{13} & \frac{3\,e^{2t}}{13} + \frac{10\,e^{-\frac{t}{6}}}{13} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{\left(3\,e^{\frac{13t}{6}}+10\right)e^{-2t}}{13} & \frac{3\left(e^{\frac{13t}{6}}-1\right)e^{-2t}}{13} \\ \frac{10\left(e^{\frac{13t}{6}}-1\right)e^{-2t}}{13} & \frac{\left(10\,e^{\frac{13t}{6}}+3\right)e^{-2t}}{13} \end{bmatrix} \int \begin{bmatrix} \frac{10\,e^{2t}}{13}+\frac{3\,e^{-\frac{t}{6}}}{13} & -\frac{3\,e^{2t}}{13}+\frac{3\,e^{-\frac{t}{6}}}{13} \\ -\frac{10\,e^{2t}}{13}+\frac{10\,e^{-\frac{t}{6}}}{13} & \frac{3\,e^{2t}}{13}+\frac{10\,e^{-\frac{t}{6}}}{13} \end{bmatrix} \begin{bmatrix} \frac{e^t}{2} \\ -\frac{2t}{3} \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{\left(3\,e^{\frac{13t}{6}}+10\right)e^{-2t}}{13} & \frac{3\left(e^{\frac{13t}{6}}-1\right)e^{-2t}}{13} \\ \frac{10\left(e^{\frac{13t}{6}}-1\right)e^{-2t}}{13} & \frac{\left(10\,e^{\frac{13t}{6}}+3\right)e^{-2t}}{13} \end{bmatrix} \begin{bmatrix} \frac{12(6+t)e^{-\frac{t}{6}}}{13}+\frac{9\,e^{\frac{5t}{6}}}{65}+\frac{(2t-1)e^{2t}}{26}+\frac{5\,e^{3t}}{39} \\ \frac{40(6+t)e^{-\frac{t}{6}}}{13}+\frac{6\,e^{\frac{5t}{6}}}{13}+\frac{(-2t+1)e^{2t}}{26}-\frac{5\,e^{3t}}{39} \end{bmatrix}$$

$$= \begin{bmatrix} t+\frac{11}{2}+\frac{4\,e^t}{15} \\ 3t+\frac{37}{2}+\frac{e^t}{3} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\ &= \begin{bmatrix} \frac{e^{-2t}\left(3(c_1+c_2)e^{\frac{13t}{6}}+13\,e^{2t}t+10c_1-3c_2+\frac{143\,e^{2t}}{2}+\frac{52\,e^{3t}}{15}\right)}{13} \\ \frac{e^{-2t}\left(10(c_1+c_2)e^{\frac{13t}{6}}+39\,e^{2t}t-10c_1+3c_2+\frac{481\,e^{2t}}{2}+\frac{13\,e^{3t}}{3}\right)}{13} \end{bmatrix} \end{aligned}$$

17.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{5}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \frac{e^t}{2} \\ -\frac{2t}{3} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t)+\vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-\frac{3}{2} & \frac{1}{2}\\ \frac{5}{3} & -\frac{1}{3}\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-\frac{3}{2}-\lambda & \frac{1}{2}\\ \frac{5}{3} & -\frac{1}{3}-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \frac{11}{6}\lambda - \frac{1}{3} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = \frac{1}{6}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
$\frac{1}{6}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-\frac{3}{2} & \frac{1}{2}\\ \frac{5}{3} & -\frac{1}{3}\end{bmatrix} - (-2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{5}{3} & \frac{5}{3}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{5}{3} & \frac{5}{3} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{10R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{6}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{5}{3} & -\frac{1}{3} \end{bmatrix} - \left(\frac{1}{6}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{5}{3} & \frac{1}{2} \\ \frac{5}{3} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{5}{3} & \frac{1}{2} & 0 \\ \frac{5}{3} & -\frac{1}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{5}{3} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{5}{3} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{10}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{10} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{10} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{10} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{10} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{10} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{10} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{10} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-2	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$\frac{1}{6}$	1	1	No	$\begin{bmatrix} \frac{3}{10} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Since eigenvalue $\frac{1}{6}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\frac{t}{6}} \\ &= \begin{bmatrix} \frac{3}{10} \\ 1 \end{bmatrix} e^{\frac{t}{6}}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{3\,\mathrm{e}^{\frac{t}{6}}}{10} \\ \mathrm{e}^{\frac{t}{6}} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{-2t} & \frac{3\,\mathrm{e}^{\frac{t}{6}}}{10} \\ \mathrm{e}^{-2t} & \mathrm{e}^{\frac{t}{6}} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{10\,\mathrm{e}^{2t}}{13} & \frac{3\,\mathrm{e}^{2t}}{13} \\ \frac{10\,\mathrm{e}^{-\frac{t}{6}}}{13} & \frac{10\,\mathrm{e}^{-\frac{t}{6}}}{13} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\mathrm{e}^{-2t} & \frac{3\,\mathrm{e}^{\frac{t}{6}}}{10} \\ \mathrm{e}^{-2t} & \mathrm{e}^{\frac{t}{6}} \end{bmatrix} \int \begin{bmatrix} -\frac{10\,\mathrm{e}^{2t}}{13} & \frac{3\,\mathrm{e}^{2t}}{13} \\ \frac{10\,\mathrm{e}^{-\frac{t}{6}}}{13} & \frac{10\,\mathrm{e}^{-\frac{t}{6}}}{13} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{e}^{t}}{2} \\ -\frac{2t}{3} \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{-2t} & \frac{3\,\mathrm{e}^{\frac{t}{6}}}{10} \\ \mathrm{e}^{-2t} & \mathrm{e}^{\frac{t}{6}} \end{bmatrix} \int \begin{bmatrix} -\frac{5\,\mathrm{e}^{3t}}{13} - \frac{2\,\mathrm{e}^{2t}t}{13} \\ \frac{5\,\mathrm{e}^{\frac{5t}{6}}}{13} - \frac{20t\,\mathrm{e}^{-\frac{t}{6}}}{39} \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{-2t} & \frac{3\,\mathrm{e}^{\frac{t}{6}}}{10} \\ \mathrm{e}^{-2t} & \mathrm{e}^{\frac{t}{6}} \end{bmatrix} \begin{bmatrix} \frac{(-6t+3)\mathrm{e}^{2t}}{78} - \frac{5\,\mathrm{e}^{3t}}{39} \\ \frac{40(6+t)\mathrm{e}^{-\frac{t}{6}}}{13} + \frac{6\,\mathrm{e}^{\frac{5t}{6}}}{13} \end{bmatrix} \\
&= \begin{bmatrix} t + \frac{11}{2} + \frac{4\,\mathrm{e}^{t}}{15} \\ 3t + \frac{37}{2} + \frac{\mathrm{e}^{t}}{3} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} -c_1\mathrm{e}^{-2t} \\ c_1\mathrm{e}^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{3c_2\mathrm{e}^{\frac{t}{6}}}{10} \\ c_2\mathrm{e}^{\frac{t}{6}} \end{bmatrix} + \begin{bmatrix} t + \frac{11}{2} + \frac{4\,\mathrm{e}^{t}}{15} \\ 3t + \frac{37}{2} + \frac{\mathrm{e}^{t}}{3} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{\left(8\,\mathrm{e}^{3t} + 9c_2\mathrm{e}^{\frac{13t}{6}} + 30\,\mathrm{e}^{2t}t + 165\,\mathrm{e}^{2t} - 30c_1\right)\mathrm{e}^{-2t}}{30} \\ \frac{\left(2\,\mathrm{e}^{3t} + 6c_2\mathrm{e}^{\frac{13t}{6}} + 18\,\mathrm{e}^{2t}t + 111\,\mathrm{e}^{2t} + 6c_1\right)\mathrm{e}^{-2t}}{6} \end{bmatrix}$$

17.5.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -\frac{3x(t)}{2} + \frac{y(t)}{2} + \frac{e^t}{2}, y'(t) = \frac{5x(t)}{3} - \frac{y(t)}{3} - \frac{2t}{3}\right]$$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t) = \left[\begin{array}{cc} -\frac{3}{2} & \frac{1}{2} \\ \frac{5}{3} & -\frac{1}{3} \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} \frac{e^t}{2} \\ -\frac{2t}{3} \end{array}\right]$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{cc} -\frac{3}{2} & \frac{1}{2} \\ \frac{5}{3} & -\frac{1}{3} \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} \frac{e^t}{2} \\ -\frac{2t}{3} \end{array}\right]$$

- Define the forcing function

$$\vec{f}(t) = \left[\begin{array}{c} \frac{e^t}{2} \\ -\frac{2t}{3} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} -\frac{3}{2} & \frac{1}{2} \\ \frac{5}{3} & -\frac{1}{3} \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right], \left[\frac{1}{6}, \left[\begin{array}{c} \frac{3}{10} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-2t} \cdot \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[\frac{1}{6}, \left[\begin{array}{c} \frac{3}{10} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{\frac{t}{6}} \cdot \left[\begin{array}{c} \frac{3}{10} \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} -e^{-2t} & \frac{3 e^{\frac{t}{6}}}{10} \\ e^{-2t} & e^{\frac{t}{6}} \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} -e^{-2t} & \frac{3 e^{\frac{t}{6}}}{10} \\ e^{-2t} & e^{\frac{t}{6}} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} -1 & \frac{3}{10} \\ 1 & 1 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} \frac{\left(3 e^{\frac{13t}{6}} + 10\right) e^{-2t}}{13} & \frac{3\left(e^{\frac{13t}{6}} - 1\right) e^{-2t}}{13} \\ \frac{10\left(e^{\frac{13t}{6}} - 1\right) e^{-2t}}{13} & \frac{\left(10 e^{\frac{13t}{6}} + 3\right) e^{-2t}}{13} \end{array}\right]$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

    $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$
  - Take the derivative of the particular solution

    $\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
  - Cancel like terms

    $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$
  - Multiply by the inverse of the fundamental matrix

    $\overrightarrow{v}'(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$
  - Integrate to solve for $\overrightarrow{v}(t)$

    $\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
  - Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

    $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

    $$\overrightarrow{x}_p(t) = \left[\begin{array}{c} \frac{\left(104\,\mathrm{e}^{3t}-2214\,\mathrm{e}^{\frac{13t}{6}}+390\,\mathrm{e}^{2t}t+2145\,\mathrm{e}^{2t}-35\right)\mathrm{e}^{-2t}}{390} \\ \frac{\left(26\,\mathrm{e}^{3t}-1476\,\mathrm{e}^{\frac{13t}{6}}+234\,\mathrm{e}^{2t}t+1443\,\mathrm{e}^{2t}+7\right)\mathrm{e}^{-2t}}{78} \end{array}\right]$$
- Plug particular solution back into general solution

  $$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \left[\begin{array}{c} \frac{\left(104\,\mathrm{e}^{3t}-2214\,\mathrm{e}^{\frac{13t}{6}}+390\,\mathrm{e}^{2t}t+2145\,\mathrm{e}^{2t}-35\right)\mathrm{e}^{-2t}}{390} \\ \frac{\left(26\,\mathrm{e}^{3t}-1476\,\mathrm{e}^{\frac{13t}{6}}+234\,\mathrm{e}^{2t}t+1443\,\mathrm{e}^{2t}+7\right)\mathrm{e}^{-2t}}{78} \end{array}\right]$$
- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{\left(104\,e^{3t}+117c_2 e^{\frac{13t}{6}}-2214\,e^{\frac{13t}{6}}+390\,e^{2t}t+2145\,e^{2t}-390c_1-35\right)e^{-2t}}{390} \\ \frac{\left(26\,e^{3t}+78c_2 e^{\frac{13t}{6}}-1476\,e^{\frac{13t}{6}}+234\,e^{2t}t+1443\,e^{2t}+78c_1+7\right)e^{-2t}}{78} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{\left(104\,e^{3t}+117c_2 e^{\frac{13t}{6}}-2214\,e^{\frac{13t}{6}}+390\,e^{2t}t+2145\,e^{2t}-390c_1-35\right)e^{-2t}}{390}, y(t) = \frac{\left(26\,e^{3t}+78c_2 e^{\frac{13t}{6}}-1476\,e^{\frac{13t}{6}}+234\,e^{2t}t\right.}{78}\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 50

```
dsolve([2*diff(x(t),t)+3*x(t)-y(t)=exp(t),5*x(t)-3*diff(y(t),t)=y(t)+2*t],singsol=all)
```

$$x(t) = -c_2 e^{-2t} + \frac{3\,e^{\frac{t}{6}}c_1}{10} + \frac{4\,e^t}{15} + \frac{11}{2} + t$$

$$y(t) = c_2 e^{-2t} + e^{\frac{t}{6}}c_1 + \frac{37}{2} + \frac{e^t}{3} + 3t$$

✓ Solution by Mathematica

Time used: 0.685 (sec). Leaf size: 105

```
DSolve[{2*x'[t]+3*x[t]-y[t]==Exp[t],5*x[t]-3*y'[t]==y[t]+2*t},{x[t],y[t]},t,IncludeSingularS
```

$$x(t) \to t + \frac{4e^t}{15} + \frac{1}{13}(10c_1 - 3c_2)e^{-2t} + \frac{3}{13}(c_1 + c_2)e^{t/6} + \frac{11}{2}$$

$$y(t) \to \frac{1}{78}e^{-2t}\left(39e^{2t}(6t+37) + 26e^{3t} + 60(c_1 + c_2)e^{13t/6} - 60c_1 + 18c_2\right)$$

17.6 problem 6

Internal problem ID [2271]
Internal file name [`OUTPUT/2271_Tuesday_February_27_2024_08_23_53_AM_59055855/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 26, page 115
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
0 &= -5y'(t) + 3x'(t) + 5y(t) + 5t \\
0 &= -3x'(t) + 5y'(t) + 2x(t)
\end{aligned}
$$

The system is

$$5y'(t) - 3x'(t) = 5y(t) + 5t \tag{1}$$

$$3x'(t) - 5y'(t) = 2x(t) \tag{2}$$

Since the left side is the same, this implies

$$
\begin{aligned}
5y(t) + 5t &= 2x(t) \\
y(t) &= \frac{2x(t)}{5} - t
\end{aligned} \tag{3}
$$

Taking derivative of the above w.r.t. t gives

$$y'(t) = \frac{2x'(t)}{5} - 1 \tag{4}$$

Substituting (3,4) in (1) to eliminate $y(t), y'(t)$ gives

$$
\begin{aligned}
-x'(t) - 5 &= 2x(t) \\
x'(t) &= -2x(t) - 5
\end{aligned} \tag{5}
$$

Which is now solved for $x(t)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{-2x - 5} dx &= \int dt \\
-\frac{\ln(-2x - 5)}{2} &= t + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2x-5}} = e^{t+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{-2x-5}} = c_2 e^{t}$$

Given now that we have the solution

$$x(t) = -\frac{e^{-2t}}{2c_2^2} - \frac{5}{2} \tag{6}$$

Then substituting (6) into (3) gives

$$y(t) = -\frac{\left(5t\, c_2^2 e^{2t} + 5c_2^2 e^{2t} + 1\right) e^{-2t}}{5c_2^2} \tag{7}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve([5*diff(y(t),t)-3*diff(x(t),t)-5*y(t)=5*t,3*diff(x(t),t)-5*diff(y(t),t)-2*x(t)=0],sin
```

$$x(t) = \frac{5}{2} + e^{\frac{2t}{5}} c_1$$
$$y(t) = -1 - \frac{2\, e^{\frac{2t}{5}} c_1}{5} - t$$

✓ Solution by Mathematica

Time used: 0.137 (sec). Leaf size: 43

```
DSolve[{5*y'[t]-3*x'[t]-5*y[t]==5*t,3*x'[t]-5*y'[t]-2*x[t]==0},{x[t],y[t]},t,IncludeSingular
```

$$x(t) \to \frac{5}{6}\left(3 + 2c_1 e^{2t/5}\right)$$
$$y(t) \to -t - \frac{2}{3} c_1 e^{2t/5} - 1$$

17.7 problem 12

Internal problem ID [2272]
Internal file name [`OUTPUT/2272_Tuesday_February_27_2024_08_23_54_AM_90256140/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 26, page 115
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 3x(t) \\
y'(t) &= 2x(t) + 3y(t) \\
z'(t) &= 3y(t) - 2z(t)
\end{aligned}
$$

17.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 2t\,e^{3t} & e^{3t} & 0 \\ \frac{(30t-6)e^{-2t}e^{5t}}{25} + \frac{6\,e^{-2t}}{25} & \frac{3(e^{5t}-1)e^{-2t}}{5} & e^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t} & 0 & 0 \\ 2t\,e^{3t} & e^{3t} & 0 \\ \frac{(30t-6)e^{-2t}e^{5t}}{25} + \frac{6\,e^{-2t}}{25} & \frac{3(e^{5t}-1)e^{-2t}}{5} & e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}c_1 \\ 2t\,e^{3t}c_1 + e^{3t}c_2 \\ \left(\frac{(30t-6)e^{-2t}e^{5t}}{25} + \frac{6\,e^{-2t}}{25}\right)c_1 + \frac{3(e^{5t}-1)e^{-2t}c_2}{5} + e^{-2t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}c_1 \\ e^{3t}(2c_1t + c_2) \\ \frac{e^{-2t}(3(2(5t-1)c_1+5c_2)e^{5t}+6c_1-15c_2+25c_3)}{25} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}3-\lambda & 0 & 0\\ 2 & 3-\lambda & 0\\ 0 & 3 & -2-\lambda\end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(3-\lambda)(3-\lambda)(-2-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}3 & 0 & 0\\ 2 & 3 & 0\\ 0 & 3 & -2\end{bmatrix} - (-2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}5 & 0 & 0\\ 2 & 5 & 0\\ 0 & 3 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}5 & 0 & 0 & 0\\ 2 & 5 & 0 & 0\\ 0 & 3 & 0 & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_2}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = \frac{5t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \frac{5t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} 0 \\ \frac{5t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
3	2	1	Yes	$\begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

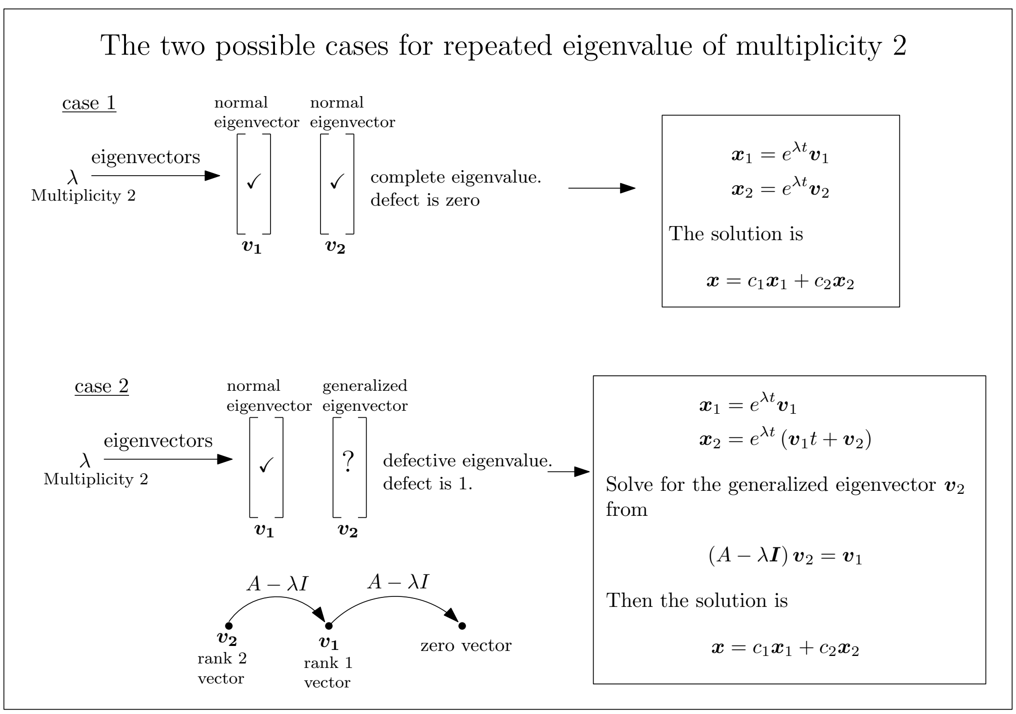


Figure 696: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{5}{6} \\ 2 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix} \mathrm{e}^{3t} \\ &= \begin{bmatrix} 0 \\ \frac{5\,\mathrm{e}^{3t}}{3} \\ \mathrm{e}^{3t} \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\left[\begin{array}{c} 0 \\ \frac{5}{3} \\ 1 \end{array}\right] t + \left[\begin{array}{c} \frac{5}{6} \\ 2 \\ 1 \end{array}\right]\right) \mathrm{e}^{3t} \\
&= \left[\begin{array}{c} \frac{5\,\mathrm{e}^{3t}}{6} \\ \frac{\mathrm{e}^{3t}(5t+6)}{3} \\ \mathrm{e}^{3t}(t+1) \end{array}\right]
\end{aligned}
$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{-2t} \\
&= \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] e^{-2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array}\right] = c_1 \left[\begin{array}{c} 0 \\ \frac{5\,\mathrm{e}^{3t}}{3} \\ \mathrm{e}^{3t} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{5\,\mathrm{e}^{3t}}{6} \\ \mathrm{e}^{3t}\left(\frac{5t}{3}+2\right) \\ \mathrm{e}^{3t}(t+1) \end{array}\right] + c_3 \left[\begin{array}{c} 0 \\ 0 \\ \mathrm{e}^{-2t} \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array}\right] = \left[\begin{array}{c} \frac{5 c_2 \mathrm{e}^{3t}}{6} \\ \frac{((5t+6)c_2+5c_1)\mathrm{e}^{3t}}{3} \\ (((t+1)\, c_2 + c_1)\, \mathrm{e}^{5t} + c_3)\, \mathrm{e}^{-2t} \end{array}\right]
$$

17.7.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t), y'(t) = 2x(t) + 3y(t), z'(t) = 3y(t) - 2z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 3 & -2 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[3, \begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

$$\vec{x}_2(t) = \mathrm{e}^{3t} \cdot \begin{bmatrix} 0 \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

$$\vec{x}_3(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_3(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_3(t)$ to be a solution to the homogeneous system

$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\begin{bmatrix}3&0&0\\2&3&0\\0&3&-2\end{bmatrix}-3\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}0\\\frac{5}{3}\\1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

- Second solution from eigenvalue 3

$$\vec{x}_3(t)=\mathrm{e}^{3t}\cdot\left(t\cdot\begin{bmatrix}0\\\frac{5}{3}\\1\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$$

- General solution to the system of ODEs

$\vec{x}=c_1\vec{x}_1+c_2\vec{x}_2(t)+c_3\vec{x}_3(t)$

- Substitute solutions into the general solution

$$\vec{x}=c_1\mathrm{e}^{-2t}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}+c_2\mathrm{e}^{3t}\cdot\begin{bmatrix}0\\\frac{5}{3}\\1\end{bmatrix}+c_3\mathrm{e}^{3t}\cdot\left(t\cdot\begin{bmatrix}0\\\frac{5}{3}\\1\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix}x(t)\\y(t)\\z(t)\end{bmatrix}=\begin{bmatrix}0\\\frac{5\,\mathrm{e}^{3t}(tc_3+c_2)}{3}\\\mathrm{e}^{-2t}\left((tc_3+c_2)\,\mathrm{e}^{5t}+c_1\right)\end{bmatrix}$$

- Solution to the system of ODEs

$\left\{x(t)=0, y(t)=\frac{5\,\mathrm{e}^{3t}(tc_3+c_2)}{3}, z(t)=\mathrm{e}^{-2t}\left((tc_3+c_2)\,\mathrm{e}^{5t}+c_1\right)\right\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 52

```
dsolve([diff(x(t),t)=3*x(t),diff(y(t),t)=2*x(t)+3*y(t),diff(z(t),t)=3*y(t)-2*z(t)],singsol=a
```

$$x(t) = c_3 \mathrm{e}^{3t}$$
$$y(t) = \mathrm{e}^{3t}(2c_3 t + c_2)$$
$$z(t) = \left(\frac{3\,\mathrm{e}^{5t}(10c_3 t + 5c_2 - 2c_3)}{25} + c_1\right)\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 78

```
DSolve[{x'[t]==3*x[t],y'[t]==2*x[t]+3*y[t],z'[t]==3*y[t]-2*z[t]},{x[t],y[t],z[t]},t,IncludeS
```

$$x(t) \to c_1 e^{3t}$$
$$y(t) \to e^{3t}(2c_1 t + c_2)$$
$$z(t) \to \frac{1}{25} e^{-2t}\left(6c_1\left(e^{5t}(5t-1)+1\right) + 5\left(3c_2\left(e^{5t}-1\right)+5c_3\right)\right)$$

18 Exercise 35, page 157

18.1 problem 1

Internal problem ID [2273]
Internal file name [`OUTPUT/2273_Tuesday_February_27_2024_08_23_54_AM_88000253/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = \cos(t)}$$

18.1.1 Solving as second order ode quadrature ode

Integrating once gives

$$y' = \sin(t) + c_1$$

Integrating again gives

$$y = -\cos(t) + c_1 x + c_2$$

Summary
The solution(s) found are the following

$$y = -\cos(t) + c_1 t + c_2 \tag{1}$$

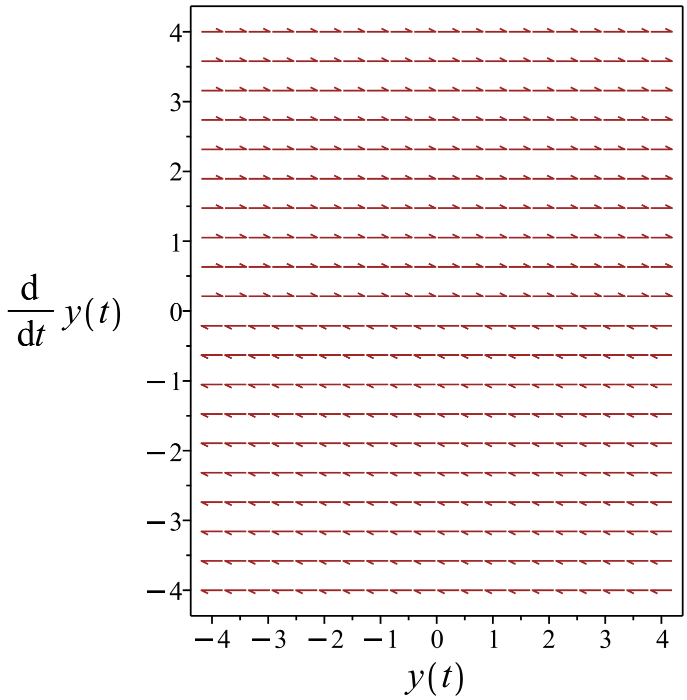


Figure 697: Slope field plot

Verification of solutions

$$y = -\cos(t) + c_1 t + c_2$$

Verified OK.

18.1.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 0, f(t) = \cos(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 t \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 t + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, t\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(t) - A_2 \sin(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(t)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 t + c_1) + (-\cos(t)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 t + c_1 - \cos(t) \tag{1}$$

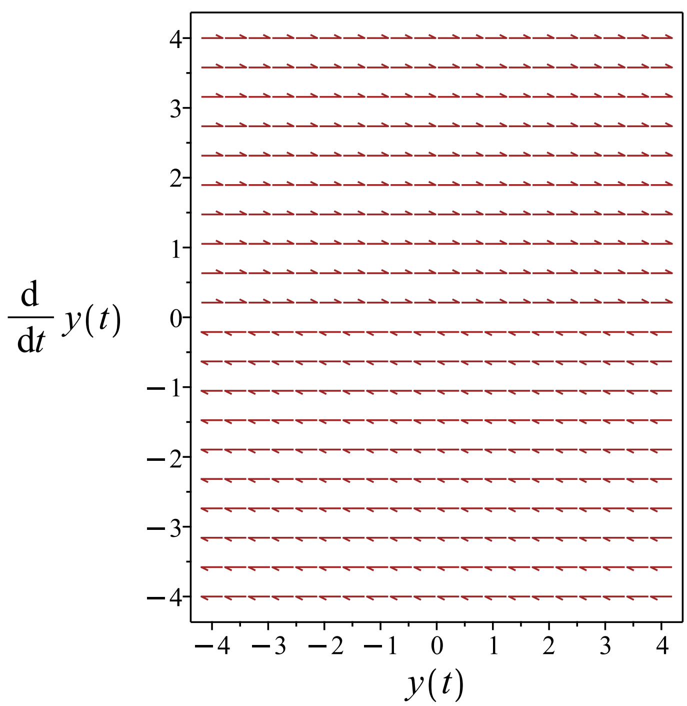


Figure 698: Slope field plot

Verification of solutions

$$y = c_2 t + c_1 - \cos(t)$$

Verified OK.

18.1.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int y'' dt = \int \cos(t)\, dt$$
$$y' = \sin(t) + c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int \sin(t) + c_1 \, \mathrm{d}t$$
$$= -\cos(t) + c_1 t + c_2$$

Summary
The solution(s) found are the following

$$y = -\cos(t) + c_1 t + c_2 \tag{1}$$

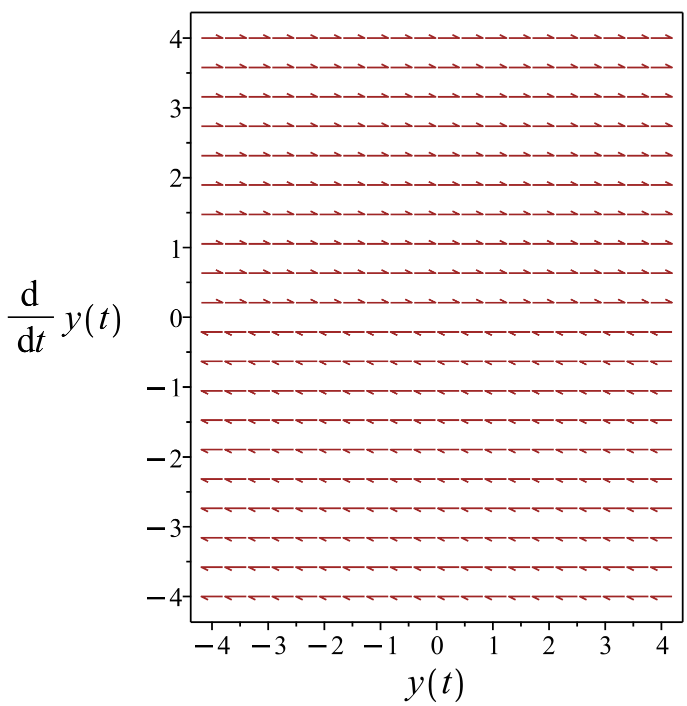


Figure 699: Slope field plot

Verification of solutions

$$y = -\cos(t) + c_1 t + c_2$$

Verified OK.

18.1.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$p'(t) - \cos(t) = 0$$

Which is now solve for $p(t)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(t) &= \int \cos(t)\, \mathrm{d}t \\ &= \sin(t) + c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \sin(t) + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \sin(t) + c_1\, \mathrm{d}t \\ &= -\cos(t) + c_1 t + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\cos(t) + c_1 t + c_2 \tag{1}$$

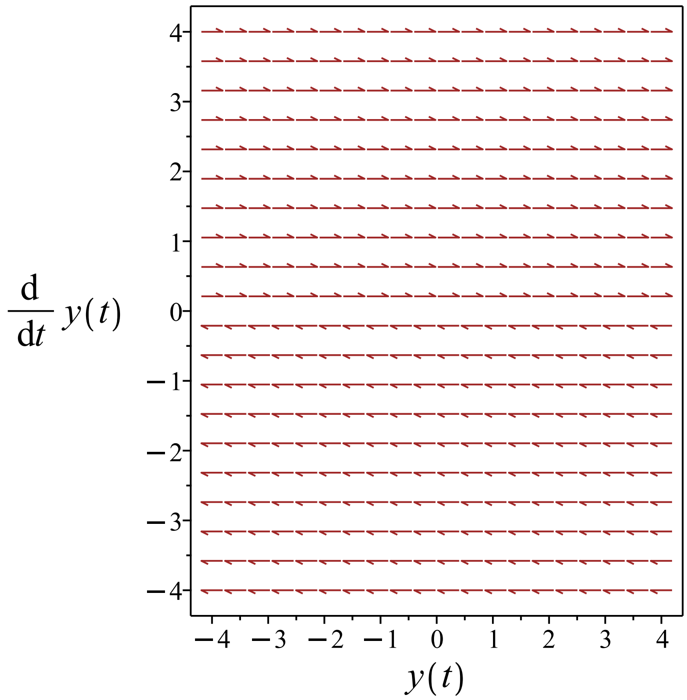


Figure 700: Slope field plot

Verification of solutions

$$y = -\cos(t) + c_1 t + c_2$$

Verified OK.

18.1.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 567: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dt \\ &= 1 \int \frac{1}{1} \, dt \\ &= 1(t) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(t)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 t + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, t\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \cos(t) - A_2 \sin(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(t)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 t + c_1) + (-\cos(t)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 t + c_1 - \cos(t) \tag{1}$$

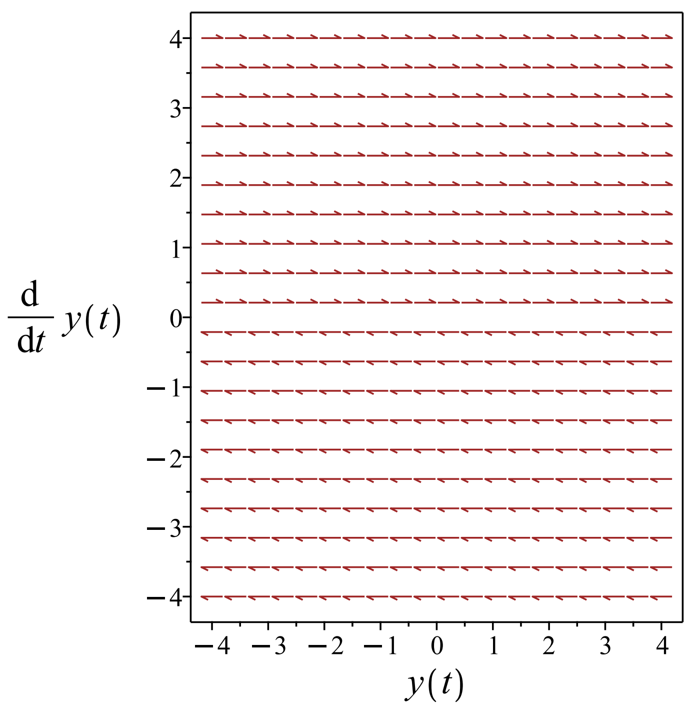


Figure 701: Slope field plot

Verification of solutions

$$y = c_2 t + c_1 - \cos(t)$$

Verified OK.

18.1.6 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= \cos(t) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\, y' + (q(t) - p'(t))\, y)' = s(x)$$

Integrating gives

$$p(t)\, y' + (q(t) - p'(t))\, y = \int s(t)\, dt$$

Substituting the above values for p, q, r, s gives

$$y' = \int \cos(t)\, dt$$

We now have a first order ode to solve which is

$$y' = \sin(t) + c_1$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \sin(t) + c_1 \, \mathrm{d}t \\
&= -\cos(t) + c_1 t + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\cos(t) + c_1 t + c_2 \tag{1}
$$

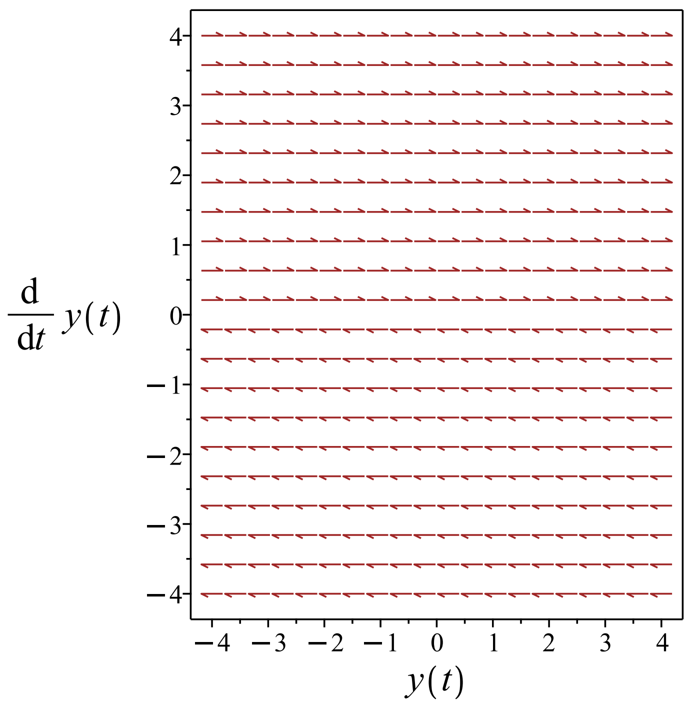


Figure 702: Slope field plot

Verification of solutions

$$
y = -\cos(t) + c_1 t + c_2
$$

Verified OK.

18.1.7 Maple step by step solution

Let's solve

$y'' = \cos(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(t) = 1$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 t + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \cos(t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 1$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = -\left(\int t \cos(t)\, dt\right) + t\left(\int \cos(t)\, dt\right)$

- Compute integrals

  $y_p(t) = -\cos(t)$

- Substitute particular solution into general solution to ODE

  $y = c_2 t + c_1 - \cos(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve(diff(y(t),t$2)=cos(t),y(t), singsol=all)
```

$$y(t) = -\cos(t) + c_1 t + c_2$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 16

```
DSolve[y''[t]==Cos[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\cos(t) + c_2 t + c_1$$

18.2 problem 2

Internal problem ID [2274]
Internal file name [`OUTPUT/2274_Tuesday_February_27_2024_08_23_55_AM_21883334/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - k^2 y = 0}$$

18.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = -k^2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - k^2 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$-k^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -k^2$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-k^2)}$$
$$= \pm\sqrt{k^2}$$

Hence

$$\lambda_1 = +\sqrt{k^2}$$
$$\lambda_2 = -\sqrt{k^2}$$

Which simplifies to

$$\lambda_1 = \sqrt{k^2}$$
$$\lambda_2 = -\sqrt{k^2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$y = c_1 e^{\left(\sqrt{k^2}\right)t} + c_2 e^{\left(-\sqrt{k^2}\right)t}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{k^2}\, t} + c_2 \mathrm{e}^{-\sqrt{k^2}\, t}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\sqrt{k^2}\, t} + c_2 \mathrm{e}^{-\sqrt{k^2}\, t} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\sqrt{k^2}\, t} + c_2 \mathrm{e}^{-\sqrt{k^2}\, t}$$

Verified OK.

18.2.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - k^2y'y = 0$$

Integrating the above w.r.t t gives

$$\int (y'y'' - k^2y'y)\,dt = 0$$
$$\frac{y'^2}{2} - \frac{k^2y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{k^2y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{k^2y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{1}{\sqrt{k^2y^2 + 2c_1}}dy = \int dt$$
$$\frac{\ln\left(\frac{k^2y}{\sqrt{k^2}} + \sqrt{k^2y^2 + 2c_1}\right)}{\sqrt{k^2}} = t + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln\left(\frac{k^2y}{\sqrt{k^2}} + \sqrt{k^2y^2+2c_1}\right)}{\sqrt{k^2}}} = \mathrm{e}^{t+c_2}$$

Which simplifies to

$$\left(ky\,\mathrm{csgn}(k) + \sqrt{k^2y^2 + 2c_1}\right)^{\frac{1}{\sqrt{k^2}}} = c_3\mathrm{e}^t$$

Simplifying the solution $y = \frac{\mathrm{csgn}(k)\left((c_3\mathrm{e}^t)^{\mathrm{csgn}(k)k} - 2(c_3\mathrm{e}^t)^{-\mathrm{csgn}(k)k}c_1\right)}{2k}$ to $y = \frac{(c_3\mathrm{e}^t)^k - 2(c_3\mathrm{e}^t)^{-k}c_1}{2k}$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int -\frac{1}{\sqrt{k^2y^2+2c_1}}dy = \int dt$$
$$-\frac{\ln\left(\frac{k^2y}{\sqrt{k^2}}+\sqrt{k^2y^2+2c_1}\right)}{\sqrt{k^2}} = t+c_4$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln\left(\frac{k^2y}{\sqrt{k^2}}+\sqrt{k^2y^2+2c_1}\right)}{\sqrt{k^2}}} = \mathrm{e}^{t+c_4}$$

Which simplifies to

$$\left(ky\,\mathrm{csgn}\left(k\right)+\sqrt{k^2y^2+2c_1}\right)^{-\frac{\mathrm{csgn}(k)}{k}} = c_5\mathrm{e}^t$$

Simplifying the solution $y = -\frac{\mathrm{csgn}(k)\left(2\left(c_5\mathrm{e}^t\right)^{\mathrm{csgn}(k)k}c_1-\left(c_5\mathrm{e}^t\right)^{-\mathrm{csgn}(k)k}\right)}{2k}$ to $y = -\frac{2\left(c_5\mathrm{e}^t\right)^kc_1-\left(c_5\mathrm{e}^t\right)^{-k}}{2k}$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_3\mathrm{e}^t\right)^k-2\left(c_3\mathrm{e}^t\right)^{-k}c_1}{2k} \tag{1}$$
$$y = -\frac{2\left(c_5\mathrm{e}^t\right)^kc_1-\left(c_5\mathrm{e}^t\right)^{-k}}{2k} \tag{2}$$

Verification of solutions

$$y = \frac{\left(c_3\mathrm{e}^t\right)^k-2\left(c_3\mathrm{e}^t\right)^{-k}c_1}{2k}$$

Verified OK.

$$y = -\frac{2\left(c_5\mathrm{e}^t\right)^kc_1-\left(c_5\mathrm{e}^t\right)^{-k}}{2k}$$

Verified OK.

18.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y''-k^2y = 0 \tag{1}$$
$$Ay''+By'+Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -k^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{k^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= k^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(k^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 569: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = k^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{\sqrt{k^2}\, t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{\sqrt{k^2}\, t}
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{\sqrt{k^2}\, t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\
&= e^{\sqrt{k^2}\, t} \int \frac{1}{e^{2\sqrt{k^2}\, t}}\, dt \\
&= e^{\sqrt{k^2}\, t} \left(-\frac{\operatorname{csgn}(k)\, e^{-2\operatorname{csgn}(k) k t}}{2k}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{\sqrt{k^2}\, t}\right) + c_2 \left(e^{\sqrt{k^2}\, t} \left(-\frac{\operatorname{csgn}(k)\, e^{-2\operatorname{csgn}(k) k t}}{2k}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are th

Simplifying the solution $y = c_1 e^{\sqrt{k^2}\, t} - \frac{c_2 \operatorname{csgn}(k) e^{-\operatorname{csgn}(k) k t}}{2k}$ to $y = c_1 e^{\sqrt{k^2}\, t} - \frac{c_2 e^{-kt}}{2k}$

Verification of solutions

$$y = c_1 e^{\sqrt{k^2}\, t} - \frac{c_2 e^{-kt}}{2k}$$

Verified OK.

18.2.4 Maple step by step solution

Let's solve

$y'' - k^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $-k^2 + r^2 = 0$

- Factor the characteristic polynomial

  $-(k - r)(k + r) = 0$

- Roots of the characteristic polynomial

  $r = (k, -k)$

- 1st solution of the ODE

  $y_1(t) = e^{kt}$

- 2nd solution of the ODE

  $y_2(t) = e^{-kt}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{kt} + c_2 e^{-kt}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(diff(y(t),t$2)=k^2*y(t),y(t), singsol=all)
```

$$y(t) = c_1 e^{kt} + c_2 e^{-kt}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 23

```
DSolve[y''[t]==k^2*y[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^{kt} + c_2 e^{-kt}$$

18.3 problem 3

Internal problem ID [2275]
Internal file name [`OUTPUT/2275_Tuesday_February_27_2024_08_23_56_AM_54582995/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + k^2 x = 0}$$

18.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = k^2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + k^2 \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$k^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = k^2$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(k^2)}$$
$$= \pm\sqrt{-k^2}$$

Hence

$$\lambda_1 = +\sqrt{-k^2}$$
$$\lambda_2 = -\sqrt{-k^2}$$

Which simplifies to

$$\lambda_1 = \sqrt{-k^2}$$
$$\lambda_2 = -\sqrt{-k^2}$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{\left(\sqrt{-k^2}\right)t} + c_2 e^{\left(-\sqrt{-k^2}\right)t}$$

Or

$$x = c_1 \mathrm{e}^{\sqrt{-k^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-k^2}\, t}$$

<u>Summary</u>
The solution(s) found are the following

$$x = c_1 \mathrm{e}^{\sqrt{-k^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-k^2}\, t} \tag{1}$$

<u>Verification of solutions</u>

$$x = c_1 \mathrm{e}^{\sqrt{-k^2}\, t} + c_2 \mathrm{e}^{-\sqrt{-k^2}\, t}$$

Verified OK.

18.3.2 Solving as second order ode can be made integrable ode

Multiplying the ode by x' gives

$$x'x'' + k^2x'x = 0$$

Integrating the above w.r.t t gives

$$\int (x'x'' + k^2x'x)\, dt = 0$$
$$\frac{x'^2}{2} + \frac{k^2x^2}{2} = c_2$$

Which is now solved for x. Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \sqrt{-k^2x^2 + 2c_1} \tag{1}$$
$$x' = -\sqrt{-k^2x^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-k^2x^2 + 2c_1}} dx = \int dt$$
$$\frac{\arctan\left(\frac{\sqrt{k^2}\, x}{\sqrt{-k^2x^2+2c_1}}\right)}{\sqrt{k^2}} = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-k^2x^2 + 2c_1}} dx = \int dt$$
$$-\frac{\arctan\left(\frac{\sqrt{k^2}\, x}{\sqrt{-k^2x^2+2c_1}}\right)}{\sqrt{k^2}} = t + c_3$$

Summary
The solution(s) found are the following

$$\frac{\arctan\left(\frac{\sqrt{k^2}\,x}{\sqrt{-k^2x^2+2c_1}}\right)}{\sqrt{k^2}} = t + c_2 \tag{1}$$

$$-\frac{\arctan\left(\frac{\sqrt{k^2}\,x}{\sqrt{-k^2x^2+2c_1}}\right)}{\sqrt{k^2}} = t + c_3 \tag{2}$$

Verification of solutions

$$\frac{\arctan\left(\frac{\sqrt{k^2}\,x}{\sqrt{-k^2x^2+2c_1}}\right)}{\sqrt{k^2}} = t + c_2$$

Verified OK.

$$-\frac{\arctan\left(\frac{\sqrt{k^2}\,x}{\sqrt{-k^2x^2+2c_1}}\right)}{\sqrt{k^2}} = t + c_3$$

Verified OK.

18.3.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + k^2x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= k^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-k^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -k^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-k^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 571: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -k^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{\sqrt{-k^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{-k^2}\, t} \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{\sqrt{-k^2}\, t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= \mathrm{e}^{\sqrt{-k^2}\, t} \int \frac{1}{\mathrm{e}^{2\sqrt{-k^2}\, t}}\, dt \\ &= \mathrm{e}^{\sqrt{-k^2}\, t} \left(\frac{\sqrt{-k^2}\, \mathrm{e}^{-2\sqrt{-k^2}\, t}}{2k^2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(\mathrm{e}^{\sqrt{-k^2}\, t}\right) + c_2\left(\mathrm{e}^{\sqrt{-k^2}\, t}\left(\frac{\sqrt{-k^2}\, \mathrm{e}^{-2\sqrt{-k^2}\, t}}{2k^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1\mathrm{e}^{\sqrt{-k^2}\, t} + \frac{c_2\sqrt{-k^2}\, \mathrm{e}^{-\sqrt{-k^2}\, t}}{2k^2} \tag{1}
$$

Verification of solutions

$$
x = c_1\mathrm{e}^{\sqrt{-k^2}\, t} + \frac{c_2\sqrt{-k^2}\, \mathrm{e}^{-\sqrt{-k^2}\, t}}{2k^2}
$$

Verified OK.

18.3.4 Maple step by step solution

Let's solve

$x'' + k^2x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $k^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm\left(\sqrt{-4k^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-k^2}, -\sqrt{-k^2}\right)$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{\sqrt{-k^2}\, t}$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-\sqrt{-k^2}\, t}$

- General solution of the ODE

  $x = c_1x_1(t) + c_2x_2(t)$

- Substitute in solutions

  $x = c_1 e^{\sqrt{-k^2}\, t} + c_2 e^{-\sqrt{-k^2}\, t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(x(t),t$2)+k^2*x(t)=0,x(t), singsol=all)
```

$$x(t) = c_1 \sin(kt) + c_2 \cos(kt)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[x''[t]+k^2*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 \cos(kt) + c_2 \sin(kt)$$

18.4 problem 4

Internal problem ID [2276]
Internal file name [`OUTPUT/2276_Tuesday_February_27_2024_08_23_56_AM_91740308/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y^3 y'' = -4}$$

18.4.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y^3 p(y) \left(\frac{d}{dy} p(y)\right) = -4$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p)\\
&= f(y)g(p)\\
&= -\frac{4}{y^3 p}
\end{aligned}$$

Where $f(y) = -\frac{4}{y^3}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{1}{p}}\,dp &= -\frac{4}{y^3}\,dy\\
\int \frac{1}{\frac{1}{p}}\,dp &= \int -\frac{4}{y^3}\,dy\\
\frac{p^2}{2} &= \frac{2}{y^2} + c_1
\end{aligned}$$

The solution is

$$\frac{p(y)^2}{2} - \frac{2}{y^2} - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^2}{2} - \frac{2}{y^2} - c_1 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{2c_1 y^2 + 4}}{y} \tag{1}$$

$$y' = -\frac{\sqrt{2c_1 y^2 + 4}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned}
\int \frac{y}{\sqrt{2c_1 y^2 + 4}}dy &= \int dx\\
\frac{c_1 y^2 + 2}{c_1\sqrt{2c_1 y^2 + 4}} &= x + c_2
\end{aligned}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y}{\sqrt{2c_1y^2+4}}dy = \int dx$$
$$-\frac{c_1y^2+2}{c_1\sqrt{2c_1y^2+4}} = x + c_3$$

Summary

The solution(s) found are the following

$$\frac{c_1y^2+2}{c_1\sqrt{2c_1y^2+4}} = x + c_2 \tag{1}$$
$$-\frac{c_1y^2+2}{c_1\sqrt{2c_1y^2+4}} = x + c_3 \tag{2}$$

Verification of solutions

$$\frac{c_1y^2+2}{c_1\sqrt{2c_1y^2+4}} = x + c_2$$

Verified OK.

$$-\frac{c_1y^2+2}{c_1\sqrt{2c_1y^2+4}} = x + c_3$$

Verified OK.

18.4.2 Maple step by step solution

Let's solve

$y^3y'' = -4$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

$y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

$u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$y^3 u(y)\left(\frac{d}{dy}u(y)\right) = -4$

- Separate variables

$u(y)\left(\frac{d}{dy}u(y)\right) = -\frac{4}{y^3}$

- Integrate both sides with respect to y

$\int u(y)\left(\frac{d}{dy}u(y)\right)dy = \int -\frac{4}{y^3}dy + c_1$

- Evaluate integral

$\frac{u(y)^2}{2} = \frac{2}{y^2} + c_1$

- Solve for $u(y)$

$\left\{u(y) = \frac{\sqrt{2c_1y^2+4}}{y}, u(y) = -\frac{\sqrt{2c_1y^2+4}}{y}\right\}$

- Solve 1st ODE for $u(y)$

$u(y) = \frac{\sqrt{2c_1y^2+4}}{y}$

- Revert to original variables with substitution $u(y) = y', y = y$

$y' = \frac{\sqrt{2c_1y^2+4}}{y}$

- Separate variables

$\frac{yy'}{\sqrt{2c_1y^2+4}} = 1$

- Integrate both sides with respect to x

$\int \frac{yy'}{\sqrt{2c_1y^2+4}}dx = \int 1dx + c_2$

- Evaluate integral

$\frac{\sqrt{2c_1y^2+4}}{2c_1} = x + c_2$

- Solve for y

$\left\{y = \frac{\sqrt{2}\sqrt{c_1(c_1^2c_2^2+2c_1^2c_2x+c_1^2x^2-1)}}{c_1}, y = -\frac{\sqrt{2}\sqrt{c_1(c_1^2c_2^2+2c_1^2c_2x+c_1^2x^2-1)}}{c_1}\right\}$

- Solve 2nd ODE for $u(y)$

  $$u(y) = -\frac{\sqrt{2c_1y^2+4}}{y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

  $$y' = -\frac{\sqrt{2c_1y^2+4}}{y}$$

- Separate variables

  $$\frac{yy'}{\sqrt{2c_1y^2+4}} = -1$$

- Integrate both sides with respect to x

  $$\int \frac{yy'}{\sqrt{2c_1y^2+4}}dx = \int (-1)\, dx + c_2$$

- Evaluate integral

  $$\frac{\sqrt{2c_1y^2+4}}{2c_1} = -x + c_2$$

- Solve for y

  $$\left\{y = \frac{\sqrt{2}\,\sqrt{c_1\,(c_1^2c_2^2-2c_1^2c_2x+c_1^2x^2-1)}}{c_1}, y = -\frac{\sqrt{2}\,\sqrt{c_1\,(c_1^2c_2^2-2c_1^2c_2x+c_1^2x^2-1)}}{c_1}\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+4/_a^3 = 0, _b(_a), HINT = [[_a,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, -_b]
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 52

```
dsolve(y(x)^3*diff(y(x),x$2)+4=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(2 + c_1 (c_2 + x)) (-2 + c_1 (c_2 + x)) c_1}}{c_1}$$
$$y(x) = -\frac{\sqrt{(2 + c_1 (c_2 + x)) (-2 + c_1 (c_2 + x)) c_1}}{c_1}$$

✓ Solution by Mathematica

Time used: 3.404 (sec). Leaf size: 93

```
DSolve[y[x]^3*y''[x]+4==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{c_1{}^2 x^2 + 2c_2 c_1{}^2 x - 4 + c_2{}^2 c_1{}^2}}{\sqrt{c_1}}$$
$$y(x) \to \frac{\sqrt{c_1{}^2 x^2 + 2c_2 c_1{}^2 x - 4 + c_2{}^2 c_1{}^2}}{\sqrt{c_1}}$$
$$y(x) \to \text{Indeterminate}$$

18.5 problem 5

Internal problem ID [2277]
Internal file name [`OUTPUT/2277_Tuesday_February_27_2024_08_23_57_AM_82117291/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{x'' - \frac{k^2}{x^2} = 0}$$

18.5.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right)x^2 = k^2$$

Which is now solved as first order ode for $p(x)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(x, p) \\ &= f(x)g(p) \\ &= \frac{k^2}{p\,x^2} \end{aligned}$$

Where $f(x) = \frac{k^2}{x^2}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\frac{1}{\frac{1}{p}}\,dp = \frac{k^2}{x^2}\,dx$$

$$\int \frac{1}{\frac{1}{p}}\,dp = \int \frac{k^2}{x^2}\,dx$$

$$\frac{p^2}{2} = -\frac{k^2}{x} + c_1$$

The solution is

$$\frac{p(x)^2}{2} + \frac{k^2}{x} - c_1 = 0$$

For solution (1) found earlier, since $p = x'$ then we now have a new first order ode to solve which is

$$\frac{x'^2}{2} + \frac{k^2}{x} - c_1 = 0$$

Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \frac{\sqrt{2}\,\sqrt{x\,(c_1 x - k^2)}}{x} \tag{1}$$

$$x' = -\frac{\sqrt{2}\,\sqrt{x\,(c_1 x - k^2)}}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{x\sqrt{2}}{2\sqrt{x\,(c_1 x - k^2)}}dx = \int dt$$

$$\frac{\sqrt{2}\,\sqrt{x^2 c_1 - k^2 x}}{2c_1} + \frac{\sqrt{2}\,k^2 \ln\left(\frac{-\frac{k^2}{2}+c_1 x}{\sqrt{c_1}} + \sqrt{x^2 c_1 - k^2 x}\right)}{4c_1^{\frac{3}{2}}} = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{x\sqrt{2}}{2\sqrt{x\left(c_1 x-k^2\right)}}dx = \int dt$$

$$-\frac{\sqrt{2}\,\sqrt{x^2 c_1-k^2 x}}{2c_1}-\frac{\sqrt{2}\,k^2 \ln\left(\frac{-\frac{k^2}{2}+c_1 x}{\sqrt{c_1}}+\sqrt{x^2 c_1-k^2 x}\right)}{4c_1^{\frac{3}{2}}} = t+c_3$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{2}\,\sqrt{x^2 c_1-k^2 x}}{2c_1}+\frac{\sqrt{2}\,k^2 \ln\left(\frac{-\frac{k^2}{2}+c_1 x}{\sqrt{c_1}}+\sqrt{x^2 c_1-k^2 x}\right)}{4c_1^{\frac{3}{2}}} = t+c_2 \tag{1}$$

$$-\frac{\sqrt{2}\,\sqrt{x^2 c_1-k^2 x}}{2c_1}-\frac{\sqrt{2}\,k^2 \ln\left(\frac{-\frac{k^2}{2}+c_1 x}{\sqrt{c_1}}+\sqrt{x^2 c_1-k^2 x}\right)}{4c_1^{\frac{3}{2}}} = t+c_3 \tag{2}$$

Verification of solutions

$$\frac{\sqrt{2}\,\sqrt{x^2 c_1-k^2 x}}{2c_1}+\frac{\sqrt{2}\,k^2 \ln\left(\frac{-\frac{k^2}{2}+c_1 x}{\sqrt{c_1}}+\sqrt{x^2 c_1-k^2 x}\right)}{4c_1^{\frac{3}{2}}} = t+c_2$$

Verified OK.

$$-\frac{\sqrt{2}\,\sqrt{x^2 c_1-k^2 x}}{2c_1}-\frac{\sqrt{2}\,k^2 \ln\left(\frac{-\frac{k^2}{2}+c_1 x}{\sqrt{c_1}}+\sqrt{x^2 c_1-k^2 x}\right)}{4c_1^{\frac{3}{2}}} = t+c_3$$

Verified OK.

18.5.2 Maple step by step solution

Let's solve

$x''x^2 = k^2$

- Highest derivative means the order of the ODE is 2

  x''

- Define new dependent variable u

  $u(t) = x'$

- Compute x''

  $u'(t) = x''$

- Use chain rule on the lhs

  $x'\left(\frac{d}{dx}u(x)\right) = x''$

- Substitute in the definition of u

  $u(x)\left(\frac{d}{dx}u(x)\right) = x''$

- Make substitutions $x' = u(x), x'' = u(x)\left(\frac{d}{dx}u(x)\right)$ to reduce order of ODE

  $u(x)\left(\frac{d}{dx}u(x)\right)x^2 = k^2$

- Separate variables

  $u(x)\left(\frac{d}{dx}u(x)\right) = \frac{k^2}{x^2}$

- Integrate both sides with respect to x

  $\int u(x)\left(\frac{d}{dx}u(x)\right)dx = \int \frac{k^2}{x^2}dx + c_1$

- Evaluate integral

  $\frac{u(x)^2}{2} = -\frac{k^2}{x} + c_1$

- Solve for $u(x)$

  $\left\{u(x) = \frac{\sqrt{2}\,\sqrt{x(c_1x-k^2)}}{x}, u(x) = -\frac{\sqrt{2}\,\sqrt{x(c_1x-k^2)}}{x}\right\}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{\sqrt{2}\,\sqrt{x(c_1x-k^2)}}{x}$

- Revert to original variables with substitution $u(x) = x', x = x$

  $x' = \frac{\sqrt{2}\,\sqrt{x(c_1x-k^2)}}{x}$

- Separate variables

$$\frac{x' x}{\sqrt{x(c_1 x - k^2)}} = \sqrt{2}$$

- Integrate both sides with respect to t

$$\int \frac{x' x}{\sqrt{x(c_1 x - k^2)}} dt = \int \sqrt{2} dt + c_2$$

- Evaluate integral

$$\frac{\sqrt{x^2 c_1 - k^2 x}}{c_1} + \frac{k^2 \ln\left(\frac{-\frac{k^2}{2} + c_1 x}{\sqrt{c_1}} + \sqrt{x^2 c_1 - k^2 x}\right)}{2 c_1^{\frac{3}{2}}} = \sqrt{2}\, t + c_2$$

- Solve for x

$$\left\{ \frac{4 c_1 \left(e^{RootOf\left(-64 \left(e^{\_Z}\right)^2 \sqrt{2}\, c_1^{\frac{5}{2}} \_Z k^2 t - 64 \left(e^{\_Z}\right)^2 c_1^{\frac{5}{2}} c_2 \_Z k^2 + 128 \left(e^{\_Z}\right)^2 \sqrt{2}\, c_1^4 c_2 t + 16 \left(e^{\_Z}\right)^2 c_1 \_Z^2 k^4 + 64 \left(e^{\_Z}\right)^2 c_1^4 c_2^2 + 128 \left(e^{\_Z}\right)^2 c_1^4 t^2 - k^8 \right.}\right.}{8\, e^{RootOf\left(-64 \left(e^{\_Z}\right)^2 \sqrt{2}\, c_1^{\frac{5}{2}} \_Z k^2 t - 64 \left(e^{\_Z}\right)^2 c \right.}} \right.$$

- Solve 2nd ODE for $u(x)$

$$u(x) = -\frac{\sqrt{2}\, \sqrt{x(c_1 x - k^2)}}{x}$$

- Revert to original variables with substitution $u(x) = x', x = x$

$$x' = -\frac{\sqrt{2}\, \sqrt{x(c_1 x - k^2)}}{x}$$

- Separate variables

$$\frac{x' x}{\sqrt{x(c_1 x - k^2)}} = -\sqrt{2}$$

- Integrate both sides with respect to t

$$\int \frac{x' x}{\sqrt{x(c_1 x - k^2)}} dt = \int -\sqrt{2} dt + c_2$$

- Evaluate integral

$$\frac{\sqrt{x^2 c_1 - k^2 x}}{c_1} + \frac{k^2 \ln\left(\frac{-\frac{k^2}{2} + c_1 x}{\sqrt{c_1}} + \sqrt{x^2 c_1 - k^2 x}\right)}{2 c_1^{\frac{3}{2}}} = -\sqrt{2}\, t + c_2$$

- Solve for x

$$\left\{ \frac{4 c_1 \left(e^{RootOf\left(-64 \left(e^{\_Z}\right)^2 \sqrt{2}\, c_1^{\frac{5}{2}} \_Z k^2 t + 64 \left(e^{\_Z}\right)^2 c_1^{\frac{5}{2}} c_2 \_Z k^2 + 128 \left(e^{\_Z}\right)^2 \sqrt{2}\, c_1^4 c_2 t - 16 \left(e^{\_Z}\right)^2 c_1 \_Z^2 k^4 - 64 \left(e^{\_Z}\right)^2 c_1^4 c_2^2 - 128 \left(e^{\_Z}\right)^2 c_1^4 t^2 + k^8 \right.}\right.}{8\, e^{RootOf\left(-64 \left(e^{\_Z}\right)^2 \sqrt{2}\, c_1^{\frac{5}{2}} \_Z k^2 t + 64 \left(e^{\_Z}\right)^2 c \right.}} \right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-k^2/_a^2 = 0, _b(_a), HINT = [[_
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, -1/2*_b]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 385

```
dsolve(diff(x(t),t$2)=k^2/x(t)^2,x(t), singsol=all)
```

$$x(t) = \frac{c_1\left(c_1^2k^4 + 2k^2c_1\mathrm{e}^{\mathrm{RootOf}\left(\mathrm{csgn}\left(\frac{1}{c_1}\right)c_1^4k^4-2\_Zc_1^3k^2\mathrm{e}^{-\_Z}-\mathrm{csgn}\left(\frac{1}{c_1}\right)\mathrm{e}^{2\_Z}c_1^2-2\,\mathrm{csgn}\left(\frac{1}{c_1}\right)\mathrm{e}^{\_Z}c_2-2\,\mathrm{csgn}\left(\frac{1}{c_1}\right)\mathrm{e}^{\_Z}t\right)} + \mathrm{e}^{2\,\mathrm{RootOf}\left(\mathrm{csgn}\right.}\right.}{}$$

$$x(t) = \frac{c_1\left(c_1^2k^4 + 2k^2c_1\mathrm{e}^{\mathrm{RootOf}\left(\mathrm{csgn}\left(\frac{1}{c_1}\right)c_1^4k^4-2\_Zc_1^3k^2\mathrm{e}^{-\_Z}-\mathrm{csgn}\left(\frac{1}{c_1}\right)\mathrm{e}^{2\_Z}c_1^2+2\,\mathrm{csgn}\left(\frac{1}{c_1}\right)\mathrm{e}^{\_Z}c_2+2\,\mathrm{csgn}\left(\frac{1}{c_1}\right)\mathrm{e}^{\_Z}t\right)} + \mathrm{e}^{2\,\mathrm{RootOf}\left(\mathrm{csgn}\right.}\right.}{}$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 71

```
DSolve[x''[t]==k^2/x[t]^2,x[t],t,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\left(\frac{2k^2\operatorname{arctanh}\left(\frac{\sqrt{-\frac{2k^2}{x(t)}+c_1}}{\sqrt{c_1}}\right)}{c_1{}^{3/2}} + \frac{x(t)\sqrt{-\frac{2k^2}{x(t)}+c_1}}{c_1}\right)^2 = (t+c_2)^2, x(t)\right]$$

18.6 problem 6

Internal problem ID [2278]
Internal file name [`OUTPUT/2278_Tuesday_February_27_2024_08_23_58_AM_84286239/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{xy'' = x^2 + 1}$$

Simplyfing the ode gives

$$y'' = \frac{x^2 + 1}{x}$$

18.6.1 Solving as second order ode quadrature ode

Integrating once gives

$$y' = \frac{x^2}{2} + \ln(x) + c_1$$

Integrating again gives

$$y = \frac{x^3}{6} + x\ln(x) - x + c_1x + c_2$$

Summary

The solution(s) found are the following

$$y = \frac{x^3}{6} + x\ln(x) - x + c_1x + c_2 \tag{1}$$

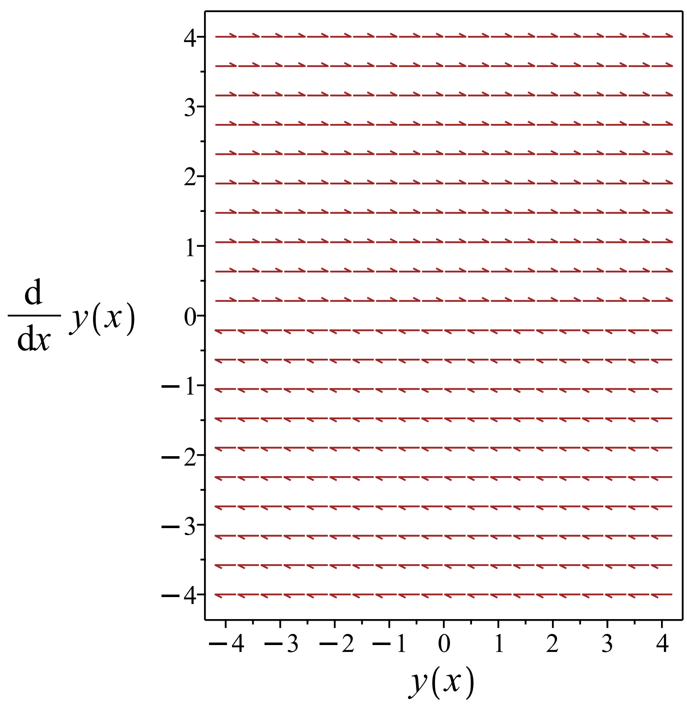


Figure 703: Slope field plot

Verification of solutions

$$y = \frac{x^3}{6} + x\ln(x) - x + c_1x + c_2$$

Verified OK.

18.6.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = x + \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\left(x + \frac{1}{x}\right)}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \left(x^2 + 1\right) dx$$

Hence

$$u_1 = -\frac{1}{3}x^3 - x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x + \frac{1}{x}}{1}\, dx$$

Which simplifies to

$$u_2 = \int \left(x + \frac{1}{x}\right) dx$$

Hence

$$u_2 = \frac{x^2}{2} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3}{3} - x + \left(\frac{x^2}{2} + \ln(x)\right) x$$

Which simplifies to

$$y_p(x) = \frac{x(x^2 + 6\ln(x) - 6)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{x(x^2 + 6\ln(x) - 6)}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + \frac{x(x^2 + 6\ln(x) - 6)}{6} \tag{1}$$

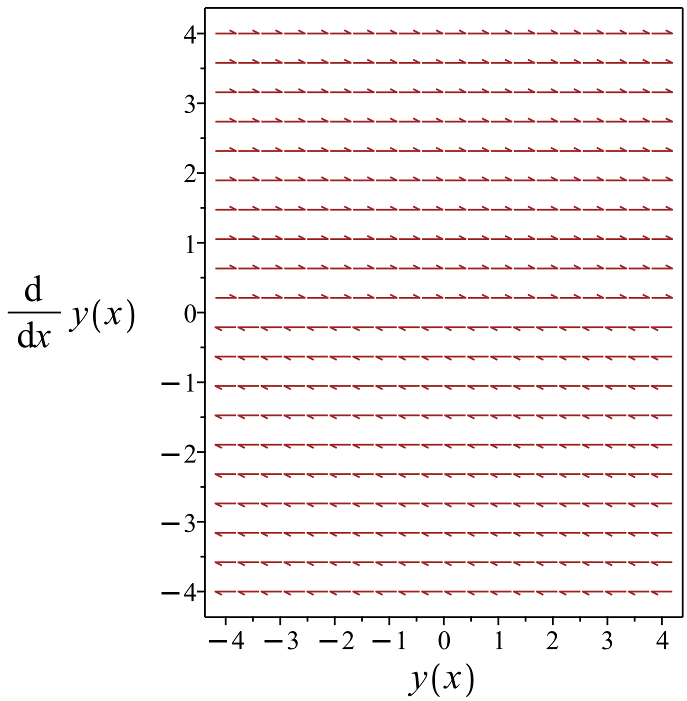


Figure 704: Slope field plot

Verification of solutions

$$y = c_2 x + c_1 + \frac{x(x^2 + 6\ln(x) - 6)}{6}$$

Verified OK.

18.6.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int \left(x + \frac{1}{x}\right) dx$$

$$y' = \frac{x^2}{2} + \ln(x) + c_1$$

Which is now solved for y. Integrating both sides gives

$$
\begin{aligned}
y &= \int \ln(x) + \frac{x^2}{2} + c_1 \, \mathrm{d}x \\
&= \frac{x^3}{6} + x\ln(x) - x + c_1 x + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^3}{6} + x\ln(x) - x + c_1 x + c_2 \tag{1}
$$

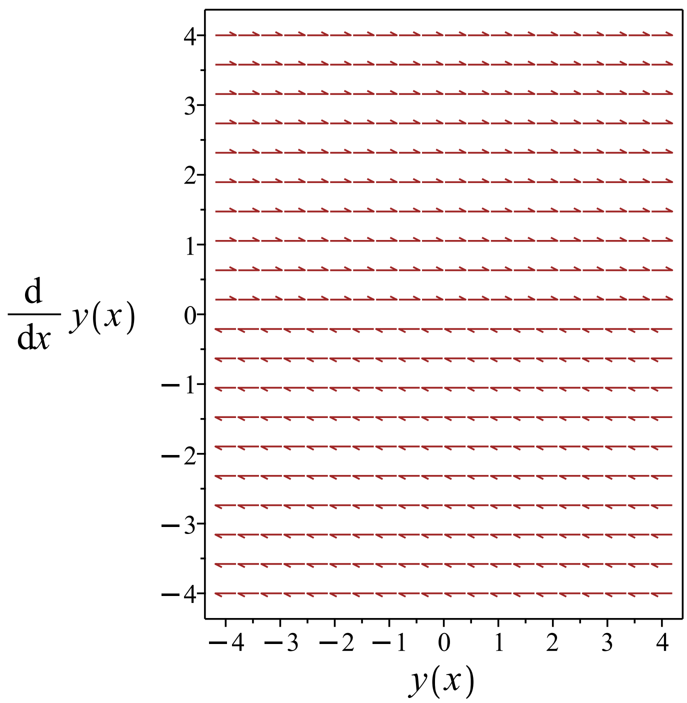


Figure 705: Slope field plot

Verification of solutions

$$
y = \frac{x^3}{6} + x\ln(x) - x + c_1 x + c_2
$$

Verified OK.

18.6.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - x - \frac{1}{x} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int \frac{x^2+1}{x}\, \mathrm{d}x \\ &= \ln(x) + \frac{x^2}{2} + c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \ln(x) + \frac{x^2}{2} + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \ln(x) + \frac{x^2}{2} + c_1\, \mathrm{d}x \\ &= \frac{x^3}{6} + x\ln(x) - x + c_1 x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^3}{6} + x\ln(x) - x + c_1 x + c_2 \tag{1}$$

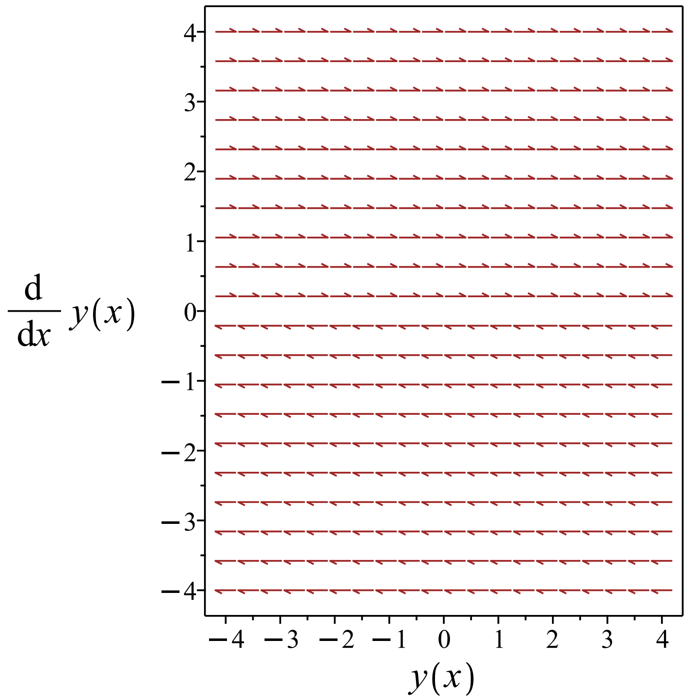


Figure 706: Slope field plot

Verification of solutions

$$y = \frac{x^3}{6} + x \ln(x) - x + c_1 x + c_2$$

Verified OK.

18.6.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 575: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x + c_1
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\left(x + \frac{1}{x}\right)}{1} \, dx$$

Which simplifies to

$$u_1 = -\int \left(x^2 + 1\right) dx$$

Hence

$$u_1 = -\frac{1}{3}x^3 - x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x + \frac{1}{x}}{1} \, dx$$

Which simplifies to

$$u_2 = \int \left(x + \frac{1}{x}\right) dx$$

Hence

$$u_2 = \frac{x^2}{2} + \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3}{3} - x + \left(\frac{x^2}{2} + \ln(x)\right) x$$

Which simplifies to

$$y_p(x) = \frac{x(x^2 + 6\ln(x) - 6)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{x(x^2 + 6\ln(x) - 6)}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + \frac{x(x^2 + 6\ln(x) - 6)}{6} \tag{1}$$

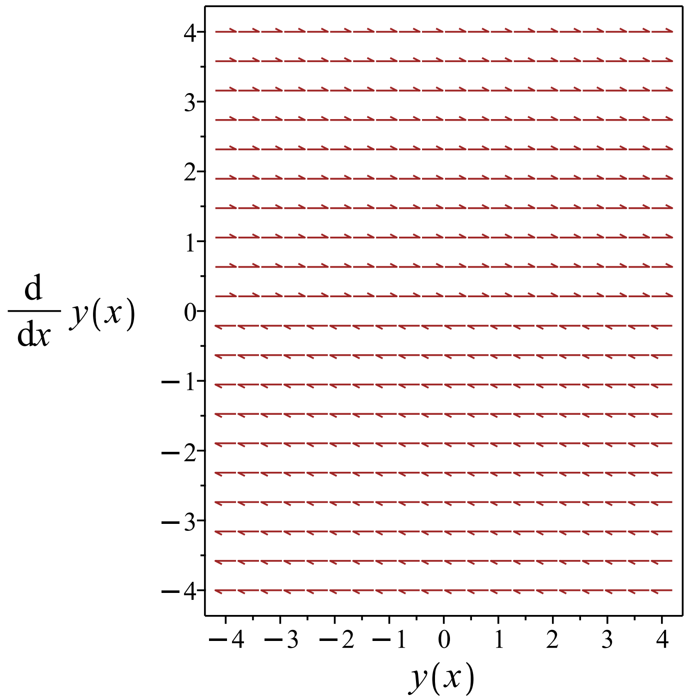


Figure 707: Slope field plot

Verification of solutions

$$y = c_2x + c_1 + \frac{x(x^2 + 6\ln(x) - 6)}{6}$$

Verified OK.

18.6.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= x + \frac{1}{x} \end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int x + \frac{1}{x}\, dx$$

We now have a first order ode to solve which is

$$y' = \ln(x) + \frac{x^2}{2} + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \ln(x) + \frac{x^2}{2} + c_1 \,\mathrm{d}x \\ &= \frac{x^3}{6} + x\ln(x) - x + c_1 x + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3}{6} + x\ln(x) - x + c_1 x + c_2 \tag{1}$$

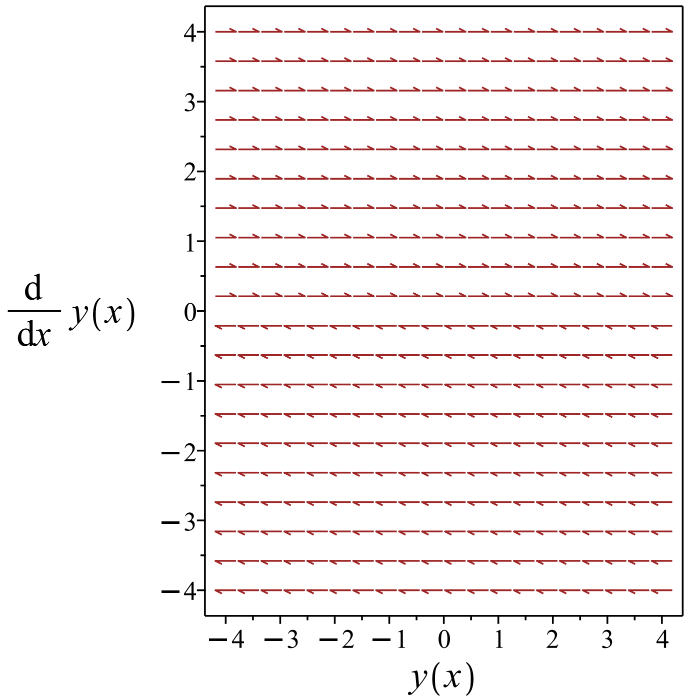


Figure 708: Slope field plot

Verification of solutions

$$y = \frac{x^3}{6} + x \ln(x) - x + c_1 x + c_2$$

Verified OK.

18.6.7 Maple step by step solution

Let's solve

$y'' = x + \frac{1}{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{x^2+1}{x}$

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

$r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{x^2+1}{x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\left(\int (x^2+1)\, dx\right) + x\left(\int \frac{x^2+1}{x} dx\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{x(x^2 + 6\ln(x) - 6)}{6}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x + c_1 + \frac{x(x^2 + 6\ln(x) - 6)}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x*diff(y(x),x$2)=1+x^2,y(x), singsol=all)
```

$$y(x) = x\ln(x) + \frac{x^3}{6} + (c_1 - 1)\,x + c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 25

```
DSolve[x*y''[x]==1+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{6} + x\log(x) + (-1 + c_2)x + c_1$$

18.7 problem 7

Internal problem ID [2279]
Internal file name [`OUTPUT/2279_Tuesday_February_27_2024_08_23_58_AM_75682043/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(1-x)\, y'' - y' = 0}$$

18.7.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int ((1-x)\, y'' - y')\, dx = 0$$
$$-(x-1)\, y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{c_1}{x-1}\,\mathrm{d}x \\
&= -c_1 \ln(x-1) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -c_1 \ln(x-1) + c_2 \tag{1}
$$

Verification of solutions

$$
y = -c_1 \ln(x-1) + c_2
$$

Verified OK.

18.7.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$
p(x) = y'
$$

Then

$$
p'(x) = y''
$$

Hence the ode becomes

$$
(1-x)\,p'(x) - p(x) = 0
$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(x,p) \\
&= f(x)g(p) \\
&= -\frac{p}{x-1}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x-1}$ and $g(p) = p$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{p}\,dp &= -\frac{1}{x-1}\,dx \\
\int \frac{1}{p}\,dp &= \int -\frac{1}{x-1}\,dx \\
\ln(p) &= -\ln(x-1) + c_1 \\
p &= \mathrm{e}^{-\ln(x-1)+c_1} \\
&= \frac{c_1}{x-1}
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_1}{x-1}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1}{x-1}\, \mathrm{d}x \\ &= c_1 \ln(x-1) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x-1) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x-1) + c_2$$

Verified OK.

18.7.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$AB u' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= 1 - x \\
B &= -1 \\
C &= 0 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (1 - x)(0) + (-1)(0) + (0)(-1) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x - 1v'' + (1) v' = 0$$

Now by applying $v' = u$ the above becomes

$$(x - 1) u'(x) + u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{u}{x - 1}
\end{aligned}$$

Where $f(x) = -\frac{1}{x-1}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u} \, du &= -\frac{1}{x - 1} \, dx \\
\int \frac{1}{u} \, du &= \int -\frac{1}{x - 1} \, dx \\
\ln(u) &= -\ln(x - 1) + c_1 \\
u &= e^{-\ln(x-1)+c_1} \\
&= \frac{c_1}{x - 1}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}v' &= u \\ &= \frac{c_1}{x-1}\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}v(x) &= \int \frac{c_1}{x-1}\, \mathrm{d}x \\ &= c_1 \ln(x-1) + c_2\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y(x) &= Bv \\ &= (-1)\left(c_1 \ln(x-1) + c_2\right) \\ &= -c_1 \ln(x-1) - c_2\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -c_1 \ln(x-1) - c_2 \tag{1}$$

Verification of solutions

$$y = -c_1 \ln(x-1) - c_2$$

Verified OK.

18.7.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(1-x)\, y'' - y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int \left((1-x)\, y'' - y'\right) dx &= 0 \\ -(x-1)\, y' &= c_1\end{aligned}$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned}y &= \int -\frac{c_1}{x-1}\, \mathrm{d}x \\ &= -c_1 \ln(x-1) + c_2\end{aligned}$$

Summary
The solution(s) found are the following

$$y = -c_1 \ln (x-1) + c_2 \tag{1}$$

Verification of solutions

$$y = -c_1 \ln (x-1) + c_2$$

Verified OK.

18.7.5 Solving using Kovacic algorithm

Writing the ode as

$$(1-x) y'' - y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x \\ B &= -1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4\,(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 577: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4(x-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x-2} + (-)\,(0) \\ &= \frac{1}{2x-2} \\ &= \frac{1}{2x-2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2\,(x-1)^2}\right) + \left(\frac{1}{2x-2}\right)^2 - \left(-\frac{1}{4\,(x-1)^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x-2} dx} \\
&= \sqrt{x-1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-1}{1-x} \, dx} \\
&= z_1 e^{-\frac{\ln(1-x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{1-x}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x-1}}{\sqrt{1-x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1-x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(1-x)}}{(y_1)^2} \, dx \\
&= y_1 (\ln (x-1))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x-1}}{\sqrt{1-x}} \right) + c_2 \left(\frac{\sqrt{x-1}}{\sqrt{1-x}} (\ln (x-1)) \right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \sqrt{x-1}}{\sqrt{1-x}} + \frac{c_2 \sqrt{x-1}\, \ln (x-1)}{\sqrt{1-x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sqrt{x-1}}{\sqrt{1-x}} + \frac{c_2 \sqrt{x-1}\, \ln (x-1)}{\sqrt{1-x}}$$

Verified OK.

18.7.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1-x \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$(1-x)\,y' = c_1$$

We now have a first order ode to solve which is

$$(1-x)\,y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{c_1}{x-1}\,\mathrm{d}x \\ &= -c_1 \ln(x-1) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -c_1 \ln(x-1) + c_2 \tag{1}$$

Verification of solutions

$$y = -c_1 \ln(x-1) + c_2$$

Verified OK.

18.7.7 Maple step by step solution

Let's solve

$(1-x)\,y'' - y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x-1}, P_3(x) = 0\right]$

  ◦ $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

$$\left((x-1)\cdot P_2(x)\right)\Big|_{x=1} = 1$$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x=1$

$$\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1} = 0$$

- $x=1$is a regular singular point

Check to see if $x_0 = 1$ is a regular singular point

$x_0 = 1$

- Multiply by denominators

$y''(x-1) + y' = 0$

- Change variables using $x = u+1$ so that the regular singular point is at $u = 0$

$$u\left(\frac{d^2}{du^2}y(u)\right) + \frac{d}{du}y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite DE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

$$\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$$

- Shift index using $k->k+1$

$$\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$$

- Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

$$u\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite DE with series expansions

$$\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)^2\,u^{k+r} = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1)^2 = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = 0$

- Recursion relation for $r = 0$

$a_{k+1} = 0$

- Solution for $r = 0$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = 0\right]$

- Revert the change of variables $u = x - 1$

$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((1-x)*diff(y(x),x$2)=diff(y(x),x),y(x), singsol=all)
```

$$y(x) = c_2 \ln(x-1) + c_1$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 18

```
DSolve[(1-x)*y''[x]==y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - c_1 \log(1-x)$$

18.8 problem 8

Internal problem ID [2280]
Internal file name [`OUTPUT/2280_Tuesday_February_27_2024_08_23_59_AM_12919399/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(x^2+1)\,y''+2x(y'+1)=0}$$

18.8.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x^2+1)\,y''+2xy'\right)dx=\int -2x dx$$
$$(x^2+1)\,y'=-x^2+c_1$$

Which is now solved for y. Integrating both sides gives

$$y=\int \frac{-x^2+c_1}{x^2+1}\,\mathrm{d}x$$
$$=-x+(c_1+1)\arctan(x)+c_2$$

Summary
The solution(s) found are the following

$$y = -x + (c_1 + 1)\arctan(x) + c_2 \tag{1}$$

Verification of solutions

$$y = -x + (c_1 + 1)\arctan(x) + c_2$$

Verified OK.

18.8.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$(x^2 + 1)\,p'(x) + 2xp(x) + 2x = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{x(-2p-2)}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{x}{x^2+1}$ and $g(p) = -2p - 2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-2p-2}\,dp &= \frac{x}{x^2+1}\,dx \\ \int \frac{1}{-2p-2}\,dp &= \int \frac{x}{x^2+1}\,dx \\ -\frac{\ln(p+1)}{2} &= \frac{\ln(x^2+1)}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{p+1}} = \mathrm{e}^{\frac{\ln(x^2+1)}{2} + c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{p+1}} = c_2\sqrt{x^2+1}$$

Which simplifies to

$$p(x) = -\frac{\left(c_2^2 \mathrm{e}^{2c_1}(x^2+1)-1\right)\mathrm{e}^{-2c_1}}{c_2^2\,(x^2+1)}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{\left(c_2^2 \mathrm{e}^{2c_1}(x^2+1)-1\right)\mathrm{e}^{-2c_1}}{c_2^2\,(x^2+1)}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{\left(\mathrm{e}^{2c_1} c_2^2 x^2 + c_2^2 \mathrm{e}^{2c_1} - 1\right)\mathrm{e}^{-2c_1}}{c_2^2\,(x^2+1)}\,\mathrm{d}x \\ &= -\frac{\mathrm{e}^{-2c_1}\left(c_2^2 \mathrm{e}^{2c_1} x - \arctan(x)\right)}{c_2^2} + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-2c_1}\left(c_2^2 \mathrm{e}^{2c_1} x - \arctan(x)\right)}{c_2^2} + c_3 \tag{1}$$

Verification of solutions

$$y = -\frac{\mathrm{e}^{-2c_1}\left(c_2^2 \mathrm{e}^{2c_1} x - \arctan(x)\right)}{c_2^2} + c_3$$

Verified OK.

18.8.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$\left(x^2+1\right) y'' + 2xy' = -2x$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int \left(\left(x^2+1\right) y'' + 2xy'\right) dx &= \int -2x dx \\ \left(x^2+1\right) y' &= -x^2 + c_1 \end{aligned}$$

Which is now solved for y. Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{-x^2 + c_1}{x^2 + 1}\, \mathrm{d}x \\
&= -x + (c_1 + 1) \arctan(x) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -x + (c_1 + 1) \arctan(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = -x + (c_1 + 1) \arctan(x) + c_2
$$

Verified OK.

18.8.4 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
(x^2 + 1)\, y'' + 2xy' &= 0 && (1) \\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 + 1 \\
B &= 2x \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = (x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 579: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{i}{4(x-i)} + \frac{i}{4x+4i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} + (-)(0) \\
&= \frac{1}{2x-2i} + \frac{1}{2x+2i} \\
&= \frac{x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)(0) + \left(\left(-\frac{1}{2(x-i)^2} - \frac{1}{2(x+i)^2}\right) + \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right)^2 - \left(\frac{1}{(x^2+1)^2}\right.\right.$$

$$0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x-2i} + \frac{1}{2x+2i}\right) dx} \\ &= \sqrt{x^2+1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1} \, dx} \\ &= z_1 e^{-\frac{\ln(x^2+1)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2} \, dx \\ &= y_1(\arctan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(1) + c_2(1(\arctan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
(x^2 + 1)\, y'' + 2xy' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + c_2 \arctan(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \arctan(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \arctan(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\arctan(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \arctan(x) \\ 0 & \frac{1}{x^2+1} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x^2+1}\right) - (\arctan(x))(0)$$

Which simplifies to

$$W = \frac{1}{x^2+1}$$

Which simplifies to

$$W = \frac{1}{x^2+1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2\arctan(x)\, x}{1}\, dx$$

Which simplifies to

$$u_1 = -\int -2\arctan(x)\, x dx$$

Hence

$$u_1 = \arctan(x)\, x^2 - x + \arctan(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2x}{1}\, dx$$

Which simplifies to

$$u_2 = \int -2x dx$$

Hence

$$u_2 = -x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x + \arctan(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 \arctan(x)) + (-x + \arctan(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \arctan(x) - x + \arctan(x) \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 \arctan(x) - x + \arctan(x)$$

Verified OK.

18.8.5 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 + 1 \\ q(x) &= 2x \\ r(x) &= 0 \\ s(x) &= -2x \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 2
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (2) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx
$$

Substituting the above values for p, q, r, s gives

$$
\left(x^2 + 1\right) y' = \int -2x\, dx
$$

We now have a first order ode to solve which is

$$
\left(x^2 + 1\right) y' = -x^2 + c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{-x^2 + c_1}{x^2 + 1}\, \mathrm{d}x \\
&= -x + (c_1 + 1) \arctan(x) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -x + (c_1 + 1) \arctan(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = -x + (c_1 + 1) \arctan(x) + c_2
$$

Verified OK.

18.8.6 Maple step by step solution

Let's solve

$(x^2+1)\,y''+2xy'=-2x$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $(x^2+1)\,u'(x)+2xu(x)=-2x$

- Integrate both sides with respect to x

  $\int \left((x^2+1)\,u'(x)+2xu(x)\right)dx=\int -2xdx+c_1$

- Evaluate integral

  $(x^2+1)\,u(x)=-x^2+c_1$

- Solve for $u(x)$

  $u(x)=\frac{-x^2+c_1}{x^2+1}$

- Solve 1st ODE for $u(x)$

  $u(x)=\frac{-x^2+c_1}{x^2+1}$

- Make substitution $u=y'$

  $y'=\frac{-x^2+c_1}{x^2+1}$

- Integrate both sides to solve for y

  $\int y'dx=\int \frac{-x^2+c_1}{x^2+1}dx+c_2$

- Compute integrals

  $y=-x+(c_1+1)\arctan(x)+c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -2*_a*(_b(_a)+1)/(_a^2+1), _b(_a)`   **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x$2)+2*x*(diff(y(x),x)+1)=0,y(x), singsol=all)
```

$$y(x) = -x + (c_1 + 1) \arctan(x) + c_2$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 18

```
DSolve[(1+x^2)*y''[x]+2*x*(y'[x]+1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (1 + c_1) \arctan(x) - x + c_2$$

18.9 problem 9

Internal problem ID [2281]
Internal file name [`OUTPUT/2281_Tuesday_February_27_2024_08_23_59_AM_24740622/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - {y'}^3 - y' = 0}$$

18.9.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + \left(-p(x)^2 - 1\right) p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p\,(p^2+1)} dp = \int dx$$
$$-\frac{\ln\left(p^2+1\right)}{2} + \ln\left(p\right) = x + c_1$$

Raising both side to exponential gives

$$e^{-\frac{\ln(p^2+1)}{2}+\ln(p)} = e^{x+c_1}$$

Which simplifies to

$$\frac{p}{\sqrt{p^2+1}} = c_2 e^x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_2 e^x \sqrt{-\frac{1}{c_2^2 e^{2x} - 1}}$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_2 e^x \sqrt{-\frac{1}{c_2^2 e^{2x} - 1}}\, dx \\ &= \frac{c_2 \sqrt{-\frac{1}{c_2^2 e^{2x}-1}}\, \sqrt{c_2^2 e^{2x} - 1}\, \ln\left(\frac{c_2^2 e^x + \sqrt{c_2^2 e^{2x}-1}\,\sqrt{c_2^2}}{\sqrt{c_2^2}}\right)}{\sqrt{c_2^2}} + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 \sqrt{-\frac{1}{c_2^2 e^{2x}-1}}\, \sqrt{c_2^2 e^{2x} - 1}\, \ln\left(\frac{c_2^2 e^x + \sqrt{c_2^2 e^{2x}-1}\,\sqrt{c_2^2}}{\sqrt{c_2^2}}\right)}{\sqrt{c_2^2}} + c_3 \tag{1}$$

Verification of solutions

$$y = \frac{c_2 \sqrt{-\frac{1}{c_2^2 e^{2x}-1}}\, \sqrt{c_2^2 e^{2x} - 1}\, \ln\left(\frac{c_2^2 e^x + \sqrt{c_2^2 e^{2x}-1}\,\sqrt{c_2^2}}{\sqrt{c_2^2}}\right)}{\sqrt{c_2^2}} + c_3$$

Verified OK.

18.9.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right)+\left(-p(y)^{2}-1\right)p(y)=0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} \int \frac{1}{p^{2}+1}dp &= y+c_{1} \\ \arctan\left(p\right) &= y+c_{1} \end{aligned}$$

Solving for p gives these solutions

$$p_{1}=\tan\left(y+c_{1}\right)$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=\tan\left(y+c_{1}\right)$$

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{\tan\left(y+c_{1}\right)}dy &= \int dx \\ -\frac{\ln\left(1+\tan\left(y+c_{1}\right)^{2}\right)}{2}+\ln\left(\tan\left(y+c_{1}\right)\right) &= x+c_{2} \end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln\left(1+\tan\left(y+c_{1}\right)^{2}\right)}{2}+\ln\left(\tan\left(y+c_{1}\right)\right)}=\mathrm{e}^{x+c_{2}}$$

Which simplifies to

$$\operatorname{csgn}\left(\sec\left(y+c_{1}\right)\right)\sin\left(y+c_{1}\right)=c_{3}\mathrm{e}^{x}$$

Simplifying the solution $y = \operatorname{RootOf}(-\operatorname{csgn}(\sec(\_Z + c_1))\sin(\_Z + c_1) + c_3 e^x)$ to $y = \operatorname{RootOf}(-\sin(\_Z + c_1) + c_3 e^x)$

Summary

The solution(s) found are the following

$$y = \operatorname{RootOf}(-\sin(\_Z + c_1) + c_3 e^x)$$

Verification of solutions

$$y = \operatorname{RootOf}(-\sin(\_Z + c_1) + c_3 e^x)$$

Verified OK.

18.9.3 Maple step by step solution

Let's solve

$y'' + (-y'^2 - 1) y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + (-u(x)^2 - 1) u(x) = 0$

- Separate variables

  $\frac{u'(x)}{(-u(x)^2-1)u(x)} = -1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{(-u(x)^2-1)u(x)} dx = \int (-1) dx + c_1$

- Evaluate integral

  $\frac{\ln(u(x)^2+1)}{2} - \ln(u(x)) = -x + c_1$

- Solve for $u(x)$

  $\left\{u(x) = \frac{1}{\sqrt{e^{-2x+2c_1}-1}}, u(x) = -\frac{1}{\sqrt{e^{-2x+2c_1}-1}}\right\}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{1}{\sqrt{e^{-2x+2c_1}-1}}$

- Make substitution $u = y'$

  $y' = \frac{1}{\sqrt{e^{-2x+2c_1}-1}}$

- Integrate both sides to solve for y

$$\int y' dx = \int \frac{1}{\sqrt{e^{-2x+2c_1}-1}} dx + c_2$$

- Compute integrals

$$y = -\arctan\left(\sqrt{e^{-2x+2c_1}-1}\right) + c_2$$

- Solve 2nd ODE for $u(x)$

$$u(x) = -\frac{1}{\sqrt{e^{-2x+2c_1}-1}}$$

- Make substitution $u = y'$

$$y' = -\frac{1}{\sqrt{e^{-2x+2c_1}-1}}$$

- Integrate both sides to solve for y

$$\int y' dx = \int -\frac{1}{\sqrt{e^{-2x+2c_1}-1}} dx + c_2$$

- Compute integrals

$$y = \arctan\left(\sqrt{e^{-2x+2c_1}-1}\right) + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)^3+_b(_a), _b(_a), HINT = [[1, 0]
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 73

```
dsolve(diff(y(x),x$2)=diff(y(x),x)^3+diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -\frac{\arctan\left(\frac{2c_1 e^{2x}-1}{2\sqrt{-(c_1 e^{2x}-1)e^{2x}c_1}}\right)}{2} + c_2$$
$$y(x) = \frac{\arctan\left(\frac{2c_1 e^{2x}-1}{2\sqrt{-(c_1 e^{2x}-1)e^{2x}c_1}}\right)}{2} + c_2$$

✓ Solution by Mathematica

Time used: 60.102 (sec). Leaf size: 71

```
DSolve[y''[x]==y'[x]^3+y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - i\log\left(\sqrt{-1+e^{2(x+c_1)}} - e^{x+c_1}\right)$$
$$y(x) \to i\log\left(\sqrt{-1+e^{2(x+c_1)}} - e^{x+c_1}\right) + c_2$$

18.10 problem 10

Internal problem ID [2282]
Internal file name [`OUTPUT/2282_Tuesday_February_27_2024_08_24_00_AM_76360333/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' - y' = -x}$$

18.10.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' - y')\,dx = \int -x dx$$

$$xy' - 2y = -\frac{x^2}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{-x^2 + 2c_1}{2x} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{-x^2 + 2c_1}{2x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{-x^2 + 2c_1}{2x}\right) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{-x^2 + 2c_1}{2x}\right) \\ d\left(\frac{y}{x^2}\right) &= \left(\frac{-x^2 + 2c_1}{2x^3}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x^2} &= \int \frac{-x^2 + 2c_1}{2x^3} dx \\ \frac{y}{x^2} &= -\frac{c_1}{2x^2} - \frac{\ln(x)}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(-\frac{c_1}{2x^2} - \frac{\ln(x)}{2}\right) + c_2x^2$$

which simplifies to

$$y = -\frac{\ln(x)\,x^2}{2} + c_2x^2 - \frac{c_1}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{\ln(x)\, x^2}{2} + c_2 x^2 - \frac{c_1}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{\ln(x)\, x^2}{2} + c_2 x^2 - \frac{c_1}{2}$$

Verified OK.

18.10.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x + x - p(x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = -1$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = -1$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x} dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)(-1) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{x}\right) &= \left(\frac{1}{x}\right)(-1) \\
\mathrm{d}\left(\frac{p}{x}\right) &= \left(-\frac{1}{x}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{p}{x} &= \int -\frac{1}{x} \,\mathrm{d}x \\
\frac{p}{x} &= -\ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
p(x) = -x\ln(x) + c_1 x
$$

which simplifies to

$$
p(x) = x(-\ln(x) + c_1)
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = x(-\ln(x) + c_1)
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int -x(\ln(x) - c_1)\,\mathrm{d}x \\
&= -\frac{\ln(x)\,x^2}{2} + \frac{x^2}{4} + \frac{c_1 x^2}{2} + c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = -\frac{\ln(x)\,x^2}{2} + \frac{x^2}{4} + \frac{c_1 x^2}{2} + c_2 \tag{1}
$$

Verification of solutions

$$
y = -\frac{\ln(x)\,x^2}{2} + \frac{x^2}{4} + \frac{c_1 x^2}{2} + c_2
$$

Verified OK.

18.10.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv = 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v = 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 0 \\ F &= -x \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x)(0) + (-1)(0) + (0)(-1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-xv'' + (1)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-xu'(x) + u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{1}{x}\, dx \\ \ln(u) &= \ln(x) + c_1 \\ u &= e^{\ln(x)+c_1} \\ &= c_1 x \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1 x \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1 x \,\mathrm{d}x \\ &= \frac{c_1 x^2}{2} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (-1)\left(\frac{c_1 x^2}{2} + c_2\right) \\ &= -\frac{c_1 x^2}{2} - c_2 \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = -1$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -1 & x^2 \\ \frac{d}{dx}(-1) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -1 & x^2 \\ 0 & 2x \end{vmatrix}$$

Therefore

$$W = (-1)(2x) - (x^2)(0)$$

Which simplifies to

$$W = -2x$$

Which simplifies to

$$W = -2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-x^3}{-2x^2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{2}dx$$

Hence

$$u_1 = -\frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x}{-2x^2}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{1}{2x}dx$$

Hence

$$u_2 = -\frac{\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2}{4} - \frac{\ln(x)\,x^2}{2}$$

Which simplifies to

$$y_p(x) = -\frac{x^2(-1 + 2\ln(x))}{4}$$

Hence the complete solution is

$$\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(-\frac{c_1 x^2}{2} - c_2\right) + \left(-\frac{x^2(-1 + 2\ln(x))}{4}\right) \\
&= -\frac{\ln(x)\, x^2}{2} + \frac{(1 - 2c_1)\, x^2}{4} - c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(x)\, x^2}{2} + \frac{(1 - 2c_1)\, x^2}{4} - c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{\ln(x)\, x^2}{2} + \frac{(1 - 2c_1)\, x^2}{4} - c_2$$

Verified OK.

18.10.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' - y' = -x$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}
\int (xy'' - y')\, dx &= \int -x dx \\
xy' - 2y &= -\frac{x^2}{2} + c_1
\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{-x^2 + 2c_1}{2x}
\end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{-x^2 + 2c_1}{2x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{-x^2 + 2c_1}{2x}\right) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{-x^2 + 2c_1}{2x}\right) \\ d\left(\frac{y}{x^2}\right) &= \left(\frac{-x^2 + 2c_1}{2x^3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int \frac{-x^2 + 2c_1}{2x^3} dx \\ \frac{y}{x^2} &= -\frac{c_1}{2x^2} - \frac{\ln(x)}{2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(-\frac{c_1}{2x^2} - \frac{\ln(x)}{2}\right) + c_2 x^2$$

which simplifies to

$$y = -\frac{\ln(x)\, x^2}{2} + c_2 x^2 - \frac{c_1}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(x)\, x^2}{2} + c_2 x^2 - \frac{c_1}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{\ln(x)\, x^2}{2} + c_2 x^2 - \frac{c_1}{2}$$

Verified OK.

18.10.5 Solving using Kovacic algorithm

Writing the ode as

$$xy'' - y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 582: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{x}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
xy'' - y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 x^2}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \frac{x^2}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{x^2}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ 0 & x \end{vmatrix}$$

Therefore

$$W = (1)\,(x) - \left(\frac{x^2}{2}\right)(0)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{x^3}{2}}{x^2}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{x}{2} dx$$

Hence

$$u_1 = \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-x}{x^2}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{1}{x} dx$$

Hence

$$u_2 = -\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2}{4} - \frac{\ln(x)\,x^2}{2}$$

Which simplifies to

$$y_p(x) = -\frac{x^2(-1+2\ln(x))}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 x^2}{2}\right) + \left(-\frac{x^2(-1+2\ln(x))}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + \frac{c_2 x^2}{2} - \frac{x^2(-1+2\ln(x))}{4} \tag{1}$$

Verification of solutions

$$y = c_1 + \frac{c_2 x^2}{2} - \frac{x^2(-1+2\ln(x))}{4}$$

Verified OK.

18.10.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= -x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$xy'-2y=\int -x\,dx$$

We now have a first order ode to solve which is

$$xy'-2y=-\frac{x^2}{2}+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=-\frac{2}{x}\\ q(x)&=\frac{-x^2+2c_1}{2x}\end{aligned}$$

Hence the ode is

$$y'-\frac{2y}{x}=\frac{-x^2+2c_1}{2x}$$

The integrating factor μ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int -\frac{2}{x}dx}\\ &=\frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\left(\frac{-x^2+2c_1}{2x}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right)&=\left(\frac{1}{x^2}\right)\left(\frac{-x^2+2c_1}{2x}\right)\\ \mathrm{d}\left(\frac{y}{x^2}\right)&=\left(\frac{-x^2+2c_1}{2x^3}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{-x^2+2c_1}{2x^3}\, \mathrm{d}x$$
$$\frac{y}{x^2} = -\frac{c_1}{2x^2} - \frac{\ln(x)}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(-\frac{c_1}{2x^2} - \frac{\ln(x)}{2}\right) + c_2 x^2$$

which simplifies to

$$y = -\frac{\ln(x)\, x^2}{2} + c_2 x^2 - \frac{c_1}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(x)\, x^2}{2} + c_2 x^2 - \frac{c_1}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{\ln(x)\, x^2}{2} + c_2 x^2 - \frac{c_1}{2}$$

Verified OK.

18.10.7 Maple step by step solution

Let's solve

$y''x - y' = -x$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)\, x - u(x) = -x$

- Isolate the derivative

  $u'(x) = -1 + \frac{u(x)}{x}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

$u'(x) - \frac{u(x)}{x} = -1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x) - \frac{u(x)}{x}\right) = -\mu(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x) - \frac{u(x)}{x}\right) = \mu'(x)\,u(x) + \mu(x)\,u'(x)$
- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,u(x))\right) dx = \int -\mu(x)\,dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,u(x) = \int -\mu(x)\,dx + c_1$
- Solve for $u(x)$

  $u(x) = \frac{\int -\mu(x)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x}$

  $u(x) = x\left(\int -\frac{1}{x}dx + c_1\right)$
- Evaluate the integrals on the rhs

  $u(x) = x(-\ln(x) + c_1)$
- Solve 1st ODE for $u(x)$

  $u(x) = x(-\ln(x) + c_1)$
- Make substitution $u = y'$

  $y' = x(-\ln(x) + c_1)$
- Integrate both sides to solve for y

  $\int y' dx = \int x(-\ln(x) + c_1)\,dx + c_2$
- Compute integrals

  $y = -\frac{\ln(x)x^2}{2} + \frac{x^2}{4} + \frac{c_1 x^2}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_b(_a)-_a)/_a, _b(_a)`   *** Sublevel
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+x=diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -\frac{\ln(x)\, x^2}{2} + \frac{(1 + 2c_1)\, x^2}{4} + c_2$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 30

```
DSolve[x*y''[x]+x==y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}x^2\log(x) + \frac{1}{4}(1 + 2c_1)x^2 + c_2$$

18.11 problem 11

Internal problem ID [2283]
Internal file name [`OUTPUT/2283_Tuesday_February_27_2024_08_24_02_AM_57528556/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x'' + x't = t^3}$$

18.11.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable x. Let

$$p(t) = x'$$

Then

$$p'(t) = x''$$

Hence the ode becomes

$$p'(t) + p(t)\,t - t^3 = 0$$

Which is now solve for $p(t)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(t) + p(t)p(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= t \\ q(t) &= t^3 \end{aligned}$$

Hence the ode is

$$p'(t) + p(t)\,t = t^3$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int t dt} \\ &= \mathrm{e}^{\frac{t^2}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu p) &= (\mu)\left(t^3\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{t^2}{2}} p\right) &= \left(\mathrm{e}^{\frac{t^2}{2}}\right)\left(t^3\right) \\ \mathrm{d}\left(\mathrm{e}^{\frac{t^2}{2}} p\right) &= \left(t^3 \mathrm{e}^{\frac{t^2}{2}}\right) \mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{\frac{t^2}{2}} p &= \int t^3 \mathrm{e}^{\frac{t^2}{2}} \,\mathrm{d}t \\ \mathrm{e}^{\frac{t^2}{2}} p &= \left(t^2 - 2\right) \mathrm{e}^{\frac{t^2}{2}} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{t^2}{2}}$ results in

$$p(t) = \mathrm{e}^{-\frac{t^2}{2}} \left(t^2 - 2\right) \mathrm{e}^{\frac{t^2}{2}} + c_1 \mathrm{e}^{-\frac{t^2}{2}}$$

which simplifies to

$$p(t) = t^2 - 2 + c_1 \mathrm{e}^{-\frac{t^2}{2}}$$

Since $p = x'$ then the new first order ode to solve is

$$x' = t^2 - 2 + c_1 \mathrm{e}^{-\frac{t^2}{2}}$$

Integrating both sides gives

$$\begin{aligned} x &= \int t^2 - 2 + c_1 \mathrm{e}^{-\frac{t^2}{2}} \,\mathrm{d}t \\ &= -2t + \frac{t^3}{3} + \frac{c_1 \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{\sqrt{2}\, t}{2}\right)}{2} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$x = -2t + \frac{t^3}{3} + \frac{c_1 \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{\sqrt{2}\, t}{2}\right)}{2} + c_2 \tag{1}$$

Verification of solutions

$$x = -2t + \frac{t^3}{3} + \frac{c_1 \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{\sqrt{2}\, t}{2}\right)}{2} + c_2$$

Verified OK.

18.11.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x't = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= t \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= t^2 + 2 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{t^2}{4} + \frac{1}{2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 584: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} + \frac{1}{2t} - \frac{1}{4t^3} + \frac{1}{4t^5} - \frac{5}{16t^7} + \frac{7}{16t^9} - \frac{21}{32t^{11}} + \frac{33}{32t^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2+2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{t^2}{4} + \frac{1}{2}\right) + (0) \\
&= \frac{t^2}{4} + \frac{1}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{t}$ in the quotient is $\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{1}{2}\right) - (0) \\
&= \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{t}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2}{4} + \frac{1}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{2}$	0	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{t}{2}\right) \\ &= \frac{t}{2} \\ &= \frac{t}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{t}{2}\right)(0) + \left(\left(\frac{1}{2}\right) + \left(\frac{t}{2}\right)^2 - \left(\frac{t^2}{4} + \frac{1}{2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \frac{t}{2} dt} \\ &= e^{\frac{t^2}{4}} \end{aligned}$$

The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{1}\,dt} \\
&= z_1 e^{-\frac{t^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = 1
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{t}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-\frac{t^2}{2}}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(1) + c_2 \left(1 \left(\frac{\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+x't=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 + \frac{c_2\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}$$

The particular solution x_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$x_p(t) = u_1x_1 + u_2x_2 \tag{1}$$

Where u_1, u_2 to be determined, and x_1, x_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$x_1 = 1$$

$$x_2 = \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{x_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{x_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of x'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ \frac{d}{dt}(1) & \frac{d}{dt}\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ 0 & \mathrm{e}^{-\frac{t^2}{2}} \end{vmatrix}$$

Therefore

$$W = (1)\left(\mathrm{e}^{-\frac{t^2}{2}}\right) - \left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2}\right)(0)$$

Which simplifies to

$$W = \mathrm{e}^{-\frac{t^2}{2}}$$

Which simplifies to

$$W = \mathrm{e}^{-\frac{t^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)t^3}{2}}{\mathrm{e}^{-\frac{t^2}{2}}}\,dt$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)t^3\mathrm{e}^{\frac{t^2}{2}}}{2}dt$$

Hence

$$u_1 = -\sqrt{\pi}\left(2\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)\sqrt{2}\left(\frac{t^2\mathrm{e}^{\frac{t^2}{2}}}{4} - \frac{\mathrm{e}^{\frac{t^2}{2}}}{2}\right) - \frac{2\sqrt{2}\left(\frac{t^3\sqrt{2}}{12} - \frac{\sqrt{2}\,t}{2}\right)}{\sqrt{\pi}}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{t^3}{\mathrm{e}^{-\frac{t^2}{2}}}\,dt$$

Which simplifies to

$$u_2 = \int t^3\mathrm{e}^{\frac{t^2}{2}}\,dt$$

Hence

$$u_2 = \left(t^2-2\right) \mathrm{e}^{\frac{t^2}{2}}$$

Which simplifies to

$$u_1 = \frac{\left(-3\sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\, t}{2}\right) \left(t^2-2\right) \mathrm{e}^{\frac{t^2}{2}} + \sqrt{2}\, t(t^2-6)\right) \sqrt{2}}{6}$$

$$u_2 = \left(t^2-2\right) \mathrm{e}^{\frac{t^2}{2}}$$

Therefore the particular solution, from equation (1) is

$$x_p(t) = \frac{\left(-3\sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\, t}{2}\right) \left(t^2-2\right) \mathrm{e}^{\frac{t^2}{2}} + \sqrt{2}\, t(t^2-6)\right) \sqrt{2}}{6} + \frac{\left(t^2-2\right) \mathrm{e}^{\frac{t^2}{2}} \sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\, t}{2}\right)}{2}$$

Which simplifies to

$$x_p(t) = \frac{1}{3}t^3 - 2t$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 + \frac{c_2\sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\, t}{2}\right)}{2}\right) + \left(\frac{1}{3}t^3 - 2t\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$x = c_1 + \frac{c_2\sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\, t}{2}\right)}{2} + \frac{t^3}{3} - 2t \tag{1}$$

<u>Verification of solutions</u>

$$x = c_1 + \frac{c_2\sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\, t}{2}\right)}{2} + \frac{t^3}{3} - 2t$$

Verified OK.

18.11.3 Maple step by step solution

Let's solve

$x'' + x't = t^3$

- Highest derivative means the order of the ODE is 2

  x''

- Make substitution $u = x'$ to reduce order of ODE

  $u'(t) + u(t)\, t = t^3$

- Isolate the derivative

  $u'(t) = -u(t)\, t + t^3$

- Group terms with $u(t)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(t) + u(t)\, t = t^3$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\, (u'(t) + u(t)\, t) = \mu(t)\, t^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, u(t))$

  $\mu(t)\, (u'(t) + u(t)\, t) = \mu'(t)\, u(t) + \mu(t)\, u'(t)$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)\, t$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{\frac{t^2}{2}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, u(t))\right) dt = \int \mu(t)\, t^3 dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, u(t) = \int \mu(t)\, t^3 dt + c_1$

- Solve for $u(t)$

  $u(t) = \frac{\int \mu(t) t^3 dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{\frac{t^2}{2}}$

  $u(t) = \frac{\int t^3 \mathrm{e}^{\frac{t^2}{2}} dt + c_1}{\mathrm{e}^{\frac{t^2}{2}}}$

- Evaluate the integrals on the rhs

$$u(t) = \frac{(t^2-2)\,e^{\frac{t^2}{2}} + c_1}{e^{\frac{t^2}{2}}}$$

- Simplify

$$u(t) = t^2 - 2 + c_1 e^{-\frac{t^2}{2}}$$

- Solve 1st ODE for $u(t)$

$$u(t) = t^2 - 2 + c_1 e^{-\frac{t^2}{2}}$$

- Make substitution $u = x'$

$$x' = t^2 - 2 + c_1 e^{-\frac{t^2}{2}}$$

- Integrate both sides to solve for x

$$\int x' dt = \int \left(t^2 - 2 + c_1 e^{-\frac{t^2}{2}}\right) dt + c_2$$

- Compute integrals

$$x = -2t + \frac{t^3}{3} + \frac{c_1 \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right)}{2} + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^3-_a*_b(_a), _b(_a)`   *** Sublevel
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(x(t),t$2)+t*diff(x(t),t)=t^3,x(t), singsol=all)
```

$$x(t) = \frac{t^3}{3} + \frac{c_1 \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)}{2} - 2t + c_2$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 38

```
DSolve[x''[t]+t*x'[t]==t^3,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \sqrt{\frac{\pi}{2}} c_1 \operatorname{erf}\left(\frac{t}{\sqrt{2}}\right) + \frac{t^3}{3} - 2t + c_2$$

18.12 problem 12

Internal problem ID [2284]
Internal file name [`OUTPUT/2284_Tuesday_February_27_2024_08_24_02_AM_55812482/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^2y'' - xy' = 1}$$

18.12.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 0, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - xy' = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + 0 = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + 0 = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 0 = 0$$

Or

$$r^2 - 2r = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 0$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^2 + c_1$$

Next, we find the particular solution to the ODE

$$x^2y'' - xy' = 1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x^2 \\ \frac{d}{dx}(1) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix}$$

Therefore

$$W = (1)(2x) - (x^2)(0)$$

Which simplifies to

$$W = 2x$$

Which simplifies to

$$W = 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2}{2x^3} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{2x} dx$$

Hence

$$u_1 = -\frac{\ln(x)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{2x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{2x^3} dx$$

Hence

$$u_2 = -\frac{1}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{2} - \frac{1}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -\frac{\ln(x)}{2} - \frac{1}{4} + c_2 x^2 + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{\ln(x)}{2} - \frac{1}{4} + c_2 x^2 + c_1 \tag{1}$$

Verification of solutions

$$y = -\frac{\ln(x)}{2} - \frac{1}{4} + c_2 x^2 + c_1$$

Verified OK.

18.12.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^2 p'(x) - x p(x) - 1 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = \frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu p) &= (\mu)\left(\frac{1}{x^2}\right) \\
\frac{d}{dx}\left(\frac{p}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right) \\
d\left(\frac{p}{x}\right) &= \frac{1}{x^3}\,dx
\end{aligned}$$

Integrating gives

$$\frac{p}{x} = \int \frac{1}{x^3}\, dx$$
$$\frac{p}{x} = -\frac{1}{2x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = -\frac{1}{2x} + c_1 x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{1}{2x} + c_1 x$$

Integrating both sides gives

$$y = \int \frac{2c_1x^2 - 1}{2x}\, dx$$
$$= \frac{c_1 x^2}{2} - \frac{\ln(x)}{2} + c_2$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^2}{2} - \frac{\ln(x)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2}{2} - \frac{\ln(x)}{2} + c_2$$

Verified OK.

18.12.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - xy' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = x^2$$
$$B = -x \tag{3}$$
$$C = 0$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 586: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - xy' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 x^2}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \frac{x^2}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{x^2}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ 0 & x \end{vmatrix}$$

Therefore

$$W = (1)(x) - \left(\frac{x^2}{2}\right)(0)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2}{2}}{x^3} dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{2x} dx$$

Hence

$$u_1 = -\frac{\ln(x)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{x^3} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^3} dx$$

Hence

$$u_2 = -\frac{1}{2x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{2} - \frac{1}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + \frac{c_2 x^2}{2}\right) + \left(-\frac{\ln(x)}{2} - \frac{1}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + \frac{c_2 x^2}{2} - \frac{\ln(x)}{2} - \frac{1}{4} \tag{1}
$$

Verification of solutions

$$
y = c_1 + \frac{c_2 x^2}{2} - \frac{\ln(x)}{2} - \frac{1}{4}
$$

Verified OK.

18.12.4 Maple step by step solution

Let's solve

$x^2 y'' - x y' = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $x^2 u'(x) - x u(x) = 1$

- Isolate the derivative

  $u'(x) = \frac{u(x)}{x} + \frac{1}{x^2}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) - \frac{u(x)}{x} = \frac{1}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x) - \frac{u(x)}{x}\right) = \frac{\mu(x)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x) - \frac{u(x)}{x}\right) = \mu'(x)\,u(x) + \mu(x)\,u'(x)$

- Isolate $\mu'(x)$

$\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

$\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)\, u(x))\right) dx = \int \frac{\mu(x)}{x^2} dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\, u(x) = \int \frac{\mu(x)}{x^2} dx + c_1$

- Solve for $u(x)$

$u(x) = \frac{\int \frac{\mu(x)}{x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

$u(x) = x\left(\int \frac{1}{x^3} dx + c_1\right)$

- Evaluate the integrals on the rhs

$u(x) = x\left(-\frac{1}{2x^2} + c_1\right)$

- Solve 1st ODE for $u(x)$

$u(x) = x\left(-\frac{1}{2x^2} + c_1\right)$

- Make substitution $u = y'$

$y' = x\left(-\frac{1}{2x^2} + c_1\right)$

- Integrate both sides to solve for y

$\int y' dx = \int x\left(-\frac{1}{2x^2} + c_1\right) dx + c_2$

- Compute integrals

$y = \frac{c_1 x^2}{2} - \frac{\ln(x)}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_a*_b(_a)+1)/_a^2, _b(_a)`   *** Suble
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x^2*diff(y(x),x$2)=x*diff(y(x),x)+1,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2}{2} - \frac{\ln(x)}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 23

```
DSolve[x^2*y''[x]==x*y'[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 x^2}{2} - \frac{\log(x)}{2} + c_2$$

18.13 problem 13

Internal problem ID [2285]
Internal file name [`OUTPUT/2285_Tuesday_February_27_2024_08_24_03_AM_7132034/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' - {y'}^2 = 1}$$

18.13.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 1 - p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p^2 + 1} dp = x + c_1$$
$$\arctan(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = \tan(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \tan(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \tan(x + c_1)\, \mathrm{d}x \\ &= \frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2$$

Verified OK.

18.13.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = 1$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int \frac{p}{p^2+1}dp = \int dy$$
$$\frac{\ln(p^2+1)}{2} = y + c_1$$

Raising both side to exponential gives

$$\sqrt{p^2+1} = \mathrm{e}^{y+c_1}$$

Which simplifies to

$$\sqrt{p^2+1} = c_2\mathrm{e}^{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \mathrm{RootOf}\left(\_Z^2 - c_2^2\mathrm{e}^{2y} + 1\right)$$

Integrating both sides gives

$$\int \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2^2\mathrm{e}^{2y} + 1\right)}dy = \int dx$$
$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2^2\mathrm{e}^{2\_a} + 1\right)}d\_a = x + c_3$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2^2\mathrm{e}^{2\_a} + 1\right)}d\_a = x + c_3 \tag{1}$$

Verification of solutions

$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 - c_2^2\mathrm{e}^{2\_a} + 1\right)}d\_a = x + c_3$$

Verified OK.

18.13.3 Maple step by step solution

Let's solve

$y'' - y'^2 = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - u(x)^2 = 1$

- Separate variables

  $\frac{u'(x)}{u(x)^2+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2+1} dx = \int 1 dx + c_1$

- Evaluate integral

  $\arctan(u(x)) = x + c_1$

- Solve for $u(x)$

  $u(x) = \tan(x + c_1)$

- Solve 1st ODE for $u(x)$

  $u(x) = \tan(x + c_1)$

- Make substitution $u = y'$

  $y' = \tan(x + c_1)$

- Integrate both sides to solve for y

  $\int y' dx = \int \tan(x + c_1) dx + c_2$

- Compute integrals

  $y = \frac{\ln\left(1+\tan(x+c_1)^2\right)}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)=1+diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -\ln(-c_2 \cos(x) + c_1 \sin(x))$$

✓ Solution by Mathematica

Time used: 1.736 (sec). Leaf size: 16

```
DSolve[y''[x]==1+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \log(\cos(x + c_1))$$

18.14 problem 14

Internal problem ID [2286]
Internal file name [`OUTPUT/2286_Tuesday_February_27_2024_08_24_04_AM_38726562/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(-x^2+1)\, y''+x y'=1}$$

18.14.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$(-x^2+1)\, p'(x)+x p(x)-1=0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x)+p(x)p(x)=q(x)$$

Where here

$$p(x) = -\frac{x}{x^2 - 1}$$
$$q(x) = -\frac{1}{x^2 - 1}$$

Hence the ode is

$$p'(x) - \frac{xp(x)}{x^2 - 1} = -\frac{1}{x^2 - 1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{x}{x^2-1}dx} \\ &= \mathrm{e}^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(-\frac{1}{x^2-1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{\sqrt{x-1}\,\sqrt{x+1}}\right) &= \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)\left(-\frac{1}{x^2-1}\right) \\ \mathrm{d}\left(\frac{p}{\sqrt{x-1}\,\sqrt{x+1}}\right) &= \left(-\frac{1}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{\sqrt{x-1}\,\sqrt{x+1}} &= \int -\frac{1}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}\,\mathrm{d}x \\ \frac{p}{\sqrt{x-1}\,\sqrt{x+1}} &= \frac{\sqrt{x-1}\,\sqrt{x+1}\,x}{x^2-1} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$ results in

$$p(x) = \frac{(x-1)(x+1)\,x}{x^2-1} + c_1\sqrt{x-1}\,\sqrt{x+1}$$

which simplifies to

$$p(x) = x + c_1\sqrt{x-1}\,\sqrt{x+1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x + c_1\sqrt{x-1}\,\sqrt{x+1}$$

Integrating both sides gives

$$\begin{aligned}y &= \int x + c_1\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{d}x\\ &= \frac{x^2}{2} + c_1\left(\frac{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}{2} - \frac{\sqrt{x-1}\,\sqrt{x+1}}{2} - \frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{2\sqrt{x+1}\,\sqrt{x-1}}\right) + c_2\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + c_1\left(\frac{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}{2} - \frac{\sqrt{x-1}\,\sqrt{x+1}}{2} - \frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{2\sqrt{x+1}\,\sqrt{x-1}}\right) + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{2} + c_1\left(\frac{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}{2} - \frac{\sqrt{x-1}\,\sqrt{x+1}}{2} - \frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{2\sqrt{x+1}\,\sqrt{x-1}}\right) + c_2$$

Verified OK.

18.14.2 Solving using Kovacic algorithm

Writing the ode as

$$\left(-x^2+1\right)y'' + xy' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= -x^2+1\\ B &= x\\ C &= 0\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 + 2}{4(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 + 2 \\ t &= 4(x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 + 2}{4(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 589: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(x+1)^2} + \frac{5}{16(x-1)^2} - \frac{1}{16(x+1)} + \frac{1}{16x-16}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2+2}{4(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{3x^2+2}{4\left(x^2-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
-1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= -\frac{1}{2}-\left(-\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(x+1\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(x+1\right)} \\
&= -\frac{x}{2x^2-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(x+1\right)}\right)(0) + \left(\left(\frac{1}{4\left(x-1\right)^2} + \frac{1}{4\left(x+1\right)^2}\right) + \left(-\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(x+1\right)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{4(x-1)} - \frac{1}{4(x+1)}\right) dx} \\
&= \frac{1}{\left(x-1\right)^{\frac{1}{4}} \left(x+1\right)^{\frac{1}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{-x^2+1} \, dx} \\
&= z_1 e^{\frac{\ln(x-1)}{4} + \frac{\ln(x+1)}{4}} \\
&= z_1 \left(\left(x-1\right)^{\frac{1}{4}} \left(x+1\right)^{\frac{1}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{x\sqrt{x^2-1}}{2} - \frac{\ln\left(x+\sqrt{x^2-1}\right)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}}\right) + c_2 \left(\frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}} \left(\frac{x\sqrt{x^2-1}}{2} - \frac{\ln\left(x+\sqrt{x^2-1}\right)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$\left(-x^2+1\right) y'' + xy' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 (x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}} - \frac{c_2 (x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}} \left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right)}{2\left(x^2-1\right)^{\frac{1}{4}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}}$$

$$y_2 = -\frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}} \left(-x\sqrt{x^2-1} + \ln\left(x+\sqrt{x^2-1}\right)\right)}{2\,(x^2-1)^{\frac{1}{4}}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}} & -\frac{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}\left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right)}{2(x^2-1)^{\frac{1}{4}}} \\ \frac{d}{dx}\left(\frac{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}}\right) & \frac{d}{dx}\left(-\frac{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}\left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right)}{2(x^2-1)^{\frac{1}{4}}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}} \\ -\frac{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}x}{2(x^2-1)^{\frac{5}{4}}} + \frac{(x+1)^{\frac{1}{4}}}{4(x^2-1)^{\frac{1}{4}}(x-1)^{\frac{3}{4}}} + \frac{(x-1)^{\frac{1}{4}}}{4(x^2-1)^{\frac{1}{4}}(x+1)^{\frac{3}{4}}} & -\frac{(x+1)^{\frac{1}{4}}\left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right)}{8(x-1)^{\frac{3}{4}}(x^2-1)^{\frac{1}{4}}} - \frac{(x-1)^{\frac{1}{4}}\left(-x\sqrt{x^2-1}+\right.}{8(x+1)^{\frac{3}{4}}(x} \end{vmatrix}$$

Therefore

$$
\begin{aligned}
W &= \left(\frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}}\right) \left(-\frac{(x+1)^{\frac{1}{4}} \left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right)}{8\,(x-1)^{\frac{3}{4}} (x^2-1)^{\frac{1}{4}}} \right. \\
&\quad - \frac{(x-1)^{\frac{1}{4}} \left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right)}{8\,(x+1)^{\frac{3}{4}} (x^2-1)^{\frac{1}{4}}} \\
&\quad - \frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}} \left(-\sqrt{x^2-1}-\frac{x^2}{\sqrt{x^2-1}}+\frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}\right)}{2\,(x^2-1)^{\frac{1}{4}}} \\
&\quad \left. + \frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}} \left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right) x}{4\,(x^2-1)^{\frac{5}{4}}}\right) \\
&\quad - \left(-\frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}} \left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right)}{2\,(x^2-1)^{\frac{1}{4}}}\right) \left(-\frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}}\, x}{2\,(x^2-1)^{\frac{5}{4}}} \right. \\
&\quad \left. + \frac{(x+1)^{\frac{1}{4}}}{4\,(x^2-1)^{\frac{1}{4}} (x-1)^{\frac{3}{4}}} + \frac{(x-1)^{\frac{1}{4}}}{4\,(x^2-1)^{\frac{1}{4}} (x+1)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
W = \sqrt{x-1}\,\sqrt{x+1}
$$

Which simplifies to

$$
W = \sqrt{x-1}\,\sqrt{x+1}
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{-\frac{(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}} \left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right)}{2(x^2-1)^{\frac{1}{4}}}}{(-x^2+1)\sqrt{x-1}\,\sqrt{x+1}}\, dx
$$

Which simplifies to

$$
u_1 = -\int \frac{-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)}{2\,(x-1)^{\frac{1}{4}} (x+1)^{\frac{1}{4}} (x^2-1)^{\frac{5}{4}}} dx
$$

Hence

$$u_1 = -\left(\int_0^x \frac{-\alpha\sqrt{\alpha^2-1}+\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{2\left(\alpha-1\right)^{\frac{1}{4}}\left(\alpha+1\right)^{\frac{1}{4}}\left(\alpha^2-1\right)^{\frac{5}{4}}}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}}}{\left(-x^2+1\right)\sqrt{x-1}\,\sqrt{x+1}}dx$$

Which simplifies to

$$u_2 = \int -\frac{1}{\left(x-1\right)^{\frac{1}{4}}\left(x+1\right)^{\frac{1}{4}}\left(x^2-1\right)^{\frac{5}{4}}}dx$$

Hence

$$u_2 = \frac{\left(x-1\right)^{\frac{3}{4}}\left(x+1\right)^{\frac{3}{4}}x}{\left(x^2-1\right)^{\frac{5}{4}}}$$

Which simplifies to

$$u_1 = \frac{\left(\int_0^x \frac{\alpha\sqrt{\alpha^2-1}-\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha-1)^{\frac{1}{4}}(\alpha+1)^{\frac{1}{4}}(\alpha^2-1)^{\frac{5}{4}}}d\alpha\right)}{2}$$

$$u_2 = \frac{\left(x-1\right)^{\frac{3}{4}}\left(x+1\right)^{\frac{3}{4}}x}{\left(x^2-1\right)^{\frac{5}{4}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\int_0^x \frac{\alpha\sqrt{\alpha^2-1}-\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha-1)^{\frac{1}{4}}(\alpha+1)^{\frac{1}{4}}(\alpha^2-1)^{\frac{5}{4}}}d\alpha\right)\left(x-1\right)^{\frac{1}{4}}\left(x+1\right)^{\frac{1}{4}}}{2\left(x^2-1\right)^{\frac{1}{4}}} - \frac{\left(x-1\right)\left(x+1\right)x\left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right)}{2\left(x^2-1\right)^{\frac{3}{2}}}$$

Which simplifies to

$$y_p(x) = \frac{\left(\int_0^x \frac{\alpha\sqrt{\alpha^2-1}-\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha-1)^{\frac{1}{4}}(\alpha+1)^{\frac{1}{4}}(\alpha^2-1)^{\frac{5}{4}}}d\alpha\right)\left(x-1\right)^{\frac{1}{4}}\left(x+1\right)^{\frac{1}{4}}\left(x^2-1\right)^{\frac{1}{4}}+x^2\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)x}{2\sqrt{x^2-1}}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(\frac{c_1(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}}{(x^2-1)^{\frac{1}{4}}} - \frac{c_2(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}\left(-x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right)}{2\left(x^2-1\right)^{\frac{1}{4}}}\right)$$

$$+ \left(\frac{\left(\int_0^x \frac{\alpha\sqrt{\alpha^2-1}-\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha-1)^{\frac{1}{4}}(\alpha+1)^{\frac{1}{4}}(\alpha^2-1)^{\frac{5}{4}}}d\alpha\right)(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}(x^2-1)^{\frac{1}{4}} + x^2\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)x}{2\sqrt{x^2-1}}\right)$$

Which simplifies to

$$y = \frac{(x+1)^{\frac{1}{4}}(x-1)^{\frac{1}{4}}\left(\sqrt{x^2-1}\,c_2x - \ln\left(x+\sqrt{x^2-1}\right)c_2 + 2c_1\right)}{2\left(x^2-1\right)^{\frac{1}{4}}}$$
$$+ \frac{\left(\int_0^x \frac{\alpha\sqrt{\alpha^2-1}-\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha-1)^{\frac{1}{4}}(\alpha+1)^{\frac{1}{4}}(\alpha^2-1)^{\frac{5}{4}}}d\alpha\right)(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}(x^2-1)^{\frac{1}{4}} + x^2\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)x}{2\sqrt{x^2-1}}$$

Summary

The solution(s) found are the following

$$y = \frac{(x+1)^{\frac{1}{4}}(x-1)^{\frac{1}{4}}\left(\sqrt{x^2-1}\,c_2x - \ln\left(x+\sqrt{x^2-1}\right)c_2 + 2c_1\right)}{2\left(x^2-1\right)^{\frac{1}{4}}} \tag{1}$$
$$+ \frac{\left(\int_0^x \frac{\alpha\sqrt{\alpha^2-1}-\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha-1)^{\frac{1}{4}}(\alpha+1)^{\frac{1}{4}}(\alpha^2-1)^{\frac{5}{4}}}d\alpha\right)(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}(x^2-1)^{\frac{1}{4}} + x^2\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)x}{2\sqrt{x^2-1}}$$

Verification of solutions

$$y = \frac{(x+1)^{\frac{1}{4}}(x-1)^{\frac{1}{4}}\left(\sqrt{x^2-1}\,c_2x - \ln\left(x+\sqrt{x^2-1}\right)c_2 + 2c_1\right)}{2\left(x^2-1\right)^{\frac{1}{4}}}$$
$$+ \frac{\left(\int_0^x \frac{\alpha\sqrt{\alpha^2-1}-\ln\left(\alpha+\sqrt{\alpha^2-1}\right)}{(\alpha-1)^{\frac{1}{4}}(\alpha+1)^{\frac{1}{4}}(\alpha^2-1)^{\frac{5}{4}}}d\alpha\right)(x-1)^{\frac{1}{4}}(x+1)^{\frac{1}{4}}(x^2-1)^{\frac{1}{4}} + x^2\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)x}{2\sqrt{x^2-1}}$$

Verified OK.

18.14.3 Maple step by step solution

Let's solve

$(-x^2+1)\,y''+xy'=1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $(-x^2+1)\,u'(x)+xu(x)=1$

- Isolate the derivative

  $u'(x)=\frac{xu(x)}{x^2-1}-\frac{1}{x^2-1}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x)-\frac{xu(x)}{x^2-1}=-\frac{1}{x^2-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x)-\frac{xu(x)}{x^2-1}\right)=-\frac{\mu(x)}{x^2-1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)-\frac{xu(x)}{x^2-1}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)x}{x^2-1}$

- Solve to find the integrating factor

  $\mu(x)=\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int-\frac{\mu(x)}{x^2-1}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int-\frac{\mu(x)}{x^2-1}dx+c_1$

- Solve for $u(x)$

  $u(x)=\frac{\int-\frac{\mu(x)}{x^2-1}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$

  $u(x)=\sqrt{x-1}\,\sqrt{x+1}\left(\int-\frac{1}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}dx+c_1\right)$

- Evaluate the integrals on the rhs

$$u(x) = \sqrt{x-1}\,\sqrt{x+1}\left(\frac{\sqrt{x-1}\,\sqrt{x+1}\,x}{x^2-1}+c_1\right)$$

- Simplify

$$u(x) = \frac{\sqrt{x-1}\,\sqrt{x+1}\,(\sqrt{x-1}\,\sqrt{x+1}\,x+c_1(x^2-1))}{x^2-1}$$

- Solve 1st ODE for $u(x)$

$$u(x) = \frac{\sqrt{x-1}\,\sqrt{x+1}\,(\sqrt{x-1}\,\sqrt{x+1}\,x+c_1(x^2-1))}{x^2-1}$$

- Make substitution $u = y'$

$$y' = \frac{\sqrt{x-1}\,\sqrt{x+1}\,(\sqrt{x-1}\,\sqrt{x+1}\,x+c_1(x^2-1))}{x^2-1}$$

- Integrate both sides to solve for y

$$\int y'dx = \int \frac{\sqrt{x-1}\,\sqrt{x+1}\,(\sqrt{x-1}\,\sqrt{x+1}\,x+c_1(x^2-1))}{x^2-1}dx + c_2$$

- Compute integrals

$$y = \frac{x^2}{2} + c_1\left(\frac{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}{2} - \frac{\sqrt{x-1}\,\sqrt{x+1}}{2} - \frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{2\sqrt{x+1}\,\sqrt{x-1}}\right) + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_a*_b(_a)-1)/((_a-1)*(_a+1)), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
dsolve((1-x^2)*diff(y(x),x$2)+x*diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = -\frac{-\sqrt{x+1}\,(x^2+2c_2)\sqrt{x-1}+c_1\left(-x^3+\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)+x\right)}{2\sqrt{x-1}\,\sqrt{x+1}}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 50

```
DSolve[(1-x^2)*y''[x]+x*y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-2c_1\text{arctanh}\left(\frac{\sqrt{x^2-1}}{x-1}\right)+x^2+c_1\sqrt{x^2-1}x+2c_2\right)$$

18.15 problem 15

Internal problem ID [2287]
Internal file name [`OUTPUT/2287_Tuesday_February_27_2024_08_24_06_AM_57231313/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 15.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - \sqrt{1 + y'^2} = 0}$$

18.15.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - \sqrt{1 + p(x)^2} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{\sqrt{p^2 + 1}} dp = x + c_1$$
$$\operatorname{arcsinh}(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = \sinh(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \sinh(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \sinh(x + c_1)\, \mathrm{d}x \\ &= \cosh(x + c_1) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cosh(x + c_1) + c_2 \tag{1}$$

Verification of solutions

$$y = \cosh(x + c_1) + c_2$$

Verified OK.

18.15.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) = \sqrt{1 + p(y)^2}$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int \frac{p}{\sqrt{p^2+1}}dp = \int dy$$
$$\sqrt{1+p(y)^2} = y + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\sqrt{1+y'^2} = y + c_1$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-1+c_1^2+2yc_1+y^2} \tag{1}$$
$$y' = -\sqrt{-1+c_1^2+2yc_1+y^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{c_1^2+2c_1y+y^2-1}}dy = \int dx$$
$$\ln\left(y+c_1+\sqrt{c_1^2+2c_1y+y^2-1}\right) = x + c_2$$

Raising both side to exponential gives

$$y+c_1+\sqrt{c_1^2+2c_1y+y^2-1} = e^{x+c_2}$$

Which simplifies to

$$y+c_1+\sqrt{(y+c_1+1)(y+c_1-1)} = c_3e^x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{c_1^2+2c_1y+y^2-1}}dy = \int dx$$
$$-\ln\left(y+c_1+\sqrt{c_1^2+2c_1y+y^2-1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y + c_1 + \sqrt{c_1^2 + 2c_1 y + y^2 - 1}} = \mathrm{e}^{x + c_4}$$

Which simplifies to

$$\frac{1}{y + c_1 + \sqrt{(y + c_1 + 1)(y + c_1 - 1)}} = c_5 \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = \frac{(c_3^2 \mathrm{e}^{2x} - 2\,\mathrm{e}^{x} c_1 c_3 + 1)\,\mathrm{e}^{-x}}{2c_3} \tag{1}$$

$$y = -\frac{(-c_5^2 \mathrm{e}^{2x} + 2c_1 c_5 \mathrm{e}^{x} - 1)\,\mathrm{e}^{-x}}{2c_5} \tag{2}$$

Verification of solutions

$$y = \frac{(c_3^2 \mathrm{e}^{2x} - 2\,\mathrm{e}^{x} c_1 c_3 + 1)\,\mathrm{e}^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{(-c_5^2 \mathrm{e}^{2x} + 2c_1 c_5 \mathrm{e}^{x} - 1)\,\mathrm{e}^{-x}}{2c_5}$$

Verified OK.

18.15.3 Maple step by step solution

Let's solve

$y'' = \sqrt{1 + y'^2}$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) = \sqrt{u(x)^2 + 1}$

- Separate variables

  $\frac{u'(x)}{\sqrt{u(x)^2+1}} = 1$

- Integrate both sides with respect to x

$$\int \frac{u'(x)}{\sqrt{u(x)^2+1}}dx = \int 1dx + c_1$$

- Evaluate integral

$$\operatorname{arcsinh}(u(x)) = x + c_1$$

- Solve for $u(x)$

$$u(x) = \sinh(x + c_1)$$

- Solve 1st ODE for $u(x)$

$$u(x) = \sinh(x + c_1)$$

- Make substitution $u = y'$

$$y' = \sinh(x + c_1)$$

- Integrate both sides to solve for y

$$\int y'dx = \int \sinh(x + c_1)\, dx + c_2$$

- Compute integrals

$$y = \cosh(x + c_1) + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x)-(diff(y(x), x)), y(x)`   *
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_b(_a)^2+1)^(1/2), _b(_a), HINT = [[1,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

✓ Solution by Maple

Time used: 0.39 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)=sqrt(1+diff(y(x),x)^2),y(x), singsol=all)
```

$$y(x) = -ix + c_1$$
$$y(x) = ix + c_1$$
$$y(x) = \cosh(c_1 + x) + c_2$$

✓ Solution by Mathematica

Time used: 0.34 (sec). Leaf size: 29

```
DSolve[y''[x]==Sqrt[1+y'[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(e^{-x-c_1} + e^{x+c_1}\right) + c_2$$

18.16 problem 16

Internal problem ID [2288]
Internal file name [`OUTPUT/2288_Tuesday_February_27_2024_08_24_08_AM_7853255/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y", "second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_xy]]
```

$$\boxed{y'' - {y'}^2 - y' = 0}$$

18.16.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + (-p(x) - 1)\, p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p\left(p+1\right)}dp = \int dx$$
$$-\ln\left(p+1\right)+\ln\left(p\right) = x+c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(p+1)+\ln(p)} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$\frac{p}{p+1} = c_2\mathrm{e}^x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{c_2\mathrm{e}^x}{-1+c_2\mathrm{e}^x}$$

Integrating both sides gives

$$y = \int -\frac{c_2\mathrm{e}^x}{-1+c_2\mathrm{e}^x}\,\mathrm{d}x$$
$$= -\ln\left(-1+c_2\mathrm{e}^x\right)+c_3$$

Summary

The solution(s) found are the following

$$y = -\ln\left(-1+c_2\mathrm{e}^x\right)+c_3 \tag{1}$$

Verification of solutions

$$y = -\ln\left(-1+c_2\mathrm{e}^x\right)+c_3$$

Verified OK.

18.16.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right) + (-p(y) - 1)\, p(y) = 0
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{p+1}dp &= \int dy \\
\ln(p+1) &= y + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
p + 1 = \mathrm{e}^{y+c_1}
$$

Which simplifies to

$$
p + 1 = c_2 \mathrm{e}^{y}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = c_2 \mathrm{e}^{y} - 1
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{c_2 \mathrm{e}^{y} - 1}dy &= \int dx \\
\ln(c_2 \mathrm{e}^{y} - 1) - \ln(\mathrm{e}^{y}) &= x + c_3
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{\ln(c_2 \mathrm{e}^{y} - 1) - \ln(\mathrm{e}^{y})} = \mathrm{e}^{x+c_3}
$$

Which simplifies to

$$
c_2 - \mathrm{e}^{-y} = c_4 \mathrm{e}^{x}
$$

Summary
The solution(s) found are the following

$$y = -\ln\left(-c_4 e^x + c_2\right) \tag{1}$$

Verification of solutions

$$y = -\ln\left(-c_4 e^x + c_2\right)$$

Verified OK.

18.16.3 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$\begin{aligned} f(x) &= -1 \\ g(y) &= -1 \end{aligned}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln\left(y'\right) \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln\left(y'\right) + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln\left(y'\right) + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = -1$ and $f = -1$, then

$$\begin{aligned}\int -g dy &= \int 1 dy \\ &= y \\ \int -f dx &= \int 1 dx \\ &= x\end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = c_2 e^y e^x$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x) g(y) \\ &= c_2 e^y e^x\end{aligned}$$

Where $f(x) = c_2 e^x$ and $g(y) = e^y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{e^y}\, dy &= c_2 e^x\, dx \\ \int \frac{1}{e^y}\, dy &= \int c_2 e^x\, dx \\ -e^{-y} &= c_2 e^x + c_3\end{aligned}$$

The solution is

$$-e^{-y} - c_2 e^x - c_3 = 0$$

Summary

The solution(s) found are the following

$$-e^{-y} - c_2 e^x - c_3 = 0 \tag{1}$$

Verification of solutions

$$-e^{-y} - c_2 e^x - c_3 = 0$$

Verified OK.

18.16.4 Maple step by step solution

Let's solve

$y'' + (-y' - 1)\, y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + (-u(x) - 1)\, u(x) = 0$

- Separate variables

  $\frac{u'(x)}{(-u(x)-1)u(x)} = -1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{(-u(x)-1)u(x)} dx = \int (-1)\, dx + c_1$

- Evaluate integral

  $\ln(u(x) + 1) - \ln(u(x)) = -x + c_1$

- Solve for $u(x)$

  $u(x) = \frac{1}{\mathrm{e}^{-x+c_1} - 1}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{1}{\mathrm{e}^{-x+c_1} - 1}$

- Make substitution $u = y'$

  $y' = \frac{1}{\mathrm{e}^{-x+c_1} - 1}$

- Integrate both sides to solve for y

  $\int y' dx = \int \frac{1}{\mathrm{e}^{-x+c_1} - 1} dx + c_2$

- Compute integrals

  $y = -\ln(\mathrm{e}^{-x+c_1} - 1) + \ln(\mathrm{e}^{-x+c_1}) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)=diff(y(x),x)^2+diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -\ln\left(-e^{x} c_1 - c_2\right)$$

✓ Solution by Mathematica

Time used: 1.815 (sec). Leaf size: 31

```
DSolve[y''[x]==y'[x]^2+y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \log\left(-1 + e^{x+c_1}\right)$$
$$y(x) \to c_2 - i\pi$$

18.17 problem 17

Internal problem ID [2289]
Internal file name [`OUTPUT/2289_Tuesday_February_27_2024_08_24_09_AM_28455994/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
  _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{y'' - yy' = 0}$$

18.17.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - yy')\, dx = 0$$
$$-\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{\frac{y^2}{2}+c_1}dy = x+c_2$$
$$\frac{\sqrt{2}\,\arctan\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x+c_2$$

Solving for y gives these solutions

$$y_1 = \tan\left(\frac{\sqrt{c_1}\,(x+c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{\sqrt{c_1}\,(x+c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tan\left(\frac{\sqrt{c_1}\,(x+c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

18.17.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - yp(y) = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int y \, \mathrm{d}y \\ &= \frac{y^2}{2} + c_1 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{y^2}{2} + c_1$$

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{\frac{y^2}{2} + c_1} dy &= x + c_2 \\ \frac{\sqrt{2}\, \arctan\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} &= x + c_2 \end{aligned}$$

Solving for y gives these solutions

$$y_1 = \tan\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tan\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

18.17.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - yy' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - yy')\, dx = 0$$
$$-\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{\frac{y^2}{2} + c_1} dy = x + c_2$$
$$\frac{\sqrt{2}\, \arctan\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \tan\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tan\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

18.17.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\frac{\partial a_2}{\partial y} = \frac{\partial a_1}{\partial y'}$$
$$\frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial y'}$$
$$\frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial y}$$

Looking at the the ode given we see that

$$a_2 = 1$$
$$a_1 = -y$$
$$a_0 = 0$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx = c_1$$
$$\int 1\, dy' + \int -y\, dy + \int 0\, dx = c_1$$

Which results in

$$-\frac{y^2}{2} + y' = c_1$$

Which is now solved Integrating both sides gives

$$\int \frac{1}{\frac{y^2}{2} + c_1} dy = x + c_2$$
$$\frac{\sqrt{2}\, \arctan\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \tan\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary
The solution(s) found are the following

$$y = \tan\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tan\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

18.17.5 Maple step by step solution

Let's solve

$y'' - yy' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right) - yu(y) = 0$

- Separate variables

  $\frac{d}{dy}u(y) = y$

- Integrate both sides with respect to y

  $\int\left(\frac{d}{dy}u(y)\right)dy = \int y dy + c_1$

- Evaluate integral

  $u(y) = \frac{y^2}{2} + c_1$
- Solve for $u(y)$

  $u(y) = \frac{y^2}{2} + c_1$
- Solve 1st ODE for $u(y)$

  $u(y) = \frac{y^2}{2} + c_1$
- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{y^2}{2} + c_1$
- Separate variables

  $\frac{y'}{\frac{y^2}{2}+c_1} = 1$
- Integrate both sides with respect to x

  $\int \frac{y'}{\frac{y^2}{2}+c_1}dx = \int 1dx + c_2$
- Evaluate integral

  $\frac{\sqrt{2}\,\arctan\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$
- Solve for y

  $y = \tan\left(\frac{\sqrt{c_1}\,(x+c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-_a*_b(_a) = 0, _b(_a), HINT = [[
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)=y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{\tan\left(\frac{(c_2+x)\sqrt{2}}{2c_1}\right)\sqrt{2}}{c_1}$$

✓ Solution by Mathematica

Time used: 15.535 (sec). Leaf size: 34

```
DSolve[y''[x]==y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{2}\sqrt{c_1}\tan\left(\frac{\sqrt{c_1}(x+c_2)}{\sqrt{2}}\right)$$

18.18 problem 18

Internal problem ID [2290]
Internal file name [`OUTPUT/2290_Tuesday_February_27_2024_08_24_09_AM_55569383/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{\left(x^2+1\right) y''+y'^2=-1}$$

18.18.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$\left(x^2+1\right) p'(x)+1+p(x)^2=0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned}
p' &= F(x,p)\\
&= f(x)g(p)\\
&= \frac{-p^2-1}{x^2+1}
\end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(p) = -p^2 - 1$. Integrating both sides gives

$$\frac{1}{-p^2-1}\,dp = \frac{1}{x^2+1}\,dx$$
$$\int \frac{1}{-p^2-1}\,dp = \int \frac{1}{x^2+1}\,dx$$
$$-\arctan(p) = \arctan(x) + c_1$$

The solution is

$$-\arctan(p(x)) - \arctan(x) - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\arctan(y') - \arctan(x) - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\tan(\arctan(x) + c_1)\,\mathrm{d}x \\ &= \frac{i\mathrm{e}^{4ic_1}x}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{4\,\mathrm{e}^{2ic_1}\ln\left((-\mathrm{e}^{2ic_1}+1)\,x + i\mathrm{e}^{2ic_1} + i\right)}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{ix}{(\mathrm{e}^{2ic_1}-1)^2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{i\mathrm{e}^{4ic_1}x}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{4\,\mathrm{e}^{2ic_1}\ln\left((-\mathrm{e}^{2ic_1}+1)\,x + i\mathrm{e}^{2ic_1} + i\right)}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{ix}{(\mathrm{e}^{2ic_1}-1)^2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{i\mathrm{e}^{4ic_1}x}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{4\,\mathrm{e}^{2ic_1}\ln\left((-\mathrm{e}^{2ic_1}+1)\,x + i\mathrm{e}^{2ic_1} + i\right)}{(\mathrm{e}^{2ic_1}-1)^2} - \frac{ix}{(\mathrm{e}^{2ic_1}-1)^2} + c_2$$

Verified OK.

18.18.2 Maple step by step solution

Let's solve

$(x^2+1)\,y''+y'^2=-1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $(x^2+1)\,u'(x)+u(x)^2=-1$

- Separate variables

  $\frac{u'(x)}{-u(x)^2-1}=\frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{-u(x)^2-1}dx=\int \frac{1}{x^2+1}dx+c_1$

- Evaluate integral

  $-\arctan(u(x))=\arctan(x)+c_1$

- Solve for $u(x)$

  $u(x)=-\tan(\arctan(x)+c_1)$

- Solve 1st ODE for $u(x)$

  $u(x)=-\tan(\arctan(x)+c_1)$

- Make substitution $u=y'$

  $y'=-\tan(\arctan(x)+c_1)$

- Integrate both sides to solve for y

  $\int y'dx=\int -\tan(\arctan(x)+c_1)\,dx+c_2$

- Compute integrals

  $y=\frac{\mathrm{I}\,\mathrm{e}^{4\,\mathrm{I}c_1}x}{(\mathrm{e}^{2\,\mathrm{I}c_1}-1)^2}-\frac{4\,\mathrm{e}^{2\,\mathrm{I}c_1}\ln((-\mathrm{e}^{2\,\mathrm{I}c_1}+1)x+\mathrm{I}\,\mathrm{e}^{2\,\mathrm{I}c_1}+\mathrm{I})}{(\mathrm{e}^{2\,\mathrm{I}c_1}-1)^2}-\frac{\mathrm{I}x}{(\mathrm{e}^{2\,\mathrm{I}c_1}-1)^2}+c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)^2+1)/(_a^2+1), _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   <- separable successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve((1+x^2)*diff(y(x),x$2)+1+diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = \frac{\ln(c_1x-1)\,c_1^2 + c_1^2c_2 + c_1x + \ln(c_1x-1)}{c_1^2}$$

✓ Solution by Mathematica

Time used: 7.025 (sec). Leaf size: 33

```
DSolve[(1+x^2)*y''[x]+1+y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\cot(c_1) + \csc^2(c_1)\log(-x\sin(c_1) - \cos(c_1)) + c_2$$

18.19 problem 19

Internal problem ID [2291]
Internal file name [`OUTPUT/2291_Tuesday_February_27_2024_08_24_10_AM_26912678/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
   _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
    _reducible, _mu_xy]]
```

$$\boxed{y'' + yy' = 0}$$

18.19.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + yy')\,dx = 0$$
$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = x + c_2$$

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary

The solution(s) found are the following

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

18.19.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + yp(y) = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int -y \, \mathrm{d}y \\ &= -\frac{y^2}{2} + c_1 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{y^2}{2} + c_1$$

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{-\frac{y^2}{2} + c_1} dy &= x + c_2 \\ \frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} &= x + c_2 \end{aligned}$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary

The solution(s) found are the following

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

18.19.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + yy' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + yy')\, dx = 0$$

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = x + c_2$$

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary

The solution(s) found are the following

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

18.19.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'}\\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'}\\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= 1\\ a_1 &= y\\ a_0 &= 0\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1\\ \int 1\, dy' + \int y\, dy + \int 0\, dx &= c_1\end{aligned}$$

Which results in

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved Integrating both sides gives

$$\begin{aligned}\int \frac{1}{-\frac{y^2}{2} + c_1} dy &= x + c_2\\ \frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} &= x + c_2\end{aligned}$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Summary
The solution(s) found are the following

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Verification of solutions

$$y = \tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Verified OK.

18.19.5 Maple step by step solution

Let's solve

$y'' + yy' = 0$

- Highest derivative means the order of the ODE is 2

  y''
- Define new dependent variable u

  $u(x) = y'$
- Compute y''

  $u'(x) = y''$
- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$
- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$
- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right) + yu(y) = 0$
- Separate variables

  $\frac{d}{dy}u(y) = -y$
- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}u(y)\right) dy = \int -y dy + c_1$

- Evaluate integral

  $u(y) = -\frac{y^2}{2} + c_1$

- Solve for $u(y)$

  $u(y) = -\frac{y^2}{2} + c_1$

- Solve 1st ODE for $u(y)$

  $u(y) = -\frac{y^2}{2} + c_1$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{y^2}{2} + c_1$

- Separate variables

  $\frac{y'}{-\frac{y^2}{2}+c_1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-\frac{y^2}{2}+c_1} dx = \int 1dx + c_2$

- Evaluate integral

  $\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$

- Solve for y

  $y = \tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_a*_b(_a) = 0, _b(_a), HINT = [[
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\tanh\left(\frac{(c_2+x)\sqrt{2}}{2c_1}\right)\sqrt{2}}{c_1}$$

✓ Solution by Mathematica

Time used: 7.104 (sec). Leaf size: 34

```
DSolve[y''[x]+y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{2}\sqrt{c_1}\tanh\left(\frac{\sqrt{c_1}(x+c_2)}{\sqrt{2}}\right)$$

18.20 problem 20

Internal problem ID [2292]
Internal file name [`OUTPUT/2292_Tuesday_February_27_2024_08_24_10_AM_5372113/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_xy]]
```

$$\boxed{y'' + 2y'^2 = 0}$$

18.20.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 2p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{2p^2}dp = x + c_1$$
$$\frac{1}{2p} = x + c_1$$

Solving for p gives these solutions

$$p_1 = \frac{1}{2x + 2c_1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{1}{2x + 2c_1}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{2x + 2c_1}\, \mathrm{d}x \\ &= \frac{\ln(x + c_1)}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x + c_1)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x + c_1)}{2} + c_2$$

Verified OK.

18.20.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right)+2p(y)^2=0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int -\frac{1}{2p}dp = \int dy$$
$$-\frac{\ln(p)}{2} = y + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{p}} = \mathrm{e}^{y+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{p}} = c_2\mathrm{e}^{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\mathrm{e}^{-2y}}{c_2^2}$$

Integrating both sides gives

$$\int c_2^2\mathrm{e}^{2y}dy = x + c_3$$
$$\frac{c_2^2\mathrm{e}^{2y}}{2} = x + c_3$$

Solving for y gives these solutions

$$y_1 = \frac{\ln\left(\frac{2c_3+2x}{c_2^2}\right)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln\left(\frac{2c_3+2x}{c_2^2}\right)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\ln\left(\frac{2c_3+2x}{c_2^2}\right)}{2}$$

Verified OK.

18.20.3 Maple step by step solution

Let's solve

$y'' + 2y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + 2u(x)^2 = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^2} = -2$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2}dx = \int (-2)\,dx + c_1$

- Evaluate integral

  $-\frac{1}{u(x)} = -2x + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{1}{-2x+c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{1}{-2x+c_1}$

- Make substitution $u = y'$

  $y' = -\frac{1}{-2x+c_1}$

- Integrate both sides to solve for y

  $\int y'dx = \int -\frac{1}{-2x+c_1}dx + c_2$

- Compute integrals

  $y = \frac{\ln(-2x+c_1)}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = \frac{\ln(2)}{2} + \frac{\ln(c_1 x + c_2)}{2}$$

✓ Solution by Mathematica

Time used: 0.183 (sec). Leaf size: 21

```
DSolve[y''[x]+2*y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\log(2x - c_1) + c_2$$

18.21 problem 21

Internal problem ID [2293]
Internal file name [`OUTPUT/2293_Tuesday_February_27_2024_08_24_11_AM_38859244/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
   _Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{yy'' + {y'}^2 = 0}$$

18.21.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + {y'}^2\right) dx = 0$$
$$yy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{y}{c_1} dy = x + c_2$$
$$\frac{y^2}{2c_1} = x + c_2$$

Solving for y gives these solutions

$$
\begin{aligned}
y_1 &= \sqrt{2c_1c_2 + 2c_1x} \\
y_2 &= -\sqrt{2c_1c_2 + 2c_1x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}
$$

$$
y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}
$$

Verification of solutions

$$
y = \sqrt{2c_1c_2 + 2c_1x}
$$

Verified OK.

$$
y = -\sqrt{2c_1c_2 + 2c_1x}
$$

Verified OK.

18.21.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$
y' = p(y)
$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
yp(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = 0
$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y, p) \\
&= f(y)g(p) \\
&= -\frac{p}{y}
\end{aligned}
$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p} dp &= -\frac{1}{y} dy \\
\int \frac{1}{p} dp &= \int -\frac{1}{y} dy \\
\ln(p) &= -\ln(y) + c_1 \\
p &= e^{-\ln(y)+c_1} \\
&= \frac{c_1}{y}
\end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{c_1}{y}$$

Integrating both sides gives

$$\begin{aligned}
\int \frac{y}{c_1} dy &= x + c_2 \\
\frac{y^2}{2c_1} &= x + c_2
\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}
y_1 &= \sqrt{2c_1c_2 + 2c_1x} \\
y_2 &= -\sqrt{2c_1c_2 + 2c_1x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$

$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

18.21.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' + y'^2 = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + y'^2\right) dx = 0$$
$$yy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{y}{c_1} dy = x + c_2$$
$$\frac{y^2}{2c_1} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \sqrt{2c_1c_2 + 2c_1x}$$
$$y_2 = -\sqrt{2c_1c_2 + 2c_1x}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$
$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

18.21.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x,y,y')\,y'' + a_1(x,y,y')\,y' + a_0(x,y,y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'}\\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'}\\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= y\\ a_1 &= y'\\ a_0 &= 0\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2\,dy' + \int a_1\,dy + \int a_0\,dx &= c_1\\ \int y\,dy' + \int y'\,dy + \int 0\,dx &= c_1\end{aligned}$$

Which results in

$$2yy' = c_1$$

Which is now solved Integrating both sides gives

$$\begin{aligned}\int \frac{2y}{c_1} dy &= x + c_2\\ \frac{y^2}{c_1} &= x + c_2\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}y_1 &= \sqrt{c_1c_2 + c_1x}\\ y_2 &= -\sqrt{c_1c_2 + c_1x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1 c_2 + c_1 x} \tag{1}$$
$$y = -\sqrt{c_1 c_2 + c_1 x} \tag{2}$$

Verification of solutions

$$y = \sqrt{c_1 c_2 + c_1 x}$$

Verified OK.

$$y = -\sqrt{c_1 c_2 + c_1 x}$$

Verified OK.

18.21.5 Maple step by step solution

Let's solve

$y y'' + y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy} u(y)\right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy} u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y) \left(\frac{d}{dy} u(y)\right)$ to reduce order of ODE

  $y u(y) \left(\frac{d}{dy} u(y)\right) + u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy} u(y)}{u(y)} = -\frac{1}{y}$

- Integrate both sides with respect to y

$\int \frac{\frac{d}{dy}u(y)}{u(y)}dy = \int -\frac{1}{y}dy + c_1$

- Evaluate integral

  $\ln(u(y)) = -\ln(y) + c_1$

- Solve for $u(y)$

  $u(y) = \frac{e^{c_1}}{y}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{e^{c_1}}{y}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{e^{c_1}}{y}$

- Separate variables

  $yy' = e^{c_1}$

- Integrate both sides with respect to x

  $\int yy'dx = \int e^{c_1}dx + c_2$

- Evaluate integral

  $\frac{y^2}{2} = e^{c_1}x + c_2$

- Solve for y

  $\left\{y = \sqrt{2\,e^{c_1}x + 2c_2}, y = -\sqrt{2\,e^{c_1}x + 2c_2}\right\}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 33

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^2=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= 0 \\
y(x) &= \sqrt{2c_1x + 2c_2} \\
y(x) &= -\sqrt{2c_1x + 2c_2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 20

```
DSolve[y[x]*y''[x]+y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to c_2\sqrt{2x - c_1}
$$

18.22 problem 22

Internal problem ID [2294]
Internal file name [`OUTPUT/2294_Tuesday_February_27_2024_08_24_11_AM_88302303/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{yy'' - y'^2 = -1}$$

18.22.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = -1$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y, p) \\
&= f(y)g(p) \\
&= \frac{p^2 - 1}{yp}
\end{aligned}
$$

Where $f(y) = \frac{1}{y}$ and $g(p) = \frac{p^2-1}{p}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{p^2-1}{p}}\,dp &= \frac{1}{y}\,dy \\
\int \frac{1}{\frac{p^2-1}{p}}\,dp &= \int \frac{1}{y}\,dy \\
\frac{\ln(p-1)}{2} + \frac{\ln(p+1)}{2} &= \ln(y) + c_1
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\left(\frac{1}{2}\right)(\ln(p-1) + \ln(p+1)) &= \ln(y) + 2c_1 \\
\ln(p-1) + \ln(p+1) &= (2)(\ln(y) + 2c_1) \\
&= 2\ln(y) + 4c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(p-1)+\ln(p+1)} = e^{2\ln(y)+2c_1}
$$

Which simplifies to

$$
\begin{aligned}
p^2 - 1 &= 2c_1 y^2 \\
&= c_2 y^2
\end{aligned}
$$

The solution is

$$
p(y)^2 - 1 = c_2 y^2
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
-1 + {y'}^2 = c_2 y^2
$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{c_2 y^2 + 1} \tag{1}$$

$$y' = -\sqrt{c_2 y^2 + 1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{c_2 y^2 + 1}} dy = \int dx$$

$$\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}} = x + c_3$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}}} = e^{x + c_3}$$

Which simplifies to

$$\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)^{\frac{1}{\sqrt{c_2}}} = c_4 e^x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{c_2 y^2 + 1}} dy = \int dx$$

$$-\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}} = x + c_5$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)}{\sqrt{c_2}}} = e^{x + c_5}$$

Which simplifies to

$$\left(y\sqrt{c_2} + \sqrt{c_2 y^2 + 1}\right)^{-\frac{1}{\sqrt{c_2}}} = c_6 e^x$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\left(c_4 \mathrm{e}^{x}\right)^{2\sqrt{c_2}}-1\right)\left(c_4 \mathrm{e}^{x}\right)^{-\sqrt{c_2}}}{2\sqrt{c_2}} \tag{1}$$

$$y = \frac{\left(\left(c_6 \mathrm{e}^{x}\right)^{-2\sqrt{c_2}}-1\right)\left(c_6 \mathrm{e}^{x}\right)^{\sqrt{c_2}}}{2\sqrt{c_2}} \tag{2}$$

Verification of solutions

$$y = \frac{\left(\left(c_4 \mathrm{e}^{x}\right)^{2\sqrt{c_2}}-1\right)\left(c_4 \mathrm{e}^{x}\right)^{-\sqrt{c_2}}}{2\sqrt{c_2}}$$

Verified OK.

$$y = \frac{\left(\left(c_6 \mathrm{e}^{x}\right)^{-2\sqrt{c_2}}-1\right)\left(c_6 \mathrm{e}^{x}\right)^{\sqrt{c_2}}}{2\sqrt{c_2}}$$

Verified OK.

18.22.2 Maple step by step solution

Let's solve

$y y'' - y'^2 = -1$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$$yu(y)\left(\frac{d}{dy}u(y)\right)-u(y)^2=-1$$

- Separate variables

$$\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{u(y)^2-1}=\frac{1}{y}$$

- Integrate both sides with respect to y

$$\int\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{u(y)^2-1}dy=\int\frac{1}{y}dy+c_1$$

- Evaluate integral

$$\frac{\ln(-1+u(y))}{2}+\frac{\ln(u(y)+1)}{2}=\ln(y)+c_1$$

- Solve for $u(y)$

$$\left\{u(y)=-\frac{e^{2c_1}\left(-1+\sqrt{(e^{c_1})^2y^2+1}\right)}{(e^{c_1})^2}-1, u(y)=\frac{e^{2c_1}\left(1+\sqrt{(e^{c_1})^2y^2+1}\right)}{(e^{c_1})^2}-1\right\}$$

- Solve 1st ODE for $u(y)$

$$u(y)=-\frac{e^{2c_1}\left(-1+\sqrt{(e^{c_1})^2y^2+1}\right)}{(e^{c_1})^2}-1$$

- Revert to original variables with substitution $u(y)=y', y=y$

$$y'=-\frac{e^{2c_1}\left(-1+\sqrt{(e^{c_1})^2y^2+1}\right)}{(e^{c_1})^2}-1$$

- Separate variables

$$\frac{y'}{-\frac{e^{2c_1}\left(-1+\sqrt{(e^{c_1})^2y^2+1}\right)}{(e^{c_1})^2}-1}=1$$

- Integrate both sides with respect to x

$$\int\frac{y'}{-\frac{e^{2c_1}\left(-1+\sqrt{(e^{c_1})^2y^2+1}\right)}{(e^{c_1})^2}-1}dx=\int 1dx+c_2$$

- Evaluate integral

$$-\left(\mathrm{e}^{c_1}\right)^2 \left(-\frac{\arctan\left(\frac{\mathrm{e}^{4c_1} y}{\sqrt{\mathrm{e}^{2c_1}\mathrm{e}^{4c_1}}}\right)}{\sqrt{\mathrm{e}^{2c_1}\mathrm{e}^{4c_1}}} + \frac{\mathrm{e}^{2c_1}\arctan\left(\frac{\mathrm{e}^{4c_1} y}{\sqrt{\mathrm{e}^{2c_1}\mathrm{e}^{4c_1}}}\right)}{\left(\mathrm{e}^{c_1}\right)^2\sqrt{\mathrm{e}^{2c_1}\mathrm{e}^{4c_1}}} - \frac{\mathrm{e}^{2c_1}\left(\frac{-\sqrt{\left(\mathrm{e}^{c_1}\right)^2\left(y-\sqrt{-\mathrm{e}^{6c_1}}\,\mathrm{e}^{-4c_1}\right)^2+2\left(\mathrm{e}^{c_1}\right)^2\sqrt{-\mathrm{e}^{6c_1}}\,\mathrm{e}^{-4c_1}\left(y-\sqrt{\right.}}}{}\right.}{}\right.$$

- Solve for y

$$y = -\frac{\left(\left(\mathrm{e}^{c_1}\right)^2-\left(\mathrm{e}^{-\mathrm{e}^{c_1}c_2-\mathrm{e}^{c_1}x+c_1}\right)^2\right)\mathrm{e}^{-3c_1+\mathrm{e}^{c_1}c_2+\mathrm{e}^{c_1}x}}{2}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = \frac{\mathrm{e}^{2c_1}\left(1+\sqrt{\left(\mathrm{e}^{c_1}\right)^2 y^2+1}\right)}{\left(\mathrm{e}^{c_1}\right)^2} - 1$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{\mathrm{e}^{2c_1}\left(1+\sqrt{\left(\mathrm{e}^{c_1}\right)^2 y^2+1}\right)}{\left(\mathrm{e}^{c_1}\right)^2} - 1$$

- Separate variables

$$\frac{y'}{\frac{\mathrm{e}^{2c_1}\left(1+\sqrt{\left(\mathrm{e}^{c_1}\right)^2 y^2+1}\right)}{\left(\mathrm{e}^{c_1}\right)^2}-1} = 1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\frac{\mathrm{e}^{2c_1}\left(1+\sqrt{\left(\mathrm{e}^{c_1}\right)^2 y^2+1}\right)}{\left(\mathrm{e}^{c_1}\right)^2}-1} dx = \int 1 dx + c_2$$

- Evaluate integral

$$(e^{c_1})^2 \left(-\frac{e^{2c_1}\arctan\left(\frac{e^{4c_1} y}{\sqrt{e^{2c_1} e^{4c_1}}}\right)}{(e^{c_1})^2\sqrt{e^{2c_1}e^{4c_1}}} + \frac{\arctan\left(\frac{e^{4c_1} y}{\sqrt{e^{2c_1} e^{4c_1}}}\right)}{\sqrt{e^{2c_1}e^{4c_1}}} - \frac{e^{2c_1}\left(\frac{-\sqrt{(e^{c_1})^2\left(y-\sqrt{-e^{6c_1}}\, e^{-4c_1}\right)^2+2(e^{c_1})^2\sqrt{-e^{6c_1}}\, e^{-4c_1}\left(y-\sqrt{-\right.}}}{}\right.}{}\right.$$

- Solve for y

$$y = \frac{\left(\left(e^{c_1}+e^{c_1}c_2+e^{c_1}x\right)^2-(e^{c_1})^2\right)e^{-3c_1-e^{c_1}c_2-e^{c_1}x}}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(_b(_a)^2-1)/_a = 0, _b(_a), HIN
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 59

```
dsolve(y(x)*diff(y(x),x$2)+1=diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{c_1 \left(e^{\frac{c_2+x}{c_1}} - e^{\frac{-x-c_2}{c_1}}\right)}{2}$$
$$y(x) = \frac{c_1 \left(e^{\frac{c_2+x}{c_1}} - e^{\frac{-x-c_2}{c_1}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 60.25 (sec). Leaf size: 85

```
DSolve[y[x]*y''[x]+1==y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{ie^{-c_1} \tanh\left(e^{c_1}(x+c_2)\right)}{\sqrt{-\operatorname{sech}^2\left(e^{c_1}(x+c_2)\right)}}$$
$$y(x) \to \frac{ie^{-c_1} \tanh\left(e^{c_1}(x+c_2)\right)}{\sqrt{-\operatorname{sech}^2\left(e^{c_1}(x+c_2)\right)}}$$

18.23 problem 23

Internal problem ID [2295]
Internal file name [`OUTPUT/2295_Tuesday_February_27_2024_08_24_12_AM_58013786/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

18.23.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\
&= \pm 1
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +1 \\
\lambda_2 &= -1
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(1)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} \tag{1}$$

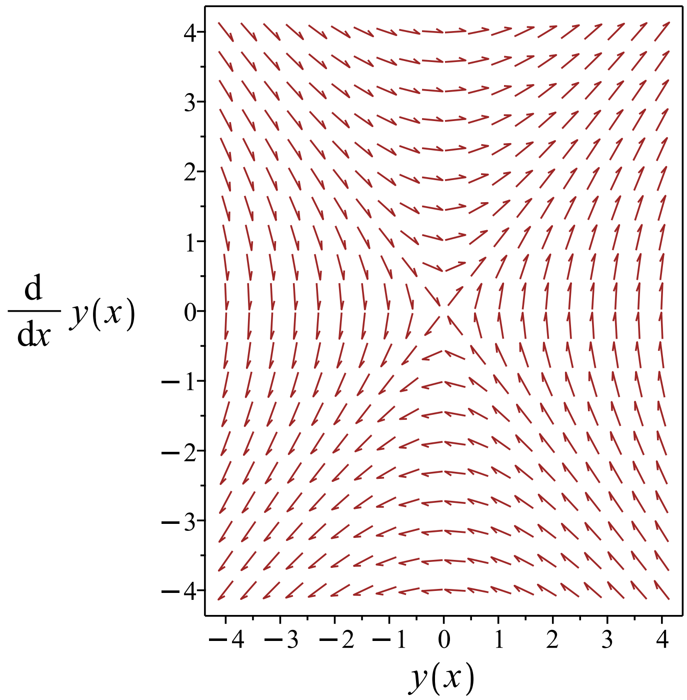


Figure 709: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x}$$

Verified OK.

18.23.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - yy')\,dx = 0$$
$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$\ln\left(y+\sqrt{y^2+2c_1}\right) = x + c_2$$

Raising both side to exponential gives

$$y+\sqrt{y^2+2c_1} = e^{x+c_2}$$

Which simplifies to

$$y+\sqrt{y^2+2c_1} = c_3 e^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$-\ln\left(y+\sqrt{y^2+2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y+\sqrt{y^2+2c_1}} = e^{x+c_4}$$

Which simplifies to

$$\frac{1}{y+\sqrt{y^2+2c_1}} = c_5 e^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_3^2 e^{2x} - 2c_1\right) e^{-x}}{2c_3} \tag{1}$$
$$y = -\frac{\left(2c_1 c_5^2 e^{2x} - 1\right) e^{-x}}{2c_5} \tag{2}$$

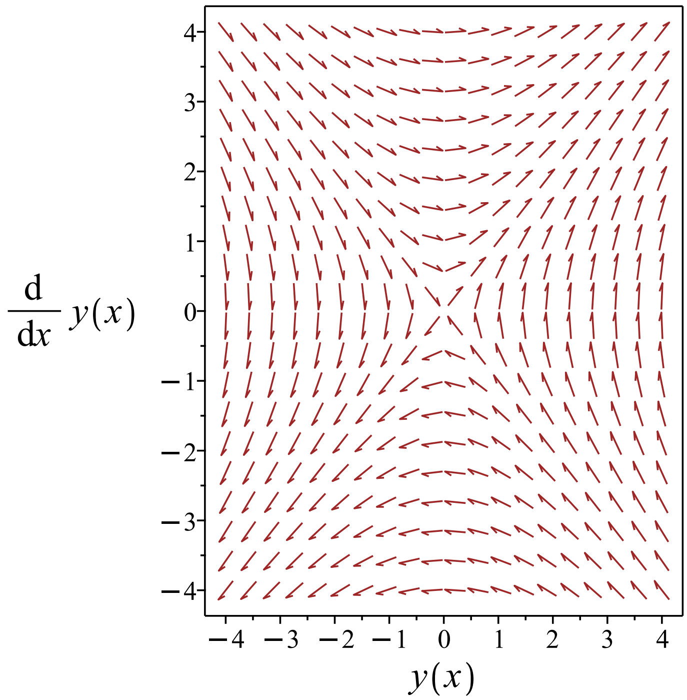


Figure 710: Slope field plot

Verification of solutions

$$y = \frac{(c_3^2 e^{2x} - 2c_1) e^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{(2c_1 c_5^2 e^{2x} - 1) e^{-x}}{2c_5}$$

Verified OK.

18.23.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 599: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\
&= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} \tag{1}$$

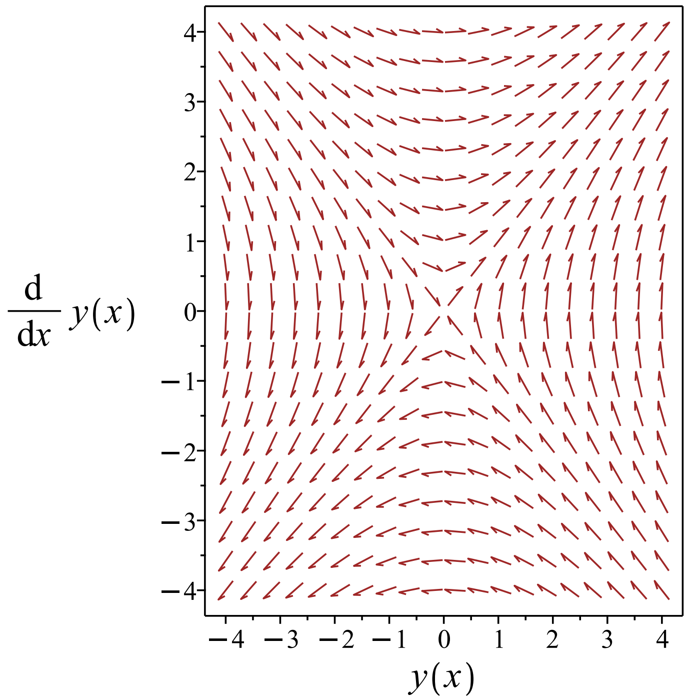


Figure 711: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

Verified OK.

18.23.4 Maple step by step solution

Let's solve

$y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-x} + c_2 e^{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=y(x),y(x), singsol=all)
```

$$y(x) = e^{-x} c_1 + c_2 e^{x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 20

```
DSolve[y''[x]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + c_2 e^{-x}$$

18.24 problem 24

Internal problem ID [2296]
Internal file name [`OUTPUT/2296_Tuesday_February_27_2024_08_24_13_AM_1427113/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode", "second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
  _Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
  _reducible, _mu_xy]]
```

$$\boxed{yy'' + {y'}^2 - yy' = 0}$$

18.24.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (yy'' + (y' - y)\,y')\,dx = 0$$
$$-\frac{y^2}{2} + yy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{2y}{y^2 + 2c_1} dy = \int dx$$
$$\ln\left(y^2 + 2c_1\right) = x + c_2$$

Raising both side to exponential gives

$$y^2 + 2c_1 = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$y^2 + 2c_1 = c_3 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{RootOf}\left(\_Z^2 - c_3 \mathrm{e}^{x} + 2c_1\right) \tag{1}$$

Verification of solutions

$$y = \mathrm{RootOf}\left(\_Z^2 - c_3 \mathrm{e}^{x} + 2c_1\right)$$

Verified OK.

18.24.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y p(y) \left(\frac{d}{dy}p(y)\right) + \left(p(y) - y\right) p(y) = 0$$

Which is now solved as first order ode for $p(y)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dy}p(y) + p(y)p(y) = q(y)$$

Where here

$$\begin{aligned} p(y) &= \frac{1}{y} \\ q(y) &= 1 \end{aligned}$$

Hence the ode is

$$\frac{d}{dy}p(y) + \frac{p(y)}{y} = 1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{y}dy} \\ &= y \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}(\mu p) &= \mu \\ \frac{\mathrm{d}}{\mathrm{d}y}(yp) &= y \\ \mathrm{d}(yp) &= y\mathrm{d}y \end{aligned}$$

Integrating gives

$$\begin{aligned} yp &= \int y \, \mathrm{d}y \\ yp &= \frac{y^2}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = y$ results in

$$p(y) = \frac{y}{2} + \frac{c_1}{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{y}{2} + \frac{c_1}{y}$$

Integrating both sides gives

$$\int \frac{2y}{y^2 + 2c_1} dy = \int dx$$
$$\ln(y^2 + 2c_1) = x + c_2$$

Raising both side to exponential gives

$$y^2 + 2c_1 = e^{x+c_2}$$

Which simplifies to

$$y^2 + 2c_1 = c_3 e^x$$

Summary

The solution(s) found are the following

$$y = \text{RootOf}\left(\_Z^2 - c_3 e^x + 2c_1\right) \tag{1}$$

Verification of solutions

$$y = \text{RootOf}\left(\_Z^2 - c_3 e^x + 2c_1\right)$$

Verified OK.

18.24.3 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = -1$$
$$g(y) = \frac{1}{y}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx} \ln (y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy} \int g dy\right) \frac{dy}{dx} \\ &= \frac{d}{dx} \int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx} \ln (y') + \frac{d}{dx} \int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln (y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = \frac{1}{y}$ and $f = -1$, then

$$\begin{aligned} \int -g dy &= \int -\frac{1}{y} dy \\ &= -\ln (y) \\ \int -f dx &= \int 1 dx \\ &= x \end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = \frac{c_2 \mathrm{e}^x}{y}$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x) g(y) \\ &= \frac{c_2 \mathrm{e}^x}{y} \end{aligned}$$

Where $f(x) = c_2 e^x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = c_2 e^x\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int c_2 e^x\,dx$$

$$\frac{y^2}{2} = c_2 e^x + c_3$$

The solution is

$$\frac{y^2}{2} - c_2 e^x - c_3 = 0$$

Summary
The solution(s) found are the following

$$\frac{y^2}{2} - c_2 e^x - c_3 = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - c_2 e^x - c_3 = 0$$

Verified OK.

18.24.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' + (y' - y)\,y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (yy'' + (y' - y)\,y')\,dx = 0$$

$$-\frac{y^2}{2} + yy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{2y}{y^2 + 2c_1}dy = \int dx$$

$$\ln\left(y^2 + 2c_1\right) = x + c_2$$

Raising both side to exponential gives

$$y^2 + 2c_1 = e^{x+c_2}$$

Which simplifies to

$$y^2 + 2c_1 = c_3 e^x$$

Summary
The solution(s) found are the following

$$y = \operatorname{RootOf}\left(\_Z^2 - c_3 e^x + 2c_1\right) \tag{1}$$

Verification of solutions

$$y = \operatorname{RootOf}\left(\_Z^2 - c_3 e^x + 2c_1\right)$$

Verified OK.

18.24.5 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\frac{\partial a_2}{\partial y} = \frac{\partial a_1}{\partial y'}$$
$$\frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial y'}$$
$$\frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial y}$$

Looking at the the ode given we see that

$$\begin{aligned} a_2 &= y \\ a_1 &= y' - y \\ a_0 &= 0 \end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx = c_1$$
$$\int y\, dy' + \int y' - y\, dy + \int 0\, dx = c_1$$

Which results in

$$2yy' - \frac{y^2}{2} = c_1$$

Which is now solved Integrating both sides gives

$$\int \frac{4y}{y^2 + 2c_1} dy = \int dx$$
$$2\ln\left(y^2 + 2c_1\right) = x + c_2$$

Raising both side to exponential gives

$$\left(y^2 + 2c_1\right)^2 = e^{x+c_2}$$

Which simplifies to

$$\left(y^2 + 2c_1\right)^2 = c_3 e^x$$

Summary

The solution(s) found are the following

$$y = \text{RootOf}\left(\_Z^2 - \sqrt{c_3 e^x} + 2c_1\right) \tag{1}$$

Verification of solutions

$$y = \text{RootOf}\left(\_Z^2 - \sqrt{c_3 e^x} + 2c_1\right)$$

Verified OK.

18.24.6 Maple step by step solution

Let's solve

$yy'' + (y' - y)\, y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

$y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

$u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$yu(y)\left(\frac{d}{dy}u(y)\right) + (u(y) - y)\,u(y) = 0$

- Isolate the derivative

$\frac{d}{dy}u(y) = 1 - \frac{u(y)}{y}$

- Group terms with $u(y)$ on the lhs of the ODE and the rest on the rhs of the ODE

$\frac{d}{dy}u(y) + \frac{u(y)}{y} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

$\mu(y)\left(\frac{d}{dy}u(y) + \frac{u(y)}{y}\right) = \mu(y)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\,u(y))$

$\mu(y)\left(\frac{d}{dy}u(y) + \frac{u(y)}{y}\right) = \left(\frac{d}{dy}\mu(y)\right)u(y) + \mu(y)\left(\frac{d}{dy}u(y)\right)$

- Isolate $\frac{d}{dy}\mu(y)$

$\frac{d}{dy}\mu(y) = \frac{\mu(y)}{y}$

- Solve to find the integrating factor

$\mu(y) = y$

- Integrate both sides with respect to y

$\int \left(\frac{d}{dy}(\mu(y)\,u(y))\right)dy = \int \mu(y)\,dy + c_1$

- Evaluate the integral on the lhs

$\mu(y)\,u(y) = \int \mu(y)\,dy + c_1$

- Solve for $u(y)$

$u(y) = \frac{\int \mu(y)dy + c_1}{\mu(y)}$

- Substitute $\mu(y) = y$

$u(y) = \frac{\int y dy + c_1}{y}$

- Evaluate the integrals on the rhs

$$u(y) = \frac{\frac{y^2}{2}+c_1}{y}$$

- Simplify

$$u(y) = \frac{y^2+2c_1}{2y}$$

- Solve 1st ODE for $u(y)$

$$u(y) = \frac{y^2+2c_1}{2y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{y^2+2c_1}{2y}$$

- Separate variables

$$\frac{y'y}{y^2+2c_1} = \frac{1}{2}$$

- Integrate both sides with respect to x

$$\int \frac{y'y}{y^2+2c_1}dx = \int \frac{1}{2}dx + c_2$$

- Evaluate integral

$$\frac{\ln(y^2+2c_1)}{2} = \frac{x}{2} + c_2$$

- Solve for y

$$\left\{y = \sqrt{-2c_1 + e^{x+2c_2}}, y = -\sqrt{-2c_1 + e^{x+2c_2}}\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 35

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^2=y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \sqrt{2\,\mathrm{e}^{x} c_1 + 2c_2}$$
$$y(x) = -\sqrt{2\,\mathrm{e}^{x} c_1 + 2c_2}$$

✓ Solution by Mathematica

Time used: 1.093 (sec). Leaf size: 41

```
DSolve[y[x]*y''[x]+y'[x]^2==y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2\sqrt{2e^x + e^{c_1}}$$
$$y(x) \to \sqrt{2}c_2\sqrt{e^x}$$

18.25 problem 25

Internal problem ID [2297]
Internal file name [`OUTPUT/2297_Tuesday_February_27_2024_08_24_14_AM_62754954/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{2yy'' - {y'}^2 = 0}$$

18.25.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$2yp(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= \frac{p}{2y}
\end{aligned}$$

Where $f(y) = \frac{1}{2y}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= \frac{1}{2y}\,dy \\
\int \frac{1}{p}\,dp &= \int \frac{1}{2y}\,dy \\
\ln(p) &= \frac{\ln(y)}{2} + c_1 \\
p &= \mathrm{e}^{\frac{\ln(y)}{2}+c_1} \\
&= c_1\sqrt{y}
\end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1\sqrt{y}$$

Integrating both sides gives

$$\begin{aligned}
\int \frac{1}{c_1\sqrt{y}}dy &= \int dx \\
\frac{2\sqrt{y}}{c_1} &= x + c_2
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{4}c_1^2c_2^2 + \frac{1}{2}c_1^2c_2x + \frac{1}{4}c_1^2x^2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{4}c_1^2c_2^2 + \frac{1}{2}c_1^2c_2x + \frac{1}{4}c_1^2x^2$$

Verified OK.

18.25.2 Maple step by step solution

Let's solve

$2yy'' - y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $2yu(y)\left(\frac{d}{dy}u(y)\right) - u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)} = \frac{1}{2y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)}dy = \int \frac{1}{2y}dy + c_1$

- Evaluate integral

  $\ln(u(y)) = \frac{\ln(y)}{2} + c_1$

- Solve for $u(y)$

  $\left\{u(y) = \frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}, u(y) = -\frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}\right\}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}$

- Separate variables

  $\frac{y'}{\sqrt{e^{-2c_1} y}} = \frac{1}{e^{-2c_1}}$
- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{e^{-2c_1} y}} dx = \int \frac{1}{e^{-2c_1}} dx + c_2$
- Evaluate integral

  $\frac{2\sqrt{e^{-2c_1} y}}{e^{-2c_1}} = \frac{x}{e^{-2c_1}} + c_2$
- Solve for y

  $y = \frac{c_2^2 (e^{-2c_1})^2 + 2c_2 e^{-2c_1} x + x^2}{4\,e^{-2c_1}}$
- Solve 2nd ODE for $u(y)$

  $u(y) = -\frac{\sqrt{e^{-2c_1} y}}{e^{-2c_1}}$
- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{\sqrt{e^{-2c_1} y}}{e^{-2c_1}}$
- Separate variables

  $\frac{y'}{\sqrt{e^{-2c_1} y}} = -\frac{1}{e^{-2c_1}}$
- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{e^{-2c_1} y}} dx = \int -\frac{1}{e^{-2c_1}} dx + c_2$
- Evaluate integral

  $\frac{2\sqrt{e^{-2c_1} y}}{e^{-2c_1}} = -\frac{x}{e^{-2c_1}} + c_2$
- Solve for y

  $y = \frac{c_2^2 (e^{-2c_1})^2 - 2c_2 e^{-2c_1} x + x^2}{4\,e^{-2c_1}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 17

```
dsolve(2*y(x)*diff(y(x),x$2)-diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \frac{(c_1 x + c_2)^2}{4}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 29

```
DSolve[2*y[x]*y''[x]-y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(c_1 x + 2c_2)^2}{4c_2}$$
$$y(x) \to \text{Indeterminate}$$

18.26 problem 26

Internal problem ID [2298]
Internal file name [`OUTPUT/2298_Tuesday_February_27_2024_08_24_14_AM_17887916/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' + 2{y'}^2 = 2}$$

18.26.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 2p(x)^2 - 2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-2p^2+2}dp = x + c_1$$
$$\frac{\operatorname{arctanh}(p)}{2} = x + c_1$$

Solving for p gives these solutions

$$p_1 = \tanh(2x + 2c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \tanh(2x + 2c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \tanh(2x + 2c_1)\, \mathrm{d}x \\ &= \frac{\ln(\cosh(2x + 2c_1))}{2} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln(\cosh(2x + 2c_1))}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{\ln(\cosh(2x + 2c_1))}{2} + c_2$$

Verified OK.

18.26.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + 2p(y)^2 = 2$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int -\frac{p}{2\left(p^2-1\right)}dp = \int dy$$
$$-\frac{\ln\left(p-1\right)}{4}-\frac{\ln\left(p+1\right)}{4} = y+c_1$$

The above can be written as

$$\left(-\frac{1}{4}\right)\left(\ln\left(p-1\right)+\ln\left(p+1\right)\right) = y+c_1$$
$$\ln\left(p-1\right)+\ln\left(p+1\right) = \left(-4\right)\left(y+c_1\right)$$
$$= -4y-4c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(p-1)+\ln(p+1)} = -4c_1\mathrm{e}^{-4y}$$

Which simplifies to

$$p^2-1 = c_2\mathrm{e}^{-4y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \operatorname{RootOf}\left(\_Z^2-c_2\mathrm{e}^{-4y}-1\right)$$

Integrating both sides gives

$$\int \frac{1}{\operatorname{RootOf}\left(\_Z^2-c_2\mathrm{e}^{-4y}-1\right)}dy = \int dx$$
$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2-c_2\mathrm{e}^{-4\_a}-1\right)}d\_a = x+c_3$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2-c_2\mathrm{e}^{-4\_a}-1\right)}d\_a = x+c_3 \tag{1}$$

Verification of solutions

$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2-c_2\mathrm{e}^{-4\_a}-1\right)}d\_a = x+c_3$$

Verified OK.

18.26.3 Maple step by step solution

Let's solve

$y'' + 2y'^2 = 2$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + 2u(x)^2 = 2$

- Separate variables

  $\frac{u'(x)}{-2u(x)^2+2} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{-2u(x)^2+2}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{\operatorname{arctanh}(u(x))}{2} = x + c_1$

- Solve for $u(x)$

  $u(x) = \tanh(2x + 2c_1)$

- Solve 1st ODE for $u(x)$

  $u(x) = \tanh(2x + 2c_1)$

- Make substitution $u = y'$

  $y' = \tanh(2x + 2c_1)$

- Integrate both sides to solve for y

  $\int y'dx = \int \tanh(2x + 2c_1)\,dx + c_2$

- Compute integrals

  $y = \frac{\ln(\cosh(2x+2c_1))}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)^2=2,y(x), singsol=all)
```

$$y(x) = -x - \frac{\ln(2)}{2} + \frac{\ln(-e^{4x}c_1 + c_2)}{2}$$

✓ Solution by Mathematica

Time used: 0.389 (sec). Leaf size: 62

```
DSolve[y''[x]+2*y'[x]^2==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(-\log(e^{2x}) + \log(e^{4x} + e^{2c_1}) + 2c_2)$$
$$y(x) \to \frac{1}{2}(-\log(e^{2x}) + \log(e^{4x}) + 2c_2)$$

18.27 problem 27

Internal problem ID [2299]
Internal file name [`OUTPUT/2299_Tuesday_February_27_2024_08_24_16_AM_22539582/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - y'^3 = 0}$$

18.27.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + \left(1 - p(x)^2\right) p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p\left(p^2 - 1\right)} dp = \int dx$$

$$\frac{\ln\left(p + 1\right)}{2} + \frac{\ln\left(p - 1\right)}{2} - \ln\left(p\right) = x + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(p+1)}{2}+\frac{\ln(p-1)}{2}-\ln(p)} = e^{x+c_1}$$

Which simplifies to

$$\frac{\sqrt{p+1}\,\sqrt{p-1}}{p} = c_2 e^x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{\sqrt{1-c_2^2 e^{2x}}-1}{\sqrt{1-c_2^2 e^{2x}}} + 1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{\sqrt{1-c_2^2 e^{2x}}}\, dx \\ &= -\operatorname{arctanh}\left(\frac{1}{\sqrt{1-c_2^2 e^{2x}}}\right) + c_3 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\operatorname{arctanh}\left(\frac{1}{\sqrt{1-c_2^2 e^{2x}}}\right) + c_3 \tag{1}$$

Verification of solutions

$$y = -\operatorname{arctanh}\left(\frac{1}{\sqrt{1-c_2^2 e^{2x}}}\right) + c_3$$

Verified OK.

18.27.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right) + \left(1 - p(y)^2\right)p(y) = 0
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{p^2 - 1}dp &= y + c_1 \\
-\operatorname{arctanh}(p) &= y + c_1
\end{aligned}
$$

Solving for p gives these solutions

$$
p_1 = -\tanh(y + c_1)
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = -\tanh(y + c_1)
$$

Integrating both sides gives

$$
\begin{aligned}
\int -\frac{1}{\tanh(y + c_1)}dy &= \int dx \\
\frac{\ln(\tanh(y + c_1) + 1)}{2} + \frac{\ln(-1 + \tanh(y + c_1))}{2} - \ln(\tanh(y + c_1)) &= x + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\frac{\ln(\tanh(y+c_1)+1)}{2} + \frac{\ln(-1+\tanh(y+c_1))}{2} - \ln(\tanh(y+c_1))} = e^{x+c_2}
$$

Which simplifies to

$$
\frac{\sqrt{\tanh(y + c_1) + 1}\,\sqrt{-1 + \tanh(y + c_1)}}{\tanh(y + c_1)} = c_3 e^x
$$

Summary
The solution(s) found are the following

$$y = -c_1 + \operatorname{arctanh}\left(\frac{\sqrt{-c_3^2 e^{2x}+1}-1}{\sqrt{-c_3^2 e^{2x}+1}}-1\right) \tag{1}$$

Verification of solutions

$$y = -c_1 + \operatorname{arctanh}\left(\frac{\sqrt{-c_3^2 e^{2x}+1}-1}{\sqrt{-c_3^2 e^{2x}+1}}-1\right)$$

Verified OK.

18.27.3 Maple step by step solution

Let's solve

$y'' + \left(1-y'^2\right) y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + \left(1-u(x)^2\right) u(x) = 0$

- Separate variables

  $\frac{u'(x)}{\left(1-u(x)^2\right)u(x)} = -1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{\left(1-u(x)^2\right)u(x)} dx = \int (-1) dx + c_1$

- Evaluate integral

  $-\frac{\ln(u(x)+1)}{2} - \frac{\ln(-1+u(x))}{2} + \ln(u(x)) = -x + c_1$

- Solve for $u(x)$

  $\left\{u(x) = \frac{\sqrt{\left(e^{-2x+2c_1}-1\right)e^{-2x+2c_1}}}{e^{-2x+2c_1}-1}, u(x) = -\frac{\sqrt{\left(e^{-2x+2c_1}-1\right)e^{-2x+2c_1}}}{e^{-2x+2c_1}-1}\right\}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{\sqrt{\left(e^{-2x+2c_1}-1\right)e^{-2x+2c_1}}}{e^{-2x+2c_1}-1}$

- Make substitution $u = y'$

$$y' = \frac{\sqrt{\left(e^{-2x+2c_1}-1\right)e^{-2x+2c_1}}}{e^{-2x+2c_1}-1}$$

- Integrate both sides to solve for y

$$\int y'dx = \int \frac{\sqrt{\left(e^{-2x+2c_1}-1\right)e^{-2x+2c_1}}}{e^{-2x+2c_1}-1}dx + c_2$$

- Compute integrals

$$y = -\frac{\sqrt{\left(e^{-2x+2c_1}-1\right)^2-1+e^{-2x+2c_1}}}{2} - \frac{\ln\left(-\frac{1}{2}+e^{-2x+2c_1}+\sqrt{\left(e^{-2x+2c_1}-1\right)^2-1+e^{-2x+2c_1}}\right)}{4} + \frac{\sqrt{\left(e^{-2x+2c_1}\right)^2-e^{-2x+2c_1}}}{2} -$$

- Solve 2nd ODE for $u(x)$

$$u(x) = -\frac{\sqrt{\left(e^{-2x+2c_1}-1\right)e^{-2x+2c_1}}}{e^{-2x+2c_1}-1}$$

- Make substitution $u = y'$

$$y' = -\frac{\sqrt{\left(e^{-2x+2c_1}-1\right)e^{-2x+2c_1}}}{e^{-2x+2c_1}-1}$$

- Integrate both sides to solve for y

$$\int y'dx = \int -\frac{\sqrt{\left(e^{-2x+2c_1}-1\right)e^{-2x+2c_1}}}{e^{-2x+2c_1}-1}dx + c_2$$

- Compute integrals

$$y = \frac{\sqrt{\left(e^{-2x+2c_1}-1\right)^2-1+e^{-2x+2c_1}}}{2} + \frac{\ln\left(-\frac{1}{2}+e^{-2x+2c_1}+\sqrt{\left(e^{-2x+2c_1}-1\right)^2-1+e^{-2x+2c_1}}\right)}{4} - \frac{\sqrt{\left(e^{-2x+2c_1}\right)^2-e^{-2x+2c_1}}}{2} + \frac{\ln}{}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)^3-_b(_a), _b(_a), HINT = [[1, 0]
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=diff(y(x),x)^3,y(x), singsol=all)
```

$$y(x) = -\operatorname{arctanh}\left(\sqrt{-c_1 e^{2x} + 1}\right) + c_2$$
$$y(x) = \operatorname{arctanh}\left(\sqrt{-c_1 e^{2x} + 1}\right) + c_2$$

✓ Solution by Mathematica

Time used: 60.1 (sec). Leaf size: 47

```
DSolve[y''[x]+y'[x]==y'[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \operatorname{arctanh}\left(\sqrt{1 + e^{2(x+c_1)}}\right)$$
$$y(x) \to \operatorname{arctanh}\left(\sqrt{1 + e^{2(x+c_1)}}\right) + c_2$$

18.28 problem 28

Internal problem ID [2300]
Internal file name [OUTPUT/2300_Tuesday_February_27_2024_08_24_18_AM_76384911/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{(y+1)\,y''-3{y'}^2=0}$$

18.28.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y'=p(y)$$

Then

$$\begin{aligned}y''&=\frac{dp}{dx}\\&=\frac{dy}{dx}\frac{dp}{dy}\\&=p\frac{dp}{dy}\end{aligned}$$

Hence the ode becomes

$$(y+1)\,p(y)\left(\frac{d}{dy}p(y)\right)-3p(y)^2=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{3p}{y+1} \end{aligned}$$

Where $f(y) = \frac{3}{y+1}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p}\,dp &= \frac{3}{y+1}\,dy \\ \int \frac{1}{p}\,dp &= \int \frac{3}{y+1}\,dy \\ \ln(p) &= 3\ln(y+1) + c_1 \\ p &= e^{3\ln(y+1)+c_1} \\ &= c_1(y+1)^3 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1(y+1)^3$$

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{c_1(y+1)^3}dy &= x + c_2 \\ -\frac{1}{2c_1(y+1)^2} &= x + c_2 \end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned} y_1 &= -\frac{\sqrt{-2c_1c_2 - 2c_1x} - 1}{\sqrt{-2c_1c_2 - 2c_1x}} \\ y_2 &= -\frac{\sqrt{-2c_1c_2 - 2c_1x} + 1}{\sqrt{-2c_1c_2 - 2c_1x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-2c_1c_2 - 2c_1x} - 1}{\sqrt{-2c_1c_2 - 2c_1x}} \tag{1}$$

$$y = -\frac{\sqrt{-2c_1c_2 - 2c_1x} + 1}{\sqrt{-2c_1c_2 - 2c_1x}} \tag{2}$$

Verification of solutions

$$y = -\frac{\sqrt{-2c_1c_2 - 2c_1x} - 1}{\sqrt{-2c_1c_2 - 2c_1x}}$$

Verified OK.

$$y = -\frac{\sqrt{-2c_1c_2 - 2c_1x} + 1}{\sqrt{-2c_1c_2 - 2c_1x}}$$

Verified OK.

18.28.2 Maple step by step solution

Let's solve

$(y+1)\,y'' - 3y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $(y+1)\,u(y)\left(\frac{d}{dy}u(y)\right) - 3u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)} = \frac{3}{y+1}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)}dy = \int \frac{3}{y+1}dy + c_1$

- Evaluate integral

$$\ln(u(y)) = 3\ln(y+1) + c_1$$

- Solve for $u(y)$

$$u(y) = e^{c_1}(y+1)^3$$

- Solve 1st ODE for $u(y)$

$$u(y) = e^{c_1}(y+1)^3$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = e^{c_1}(y+1)^3$$

- Separate variables

$$\frac{y'}{(y+1)^3} = e^{c_1}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{(y+1)^3}dx = \int e^{c_1}dx + c_2$$

- Evaluate integral

$$-\frac{1}{2(y+1)^2} = e^{c_1}x + c_2$$

- Solve for y

$$\left\{y = -\frac{\sqrt{-2\,e^{c_1}x-2c_2}-1}{\sqrt{-2\,e^{c_1}x-2c_2}}, y = -\frac{\sqrt{-2\,e^{c_1}x-2c_2}+1}{\sqrt{-2\,e^{c_1}x-2c_2}}\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 59

```
dsolve((y(x)+1)*diff(y(x),x$2)=3*diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -1$$
$$y(x) = -\frac{\sqrt{-2c_1x - 2c_2} - 1}{\sqrt{-2c_1x - 2c_2}}$$
$$y(x) = -\frac{\sqrt{-2c_1x - 2c_2} + 1}{\sqrt{-2c_1x - 2c_2}}$$

✓ Solution by Mathematica

Time used: 1.378 (sec). Leaf size: 107

```
DSolve[(y[x]+1)*y''[x]==3*y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2c_1x + \sqrt{2}\sqrt{-c_1(x + c_2)} + 2c_2c_1}{2c_1(x + c_2)}$$
$$y(x) \to \frac{-2c_1x + \sqrt{2}\sqrt{-c_1(x + c_2)} - 2c_2c_1}{2c_1(x + c_2)}$$
$$y(x) \to -1$$
$$y(x) \to \text{Indeterminate}$$

18.29 problem 29

Internal problem ID [2301]
Internal file name [`OUTPUT/2301_Tuesday_February_27_2024_08_24_18_AM_73998394/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = \sec(x)\tan(x)}$$

With initial conditions

$$\left[y(0) = \frac{\pi}{4}, y'(0) = 1\right]$$

18.29.1 Solving as second order ode quadrature ode

Integrating once gives

$$y' = \sec(x) + c_1$$

Integrating again gives

$$y = \ln(\sec(x) + \tan(x)) + c_1 x + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \ln(\sec(x) + \tan(x)) + c_1 x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{\pi}{4}$ and $x = 0$ in the above gives

$$\frac{\pi}{4} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\sec(x)\tan(x) + 1 + \tan(x)^2}{\sec(x) + \tan(x)} + c_1$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

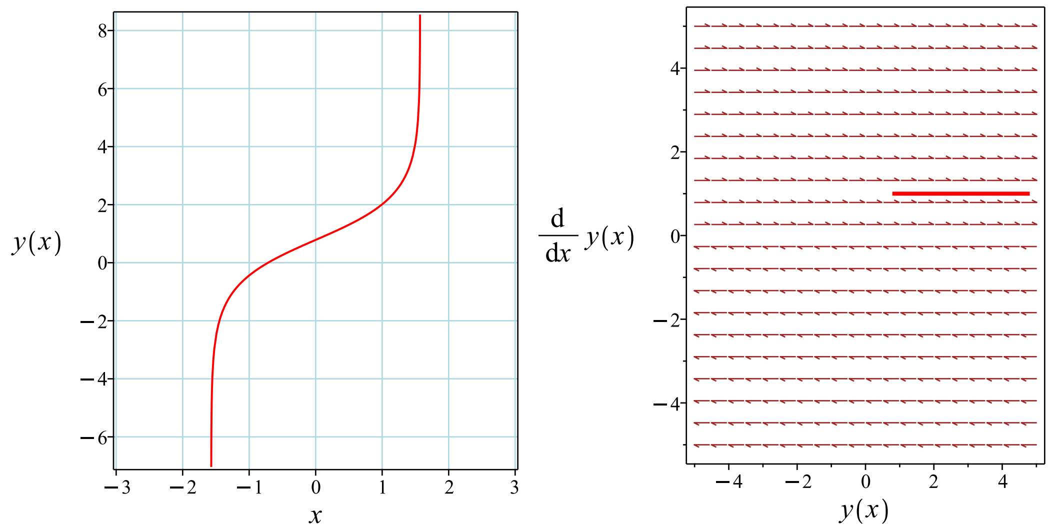
$$c_1 = 0$$
$$c_2 = \frac{\pi}{4}$$

Substituting these values back in above solution results in

$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$$

Summary
The solution(s) found are the following

$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$$

Verified OK.

18.29.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = \sec(x)\tan(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)}\\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= 1\\ y_2 &= x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x \sec(x) \tan(x)}{1} \, dx$$

Which simplifies to

$$u_1 = -\int x \sec(x) \tan(x) \, dx$$

Hence

$$u_1 = -\frac{x}{\cos(x)} + \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sec(x) \tan(x)}{1} \, dx$$

Which simplifies to

$$u_2 = \int \sec(x)\tan(x)\,dx$$

Hence

$$u_2 = \sec(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x}{\cos(x)} + \ln(\sec(x) + \tan(x)) + \sec(x)\,x$$

Which simplifies to

$$y_p(x) = \ln(\sec(x) + \tan(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (\ln(\sec(x) + \tan(x))) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 + \ln(\sec(x) + \tan(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{\pi}{4}$ and $x = 0$ in the above gives

$$\frac{\pi}{4} = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 + \frac{\sec(x)\tan(x) + 1 + \tan(x)^2}{\sec(x) + \tan(x)}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

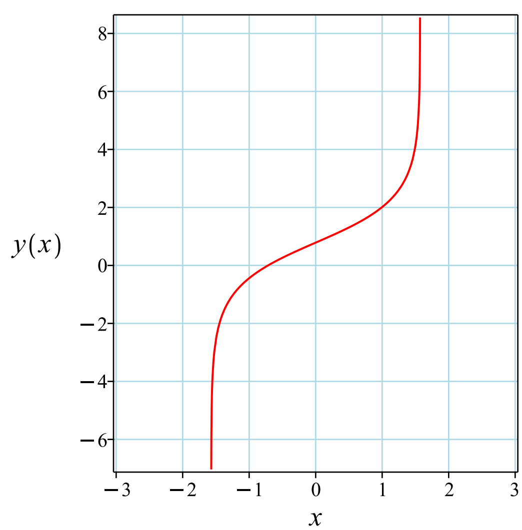
$$
\begin{aligned}
c_1 &= \frac{\pi}{4} \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

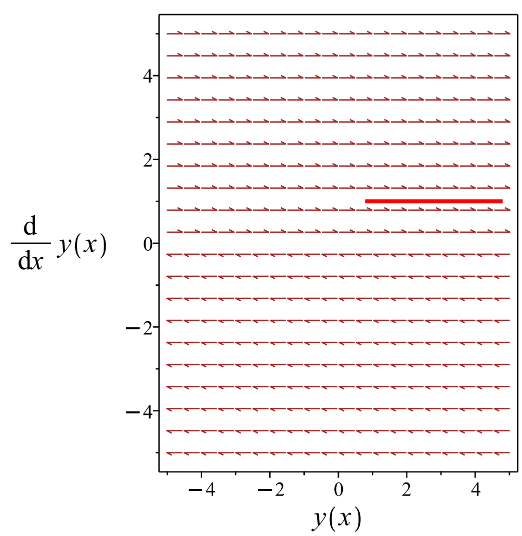
$$
y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}
$$

Summary
The solution(s) found are the following

$$
y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}
$$

Verified OK.

18.29.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int y'' dx &= \int \sec(x) \tan(x)\, dx \\
y' &= \sec(x) + c_1
\end{aligned}
$$

Which is now solved for y. Integrating both sides gives

$$
\begin{aligned}
y &= \int \sec(x) + c_1 \, \mathrm{d}x \\
&= \ln(\sec(x) + \tan(x)) + c_1 x + c_2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \ln(\sec(x) + \tan(x)) + c_1 x + c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{\pi}{4}$ and $x = 0$ in the above gives

$$
\frac{\pi}{4} = c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{\sec(x)\tan(x) + 1 + \tan(x)^2}{\sec(x) + \tan(x)} + c_1
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + 1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

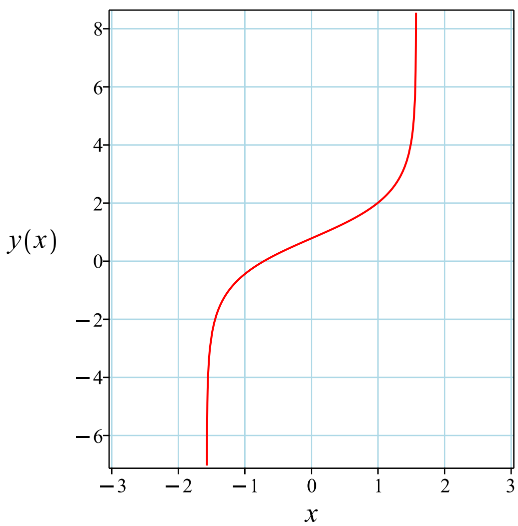
$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= \frac{\pi}{4}
\end{aligned}
$$

Substituting these values back in above solution results in

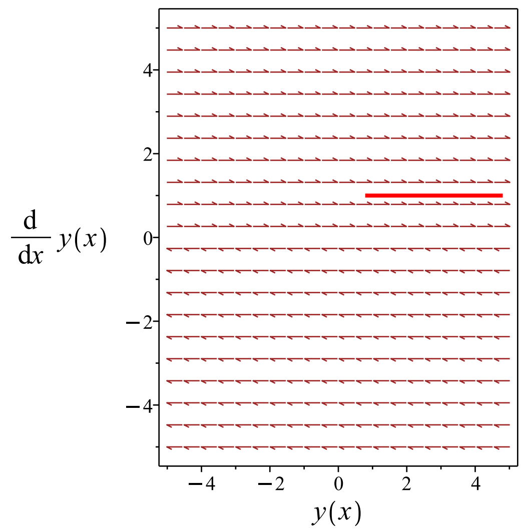
$$
y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}
$$

Summary
The solution(s) found are the following

$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$$

Verified OK.

18.29.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - \sec(x)\tan(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$
\begin{aligned}
p(x) &= \int \sec(x)\tan(x)\, \mathrm{d}x \\
&= \sec(x) + c_1
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + 1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$p(x) = \sec(x)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \sec(x)$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \sec(x)\, \mathrm{d}x \\
&= \ln(\sec(x) + \tan(x)) + c_2
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = c_2$$

$$c_2 = \frac{\pi}{4}$$

Substituting c_2 found above in the general solution gives

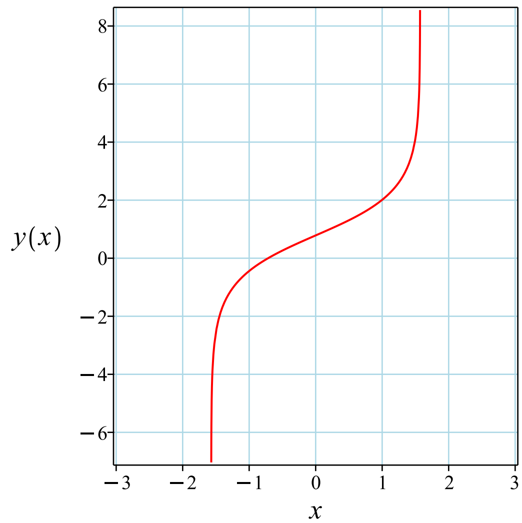
$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$$

Initial conditions are used to solve for the constants of integration.

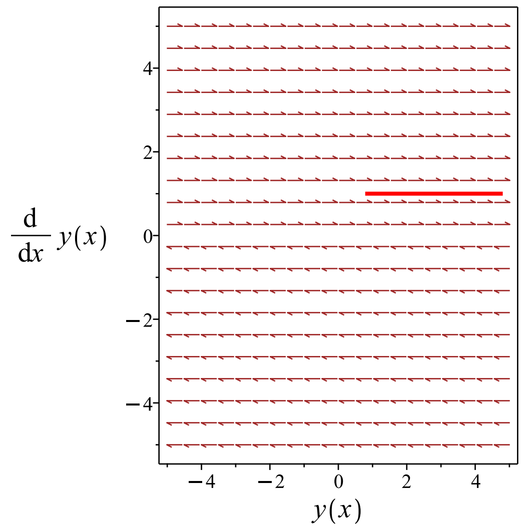
Summary

The solution(s) found are the following

$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$$

Verified OK.

18.29.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 606: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 x + c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x \sec(x) \tan(x)}{1} \, dx$$

Which simplifies to

$$u_1 = -\int x \sec(x) \tan(x) \, dx$$

Hence

$$u_1 = -\frac{x}{\cos(x)} + \ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sec(x) \tan(x)}{1} \, dx$$

Which simplifies to

$$u_2 = \int \sec(x)\tan(x)\,dx$$

Hence

$$u_2 = \sec(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x}{\cos(x)} + \ln(\sec(x) + \tan(x)) + \sec(x)\,x$$

Which simplifies to

$$y_p(x) = \ln(\sec(x) + \tan(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (\ln(\sec(x) + \tan(x))) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 + \ln(\sec(x) + \tan(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{\pi}{4}$ and $x = 0$ in the above gives

$$\frac{\pi}{4} = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 + \frac{\sec(x)\tan(x) + 1 + \tan(x)^2}{\sec(x) + \tan(x)}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

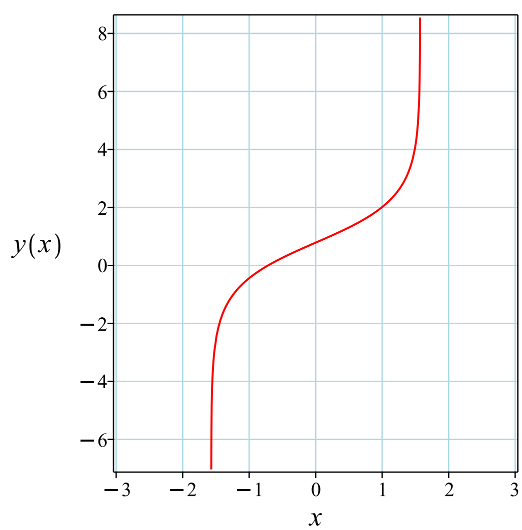
$$c_1 = \frac{\pi}{4}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

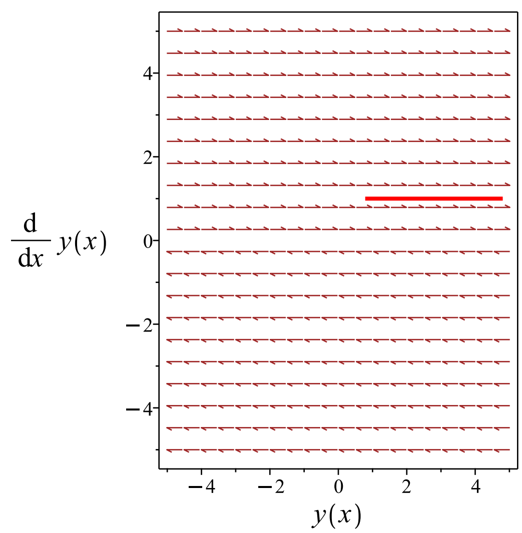
$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$$

Summary
The solution(s) found are the following

$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$$

Verified OK.

18.29.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= \sec(x)\tan(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int \sec(x)\tan(x)\;dx$$

We now have a first order ode to solve which is

$$y' = \sec(x) + c_1$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \sec(x) + c_1 \, dx \\
&= \ln(\sec(x) + \tan(x)) + c_1 x + c_2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \ln(\sec(x) + \tan(x)) + c_1 x + c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{\pi}{4}$ and $x = 0$ in the above gives

$$
\frac{\pi}{4} = c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{\sec(x)\tan(x) + 1 + \tan(x)^2}{\sec(x) + \tan(x)} + c_1
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + 1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= \frac{\pi}{4}
\end{aligned}
$$

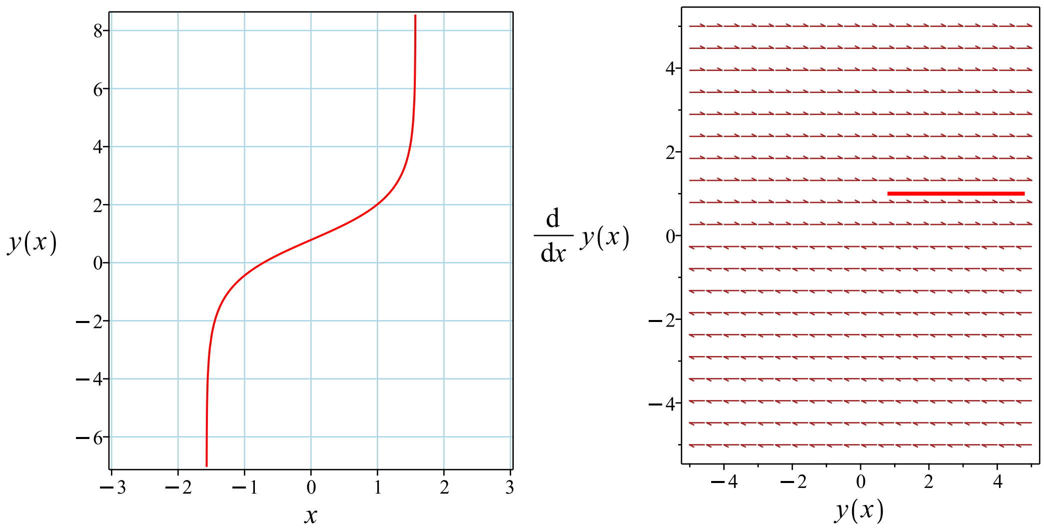
Substituting these values back in above solution results in

$$
y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}
$$

Summary

The solution(s) found are the following

$$
y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$$

Verified OK.

18.29.7 Maple step by step solution

Let's solve

$$\left[y'' = \sec(x)\tan(x), y(0) = \tfrac{\pi}{4}, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

$y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$y_2(x) = x$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 + c_2 x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)\tan(x)\right]$

  - Wronskian of solutions of the homogeneous equation

$W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

$W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

$y_p(x) = -\left(\int x\sec(x)\tan(x)\,dx\right) + x\left(\int \sec(x)\tan(x)\,dx\right)$

  - Compute integrals

$y_p(x) = \ln(\sec(x) + \tan(x))$

- Substitute particular solution into general solution to ODE

$y = c_2 x + c_1 + \ln(\sec(x) + \tan(x))$

□ Check validity of solution $y = c_2 x + c_1 + \ln(\sec(x) + \tan(x))$

  - Use initial condition $y(0) = \frac{\pi}{4}$

$\frac{\pi}{4} = c_1$

  - Compute derivative of the solution

$y' = c_2 + \frac{\sec(x)\tan(x) + 1 + \tan(x)^2}{\sec(x) + \tan(x)}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 1$

$1 = c_2 + 1$

- Solve for c_1 and c_2

  $\{c_1 = \frac{\pi}{4}, c_2 = 0\}$
- Substitute constant values into general solution and simplify

  $y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$

- Solution to the IVP

  $y = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 14

```
dsolve([diff(y(x),x$2)=sec(x)*tan(x),y(0) = 1/4*Pi, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = \ln(\sec(x) + \tan(x)) + \frac{\pi}{4}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 20

```
DSolve[{y''[x]==Sec[x]*Tan[x],{y[0]==Pi/4,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \rightarrow \frac{1}{4}\left(8\operatorname{arctanh}\left(\tan\left(\frac{x}{2}\right)\right) + \pi\right)$$

18.30 problem 30

Internal problem ID [2302]
Internal file name [`OUTPUT/2302_Tuesday_February_27_2024_08_24_20_AM_63640428/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

$$\boxed{2y'' - \mathrm{e}^{y} = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

18.30.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$2y'y'' - y'\mathrm{e}^{y} = 0$$

Integrating the above w.r.t x gives

$$\int (2y'y'' - y'\mathrm{e}^{y})\, dx = 0$$
$$y'^2 - \mathrm{e}^{y} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{e^y + c_1} \tag{1}$$
$$y' = -\sqrt{e^y + c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{e^y + c_1}} dy = \int dx$$
$$-\frac{2 \operatorname{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{e^y + c_1}} dy = \int dx$$
$$\frac{2 \operatorname{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$-\frac{2 \operatorname{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$-\frac{2 \operatorname{arctanh}\left(\frac{\sqrt{c_1+1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\tanh\left(\frac{c_2\sqrt{c_1}}{2}+\frac{x\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}} \left(1-\tanh\left(\frac{c_2\sqrt{c_1}}{2}+\frac{x\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_2\sqrt{c_1}}{2}+\frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{\left(-\mathrm{e}^{c_2\sqrt{c_1}}+1\right)\sqrt{c_1}}{\mathrm{e}^{c_2\sqrt{c_1}}+1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{\mathrm{e}^y+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{c_1+1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\tanh\left(\frac{c_3\sqrt{c_1}}{2}+\frac{x\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}} \left(1-\tanh\left(\frac{c_3\sqrt{c_1}}{2}+\frac{x\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_3\sqrt{c_1}}{2}+\frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{\left(-\mathrm{e}^{c_3\sqrt{c_1}}+1\right)\sqrt{c_1}}{\mathrm{e}^{c_3\sqrt{c_1}}+1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.30.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$2p(y)\left(\frac{d}{dy}p(y)\right) = e^{y}$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{e^{y}}{2p} \end{aligned}$$

Where $f(y) = \frac{e^{y}}{2}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\frac{1}{\frac{1}{p}}\,dp = \frac{e^{y}}{2}\,dy$$

$$\int \frac{1}{\frac{1}{p}}\,dp = \int \frac{e^{y}}{2}\,dy$$

$$\frac{p^{2}}{2} = \frac{e^{y}}{2} + c_{1}$$

The solution is

$$\frac{p(y)^{2}}{2} - \frac{e^{y}}{2} - c_{1} = 0$$

Initial conditions are used to solve for c_1. Substituting $y = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-c_1 = 0$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2}{2} - \frac{e^y}{2} = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = e^{\frac{y}{2}}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = e^{\frac{y}{2}}$$

Integrating both sides gives

$$\int e^{-\frac{y}{2}} dy = x + c_2$$

$$-2\,e^{-\frac{y}{2}} = x + c_2$$

Solving for y gives these solutions

$$y_1 = 2\ln\left(-\frac{2}{x + c_2}\right)$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 2\ln(2) + 2\ln\left(-\frac{1}{c_2}\right)$$

$$c_2 = -2$$

Substituting c_2 found above in the general solution gives

$$y = 2\ln(2) + 2\ln\left(-\frac{1}{x - 2}\right)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = 2\ln(2) + 2\ln\left(-\frac{1}{x-2}\right) \tag{1}$$

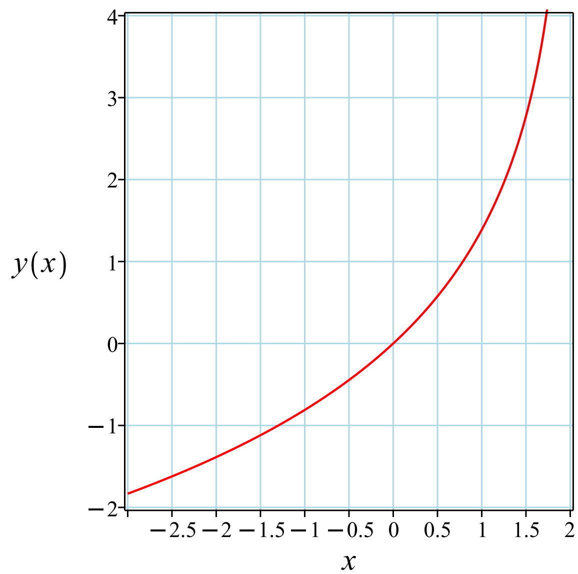


Figure 718: Solution plot

Verification of solutions

$$y = 2\ln(2) + 2\ln\left(-\frac{1}{x-2}\right)$$

Verified OK.

18.30.3 Maple step by step solution

Let's solve

$$\left[2y'' = e^{y}, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

$u(x) = y'$

- Compute y''

$u'(x) = y''$

- Use chain rule on the lhs

$y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

$u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$2u(y)\left(\frac{d}{dy}u(y)\right) = \mathrm{e}^{y}$

- Integrate both sides with respect to y

$\int 2u(y)\left(\frac{d}{dy}u(y)\right)dy = \int \mathrm{e}^{y}dy + c_1$

- Evaluate integral

$u(y)^2 = \mathrm{e}^{y} + c_1$

- Solve for $u(y)$

$\left\{u(y) = \sqrt{\mathrm{e}^{y} + c_1}, u(y) = -\sqrt{\mathrm{e}^{y} + c_1}\right\}$

- Solve 1st ODE for $u(y)$

$u(y) = \sqrt{\mathrm{e}^{y} + c_1}$

- Revert to original variables with substitution $u(y) = y', y = y$

$y' = \sqrt{\mathrm{e}^{y} + c_1}$

- Separate variables

$\frac{y'}{\sqrt{\mathrm{e}^{y}+c_1}} = 1$

- Integrate both sides with respect to x

$\int \frac{y'}{\sqrt{\mathrm{e}^{y}+c_1}}dx = \int 1dx + c_2$

- Evaluate integral

$-\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{\mathrm{e}^{y}+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$

- Solve for y

$$y = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}+\frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\sqrt{e^y + c_1}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\sqrt{e^y + c_1}$$

- Separate variables

$$\frac{y'}{\sqrt{e^y + c_1}} = -1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{e^y + c_1}}dx = \int (-1)\, dx + c_2$$

- Evaluate integral

$$-\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{e^y+c_1}}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = -x + c_2$$

- Solve for y

$$y = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}-\frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$$

☐ Check validity of solution $y = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}-\frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$

- ○ Use initial condition $y(0) = 0$

$$0 = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$$

- ○ Compute derivative of the solution

$$y' = -\frac{\tanh\left(\frac{c_2\sqrt{c_1}}{2}-\frac{x\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}}\left(1-\tanh\left(\frac{c_2\sqrt{c_1}}{2}-\frac{x\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_2\sqrt{c_1}}{2}-\frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$$

- ○ Use the initial condition $y'\Big|_{\{x=0\}} = 1$

$$1 = -\frac{\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}}\left(1-\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$$

- ○ Solve for c_1 and c_2
- ○ The solution does not satisfy the initial conditions

☐ Check validity of solution $y = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}+\frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$

- Use initial condition $y(0) = 0$

$$0 = \ln\left(\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2 c_1 - c_1\right)$$

- Compute derivative of the solution

$$y' = \frac{\tanh\left(\frac{c_2\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}} \left(1 - \tanh\left(\frac{c_2\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_2\sqrt{c_1}}{2} + \frac{x\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$$

- Use the initial condition $y'\big|_{\{x=0\}} = 1$

$$1 = \frac{\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right) c_1^{\frac{3}{2}} \left(1 - \tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2\right)}{\tanh\left(\frac{c_2\sqrt{c_1}}{2}\right)^2 c_1 - c_1}$$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(1/2)*exp(_a) = 0, _b(_a), HINT
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 1/2*_b]
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 15

```
dsolve([2*diff(y(x),x$2)=exp(y(x)),y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = 2\ln(2) + \ln\left(\frac{1}{(-2+x)^2}\right)$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 15

```
DSolve[{2*y''[x]==Exp[y[x]],{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2\log\left(1-\frac{x}{2}\right)$$

18.31 problem 31

Internal problem ID [2303]
Internal file name [`OUTPUT/2303_Tuesday_February_27_2024_08_25_35_AM_38217236/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y'' - y^3 = 0}$$

With initial conditions

$$\left[y(0) = -1, y'(0) = \frac{\sqrt{2}}{2}\right]$$

18.31.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - y^3y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - y^3y')\, dx = 0$$

$$\frac{y'^2}{2} - \frac{y^4}{4} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{2y^4 + 8c_1}}{2} \tag{1}$$

$$y' = -\frac{\sqrt{2y^4 + 8c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2}{\sqrt{2y^4 + 8c_1}} dy = \int dx$$

$$\frac{\sqrt{2}\,\sqrt{4 - \frac{2iy^2}{\sqrt{c_1}}}\,\sqrt{4 + \frac{2iy^2}{\sqrt{c_1}}}\,\text{EllipticF}\left(\frac{y\sqrt{2}\,\sqrt{\frac{i}{\sqrt{c_1}}}}{2}, i\right)}{2\sqrt{\frac{i}{\sqrt{c_1}}}\,\sqrt{2y^4 + 8c_1}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2}{\sqrt{2y^4 + 8c_1}} dy = \int dx$$

$$-\frac{\sqrt{2}\,\sqrt{4 - \frac{2iy^2}{\sqrt{c_1}}}\,\sqrt{4 + \frac{2iy^2}{\sqrt{c_1}}}\,\text{EllipticF}\left(\frac{y\sqrt{2}\,\sqrt{\frac{i}{\sqrt{c_1}}}}{2}, i\right)}{2\sqrt{\frac{i}{\sqrt{c_1}}}\,\sqrt{2y^4 + 8c_1}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\sqrt{2}\,\sqrt{4 - \frac{2iy^2}{\sqrt{c_1}}}\,\sqrt{4 + \frac{2iy^2}{\sqrt{c_1}}}\,\text{EllipticF}\left(\frac{y\sqrt{2}\,\sqrt{\frac{i}{\sqrt{c_1}}}}{2}, i\right)}{2\sqrt{\frac{i}{\sqrt{c_1}}}\,\sqrt{2y^4 + 8c_1}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$\frac{\left(-\frac{1}{2} - \frac{i}{2}\right)\sqrt{\frac{2\sqrt{c_1} - i}{\sqrt{c_1}}}\,\sqrt{\frac{2\sqrt{c_1} + i}{\sqrt{c_1}}}\,\sqrt{2}\,c_1^{\frac{1}{4}}\,\text{EllipticF}\left(\frac{\frac{1}{2} - \frac{i}{2}}{c_1^{\frac{1}{4}}}, i\right)}{\sqrt{4c_1 + 1}} = c_2 \tag{1A}$$

Unable to solve for y to solve for constant of integration

Looking at the Second solution

$$-\frac{\sqrt{2}\,\sqrt{4-\frac{2iy^2}{\sqrt{c_1}}}\,\sqrt{4+\frac{2iy^2}{\sqrt{c_1}}}\,\operatorname{EllipticF}\left(\frac{y\sqrt{2}\,\sqrt{\frac{i}{\sqrt{c_1}}}}{2}, i\right)}{2\sqrt{\frac{i}{\sqrt{c_1}}}\,\sqrt{2y^4+8c_1}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{\frac{2\sqrt{c_1}-i}{\sqrt{c_1}}}\,\sqrt{\frac{2\sqrt{c_1}+i}{\sqrt{c_1}}}\,\sqrt{2}\,c_1^{\frac{1}{4}}\operatorname{EllipticF}\left(\frac{\frac{1}{2}-\frac{i}{2}}{c_1^{\frac{1}{4}}}, i\right)}{\sqrt{4c_1+1}} = c_3 \tag{1A}$$

Unable to solve for y to solve for constant of integration

Verification of solutions N/A

18.31.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - y^3 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{y^3}{p} \end{aligned}$$

Where $f(y) = y^3$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\frac{1}{\frac{1}{p}}\,dp = y^3\,dy$$

$$\int \frac{1}{\frac{1}{p}}\,dp = \int y^3\,dy$$

$$\frac{p^2}{2} = \frac{y^4}{4} + c_1$$

The solution is

$$\frac{p(y)^2}{2} - \frac{y^4}{4} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $y = -1$ and $p = \frac{\sqrt{2}}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-c_1 = 0$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2}{2} - \frac{y^4}{4} = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = \frac{\sqrt{2}\,y^2}{2}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{2}\,y^2}{2}$$

Integrating both sides gives

$$\int \frac{\sqrt{2}}{y^2} dy = x + c_2$$

$$-\frac{\sqrt{2}}{y} = x + c_2$$

Solving for y gives these solutions

$$y_1 = -\frac{\sqrt{2}}{x + c_2}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{\sqrt{2}}{c_2}$$

$$c_2 = \sqrt{2}$$

Substituting c_2 found above in the general solution gives

$$y = -\frac{\sqrt{2}}{x + \sqrt{2}}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{2}}{x + \sqrt{2}} \tag{1}$$

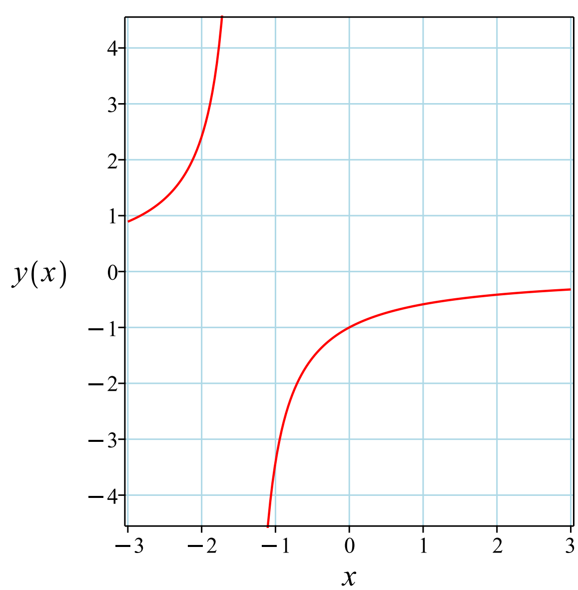


Figure 719: Solution plot

Verification of solutions

$$y = -\frac{\sqrt{2}}{x+\sqrt{2}}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
<- 2nd_order JacobiSN successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)=y(x)^3,y(0) = -1, D(y)(0) = 1/2*sqrt(2)],y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2}}{x+\sqrt{2}}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 18

```
DSolve[{y''[x]==y[x]^3,{y[0]==1,y'[0]==Sqrt[2]/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{\sqrt{2}x-2}$$

18.32 problem 32

Internal problem ID [2304]
Internal file name [`OUTPUT/2304_Tuesday_February_27_2024_08_25_37_AM_70736289/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{y'' - {y'}^2 \cos(x) = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

18.32.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - p(x)^2 \cos(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned}
p' &= F(x,p) \\
&= f(x)g(p) \\
&= p^2 \cos(x)
\end{aligned}$$

Where $f(x) = \cos(x)$ and $g(p) = p^2$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p^2}\,dp &= \cos(x)\,dx \\
\int \frac{1}{p^2}\,dp &= \int \cos(x)\,dx \\
-\frac{1}{p} &= \sin(x) + c_1
\end{aligned}$$

The solution is

$$-\frac{1}{p(x)} - \sin(x) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 - c_1 = 0$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$-\frac{\sin(x)\,p + 1 - p}{p} = 0$$

The above simplifies to

$$-\sin(x)\,p + p - 1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-y'\sin(x) + y' - 1 = 0$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{1}{\sin (x) - 1}\, \mathrm{d}x \\
&= -\frac{2}{-1 + \tan \left(\frac{x}{2}\right)} + c_2
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = c_2 + 2
$$

$$
c_2 = 0
$$

Substituting c_2 found above in the general solution gives

$$
y = -\frac{2}{-1 + \tan \left(\frac{x}{2}\right)}
$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$
y = -\frac{2}{-1 + \tan \left(\frac{x}{2}\right)} \tag{1}
$$

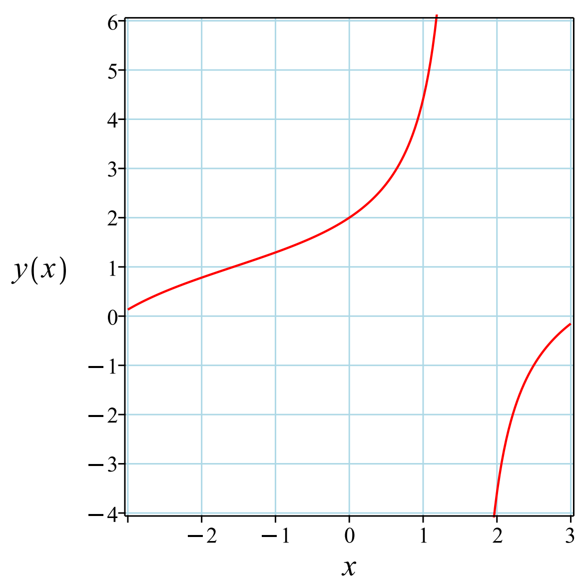


Figure 720: Solution plot

Verification of solutions

$$y = -\frac{2}{-1 + \tan\left(\frac{x}{2}\right)}$$

Verified OK.

18.32.2 Maple step by step solution

Let's solve

$\left[y'' - y'^2 \cos(x) = 0, y(0) = 2, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - u(x)^2 \cos(x) = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^2} = \cos(x)$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2} dx = \int \cos(x) dx + c_1$

- Evaluate integral

  $-\frac{1}{u(x)} = \sin(x) + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{1}{\sin(x)+c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{1}{\sin(x)+c_1}$

- Make substitution $u = y'$

  $y' = -\frac{1}{\sin(x)+c_1}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{1}{\sin(x)+c_1} dx + c_2$

- Compute integrals

$$y = -\frac{2\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)c_1+2}{2\sqrt{c_1^2-1}}\right)}{\sqrt{c_1^2-1}} + c_2$$

☐ Check validity of solution $y = -\frac{2\arctan\left(\frac{2\tan\left(\frac{x}{2}\right)c_1+2}{2\sqrt{c_1^2-1}}\right)}{\sqrt{c_1^2-1}} + c_2$

- Use initial condition $y(0) = 2$

$$2 = -\frac{2\arctan\left(\frac{1}{\sqrt{c_1^2-1}}\right)}{\sqrt{c_1^2-1}} + c_2$$

- Compute derivative of the solution

$$y' = -\frac{2\left(\frac{1}{2}+\frac{\tan\left(\frac{x}{2}\right)^2}{2}\right)c_1}{(c_1^2-1)\left(\frac{\left(2\tan\left(\frac{x}{2}\right)c_1+2\right)^2}{4\left(c_1^2-1\right)}+1\right)}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

$$1 = -\frac{c_1}{(c_1^2-1)\left(\frac{1}{c_1^2-1}+1\right)}$$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)^2*cos(_a), _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✗ Solution by Maple

```
dsolve([diff(y(x),x$2)=diff(y(x),x)^2*cos(x),y(0) = 2, D(y)(0) = 1],y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]==y'[x]^2*Cos[x],{y[0]==2,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

{}

18.33 problem 33

Internal problem ID [2305]
Internal file name [`OUTPUT/2305_Tuesday_February_27_2024_08_25_37_AM_29486375/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order,
   _with_potential_symmetries], [_2nd_order, _reducible, _mu_xy
   ]]
```

$$\boxed{yy'' - y^2y' - y'^2 = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

18.33.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right)+\left(-y^2-p(y)\right)p(y)=0$$

Which is now solved as first order ode for $p(y)$.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{y}dy}\\ &= \frac{1}{y}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}(\mu p) &= (\mu)(y)\\ \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{p}{y}\right) &= \left(\frac{1}{y}\right)(y)\\ \mathrm{d}\left(\frac{p}{y}\right) &= \mathrm{d}y\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{y} &= \int \mathrm{d}y\\ \frac{p}{y} &= y+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=\frac{1}{y}$ results in

$$p(y)=c_1y+y^2$$

which simplifies to

$$p(y)=y(y+c_1)$$

Initial conditions are used to solve for c_1. Substituting $y=2$ and $p=1$ in the above solution gives an equation to solve for the constant of integration.

$$1=4+2c_1$$

$$c_1=-\frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$p(y) = \frac{y(2y-3)}{2}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{y(2y-3)}{2}$$

Integrating both sides gives

$$\int \frac{2}{y(2y-3)} dy = \int dx$$
$$\frac{2\ln(2y-3)}{3} - \frac{2\ln(y)}{3} = x + c_2$$

The above can be written as

$$\left(\frac{2}{3}\right)(\ln(2y-3) - \ln(y)) = x + c_2$$
$$\ln(2y-3) - \ln(y) = \left(\frac{3}{2}\right)(x + c_2)$$
$$= \frac{3x}{2} + \frac{3c_2}{2}$$

Raising both side to exponential gives

$$e^{\ln(2y-3) - \ln(y)} = \frac{3c_2 e^{\frac{3x}{2}}}{2}$$

Which simplifies to

$$\frac{2y-3}{y} = c_3 e^{\frac{3x}{2}}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{3}{-2 + c_3}$$

$$c_3 = \frac{1}{2}$$

Substituting c_3 found above in the general solution gives

$$y = -\frac{6}{e^{\frac{3x}{2}} - 4}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = -\frac{6}{e^{\frac{3x}{2}} - 4} \tag{1}$$

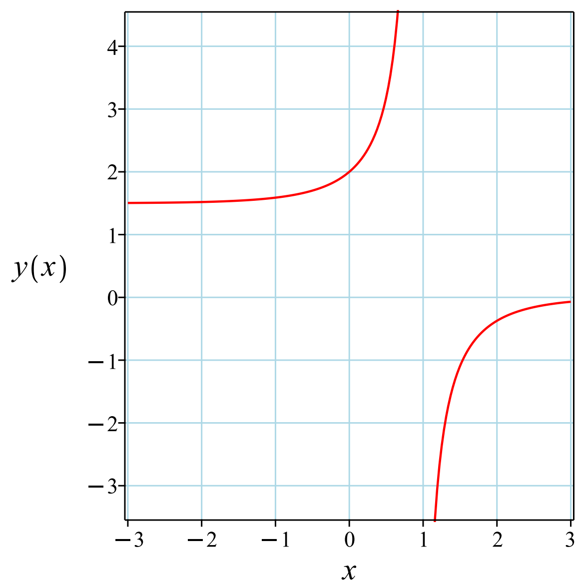


Figure 721: Solution plot

Verification of solutions

$$y = -\frac{6}{e^{\frac{3x}{2}} - 4}$$

Verified OK.

18.33.2 Maple step by step solution

Let's solve

$$\left[y y'' + \left(-y^2 - y'\right) y' = 0, y(0) = 2, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

$u(x) = y'$

- Compute y''

$u'(x) = y''$

- Use chain rule on the lhs

$y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

$u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$yu(y)\left(\frac{d}{dy}u(y)\right) + (-y^2 - u(y))\,u(y) = 0$

- Isolate the derivative

$\frac{d}{dy}u(y) = \frac{u(y)}{y} + y$

- Group terms with $u(y)$ on the lhs of the ODE and the rest on the rhs of the ODE

$\frac{d}{dy}u(y) - \frac{u(y)}{y} = y$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

$\mu(y)\left(\frac{d}{dy}u(y) - \frac{u(y)}{y}\right) = \mu(y)\,y$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\,u(y))$

$\mu(y)\left(\frac{d}{dy}u(y) - \frac{u(y)}{y}\right) = \left(\frac{d}{dy}\mu(y)\right)u(y) + \mu(y)\left(\frac{d}{dy}u(y)\right)$

- Isolate $\frac{d}{dy}\mu(y)$

$\frac{d}{dy}\mu(y) = -\frac{\mu(y)}{y}$

- Solve to find the integrating factor

$\mu(y) = \frac{1}{y}$

- Integrate both sides with respect to y

$\int \left(\frac{d}{dy}(\mu(y)\,u(y))\right) dy = \int \mu(y)\,y dy + c_1$

- Evaluate the integral on the lhs

$\mu(y)\,u(y) = \int \mu(y)\,y dy + c_1$

- Solve for $u(y)$

$u(y) = \frac{\int \mu(y) y dy + c_1}{\mu(y)}$

- Substitute $\mu(y) = \frac{1}{y}$

  $u(y) = y\left(\int 1 dy + c_1\right)$

- Evaluate the integrals on the rhs

  $u(y) = y(y + c_1)$

- Solve 1st ODE for $u(y)$

  $u(y) = y(y + c_1)$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = y(y + c_1)$

- Separate variables

  $\frac{y'}{y(y+c_1)} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y+c_1)} dx = \int 1 dx + c_2$

- Evaluate integral

  $-\frac{\ln(y+c_1)}{c_1} + \frac{\ln(y)}{c_1} = x + c_2$

- Solve for y

  $y = -\frac{c_1 \mathrm{e}^{c_1 c_2 + c_1 x}}{-1 + \mathrm{e}^{c_1 c_2 + c_1 x}}$

☐ Check validity of solution $y = -\frac{c_1 \mathrm{e}^{c_1 c_2 + c_1 x}}{-1 + \mathrm{e}^{c_1 c_2 + c_1 x}}$

- ∘ Use initial condition $y(0) = 2$

  $2 = -\frac{c_1 \mathrm{e}^{c_1 c_2}}{-1 + \mathrm{e}^{c_1 c_2}}$

- ∘ Compute derivative of the solution

  $y' = \frac{c_1^2 \left(\mathrm{e}^{c_1 c_2 + c_1 x}\right)^2}{\left(-1 + \mathrm{e}^{c_1 c_2 + c_1 x}\right)^2} - \frac{c_1^2 \mathrm{e}^{c_1 c_2 + c_1 x}}{-1 + \mathrm{e}^{c_1 c_2 + c_1 x}}$

- ∘ Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = \frac{c_1^2 \left(\mathrm{e}^{c_1 c_2}\right)^2}{\left(-1 + \mathrm{e}^{c_1 c_2}\right)^2} - \frac{c_1^2 \mathrm{e}^{c_1 c_2}}{-1 + \mathrm{e}^{c_1 c_2}}$

- ∘ Solve for c_1 and c_2

  $\left\{c_1 = -\frac{3}{2}, c_2 = -\frac{4 \ln(2)}{3}\right\}$

- ∘ Substitute constant values into general solution and simplify

$$y = \frac{6\,\mathrm{e}^{-\frac{3x}{2}}}{-1+4\,\mathrm{e}^{-\frac{3x}{2}}}$$

- Solution to the IVP

$$y = \frac{6\,\mathrm{e}^{-\frac{3x}{2}}}{-1+4\,\mathrm{e}^{-\frac{3x}{2}}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-_b(_a)*(_a^2+_b(_a))/_a = 0, _b(
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 14

```
dsolve([y(x)*diff(y(x),x$2)-y(x)^2*diff(y(x),x)=diff(y(x),x)^2,y(0) = 2, D(y)(0) = 1],y(x),
```

$$y(x) = -\frac{6}{\mathrm{e}^{\frac{3x}{2}} - 4}$$

✓ Solution by Mathematica

Time used: 1.908 (sec). Leaf size: 18

```
DSolve[{y[x]*y''[x]-y[x]^2*y'[x]==y'[x]^2,{y[0]==2,y'[0]==1}},y[x],x,IncludeSingularSolution
```

$$y(x) \to -\frac{6}{e^{3x/2} - 4}$$

18.34 problem 34

Internal problem ID [2306]
Internal file name [`OUTPUT/2306_Tuesday_February_27_2024_08_25_39_AM_65165895/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{(x^2+1)\,y''+y'^2=-1}$$

With initial conditions

$$[y(0)=1, y'(0)=1]$$

18.34.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$(x^2+1)\,p'(x)+1+p(x)^2=0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{-p^2-1}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(p) = -p^2-1$. Integrating both sides gives

$$\frac{1}{-p^2-1}\,dp = \frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{-p^2-1}\,dp = \int \frac{1}{x^2+1}\,dx$$

$$-\arctan(p) = \arctan(x) + c_1$$

The solution is

$$-\arctan(p(x)) - \arctan(x) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\pi}{4} - c_1 = 0$$

$$c_1 = -\frac{\pi}{4}$$

Substituting c_1 found above in the general solution gives

$$-\arctan(p) - \arctan(x) + \frac{\pi}{4} = 0$$

Solving for $p(x)$ from the above gives

$$p(x) = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \cot\left(\arctan(x) + \frac{\pi}{4}\right)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \cot\left(\arctan(x) + \frac{\pi}{4}\right) dx \\ &= -x + 2\ln(x+1) + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2$$

$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$y = -x + 2\ln(x+1) + 1$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = -x + 2\ln(x+1) + 1 \tag{1}$$

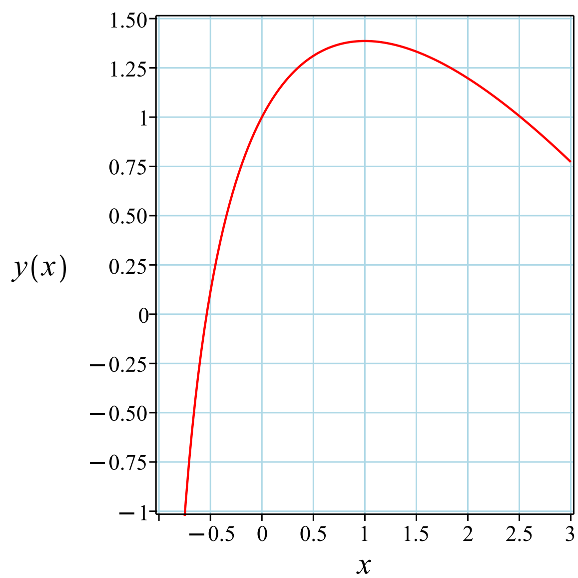


Figure 722: Solution plot

Verification of solutions

$$y = -x + 2\ln(x+1) + 1$$

Verified OK.

18.34.2 Maple step by step solution

Let's solve

$$\left[(x^2+1)\, y''+y'^2=-1, y(0)=1, y'\Big|_{\{x=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $(x^2+1)\, u'(x)+u(x)^2=-1$

- Separate variables

  $\frac{u'(x)}{-u(x)^2-1}=\frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{-u(x)^2-1}dx=\int \frac{1}{x^2+1}dx+c_1$

- Evaluate integral

  $-\arctan\left(u(x)\right)=\arctan\left(x\right)+c_1$

- Solve for $u(x)$

  $u(x)=-\tan\left(\arctan\left(x\right)+c_1\right)$

- Solve 1st ODE for $u(x)$

  $u(x)=-\tan\left(\arctan\left(x\right)+c_1\right)$

- Make substitution $u=y'$

  $y'=-\tan\left(\arctan\left(x\right)+c_1\right)$

- Integrate both sides to solve for y

  $\int y'dx=\int -\tan\left(\arctan\left(x\right)+c_1\right)dx+c_2$

- Compute integrals

  $y=\frac{\mathrm{I}\, \mathrm{e}^{4\, \mathrm{I} c_1} x}{\left(\mathrm{e}^{2\, \mathrm{I} c_1}-1\right)^2}-\frac{4\, \mathrm{e}^{2\, \mathrm{I} c_1} \ln\left(\left(-\mathrm{e}^{2\, \mathrm{I} c_1}+1\right)x+\mathrm{I}\, \mathrm{e}^{2\, \mathrm{I} c_1}+\mathrm{I}\right)}{\left(\mathrm{e}^{2\, \mathrm{I} c_1}-1\right)^2}-\frac{\mathrm{I} x}{\left(\mathrm{e}^{2\, \mathrm{I} c_1}-1\right)^2}+c_2$

□ Check validity of solution $y=\frac{\mathrm{I}\mathrm{e}^{4\, \mathrm{I} c_1} x}{\left(\mathrm{e}^{2\, \mathrm{I} c_1}-1\right)^2}-\frac{4\mathrm{e}^{2\, \mathrm{I} c_1} \ln\left(\left(-\mathrm{e}^{2\, \mathrm{I} c_1}+1\right)x+\mathrm{I}\mathrm{e}^{2\, \mathrm{I} c_1}+\mathrm{I}\right)}{\left(\mathrm{e}^{2\, \mathrm{I} c_1}-1\right)^2}-\frac{\mathrm{I} x}{\left(\mathrm{e}^{2\, \mathrm{I} c_1}-1\right)^2}+c_2$

  - Use initial condition $y(0)=1$

    $1=-\frac{4\, \mathrm{e}^{2\, \mathrm{I} c_1} \ln\left(\mathrm{I}+\mathrm{I}\, \mathrm{e}^{2\, \mathrm{I} c_1}\right)}{\left(\mathrm{e}^{2\, \mathrm{I} c_1}-1\right)^2}+c_2$

  - Compute derivative of the solution

$$y' = \frac{\mathrm{I}\,\mathrm{e}^{4\,\mathrm{I}c_1}}{\left(\mathrm{e}^{2\,\mathrm{I}c_1}-1\right)^2} - \frac{4\,\mathrm{e}^{2\,\mathrm{I}c_1}\left(-\mathrm{e}^{2\,\mathrm{I}c_1}+1\right)}{\left(\mathrm{e}^{2\,\mathrm{I}c_1}-1\right)^2\left(\left(-\mathrm{e}^{2\,\mathrm{I}c_1}+1\right)x+\mathrm{I}\,\mathrm{e}^{2\,\mathrm{I}c_1}+\mathrm{I}\right)} - \frac{\mathrm{I}}{\left(\mathrm{e}^{2\,\mathrm{I}c_1}-1\right)^2}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

$$1 = \frac{\mathrm{I}\,\mathrm{e}^{4\,\mathrm{I}c_1}}{\left(\mathrm{e}^{2\,\mathrm{I}c_1}-1\right)^2} - \frac{4\,\mathrm{e}^{2\,\mathrm{I}c_1}\left(-\mathrm{e}^{2\,\mathrm{I}c_1}+1\right)}{\left(\mathrm{e}^{2\,\mathrm{I}c_1}-1\right)^2\left(\mathrm{I}+\mathrm{I}\,\mathrm{e}^{2\,\mathrm{I}c_1}\right)} - \frac{\mathrm{I}}{\left(\mathrm{e}^{2\,\mathrm{I}c_1}-1\right)^2}$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{\pi}{4}, c_2 = 1 - 2\ln\left(1+\mathrm{I}\right)\right\}$$

- Substitute constant values into general solution and simplify

$$y = -x + 2\ln\left(\left(1+\mathrm{I}\right)\left(x+1\right)\right) + 1 - \ln\left(2\right) - \frac{\mathrm{I}\pi}{2}$$

- Solution to the IVP

$$y = -x + 2\ln\left(\left(1+\mathrm{I}\right)\left(x+1\right)\right) + 1 - \ln\left(2\right) - \frac{\mathrm{I}\pi}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 21

```
dsolve([(1+x^2)*diff(y(x),x$2)+1+diff(y(x),x)^2=0,y(0) = 1, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = -x + 2\ln\left(-x-1\right) - 2i\pi + 1$$

✓ Solution by Mathematica

Time used: 6.806 (sec). Leaf size: 23

```
DSolve[{(1+x^2)*y''[x]+1+y'[x]^2==0,{y[0]==1,y'[0]==1}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -x + 2\log(-x-1) - 2i\pi + 1$$

18.35 problem 35

Internal problem ID [2307]
Internal file name [OUTPUT/2307_Tuesday_February_27_2024_08_25_39_AM_11739347/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{yy'' - y^3 - y'^2 = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

18.35.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y) \left(\frac{d}{dy}p(y)\right) - y^3 - p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. Using the change of variables $p(y) = u(y)\,y$ on the above ode results in new ode in $u(y)$

$$y^2u(y) \left(\left(\frac{d}{dy}u(y)\right) y + u(y)\right) - u(y)^2\, y^2 = y^3$$

Integrating both sides gives

$$\int u du = y + c_2$$
$$\frac{u^2}{2} = y + c_2$$

Solving for u gives these solutions

$$u_1 = \sqrt{2c_2 + 2y}$$
$$u_2 = -\sqrt{2c_2 + 2y}$$

Therefore the solution $p(y)$ is

$$\begin{aligned} p(y) &= yu \\ &= y\sqrt{2c_2 + 2y} \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $y = 1$ and $p = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \sqrt{2c_2 + 2}$$

$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$p(y) = \sqrt{2 + 2y}\, y$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \sqrt{2 + 2y}\, y$$

Integrating both sides gives

$$\int \frac{1}{\sqrt{2+2y}\, y} dy = \int dx$$

$$-\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\, \sqrt{2+2y}}{2}\right) = x + c_3$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\operatorname{arccoth}\left(\sqrt{2}\right) \sqrt{2} + \frac{i\pi\sqrt{2}}{2} = c_3$$

$$c_3 = -\operatorname{arccoth}\left(\sqrt{2}\right) \sqrt{2} + \frac{i\pi\sqrt{2}}{2}$$

Substituting c_3 found above in the general solution gives

$$-\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\, \sqrt{2+2y}}{2}\right) = x - \operatorname{arccoth}\left(\sqrt{2}\right) \sqrt{2} + \frac{i\pi\sqrt{2}}{2}$$

Initial conditions are used to solve for the constants of integration.

<u>Summary</u>

The solution(s) found are the following

$$y = \tanh\left(\frac{\left(-i\pi\sqrt{2} + 2 \operatorname{arccoth}\left(\sqrt{2}\right) \sqrt{2} - 2x\right) \sqrt{2}}{4}\right)^2 - 1 \tag{1}$$

<u>Verification of solutions</u>

$$y = \tanh\left(\frac{\left(-i\pi\sqrt{2} + 2 \operatorname{arccoth}\left(\sqrt{2}\right) \sqrt{2} - 2x\right) \sqrt{2}}{4}\right)^2 - 1$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(_a^3+_b(_a)^2)/_a = 0, _b(_a),
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 3/2*_b]
```

✓ Solution by Maple

Time used: 18.109 (sec). Leaf size: 25

```
dsolve([y(x)*diff(y(x),x$2)=y(x)^3+diff(y(x),x)^2,y(0) = 1, D(y)(0) = 2],y(x), singsol=all)
```

$$y(x) = -\operatorname{sech}\left(\frac{\sqrt{2}\,(x-\sqrt{2}\operatorname{arctanh}(\sqrt{2}))}{2}\right)^2$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y[x]*y''[x]==y[x]^3+y'[x]^2,{y[0]==1,y'[0]==2}},y[x],x,IncludeSingularSolutions -> T
```

{}

18.36 problem 36

Internal problem ID [2308]
Internal file name [`OUTPUT/2308_Tuesday_February_27_2024_08_25_48_AM_89056847/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{\left(1+y'^2\right)^2 - y^2 y'' = 0}$$

With initial conditions

$$\left[y(0) = 3, y'(0) = \sqrt{2}\right]$$

18.36.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$-y^2p(y)\left(\frac{d}{dy}p(y)\right)+\left(p(y)^3+2p(y)\right)p(y)=-1$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}p' &= F(y,p)\\ &= f(y)g(p)\\ &= \frac{(p^2+1)^2}{y^2p}\end{aligned}$$

Where $f(y)=\frac{1}{y^2}$ and $g(p)=\frac{(p^2+1)^2}{p}$. Integrating both sides gives

$$\frac{1}{\frac{(p^2+1)^2}{p}}\,dp=\frac{1}{y^2}\,dy$$

$$\int\frac{1}{\frac{(p^2+1)^2}{p}}\,dp=\int\frac{1}{y^2}\,dy$$

$$-\frac{1}{2(p^2+1)}=-\frac{1}{y}+c_1$$

The solution is

$$-\frac{1}{2\left(p(y)^2+1\right)}+\frac{1}{y}-c_1=0$$

Initial conditions are used to solve for c_1. Substituting $y=3$ and $p=\sqrt{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-c_1+\frac{1}{6}=0$$

$$c_1=\frac{1}{6}$$

Substituting c_1 found above in the general solution gives

$$-\frac{p^2y-6p^2+4y-6}{6(p^2+1)y}=0$$

The above simplifies to

$$-p^2y+6p^2-4y+6=0$$

Solving for $p(y)$ from the above gives

$$p(y) = -\frac{\sqrt{2}\,\sqrt{-2y^2+15y-18}}{y-6}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\sqrt{2}\,\sqrt{-2y^2+15y-18}}{y-6}$$

Integrating both sides gives

$$\int -\frac{(y-6)\sqrt{2}}{2\sqrt{-2y^2+15y-18}}dy = \int dx$$

$$-\frac{\sqrt{2}\left(-\frac{\sqrt{-2y^2+15y-18}}{2} - \frac{9\sqrt{2}\arcsin\left(\frac{4y}{9}-\frac{5}{3}\right)}{8}\right)}{2} = x + c_2$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{9\arcsin\left(\frac{1}{3}\right)}{8} + \frac{3\sqrt{2}}{4} = c_2$$

$$c_2 = -\frac{9\arcsin\left(\frac{1}{3}\right)}{8} + \frac{3\sqrt{2}}{4}$$

Substituting c_2 found above in the general solution gives

$$-\frac{\sqrt{2}\left(-\frac{\sqrt{-2y^2+15y-18}}{2} - \frac{9\sqrt{2}\arcsin\left(\frac{4y}{9}-\frac{5}{3}\right)}{8}\right)}{2} = x - \frac{9\arcsin\left(\frac{1}{3}\right)}{8} + \frac{3\sqrt{2}}{4}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$\frac{9\arcsin\left(\frac{4y}{9}-\frac{5}{3}\right)}{8} + \frac{\sqrt{2}\,\sqrt{-2y^2+15y-18}}{4} = x - \frac{9\arcsin\left(\frac{1}{3}\right)}{8} + \frac{3\sqrt{2}}{4} \tag{1}$$

Verification of solutions

$$\frac{9\arcsin\left(\frac{4y}{9}-\frac{5}{3}\right)}{8} + \frac{\sqrt{2}\,\sqrt{-2y^2+15y-18}}{4} = x - \frac{9\arcsin\left(\frac{1}{3}\right)}{8} + \frac{3\sqrt{2}}{4}$$

Verified OK.

18.36.2 Maple step by step solution

Let's solve

$$\left[-y^2y''+\left(y'^3+2y'\right)y'=-1, y(0)=3, y'\Big|_{\{x=0\}}=\sqrt{2}\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x)=y'$

- Compute y''

  $u'(x)=y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right)=y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right)=y''$

- Make substitutions $y'=u(y), y''=u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $-y^2u(y)\left(\frac{d}{dy}u(y)\right)+\left(u(y)^3+2u(y)\right)u(y)=-1$

- Separate variables

  $\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{-\left(u(y)^3+2u(y)\right)u(y)-1}=-\frac{1}{y^2}$

- Integrate both sides with respect to y

  $\int\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{-\left(u(y)^3+2u(y)\right)u(y)-1}dy=\int-\frac{1}{y^2}dy+c_1$

- Evaluate integral

  $\frac{1}{2\left(u(y)^2+1\right)}=\frac{1}{y}+c_1$

- Solve for $u(y)$

  $\left\{u(y)=-\frac{\sqrt{-2(yc_1+1)(2yc_1-y+2)}}{2(yc_1+1)}, u(y)=\frac{\sqrt{-2(yc_1+1)(2yc_1-y+2)}}{2(yc_1+1)}\right\}$

- Solve 1st ODE for $u(y)$

  $u(y)=-\frac{\sqrt{-2(yc_1+1)(2yc_1-y+2)}}{2(yc_1+1)}$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\frac{\sqrt{-2(yc_1+1)(2yc_1-y+2)}}{2(yc_1+1)}$$

- Separate variables

$$\frac{y'(yc_1+1)}{\sqrt{-2(yc_1+1)(2yc_1-y+2)}} = -\frac{1}{2}$$

- Integrate both sides with respect to x

$$\int \frac{y'(yc_1+1)}{\sqrt{-2(yc_1+1)(2yc_1-y+2)}}dx = \int -\frac{1}{2}dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2}\arctan\left(\frac{\sqrt{2}\sqrt{c_1(-1+2c_1)}\left(y-\frac{-8c_1+2}{4c_1(-1+2c_1)}\right)}{\sqrt{-4-2c_1(-1+2c_1)y^2+(-8c_1+2)y}}\right)}{2\sqrt{c_1(-1+2c_1)}} + c_1\left(-\frac{\sqrt{-4-2c_1(-1+2c_1)y^2+(-8c_1+2)y}}{2c_1(-1+2c_1)} + \frac{(-8c_1+2)\sqrt{2}\arctan\left(\frac{\sqrt{2}\sqrt{c_1}}{\sqrt{-4-}}\right.}{8c_1(-1+2c_1)}\right.$$

- Solve for y

$$\left\{\frac{RootOf\left(\_Z^2+(16c_1-4)\_Z+256c_1^4c_2^2-256c_1^4c_2x+64c_1^4x^2+32\sqrt{2}\sqrt{2c_1^2-c_1}\,c_1c_2RootOf\left((-1+2c_1)c_1\left(-1+48c_1^2c_2^2-4\sqrt{2}\sqrt{2c_1^2-}\right.\right.\right.}{}\right.$$

- Solve 2nd ODE for $u(y)$

$$u(y) = \frac{\sqrt{-2(yc_1+1)(2yc_1-y+2)}}{2(yc_1+1)}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{\sqrt{-2(yc_1+1)(2yc_1-y+2)}}{2(yc_1+1)}$$

- Separate variables

$$\frac{y'(yc_1+1)}{\sqrt{-2(yc_1+1)(2yc_1-y+2)}} = \frac{1}{2}$$

- Integrate both sides with respect to x

$$\int \frac{y'(yc_1+1)}{\sqrt{-2(yc_1+1)(2yc_1-y+2)}}dx = \int \frac{1}{2}dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2}\arctan\left(\frac{\sqrt{2}\sqrt{c_1(-1+2c_1)}\left(y-\frac{-8c_1+2}{4c_1(-1+2c_1)}\right)}{\sqrt{-4-2c_1(-1+2c_1)y^2+(-8c_1+2)y}}\right)}{2\sqrt{c_1(-1+2c_1)}} + c_1\left(-\frac{\sqrt{-4-2c_1(-1+2c_1)y^2+(-8c_1+2)y}}{2c_1(-1+2c_1)} + \frac{(-8c_1+2)\sqrt{2}\arctan\left(\frac{\sqrt{2}\sqrt{c_1}}{\sqrt{-4-}}\right.}{8c_1(-1+2c_1)}\right.$$

- Solve for y

$$\left\{\frac{RootOf\left(\_Z^2+(16c_1-4)\_Z+256c_1^4c_2^2+256c_1^4c_2x+64c_1^4x^2+32\sqrt{2c_1^2-c_1}\,\sqrt{2}\,c_1c_2\,RootOf\left((-1+2c_1)c_1\left(-1+48c_1^2c_2^2-4\sqrt{2}\,\sqrt{2c_1^2-\right.\right.\right.}{}\right.$$

- Check validity of solution $\frac{RootOf\left(\_Z^2+(16c_1-4)\_Z+256c_1^4c_2^2-256c_1^4c_2x+64c_1^4x^2+32\sqrt{2}\,\sqrt{2c_1^2-c_1}\,c_1c_2\,RootOf\left((-1\right.\right.}{}$
  - Use initial condition $y(0)=3$

    $$\frac{RootOf\left(\_Z^2+(16c_1-4)\_Z+256c_1^4c_2^2+32\sqrt{2}\,\sqrt{2c_1^2-c_1}\,c_1c_2\,RootOf\left((-1+2c_1)c_1\left(-1+48c_1^2c_2^2-4\sqrt{2}\,\sqrt{2c_1^2-c_1}\,c_2\_Z-4\sqrt{2c_1^2-c_1}\right.\right.\right.}{}$$

  - Compute derivative of the solution
  - Use the initial condition $y'\Big|_{\{x=0\}}=\sqrt{2}$
  - Solve for c_1 and c_2

    $$\left\{c_1=c_1, c_2=\frac{2\,RootOf\left(-1+(64c_1^2-32c_1)\_Z^2\right)}{-1+2c_1}\right\}$$

  - Substitute constant values into general solution and simplify

    $$\frac{RootOf\left(\_Z^2+(16c_1-4)\_Z+4\cos\left(RootOf\left(\left(-16c_1^4x^2+4\sqrt{2}\,\sqrt{2c_1^2-c_1}\,c_1x\_Z+64c_1^3\,RootOf\left(-1+(64c_1^2-32c_1)\_Z^2\right)x+24c_1^3x^2\right.\right.\right.\right.}{}$$

- Check validity of solution $\frac{RootOf\left(\_Z^2+(16c_1-4)\_Z+256c_1^4c_2^2+256c_1^4c_2x+64c_1^4x^2+32\sqrt{2c_1^2-c_1}\,\sqrt{2}\,c_1c_2\,RootOf\left((-1\right.\right.}{}$
  - Use initial condition $y(0)=3$

    $$\frac{RootOf\left(\_Z^2+(16c_1-4)\_Z+256c_1^4c_2^2+32\sqrt{2}\,\sqrt{2c_1^2-c_1}\,c_1c_2\,RootOf\left((-1+2c_1)c_1\left(-1+48c_1^2c_2^2-4\sqrt{2}\,\sqrt{2c_1^2-c_1}\,c_2\_Z-4\sqrt{2c_1^2-c_1}\right.\right.\right.}{}$$

  - Compute derivative of the solution
  - Use the initial condition $y'\Big|_{\{x=0\}}=\sqrt{2}$
  - Solve for c_1 and c_2

    $$\left\{c_1=c_1, c_2=\frac{2\,RootOf\left(-1+(64c_1^2-32c_1)\_Z^2\right)}{-1+2c_1}\right\}$$

  - Substitute constant values into general solution and simplify

    $$\frac{RootOf\left(\_Z^2+(16c_1-4)\_Z+4\cos\left(RootOf\left((-1+2c_1)\left(16c_1^4x^2+4\sqrt{2}\,\sqrt{2c_1^2-c_1}\,c_1x\_Z+64c_1^3\,RootOf\left(-1+(64c_1^2-32c_1)\_Z^2\right)x\right.\right.\right.\right.}{}$$

- Solutions to the IVP

$$\begin{cases} \frac{RootOf\left(\_Z^2+(16c_1-4)\_Z+4\cos\left(RootOf\left((-1+2c_1)\left(16c_1^4x^2+4\sqrt{2}\sqrt{2c_1^2-c_1}\,c_1x\_Z+64c_1^3RootOf\left(-1+(64c_1^2-32c_1)\_Z^2\right.\right.\right.\right.\right.}{} \end{cases}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(_b(_a)^2+1)^2/_a^2 = 0, _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   <- separable successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 75.984 (sec). Leaf size: 221

```
dsolve([(1+diff(y(x),x)^2)^2=y(x)^2*diff(y(x),x$2),y(0) = 3, D(y)(0) = sqrt(2)],y(x), singso
```

$y(x)$

$$= \mathrm{RootOf}\left(\sqrt{2}\left(\int_{\_Z}^{3} \frac{\mathrm{RootOf}\left(\left(-\sqrt{-(3\_Z-1)(6\_Z+1)}+6\right.\right.}{\sqrt{-\left(\mathrm{RootOf}\left(\left(-\sqrt{-(3\_Z-1)(6\_Z+1)}+6\_Z-2\right)\sqrt{2}\right)\_a-1\right)\left(2\,\mathrm{RootOf}\right.}} \right.\right.$$

$$\left. + x\right)$$

$y(x)$

$$= \mathrm{RootOf}\left(\sqrt{2}\left(\int_{3}^{\_Z} \frac{\mathrm{RootOf}\left(\left(\sqrt{-(3\_Z-1)(6\_Z+1)}+6\_\right.\right.}{\sqrt{-\left(\mathrm{RootOf}\left(\left(\sqrt{-(3\_Z-1)(6\_Z+1)}+6\_Z-2\right)\sqrt{2}\right)\_a-1\right)\left(2\,\mathrm{RootOf}\left(\right.\right.}} \right.\right.$$

$$\left. + x\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(1+y'[x]^2)^2==y[x]^2*y''[x],{y[0]==3,y'[0]==Sqrt[2]}},y[x],x,IncludeSingularSolutio
```

Timed out

18.37 problem 37

Internal problem ID [2309]
Internal file name [`OUTPUT/2309_Tuesday_February_27_2024_08_25_49_AM_20872197/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{y'' - {y'}^2 \sin(x) = 0}$$

With initial conditions

$$\left[y(0) = 0, y'(0) = \frac{1}{2}\right]$$

18.37.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - p(x)^2 \sin(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= p^2 \sin(x) \end{aligned}$$

Where $f(x) = \sin(x)$ and $g(p) = p^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p^2}\,dp &= \sin(x)\,dx \\ \int \frac{1}{p^2}\,dp &= \int \sin(x)\,dx \\ -\frac{1}{p} &= -\cos(x) + c_1 \end{aligned}$$

The solution is

$$-\frac{1}{p(x)} + \cos(x) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-1 - c_1 = 0$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$\frac{\cos(x)\,p - 1 + p}{p} = 0$$

The above simplifies to

$$\cos(x)\,p - 1 + p = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y'\cos(x) - 1 + y' = 0$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{1}{\cos(x)+1}\, dx \\
&= \tan\left(\frac{x}{2}\right)+c_2
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $x=0$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = c_2
$$

$$
c_2 = 0
$$

Substituting c_2 found above in the general solution gives

$$
y = \tan\left(\frac{x}{2}\right)
$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$
y = \tan\left(\frac{x}{2}\right) \tag{1}
$$

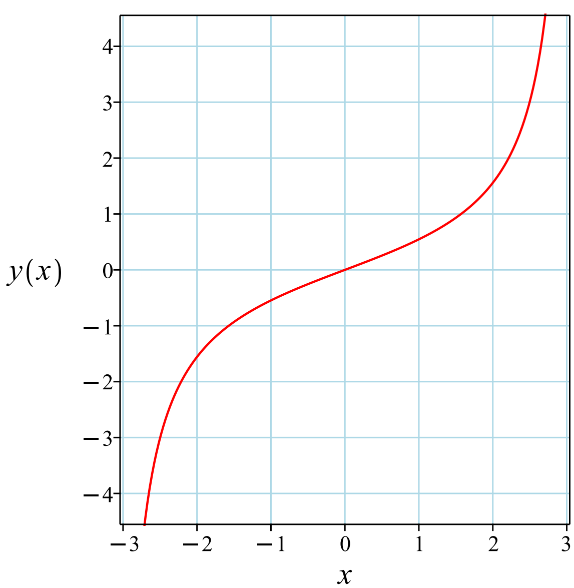


Figure 723: Solution plot

Verification of solutions

$$y = \tan\left(\frac{x}{2}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)^2*sin(_a), _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✗ Solution by Maple

```
dsolve([diff(y(x),x$2)=diff(y(x),x)^2*sin(x),y(0) = 0, D(y)(0) = 1/2],y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 1.773 (sec). Leaf size: 6

```
DSolve[{y''[x]==y'[x]^2*Sin[x],{y[0]==0,y'[0]==1/2}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \text{Indeterminate}$$

18.38 problem 38

Internal problem ID [2310]
Internal file name [OUTPUT/2310_Tuesday_February_27_2024_08_25_49_AM_38444091/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2yy'' - y^3 - 2y'^2 = 0}$$

With initial conditions

$$[y(0) = -1, y'(0) = 0]$$

18.38.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$2yp(y)\left(\frac{d}{dy}p(y)\right)-y^3-2p(y)^2=0$$

Which is now solved as first order ode for $p(y)$. Using the change of variables $p(y) = u(y)\,y$ on the above ode results in new ode in $u(y)$

$$2y^2u(y)\left(\left(\frac{d}{dy}u(y)\right)y+u(y)\right)-2u(y)^2\,y^2=y^3$$

Integrating both sides gives

$$\begin{aligned}\int 2udu &= y+c_2\\ u^2 &= y+c_2\end{aligned}$$

Solving for u gives these solutions

$$\begin{aligned}u_1 &= \sqrt{y+c_2}\\ u_2 &= -\sqrt{y+c_2}\end{aligned}$$

Therefore the solution $p(y)$ is

$$\begin{aligned}p(y) &= yu\\ &= y\sqrt{y+c_2}\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $y=-1$ and $p=0$ in the above solution gives an equation to solve for the constant of integration.

$$0=-\sqrt{c_2-1}$$

$$c_2=1$$

Substituting c_2 found above in the general solution gives

$$p(y)=\sqrt{y+1}\,y$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=\sqrt{y+1}\,y$$

Integrating both sides gives

$$\int \frac{1}{\sqrt{y+1}\, y} dy = \int dx$$

$$-2 \operatorname{arctanh}\left(\sqrt{y+1}\right) = x + c_3$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_3$$

$$c_3 = 0$$

Substituting c_3 found above in the general solution gives

$$-2 \operatorname{arctanh}\left(\sqrt{y+1}\right) = x$$

Solving for y from the above gives

$$y = -\operatorname{sech}\left(\frac{x}{2}\right)^2$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = -\operatorname{sech}\left(\frac{x}{2}\right)^2 \tag{1}$$

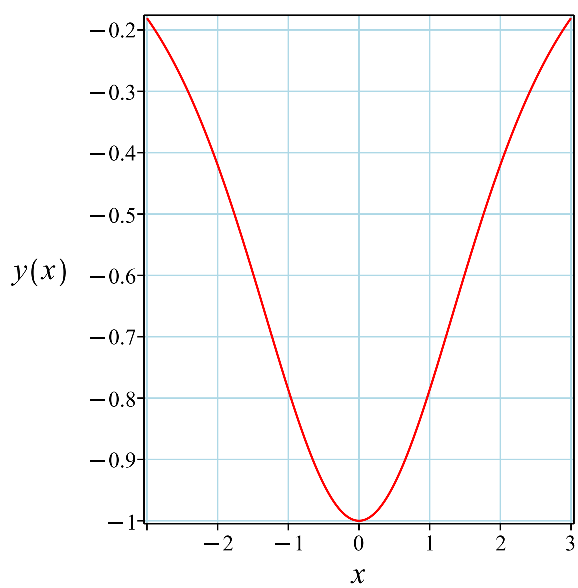


Figure 724: Solution plot

Verification of solutions

$$y = -\operatorname{sech}\left(\frac{x}{2}\right)^2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(1/2)*(_a^3+2*_b(_a)^2)/_a = 0,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 3/2*_b]
```

✓ Solution by Maple

Time used: 0.859 (sec). Leaf size: 15

```
dsolve([2*y(x)*diff(y(x),x$2)=y(x)^3+2*diff(y(x),x)^2,y(0) = -1, D(y)(0) = 0],y(x), singsol=
```

$$y(x) = \operatorname{RootOf}\left(2 \operatorname{arctanh}\left(\sqrt{\_Z+1}\right) + x\right)$$

✓ Solution by Mathematica

Time used: 60.205 (sec). Leaf size: 15

```
DSolve[{2*y[x]*y''[x]==y[x]^3+2*y'[x]^2,{y[0]==-1,y'[0]==0}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\operatorname{sech}^2\left(\frac{x}{2}\right)$$

18.39 problem 39

Internal problem ID [2311]
Internal file name [`OUTPUT/2311_Tuesday_February_27_2024_08_25_50_AM_28011907/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{x'' - k^2 x = 0}$$

With initial conditions

$$[x(0) = 0, x'(0) = v_0]$$

18.39.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= -k^2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' - k^2 x = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -k^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.39.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = -k^2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - k^2 \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$-k^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -k^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-k^2)} \\ &= \pm\sqrt{k^2}\end{aligned}$$

Hence

$$\lambda_1 = +\sqrt{k^2}$$
$$\lambda_2 = -\sqrt{k^2}$$

Which simplifies to

$$\lambda_1 = \sqrt{k^2}$$
$$\lambda_2 = -\sqrt{k^2}$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{\left(\sqrt{k^2}\right)t} + c_2 e^{\left(-\sqrt{k^2}\right)t}$$

Or

$$x = c_1 \mathrm{e}^{\sqrt{k^2}\, t} + c_2 \mathrm{e}^{-\sqrt{k^2}\, t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{\sqrt{k^2}\, t} + c_2 \mathrm{e}^{-\sqrt{k^2}\, t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1 \sqrt{k^2}\, \mathrm{e}^{\sqrt{k^2}\, t} - c_2 \sqrt{k^2}\, \mathrm{e}^{-\sqrt{k^2}\, t}$$

substituting $x' = v_0$ and $t = 0$ in the above gives

$$v_0 = \operatorname{csgn}(k)\, k(c_1 - c_2) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{\operatorname{csgn}(k)\, v_0}{2k}$$
$$c_2 = -\frac{\operatorname{csgn}(k)\, v_0}{2k}$$

Substituting these values back in above solution results in

$$x = \frac{\mathrm{e}^{\sqrt{k^2}\, t} \operatorname{csgn}(k)\, v_0 - \operatorname{csgn}(k)\, v_0 \mathrm{e}^{-\sqrt{k^2}\, t}}{2k}$$

Which simplifies to

$$x = \frac{\operatorname{csgn}(k)\, v_0 \left(e^{\operatorname{csgn}(k) k t} - e^{-\operatorname{csgn}(k) k t}\right)}{2k}$$

Summary

The solution(s) found are the following

$$x = \frac{\operatorname{csgn}(k)\, v_0 \left(e^{\operatorname{csgn}(k) k t} - e^{-\operatorname{csgn}(k) k t}\right)}{2k} \tag{1}$$

Verification of solutions

$$x = \frac{\operatorname{csgn}(k)\, v_0 \left(e^{\operatorname{csgn}(k) k t} - e^{-\operatorname{csgn}(k) k t}\right)}{2k}$$

Verified OK.

18.39.3 Solving as second order ode can be made integrable ode

Multiplying the ode by x' gives

$$x'x'' - k^2 x' x = 0$$

Integrating the above w.r.t t gives

$$\int \left(x'x'' - k^2 x' x\right) dt = 0$$
$$\frac{x'^2}{2} - \frac{k^2 x^2}{2} = c_2$$

Which is now solved for x. Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = \sqrt{k^2 x^2 + 2c_1} \tag{1}$$
$$x' = -\sqrt{k^2 x^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{k^2 x^2 + 2c_1}} dx = \int dt$$
$$\frac{\ln\left(\frac{k^2 x}{\sqrt{k^2}} + \sqrt{k^2 x^2 + 2c_1}\right)}{\sqrt{k^2}} = t + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(\frac{k^2x}{\sqrt{k^2}}+\sqrt{k^2x^2+2c_1}\right)}{\sqrt{k^2}}} = e^{t+c_2}$$

Which simplifies to

$$\left(kx\ \mathrm{csgn}\left(k\right)+\sqrt{k^2x^2+2c_1}\right)^{\frac{1}{\sqrt{k^2}}} = c_3e^t$$

Simplifying the solution $x = \frac{\mathrm{csgn}(k)\left(\left(c_3e^t\right)^{\mathrm{csgn}(k)k}-2\left(c_3e^t\right)^{-\mathrm{csgn}(k)k}c_1\right)}{2k}$ to $x = \frac{\left(c_3e^t\right)^k-2\left(c_3e^t\right)^{-k}c_1}{2k}$
Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{k^2x^2+2c_1}}dx = \int dt$$

$$-\frac{\ln\left(\frac{k^2x}{\sqrt{k^2}}+\sqrt{k^2x^2+2c_1}\right)}{\sqrt{k^2}} = t+c_4$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(\frac{k^2x}{\sqrt{k^2}}+\sqrt{k^2x^2+2c_1}\right)}{\sqrt{k^2}}} = e^{t+c_4}$$

Which simplifies to

$$\left(kx\ \mathrm{csgn}\left(k\right)+\sqrt{k^2x^2+2c_1}\right)^{-\frac{\mathrm{csgn}(k)}{k}} = c_5e^t$$

Simplifying the solution $x = -\frac{\mathrm{csgn}(k)\left(2\left(c_5e^t\right)^{\mathrm{csgn}(k)k}c_1-\left(c_5e^t\right)^{-\mathrm{csgn}(k)k}\right)}{2k}$ to $x = -\frac{2\left(c_5e^t\right)^kc_1-\left(c_5e^t\right)^{-k}}{2k}$
Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$x = \frac{\left(c_3e^t\right)^k-2\left(c_3e^t\right)^{-k}c_1}{2k} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{c_3^k-2c_3^{-k}c_1}{2k} \tag{1A}$$

Taking derivative of the solution gives

$$x' = \frac{(c_3 e^t)^k k + 2(c_3 e^t)^{-k} k c_1}{2k}$$

substituting $x' = v_0$ and $t = 0$ in the above gives

$$v_0 = \frac{c_3^k}{2} + c_3^{-k} c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = \frac{v_0^2}{2}$$
$$c_3 = v_0^{\frac{1}{k}}$$

Substituting these values back in above solution results in

$$x = \frac{\left(v_0^{\frac{1}{k}} e^t\right)^k - \left(v_0^{\frac{1}{k}} e^t\right)^{-k} v_0^2}{2k}$$

Looking at the Second solution

$$x = -\frac{2(c_5 e^t)^k c_1 - (c_5 e^t)^{-k}}{2k} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = \frac{-c_5^k c_1 + \frac{c_5^{-k}}{2}}{k} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{2(c_5 e^t)^k k c_1 + (c_5 e^t)^{-k} k}{2k}$$

substituting $x' = v_0$ and $t = 0$ in the above gives

$$v_0 = -c_5^k c_1 - \frac{c_5^{-k}}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Solving for the constants gives

$$c_1 = \frac{v_0^2}{2}$$
$$c_5 = \left(-\frac{1}{v_0}\right)^{\frac{1}{k}}$$

Substituting these values back in above solution results in

$$x = \frac{-\left(\left(-\frac{1}{v_0}\right)^{\frac{1}{k}} \mathrm{e}^t\right)^k v_0^2 + \left(\left(-\frac{1}{v_0}\right)^{\frac{1}{k}} \mathrm{e}^t\right)^{-k}}{2k}$$

Summary

The solution(s) found are the following

$$x = \frac{\left(v_0^{\frac{1}{k}} \mathrm{e}^t\right)^k - \left(v_0^{\frac{1}{k}} \mathrm{e}^t\right)^{-k} v_0^2}{2k} \tag{1}$$

$$x = \frac{-\left(\left(-\frac{1}{v_0}\right)^{\frac{1}{k}} \mathrm{e}^t\right)^k v_0^2 + \left(\left(-\frac{1}{v_0}\right)^{\frac{1}{k}} \mathrm{e}^t\right)^{-k}}{2k} \tag{2}$$

Verification of solutions

$$x = \frac{\left(v_0^{\frac{1}{k}} \mathrm{e}^t\right)^k - \left(v_0^{\frac{1}{k}} \mathrm{e}^t\right)^{-k} v_0^2}{2k}$$

Verified OK.

$$x = \frac{-\left(\left(-\frac{1}{v_0}\right)^{\frac{1}{k}} \mathrm{e}^t\right)^k v_0^2 + \left(\left(-\frac{1}{v_0}\right)^{\frac{1}{k}} \mathrm{e}^t\right)^{-k}}{2k}$$

Verified OK.

18.39.4 Solving using Kovacic algorithm

Writing the ode as

$$x'' - k^2 x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -k^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{k^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = k^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = \left(k^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 613: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = k^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = e^{\sqrt{k^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}x_1 &= z_1 \\ &= e^{\sqrt{k^2}\, t}\end{aligned}$$

Which simplifies to

$$x_1 = e^{\sqrt{k^2}\, t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned}x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= e^{\sqrt{k^2}\, t} \int \frac{1}{e^{2\sqrt{k^2}\, t}}\, dt \\ &= e^{\sqrt{k^2}\, t} \left(-\frac{\operatorname{csgn}(k)\, e^{-2\operatorname{csgn}(k) k t}}{2k} \right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(e^{\sqrt{k^2}\, t}\right) + c_2 \left(e^{\sqrt{k^2}\, t} \left(-\frac{\operatorname{csgn}(k)\, e^{-2 \operatorname{csgn}(k) k t}}{2k}\right)\right)
\end{aligned}
$$

Simplifying the solution $x = c_1 e^{\sqrt{k^2}\, t} - \frac{c_2 \operatorname{csgn}(k) e^{-\operatorname{csgn}(k) k t}}{2k}$ to $x = c_1 e^{\sqrt{k^2}\, t} - \frac{c_2 e^{-kt}}{2k}$ Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 e^{\sqrt{k^2}\, t} - \frac{c_2 e^{-kt}}{2k} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$
0 = \frac{2c_1 k - c_2}{2k} \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = c_1 \sqrt{k^2}\, e^{\sqrt{k^2}\, t} + \frac{c_2 e^{-kt}}{2}
$$

substituting $x' = v_0$ and $t = 0$ in the above gives

$$
v_0 = \operatorname{csgn}(k)\, k c_1 + \frac{c_2}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{v_0}{k\left(\operatorname{csgn}(k) + 1\right)} \\
c_2 &= \frac{2v_0}{\operatorname{csgn}(k) + 1}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = \frac{e^{\sqrt{k^2}\, t} v_0 - e^{-kt} v_0}{\operatorname{csgn}(k)\, k + k}
$$

Which simplifies to

$$
x = -\frac{v_0 \left(e^{-kt} - e^{\operatorname{csgn}(k) k t}\right)}{k\left(\operatorname{csgn}(k) + 1\right)}
$$

Summary

The solution(s) found are the following

$$x = -\frac{v_0 \left(\mathrm{e}^{-kt} - \mathrm{e}^{\mathrm{csgn}(k) k t}\right)}{k \left(\mathrm{csgn}\left(k\right)+1\right)} \tag{1}$$

Verification of solutions

$$x = -\frac{v_0 \left(\mathrm{e}^{-kt} - \mathrm{e}^{\mathrm{csgn}(k) k t}\right)}{k \left(\mathrm{csgn}\left(k\right)+1\right)}$$

Verified OK.

18.39.5 Maple step by step solution

Let's solve

$\left[x'' - k^2 x = 0, x(0) = 0, x'\Big|_{\{t=0\}} = v_0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $-k^2 + r^2 = 0$

- Factor the characteristic polynomial

  $-(k-r)(k+r) = 0$

- Roots of the characteristic polynomial

  $r = (k, -k)$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{kt}$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-kt}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{kt} + c_2 \mathrm{e}^{-kt}$

☐ Check validity of solution $x = c_1 \mathrm{e}^{kt} + c_2 \mathrm{e}^{-kt}$

- Use initial condition $x(0) = 0$

  $0 = c_1 + c_2$
- Compute derivative of the solution

  $x' = c_1 k\,\mathrm{e}^{kt} - c_2 k\,\mathrm{e}^{-kt}$
- Use the initial condition $x'\Big|_{\{t=0\}} = v_0$

  $v_0 = c_1 k - c_2 k$
- Solve for c_1 and c_2

  $\left\{c_1 = \frac{v_0}{2k}, c_2 = -\frac{v_0}{2k}\right\}$
- Substitute constant values into general solution and simplify

  $x = -\frac{v_0(-\mathrm{e}^{kt}+\mathrm{e}^{-kt})}{2k}$

- Solution to the IVP

  $x = -\frac{v_0(-\mathrm{e}^{kt}+\mathrm{e}^{-kt})}{2k}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([diff(x(t),t$2)-k^2*x(t)=0,x(0) = 0, D(x)(0) = v__0],x(t), singsol=all)
```

$$x(t) = -\frac{v_0(\mathrm{e}^{-kt} - \mathrm{e}^{kt})}{2k}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 27

```
DSolve[{x''[t]-k^2*x[t]==0,{x[0]==0,x'[0]==v0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{\text{v0}e^{-kt}(e^{2kt} - 1)}{2k}$$

18.40 problem 40

Internal problem ID [2312]
Internal file name [OUTPUT/2312_Tuesday_February_27_2024_08_25_51_AM_92447401/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{yy'' - 2y'^2 - y^2 = 0}$$

With initial conditions

$$\left[y(0) = 1, y'(0) = \sqrt{3}\right]$$

18.40.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right)-2p(y)^2-y^2=0$$

Which is now solved as first order ode for $p(y)$. Using the change of variables $p(y) = u(y)\, y$ on the above ode results in new ode in $u(y)$

$$y^2u(y)\left(\left(\frac{d}{dy}u(y)\right)y+u(y)\right)-2u(y)^2\,y^2=y^2$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(y,u)\\
&= f(y)g(u)\\
&= \frac{u^2+1}{uy}
\end{aligned}$$

Where $f(y)=\frac{1}{y}$ and $g(u)=\frac{u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{u^2+1}{u}}\,du &= \frac{1}{y}\,dy\\
\int\frac{1}{\frac{u^2+1}{u}}\,du &= \int\frac{1}{y}\,dy\\
\frac{\ln\left(u^2+1\right)}{2} &= \ln\left(y\right)+c_2
\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2+1}=\mathrm{e}^{\ln(y)+c_2}$$

Which simplifies to

$$\sqrt{u^2+1}=c_3y$$

Which simplifies to

$$\sqrt{u\left(y\right)^2+1}=c_3\mathrm{e}^{c_2}y$$

The solution is

$$\sqrt{u\left(y\right)^2+1}=c_3\mathrm{e}^{c_2}y$$

Replacing $u(y)$ in the above solution by $\frac{p(y)}{y}$ results in the solution for $p(y)$ in implicit form

$$\sqrt{\frac{p(y)^2}{y^2}+1} = c_3 e^{c_2} y$$
$$\sqrt{\frac{p(y)^2+y^2}{y^2}} = c_3 e^{c_2} y$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{4}{c_3^2}\right)}{2}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $y = 1$ and $p = \sqrt{3}$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 2c_3\sqrt{\frac{1}{c_3^2}}$$

This solution is valid for any c_3. Hence there are infinite number of solutions.

Solving for $p(y)$ from the above gives

$$p(y) = \sqrt{4y^2-1}\,y$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \sqrt{4y^2-1}\,y$$

Integrating both sides gives

$$\int \frac{1}{\sqrt{4y^2-1}\,y} dy = \int dx$$
$$-\arctan\left(\frac{1}{\sqrt{4y^2-1}}\right) = x + c_4$$

Initial conditions are used to solve for c_4. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\pi}{6} = c_4$$

$$c_4 = -\frac{\pi}{6}$$

Substituting c_4 found above in the general solution gives

$$-\arctan\left(\frac{1}{\sqrt{4y^2-1}}\right) = x - \frac{\pi}{6}$$

Solving for y from the above gives

$$y = \frac{\sqrt{\cot\left(x+\frac{\pi}{3}\right)^2+1}}{2\cot\left(x+\frac{\pi}{3}\right)}$$

Simplifying the solution $y = \frac{\sec\left(x+\frac{\pi}{3}\right)\operatorname{csgn}\left(\csc\left(x+\frac{\pi}{3}\right)\right)}{2}$ to $y = \frac{\sec\left(x+\frac{\pi}{3}\right)}{2}$ Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{\sec\left(x+\frac{\pi}{3}\right)}{2} \tag{1}$$

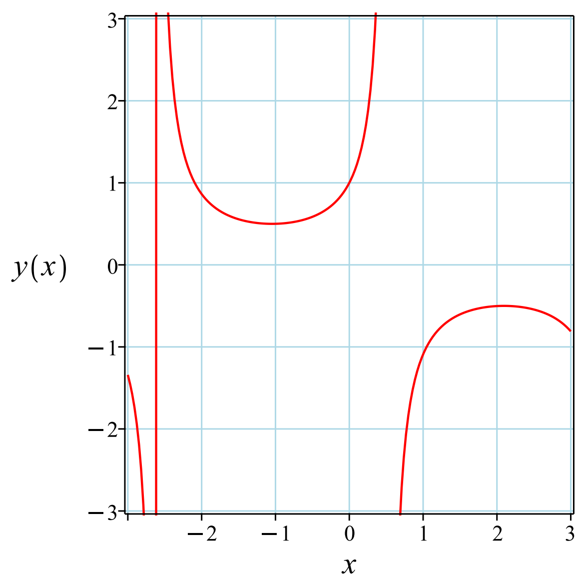


Figure 725: Solution plot

Verification of solutions

$$y = \frac{\sec\left(x+\frac{\pi}{3}\right)}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 16

```
dsolve([y(x)*diff(y(x),x$2)=2*diff(y(x),x)^2+y(x)^2,y(0) = 1, D(y)(0) = sqrt(3)],y(x), sings
```

$$y(x) = \frac{1}{-\sqrt{3}\,\sin(x) + \cos(x)}$$

✓ Solution by Mathematica

Time used: 0.175 (sec). Leaf size: 19

```
DSolve[{y[x]*y''[x]==2*y'[x]^2+y[x]^2,{y[0]==1,y'[0]==Sqrt[3]}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{1}{2}\csc\left(\frac{1}{6}(\pi - 6x)\right)$$

18.41 problem 41

Internal problem ID [2313]
Internal file name [`OUTPUT/2313_Tuesday_February_27_2024_08_25_56_AM_14896452/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 35, page 157
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(1-\mathrm{e}^{x})\, y''-\mathrm{e}^{x} y'=0}$$

With initial conditions

$$[y(1)=0, y'(1)=1]$$

18.41.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{e^x}{1 - e^x} \\ q(x) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{e^x y'}{1 - e^x} = 0$$

The domain of $p(x) = -\frac{e^x}{1-e^x}$ is

$$\{2i\pi\_Z82 < x\}$$

But the point $x_0 = 1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

18.41.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int ((1 - e^x) y'' - e^x y') \, dx &= 0 \\ -(-1 + e^x) y' &= c_1 \end{aligned}$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{c_1}{-1 + e^x} \, dx \\ &= -c_1(\ln(-1 + e^x) - \ln(e^x)) + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -c_1(\ln(-1 + e^x) - \ln(e^x)) + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = -c_1 \ln (-1 + \mathrm{e}) + c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \left(\frac{\mathrm{e}^x}{-1 + \mathrm{e}^x} - 1 \right)$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -\frac{c_1}{-1 + \mathrm{e}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 - \mathrm{e} \\ c_2 &= -(-1 + \mathrm{e}) (\ln (-1 + \mathrm{e}) - 1) \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\ln (-1 + \mathrm{e}) \mathrm{e} + \mathrm{e} \ln (-1 + \mathrm{e}^x) - \mathrm{e} \ln (\mathrm{e}^x) + \ln (-1 + \mathrm{e}) + \mathrm{e} - \ln (-1 + \mathrm{e}^x) + \ln (\mathrm{e}^x) - 1$$

Which simplifies to

$$y = -(\ln (\mathrm{e}^x) + \ln (-1 + \mathrm{e}) - \ln (-1 + \mathrm{e}^x) - 1) (-1 + \mathrm{e})$$

Summary

The solution(s) found are the following

$$y = -(\ln (\mathrm{e}^x) + \ln (-1 + \mathrm{e}) - \ln (-1 + \mathrm{e}^x) - 1) (-1 + \mathrm{e}) \tag{1}$$

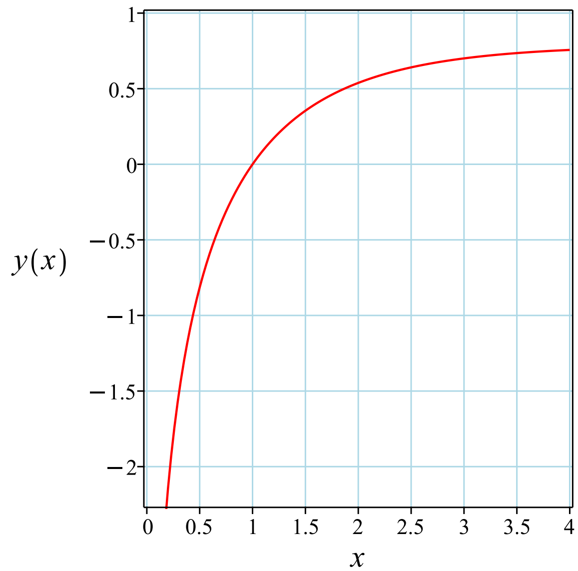


Figure 726: Solution plot

Verification of solutions

$$y = -(\ln(\mathrm{e}^{x}) + \ln(-1+\mathrm{e}) - \ln(-1+\mathrm{e}^{x}) - 1)(-1+\mathrm{e})$$

Verified OK.

18.41.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$(1-\mathrm{e}^{x})\, p'(x) - \mathrm{e}^{x} p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= -\frac{\mathrm{e}^{x} p}{-1+\mathrm{e}^{x}} \end{aligned}$$

Where $f(x) = -\frac{e^x}{-1+e^x}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\, dp &= -\frac{e^x}{-1+e^x}\, dx \\
\int \frac{1}{p}\, dp &= \int -\frac{e^x}{-1+e^x}\, dx \\
\ln(p) &= -\ln(-1+e^x) + c_1 \\
p &= e^{-\ln(-1+e^x)+c_1} \\
&= \frac{c_1}{-1+e^x}
\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{-1+e}$$

$$c_1 = -1 + e$$

Substituting c_1 found above in the general solution gives

$$p(x) = \frac{-1+e}{-1+e^x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{-1+e}{-1+e^x}$$

Integrating both sides gives

$$\begin{aligned}
y &= \int \frac{-1+e}{-1+e^x}\, \mathrm{d}x \\
&= (-1+e)\left(\ln(-1+e^x) - \ln(e^x)\right) + c_2
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(-1+e)\, e - \ln(-1+e) - e + 1 + c_2$$

$$c_2 = -\ln(-1+e)\, e + \ln(-1+e) + e - 1$$

Substituting c_2 found above in the general solution gives

$$y = -\ln(-1+e)\, e + e\ln(-1+e^x) - e\ln(e^x) + \ln(-1+e) + e - \ln(-1+e^x) + \ln(e^x) - 1$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &-\ln(-1+\mathrm{e})\,\mathrm{e}+\mathrm{e}\ln(-1+\mathrm{e}^{x})-\mathrm{e}\ln(\mathrm{e}^{x}) \\
&+\ln(-1+\mathrm{e})+\mathrm{e}-\ln(-1+\mathrm{e}^{x})+\ln(\mathrm{e}^{x})-1
\end{aligned} \tag{1}
$$

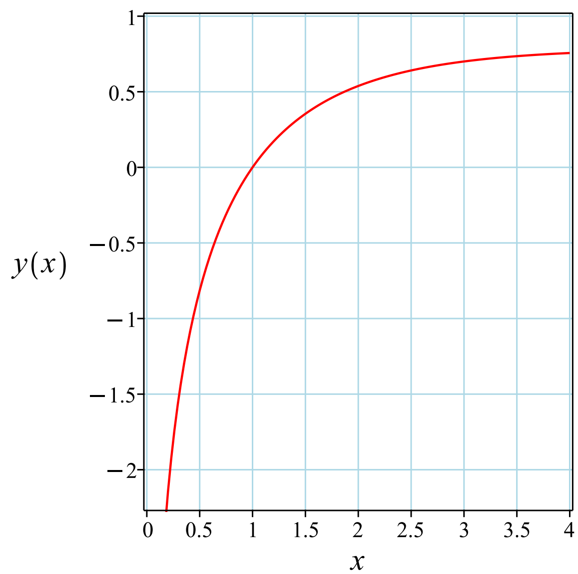


Figure 727: Solution plot

Verification of solutions

$$
y=-\ln(-1+\mathrm{e})\,\mathrm{e}+\mathrm{e}\ln(-1+\mathrm{e}^{x})-\mathrm{e}\ln(\mathrm{e}^{x})+\ln(-1+\mathrm{e})+\mathrm{e}-\ln(-1+\mathrm{e}^{x})+\ln(\mathrm{e}^{x})-1
$$

Verified OK.

18.41.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
(1-\mathrm{e}^{x})\,y''-\mathrm{e}^{x}y'=0
$$

Integrating both sides of the ODE w.r.t x gives

$$
\int \left((1-\mathrm{e}^{x})\,y''-\mathrm{e}^{x}y'\right)dx = 0
$$

$$
-(-1+\mathrm{e}^{x})\,y' = c_1
$$

Which is now solved for y. Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{c_1}{-1+\mathrm{e}^x}\,\mathrm{d}x \\
&= -c_1(\ln(-1+\mathrm{e}^x) - \ln(\mathrm{e}^x)) + c_2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = -c_1(\ln(-1+\mathrm{e}^x) - \ln(\mathrm{e}^x)) + c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$
0 = -c_1 \ln(-1+\mathrm{e}) + c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1\left(\frac{\mathrm{e}^x}{-1+\mathrm{e}^x} - 1\right)
$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$
1 = -\frac{c_1}{-1+\mathrm{e}} \tag{2A}
$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1-\mathrm{e} \\
c_2 &= -(-1+\mathrm{e})(\ln(-1+\mathrm{e}) - 1)
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -\ln(-1+\mathrm{e})\,\mathrm{e} + \mathrm{e}\ln(-1+\mathrm{e}^x) - \mathrm{e}\ln(\mathrm{e}^x) + \ln(-1+\mathrm{e}) + \mathrm{e} - \ln(-1+\mathrm{e}^x) + \ln(\mathrm{e}^x) - 1
$$

Which simplifies to

$$
y = -(\ln(\mathrm{e}^x) + \ln(-1+\mathrm{e}) - \ln(-1+\mathrm{e}^x) - 1)(-1+\mathrm{e})
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = -(\ln(\mathrm{e}^x) + \ln(-1+\mathrm{e}) - \ln(-1+\mathrm{e}^x) - 1)(-1+\mathrm{e}) \tag{1}
$$

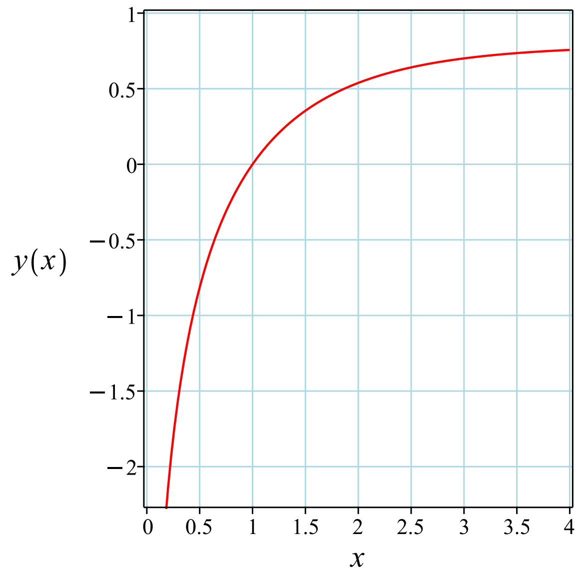


Figure 728: Solution plot

Verification of solutions

$$y = -(\ln(e^x) + \ln(-1+e) - \ln(-1+e^x) - 1)(-1+e)$$

Verified OK.

18.41.5 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 - e^x \\ q(x) &= -e^x \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= -e^x \\ q'(x) &= -e^x \end{aligned}$$

Therefore (1) becomes

$$-e^x - (-e^x) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$(1 - e^x)\, y' = c_1$$

We now have a first order ode to solve which is

$$(1 - e^x)\, y' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{c_1}{-1 + e^x}\, \mathrm{d}x \\ &= -c_1(\ln(-1 + e^x) - \ln(e^x)) + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -c_1(\ln(-1 + e^x) - \ln(e^x)) + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = -c_1 \ln(-1 + e) + c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1\left(\frac{e^x}{-1 + e^x} - 1\right)$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -\frac{c_1}{-1 + e} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1 - \mathrm{e}$$
$$c_2 = -(-1 + \mathrm{e})(\ln(-1 + \mathrm{e}) - 1)$$

Substituting these values back in above solution results in

$$y = -\ln(-1 + \mathrm{e})\,\mathrm{e} + \mathrm{e}\ln(-1 + \mathrm{e}^x) - \mathrm{e}\ln(\mathrm{e}^x) + \ln(-1 + \mathrm{e}) + \mathrm{e} - \ln(-1 + \mathrm{e}^x) + \ln(\mathrm{e}^x) - 1$$

Which simplifies to

$$y = -(\ln(\mathrm{e}^x) + \ln(-1 + \mathrm{e}) - \ln(-1 + \mathrm{e}^x) - 1)(-1 + \mathrm{e})$$

Summary

The solution(s) found are the following

$$y = -(\ln(\mathrm{e}^x) + \ln(-1 + \mathrm{e}) - \ln(-1 + \mathrm{e}^x) - 1)(-1 + \mathrm{e}) \tag{1}$$

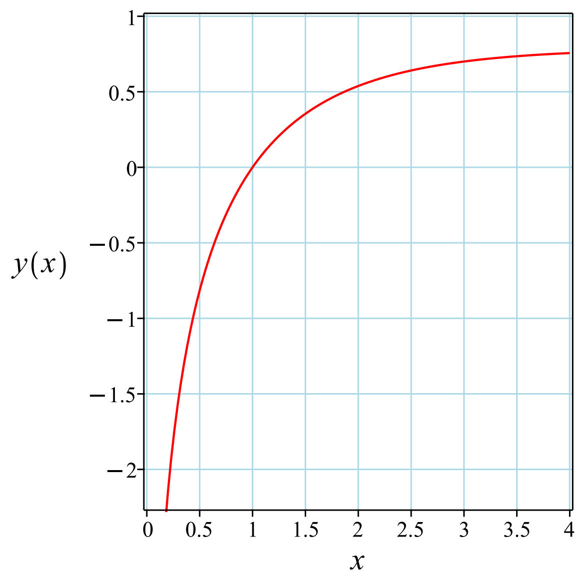


Figure 729: Solution plot

Verification of solutions

$$y = -(\ln(\mathrm{e}^x) + \ln(-1 + \mathrm{e}) - \ln(-1 + \mathrm{e}^x) - 1)(-1 + \mathrm{e})$$

Verified OK.

18.41.6 Maple step by step solution

Let's solve

$$\left[(1-e^x)\,y''-e^x y'=0, y(1)=0, y'\Big|_{\{x=1\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $(1-e^x)\,u'(x)-e^x u(x)=0$

- Integrate both sides with respect to x

  $\int ((1-e^x)\,u'(x)-e^x u(x))\,dx=\int 0dx+c_1$

- Evaluate integral

  $-(-1+e^x)\,u(x)=c_1$

- Solve for $u(x)$

  $u(x)=-\frac{c_1}{-1+e^x}$

- Solve 1st ODE for $u(x)$

  $u(x)=-\frac{c_1}{-1+e^x}$

- Make substitution $u=y'$

  $y'=-\frac{c_1}{-1+e^x}$

- Integrate both sides to solve for y

  $\int y'dx=\int -\frac{c_1}{-1+e^x}dx+c_2$

- Compute integrals

  $y=-c_1(\ln(-1+e^x)-\ln(e^x))+c_2$

□ Check validity of solution $y=-c_1(\ln(-1+e^x)-\ln(e^x))+c_2$

- ○ Use initial condition $y(1)=0$

  $0=-(\ln(-1+e)-1)\,c_1+c_2$

- ○ Compute derivative of the solution

  $y'=-c_1\left(\frac{e^x}{-1+e^x}-1\right)$

- ○ Use the initial condition $y'\Big|_{\{x=1\}}=1$

$$1=-c_1\left(\frac{\mathrm{e}}{-1+\mathrm{e}}-1\right)$$

- Solve for c_1 and c_2

$$\{c_1=1-\mathrm{e}, c_2=-\ln(-1+\mathrm{e})\,\mathrm{e}+\ln(-1+\mathrm{e})+\mathrm{e}-1\}$$

- Substitute constant values into general solution and simplify

$$y=-(\ln(\mathrm{e}^x)+\ln(-1+\mathrm{e})-\ln(-1+\mathrm{e}^x)-1)(-1+\mathrm{e})$$

- Solution to the IVP

$$y=-(\ln(\mathrm{e}^x)+\ln(-1+\mathrm{e})-\ln(-1+\mathrm{e}^x)-1)(-1+\mathrm{e})$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 27

```
dsolve([(1-exp(x))*diff(y(x),x$2)=exp(x)*diff(y(x),x),y(1) = 0, D(y)(1) = 1],y(x), singsol=a
```

$$y(x)=-(\ln(\mathrm{e}^x)+\ln(-1+\mathrm{e})-\ln(\mathrm{e}^x-1)-1)(-1+\mathrm{e})$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 27

```
DSolve[{(1-Exp[x])*y''[x]==Exp[x]*y'[x],{y[1]==0,y'[1]==1}},y[x],x,IncludeSingularSolutions
```

$$y(x)\to -2(e-1)\left(\operatorname{arctanh}(1-2e)-\operatorname{arctanh}(1-2e^x)\right)$$

19 Exercise 37, page 171

19.1 problem 1

Internal problem ID [2314]
Internal file name [`OUTPUT/2314_Tuesday_February_27_2024_08_25_57_AM_5814105/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{4y^2 - y'^2 x^2 = 0}$$

19.1.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{4}{x^2}, g = y^2$. Hence the ode is

$$(y')^2 = \frac{4y^2}{x^2}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{4}{x^2} > 0$$
$$y^2 > 0$$

Under the above assumption the differential equations become separable and can be written as

$$
\begin{aligned}
y' &= \sqrt{f}\, \sqrt{g} \\
y' &= -\sqrt{f}\, \sqrt{g}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\, dy &= \left(\sqrt{f}\right)\, dx \\
-\frac{1}{\sqrt{g}}\, dy &= \left(\sqrt{f}\right)\, dx
\end{aligned}
$$

Replacing $f(x), g(y)$ by their values gives

$$
\begin{aligned}
\frac{1}{\sqrt{y^2}}\, dy &= \left(\sqrt{4}\, \sqrt{\frac{1}{x^2}}\right)\, dx \\
-\frac{1}{\sqrt{y^2}}\, dy &= \left(\sqrt{4}\, \sqrt{\frac{1}{x^2}}\right)\, dx
\end{aligned}
$$

Integrating now gives the solutions.

$$
\begin{aligned}
\int \frac{1}{\sqrt{y^2}} dy &= \int \sqrt{4}\, \sqrt{\frac{1}{x^2}} dx + c_1 \\
\int -\frac{1}{\sqrt{y^2}} dy &= \int \sqrt{4}\, \sqrt{\frac{1}{x^2}} dx + c_1
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y \ln (y)}{\sqrt{y^2}} &= \sqrt{4}\, \sqrt{\frac{1}{x^2}}\, x \ln (x) + c_1 \\
-\frac{y \ln (y)}{\sqrt{y^2}} &= \sqrt{4}\, \sqrt{\frac{1}{x^2}}\, x \ln (x) + c_1
\end{aligned}
$$

Therefore

$$
\frac{y \ln (y)}{\sqrt{y^2}} = \sqrt{4}\, \sqrt{\frac{1}{x^2}}\, x \ln (x) + c_1
$$

$$
-\frac{y \ln (y)}{\sqrt{y^2}} = \sqrt{4}\, \sqrt{\frac{1}{x^2}}\, x \ln (x) + c_1
$$

Summary
The solution(s) found are the following

$$\frac{y \ln(y)}{\sqrt{y^2}} = \sqrt{4}\,\sqrt{\frac{1}{x^2}}\, x \ln(x) + c_1 \tag{1}$$

$$-\frac{y \ln(y)}{\sqrt{y^2}} = \sqrt{4}\,\sqrt{\frac{1}{x^2}}\, x \ln(x) + c_1 \tag{2}$$

Verification of solutions

$$\frac{y \ln(y)}{\sqrt{y^2}} = \sqrt{4}\,\sqrt{\frac{1}{x^2}}\, x \ln(x) + c_1$$

Verified OK. `{0 < y^2, 0 < 4/x^2}`

$$-\frac{y \ln(y)}{\sqrt{y^2}} = \sqrt{4}\,\sqrt{\frac{1}{x^2}}\, x \ln(x) + c_1$$

Verified OK. `{0 < y^2, 0 < 4/x^2}`

19.1.2 Maple step by step solution

Let's solve

$4y^2 - y'^2x^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{2}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{2}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = 2\ln(x) + c_1$

- Solve for y

  $y = e^{c_1}x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*y(x)^2=diff(y(x),x)^2*x^2,y(x), singsol=all)
```

$$y(x) = c_1 x^2$$
$$y(x) = \frac{c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 26

```
DSolve[4*y[x]^2==y'[x]^2*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^2}$$
$$y(x) \to c_1 x^2$$
$$y(x) \to 0$$

19.2 problem 2

Internal problem ID [2315]
Internal file name [OUTPUT/2315_Tuesday_February_27_2024_08_25_57_AM_60061494/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{xyy'^2 + (y + x)\, y' = -1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{1}{y} \tag{1}$$

$$y' = -\frac{1}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -y dy = x + c_1$$
$$-\frac{y^2}{2} = x + c_1$$

Solving for y gives these solutions

$$y_1 = \sqrt{-2x - 2c_1}$$
$$y_2 = -\sqrt{-2x - 2c_1}$$

Summary
The solution(s) found are the following

$$y = \sqrt{-2x - 2c_1} \tag{1}$$
$$y = -\sqrt{-2x - 2c_1} \tag{2}$$

Verification of solutions

$$y = \sqrt{-2x - 2c_1}$$

Verified OK.

$$y = -\sqrt{-2x - 2c_1}$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{x}\, \mathrm{d}x \\ &= -\ln(x) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = -\ln(x) + c_2$$

Verified OK.

19.2.1 Maple step by step solution

Let's solve

$xyy'^2 + (y + x)\, y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(xyy'^2 + (y + x)\, y'\right) dx = \int (-1)\, dx + c_1$

- Cannot compute integral

$$\int \left(xyy'^2 + (y+x)\, y'\right) dx = -x + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(x*y(x)*diff(y(x),x)^2+(x+y(x))*diff(y(x),x)+1=0,y(x), singsol=all)
```

$$y(x) = -\ln(x) + c_1$$
$$y(x) = \sqrt{-2x + c_1}$$
$$y(x) = -\sqrt{-2x + c_1}$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 53

```
DSolve[x*y[x]*y'[x]^2+(x+y[x])*y'[x]+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt{-x + c_1}$$
$$y(x) \to \sqrt{2}\sqrt{-x + c_1}$$
$$y(x) \to -\log(x) + c_1$$

19.3 problem 3

Internal problem ID [2316]
Internal file name [OUTPUT/2316_Tuesday_February_27_2024_08_25_57_AM_40695691/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{(2y - x^2)\, y'^2 - 2x^2 y y'^2 = -1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1}{\sqrt{2x^2 y + x^2 - 2y}} \tag{1}$$
$$y' = -\frac{1}{\sqrt{2x^2 y + x^2 - 2y}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (2)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
-> Calling odsolve with the ODE`, diff(y(x), x) = (-2*y(x)-1)/(x*(2*((2*y(x)+1)*(2*y(x)*x^2+
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(y(x), x) = (-y(x)^2+1)/(y(x)^4*x^4-2*y(x)^2*x^4-y(x)*
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   trying exact
   Looking for potential symmetries
   trying inverse Riccati
```

✗ Solution by Maple

```
dsolve(1+(2*y(x)-x^2)*diff(y(x),x)^2-2*x^2*y(x)*diff(y(x),x)^2=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[1+(2*y[x]-x^2)*y'[x]^2-2*x^2*y[x]*y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

19.4 problem 4

Internal problem ID [2317]
Internal file name [`OUTPUT/2317_Tuesday_February_27_2024_08_25_59_AM_8183189/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 4.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{x\left(-1+y'^2\right)-2yy'=0}$$

19.4.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$x(p^2-1)-2yp=0$$

Solving for y from the above results in

$$y=\frac{x(p^2-1)}{2p} \tag{1A}$$

This has the form

$$y = xf(p)+g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f+(xf'+g')\frac{dp}{dx}$$
$$p-f = (xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2 - 1}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2 - 1}{2p} = x\left(1 - \frac{p^2 - 1}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2 - 1}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2 - 1}{2p(x)}}{x\left(1 - \frac{p(x)^2 - 1}{2p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}\mu p = 0$$
$$\frac{d}{dx}\left(\frac{p}{x}\right) = 0$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -ix \tag{1}$$
$$y = ix \tag{2}$$
$$y = \frac{c_1^2 x^2 - 1}{2c_1} \tag{3}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 32

```
dsolve(x*(diff(y(x),x)^2-1)=2*y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -ix$$
$$y(x) = ix$$
$$y(x) = \frac{-c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 71

```
DSolve[x*(y'[x]^2-1)==2*y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-c_1}\left(-x^2+e^{2c_1}\right)$$
$$y(x) \to \frac{1}{2}e^{-c_1}\left(-1+e^{2c_1}x^2\right)$$
$$y(x) \to -ix$$
$$y(x) \to ix$$

19.5 problem 5

Internal problem ID [2318]
Internal file name [`OUTPUT/2318_Tuesday_February_27_2024_08_25_59_AM_53818986/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 5.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{(1-y^2)\,{y'}^2=1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y'=-\frac{1}{\sqrt{1-y^2}} \tag{1}$$

$$y'=\frac{1}{\sqrt{1-y^2}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\sqrt{-y^2+1}dy=\int dx$$

$$-\frac{y\sqrt{1-y^2}}{2}-\frac{\arcsin(y)}{2}=x+c_1$$

Summary
The solution(s) found are the following

$$-\frac{y\sqrt{1-y^2}}{2}-\frac{\arcsin(y)}{2}=x+c_1 \tag{1}$$

Verification of solutions

$$-\frac{y\sqrt{1-y^2}}{2}-\frac{\arcsin\left(y\right)}{2}=x+c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int \sqrt{-y^2+1}dy=\int dx$$

$$\frac{y\sqrt{1-y^2}}{2}+\frac{\arcsin\left(y\right)}{2}=x+c_2$$

Summary

The solution(s) found are the following

$$\frac{y\sqrt{1-y^2}}{2}+\frac{\arcsin\left(y\right)}{2}=x+c_2 \tag{1}$$

Verification of solutions

$$\frac{y\sqrt{1-y^2}}{2}+\frac{\arcsin\left(y\right)}{2}=x+c_2$$

Verified OK.

19.5.1 Maple step by step solution

Let's solve

$(1-y^2)\,y'^2=1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'\sqrt{1-y^2}=1$

- Integrate both sides with respect to x

  $\int y'\sqrt{1-y^2}dx=\int 1dx+c_1$

- Evaluate integral

  $\frac{y\sqrt{1-y^2}}{2}+\frac{\arcsin(y)}{2}=x+c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 46

```
dsolve((1-y(x)^2)*diff(y(x),x)^2=1,y(x), singsol=all)
```

$$y(x) = \sin(\operatorname{RootOf}(\sin(\_Z)\operatorname{csgn}(\cos(\_Z))\cos(\_Z) + \_Z + 2c_1 - 2x))$$
$$y(x) = \sin(\operatorname{RootOf}(-\sin(\_Z)\operatorname{csgn}(\cos(\_Z))\cos(\_Z) - \_Z + 2c_1 - 2x))$$

✓ Solution by Mathematica

Time used: 0.089 (sec). Leaf size: 105

```
DSolve[(1-y[x]^2)*y'[x]^2==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{1}{2}\#1\sqrt{1-\#1^2} - \arctan\left(\frac{\sqrt{1-\#1^2}}{\#1+1}\right)\&\right][-x+c_1]$$
$$y(x) \to \text{InverseFunction}\left[\frac{1}{2}\#1\sqrt{1-\#1^2} - \arctan\left(\frac{\sqrt{1-\#1^2}}{\#1+1}\right)\&\right][x+c_1]$$

19.6 problem 6

Internal problem ID [2319]
Internal file name [`OUTPUT/2319_Tuesday_February_27_2024_08_25_59_AM_94414629/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 6.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{xyy'^2 + (yx - 1)\, y' - y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{yx - \sqrt{y^2x^2 + 4y^2x - 2yx + 1} - 1}{2yx} \tag{1}$$

$$y' = -\frac{yx + \sqrt{y^2x^2 + 4y^2x - 2yx + 1} - 1}{2yx} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (2)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
-> Calling odsolve with the ODE`, diff(y(x), x) = (-2*x*y(x)^2-y(x)*x^2-y(x)^2)/(y(x)^2*x^3+
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   trying exact
   Looking for potential symmetries
   trying inverse_Riccati
   trying an equivalence to an Abel ODE
   differential order: 1; trying a linearization to 2nd order
   --- trying a change of variables {x -> y(x), y(x) -> x}
   differential order: 1; trying a linearization to 2nd order
   trying 1st order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(y(x), x) = ((y(x)*x^2+y(x)*x-1)/x-2*y(x)*x-y(x))/(y(x
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
```

✗ Solution by Maple

```
dsolve(x*y(x)*diff(y(x),x)^2+(x*y(x)-1)*diff(y(x),x)=y(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y[x]*y'[x]^2+(x*y[x]-1)*y'[x]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

19.7 problem 7

Internal problem ID [2320]
Internal file name [`OUTPUT/2320_Tuesday_February_27_2024_08_26_04_AM_83703145/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 7.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 {y'}^2 + xyy' = 2x^2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{x}{y} \tag{1}$$

$$y' = -\frac{2x}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x}{y} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = x\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{x^2 + 2c_1}$$

$$y = -\sqrt{x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{x^2 + 2c_1} \tag{2}$$

Verification of solutions

$$y = \sqrt{x^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1}$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{2x}{y} \end{aligned}$$

Where $f(x) = -2x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = -2x\,dx$$
$$\int \frac{1}{\frac{1}{y}}\,dy = \int -2x\,dx$$
$$\frac{y^2}{2} = -x^2 + c_2$$

Which results in

$$y = \sqrt{-2x^2 + 2c_2}$$
$$y = -\sqrt{-2x^2 + 2c_2}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-2x^2 + 2c_2} \tag{1}$$
$$y = -\sqrt{-2x^2 + 2c_2} \tag{2}$$

Verification of solutions

$$y = \sqrt{-2x^2 + 2c_2}$$

Verified OK.

$$y = -\sqrt{-2x^2 + 2c_2}$$

Verified OK.

19.7.1 Maple step by step solution

Let's solve

$y^2 y'^2 + xyy' = 2x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = x$

- Integrate both sides with respect to x

$$\int yy'dx = \int xdx + c_1$$

- Evaluate integral

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

- Solve for y

$$\left\{y = \sqrt{x^2 + 2c_1}, y = -\sqrt{x^2 + 2c_1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
dsolve(y(x)^2*diff(y(x),x)^2+x*y(x)*diff(y(x),x)-2*x^2=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x^2 + c_1}$$
$$y(x) = -\sqrt{x^2 + c_1}$$
$$y(x) = \sqrt{-2x^2 + c_1}$$
$$y(x) = -\sqrt{-2x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 80

```
DSolve[y[x]^2*y'[x]^2+x*y[x]*y'[x]-2*x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt{-x^2 + c_1}$$
$$y(x) \to \sqrt{2}\sqrt{-x^2 + c_1}$$
$$y(x) \to -\sqrt{x^2 + 2c_1}$$
$$y(x) \to \sqrt{x^2 + 2c_1}$$

19.8 problem 8

Internal problem ID [2321]
Internal file name [`OUTPUT/2321_Tuesday_February_27_2024_08_26_04_AM_96513986/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 8.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{y^2y'^2 - 2xyy' + 2y^2 = x^2}$$

19.8.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y^2p^2 - 2xyp + 2y^2 = x^2$$

Solving for y from the above results in

$$y = \frac{\left(p + \sqrt{2p^2 + 2}\right)x}{p^2 + 2} \tag{1A}$$

$$y = -\frac{\left(-p + \sqrt{2p^2 + 2}\right)x}{p^2 + 2} \tag{2A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p + \sqrt{2p^2 + 2}}{p^2 + 2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p + \sqrt{2p^2 + 2}}{p^2 + 2} = x \left(\frac{1 + \frac{2p}{\sqrt{2p^2+2}}}{p^2 + 2} - \frac{2\left(p + \sqrt{2p^2 + 2}\right) p}{\left(p^2 + 2\right)^2} \right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p + \sqrt{2p^2 + 2}}{p^2 + 2} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 1 \\
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= x \\
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)+\sqrt{2p(x)^2+2}}{p(x)^2+2}}{x \left(\frac{1+\frac{2p(x)}{\sqrt{2p(x)^2+2}}}{p(x)^2+2} - \frac{2\left(p(x)+\sqrt{2p(x)^2+2}\right)p(x)}{\left(p(x)^2+2\right)^2} \right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p) \left(\frac{1+\frac{2p}{\sqrt{2p^2+2}}}{p^2+2} - \frac{2\left(p+\sqrt{2p^2+2}\right)p}{\left(p^2+2\right)^2} \right)}{p - \frac{p+\sqrt{2p^2+2}}{p^2+2}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{p^2\sqrt{2p^2+2} + 2p^3 - 2\sqrt{2p^2+2}}{\left(p^3 - \sqrt{2p^2+2} + p\right)\sqrt{2p^2+2}\,(p^2+2)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2\left(p^3 + \frac{p^2\sqrt{2p^2+2}}{2} - \sqrt{2p^2+2}\right)x(p)}{\sqrt{2p^2+2}\,\left(p^3 - \sqrt{2p^2+2} + p\right)(p^2+2)} = 0$$

The integrating factor μ is

$$\mu = e^{\int \frac{p^2\sqrt{2p^2+2}+2p^3-2\sqrt{2p^2+2}}{\left(p^3-\sqrt{2p^2+2}+p\right)\sqrt{2p^2+2}\,(p^2+2)}dp}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(e^{\int \frac{p^2\sqrt{2p^2+2}+2p^3-2\sqrt{2p^2+2}}{\left(p^3-\sqrt{2p^2+2}+p\right)\sqrt{2p^2+2}\,(p^2+2)}dp}x\right) = 0$$

Integrating gives

$$e^{\int \frac{p^2\sqrt{2p^2+2}+2p^3-2\sqrt{2p^2+2}}{\left(p^3-\sqrt{2p^2+2}+p\right)\sqrt{2p^2+2}\,(p^2+2)}dp}x = c_2$$

Dividing both sides by the integrating factor $\mu = e^{\int \frac{p^2\sqrt{2p^2+2}+2p^3-2\sqrt{2p^2+2}}{\left(p^3-\sqrt{2p^2+2}+p\right)\sqrt{2p^2+2}\,(p^2+2)}dp}$ results in

$$x(p) = c_2 e^{\int -\frac{2\left(p^3+\frac{p^2\sqrt{2p^2+2}}{2}-\sqrt{2p^2+2}\right)}{\sqrt{2p^2+2}\,\left(p^3-\sqrt{2p^2+2}+p\right)(p^2+2)}dp}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Solving ode 2A Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx}
\end{aligned}
\tag{2}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p - \sqrt{2p^2 + 2}}{p^2 + 2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p - \sqrt{2p^2 + 2}}{p^2 + 2} = x\left(\frac{1 - \frac{2p}{\sqrt{2p^2+2}}}{p^2 + 2} - \frac{2\left(p - \sqrt{2p^2 + 2}\right)p}{\left(p^2 + 2\right)^2}\right)p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p - \sqrt{2p^2 + 2}}{p^2 + 2} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= -1 \\
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -x \\
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x) - \sqrt{2p(x)^2+2}}{p(x)^2+2}}{x\left(\frac{1 - \frac{2p(x)}{\sqrt{2p(x)^2+2}}}{p(x)^2+2} - \frac{2\left(p(x) - \sqrt{2p(x)^2+2}\right)p(x)}{\left(p(x)^2+2\right)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{1-\frac{2p}{\sqrt{2p^2+2}}}{p^2+2} - \frac{2\left(p-\sqrt{2p^2+2}\right)p}{\left(p^2+2\right)^2}\right)}{p - \frac{p-\sqrt{2p^2+2}}{p^2+2}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{-p^2\sqrt{2p^2+2} + 2p^3 + 2\sqrt{2p^2+2}}{\left(p^3+\sqrt{2p^2+2}+p\right)\sqrt{2p^2+2}\,\left(p^2+2\right)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{\left(-p^2\sqrt{2p^2+2} + 2p^3 + 2\sqrt{2p^2+2}\right)x(p)}{\left(p^3+\sqrt{2p^2+2}+p\right)\sqrt{2p^2+2}\,\left(p^2+2\right)} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-p^2\sqrt{2p^2+2}+2p^3+2\sqrt{2p^2+2}}{\left(p^3+\sqrt{2p^2+2}+p\right)\sqrt{2p^2+2}\,\left(p^2+2\right)}dp}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathrm{e}^{\int -\frac{-p^2\sqrt{2p^2+2}+2p^3+2\sqrt{2p^2+2}}{\left(p^3+\sqrt{2p^2+2}+p\right)\sqrt{2p^2+2}\,\left(p^2+2\right)}dp}x\right) = 0$$

Integrating gives

$$\mathrm{e}^{\int -\frac{-p^2\sqrt{2p^2+2}+2p^3+2\sqrt{2p^2+2}}{\left(p^3+\sqrt{2p^2+2}+p\right)\sqrt{2p^2+2}\,\left(p^2+2\right)}dp}x = c_4$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-p^2\sqrt{2p^2+2}+2p^3+2\sqrt{2p^2+2}}{\left(p^3+\sqrt{2p^2+2}+p\right)\sqrt{2p^2+2}\,(p^2+2)}dp}$ results in

$$x(p) = c_4 \mathrm{e}^{-\left(\int -\frac{2\left(p^3-\frac{p^2\sqrt{2p^2+2}}{2}+\sqrt{2p^2+2}\right)}{\sqrt{2p^2+2}\left(p^3+\sqrt{2p^2+2}+p\right)(p^2+2)}dp\right)}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Summary

The solution(s) found are the following

$$y = x \tag{1}$$
$$y = -ix \tag{2}$$
$$y = ix \tag{3}$$
$$y = -x \tag{4}$$
$$y = -ix \tag{5}$$
$$y = ix \tag{6}$$

Verification of solutions

$$y = x$$

Verified OK.

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = -x$$

Verified OK.

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 107

```
dsolve(y(x)^2*diff(y(x),x)^2-2*x*y(x)*diff(y(x),x)+2*y(x)^2=x^2,y(x), singsol=all)
```

$$y(x) = -x$$
$$y(x) = x$$
$$y(x) = \sqrt{-2\sqrt{2}\, c_1 x - c_1^2 - x^2}$$
$$y(x) = \sqrt{2\sqrt{2}\, c_1 x - c_1^2 - x^2}$$
$$y(x) = -\sqrt{-2\sqrt{2}\, c_1 x - c_1^2 - x^2}$$
$$y(x) = -\sqrt{2\sqrt{2}\, c_1 x - c_1^2 - x^2}$$

✓ Solution by Mathematica

Time used: 6.072 (sec). Leaf size: 233

```
DSolve[y[x]^2*y'[x]^2-2*x*y[x]*y'[x]+2*y[x]^2==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2 - 4\sqrt{2}x\cosh(c_1) - 4\sqrt{2}x\sinh(c_1) - 4\cosh(2c_1) - 4\sinh(2c_1)}$$
$$y(x) \to \sqrt{-x^2 - 4\sqrt{2}x\cosh(c_1) - 4\sqrt{2}x\sinh(c_1) - 4\cosh(2c_1) - 4\sinh(2c_1)}$$
$$y(x) \to -\sqrt{-x^2 + 4\sqrt{2}x\cosh(c_1) + 4\sqrt{2}x\sinh(c_1) - 4\cosh(2c_1) - 4\sinh(2c_1)}$$
$$y(x) \to \sqrt{-x^2 + 4\sqrt{2}x\cosh(c_1) + 4\sqrt{2}x\sinh(c_1) - 4\cosh(2c_1) - 4\sinh(2c_1)}$$
$$y(x) \to -\sqrt{-x^2}$$
$$y(x) \to \sqrt{-x^2}$$

19.9 problem 9

Internal problem ID [2322]
Internal file name [OUTPUT/2322_Tuesday_February_27_2024_08_26_05_AM_72160901/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 9.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^3 + (x + y - 2yx)\, y'^2 - 2y'xy(y + x) = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 0 \tag{1}$$
$$y' = 2yx \tag{2}$$
$$y' = -y - x \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \,\mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

Verification of solutions

$$y = c_1$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= 2xy
\end{aligned}$$

Where $f(x) = 2x$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= 2x\,dx \\
\int \frac{1}{y}\,dy &= \int 2x\,dx \\
\ln(y) &= x^2 + c_2 \\
y &= e^{x^2+c_2} \\
&= c_2 e^{x^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 e^{x^2} \tag{1}$$

Verification of solutions

$$y = c_2 e^{x^2}$$

Verified OK.

Solving equation (3)

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= 1 \\
q(x) &= -x
\end{aligned}$$

Hence the ode is

$$y' + y = -x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(-x) \\ \frac{d}{dx}(e^{x} y) &= (e^{x})(-x) \\ d(e^{x} y) &= (-x\, e^{x})\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{x} y &= \int -x\, e^{x}\, dx \\ e^{x} y &= -(x-1)\, e^{x} + c_3\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = -e^{-x}(x-1)\, e^{x} + c_3 e^{-x}$$

which simplifies to

$$y = 1 - x + c_3 e^{-x}$$

Summary
The solution(s) found are the following

$$y = 1 - x + c_3 e^{-x} \tag{1}$$

Verification of solutions

$$y = 1 - x + c_3 e^{-x}$$

Verified OK.

19.9.1 Maple step by step solution

Let's solve

$y'^3 + (x + y - 2yx)\, y'^2 - 2y'xy(y + x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(y'^3 + (x + y - 2yx)\, y'^2 - 2y'xy(y + x)\right) dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int \left(y'^3 + (x + y - 2yx)\, y'^2 - 2y'xy(y + x)\right) dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x)^3+(x+y(x)-2*x*y(x))*diff(y(x),x)^2-2*diff(y(x),x)*x*y(x)*(x+y(x))=0,y(x)
```

$$y(x) = e^{x^2} c_1$$
$$y(x) = 1 + e^{-x} c_1 - x$$
$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 36

```
DSolve[y'[x]^3+(x+y[x]-2*x*y[x])*y'[x]^2-2*y'[x]*x*y[x]*(x+y[x])==0,y[x],x,IncludeSingularSo
```

$$y(x) \to c_1$$
$$y(x) \to c_1 e^{x^2}$$
$$y(x) \to -x + c_1 e^{-x} + 1$$

19.10 problem 10

Internal problem ID [2323]
Internal file name [OUTPUT/2323_Tuesday_February_27_2024_08_26_06_AM_46129370/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 10.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "quadrature", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y{y'}^{2}+\left(y^{2}-x^{3}-y^{2}x\right)y'-xy\left(y^{2}+x^{2}\right)=0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -y \tag{1}$$

$$y' = \frac{x(y^2+x^2)}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{1}{y}dy = \int dx$$

$$-\ln\left(y\right) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{y} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$\frac{1}{y} = c_2 e^{x}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-x}}{c_2} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-x}}{c_2}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$\begin{aligned} y' &= \frac{x(x^2+y^2)}{y} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 621: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{e^{x^2}}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{x^2}}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 e^{-x^2}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x(x^2 + y^2)}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y^2 x\, e^{-x^2} \\ S_y &= y\, e^{-x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-x^2} x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-R^2} R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{(R^2+1)\,\mathrm{e}^{-R^2}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2\mathrm{e}^{-x^2}}{2} = -\frac{(x^2+1)\,\mathrm{e}^{-x^2}}{2} + c_1$$

Which simplifies to

$$\frac{(x^2+y^2+1)\,\mathrm{e}^{-x^2}}{2} - c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{(x^2+y^2+1)\,\mathrm{e}^{-x^2}}{2} - c_1 = 0 \tag{1}$$

Verification of solutions

$$\frac{(x^2+y^2+1)\,\mathrm{e}^{-x^2}}{2} - c_1 = 0$$

Verified OK.

19.10.1 Maple step by step solution

Let's solve

$yy'^2 + (y^2 - x^3 - y^2x)\,y' - xy(y^2+x^2) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int (-1)\,dx + c_1$

- Evaluate integral

  $\ln(y) = -x + c_1$
- Solve for y

  $y = e^{-x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(y(x)*diff(y(x),x)^2+(y(x)^2-x^3-x*y(x)^2)*diff(y(x),x)-x*y(x)*(x^2+y(x)^2)=0,y(x), si
```

$$y(x) = e^{-x}c_1$$
$$y(x) = \sqrt{e^{x^2}c_1 - x^2 - 1}$$
$$y(x) = -\sqrt{e^{x^2}c_1 - x^2 - 1}$$

✓ Solution by Mathematica

Time used: 14.233 (sec). Leaf size: 61

```
DSolve[y[x]*y'[x]^2+(y[x]^2-x^3-x*y[x]^2)*y'[x]-x*y[x]*(x^2+y[x]^2)==0,y[x],x,IncludeSingula
```

$$y(x) \to c_1 e^{-x}$$
$$y(x) \to -\sqrt{-x^2 + c_1 e^{x^2} - 1}$$
$$y(x) \to \sqrt{-x^2 + c_1 e^{x^2} - 1}$$

19.11 problem 11

Internal problem ID [2324]
Internal file name [`OUTPUT/2324_Tuesday_February_27_2024_08_26_06_AM_57096824/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 11.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y - y'x(y'+1) = 0}$$

19.11.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - px(p+1) = 0$$

Solving for y from the above results in

$$y = px(p+1) \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p(p+1) \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - p(p+1) = x(2p+1)\,p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - p(p+1) = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= 0
\end{aligned}
$$

Removing solutions for p which leads to undefined results and substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - p(x)\,(p(x)+1)}{x\,(2p\,(x)+1)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\,(2p+1)}{p - p\,(p+1)} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{-2p-1}{p^2} \\
q(p) &= 0
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{(-2p-1)\,x(p)}{p^2} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{-2p-1}{p^2}dp} \\ &= e^{-\frac{1}{p}+2\ln(p)}\end{aligned}$$

Which simplifies to

$$\mu = p^2 e^{-\frac{1}{p}}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(p^2 e^{-\frac{1}{p}} x\right) &= 0\end{aligned}$$

Integrating gives

$$p^2 e^{-\frac{1}{p}} x = c_2$$

Dividing both sides by the integrating factor $\mu = p^2 e^{-\frac{1}{p}}$ results in

$$x(p) = \frac{c_2 e^{\frac{1}{p}}}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= \frac{-x+\sqrt{x^2+4yx}}{2x} \\ p &= -\frac{x+\sqrt{x^2+4yx}}{2x}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$x = \frac{4c_2 x^2 e^{\frac{2x}{-x+\sqrt{x(4y+x)}}}}{\left(-x+\sqrt{x\,(4y+x)}\right)^2}$$

$$x = \frac{4c_2 x^2 e^{-\frac{2x}{x+\sqrt{x(4y+x)}}}}{\left(x+\sqrt{x\,(4y+x)}\right)^2}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{4c_2 x^2 \mathrm{e}^{\frac{2x}{-x+\sqrt{x(4y+x)}}}}{\left(-x+\sqrt{x\left(4y+x\right)}\right)^2} \tag{2}$$

$$x = \frac{4c_2 x^2 \mathrm{e}^{-\frac{2x}{x+\sqrt{x(4y+x)}}}}{\left(x+\sqrt{x\left(4y+x\right)}\right)^2} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{4c_2 x^2 \mathrm{e}^{\frac{2x}{-x+\sqrt{x(4y+x)}}}}{\left(-x+\sqrt{x\left(4y+x\right)}\right)^2}$$

Verified OK.

$$x = \frac{4c_2 x^2 \mathrm{e}^{-\frac{2x}{x+\sqrt{x(4y+x)}}}}{\left(x+\sqrt{x\left(4y+x\right)}\right)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 65

```
dsolve(y(x)=diff(y(x),x)*x*(1+diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \frac{x\left(1+2\,\mathrm{LambertW}\left(-\frac{1}{2\sqrt{\frac{c_1}{x}}}\right)\right)}{4\,\mathrm{LambertW}\left(-\frac{1}{2\sqrt{\frac{c_1}{x}}}\right)^2}$$

$$y(x) = \frac{x\left(1+2\,\mathrm{LambertW}\left(\frac{1}{2\sqrt{\frac{c_1}{x}}}\right)\right)}{4\,\mathrm{LambertW}\left(\frac{1}{2\sqrt{\frac{c_1}{x}}}\right)^2}$$

✓ Solution by Mathematica

Time used: 0.566 (sec). Leaf size: 102

```
DSolve[y[x]==y'[x]*x*(1+y'[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{\sqrt{\frac{4y(x)}{x}+1}-1}-\log\left(\sqrt{\frac{4y(x)}{x}+1}-1\right)=\frac{\log(x)}{2}+c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{1}{\sqrt{\frac{4y(x)}{x}+1}+1}+\log\left(\sqrt{\frac{4y(x)}{x}+1}+1\right)=-\frac{\log(x)}{2}+c_1,y(x)\right]$$

$$y(x)\to 0$$

19.12 problem 12

Internal problem ID [2325]
Internal file name [`OUTPUT/2325_Tuesday_February_27_2024_08_26_06_AM_13453402/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 12.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"dAlembert", "separable"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y - 3\ln(y') = x}$$

19.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= e^{\frac{y}{3}} e^{-\frac{x}{3}} \end{aligned}$$

Where $f(x) = e^{-\frac{x}{3}}$ and $g(y) = e^{\frac{y}{3}}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{e^{\frac{y}{3}}}\, dy &= e^{-\frac{x}{3}}\, dx \\ \int \frac{1}{e^{\frac{y}{3}}}\, dy &= \int e^{-\frac{x}{3}}\, dx \\ -3\, e^{-\frac{y}{3}} &= -3\, e^{-\frac{x}{3}} + c_1 \end{aligned}$$

Which results in

$$y = 3\ln\left(\frac{3}{3\,e^{-\frac{x}{3}} - c_1}\right)$$

Summary

The solution(s) found are the following

$$y = 3\ln\left(\frac{3}{3\,e^{-\frac{x}{3}} - c_1}\right) \tag{1}$$

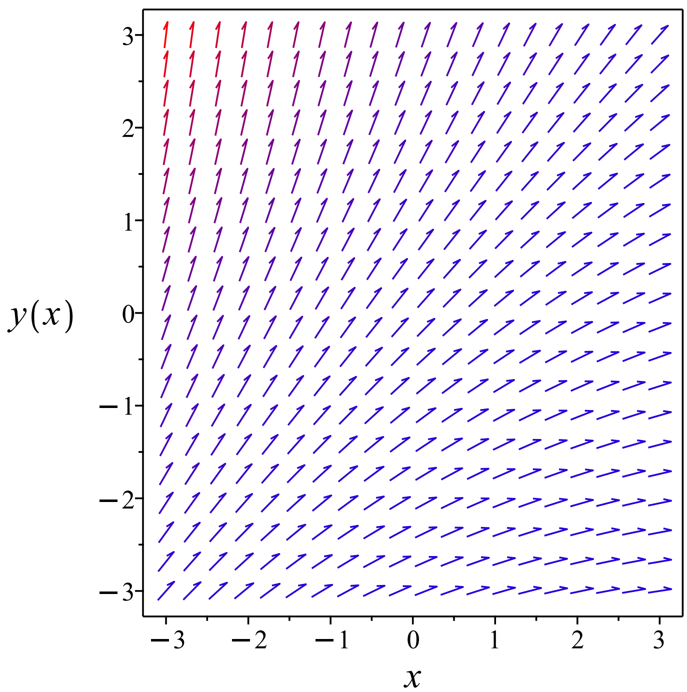


Figure 730: Slope field plot

Verification of solutions

$$y = 3\ln\left(\frac{3}{3\,e^{-\frac{x}{3}} - c_1}\right)$$

Verified OK.

19.12.2 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - 3\ln(p) = x$$

Solving for y from the above results in

$$y = x + 3\ln(p) \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 1$$
$$g = 3\ln(p)$$

Hence (2) becomes

$$p - 1 = \frac{3p'(x)}{p} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - 1 = 0$$

Solving for p from the above gives

$$p = 1$$

Substituting these in (1A) gives

$$y = x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{(p(x) - 1)\,p(x)}{3} \tag{3}$$

This ODE is now solved for $p(x)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{3}{(p-1)\,p} dp &= \int dx \\
3\ln(p-1) - 3\ln(p) &= x + c_1
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
(3)\left(\ln(p-1) - \ln(p)\right) &= x + c_1 \\
\ln(p-1) - \ln(p) &= \left(\frac{1}{3}\right)(x + c_1) \\
&= \frac{x}{3} + \frac{c_1}{3}
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{\ln(p-1) - \ln(p)} = \frac{c_1 \mathrm{e}^{\frac{x}{3}}}{3}
$$

Which simplifies to

$$
\frac{p-1}{p} = c_2 \mathrm{e}^{\frac{x}{3}}
$$

Substituing the above solution for p in (2A) gives

$$
y = x + 3\ln\left(-\frac{1}{-1 + c_2 \mathrm{e}^{\frac{x}{3}}}\right)
$$

Summary

The solution(s) found are the following

$$
y = x \tag{1}
$$

$$
y = x + 3\ln\left(-\frac{1}{-1 + c_2 \mathrm{e}^{\frac{x}{3}}}\right) \tag{2}
$$

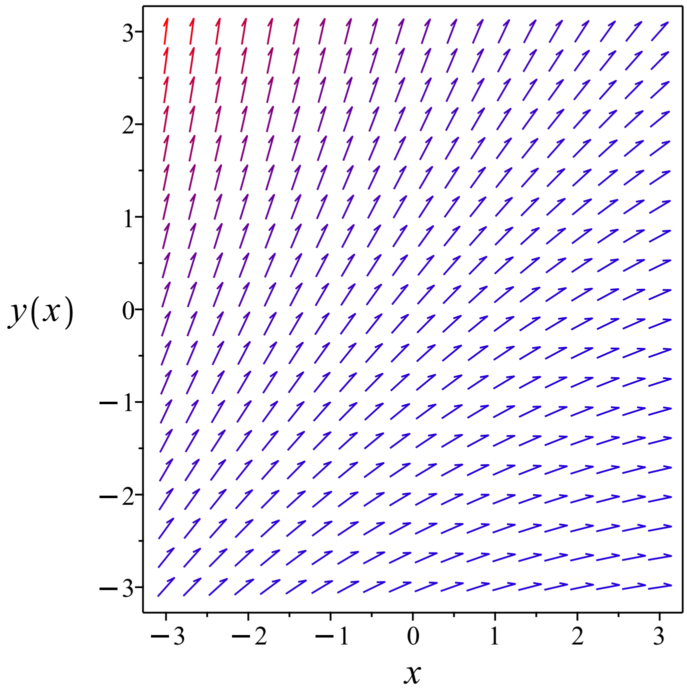


Figure 731: Slope field plot

Verification of solutions

$$y = x$$

Verified OK.

$$y = x + 3\ln\left(-\frac{1}{-1 + c_2 \mathrm{e}^{\frac{x}{3}}}\right)$$

Verified OK.

19.12.3 Maple step by step solution

Let's solve

$y - 3\ln(y') = x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\mathrm{e}^{\frac{y}{3}}} = \mathrm{e}^{-\frac{x}{3}}$

- Integrate both sides with respect to x

$$\int \frac{y'}{e^{\frac{y}{3}}}dx = \int e^{-\frac{x}{3}}dx + c_1$$

- Evaluate integral

$$-\frac{3}{e^{\frac{y}{3}}} = -3\,e^{-\frac{x}{3}} + c_1$$

- Solve for y

$$y = 3\ln\left(\frac{3}{3\,e^{-\frac{x}{3}} - c_1}\right)$$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
dsolve(y(x)=x+3*ln(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = x$$
$$y(x) = x + 3\ln\left(\frac{e^{-\frac{x}{3}}c_1}{-1 + c_1 e^{-\frac{x}{3}}}\right)$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 22

```
DSolve[y[x]==x+3*Log[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3\log\left(e^{-x/3} - \frac{c_1}{3}\right)$$

19.13 problem 13

Internal problem ID [2326]
Internal file name [OUTPUT/2326_Tuesday_February_27_2024_08_26_07_AM_71791739/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 13.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y\left(1+y'^2\right) = 2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{-y\left(y-2\right)}}{y} \tag{1}$$

$$y' = -\frac{\sqrt{-y\left(y-2\right)}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y}{\sqrt{-y\left(y-2\right)}}dy = \int dx$$

$$-\sqrt{-y^2+2y}+\arcsin\left(y-1\right) = x+c_1$$

Summary
The solution(s) found are the following

$$-\sqrt{-y^2+2y}+\arcsin\left(y-1\right) = x+c_1 \tag{1}$$

Verification of solutions

$$-\sqrt{-y^2+2y}+\arcsin(y-1)=x+c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y}{\sqrt{-y(y-2)}}dy=\int dx$$
$$\sqrt{-y^2+2y}-\arcsin(y-1)=x+c_2$$

Summary

The solution(s) found are the following

$$\sqrt{-y^2+2y}-\arcsin(y-1)=x+c_2 \tag{1}$$

Verification of solutions

$$\sqrt{-y^2+2y}-\arcsin(y-1)=x+c_2$$

Verified OK.

19.13.1 Maple step by step solution

Let's solve

$y(1+y'^2)=2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{\sqrt{-y(y-2)}}=1$

- Integrate both sides with respect to x

  $\int \frac{y'y}{\sqrt{-y(y-2)}}dx=\int 1dx+c_1$

- Evaluate integral

  $-\sqrt{-y^2+2y}+\arcsin(y-1)=x+c_1$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 50

```
dsolve(y(x)*(1+diff(y(x),x)^2)=2,y(x), singsol=all)
```

$$y(x) = 2$$
$$y(x) = -\sin\left(\operatorname{RootOf}\left(-\_Z - x - \operatorname{csgn}\left(\cos\left(\_Z\right)\right)\cos\left(\_Z\right) + c_1\right)\right) + 1$$
$$y(x) = \sin\left(\operatorname{RootOf}\left(-\_Z - x + \operatorname{csgn}\left(\cos\left(\_Z\right)\right)\cos\left(\_Z\right) + c_1\right)\right) + 1$$

✓ Solution by Mathematica

Time used: 0.359 (sec). Leaf size: 118

```
DSolve[y[x]*(1+y'[x]^2)==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[-4\arctan\left(\frac{\sqrt{\#1}}{\sqrt{2}-\sqrt{2-\#1}}\right) - \sqrt{-((\#1-2)\#1)}\&\right][-x+c_1]$$
$$y(x) \to \text{InverseFunction}\left[-4\arctan\left(\frac{\sqrt{\#1}}{\sqrt{2}-\sqrt{2-\#1}}\right) - \sqrt{-((\#1-2)\#1)}\&\right][x+c_1]$$
$$y(x) \to 2$$

19.14 problem 14

Internal problem ID [2327]
Internal file name [`OUTPUT/2327_Tuesday_February_27_2024_08_26_07_AM_81855955/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 14.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{yy'^2 - 2xy' + y = 0}$$

19.14.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y\,p^2 - 2xp + y = 0$$

Solving for y from the above results in

$$y = \frac{2xp}{p^2 + 1} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{2p}{p^2+1}$$
$$g = 0$$

Hence (2) becomes

$$p - \frac{2p}{p^2+1} = x\left(\frac{2}{p^2+1} - \frac{4p^2}{\left(p^2+1\right)^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{2p}{p^2+1} = 0$$

Solving for p from the above gives

$$p = 0$$
$$p = 1$$
$$p = -1$$

Substituting these in (1A) gives

$$y = -x$$
$$y = 0$$
$$y = x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{2p(x)}{p(x)^2+1}}{x\left(\frac{2}{p(x)^2+1} - \frac{4p(x)^2}{\left(p(x)^2+1\right)^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{2}{p^2+1} - \frac{4p^2}{\left(p^2+1\right)^2}\right)}{p - \frac{2p}{p^2+1}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{p(p^2+1)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p(p^2+1)} = 0$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{p(p^2+1)}dp}$$
$$= e^{-\ln(p^2+1)+2\ln(p)}$$

Which simplifies to

$$\mu = \frac{p^2}{p^2+1}$$

The ode becomes

$$\frac{d}{dp}\mu x = 0$$
$$\frac{d}{dp}\left(\frac{p^2 x}{p^2+1}\right) = 0$$

Integrating gives

$$\frac{p^2 x}{p^2+1} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{p^2}{p^2+1}$ results in

$$x(p) = \frac{c_3(p^2+1)}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{x+\sqrt{x^2-y^2}}{y}$$
$$p = -\frac{-x+\sqrt{x^2-y^2}}{y}$$

Substituting the above in the solution for x found above gives

$$x = \frac{2c_3 x}{x + \sqrt{x^2 - y^2}}$$
$$x = -\frac{2c_3 x}{-x + \sqrt{x^2 - y^2}}$$

Summary

The solution(s) found are the following

$$y = -x \tag{1}$$
$$y = 0 \tag{2}$$
$$y = x \tag{3}$$
$$x = \frac{2c_3 x}{x + \sqrt{x^2 - y^2}} \tag{4}$$
$$x = -\frac{2c_3 x}{-x + \sqrt{x^2 - y^2}} \tag{5}$$

Verification of solutions

$$y = -x$$

Verified OK.

$$y = 0$$

Verified OK.

$$y = x$$

Verified OK.

$$x = \frac{2c_3 x}{x + \sqrt{x^2 - y^2}}$$

Verified OK.

$$x = -\frac{2c_3 x}{-x + \sqrt{x^2 - y^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 71

```
dsolve(y(x)*diff(y(x),x)^2-2*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -x$$
$$y(x) = x$$
$$y(x) = 0$$
$$y(x) = \sqrt{c_1 (-2ix + c_1)}$$
$$y(x) = \sqrt{c_1 (2ix + c_1)}$$
$$y(x) = -\sqrt{c_1 (-2ix + c_1)}$$
$$y(x) = -\sqrt{c_1 (2ix + c_1)}$$

✓ Solution by Mathematica

Time used: 2.388 (sec). Leaf size: 174

```
DSolve[y[x]*y'[x]^2-2*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{4}\left(\cosh\left(\frac{c_1}{2}\right)+\sinh\left(\frac{c_1}{2}\right)\right)\sqrt{-8ix+\cosh(c_1)+\sinh(c_1)}$$
$$y(x) \to \frac{1}{4}\left(\cosh\left(\frac{c_1}{2}\right)+\sinh\left(\frac{c_1}{2}\right)\right)\sqrt{-8ix+\cosh(c_1)+\sinh(c_1)}$$
$$y(x) \to -\frac{1}{4}\left(\cosh\left(\frac{c_1}{2}\right)+\sinh\left(\frac{c_1}{2}\right)\right)\sqrt{8ix+\cosh(c_1)+\sinh(c_1)}$$
$$y(x) \to \frac{1}{4}\left(\cosh\left(\frac{c_1}{2}\right)+\sinh\left(\frac{c_1}{2}\right)\right)\sqrt{8ix+\cosh(c_1)+\sinh(c_1)}$$
$$y(x) \to 0$$
$$y(x) \to -x$$
$$y(x) \to x$$

19.15 problem 15

Internal problem ID [2328]
Internal file name [`OUTPUT/2328_Tuesday_February_27_2024_08_26_08_AM_51440526/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 15.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 + y^2 = 1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{1 - y^2} \tag{1}$$
$$y' = -\sqrt{1 - y^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2 + 1}} dy = x + c_1$$
$$\arcsin(y) = x + c_1$$

Solving for y gives these solutions

$$y_1 = \sin(x + c_1)$$

Summary
The solution(s) found are the following

$$y = \sin(x + c_1) \tag{1}$$

Verification of solutions

$$y = \sin(x + c_1)$$

Verified OK.
Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2+1}} dy = x + c_2$$
$$-\arcsin(y) = x + c_2$$

Solving for y gives these solutions

$$y_1 = -\sin(x + c_2)$$

Summary
The solution(s) found are the following

$$y = -\sin(x + c_2) \tag{1}$$

Verification of solutions

$$y = -\sin(x + c_2)$$

Verified OK.

19.15.1 Maple step by step solution

Let's solve

$y'^2 + y^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = 1$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{1-y^2}}dx = \int 1dx + c_1$$

- Evaluate integral

$$\arcsin(y) = x + c_1$$

- Solve for y

$$y = \sin(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)^2+y(x)^2=1,y(x), singsol=all)
```

$$y(x) = -1$$
$$y(x) = 1$$
$$y(x) = -\sin(c_1 - x)$$
$$y(x) = \sin(c_1 - x)$$

✓ Solution by Mathematica

Time used: 0.175 (sec). Leaf size: 39

```
DSolve[y'[x]^2+y[x]^2==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x + c_1)$$
$$y(x) \to \cos(x - c_1)$$
$$y(x) \to -1$$
$$y(x) \to 1$$
$$y(x) \to \text{Interval}[\{-1, 1\}]$$

19.16 problem 16

Internal problem ID [2329]
Internal file name [`OUTPUT/2329_Tuesday_February_27_2024_08_26_08_AM_10569062/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 16.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{x\left(-1+y'^2\right)-2yy'=0}$$

19.16.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$x(p^2-1)-2yp=0$$

Solving for y from the above results in

$$y=\frac{x(p^2-1)}{2p} \tag{1A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$
$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2 - 1}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2 - 1}{2p} = x\left(1 - \frac{p^2 - 1}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2 - 1}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2 - 1}{2p(x)}}{x\left(1 - \frac{p(x)^2 - 1}{2p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}\mu p = 0$$
$$\frac{d}{dx}\left(\frac{p}{x}\right) = 0$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Summary
The solution(s) found are the following

$$y = -ix \tag{1}$$
$$y = ix \tag{2}$$
$$y = \frac{c_1^2 x^2 - 1}{2c_1} \tag{3}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 32

```
dsolve((diff(y(x),x)^2-1)*x=2*diff(y(x),x)*y(x),y(x), singsol=all)
```

$$y(x) = -ix$$
$$y(x) = ix$$
$$y(x) = \frac{-c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 71

```
DSolve[(y'[x]^2-1)*x==2*y'[x]*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-c_1}\left(-x^2 + e^{2c_1}\right)$$
$$y(x) \to \frac{1}{2}e^{-c_1}\left(-1 + e^{2c_1}x^2\right)$$
$$y(x) \to -ix$$
$$y(x) \to ix$$

19.17 problem 17

Internal problem ID [2330]
Internal file name [`OUTPUT/2330_Tuesday_February_27_2024_08_26_08_AM_95469468/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 17.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{-2yy' + {y'}^2 x = -4x}$$

19.17.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 x - 2yp = -4x$$

Solving for y from the above results in

$$y = \frac{x(p^2 + 4)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2 + 4}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2 + 4}{2p} = x\left(1 - \frac{p^2 + 4}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2 + 4}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 2 \\
p &= -2
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -2x \\
y &= 2x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2+4}{2p(x)}}{x\left(1 - \frac{p(x)^2+4}{2p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}\mu p = 0$$
$$\frac{d}{dx}\left(\frac{p}{x}\right) = 0$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Summary
The solution(s) found are the following

$$y = -2x \tag{1}$$
$$y = 2x \tag{2}$$
$$y = \frac{c_1^2 x^2 + 4}{2c_1} \tag{3}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = 2x$$

Verified OK.

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 30

```
dsolve(4*x-2*diff(y(x),x)*y(x)+diff(y(x),x)^2*x=0,y(x), singsol=all)
```

$$y(x) = -2x$$
$$y(x) = 2x$$
$$y(x) = \frac{4c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.295 (sec). Leaf size: 43

```
DSolve[4*x-2*y'[x]*y[x]+y'[x]^2*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x\cosh(-\log(x) + c_1)$$
$$y(x) \to -2x\cosh(\log(x) + c_1)$$
$$y(x) \to -2x$$
$$y(x) \to 2x$$

19.18 problem 18

Internal problem ID [2331]
Internal file name [`OUTPUT/2331_Tuesday_February_27_2024_08_26_08_AM_65434278/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 18.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y + y'^2 - y'x^3 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \left(\frac{x^2}{2} + \frac{\sqrt{x^4 - 8y}}{2}\right) x \tag{1}$$

$$y' = \left(\frac{x^2}{2} - \frac{\sqrt{x^4 - 8y}}{2}\right) x \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\left(x^2 + \sqrt{x^4 - 8y}\right) x}{2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(x^2 + \sqrt{x^4 - 8y}\right) x(b_3 - a_2)}{2} - \frac{\left(x^2 + \sqrt{x^4 - 8y}\right)^2 x^2 a_3}{4} - \left(\frac{\left(2x + \frac{2x^3}{\sqrt{x^4 - 8y}}\right) x}{2} + \frac{x^2}{2} + \frac{\sqrt{x^4 - 8y}}{2}\right)(xa_2 + ya_3 + a_1) + \frac{2x(xb_2 + yb_3 + b_1)}{\sqrt{x^4 - 8y}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^8a_3 + \sqrt{x^4 - 8y}\, x^6 a_3 + (x^4 - 8y)^{\frac{3}{2}} x^2 a_3 + 8x^5 a_2 - 2x^5 b_3 - 10x^4 y a_3 + 8\sqrt{x^4 - 8y}\, x^3 a_2 - 2\sqrt{x^4 - 8y}\, x}{} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-2x^8a_3 - \sqrt{x^4 - 8y}\, x^6 a_3 - \left(x^4 - 8y\right)^{\frac{3}{2}} x^2 a_3 - 8x^5 a_2 \\ &+ 2x^5 b_3 + 10x^4 y a_3 - 8\sqrt{x^4 - 8y}\, x^3 a_2 + 2\sqrt{x^4 - 8y}\, x^3 b_3 \\ &- 6\sqrt{x^4 - 8y}\, x^2 y a_3 - 6x^4 a_1 - 6\sqrt{x^4 - 8y}\, x^2 a_1 + 8x^2 b_2 \\ &+ 32xya_2 - 8xyb_3 + 16y^2 a_3 + 4b_2\sqrt{x^4 - 8y} + 8xb_1 + 16ya_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-\sqrt{x^4 - 8y}\, x^6 a_3 - 2\left(x^4 - 8y\right) x^4 a_3 - \left(x^4 - 8y\right)^{\frac{3}{2}} x^2 a_3 - 4x^5 a_2 - 4x^4 y a_3 \\ &- 8\sqrt{x^4 - 8y}\, x^3 a_2 + 2\sqrt{x^4 - 8y}\, x^3 b_3 - 6\sqrt{x^4 - 8y}\, x^2 y a_3 - 4x^4 a_1 \\ &- 4\left(x^4 - 8y\right) x a_2 + 2\left(x^4 - 8y\right) x b_3 - 2\left(x^4 - 8y\right) y a_3 - 6\sqrt{x^4 - 8y}\, x^2 a_1 \\ &- 2\left(x^4 - 8y\right) a_1 + 8x^2 b_2 + 8xyb_3 + 4b_2\sqrt{x^4 - 8y} + 8xb_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
&-2x^8a_3 - 2\sqrt{x^4-8y}\,x^6a_3 - 8x^5a_2 + 2x^5b_3 + 10x^4ya_3 - 6x^4a_1 \\
&- 8\sqrt{x^4-8y}\,x^3a_2 + 2\sqrt{x^4-8y}\,x^3b_3 + 2\sqrt{x^4-8y}\,x^2ya_3 - 6\sqrt{x^4-8y}\,x^2a_1 \\
&+ 8x^2b_2 + 32xya_2 - 8xyb_3 + 16y^2a_3 + 8xb_1 + 4b_2\sqrt{x^4-8y} + 16ya_1 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^4-8y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^4-8y} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-2v_1^8a_3 - 2v_3v_1^6a_3 - 8v_1^5a_2 + 10v_1^4v_2a_3 + 2v_1^5b_3 - 6v_1^4a_1 \\
&- 8v_3v_1^3a_2 + 2v_3v_1^2v_2a_3 + 2v_3v_1^3b_3 - 6v_3v_1^2a_1 + 32v_1v_2a_2 \\
&+ 16v_2^2a_3 + 8v_1^2b_2 - 8v_1v_2b_3 + 16v_2a_1 + 8v_1b_1 + 4b_2v_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-2v_1^8a_3 - 2v_3v_1^6a_3 + (-8a_2 + 2b_3)\,v_1^5 + 10v_1^4v_2a_3 - 6v_1^4a_1 \\
&+ (-8a_2 + 2b_3)\,v_1^3v_3 + 2v_3v_1^2v_2a_3 - 6v_3v_1^2a_1 + 8v_1^2b_2 \\
&+ (32a_2 - 8b_3)\,v_1v_2 + 8v_1b_1 + 16v_2^2a_3 + 16v_2a_1 + 4b_2v_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-6a_1 &= 0 \\
16a_1 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
10a_3 &= 0 \\
16a_3 &= 0 \\
8b_1 &= 0 \\
4b_2 &= 0 \\
8b_2 &= 0 \\
-8a_2 + 2b_3 &= 0 \\
32a_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 4a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 4y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 4y - \left(\frac{\left(x^2 + \sqrt{x^4 - 8y}\right) x}{2}\right)(x) \\
&= -\frac{x^4}{2} - \frac{\sqrt{x^4 - 8y}\, x^2}{2} + 4y \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{x^4}{2} - \frac{\sqrt{x^4-8y}\, x^2}{2} + 4y} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{4} + \frac{\ln\left(x^2 + \sqrt{x^4 - 8y}\right)}{4} - \frac{\ln\left(-x^2 + \sqrt{x^4 - 8y}\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\left(x^2 + \sqrt{x^4 - 8y}\right) x}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{\sqrt{x^4 - 8y}} \\ S_y &= -\frac{2}{\left(x^2 + \sqrt{x^4 - 8y}\right) \sqrt{x^4 - 8y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{4} + \frac{\ln\left(x^2 + \sqrt{x^4 - 8y}\right)}{4} - \frac{\ln\left(-x^2 + \sqrt{x^4 - 8y}\right)}{4} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{4} + \frac{\ln\left(x^2 + \sqrt{x^4 - 8y}\right)}{4} - \frac{\ln\left(-x^2 + \sqrt{x^4 - 8y}\right)}{4} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{4} + \frac{\ln\left(x^2 + \sqrt{x^4 - 8y}\right)}{4} - \frac{\ln\left(-x^2 + \sqrt{x^4 - 8y}\right)}{4} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{4} + \frac{\ln\left(x^2 + \sqrt{x^4 - 8y}\right)}{4} - \frac{\ln\left(-x^2 + \sqrt{x^4 - 8y}\right)}{4} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{\left(-x^2 + \sqrt{x^4 - 8y}\right)x}{2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &- \frac{\left(-x^2+\sqrt{x^4-8y}\right)x(b_3-a_2)}{2} - \frac{\left(-x^2+\sqrt{x^4-8y}\right)^2 x^2 a_3}{4} \\ &- \left(-\frac{\left(-2x+\frac{2x^3}{\sqrt{x^4-8y}}\right)x}{2} + \frac{x^2}{2} - \frac{\sqrt{x^4-8y}}{2}\right)(xa_2+ya_3+a_1) \\ &- \frac{2x(xb_2+yb_3+b_1)}{\sqrt{x^4-8y}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-2x^8a_3 + \sqrt{x^4-8y}\,x^6a_3 + (x^4-8y)^{\frac{3}{2}}x^2a_3 - 8x^5a_2 + 2x^5b_3 + 10x^4ya_3 + 8\sqrt{x^4-8y}\,x^3a_2 - 2\sqrt{x^4-8y}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} 2x^8a_3 &- \sqrt{x^4-8y}\,x^6a_3 - \left(x^4-8y\right)^{\frac{3}{2}}x^2a_3 + 8x^5a_2 - 2x^5b_3 \\ &- 10x^4ya_3 - 8\sqrt{x^4-8y}\,x^3a_2 + 2\sqrt{x^4-8y}\,x^3b_3 \\ &- 6\sqrt{x^4-8y}\,x^2ya_3 + 6x^4a_1 - 6\sqrt{x^4-8y}\,x^2a_1 - 8x^2b_2 \\ &- 32xa_2y + 8xyb_3 - 16y^2a_3 + 4b_2\sqrt{x^4-8y} - 8xb_1 - 16a_1y = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-\sqrt{x^4-8y}\,x^6a_3+2\left(x^4-8y\right)x^4a_3-\left(x^4-8y\right)^{\frac{3}{2}}x^2a_3+4x^5a_2+4x^4ya_3\\
&-8\sqrt{x^4-8y}\,x^3a_2+2\sqrt{x^4-8y}\,x^3b_3-6\sqrt{x^4-8y}\,x^2ya_3+4x^4a_1\\
&+4\left(x^4-8y\right)xa_2-2\left(x^4-8y\right)xb_3+2\left(x^4-8y\right)ya_3-6\sqrt{x^4-8y}\,x^2a_1\\
&+2\left(x^4-8y\right)a_1-8x^2b_2-8xyb_3+4b_2\sqrt{x^4-8y}-8xb_1=0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x^8a_3-2\sqrt{x^4-8y}\,x^6a_3+8x^5a_2-2x^5b_3-10x^4ya_3+6x^4a_1-8\sqrt{x^4-8y}\,x^3a_2\\
&+2\sqrt{x^4-8y}\,x^3b_3+2\sqrt{x^4-8y}\,x^2ya_3-6\sqrt{x^4-8y}\,x^2a_1-8x^2b_2\\
&-32xa_2y+8xyb_3-16y^2a_3-8xb_1+4b_2\sqrt{x^4-8y}-16a_1y=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^4-8y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{x^4-8y}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_1^8a_3-2v_3v_1^6a_3+8v_1^5a_2-10v_1^4v_2a_3-2v_1^5b_3+6v_1^4a_1\\
&-8v_3v_1^3a_2+2v_3v_1^2v_2a_3+2v_3v_1^3b_3-6v_3v_1^2a_1-32v_1a_2v_2\\
&-16v_2^2a_3-8v_1^2b_2+8v_1v_2b_3-16a_1v_2-8v_1b_1+4b_2v_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&2v_1^8a_3-2v_3v_1^6a_3+(8a_2-2b_3)\,v_1^5-10v_1^4v_2a_3+6v_1^4a_1\\
&+(-8a_2+2b_3)\,v_1^3v_3+2v_3v_1^2v_2a_3-6v_3v_1^2a_1-8v_1^2b_2\\
&+(-32a_2+8b_3)\,v_1v_2-8v_1b_1-16v_2^2a_3-16a_1v_2+4b_2v_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-16a_1 &= 0 \\
-6a_1 &= 0 \\
6a_1 &= 0 \\
-16a_3 &= 0 \\
-10a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
-8b_1 &= 0 \\
-8b_2 &= 0 \\
4b_2 &= 0 \\
-32a_2 + 8b_3 &= 0 \\
-8a_2 + 2b_3 &= 0 \\
8a_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 4a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= 4y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 4y - \left(-\frac{\left(-x^2 + \sqrt{x^4 - 8y}\right)x}{2}\right)(x) \\
&= -\frac{x^4}{2} + \frac{\sqrt{x^4 - 8y}\,x^2}{2} + 4y \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{x^4}{2} + \frac{\sqrt{x^4 - 8y}\, x^2}{2} + 4y} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{4} - \frac{\ln\left(x^2 + \sqrt{x^4 - 8y}\right)}{4} + \frac{\ln\left(-x^2 + \sqrt{x^4 - 8y}\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\left(-x^2 + \sqrt{x^4 - 8y}\right) x}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{\sqrt{x^4 - 8y}} \\ S_y &= -\frac{2}{\sqrt{x^4 - 8y}\, \left(-x^2 + \sqrt{x^4 - 8y}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{4} - \frac{\ln\left(x^2+\sqrt{x^4-8y}\right)}{4} + \frac{\ln\left(-x^2+\sqrt{x^4-8y}\right)}{4} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{4} - \frac{\ln\left(x^2+\sqrt{x^4-8y}\right)}{4} + \frac{\ln\left(-x^2+\sqrt{x^4-8y}\right)}{4} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{4} - \frac{\ln\left(x^2+\sqrt{x^4-8y}\right)}{4} + \frac{\ln\left(-x^2+\sqrt{x^4-8y}\right)}{4} = c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{4} - \frac{\ln\left(x^2+\sqrt{x^4-8y}\right)}{4} + \frac{\ln\left(-x^2+\sqrt{x^4-8y}\right)}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      -> Calling odsolve with the ODE`, diff(diff(y(x), x), x)-(diff(y(x), x))/x, y(x)`
         Methods for second order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         <- LODE of Euler type successful
      <- 1st order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      <- 1st order ODE linearizable_by_differentiation successful`
```

✓ Solution by Maple

Time used: 0.218 (sec). Leaf size: 21

```
dsolve(2*x^2*y(x)+diff(y(x),x)^2=diff(y(x),x)*x^3,y(x), singsol=all)
```

$$y(x) = \frac{x^4}{8}$$
$$y(x) = c_1(x^2 - 2c_1)$$

✓ Solution by Mathematica

Time used: 2.642 (sec). Leaf size: 216

```
DSolve[2*x^2*y[x]+y'[x]^2==y'[x]*x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\sqrt{x^6-8x^2y(x)}\log\left(\sqrt{x^4-8y(x)}+x^2\right)}{2x\sqrt{x^4-8y(x)}} - \frac{\sqrt{x^6-8x^2y(x)}\log(y(x))}{4x\sqrt{x^4-8y(x)}} + \frac{1}{4}\log(y(x)) = c_1, y(x)\right]$$
$$\text{Solve}\left[-\frac{\sqrt{x^6-8x^2y(x)}\log\left(\sqrt{x^4-8y(x)}+x^2\right)}{2x\sqrt{x^4-8y(x)}} + \frac{\sqrt{x^6-8x^2y(x)}\log(y(x))}{4x\sqrt{x^4-8y(x)}} + \frac{1}{4}\log(y(x)) = c_1, y(x)\right]$$
$$y(x) \to \frac{x^4}{8}$$

19.19 problem 19

Internal problem ID [2332]
Internal file name [`OUTPUT/2332_Tuesday_February_27_2024_08_26_12_AM_2282511/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 19.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{yy'^2 - 3xy' - y = 0}$$

19.19.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y\,p^2 - 3xp - y = 0$$

Solving for y from the above results in

$$y = \frac{3xp}{p^2 - 1} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{3p}{p^2 - 1} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{3p}{p^2 - 1} = x\left(\frac{3}{p^2 - 1} - \frac{6p^2}{(p^2 - 1)^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{3p}{p^2 - 1} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= 2 \\
p &= -2
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -2x \\
y &= 0 \\
y &= 2x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{3p(x)}{p(x)^2 - 1}}{x\left(\frac{3}{p(x)^2 - 1} - \frac{6p(x)^2}{\left(p(x)^2 - 1\right)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{3}{p^2 - 1} - \frac{6p^2}{(p^2 - 1)^2}\right)}{p - \frac{3p}{p^2 - 1}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{-3p^2 - 3}{p^5 - 5p^3 + 4p}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{(-3p^2 - 3)\, x(p)}{p^5 - 5p^3 + 4p} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-3p^2-3}{p^5-5p^3+4p}dp}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(e^{\int -\frac{-3p^2-3}{p^5-5p^3+4p}dp} x\right) = 0$$

Integrating gives

$$e^{\int -\frac{-3p^2-3}{p^5-5p^3+4p}dp} x = c_3$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{-3p^2-3}{p^5-5p^3+4p}dp}$ results in

$$x(p) = c_3 e^{-3\left(\int \frac{p^2+1}{p^5-5p^3+4p}dp\right)}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$
$$y = 0 \tag{2}$$
$$y = 2x \tag{3}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = 0$$

Verified OK.

$$y = 2x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 1.922 (sec). Leaf size: 273

```
dsolve(diff(y(x),x)^2*y(x)=3*diff(y(x),x)*x+y(x),y(x), singsol=all)
```

$$y(x) = 0$$

$$\ln(x) - \frac{3\,\mathrm{arctanh}\left(\frac{3}{\sqrt{\frac{9x^2+4y(x)^2}{x^2}}}\right)}{8} + \frac{5\,\mathrm{arctanh}\left(\frac{9x+8y(x)}{5x\sqrt{\frac{9x^2+4y(x)^2}{x^2}}}\right)}{16} - \frac{5\,\mathrm{arctanh}\left(\frac{-9x+8y(x)}{5x\sqrt{\frac{9x^2+4y(x)^2}{x^2}}}\right)}{16} + \frac{5\ln\left(\frac{y(x)+2x}{x}\right)}{16} + \frac{5\ln\left(\frac{-2x+y(x)}{x}\right)}{16} + \frac{3\ln\left(\frac{y(x)}{x}\right)}{8} - c_1 = 0$$

$$\ln(x) + \frac{3\,\mathrm{arctanh}\left(\frac{3}{\sqrt{\frac{9x^2+4y(x)^2}{x^2}}}\right)}{8} - \frac{5\,\mathrm{arctanh}\left(\frac{9x+8y(x)}{5x\sqrt{\frac{9x^2+4y(x)^2}{x^2}}}\right)}{16} + \frac{5\,\mathrm{arctanh}\left(\frac{-9x+8y(x)}{5x\sqrt{\frac{9x^2+4y(x)^2}{x^2}}}\right)}{16} + \frac{5\ln\left(\frac{y(x)+2x}{x}\right)}{16} + \frac{5\ln\left(\frac{-2x+y(x)}{x}\right)}{16} + \frac{3\ln\left(\frac{y(x)}{x}\right)}{8} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 75.255 (sec). Leaf size: 2113

```
DSolve[y'[x]^2*y[x]==3*y'[x]*x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

19.20 problem 20

Internal problem ID [2333]
Internal file name [`OUTPUT/2333_Tuesday_February_27_2024_08_27_12_AM_43408919/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 20.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, _dAlembert]
```

$$\boxed{-y{y'}^{2} = -8x - 1}$$

19.20.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$-y\,p^{2} = -8x - 1$$

Solving for y from the above results in

$$y = \frac{8x}{p^{2}} + \frac{1}{p^{2}} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{8}{p^2}$$
$$g = \frac{1}{p^2}$$

Hence (2) becomes

$$p - \frac{8}{p^2} = \left(-\frac{16x}{p^3} - \frac{2}{p^3}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{8}{p^2} = 0$$

Solving for p from the above gives

$$p = 2$$
$$p = -1 - i\sqrt{3}$$
$$p = -1 + i\sqrt{3}$$

Substituting these in (1A) gives

$$y = 2x + \frac{1}{4}$$
$$y = \frac{8x + 1}{-2 + 2i\sqrt{3}}$$
$$y = \frac{8x + 1}{-2 - 2i\sqrt{3}}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{8}{p(x)^2}}{-\frac{16x}{p(x)^3} - \frac{2}{p(x)^3}} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{-\frac{16x(p)}{p^3} - \frac{2}{p^3}}{p - \frac{8}{p^2}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{16}{p^4 - 8p}$$
$$q(p) = -\frac{2}{p^4 - 8p}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{16x(p)}{p^4 - 8p} = -\frac{2}{p^4 - 8p}$$

The integrating factor μ is

$$\mu = e^{\int \frac{16}{p^4 - 8p}dp}$$
$$= e^{\frac{2\ln\left(p^2+2p+4\right)}{3} + \frac{2\ln(p-2)}{3} - 2\ln(p)}$$

Which simplifies to

$$\mu = \frac{\left(p^2 + 2p + 4\right)^{\frac{2}{3}} (p-2)^{\frac{2}{3}}}{p^2}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) = (\mu)\left(-\frac{2}{p^4 - 8p}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\left(p^2 + 2p + 4\right)^{\frac{2}{3}} (p-2)^{\frac{2}{3}} x}{p^2}\right) = \left(\frac{\left(p^2 + 2p + 4\right)^{\frac{2}{3}} (p-2)^{\frac{2}{3}}}{p^2}\right)\left(-\frac{2}{p^4 - 8p}\right)$$
$$\mathrm{d}\left(\frac{\left(p^2 + 2p + 4\right)^{\frac{2}{3}} (p-2)^{\frac{2}{3}} x}{p^2}\right) = \left(-\frac{2\left(p^2 + 2p + 4\right)^{\frac{2}{3}} (p-2)^{\frac{2}{3}}}{p^3 \left(p^3 - 8\right)}\right) \mathrm{d}p$$

Integrating gives

$$\frac{\left(p^2 + 2p + 4\right)^{\frac{2}{3}} (p-2)^{\frac{2}{3}} x}{p^2} = \int -\frac{2\left(p^2 + 2p + 4\right)^{\frac{2}{3}} (p-2)^{\frac{2}{3}}}{p^3 \left(p^3 - 8\right)} \mathrm{d}p$$
$$\frac{\left(p^2 + 2p + 4\right)^{\frac{2}{3}} (p-2)^{\frac{2}{3}} x}{p^2} = -\frac{(p-2)^{\frac{5}{3}} \left(p^2 + 2p + 4\right)^{\frac{5}{3}}}{8p^2 \left(p^3 - 8\right)} + c_3$$

Dividing both sides by the integrating factor $\mu = \frac{(p^2+2p+4)^{\frac{2}{3}}(p-2)^{\frac{2}{3}}}{p^2}$ results in

$$x(p) = -\frac{(p^2+2p+4)(p-2)}{8(p^3-8)} + \frac{c_3 p^2}{(p^2+2p+4)^{\frac{2}{3}}(p-2)^{\frac{2}{3}}}$$

which simplifies to

$$x(p) = -\frac{1}{8} + \frac{c_3 p^2}{(p^2+2p+4)^{\frac{2}{3}}(p-2)^{\frac{2}{3}}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\sqrt{y(8x+1)}}{y}$$
$$p = -\frac{\sqrt{y(8x+1)}}{y}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{1}{8} + \frac{c_3(8x+1)}{\left(\frac{8x+1+2\sqrt{y(8x+1)}+4y}{y}\right)^{\frac{2}{3}}\left(\frac{\sqrt{y(8x+1)}-2y}{y}\right)^{\frac{2}{3}} y}$$
$$x = -\frac{1}{8} + \frac{c_3(8x+1)}{\left(\frac{8x+1-2\sqrt{y(8x+1)}+4y}{y}\right)^{\frac{2}{3}}\left(\frac{-\sqrt{y(8x+1)}-2y}{y}\right)^{\frac{2}{3}} y}$$

Summary

The solution(s) found are the following

$$y = 2x + \frac{1}{4} \tag{1}$$
$$y = \frac{8x+1}{-2+2i\sqrt{3}} \tag{2}$$
$$y = \frac{8x+1}{-2-2i\sqrt{3}} \tag{3}$$
$$x = -\frac{1}{8} + \frac{c_3(8x+1)}{\left(\frac{8x+1+2\sqrt{y(8x+1)}+4y}{y}\right)^{\frac{2}{3}}\left(\frac{\sqrt{y(8x+1)}-2y}{y}\right)^{\frac{2}{3}} y} \tag{4}$$
$$x = -\frac{1}{8} + \frac{c_3(8x+1)}{\left(\frac{8x+1-2\sqrt{y(8x+1)}+4y}{y}\right)^{\frac{2}{3}}\left(\frac{-\sqrt{y(8x+1)}-2y}{y}\right)^{\frac{2}{3}} y} \tag{5}$$

Verification of solutions

$$y = 2x + \frac{1}{4}$$

Verified OK.

$$y = \frac{8x + 1}{-2 + 2i\sqrt{3}}$$

Verified OK.

$$y = \frac{8x + 1}{-2 - 2i\sqrt{3}}$$

Verified OK.

$$x = -\frac{1}{8} + \frac{c_3(8x + 1)}{\left(\frac{8x+1+2\sqrt{y(8x+1)}+4y}{y}\right)^{\frac{2}{3}} \left(\frac{\sqrt{y(8x+1)}-2y}{y}\right)^{\frac{2}{3}} y}$$

Verified OK.

$$x = -\frac{1}{8} + \frac{c_3(8x + 1)}{\left(\frac{8x+1-2\sqrt{y(8x+1)}+4y}{y}\right)^{\frac{2}{3}} \left(\frac{-\sqrt{y(8x+1)}-2y}{y}\right)^{\frac{2}{3}} y}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 137

```
dsolve(8*x+1=diff(y(x),x)^2*y(x),y(x), singsol=all)
```

$$-\frac{8c_1(8x+1)}{\left(\frac{-2y(x)-\sqrt{y(x)(8x+1)}}{y(x)}\right)^{\frac{2}{3}}\left(\frac{8x+1-2\sqrt{y(x)(8x+1)}+4y(x)}{y(x)}\right)^{\frac{2}{3}}y(x)}+x+\frac{1}{8}=0$$

$$\frac{1}{8}-\frac{8c_1(8x+1)}{\left(\frac{-2y(x)+\sqrt{y(x)(8x+1)}}{y(x)}\right)^{\frac{2}{3}}\left(\frac{8x+1+2\sqrt{y(x)(8x+1)}+4y(x)}{y(x)}\right)^{\frac{2}{3}}y(x)}+x=0$$

✓ Solution by Mathematica

Time used: 3.669 (sec). Leaf size: 79

```
DSolve[8*x+1==y'[x]^2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{4}\left(-8\sqrt{8x+1}x-\sqrt{8x+1}+12c_1\right)^{2/3}$$

$$y(x)\to\frac{1}{4}\left(8\sqrt{8x+1}x+\sqrt{8x+1}+12c_1\right)^{2/3}$$

19.21 problem 21

Internal problem ID [2334]
Internal file name [`OUTPUT/2334_Tuesday_February_27_2024_08_27_13_AM_75189765/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 21.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{yy'^2 + 2y' = -1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-1 + \sqrt{1-y}}{y} \tag{1}$$

$$y' = -\frac{1 + \sqrt{1-y}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y}{-1 + \sqrt{1-y}} dy = \int dx$$

$$\frac{2(1-y)^{\frac{3}{2}}}{3} + 1 - y = x + c_1$$

Summary
The solution(s) found are the following

$$\frac{2(1-y)^{\frac{3}{2}}}{3}+1-y=x+c_1 \tag{1}$$

Verification of solutions

$$\frac{2(1-y)^{\frac{3}{2}}}{3}+1-y=x+c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y}{1+\sqrt{1-y}}dy=\int dx$$

$$-\frac{2(1-y)^{\frac{3}{2}}}{3}+1-y=x+c_2$$

Summary
The solution(s) found are the following

$$-\frac{2(1-y)^{\frac{3}{2}}}{3}+1-y=x+c_2 \tag{1}$$

Verification of solutions

$$-\frac{2(1-y)^{\frac{3}{2}}}{3}+1-y=x+c_2$$

Verified OK.

19.21.1 Maple step by step solution

Let's solve

$yy'^2+2y'=-1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{-1+\sqrt{1-y}}=1$

- Integrate both sides with respect to x

  $\int \frac{y' y}{-1+\sqrt{1-y}} dx = \int 1 dx + c_1$

- Evaluate integral

  $\frac{2(1-y)^{\frac{3}{2}}}{3} + 1 - y = x + c_1$

- Solve for y

  $y = -\left(\frac{\left(-1+6c_1+6x+2\sqrt{9c_1^2+18c_1x+9x^2-3c_1-3x}\right)^{\frac{1}{3}}}{2} + \frac{1}{2\left(-1+6c_1+6x+2\sqrt{9c_1^2+18c_1x+9x^2-3c_1-3x}\right)^{\frac{1}{3}}} - \frac{1}{2}\right)^2 + 1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 53

```
dsolve(diff(y(x),x)^2*y(x)+2*diff(y(x),x)+1=0,y(x), singsol=all)
```

$$\frac{(2y(x)-2)\sqrt{1-y(x)}}{3} + x - c_1 + y(x) - 1 = 0$$
$$\frac{(-2y(x)+2)\sqrt{1-y(x)}}{3} + x - c_1 + y(x) - 1 = 0$$

✓ Solution by Mathematica

Time used: 23.957 (sec). Leaf size: 1098

```
DSolve[y'[x]^2*y[x]+2*y'[x]+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{3}\left(-24x^2+36x-48c_1x+8\sqrt{3}\sqrt{(x+c_1)^3(3x-1+3c_1)}-9-24{c_1}^2+36c_1\right)^{2/3}+\sqrt[3]{-24x^2+36x}}{4\sqrt[3]{-24x^2+(36-48c_1)x+8\sqrt{3}\sqrt{(x+c_1)^3(}}}$$

$$y(x) \to \frac{1}{8}\left(\frac{3^{2/3}\left(1+i\sqrt{3}\right)(8x-3+8c_1)}{\sqrt[3]{-24x^2+(36-48c_1)x+8\sqrt{3}\sqrt{(x+c_1)^3(3x-1+3c_1)}-9-24{c_1}^2+36c_1}} + i\sqrt[3]{3}\left(\sqrt{3}+i\right)\sqrt[3]{-24x^2+36x-48c_1x+8\sqrt{3}\sqrt{(x+c_1)^3(3x-1+3c_1)}-9-24{c_1}^2+36c_1} + 2\right)$$

$$y(x) \to \frac{1}{8}\left(\frac{3^{2/3}\left(1-i\sqrt{3}\right)(8x-3+8c_1)}{\sqrt[3]{-24x^2+(36-48c_1)x+8\sqrt{3}\sqrt{(x+c_1)^3(3x-1+3c_1)}-9-24{c_1}^2+36c_1}} - \sqrt[3]{3}\left(1+i\sqrt{3}\right)\sqrt[3]{-24x^2+36x-48c_1x+8\sqrt{3}\sqrt{(x+c_1)^3(3x-1+3c_1)}-9-24{c_1}^2+36c_1} + 2\right)$$

$$y(x) \to \frac{\sqrt[3]{3}\left(-24x^2+36x+48c_1x+8\sqrt{3}\sqrt{(-x+c_1)^3(-3x+1+3c_1)}-9-24{c_1}^2-36c_1\right)^{2/3}+\sqrt[3]{-24x^2+3}}{4\sqrt[3]{-24x^2+12(3+4c_1)x+8\sqrt{3}\sqrt{(-x+c}}}$$

$$y(x) \to \frac{1}{8}\left(\frac{3^{2/3}\left(1+i\sqrt{3}\right)(8x-3-8c_1)}{\sqrt[3]{-24x^2+12(3+4c_1)x+8\sqrt{3}\sqrt{(-x+c_1)^3(-3x+1+3c_1)}-9-24{c_1}^2-36c_1}} + i\sqrt[3]{3}\left(\sqrt{3}+i\right)\sqrt[3]{-24x^2+12(3+4c_1)x+8\sqrt{3}\sqrt{(-x+c_1)^3(-3x+1+3c_1)}-9-24{c_1}^2-36c_1} + 2\right)$$

$$y(x) \to \frac{1}{8}\left(\frac{3^{2/3}\left(1-i\sqrt{3}\right)(8x-3-8c_1)}{\sqrt[3]{-24x^2+12(3+4c_1)x+8\sqrt{3}\sqrt{(-x+c_1)^3(-3x+1+3c_1)}-9-24{c_1}^2-36c_1}}\right.$$

19.22 problem 22

Internal problem ID [2335]
Internal file name [`OUTPUT/2335_Tuesday_February_27_2024_08_27_13_AM_52098146/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 22.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{\left(1+y'^2\right) x-(y+x)\, y'=0}$$

19.22.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$\left(p^2+1\right) x-(y+x)\, p=0$$

Solving for y from the above results in

$$y = \frac{x(p^2-p+1)}{p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2 - p + 1}{p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2 - p + 1}{p} = x\left(\frac{2p - 1}{p} - \frac{p^2 - p + 1}{p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2 - p + 1}{p} = 0
$$

Solving for p from the above gives

$$
p = 1
$$

Substituting these in (1A) gives

$$
y = x
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2 - p(x) + 1}{p(x)}}{x\left(\frac{2p(x) - 1}{p(x)} - \frac{p(x)^2 - p(x) + 1}{p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{2p-1}{p} - \frac{p^2 - p + 1}{p^2}\right)}{p - \frac{p^2 - p + 1}{p}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$p(p) = -\frac{p+1}{p}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{(p+1)\,x(p)}{p} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{p+1}{p}dp}$$
$$= e^{-p-\ln(p)}$$

Which simplifies to

$$\mu = \frac{e^{-p}}{p}$$

The ode becomes

$$\frac{d}{dp}\mu x = 0$$
$$\frac{d}{dp}\left(\frac{e^{-p}x}{p}\right) = 0$$

Integrating gives

$$\frac{e^{-p}x}{p} = c_2$$

Dividing both sides by the integrating factor $\mu = \frac{e^{-p}}{p}$ results in

$$x(p) = c_2 p\, e^{p}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{y + x + \sqrt{y^2 + 2yx - 3x^2}}{2x}$$
$$p = -\frac{-y - x + \sqrt{y^2 + 2yx - 3x^2}}{2x}$$

Substituting the above in the solution for x found above gives

$$x = \frac{c_2\left(y + x + \sqrt{y^2 + 2yx - 3x^2}\right) e^{\frac{y+x+\sqrt{y^2+2yx-3x^2}}{2x}}}{2x}$$

$$x = \frac{c_2\left(y + x - \sqrt{y^2 + 2yx - 3x^2}\right) e^{\frac{y+x-\sqrt{y^2+2yx-3x^2}}{2x}}}{2x}$$

Summary

The solution(s) found are the following

$$y = x \tag{1}$$

$$x = \frac{c_2\left(y + x + \sqrt{y^2 + 2yx - 3x^2}\right) e^{\frac{y+x+\sqrt{y^2+2yx-3x^2}}{2x}}}{2x} \tag{2}$$

$$x = \frac{c_2\left(y + x - \sqrt{y^2 + 2yx - 3x^2}\right) e^{\frac{y+x-\sqrt{y^2+2yx-3x^2}}{2x}}}{2x} \tag{3}$$

Verification of solutions

$$y = x$$

Verified OK.

$$x = \frac{c_2\left(y + x + \sqrt{y^2 + 2yx - 3x^2}\right) e^{\frac{y+x+\sqrt{y^2+2yx-3x^2}}{2x}}}{2x}$$

Verified OK.

$$x = \frac{c_2\left(y + x - \sqrt{y^2 + 2yx - 3x^2}\right) e^{\frac{y+x-\sqrt{y^2+2yx-3x^2}}{2x}}}{2x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 36

```
dsolve((diff(y(x),x)^2+1)*x=diff(y(x),x)*(x+y(x)),y(x), singsol=all)
```

$$y(x) = x$$

$$y(x) = \frac{x\left(\operatorname{LambertW}\left(\frac{x}{c_1}\right)^2 - \operatorname{LambertW}\left(\frac{x}{c_1}\right) + 1\right)}{\operatorname{LambertW}\left(\frac{x}{c_1}\right)}$$

✓ Solution by Mathematica

Time used: 3.512 (sec). Leaf size: 162

```
DSolve[(y'[x]^2+1)*x==y'[x]*(x+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{4}\left(-\frac{y(x)}{x}+\sqrt{\frac{y(x)}{x}-1}\sqrt{\frac{y(x)}{x}+3}-4\log\left(\sqrt{\frac{y(x)}{x}-1}-\sqrt{\frac{y(x)}{x}+3}\right)-3\right)=-\frac{\log(x)}{2}+c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{1}{4}\left(\frac{y(x)}{x}+\sqrt{\frac{y(x)}{x}-1}\sqrt{\frac{y(x)}{x}+3}-4\log\left(\sqrt{\frac{y(x)}{x}-1}-\sqrt{\frac{y(x)}{x}+3}\right)+3\right)=\frac{\log(x)}{2}+c_1,y(x)\right]$$

19.23 problem 23

Internal problem ID [2336]
Internal file name [`OUTPUT/2336_Tuesday_February_27_2024_08_27_14_AM_15775475/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 23.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{-3yy' + y'^2x = -x^2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{3y + \sqrt{9y^2 - 4x^3}}{2x} \tag{1}$$
$$y' = \frac{3y - \sqrt{9y^2 - 4x^3}}{2x} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Writing the ode as

$$y' = \frac{3y + \sqrt{-4x^3 + 9y^2}}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(3y + \sqrt{-4x^3 + 9y^2}\right)(b_3 - a_2)}{2x} - \frac{\left(3y + \sqrt{-4x^3 + 9y^2}\right)^2 a_3}{4x^2} \\ &- \left(-\frac{3y + \sqrt{-4x^3 + 9y^2}}{2x^2} - \frac{3x}{\sqrt{-4x^3 + 9y^2}}\right)(xa_2 + ya_3 + a_1) \\ &- \frac{\left(3 + \frac{9y}{\sqrt{-4x^3+9y^2}}\right)(xb_2 + yb_3 + b_1)}{2x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-12x^4a_2 + 8x^4b_3 - 28x^3ya_3 + (-4x^3 + 9y^2)^{\frac{3}{2}} a_3 + 2b_2x^2\sqrt{-4x^3 + 9y^2} + 3\sqrt{-4x^3 + 9y^2}\, y^2a_3 - 4x^3a_1 +}{4x^2\sqrt{-4x^3 + 9y^2}}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &12x^4a_2 - 8x^4b_3 + 28x^3ya_3 - \left(-4x^3 + 9y^2\right)^{\frac{3}{2}} a_3 - 2b_2x^2\sqrt{-4x^3 + 9y^2} \\ &- 3\sqrt{-4x^3 + 9y^2}\, y^2a_3 + 4x^3a_1 - 18x^2yb_2 - 36y^3a_3 \\ &- 6\sqrt{-4x^3 + 9y^2}\, xb_1 + 6\sqrt{-4x^3 + 9y^2}\, ya_1 - 18xyb_1 + 18y^2a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &12x^4a_2 + 12x^3ya_3 - \left(-4x^3 + 9y^2\right)^{\frac{3}{2}} a_3 + 2\left(-4x^3 + 9y^2\right)xb_3 - 4\left(-4x^3\right. \\ &\left.+ 9y^2\right)ya_3 - 2b_2x^2\sqrt{-4x^3 + 9y^2} - 3\sqrt{-4x^3 + 9y^2}\, y^2a_3 + 12x^3a_1 - 18x^2yb_2 \\ &- 18x\,y^2b_3 + 2\left(-4x^3 + 9y^2\right)a_1 - 6\sqrt{-4x^3 + 9y^2}\, xb_1 + 6\sqrt{-4x^3 + 9y^2}\, ya_1 \\ &- 18xyb_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&12x^4a_2 - 8x^4b_3 + 4x^3\sqrt{-4x^3+9y^2}\,a_3 + 28x^3ya_3 + 4x^3a_1 \\
&- 2b_2x^2\sqrt{-4x^3+9y^2} - 18x^2yb_2 - 12\sqrt{-4x^3+9y^2}\,y^2a_3 - 36y^3a_3 \\
&- 6\sqrt{-4x^3+9y^2}\,xb_1 - 18xyb_1 + 6\sqrt{-4x^3+9y^2}\,ya_1 + 18y^2a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{-4x^3+9y^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{-4x^3+9y^2} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&12v_1^4a_2 + 28v_1^3v_2a_3 + 4v_1^3v_3a_3 - 8v_1^4b_3 + 4v_1^3a_1 - 36v_2^3a_3 - 12v_3v_2^2a_3 \\
&- 18v_1^2v_2b_2 - 2b_2v_1^2v_3 + 18v_2^2a_1 + 6v_3v_2a_1 - 18v_1v_2b_1 - 6v_3v_1b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(12a_2 - 8b_3)\,v_1^4 + 28v_1^3v_2a_3 + 4v_1^3v_3a_3 + 4v_1^3a_1 - 18v_1^2v_2b_2 - 2b_2v_1^2v_3 \\
&- 18v_1v_2b_1 - 6v_3v_1b_1 - 36v_2^3a_3 - 12v_3v_2^2a_3 + 18v_2^2a_1 + 6v_3v_2a_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
4a_1 &= 0 \\
6a_1 &= 0 \\
18a_1 &= 0 \\
-36a_3 &= 0 \\
-12a_3 &= 0 \\
4a_3 &= 0 \\
28a_3 &= 0 \\
-18b_1 &= 0 \\
-6b_1 &= 0 \\
-18b_2 &= 0 \\
-2b_2 &= 0 \\
12a_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= \frac{3a_2}{2}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= \frac{3y}{2}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= \frac{3y}{2} - \left(\frac{3y + \sqrt{-4x^3 + 9y^2}}{2x}\right)(x) \\
&= -\frac{\sqrt{-4x^3 + 9y^2}}{2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{\sqrt{-4x^3+9y^2}}{2}} dy \end{aligned}$$

Which results in

$$S = -\frac{2 \ln\left(y\sqrt{9} + \sqrt{-4x^3 + 9y^2}\right) \sqrt{9}}{9}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{3y + \sqrt{-4x^3 + 9y^2}}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{4x^2}{\sqrt{-4x^3 + 9y^2}\,\left(3y + \sqrt{-4x^3 + 9y^2}\right)} \\ S_y &= -\frac{2}{\sqrt{-4x^3 + 9y^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{8x^3 - 18y^2 - 6\sqrt{-4x^3 + 9y^2}\, y}{x\sqrt{-4x^3 + 9y^2}\, \left(3y + \sqrt{-4x^3 + 9y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2\ln\left(3y + \sqrt{9y^2 - 4x^3}\right)}{3} = -2\ln(x) + c_1$$

Which simplifies to

$$-\frac{2\ln\left(3y + \sqrt{9y^2 - 4x^3}\right)}{3} = -2\ln(x) + c_1$$

Which gives

$$y = \frac{\left(4\,\mathrm{e}^{3c_1} + x^3\right)\mathrm{e}^{-\frac{3c_1}{2}}}{6}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(4\,\mathrm{e}^{3c_1} + x^3\right)\mathrm{e}^{-\frac{3c_1}{2}}}{6} \tag{1}$$

Verification of solutions

$$y = \frac{\left(4\,\mathrm{e}^{3c_1} + x^3\right)\mathrm{e}^{-\frac{3c_1}{2}}}{6}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{\sqrt{-4x^3+9y^2}-3y}{2x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(\sqrt{-4x^3+9y^2}-3y\right)(b_3-a_2)}{2x} - \frac{\left(\sqrt{-4x^3+9y^2}-3y\right)^2 a_3}{4x^2} - \left(\frac{3x}{\sqrt{-4x^3+9y^2}} + \frac{\sqrt{-4x^3+9y^2}-3y}{2x^2}\right)(xa_2+ya_3+a_1) + \frac{\left(\frac{9y}{\sqrt{-4x^3+9y^2}}-3\right)(xb_2+yb_3+b_1)}{2x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{12x^4a_2 - 8x^4b_3 + 28x^3ya_3 + (-4x^3+9y^2)^{\frac{3}{2}}a_3 + 2b_2\sqrt{-4x^3+9y^2}\,x^2 + 3\sqrt{-4x^3+9y^2}\,y^2a_3 + 4x^3a_1 - 1}{4\sqrt{-4x^3+9y^2}\,x^2} = 0$$

Setting the numerator to zero gives

$$-12x^4a_2 + 8x^4b_3 - 28x^3ya_3 - \left(-4x^3+9y^2\right)^{\frac{3}{2}}a_3 - 2b_2\sqrt{-4x^3+9y^2}\,x^2 - 3\sqrt{-4x^3+9y^2}\,y^2a_3 - 4x^3a_1 + 18x^2yb_2 + 36y^3a_3 - 6\sqrt{-4x^3+9y^2}\,xb_1 + 6\sqrt{-4x^3+9y^2}\,ya_1 + 18xyb_1 - 18y^2a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-12x^4a_2 - 12x^3ya_3 - \left(-4x^3+9y^2\right)^{\frac{3}{2}} a_3 - 2\left(-4x^3+9y^2\right) xb_3 \\
&+ 4\left(-4x^3+9y^2\right) ya_3 - 2b_2\sqrt{-4x^3+9y^2}\, x^2 - 3\sqrt{-4x^3+9y^2}\, y^2a_3 \\
&- 12x^3a_1 + 18x^2yb_2 + 18x\,y^2b_3 - 2\left(-4x^3+9y^2\right) a_1 \\
&- 6\sqrt{-4x^3+9y^2}\, xb_1 + 6\sqrt{-4x^3+9y^2}\, ya_1 + 18xyb_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-12x^4a_2 + 8x^4b_3 + 4x^3\sqrt{-4x^3+9y^2}\, a_3 - 28x^3ya_3 - 4x^3a_1 \\
&- 2b_2\sqrt{-4x^3+9y^2}\, x^2 + 18x^2yb_2 - 12\sqrt{-4x^3+9y^2}\, y^2a_3 + 36y^3a_3 \\
&- 6\sqrt{-4x^3+9y^2}\, xb_1 + 18xyb_1 + 6\sqrt{-4x^3+9y^2}\, ya_1 - 18y^2a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{-4x^3+9y^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{-4x^3+9y^2} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-12v_1^4a_2 - 28v_1^3v_2a_3 + 4v_1^3v_3a_3 + 8v_1^4b_3 - 4v_1^3a_1 + 36v_2^3a_3 - 12v_3v_2^2a_3 \\
&+ 18v_1^2v_2b_2 - 2b_2v_3v_1^2 - 18v_2^2a_1 + 6v_3v_2a_1 + 18v_1v_2b_1 - 6v_3v_1b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-12a_2 + 8b_3)\, v_1^4 - 28v_1^3v_2a_3 + 4v_1^3v_3a_3 - 4v_1^3a_1 + 18v_1^2v_2b_2 - 2b_2v_3v_1^2 \\
&+ 18v_1v_2b_1 - 6v_3v_1b_1 + 36v_2^3a_3 - 12v_3v_2^2a_3 - 18v_2^2a_1 + 6v_3v_2a_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-18a_1 &= 0 \\
-4a_1 &= 0 \\
6a_1 &= 0 \\
-28a_3 &= 0 \\
-12a_3 &= 0 \\
4a_3 &= 0 \\
36a_3 &= 0 \\
-6b_1 &= 0 \\
18b_1 &= 0 \\
-2b_2 &= 0 \\
18b_2 &= 0 \\
-12a_2 + 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= \frac{3a_2}{2}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= \frac{3y}{2}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= \frac{3y}{2} - \left(-\frac{\sqrt{-4x^3 + 9y^2} - 3y}{2x}\right)(x) \\
&= \frac{\sqrt{-4x^3 + 9y^2}}{2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sqrt{-4x^3+9y^2}}{2}} dy \end{aligned}$$

Which results in

$$S = \frac{2 \ln \left(y\sqrt{9} + \sqrt{-4x^3 + 9y^2}\right) \sqrt{9}}{9}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\sqrt{-4x^3 + 9y^2} - 3y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{4x^2}{\sqrt{-4x^3 + 9y^2}\,\left(3y + \sqrt{-4x^3 + 9y^2}\right)} \\ S_y &= \frac{2}{\sqrt{-4x^3 + 9y^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln\left(3y + \sqrt{9y^2 - 4x^3}\right)}{3} = c_1$$

Which simplifies to

$$\frac{2\ln\left(3y + \sqrt{9y^2 - 4x^3}\right)}{3} = c_1$$

Which gives

$$y = \frac{(4x^3 + e^{3c_1})\, e^{-\frac{3c_1}{2}}}{6}$$

Summary

The solution(s) found are the following

$$y = \frac{(4x^3 + e^{3c_1})\, e^{-\frac{3c_1}{2}}}{6} \tag{1}$$

Verification of solutions

$$y = \frac{(4x^3 + e^{3c_1})\, e^{-\frac{3c_1}{2}}}{6}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 51

```
dsolve(x^2-3*diff(y(x),x)*y(x)+x*diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = -\frac{2x^{\frac{3}{2}}}{3}$$
$$y(x) = \frac{2x^{\frac{3}{2}}}{3}$$
$$y(x) = \frac{4x^3 + c_1^2}{6c_1}$$
$$y(x) = \frac{c_1^2 x^3 + 4}{6c_1}$$

✓ Solution by Mathematica

Time used: 0.293 (sec). Leaf size: 83

```
DSolve[x^2-3*y'[x]*y[x]+x*y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{6}e^{-\frac{3c_1}{2}}\left(4x^3 + e^{3c_1}\right)$$
$$y(x) \to \frac{1}{6}e^{-\frac{3c_1}{2}}\left(4x^3 + e^{3c_1}\right)$$
$$y(x) \to -\frac{2x^{3/2}}{3}$$
$$y(x) \to \frac{2x^{3/2}}{3}$$

19.24 problem 24

Internal problem ID [2337]
Internal file name [`OUTPUT/2337_Tuesday_February_27_2024_08_27_20_AM_3136464/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 37, page 171
Problem number: 24.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y + 2xy' - y'^2x = 0}$$

19.24.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$-p^2x + 2xp + y = 0$$

Solving for y from the above results in

$$y = (p^2 - 2p)\,x \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = p^2 - 2p$$
$$g = 0$$

Hence (2) becomes

$$-p^2 + 3p = x(2p - 2)\,p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 + 3p = 0$$

Solving for p from the above gives

$$p = 0$$
$$p = 3$$

Substituting these in (1A) gives

$$y = 0$$
$$y = 3x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 + 3p(x)}{x\,(2p\,(x) - 2)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\,(2p - 2)}{-p^2 + 3p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{-2p + 2}{p\,(p - 3)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{(-2p+2)\,x(p)}{p\,(p-3)} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-2p+2}{p(p-3)}dp} \\ &= \mathrm{e}^{\frac{4\ln(p-3)}{3}+\frac{2\ln(p)}{3}}\end{aligned}$$

Which simplifies to

$$\mu = (p-3)^{\frac{4}{3}}\,p^{\frac{2}{3}}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}p}\left((p-3)^{\frac{4}{3}}\,p^{\frac{2}{3}}x\right) &= 0\end{aligned}$$

Integrating gives

$$(p-3)^{\frac{4}{3}}\,p^{\frac{2}{3}}x = c_3$$

Dividing both sides by the integrating factor $\mu = (p-3)^{\frac{4}{3}}\,p^{\frac{2}{3}}$ results in

$$x(p) = \frac{c_3}{(p-3)^{\frac{4}{3}}\,p^{\frac{2}{3}}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= \frac{x+\sqrt{x^2+yx}}{x} \\ p &= -\frac{-x+\sqrt{x^2+yx}}{x}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$x = \frac{c_3 x}{\left(-2x+\sqrt{x\,(y+x)}\right)\left(\frac{-2x+\sqrt{x(y+x)}}{x}\right)^{\frac{1}{3}}\left(\frac{x+\sqrt{x(y+x)}}{x}\right)^{\frac{2}{3}}}$$

$$x = -\frac{c_3 x}{\left(2x+\sqrt{x\,(y+x)}\right)\left(\frac{-2x-\sqrt{x(y+x)}}{x}\right)^{\frac{1}{3}}\left(\frac{x-\sqrt{x(y+x)}}{x}\right)^{\frac{2}{3}}}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$
$$y = 3x \tag{2}$$
$$x = \frac{c_3 x}{\left(-2x + \sqrt{x(y+x)}\right)\left(\frac{-2x+\sqrt{x(y+x)}}{x}\right)^{\frac{1}{3}}\left(\frac{x+\sqrt{x(y+x)}}{x}\right)^{\frac{2}{3}}} \tag{3}$$
$$x = -\frac{c_3 x}{\left(2x + \sqrt{x(y+x)}\right)\left(\frac{-2x-\sqrt{x(y+x)}}{x}\right)^{\frac{1}{3}}\left(\frac{x-\sqrt{x(y+x)}}{x}\right)^{\frac{2}{3}}} \tag{4}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = 3x$$

Verified OK.

$$x = \frac{c_3 x}{\left(-2x + \sqrt{x(y+x)}\right)\left(\frac{-2x+\sqrt{x(y+x)}}{x}\right)^{\frac{1}{3}}\left(\frac{x+\sqrt{x(y+x)}}{x}\right)^{\frac{2}{3}}}$$

Verified OK.

$$x = -\frac{c_3 x}{\left(2x + \sqrt{x(y+x)}\right)\left(\frac{-2x-\sqrt{x(y+x)}}{x}\right)^{\frac{1}{3}}\left(\frac{x-\sqrt{x(y+x)}}{x}\right)^{\frac{2}{3}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 118

```
dsolve(y(x)+2*diff(y(x),x)*x=diff(y(x),x)^2*x,y(x), singsol=all)
```

$$x\left(1-\frac{c_1}{\left(-2x+\sqrt{x\left(x+y\left(x\right)\right)}\right)\left(\frac{-2x+\sqrt{x(x+y(x))}}{x}\right)^{\frac{1}{3}}\left(\frac{x+\sqrt{x(x+y(x))}}{x}\right)^{\frac{2}{3}}}\right)=0$$

$$x\left(1+\frac{c_1}{\left(2x+\sqrt{x\left(x+y\left(x\right)\right)}\right)\left(\frac{-2x-\sqrt{x(x+y(x))}}{x}\right)^{\frac{1}{3}}\left(\frac{x-\sqrt{x(x+y(x))}}{x}\right)^{\frac{2}{3}}}\right)=0$$

✓ Solution by Mathematica

Time used: 60.106 (sec). Leaf size: 1178

```
DSolve[y[x]+2*y'[x]*x==y'[x]^2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{2}\sqrt{x}\left(x^{3/2}-2e^{\frac{3c_1}{2}}\right)}{\sqrt[3]{-10e^{\frac{3c_1}{2}}x^{3/2}+\sqrt{e^{\frac{3c_1}{2}}\left(4x^{3/2}+e^{\frac{3c_1}{2}}\right)^3}-2x^3+e^{3c_1}}} + \frac{\sqrt[3]{-10e^{\frac{3c_1}{2}}x^{3/2}+\sqrt{e^{\frac{3c_1}{2}}\left(4x^{3/2}+e^{\frac{3c_1}{2}}\right)^3}-2x^3+e^{3c_1}}}{\sqrt[3]{2}} + 2x$$

$$y(x) \to \frac{\left(1+i\sqrt{3}\right)\sqrt{x}\left(-x^{3/2}+2e^{\frac{3c_1}{2}}\right)}{2^{2/3}\sqrt[3]{-10e^{\frac{3c_1}{2}}x^{3/2}+\sqrt{e^{\frac{3c_1}{2}}\left(4x^{3/2}+e^{\frac{3c_1}{2}}\right)^3}-2x^3+e^{3c_1}}} + \frac{i\left(\sqrt{3}+i\right)\sqrt[3]{-10e^{\frac{3c_1}{2}}x^{3/2}+\sqrt{e^{\frac{3c_1}{2}}\left(4x^{3/2}+e^{\frac{3c_1}{2}}\right)^3}-2x^3+e^{3c_1}}}{2\sqrt[3]{2}} + 2x$$

$$y(x) \to \frac{i\left(\sqrt{3}+i\right)\sqrt{x}\left(x^{3/2}-2e^{\frac{3c_1}{2}}\right)}{2^{2/3}\sqrt[3]{-10e^{\frac{3c_1}{2}}x^{3/2}+\sqrt{e^{\frac{3c_1}{2}}\left(4x^{3/2}+e^{\frac{3c_1}{2}}\right)^3}-2x^3+e^{3c_1}}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-10e^{\frac{3c_1}{2}}x^{3/2}+\sqrt{e^{\frac{3c_1}{2}}\left(4x^{3/2}+e^{\frac{3c_1}{2}}\right)^3}-2x^3+e^{3c_1}}}{2\sqrt[3]{2}} + 2x$$

$$y(x) \to \frac{\sqrt[3]{2}e^{\frac{3c_1}{2}}\sqrt{x}\left(2+e^{\frac{3c_1}{2}}x^{3/2}\right)}{\sqrt[3]{10e^{\frac{15c_1}{2}}x^{3/2}+\sqrt{-e^{12c_1}\left(-1+4e^{\frac{3c_1}{2}}x^{3/2}\right)^3}-2e^{9c_1}x^3+e^{6c_1}}} + \frac{e^{-3c_1}\sqrt[3]{10e^{\frac{15c_1}{2}}x^{3/2}+\sqrt{-e^{12c_1}\left(-1+4e^{\frac{3c_1}{2}}x^{3/2}\right)^3}-2e^{9c_1}x^3+e^{6c_1}}}{\sqrt[3]{2}} + 2x$$

$$y(x) \to \frac{1}{4}\left(-\frac{2\sqrt[3]{2}\left(1+i\sqrt{3}\right)e^{\frac{3c_1}{2}}\sqrt{x}\left(2+e^{\frac{3c_1}{2}}x^{3/2}\right)}{\sqrt[3]{10e^{\frac{15c_1}{2}}x^{3/2}+\sqrt{-e^{12c_1}\left(-1+4e^{\frac{3c_1}{2}}x^{3/2}\right)^3}-2e^{9c_1}x^3+e^{6c_1}}}\right.$$

$$\left.+i2^{2/3}\left(\sqrt{3}+i\right)e^{-3c_1}\sqrt[3]{10e^{\frac{15c_1}{2}}x^{3/2}+\sqrt{-e^{12c_1}\left(-1+4e^{\frac{3c_1}{2}}x^{3/2}\right)^3}-2e^{9c_1}x^3+e^{6c_1}}+8x\right)$$

20 Exercise 38, page 173

20.1 problem 1

Internal problem ID [2338]
Internal file name [`OUTPUT/2338_Tuesday_February_27_2024_08_27_22_AM_6144563/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{-{y'}^{2}-y' = -x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{1}{2}+\frac{\sqrt{1+4x}}{2} \tag{1}$$

$$y' = -\frac{1}{2}-\frac{\sqrt{1+4x}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{2}+\frac{\sqrt{1+4x}}{2}\,\mathrm{d}x \\ &= -\frac{x}{2}+\frac{(1+4x)^{\frac{3}{2}}}{12}+c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{x}{2}+\frac{(1+4x)^{\frac{3}{2}}}{12}+c_1 \tag{1}$$

Verification of solutions

$$y = -\frac{x}{2} + \frac{(1+4x)^{\frac{3}{2}}}{12} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{2} - \frac{\sqrt{1+4x}}{2}\,\mathrm{d}x \\ &= -\frac{x}{2} - \frac{(1+4x)^{\frac{3}{2}}}{12} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{2} - \frac{(1+4x)^{\frac{3}{2}}}{12} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{x}{2} - \frac{(1+4x)^{\frac{3}{2}}}{12} + c_2$$

Verified OK.

20.1.1 Maple step by step solution

Let's solve

$-y'^2 - y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(-y'^2 - y'\right) dx = \int -x dx + c_1$

- Cannot compute integral

  $\int \left(-y'^2 - y'\right) dx = -\frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 40

```
dsolve(x=diff(y(x),x)^2+diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{(-4x-1)\sqrt{4x+1}}{12} - \frac{x}{2} + c_1$$

$$y(x) = -\frac{x}{2} + \frac{(4x+1)^{\frac{3}{2}}}{12} + c_1$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 54

```
DSolve[x==y'[x]^2+y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-\frac{1}{6}(4x+1)^{3/2} - x\right) + c_1$$

$$y(x) \to \frac{1}{12}\left((4x+1)^{3/2} - 6x + 12c_1\right)$$

20.2 problem 2

Internal problem ID [2339]
Internal file name [`OUTPUT/2339_Tuesday_February_27_2024_08_27_22_AM_80859933/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 2.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert", "first_order_nonlinear_p_but_linear_in_x_y"**

Maple gives the following as the ode type

`[[_homogeneous, `class C`], _dAlembert]`

$$\boxed{-y + y'^3 = -x}$$

20.2.1 Solving as first order nonlinear p but linear in x y ode

The ode has the form

$$(y')^{\frac{n}{m}} = ax + by + c \tag{1}$$

Where $n = 3, m = 1, a = -1, b = 1, c = 0$. Hence the ode is

$$(y')^3 = y - x$$

Let

$$u = ax + by + c$$

Hence

$$u' = a + by'$$
$$y' = \frac{u' - a}{b}$$

Substituting the above in (1) gives

$$\left(\frac{u' - a}{b}\right)^{\frac{n}{m}} = u$$
$$\left(\frac{u' - a}{b}\right)^{n} = u^{m}$$

Plugging in the above the values for n, m, a, b, c gives

$$(u'(x) + 1)^3 = u$$

Therefore the solutions are

$$u'(x) + 1 = u^{\frac{1}{3}}$$
$$u'(x) + 1 = -\frac{u^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\, u^{\frac{1}{3}}}{2}$$
$$u'(x) + 1 = -\frac{u^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\, u^{\frac{1}{3}}}{2}$$

Rewriting the above gives

$$u'(x) = u^{\frac{1}{3}} - 1$$
$$u'(x) = -\frac{u^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\, u^{\frac{1}{3}}}{2} - 1$$
$$u'(x) = -\frac{u^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\, u^{\frac{1}{3}}}{2} - 1$$

Each of the above is a separable ODE in $u(x)$. This results in

$$\frac{du}{u^{\frac{1}{3}} - 1} = dx$$
$$\frac{du}{-\frac{u^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\, u^{\frac{1}{3}}}{2} - 1} = dx$$
$$\frac{du}{-\frac{u^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\, u^{\frac{1}{3}}}{2} - 1} = dx$$

Integrating each of the above solutions gives

$$\int \frac{du}{u^{\frac{1}{3}} - 1} = x + c_1$$
$$\int \frac{du}{-\frac{u^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\, u^{\frac{1}{3}}}{2} - 1} = x + c_1$$
$$\int \frac{du}{-\frac{u^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\, u^{\frac{1}{3}}}{2} - 1} = x + c_1$$

But since

$$
\begin{aligned}
u &= ax + by + c \\
&= y - x
\end{aligned}
$$

Then the solutions can be written as

$$
\begin{aligned}
\int^{y-x} \frac{1}{\tau^{\frac{1}{3}} - 1} d\tau &= x + c_1 \\
\int^{y-x} \frac{1}{-\frac{\tau^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,\tau^{\frac{1}{3}}}{2} - 1} d\tau &= x + c_1 \\
\int^{y-x} \frac{1}{-\frac{\tau^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,\tau^{\frac{1}{3}}}{2} - 1} d\tau &= x + c_1
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\int^{y-x} \frac{1}{\tau^{\frac{1}{3}} - 1} d\tau = x + c_1 \tag{1}
$$

$$
\int^{y-x} \frac{1}{-\frac{\tau^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,\tau^{\frac{1}{3}}}{2} - 1} d\tau = x + c_1 \tag{2}
$$

$$
\int^{y-x} \frac{1}{-\frac{\tau^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,\tau^{\frac{1}{3}}}{2} - 1} d\tau = x + c_1 \tag{3}
$$

Verification of solutions

$$
\int^{y-x} \frac{1}{\tau^{\frac{1}{3}} - 1} d\tau = x + c_1
$$

Verified OK.

$$
\int^{y-x} \frac{1}{-\frac{\tau^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,\tau^{\frac{1}{3}}}{2} - 1} d\tau = x + c_1
$$

Verified OK.

$$
\int^{y-x} \frac{1}{-\frac{\tau^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,\tau^{\frac{1}{3}}}{2} - 1} d\tau = x + c_1
$$

Verified OK.

20.2.2 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^3 - y = -x$$

Solving for y from the above results in

$$y = p^3 + x \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= 1 \\ g &= p^3 \end{aligned}$$

Hence (2) becomes

$$p - 1 = 3p^2p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - 1 = 0$$

Solving for p from the above gives

$$p = 1$$

Substituting these in (1A) gives

$$y = x + 1$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - 1}{3p(x)^2} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{3p^2}{p-1} \tag{4}$$

This ODE is now solved for $x(p)$. Integrating both sides gives

$$\begin{aligned} x(p) &= \int \frac{3p^2}{p-1}\, \mathrm{d}p \\ &= \frac{3p^2}{2} + 3p + 3\ln(p-1) + c_2 \end{aligned}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned} p &= (y-x)^{\frac{1}{3}} \\ p &= -\frac{(y-x)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(y-x)^{\frac{1}{3}}}{2} \\ p &= -\frac{(y-x)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(y-x)^{\frac{1}{3}}}{2} \end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned} x &= \frac{3(y-x)^{\frac{2}{3}}}{2} + 3(y-x)^{\frac{1}{3}} + 3\ln\left((y-x)^{\frac{1}{3}} - 1\right) + c_2 \\ x &= -\frac{3(y-x)^{\frac{2}{3}}}{4} + \frac{3i\sqrt{3}\,(y-x)^{\frac{2}{3}}}{4} - \frac{3(y-x)^{\frac{1}{3}}}{2} - \frac{3i\sqrt{3}\,(y-x)^{\frac{1}{3}}}{2} \\ &\quad - 3\ln(2) + 3\ln\left(-(y-x)^{\frac{1}{3}} - i\sqrt{3}\,(y-x)^{\frac{1}{3}} - 2\right) + c_2 \\ x &= -\frac{3(y-x)^{\frac{2}{3}}}{4} - \frac{3i\sqrt{3}\,(y-x)^{\frac{2}{3}}}{4} - \frac{3(y-x)^{\frac{1}{3}}}{2} + \frac{3i\sqrt{3}\,(y-x)^{\frac{1}{3}}}{2} \\ &\quad - 3\ln(2) + 3\ln\left(i\sqrt{3}\,(y-x)^{\frac{1}{3}} - (y-x)^{\frac{1}{3}} - 2\right) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x + 1 \tag{1}$$

$$x = \frac{3(y-x)^{\frac{2}{3}}}{2} + 3(y-x)^{\frac{1}{3}} + 3\ln\left((y-x)^{\frac{1}{3}} - 1\right) + c_2 \tag{2}$$

$$\begin{aligned} x = &-\frac{3(y-x)^{\frac{2}{3}}}{4} + \frac{3i\sqrt{3}\,(y-x)^{\frac{2}{3}}}{4} - \frac{3(y-x)^{\frac{1}{3}}}{2} - \frac{3i\sqrt{3}\,(y-x)^{\frac{1}{3}}}{2} \\ &- 3\ln(2) + 3\ln\left(-(y-x)^{\frac{1}{3}} - i\sqrt{3}\,(y-x)^{\frac{1}{3}} - 2\right) + c_2 \end{aligned} \tag{3}$$

$$\begin{aligned} x = &-\frac{3(y-x)^{\frac{2}{3}}}{4} - \frac{3i\sqrt{3}\,(y-x)^{\frac{2}{3}}}{4} - \frac{3(y-x)^{\frac{1}{3}}}{2} + \frac{3i\sqrt{3}\,(y-x)^{\frac{1}{3}}}{2} \\ &- 3\ln(2) + 3\ln\left(i\sqrt{3}\,(y-x)^{\frac{1}{3}} - (y-x)^{\frac{1}{3}} - 2\right) + c_2 \end{aligned} \tag{4}$$

Verification of solutions

$$y = x + 1$$

Verified OK.

$$x = \frac{3(y-x)^{\frac{2}{3}}}{2} + 3(y-x)^{\frac{1}{3}} + 3\ln\left((y-x)^{\frac{1}{3}} - 1\right) + c_2$$

Verified OK.

$$\begin{aligned} x = &-\frac{3(y-x)^{\frac{2}{3}}}{4} + \frac{3i\sqrt{3}\,(y-x)^{\frac{2}{3}}}{4} - \frac{3(y-x)^{\frac{1}{3}}}{2} - \frac{3i\sqrt{3}\,(y-x)^{\frac{1}{3}}}{2} \\ &- 3\ln(2) + 3\ln\left(-(y-x)^{\frac{1}{3}} - i\sqrt{3}\,(y-x)^{\frac{1}{3}} - 2\right) + c_2 \end{aligned}$$

Verified OK.

$$\begin{aligned} x = &-\frac{3(y-x)^{\frac{2}{3}}}{4} - \frac{3i\sqrt{3}\,(y-x)^{\frac{2}{3}}}{4} - \frac{3(y-x)^{\frac{1}{3}}}{2} + \frac{3i\sqrt{3}\,(y-x)^{\frac{1}{3}}}{2} \\ &- 3\ln(2) + 3\ln\left(i\sqrt{3}\,(y-x)^{\frac{1}{3}} - (y-x)^{\frac{1}{3}} - 2\right) + c_2 \end{aligned}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 217

```
dsolve(x=y(x)-diff(y(x),x)^3,y(x), singsol=all)
```

$$x-\frac{3(-x+y(x))^{\frac{2}{3}}}{2}-3(-x+y(x))^{\frac{1}{3}}-3\ln\left((-x+y(x))^{\frac{1}{3}}-1\right)-c_1=0$$

$$x+\frac{3(-x+y(x))^{\frac{2}{3}}}{4}-\frac{3i\sqrt{3}\,(-x+y(x))^{\frac{2}{3}}}{4}+\frac{3(-x+y(x))^{\frac{1}{3}}}{2}+\frac{3i\sqrt{3}\,(-x+y(x))^{\frac{1}{3}}}{2}+6\ln(2)-3\ln\left(-4-2i\sqrt{3}\,(-x+y(x))^{\frac{1}{3}}-2(-x+y(x))^{\frac{1}{3}}\right)-c_1=0$$

$$x+\frac{3(-x+y(x))^{\frac{2}{3}}}{4}+\frac{3i\sqrt{3}\,(-x+y(x))^{\frac{2}{3}}}{4}+\frac{3(-x+y(x))^{\frac{1}{3}}}{2}-\frac{3i\sqrt{3}\,(-x+y(x))^{\frac{1}{3}}}{2}+6\ln(2)-3\ln\left(2i\sqrt{3}\,(-x+y(x))^{\frac{1}{3}}-2(-x+y(x))^{\frac{1}{3}}-4\right)-c_1=0$$

✓ Solution by Mathematica

Time used: 11.095 (sec). Leaf size: 298

```
DSolve[x==y[x]-y'[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{3}{2}(y(x)-x)^{2/3}+3\sqrt[3]{y(x)-x}+3\log\left(\sqrt[3]{y(x)-x}-1\right)-x=c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{1}{2}\left(\frac{1}{2}\sqrt[3]{y(x)-x}\left(4i(y(x)-x)^{2/3}+3\sqrt{3}\sqrt[3]{y(x)-x}-3i\sqrt[3]{y(x)-x}-6\sqrt{3}-6i\right)+6i\log\left(\sqrt{2-2i\sqrt{3}}\right.\right.\right.$$

$$\left.-i(y(x)-x)=c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{y(x)}{2}\right.$$

$$+\frac{1}{4}\left(-\frac{1}{2}\sqrt[3]{y(x)-x}\left(4(y(x)-x)^{2/3}+3i\sqrt{3}\sqrt[3]{y(x)-x}-3\sqrt[3]{y(x)-x}-6i\sqrt{3}-6\right)-6\log\left(2i\sqrt[3]{y(x)-x}+\sqrt{}\right.\right.$$

20.3 problem 3

Internal problem ID [2340]
Internal file name [`OUTPUT/2340_Tuesday_February_27_2024_08_27_22_AM_55605451/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2yy' - {y'}^2 x = -x}$$

20.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$-p^2 x + 2yp = -x$$

Solving for y from the above results in

$$y = \frac{x(p^2 - 1)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2 - 1}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2 - 1}{2p} = x\left(1 - \frac{p^2 - 1}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2 - 1}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2 - 1}{2p(x)}}{x\left(1 - \frac{p(x)^2 - 1}{2p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu p &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -ix \tag{1}$$

$$y = ix \tag{2}$$

$$y = \frac{c_1^2 x^2 - 1}{2c_1} \tag{3}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 32

```
dsolve(x+2*diff(y(x),x)*y(x)=diff(y(x),x)^2*x,y(x), singsol=all)
```

$$y(x) = -ix$$
$$y(x) = ix$$
$$y(x) = \frac{-c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.133 (sec). Leaf size: 71

```
DSolve[x+2*y'[x]*y[x]==y'[x]^2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-c_1}\left(-x^2 + e^{2c_1}\right)$$
$$y(x) \to \frac{1}{2}e^{-c_1}\left(-1 + e^{2c_1}x^2\right)$$
$$y(x) \to -ix$$
$$y(x) \to ix$$

20.4 problem 4

Internal problem ID [2341]
Internal file name [`OUTPUT/2341_Tuesday_February_27_2024_08_27_23_AM_13739999/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 4.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{-2yy' + {y'}^2 x = -4x}$$

20.4.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 x - 2yp = -4x$$

Solving for y from the above results in

$$y = \frac{x(p^2 + 4)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2+4}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2+4}{2p} = x\left(1 - \frac{p^2+4}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2+4}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 2 \\
p &= -2
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -2x \\
y &= 2x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2+4}{2p(x)}}{x\left(1 - \frac{p(x)^2+4}{2p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu p &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$

$$y = 2x \tag{2}$$

$$y = \frac{c_1^2 x^2 + 4}{2c_1} \tag{3}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = 2x$$

Verified OK.

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(4*x-2*diff(y(x),x)*y(x)+x*diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = -2x$$
$$y(x) = 2x$$
$$y(x) = \frac{4c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.172 (sec). Leaf size: 43

```
DSolve[4*x-2*y'[x]*y[x]+x*y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x\cosh(-\log(x) + c_1)$$
$$y(x) \to -2x\cosh(\log(x) + c_1)$$
$$y(x) \to -2x$$
$$y(x) \to 2x$$

20.5 problem 5

Internal problem ID [2342]
Internal file name [OUTPUT/2342_Tuesday_February_27_2024_08_27_23_AM_77458122/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 5.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_dAlembert]`

$$\boxed{xy'^3 - yy' = 1}$$

20.5.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$x\,p^3 - yp = 1$$

Solving for y from the above results in

$$y = p^2 x - \frac{1}{p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = p^2$$
$$g = -\frac{1}{p}$$

Hence (2) becomes

$$-p^2 + p = \left(2xp + \frac{1}{p^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 + p = 0$$

Solving for p from the above gives

$$p = 0$$
$$p = 1$$

Removing solutions for p which leads to undefined results and substituting these in (1A) gives

$$y = x - 1$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 + p(x)}{2p(x)\,x + \frac{1}{p(x)^2}} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{2x(p)\,p + \frac{1}{p^2}}{-p^2 + p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$
\begin{aligned}
p(p) &= \frac{2}{p-1} \\
q(p) &= -\frac{1}{p^3 (p-1)}
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) + \frac{2x(p)}{p-1} = -\frac{1}{p^3 (p-1)}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2}{p-1}dp} \\
&= (p-1)^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{1}{p^3 (p-1)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}p}\left((p-1)^2 x\right) &= \left((p-1)^2\right)\left(-\frac{1}{p^3 (p-1)}\right) \\
\mathrm{d}\left((p-1)^2 x\right) &= \left(\frac{-p+1}{p^3}\right) \mathrm{d}p
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(p-1)^2 x &= \int \frac{-p+1}{p^3} \,\mathrm{d}p \\
(p-1)^2 x &= \frac{1}{p} - \frac{1}{2p^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = (p-1)^2$ results in

$$
x(p) = \frac{\frac{1}{p} - \frac{1}{2p^2}}{(p-1)^2} + \frac{c_1}{(p-1)^2}
$$

which simplifies to

$$
x(p) = \frac{2c_1 p^2 + 2p - 1}{2 (p-1)^2 p^2}
$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}}{6x} + \frac{2y}{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}}$$

$$p = -\frac{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}}{12x} - \frac{y}{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+\cdots}{x}}\cdots\right.\right.}{6x}\cdots\right.}{\cdots}$$

$$p = -\frac{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}}{12x} - \frac{y}{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+\cdots}{x}}\cdots\right.\right.}{6x}\cdots\right.}{\cdots}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{54\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}} 3^{\frac{1}{3}}\left(-\frac{2^{\frac{1}{3}}\left(\sqrt{\frac{-4y^3+27x}{x}}\,c_1 3^{\frac{1}{6}}+2\left(y+\frac{3c_1}{2}\right)3^{\frac{2}{3}}\right)x\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}}{3} - \frac{2^{\frac{2}{3}}3^{\frac{5}{6}}x^2\cdots}{\cdots}\cdots\right.}{\left(2^{\frac{2}{3}}3^{\frac{1}{3}}xy+\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}\right)^2\left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}}xy+2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(\sqrt{3}\,\sqrt{\cdots}\cdots\right.\right.\right.}$$

$$x = -\frac{36\left(\left(\frac{8yc_1}{9}-\frac{2x}{3}\right)\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+\left(-\frac{\left(c_1\left(i3^{\frac{2}{3}}+3^{\frac{1}{6}}\right)\sqrt{\frac{-4y^3+27x}{x}}+6\left(i3^{\frac{1}{6}}+\frac{3^{\frac{2}{3}}}{3}\right)\left(y+\frac{3c_1}{2}\right)\right)2^{\frac{1}{3}}\left(\left(\sqrt{3}\,\cdots\right.\right.}{9}\cdots\right.\right.}{\left(-\frac{\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}}{6}+\left(2\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}\cdots\right.\right.}$$

$$x = \frac{36\, 3^{\frac{1}{3}} \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{2}{3}} 2^{\frac{2}{3}} \Bigg(\left(-\frac{8yc_1}{9} + \frac{2x}{3}\right) \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{2}{3}} + x \Bigg(-\frac{\left(\left(i 3^{\frac{2}{3}} - 3^{\frac{1}{6}}\right) c_1 \sqrt{\cdots}\right.}{\cdots}}{\left(\left(i - \sqrt{3}\right) \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{2}{3}} + \left(3^{\frac{5}{6}} + i 3^{\frac{1}{3}}\right) y 2^{\frac{2}{3}} x\right)^2 \Bigg(-\frac{\left(3^{\frac{1}{3}} + i 3^{\frac{5}{6}}\right) \cdots}{\cdots}}$$

Summary

The solution(s) found are the following

$$y = x - 1 \tag{1}$$

$$x = -\frac{54 \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{2}{3}} 3^{\frac{1}{3}} \Bigg(-\frac{2^{\frac{1}{3}} \left(\sqrt{\frac{-4y^3+27x}{x}}\, c_1 3^{\frac{1}{6}} + 2\left(y + \frac{3c_1}{2}\right) 3^{\frac{2}{3}}\right) x \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{1}{3}}}{3} - \frac{2^{\frac{2}{3}} 3^{\frac{5}{6}} x^2 \cdots}{\cdots}}{\left(2^{\frac{2}{3}} 3^{\frac{1}{3}} xy + \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{2}{3}}\right)^2 \Bigg(2\, 2^{\frac{1}{3}} 3^{\frac{2}{3}} xy + 2^{\frac{2}{3}} 3^{\frac{1}{3}} \Bigg(\Bigg(\sqrt{3}\, \sqrt{\cdots}} \tag{2}$$

$$x = -\frac{36 \Bigg(\left(\frac{8yc_1}{9} - \frac{2x}{3}\right) \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{2}{3}} + \Bigg(-\frac{\left(c_1 \left(i 3^{\frac{2}{3}} + 3^{\frac{1}{6}}\right) \sqrt{\frac{-4y^3+27x}{x}} + 6\left(i 3^{\frac{1}{6}} + \frac{3^{\frac{2}{3}}}{3}\right)\left(y + \frac{3c_1}{2}\right)\right) 2^{\frac{1}{3}} \left(\left(\sqrt{3}\, \cdots\right.\right.}{9}}{\Bigg(-\frac{\left(i 3^{\frac{5}{6}} - 3^{\frac{1}{3}}\right) 2^{\frac{2}{3}} \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{2}{3}}}{6} + \Bigg(2 \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{1}{3}} \cdots} \tag{3}$$

$$x = \frac{36\, 3^{\frac{1}{3}} \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{2}{3}} 2^{\frac{2}{3}} \Bigg(\left(-\frac{8yc_1}{9} + \frac{2x}{3}\right) \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{2}{3}} + x \Bigg(-\frac{\left(\left(i 3^{\frac{2}{3}} - 3^{\frac{1}{6}}\right) c_1 \sqrt{\cdots}\right.}{\cdots}}{\left(\left(i - \sqrt{3}\right) \left(\left(\sqrt{3}\, \sqrt{\frac{-4y^3+27x}{x}} + 9\right) x^2\right)^{\frac{2}{3}} + \left(3^{\frac{5}{6}} + i 3^{\frac{1}{3}}\right) y 2^{\frac{2}{3}} x\right)^2 \Bigg(-\frac{\left(3^{\frac{1}{3}} + i 3^{\frac{5}{6}}\right) \cdots}{\cdots}} \tag{4}$$

Verification of solutions

$$y = x - 1$$

Verified OK.

$$x = -\frac{54\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}} 3^{\frac{1}{3}}\left(-\frac{2^{\frac{1}{3}}\left(\sqrt{\frac{-4y^3+27x}{x}}\,c_1 3^{\frac{1}{6}}+2\left(y+\frac{3c_1}{2}\right)3^{\frac{2}{3}}\right)x\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}}{3} - \frac{2^{\frac{2}{3}}3^{\frac{5}{6}}x^2 \cdots}{\cdots}\right.}{\left(2^{\frac{2}{3}}3^{\frac{1}{3}}xy+\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}\right)^2\left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}}xy+2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(\sqrt{3}\,\sqrt{\cdots}\right.\right.\right.}$$

Warning, solution could not be verified

$$x = -\frac{36\left(\left(\frac{8yc_1}{9}-\frac{2x}{3}\right)\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+\left(-\frac{\left(c_1\left(i3^{\frac{2}{3}}+3^{\frac{1}{6}}\right)\sqrt{\frac{-4y^3+27x}{x}}+6\left(i3^{\frac{1}{6}}+\frac{3^{\frac{2}{3}}}{3}\right)\left(y+\frac{3c_1}{2}\right)\right)2^{\frac{1}{3}}\left(\left(\sqrt{3}\,\cdots\right.\right.}{9}\right.\right.}{\left(-\frac{\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}}{6}+\left(2\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}\cdots\right.\right.}$$

Warning, solution could not be verified

$$x = \frac{36\,3^{\frac{1}{3}}\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}2^{\frac{2}{3}}\left(\left(-\frac{8yc_1}{9}+\frac{2x}{3}\right)\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+x\left(-\frac{\left(\left(i3^{\frac{2}{3}}-3^{\frac{1}{6}}\right)c_1\sqrt{\cdots}\right.}{\cdots}\right.\right.}{\left(\left(i-\sqrt{3}\right)\left(\left(\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+\left(3^{\frac{5}{6}}+i3^{\frac{1}{3}}\right)y2^{\frac{2}{3}}x\right)^2\left(-\frac{\left(3^{\frac{1}{3}}+i3^{\frac{5}{6}}\right)\cdots}{\cdots}\right.}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 2049

```
dsolve(x*diff(y(x),x)^3=y(x)*diff(y(x),x)+1,y(x), singsol=all)
```

$$\frac{12\left(2\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}y(x)+x\left(\frac{2^{\frac{1}{3}}\left(3^{\frac{1}{6}}\sqrt{\frac{-4y(x)^3+27x}{x}}+3\,3^{\frac{2}{3}}\right)\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}}{2}+2^{\frac{2}{3}}3^{\frac{1}{3}}y(\right.\right.}{\left(y(x)\,2^{\frac{2}{3}}3^{\frac{1}{3}}x+\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}\right)^2\left(2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)^2x^4\right)^{\frac{1}{3}}+2x\left(y(x)\,3\right.\right.}$$

$$+x$$

$$-\frac{18x^4\left(2^{\frac{2}{3}}3^{\frac{5}{6}}\sqrt{\frac{-4y(x)^3+27x}{x}}\,x+2y(x)\,3^{\frac{2}{3}}2^{\frac{1}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}+9\,3^{\frac{1}{3}}2^{\frac{2}{3}}x-3\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3-}{x}}\right.\right.\right.}{\left(2y(x)\,3^{\frac{2}{3}}2^{\frac{1}{3}}x+2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)^2x^4\right)^{\frac{1}{3}}-6x\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}\right)^2\left(\right.}$$

$$=0$$

$$-\frac{3x^3\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}c_1\left(\frac{8\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}y(x)}{9}+x\left(2^{\frac{1}{3}}\left(\left(\frac{i3^{\frac{2}{3}}}{9}-\frac{3^{\frac{1}{6}}}{9}\right)\sqrt{\frac{-4y(x)^3}{x}}\right.\right.\right.}{2\left(\left(i-\sqrt{3}\right)\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+2^{\frac{2}{3}}y(x)\,x\left(i3^{\frac{1}{3}}+3^{\frac{5}{6}}\right)\right)^2\left(-\frac{2^{\frac{2}{3}}\left(i3^{\frac{5}{6}}+3\,3^{\frac{1}{3}}\right)\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+\right.\right.}{6}\right.}$$

$$+x$$

$$+\frac{216x^4 2^{\frac{2}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)^2x^4\right)^{\frac{1}{3}}3^{\frac{1}{3}}\left(-\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+y(x)\left(i3^{\frac{1}{6}}-\frac{3^{\frac{2}{3}}}{3}\right)2^{\frac{1}{3}}\left(\right.\right.}{\left(\left(1+i\sqrt{3}\right)\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+\left(-i3^{\frac{5}{6}}+3^{\frac{1}{3}}\right)x2^{\frac{2}{3}}y(x)\right)^2\left(\frac{\left(-3^{\frac{5}{6}}+i3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+\right.\right.}{2}\right.}$$

$$=0$$

Expression too large to display

✓ Solution by Mathematica

Time used: 144.072 (sec). Leaf size: 21579

```
DSolve[x*y'[x]^3==y[x]*y'[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

20.6 problem 6

Internal problem ID [2343]
Internal file name [`OUTPUT/2343_Tuesday_February_27_2024_08_30_24_AM_81408897/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 6.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{y\left(1+y'^2\right)-2xy'=0}$$

20.6.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$y(p^2+1)-2xp=0$$

Solving for y from the above results in

$$y=\frac{2xp}{p^2+1} \tag{1A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$
$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{2p}{p^2+1} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{2p}{p^2+1} = x\left(\frac{2}{p^2+1} - \frac{4p^2}{\left(p^2+1\right)^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{2p}{p^2+1} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= 1 \\
p &= -1
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -x \\
y &= 0 \\
y &= x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{2p(x)}{p(x)^2+1}}{x\left(\frac{2}{p(x)^2+1} - \frac{4p(x)^2}{\left(p(x)^2+1\right)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{2}{p^2+1} - \frac{4p^2}{\left(p^2+1\right)^2}\right)}{p - \frac{2p}{p^2+1}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{p(p^2+1)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p(p^2+1)} = 0$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{p(p^2+1)}dp}$$
$$= e^{-\ln(p^2+1)+2\ln(p)}$$

Which simplifies to

$$\mu = \frac{p^2}{p^2+1}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p^2 x}{p^2+1}\right) = 0$$

Integrating gives

$$\frac{p^2 x}{p^2+1} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{p^2}{p^2+1}$ results in

$$x(p) = \frac{c_3(p^2+1)}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{x+\sqrt{x^2-y^2}}{y}$$
$$p = -\frac{-x+\sqrt{x^2-y^2}}{y}$$

Substituting the above in the solution for x found above gives

$$
\begin{aligned}
x &= \frac{2c_3 x}{x + \sqrt{x^2 - y^2}} \\
x &= -\frac{2c_3 x}{-x + \sqrt{x^2 - y^2}}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$y = -x \tag{1}$$
$$y = 0 \tag{2}$$
$$y = x \tag{3}$$
$$x = \frac{2c_3 x}{x + \sqrt{x^2 - y^2}} \tag{4}$$
$$x = -\frac{2c_3 x}{-x + \sqrt{x^2 - y^2}} \tag{5}$$

<u>Verification of solutions</u>

$$y = -x$$

Verified OK.

$$y = 0$$

Verified OK.

$$y = x$$

Verified OK.

$$x = \frac{2c_3 x}{x + \sqrt{x^2 - y^2}}$$

Verified OK.

$$x = -\frac{2c_3 x}{-x + \sqrt{x^2 - y^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 71

```
dsolve((diff(y(x),x)^2+1)*y(x)=2*diff(y(x),x)*x,y(x), singsol=all)
```

$$y(x) = -x$$
$$y(x) = x$$
$$y(x) = 0$$
$$y(x) = \sqrt{c_1 (-2ix + c_1)}$$
$$y(x) = \sqrt{c_1 (2ix + c_1)}$$
$$y(x) = -\sqrt{c_1 (-2ix + c_1)}$$
$$y(x) = -\sqrt{c_1 (2ix + c_1)}$$

✓ Solution by Mathematica

Time used: 1.091 (sec). Leaf size: 174

```
DSolve[(y'[x]^2+1)*y[x]==2*y'[x]*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{4}\left(\cosh\left(\frac{c_1}{2}\right)+\sinh\left(\frac{c_1}{2}\right)\right)\sqrt{-8ix+\cosh(c_1)+\sinh(c_1)}$$
$$y(x) \to \frac{1}{4}\left(\cosh\left(\frac{c_1}{2}\right)+\sinh\left(\frac{c_1}{2}\right)\right)\sqrt{-8ix+\cosh(c_1)+\sinh(c_1)}$$
$$y(x) \to -\frac{1}{4}\left(\cosh\left(\frac{c_1}{2}\right)+\sinh\left(\frac{c_1}{2}\right)\right)\sqrt{8ix+\cosh(c_1)+\sinh(c_1)}$$
$$y(x) \to \frac{1}{4}\left(\cosh\left(\frac{c_1}{2}\right)+\sinh\left(\frac{c_1}{2}\right)\right)\sqrt{8ix+\cosh(c_1)+\sinh(c_1)}$$
$$y(x) \to 0$$
$$y(x) \to -x$$
$$y(x) \to x$$

20.7 problem 7

Internal problem ID [2344]
Internal file name [OUTPUT/2344_Tuesday_February_27_2024_08_30_24_AM_7941267/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 7.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{-2yy' + {y'}^2 x = -2x}$$

20.7.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 x - 2yp = -2x$$

Solving for y from the above results in

$$y = \frac{x(p^2 + 2)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{p^2+2}{2p} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{p^2+2}{2p} = x\left(1 - \frac{p^2+2}{2p^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p^2+2}{2p} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= \sqrt{2} \\ p &= -\sqrt{2} \end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned} y &= \sqrt{2}\, x \\ y &= -\sqrt{2}\, x \end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)^2+2}{2p(x)}}{x\left(1 - \frac{p(x)^2+2}{2p(x)^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}\mu p = 0$$
$$\frac{d}{dx}\left(\frac{p}{x}\right) = 0$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2x^2 + 2}{2c_1}$$

Summary
The solution(s) found are the following

$$y = \sqrt{2}\, x \tag{1}$$
$$y = -\sqrt{2}\, x \tag{2}$$
$$y = \frac{c_1^2x^2 + 2}{2c_1} \tag{3}$$

Verification of solutions

$$y = \sqrt{2}\, x$$

Verified OK.

$$y = -\sqrt{2}\, x$$

Verified OK.

$$y = \frac{c_1^2 x^2 + 2}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 35

```
dsolve(2*x+diff(y(x),x)^2*x=2*diff(y(x),x)*y(x),y(x), singsol=all)
```

$$y(x) = \sqrt{2}\, x$$
$$y(x) = -\sqrt{2}\, x$$
$$y(x) = \frac{2c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 63

```
DSolve[2*x+y'[x]*x==2*y'[x]*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{34}\left(\left(17+\sqrt{17}\right)\log\left(\frac{4y(x)}{x}+\sqrt{17}-1\right)\right.\right.$$
$$\left.\left.-\left(\sqrt{17}-17\right)\log\left(-\frac{4y(x)}{x}+\sqrt{17}+1\right)\right)=-\log(x)+c_1,y(x)\right]$$

20.8 problem 8

Internal problem ID [2345]
Internal file name [`OUTPUT/2345_Tuesday_February_27_2024_08_30_24_AM_99185590/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 8.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_dAlembert]`

$$\boxed{-yy' - y'^2 = -x}$$

20.8.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$-p^2 - yp = -x$$

Solving for y from the above results in

$$y = \frac{x}{p} - p \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{1}{p} \\
g &= -p
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{1}{p} = \left(-\frac{x}{p^2} - 1\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{1}{p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 1 \\
p &= -1
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 1 - x \\
y &= x - 1
\end{aligned}
$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{1}{p(x)}}{-\frac{x}{p(x)^2} - 1} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{-\frac{x(p)}{p^2} - 1}{p - \frac{1}{p}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{1}{p^3 - p} \\
q(p) &= -\frac{p}{p^2 - 1}
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) + \frac{x(p)}{p^3 - p} = -\frac{p}{p^2 - 1}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{p^3 - p}dp} \\
&= \mathrm{e}^{\frac{\ln(p+1)}{2} + \frac{\ln(p-1)}{2} - \ln(p)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\sqrt{p+1}\,\sqrt{p-1}}{p}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{p}{p^2 - 1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p}\right) &= \left(\frac{\sqrt{p+1}\,\sqrt{p-1}}{p}\right)\left(-\frac{p}{p^2 - 1}\right) \\
\mathrm{d}\left(\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p}\right) &= \left(-\frac{\sqrt{p+1}\,\sqrt{p-1}}{p^2 - 1}\right)\mathrm{d}p
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p} &= \int -\frac{\sqrt{p+1}\,\sqrt{p-1}}{p^2 - 1}\,\mathrm{d}p \\
\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p} &= -\frac{\sqrt{p-1}\,\sqrt{p+1}\,\ln\left(p + \sqrt{p^2 - 1}\right)}{\sqrt{p^2 - 1}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p+1}\,\sqrt{p-1}}{p}$ results in

$$
x(p) = -\frac{p\ln\left(p + \sqrt{p^2 - 1}\right)}{\sqrt{p^2 - 1}} + \frac{c_1 p}{\sqrt{p+1}\,\sqrt{p-1}}
$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$
\begin{aligned}
p &= -\frac{y}{2}+\frac{\sqrt{y^{2}+4x}}{2} \\
p &= -\frac{y}{2}-\frac{\sqrt{y^{2}+4x}}{2}
\end{aligned}
$$

Substituting the above in the solution for x found above gives

$$
x = \left(-y + \sqrt{y^{2}+4x}\right)\left(-\frac{-\ln\left(2\right)+\ln\left(-y+\sqrt{y^{2}+4x}+\sqrt{2y^{2}+4x-2y\sqrt{y^{2}+4x}-4}\right)}{\sqrt{2y^{2}+4x-2y\sqrt{y^{2}+4x}-4}} + \frac{2c_{1}}{\sqrt{-2y+2\sqrt{y^{2}+4x}+4}\,\sqrt{-2y+2\sqrt{y^{2}+4x}-4}}\right)
$$

$$
x = \left(y+\sqrt{y^{2}+4x}\right)\left(\frac{-\ln\left(2\right)+\ln\left(-y-\sqrt{y^{2}+4x}+\sqrt{2y^{2}+4x+2y\sqrt{y^{2}+4x}-4}\right)}{\sqrt{2y^{2}+4x+2y\sqrt{y^{2}+4x}-4}} - \frac{2c_{1}}{\sqrt{-2y-2\sqrt{y^{2}+4x}+4}\,\sqrt{-2y-2\sqrt{y^{2}+4x}-4}}\right)
$$

Summary
The solution(s) found are the following

$$y = 1 - x \tag{1}$$

$$y = x - 1 \tag{2}$$

$$x = \left(-y + \sqrt{y^2 + 4x}\right)\left(-\frac{-\ln(2) + \ln\left(-y + \sqrt{y^2 + 4x} + \sqrt{2y^2 + 4x - 2y\sqrt{y^2 + 4x} - 4}\right)}{\sqrt{2y^2 + 4x - 2y\sqrt{y^2 + 4x} - 4}} + \frac{2c_1}{\sqrt{-2y + 2\sqrt{y^2 + 4x} + 4}\,\sqrt{-2y + 2\sqrt{y^2 + 4x} - 4}}\right) \tag{3}$$

$$x = \left(y + \sqrt{y^2 + 4x}\right)\left(\frac{-\ln(2) + \ln\left(-y - \sqrt{y^2 + 4x} + \sqrt{2y^2 + 4x + 2y\sqrt{y^2 + 4x} - 4}\right)}{\sqrt{2y^2 + 4x + 2y\sqrt{y^2 + 4x} - 4}} - \frac{2c_1}{\sqrt{-2y - 2\sqrt{y^2 + 4x} + 4}\,\sqrt{-2y - 2\sqrt{y^2 + 4x} - 4}}\right) \tag{4}$$

Verification of solutions

$$y = 1 - x$$

Verified OK.

$$y = x - 1$$

Verified OK.

$$x = \left(-y + \sqrt{y^2 + 4x}\right)\left(-\frac{-\ln(2) + \ln\left(-y + \sqrt{y^2 + 4x} + \sqrt{2y^2 + 4x - 2y\sqrt{y^2 + 4x} - 4}\right)}{\sqrt{2y^2 + 4x - 2y\sqrt{y^2 + 4x} - 4}} + \frac{2c_1}{\sqrt{-2y + 2\sqrt{y^2 + 4x} + 4}\,\sqrt{-2y + 2\sqrt{y^2 + 4x} - 4}}\right)$$

Verified OK.

$$x = \left(y + \sqrt{y^2 + 4x}\right)\left(\frac{-\ln(2) + \ln\left(-y - \sqrt{y^2 + 4x} + \sqrt{2y^2 + 4x + 2y\sqrt{y^2 + 4x} - 4}\right)}{\sqrt{2y^2 + 4x + 2y\sqrt{y^2 + 4x} - 4}} - \frac{2c_1}{\sqrt{-2y - 2\sqrt{y^2 + 4x} + 4}\,\sqrt{-2y - 2\sqrt{y^2 + 4x} - 4}}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 304

```
dsolve(x=diff(y(x),x)*y(x)+diff(y(x),x)^2,y(x), singsol=all)
```

$$\frac{\left(-y(x)+\sqrt{y(x)^{2}+4x}\right)c_{1}}{\sqrt{-2y(x)+2\sqrt{y(x)^{2}+4x}+4}\,\sqrt{-2y(x)+2\sqrt{y(x)^{2}+4x}-4}}+x$$

$$+\frac{\left(-y(x)+\sqrt{y(x)^{2}+4x}\right)\left(-\ln(2)+\ln\left(-y(x)+\sqrt{y(x)^{2}+4x}+\sqrt{2y(x)^{2}-2y(x)\sqrt{y(x)^{2}+4x}+\right.\right.}{\sqrt{2y(x)^{2}-2y(x)\sqrt{y(x)^{2}+4x}+4x-4}}$$

$$=0$$

$$\frac{\left(y(x)+\sqrt{y(x)^{2}+4x}\right)c_{1}}{\sqrt{-2y(x)-2\sqrt{y(x)^{2}+4x}+4}\,\sqrt{-2y(x)-2\sqrt{y(x)^{2}+4x}-4}}+x$$

$$-\frac{\left(y(x)+\sqrt{y(x)^{2}+4x}\right)\left(-\ln(2)+\ln\left(-y(x)-\sqrt{y(x)^{2}+4x}+\sqrt{2y(x)^{2}+2y(x)\sqrt{y(x)^{2}+4x}+4\right.\right.}{\sqrt{2y(x)^{2}+2y(x)\sqrt{y(x)^{2}+4x}+4x-4}}$$

$$=0$$

✓ Solution by Mathematica

Time used: 0.497 (sec). Leaf size: 77

```
DSolve[x==y'[x]*y[x]+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\left\{x=-\frac{2K[1]\arctan\left(\frac{\sqrt{1-K[1]^{2}}}{K[1]+1}\right)}{\sqrt{1-K[1]^{2}}}+\frac{c_{1}K[1]}{\sqrt{1-K[1]^{2}}},y(x)=\frac{x}{K[1]}-K[1]\right\},\{y(x),K[1]\}\right]$$

20.9 problem 9

Internal problem ID [2346]
Internal file name [OUTPUT/2346_Tuesday_February_27_2024_08_30_26_AM_72857342/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 9.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{4{y'}^2x+2xy'-y=0}$$

20.9.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$4p^2x+2xp-y=0$$

Solving for y from the above results in

$$y=\left(4p^2+2p\right)x \tag{1A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$
$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 4p^2 + 2p \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
-4p^2 - p = x(8p + 2)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-4p^2 - p = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= -\frac{1}{4}
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= -\frac{x}{4}
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{-4p(x)^2 - p(x)}{x\,(8p\,(x) + 2)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{2x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$
p'(x) + \frac{p(x)}{2x} = 0
$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{2x} dx}$$
$$= \sqrt{x}$$

The ode becomes

$$\frac{d}{dx}\mu p = 0$$
$$\frac{d}{dx}\left(\sqrt{x}\, p\right) = 0$$

Integrating gives

$$\sqrt{x}\, p = c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$p(x) = \frac{c_1}{\sqrt{x}}$$

Substituing the above solution for p in (2A) gives

$$y = \left(\frac{4c_1^2}{x} + \frac{2c_1}{\sqrt{x}}\right) x$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$
$$y = -\frac{x}{4} \tag{2}$$
$$y = \left(\frac{4c_1^2}{x} + \frac{2c_1}{\sqrt{x}}\right) x \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = -\frac{x}{4}$$

Verified OK.

$$y = \left(\frac{4c_1^2}{x} + \frac{2c_1}{\sqrt{x}}\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
dsolve(4*diff(y(x),x)^2*x+2*diff(y(x),x)*x=y(x),y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -\frac{x}{4} \\
y(x) &= 4c_1 + 2\sqrt{c_1 x} \\
y(x) &= 4c_1 - 2\sqrt{c_1 x}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.152 (sec). Leaf size: 72

```
DSolve[4*y'[x]^2*x+2*y'[x]*x==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \frac{1}{4}e^{2c_1}\left(-2\sqrt{x} + e^{2c_1}\right) \\
y(x) &\to \frac{1}{4}e^{-4c_1}\left(1 + 2e^{2c_1}\sqrt{x}\right) \\
y(x) &\to 0 \\
y(x) &\to -\frac{x}{4}
\end{aligned}
$$

20.10 problem 10

Internal problem ID [2347]
Internal file name [`OUTPUT/2347_Tuesday_February_27_2024_08_30_26_AM_72406962/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 10.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y - y'x(y'+1) = 0}$$

20.10.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - px(p+1) = 0$$

Solving for y from the above results in

$$y = px(p+1) \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = p(p+1)$$
$$g = 0$$

Hence (2) becomes

$$p - p(p+1) = x(2p+1)\,p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - p(p+1) = 0$$

Solving for p from the above gives

$$p = 0$$
$$p = 0$$

Removing solutions for p which leads to undefined results and substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - p(x)\,(p(x)+1)}{x\,(2p\,(x)+1)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\,(2p+1)}{p - p\,(p+1)} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{-2p-1}{p^2}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{(-2p-1)\,x(p)}{p^2} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-2p-1}{p^2}dp}$$
$$= e^{-\frac{1}{p}+2\ln(p)}$$

Which simplifies to

$$\mu = p^2 e^{-\frac{1}{p}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(p^2 e^{-\frac{1}{p}} x\right) = 0$$

Integrating gives

$$p^2 e^{-\frac{1}{p}} x = c_2$$

Dividing both sides by the integrating factor $\mu = p^2 e^{-\frac{1}{p}}$ results in

$$x(p) = \frac{c_2 e^{\frac{1}{p}}}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{-x + \sqrt{x^2 + 4yx}}{2x}$$
$$p = -\frac{x + \sqrt{x^2 + 4yx}}{2x}$$

Substituting the above in the solution for x found above gives

$$x = \frac{4c_2 x^2 e^{\frac{2x}{-x+\sqrt{x(4y+x)}}}}{\left(-x + \sqrt{x\,(4y+x)}\right)^2}$$

$$x = \frac{4c_2 x^2 e^{-\frac{2x}{x+\sqrt{x(4y+x)}}}}{\left(x + \sqrt{x\,(4y+x)}\right)^2}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{4c_2 x^2 e^{\frac{2x}{-x+\sqrt{x(4y+x)}}}}{\left(-x+\sqrt{x\,(4y+x)}\right)^2} \tag{2}$$

$$x = \frac{4c_2 x^2 e^{-\frac{2x}{x+\sqrt{x(4y+x)}}}}{\left(x+\sqrt{x\,(4y+x)}\right)^2} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{4c_2 x^2 e^{\frac{2x}{-x+\sqrt{x(4y+x)}}}}{\left(-x+\sqrt{x\,(4y+x)}\right)^2}$$

Verified OK.

$$x = \frac{4c_2 x^2 e^{-\frac{2x}{x+\sqrt{x(4y+x)}}}}{\left(x+\sqrt{x\,(4y+x)}\right)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 65

```
dsolve(y(x)=diff(y(x),x)*x*(diff(y(x),x)+1),y(x), singsol=all)
```

$$y(x) = \frac{x\left(1+2\,\mathrm{LambertW}\left(-\frac{1}{2\sqrt{\frac{c_1}{x}}}\right)\right)}{4\,\mathrm{LambertW}\left(-\frac{1}{2\sqrt{\frac{c_1}{x}}}\right)^2}$$

$$y(x) = \frac{x\left(1+2\,\mathrm{LambertW}\left(\frac{1}{2\sqrt{\frac{c_1}{x}}}\right)\right)}{4\,\mathrm{LambertW}\left(\frac{1}{2\sqrt{\frac{c_1}{x}}}\right)^2}$$

✓ Solution by Mathematica

Time used: 0.523 (sec). Leaf size: 102

```
DSolve[y[x]==y'[x]*x*(y'[x]+1),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{\sqrt{\frac{4y(x)}{x}+1}-1}-\log\left(\sqrt{\frac{4y(x)}{x}+1}-1\right)=\frac{\log(x)}{2}+c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{1}{\sqrt{\frac{4y(x)}{x}+1}+1}+\log\left(\sqrt{\frac{4y(x)}{x}+1}+1\right)=-\frac{\log(x)}{2}+c_1,y(x)\right]$$

$$y(x)\to 0$$

20.11 problem 11

Internal problem ID [2348]
Internal file name [`OUTPUT/2348_Tuesday_February_27_2024_08_30_26_AM_23240955/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 11.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _dAlembert]
```

$$\boxed{2xy'^3 - yy'^2 = -1}$$

20.11.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$2x\,p^3 - y\,p^2 = -1$$

Solving for y from the above results in

$$y = 2px + \frac{1}{p^2} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 2p$$
$$g = \frac{1}{p^2}$$

Hence (2) becomes

$$-p = \left(2x - \frac{2}{p^3}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = \infty$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x)}{2x - \frac{2}{p(x)^3}} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{2x(p) - \frac{2}{p^3}}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{p}$$
$$q(p) = \frac{2}{p^4}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p} = \frac{2}{p^4}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{p}dp} \\ &= p^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(\frac{2}{p^4}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}(p^2 x) &= (p^2)\left(\frac{2}{p^4}\right) \\ \mathrm{d}(p^2 x) &= \left(\frac{2}{p^2}\right)\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^2 x &= \int \frac{2}{p^2}\,\mathrm{d}p \\ p^2 x &= -\frac{2}{p} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^2$ results in

$$x(p) = -\frac{2}{p^3} + \frac{c_1}{p^2}$$

which simplifies to

$$x(p) = \frac{c_1 p - 2}{p^3}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{1}{3}}}{6x} + \frac{y^2}{6x\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{1}{3}}} + \frac{y}{6x}$$

$$p = -\frac{\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{1}{3}}}{12x} - \frac{y^2}{12x\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{1}{3}}} + \frac{y}{6x} + \frac{i\sqrt{3}\left(\frac{(6\sqrt{}}{}\right.}{}$$

$$p = -\frac{\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{1}{3}}}{12x} - \frac{y^2}{12x\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{1}{3}}} + \frac{y}{6x} - \frac{i\sqrt{3}\left(\frac{(6\sqrt{}}{}\right.}{}$$

Substituting the above in the solution for x found above gives

$$x = \frac{36\left(\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}} c_1+\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{1}{3}} c_1 y+c_1 y^2-12x\left(6\sqrt{\cdots}\right.\right.}{\left(\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}+y\left(6\sqrt{3}\,\sqrt{-y^3+2\cdots}\right.\right.}$$

$$x = \frac{144\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}\left(\left(-i\sqrt{3}\,c_1+c_1\right)\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}+\left(-2yc_1\cdots\right.\right.}{\left(-i\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}\sqrt{3}+i\sqrt{3}\,y^2+\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x\cdots\right.\right.}$$

$$x = \frac{144\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}} x^2\left(c_1\left(-1-i\sqrt{3}\right)\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}+2(yc_1\cdots\right.}{\left(-i\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}\sqrt{3}+i\sqrt{3}\,y^2-\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\cdots\right.\right.}$$

Summary

The solution(s) found are the following

$$y = \infty \tag{1}$$

$$x = \frac{36\left(\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}} c_1+\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{1}{3}} c_1 y+c_1 y^2-12x\left(6\sqrt{\cdots}\right.\right.}{\left(\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}+y\left(6\sqrt{3}\,\sqrt{-y^3+2\cdots}\right.\right.} \tag{2}$$

$$x = \frac{144\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}\left(\left(-i\sqrt{3}\,c_1+c_1\right)\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}+\left(-2yc_1\cdots\right.\right.}{\left(-i\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}\sqrt{3}+i\sqrt{3}\,y^2+\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x\cdots\right.\right.} \tag{3}$$

$$x = \frac{144\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}} x^2\left(c_1\left(-1-i\sqrt{3}\right)\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}+2(yc_1\cdots\right.}{\left(-i\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}\sqrt{3}+i\sqrt{3}\,y^2-\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\cdots\right.\right.} \tag{4}$$

Verification of solutions

$$y = \infty$$

Verified OK.

$$x = \frac{36\Big(\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}c_1+\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{1}{3}}c_1y+c_1y^2-12x\big(6\sqrt{}}{\Big(\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}+y\left(6\sqrt{3}\,\sqrt{-y^3+2}\right.}$$

Verified OK.

$$x = \frac{144\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}\Big(\left(-i\sqrt{3}\,c_1+c_1\right)\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}+(-2yc_1}{\Big(-i\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}\sqrt{3}+i\sqrt{3}\,y^2+\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x\right.}$$

Warning, solution could not be verified

$$x = \frac{144\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}x^2\Big(c_1\left(-1-i\sqrt{3}\right)\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}+2(yc_1}{\Big(-i\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right)^{\frac{2}{3}}\sqrt{3}+i\sqrt{3}\,y^2-\left(6\sqrt{3}\,\sqrt{-y^3+27x^2}\,x+y^3-54x^2\right.}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 513

```
dsolve(2*diff(y(x),x)^3*x+1=diff(y(x),x)^2*y(x),y(x), singsol=all)
```

$$y(x) = -\frac{9\left(-\frac{2\,3^{\frac{2}{3}}\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{2}{3}}c_1^2}{9}+\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)\left(-\frac{2\,3^{\frac{1}{3}}\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{1}{3}}c_1}{9}+x\right)x\right)3^{\frac{1}{3}}x^2}{\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{1}{3}}\left(c_1 3^{\frac{1}{3}}x+\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{2}{3}}\right)^2}$$

$$y(x) = \frac{4\left(3\left(-i3^{\frac{1}{6}}+\frac{3^{\frac{2}{3}}}{3}\right)c_1^2\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{2}{3}}+\left(c_1\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}\right)\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{1}{3}}+9x\right)x}{\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{1}{3}}\left(\left(i-\sqrt{3}\right)\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{2}{3}}+c_1x\left(i3^{\frac{1}{3}}+3\right.\right.}$$

$$y(x) = -\frac{4\,3^{\frac{1}{3}}\left(-3c_1^2\left(i3^{\frac{1}{6}}+\frac{3^{\frac{2}{3}}}{3}\right)\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{2}{3}}+\left(c_1\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{1}{3}}-9\right.\right.}{\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{1}{3}}\left(\left(\sqrt{3}+i\right)\left(\left(-9+\sqrt{\frac{-3c_1^3+81x}{x}}\right)x^2\right)^{\frac{2}{3}}+c_1\left(-3^{\frac{5}{6}}+\right.\right.}$$

✓ Solution by Mathematica

Time used: 151.15 (sec). Leaf size: 17695

```
DSolve[2*y'[x]^3*x+1==y'[x]^2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

20.12 problem 12

Internal problem ID [2349]
Internal file name [`OUTPUT/2349_Tuesday_February_27_2024_08_32_22_AM_75913950/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 12.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries]]`

$$\boxed{{y'}^3 + xyy' - 2y^2 = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(27y^2 + 3\sqrt{3y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{3} - \frac{yx}{\left(27y^2 + 3\sqrt{3y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} \tag{1}$$

$$y' = -\frac{\left(27y^2 + 3\sqrt{3y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{6} + \frac{yx}{2\left(27y^2 + 3\sqrt{3y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(27y^2+3\sqrt{3y^3x^3+81y^4}\right)^{\frac{1}{3}}}{3} + \frac{\cdots}{\left(27y^2+3\cdots\right.}\right.}{2} \tag{2}$$

$$y' = -\frac{\left(27y^2 + 3\sqrt{3y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{6} + \frac{yx}{2\left(27y^2 + 3\sqrt{3y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(27y^2+3\sqrt{3y^3x^3+81y^4}\right)^{\frac{1}{3}}}{3} + \frac{\cdots}{\left(27y^2+3\cdots\right.}\right.}{2} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\left(27y^2 + 3\sqrt{3x^3y^3 + 81y^4}\right)^{\frac{2}{3}} - 3xy}{3\left(27y^2 + 3\sqrt{3x^3y^3 + 81y^4}\right)^{\frac{1}{3}}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2 + ya_3 + a_1 \tag{1E}
$$

$$
\eta = xb_2 + yb_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{\left(\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}-3xy\right)(b_3-a_2)}{3\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}}-\frac{\left(\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}-3xy\right)^2 a_3}{9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}}-\left(\frac{\frac{9y^3x^2}{\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}\sqrt{3x^3y^3+81y^4}}-3y}{3\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}}-\frac{3\left(\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}-3xy\right)y^3x^2}{2\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{4}{3}}\sqrt{3x^3y^3+81y^4}}\right)(xa_2+ya_3+a_1)-\left(\frac{\frac{36y+\frac{2\left(\frac{27}{2}x^3y^2+486y^3\right)}{3\sqrt{3x^3y^3+81y^4}}}{\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}}-3x}{3\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}}-\frac{\left(\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}-3xy\right)\left(54y+\frac{\frac{27}{2}x^3y^2+486y^3}{\sqrt{3x^3y^3+81y^4}}\right)}{9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{4}{3}}}\right)(xb_2+yb_3+b_1)=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}x^4y^2b_2+9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}x^3y^3a_2+9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)}{}$$

$$=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} x^4y^2b_2 \\
&-9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} x^3y^3a_2 \\
&-9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} x^3y^3b_3 \\
&-9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} x^2y^4a_3 \\
&-9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} x^3y^2b_1 \\
&-9\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} x^2y^3a_1 \\
&-324\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} x\,y^3b_2 \\
&-36\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} \sqrt{3x^3y^3+81y^4}\,y^2b_3 \\
&+54\sqrt{3x^3y^3+81y^4}\,x^2y^2b_2-108\sqrt{3x^3y^3+81y^4}\,x\,y^3b_3 \\
&-36\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} \sqrt{3x^3y^3+81y^4}\,yb_1 \\
&+54\sqrt{3x^3y^3+81y^4}\,x\,y^2b_1+324\sqrt{3x^3y^3+81y^4}\,x\,y^3a_2 \\
&+4\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{4}{3}} \sqrt{3x^3y^3+81y^4}\,xya_3 \\
&-6\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} \sqrt{3x^3y^3+81y^4}\,x^2y^2a_3 \\
&-36\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} \sqrt{3x^3y^3+81y^4}\,xyb_2 \\
&-324\sqrt{3x^3y^3+81y^4}\,y^4a_3+162\sqrt{3x^3y^3+81y^4}\,y^3a_1 \\
&+2916x\,y^5a_2+81x^4y^4a_2-297x^3y^5a_3 \\
&+27x^3y^4a_1+27x^5y^3b_2-27x^4y^4b_3+27x^4y^3b_1 \\
&+486x^2y^4b_2-972x\,y^5b_3+486x\,y^4b_1 \\
&-2\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{5}{3}} \sqrt{3x^3y^3+81y^4}\,a_2 \\
&+2\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{5}{3}} \sqrt{3x^3y^3+81y^4}\,b_3 \\
&-324\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} y^4b_3 \\
&+6b_2\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{4}{3}} \sqrt{3x^3y^3+81y^4} \\
&-324\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} y^3b_1 \\
&-6\left(3x^3y^3+81y^4\right)^{\frac{3}{2}} a_3-7290y^6a_3+1458y^5a_1=0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{y^3\left(x^3+27y\right)}, \left(27y^2+3\sqrt{3}\,\sqrt{y^3\left(x^3+27y\right)}\right)^{\frac{1}{3}}, \left(27y^2+3\sqrt{3}\,\sqrt{y^3\left(x^3+27y\right)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{y^3\left(x^3+27y\right)} = v_3, \left(27y^2 + 3\sqrt{3}\,\sqrt{y^3\left(x^3+27y\right)}\right)^{\frac{1}{3}} = v_4, \left(27y^2+3\sqrt{3}\,\sqrt{y^3\left(x^3+27y\right)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
-9v_2\Big(&6\sqrt{3}\,v_3v_1^3v_2^2a_3 + 12\sqrt{3}\,v_5v_3v_1b_2 + 18\sqrt{3}\,v_5v_3v_2a_2 \\
&- 6\sqrt{3}\,v_5v_3v_2b_3 - 18\sqrt{3}\,v_3v_1^2v_2b_2 - 108\sqrt{3}\,v_3v_1v_2^2a_2 + 36\sqrt{3}\,v_3v_1v_2^2b_3 \\
&- 54\sqrt{3}\,v_4v_3v_2b_2 - 18\sqrt{3}\,v_3v_1v_2b_1 + 3v_5v_1^2v_2^2a_1 - 18v_4v_1^3v_2^2b_2 \\
&- 324v_4v_1v_2^4a_3 + 270\sqrt{3}\,v_3v_2^3a_3 + 108v_5v_1v_2^2b_2 + 12\sqrt{3}\,v_5v_3b_1 \\
&- 54\sqrt{3}\,v_3v_2^2a_1 - 12v_4v_1^4v_2^3a_3 + 3v_5v_1^4v_2b_2 + 9v_5v_1^3v_2^2a_2 - 3v_5v_1^3v_2^2b_3 \\
&+ 3v_5v_1^2v_2^3a_3 + 2\sqrt{3}\,v_5v_3v_1^2v_2a_3 - 36\sqrt{3}\,v_4v_3v_1v_2^2a_3 + 2430v_2^5a_3 \\
&- 486v_2^4a_1 + 3v_5v_1^3v_2b_1 - 9v_1^5v_2^2b_2 - 27v_1^4v_2^3a_2 + 9v_1^4v_2^3b_3 + 99v_1^3v_2^4a_3 \\
&- 9v_1^4v_2^2b_1 - 9v_1^3v_2^3a_1 + 324v_1v_2^4b_3 + 162v_5v_2^3a_2 - 54v_5v_2^3b_3 \\
&+ 108v_5v_2^2b_1 - 486v_4v_2^3b_2 - 162v_2^3v_1^2b_2 - 972v_2^4v_1a_2 - 162v_2^3v_1b_1\Big) = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&1458 b_1 v_1 v_2^4 + 4374 b_2 v_4 v_2^4 + (-1458 a_2 + 486 b_3)\, v_5 v_2^4 - 972 b_1 v_5 v_2^3 \\
&- 54\sqrt{3}\, a_3 v_3 v_1^3 v_2^3 + 162\sqrt{3}\, b_2 v_3 v_1^2 v_2^2 + 162\sqrt{3}\, b_1 v_3 v_1 v_2^2 \\
&- 21870 a_3 v_2^6 + 4374 a_1 v_2^5 - 18\sqrt{3}\, a_3 v_3 v_5 v_1^2 v_2^2 + 324\sqrt{3}\, a_3 v_3 v_4 v_1 v_2^3 \\
&- 108\sqrt{3}\, b_2 v_3 v_5 v_1 v_2 + 2916 a_3 v_4 v_1 v_2^5 + (8748 a_2 - 2916 b_3)\, v_1 v_2^5 \\
&+ \left(972\sqrt{3}\, a_2 - 324\sqrt{3}\, b_3\right) v_3 v_1 v_2^3 - 972 b_2 v_5 v_1 v_2^3 - 2430\sqrt{3}\, a_3 v_3 v_2^4 \\
&+ 486\sqrt{3}\, a_1 v_3 v_2^3 + \left(-162\sqrt{3}\, a_2 + 54\sqrt{3}\, b_3\right) v_3 v_5 v_2^2 \\
&+ 108 a_3 v_4 v_1^4 v_2^4 + (243 a_2 - 81 b_3)\, v_1^4 v_2^4 - 27 b_2 v_5 v_1^4 v_2^2 + 162 b_2 v_4 v_1^3 v_2^3 \\
&+ (-81 a_2 + 27 b_3)\, v_5 v_1^3 v_2^3 - 27 b_1 v_5 v_1^3 v_2^2 - 27 a_3 v_5 v_1^2 v_2^4 \\
&- 27 a_1 v_5 v_1^2 v_2^3 + 486\sqrt{3}\, b_2 v_3 v_4 v_2^2 - 108\sqrt{3}\, v_5 v_3 b_1 v_2 + 81 v_2^3 b_2 v_1^5 \\
&+ 81 b_1 v_1^4 v_2^3 - 891 a_3 v_1^3 v_2^5 + 81 a_1 v_1^3 v_2^4 + 1458 b_2 v_1^2 v_2^4 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-27a_1 &= 0\\
81a_1 &= 0\\
4374a_1 &= 0\\
-21870a_3 &= 0\\
-891a_3 &= 0\\
-27a_3 &= 0\\
108a_3 &= 0\\
2916a_3 &= 0\\
-972b_1 &= 0\\
-27b_1 &= 0\\
81b_1 &= 0\\
1458b_1 &= 0\\
-972b_2 &= 0\\
-27b_2 &= 0\\
81b_2 &= 0\\
162b_2 &= 0\\
1458b_2 &= 0\\
4374b_2 &= 0\\
486\sqrt{3}\, a_1 &= 0\\
-2430\sqrt{3}\, a_3 &= 0\\
-54\sqrt{3}\, a_3 &= 0\\
-18\sqrt{3}\, a_3 &= 0\\
324\sqrt{3}\, a_3 &= 0\\
-108\sqrt{3}\, b_1 &= 0\\
162\sqrt{3}\, b_1 &= 0\\
-108\sqrt{3}\, b_2 &= 0\\
162\sqrt{3}\, b_2 &= 0\\
486\sqrt{3}\, b_2 &= 0\\
-1458a_2 + 486b_3 &= 0\\
-81a_2 + 27b_3 &= 0\\
243a_2 - 81b_3 &= 0\\
8748a_2 - 2916b_3 &= 0\\
-162\sqrt{3}\, a_2 + 54\sqrt{3}\, b_3 &= 0\\
972\sqrt{3}\, a_2 - 324\sqrt{3}\, b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{\frac{x}{3}} \\
&= \frac{3y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^3
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^3}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{\frac{x}{3}} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= 3\ln(x) \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\left(27y^2 + 3\sqrt{3x^3y^3 + 81y^4}\right)^{\frac{2}{3}} - 3xy}{3\left(27y^2 + 3\sqrt{3x^3y^3 + 81y^4}\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{3y}{x^4} \\ R_y &= \frac{1}{x^3} \\ S_x &= \frac{3}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{9x^3\left(27y^2 + 3\sqrt{3}\,\sqrt{y^3(x^3+27y)}\right)^{\frac{1}{3}}}{-\left(27y^2 + 3\sqrt{3}\,\sqrt{y^3(x^3+27y)}\right)^{\frac{2}{3}} x + 3y\left(x^2 + 3\left(27y^2 + 3\sqrt{3}\,\sqrt{y^3(x^3+27y)}\right)^{\frac{1}{3}}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{9\,3^{\frac{1}{3}} \left(\sqrt{3}\,\sqrt{27R+1}+9\sqrt{R}\right)^{\frac{1}{3}}}{\sqrt{R}\,\left(3^{\frac{2}{3}} \left(\sqrt{3}\,\sqrt{27R+1}+9\sqrt{R}\right)^{\frac{2}{3}} - 9\,3^{\frac{1}{3}}\sqrt{R}\,\left(\sqrt{3}\,\sqrt{27R+1}+9\sqrt{R}\right)^{\frac{1}{3}} - 3\right)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{9\left(3\sqrt{81R+3}+27\sqrt{R}\right)^{\frac{1}{3}}}{\left(9^{\frac{1}{3}}\left(\left(\sqrt{81R+3}+9\sqrt{R}\right)^{2}\right)^{\frac{1}{3}} - 9\sqrt{R}\,\left(3\sqrt{81R+3}+27\sqrt{R}\right)^{\frac{1}{3}} - 3\right)\sqrt{R}} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9(3\sqrt{81\_a+3}+27\sqrt{\_a})^{\frac{1}{3}}}{\left(9^{\frac{1}{3}}\left((\sqrt{81\_a+3}+9\sqrt{\_a})^{2}\right)^{\frac{1}{3}} - 9\sqrt{\_a}\,(3\sqrt{81\_a+3}+27\sqrt{\_a})^{\frac{1}{3}} - 3\right)\sqrt{\_a}} d\_a + c_1$$

Which simplifies to

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9(3\sqrt{81\_a+3}+27\sqrt{\_a})^{\frac{1}{3}}}{\left(9^{\frac{1}{3}}\left((\sqrt{81\_a+3}+9\sqrt{\_a})^{2}\right)^{\frac{1}{3}} - 9\sqrt{\_a}\,(3\sqrt{81\_a+3}+27\sqrt{\_a})^{\frac{1}{3}} - 3\right)\sqrt{\_a}} d\_a + c_1$$

Summary

The solution(s) found are the following

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9(3\sqrt{81\_a+3}+27\sqrt{\_a})^{\frac{1}{3}}}{\left(9^{\frac{1}{3}}\left((\sqrt{81\_a+3}+9\sqrt{\_a})^{2}\right)^{\frac{1}{3}} - 9\sqrt{\_a}\,(3\sqrt{81\_a+3}+27\sqrt{\_a})^{\frac{1}{3}} - 3\right)\sqrt{\_a}} d\_a + c_1 \tag{1}$$

Verification of solutions

$$3 \ln(x) = \int^{\frac{y}{x^3}} \frac{9\left(3\sqrt{81\_a+3}+27\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(9^{\frac{1}{3}}\left(\left(\sqrt{81\_a+3}+9\sqrt{\_a}\right)^2\right)^{\frac{1}{3}}-9\sqrt{\_a}\left(3\sqrt{81\_a+3}+27\sqrt{\_a}\right)^{\frac{1}{3}}-3\right)\sqrt{\_a}}d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}+3i\sqrt{3}\,yx-\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}+3xy}{6\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 \\
& + \frac{\left(i\sqrt{3}\,\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}+3i\sqrt{3}\,yx-\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}+3xy\right)(b_3-a_2)}{6\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}} \\
& - \frac{\left(i\sqrt{3}\,\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}+3i\sqrt{3}\,yx-\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}+3xy\right)^2 a_3}{36\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}} \\
& - \left(\frac{\frac{9i\sqrt{3}\,y^3x^2}{\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}\sqrt{3x^3y^3+81y^4}}+3i\sqrt{3}\,y-\frac{9y^3x^2}{\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}\sqrt{3x^3y^3+81y^4}}+3y}{6\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}}\right. \\
& \left. - \frac{3\left(i\sqrt{3}\,\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}+3i\sqrt{3}\,yx-\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}+3xy\right)y^3x^2}{4\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{4}{3}}\sqrt{3x^3y^3+81y^4}}\right)(xa_2 \\
& + ya_3+a_1) - \left(\frac{\frac{2i\sqrt{3}\,\left(54y+\frac{\frac{27}{2}x^3y^2+486y^3}{\sqrt{3x^3y^3+81y^4}}\right)}{3\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}}+3i\sqrt{3}\,x-\frac{2\left(54y+\frac{\frac{27}{2}x^3y^2+486y^3}{\sqrt{3x^3y^3+81y^4}}\right)}{3\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}}+3x}{6\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}}\right. \\
& - \frac{\left(i\sqrt{3}\,\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}+3i\sqrt{3}\,yx-\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}+3xy\right)\left(54y+\frac{\frac{27}{2}x^3y^2+486y^3}{\sqrt{3x^3y^3+81y^4}}\right)}{18\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{4}{3}}} \\
& + yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{y^3\left(x^3+27y\right)}, \left(27y^2+3\sqrt{3}\,\sqrt{y^3\left(x^3+27y\right)}\right)^{\frac{1}{3}}, \left(27y^2+3\sqrt{3}\,\sqrt{y^3\left(x^3+27y\right)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{y^3\left(x^3+27y\right)} = v_3, \left(27y^2 + 3\sqrt{3}\,\sqrt{y^3\left(x^3+27y\right)}\right)^{\frac{1}{3}} = v_4, \left(27y^2+3\sqrt{3}\,\sqrt{y^3\left(x^3+27y\right)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
-9v_2\Big(&-2\sqrt{3}\,v_5v_3v_1^2v_2a_3 - 72\sqrt{3}\,v_4v_3v_1v_2^2a_3 + 162v_2^3v_1^2b_2\\
&+ 972v_2^4v_1a_2 + 162v_2^3v_1b_1 + 9v_1^5v_2^2b_2 + 27v_1^4v_2^3a_2 - 9v_1^4v_2^3b_3\\
&- 99v_1^3v_2^4a_3 + 9v_1^4v_2^2b_1 + 9v_1^3v_2^3a_1 - 324v_1v_2^4b_3 - 162v_5v_2^3a_2\\
&+ 54v_5v_2^3b_3 - 108v_5v_2^2b_1 - 972v_4v_2^3b_2 + 972i\sqrt{3}\,v_2^4v_1a_2\\
&- 324i\sqrt{3}\,v_1v_2^4b_3 - 18iv_3v_1^3v_2^2a_3 + 108i\sqrt{3}\,v_5v_2^2b_1\\
&+ 162i\sqrt{3}\,v_2^3v_1b_1 + 36iv_5v_3v_1b_2 + 54iv_5v_3v_2a_2 - 18iv_5v_3v_2b_3\\
&+ 54iv_3v_1^2v_2b_2 + 324iv_3v_1v_2^2a_2 - 108iv_3v_1v_2^2b_3 + 54iv_3v_1v_2b_1\\
&- 24v_4v_1^4v_2^3a_3 - 3v_5v_1^4v_2b_2 - 9v_5v_1^3v_2^2a_2 + 3v_5v_1^3v_2^2b_3\\
&- 3v_5v_1^2v_2^3a_3 - 3v_5v_1^3v_2b_1 - 3v_5v_1^2v_2^2a_1 - 36v_4v_1^3v_2^2b_2\\
&- 648v_4v_1v_2^4a_3 - 270\sqrt{3}\,v_3v_2^3a_3 - 108v_5v_1v_2^2b_2\\
&- 12\sqrt{3}\,v_5v_3b_1 + 54\sqrt{3}\,v_3v_2^2a_1 - 2430i\sqrt{3}\,v_2^5a_3\\
&+ 486i\sqrt{3}\,v_2^4a_1 + 3i\sqrt{3}\,v_5v_1^4v_2b_2 + 9i\sqrt{3}\,v_5v_1^3v_2^2a_2\\
&- 3i\sqrt{3}\,v_5v_1^3v_2^2b_3 + 3i\sqrt{3}\,v_5v_1^2v_2^3a_3 + 3i\sqrt{3}\,v_5v_1^3v_2b_1\\
&+ 3i\sqrt{3}\,v_5v_1^2v_2^2a_1 + 108i\sqrt{3}\,v_5v_1v_2^2b_2 + 6iv_5v_3v_1^2v_2a_3\\
&+ 9i\sqrt{3}\,v_1^5v_2^2b_2 + 27i\sqrt{3}\,v_1^4v_2^3a_2 - 9i\sqrt{3}\,v_1^4v_2^3b_3\\
&- 99i\sqrt{3}\,v_1^3v_2^4a_3 + 9i\sqrt{3}\,v_1^4v_2^2b_1 + 9i\sqrt{3}\,v_1^3v_2^3a_1\\
&+ 162i\sqrt{3}\,v_5v_2^3a_2 - 54i\sqrt{3}\,v_5v_2^3b_3 + 162i\sqrt{3}\,v_2^3v_1^2b_2\\
&- 810iv_3v_2^3a_3 + 36iv_5v_3b_1 + 162iv_3v_2^2a_1 - 2430v_2^5a_3\\
&+ 486v_2^4a_1 - 6\sqrt{3}\,v_3v_1^3v_2^2a_3 - 12\sqrt{3}\,v_5v_3v_1b_2 - 18\sqrt{3}\,v_5v_3v_2a_2\\
&+ 6\sqrt{3}\,v_5v_3v_2b_3 + 18\sqrt{3}\,v_3v_1^2v_2b_2 + 108\sqrt{3}\,v_3v_1v_2^2a_2\\
&- 36\sqrt{3}\,v_3v_1v_2^2b_3 - 108\sqrt{3}\,v_4v_3v_2b_2 + 18\sqrt{3}\,v_3v_1v_2b_1\Big) = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&5832a_3v_4v_1v_2^5 + \left(-54ia_3 + 18\sqrt{3}\,a_3\right) v_3v_5v_1^2v_2^2\\
&+ \left(-324ib_2 + 108\sqrt{3}\,b_2\right) v_3v_5v_1v_2\\
&+ 8748b_2v_4v_2^4 + 648\sqrt{3}\,a_3v_3v_4v_1v_2^3\\
&+ \left(-8748i\sqrt{3}\,a_2 + 2916i\sqrt{3}\,b_3 - 8748a_2 + 2916b_3\right) v_1v_2^5\\
&+ \left(-1458i\sqrt{3}\,b_1 - 1458b_1\right) v_1v_2^4\\
&+ \left(7290ia_3 + 2430\sqrt{3}\,a_3\right) v_3v_2^4\\
&+ \left(-1458i\sqrt{3}\,a_2 + 486i\sqrt{3}\,b_3 + 1458a_2 - 486b_3\right) v_5v_2^4\\
&+ \left(-1458ia_1 - 486\sqrt{3}\,a_1\right) v_3v_2^3\\
&+ \left(-972i\sqrt{3}\,b_1 + 972b_1\right) v_5v_2^3\\
&+ \left(-1458i\sqrt{3}\,b_2 - 1458b_2\right) v_1^2v_2^4\\
&+ \left(891i\sqrt{3}\,a_3 + 891a_3\right) v_1^3v_2^5 + \left(-81i\sqrt{3}\,a_1 - 81a_1\right) v_1^3v_2^4\\
&+ \left(-243i\sqrt{3}\,a_2 + 81i\sqrt{3}\,b_3 - 243a_2 + 81b_3\right) v_1^4v_2^4\\
&+ \left(-81i\sqrt{3}\,b_1 - 81b_1\right) v_1^4v_2^3 + \left(-81i\sqrt{3}\,b_2 - 81b_2\right) v_1^5v_2^3\\
&+ 324b_2v_4v_1^3v_2^3 + \left(21870i\sqrt{3}\,a_3 + 21870a_3\right) v_2^6\\
&+ \left(-4374i\sqrt{3}\,a_1 - 4374a_1\right) v_2^5\\
&+ 972\sqrt{3}\,b_2v_3v_4v_2^2 + 216a_3v_4v_1^4v_2^4\\
&+ \left(-2916ia_2 + 972ib_3 - 972\sqrt{3}\,a_2 + 324\sqrt{3}\,b_3\right) v_3v_1v_2^3\\
&+ \left(-972i\sqrt{3}\,b_2 + 972b_2\right) v_5v_1v_2^3\\
&+ \left(-486ib_1 - 162\sqrt{3}\,b_1\right) v_3v_1v_2^2\\
&+ \left(-27i\sqrt{3}\,a_3 + 27a_3\right) v_5v_1^2v_2^4\\
&+ \left(-27i\sqrt{3}\,a_1 + 27a_1\right) v_5v_1^2v_2^3\\
&+ \left(-486ib_2 - 162\sqrt{3}\,b_2\right) v_3v_1^2v_2^2\\
&+ \left(162ia_3 + 54\sqrt{3}\,a_3\right) v_3v_1^3v_2^3\\
&+ \left(-81i\sqrt{3}\,a_2 + 27i\sqrt{3}\,b_3 + 81a_2 - 27b_3\right) v_5v_1^3v_2^3\\
&+ \left(-27i\sqrt{3}\,b_1 + 27b_1\right) v_5v_1^3v_2^2 + \left(-27i\sqrt{3}\,b_2 + 27b_2\right) v_5v_1^4v_2^2\\
&+ \left(-486ia_2 + 162ib_3 + 162\sqrt{3}\,a_2 - 54\sqrt{3}\,b_3\right) v_3v_5v_2^2\\
&+ \left(-324ib_1 + 108\sqrt{3}\,b_1\right) v_3v_5v_2 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
216a_3 &= 0\\
5832a_3 &= 0\\
324b_2 &= 0\\
8748b_2 &= 0\\
648\sqrt{3}\,a_3 &= 0\\
972\sqrt{3}\,b_2 &= 0\\
-1458ia_1 - 486\sqrt{3}\,a_1 &= 0\\
-486ib_1 - 162\sqrt{3}\,b_1 &= 0\\
-486ib_2 - 162\sqrt{3}\,b_2 &= 0\\
-324ib_1 + 108\sqrt{3}\,b_1 &= 0\\
-324ib_2 + 108\sqrt{3}\,b_2 &= 0\\
-54ia_3 + 18\sqrt{3}\,a_3 &= 0\\
162ia_3 + 54\sqrt{3}\,a_3 &= 0\\
7290ia_3 + 2430\sqrt{3}\,a_3 &= 0\\
-4374i\sqrt{3}\,a_1 - 4374a_1 &= 0\\
-1458i\sqrt{3}\,b_1 - 1458b_1 &= 0\\
-1458i\sqrt{3}\,b_2 - 1458b_2 &= 0\\
-972i\sqrt{3}\,b_1 + 972b_1 &= 0\\
-972i\sqrt{3}\,b_2 + 972b_2 &= 0\\
-81i\sqrt{3}\,a_1 - 81a_1 &= 0\\
-81i\sqrt{3}\,b_1 - 81b_1 &= 0\\
-81i\sqrt{3}\,b_2 - 81b_2 &= 0\\
-27i\sqrt{3}\,a_1 + 27a_1 &= 0\\
-27i\sqrt{3}\,a_3 + 27a_3 &= 0\\
-27i\sqrt{3}\,b_1 + 27b_1 &= 0\\
-27i\sqrt{3}\,b_2 + 27b_2 &= 0\\
891i\sqrt{3}\,a_3 + 891a_3 &= 0\\
21870i\sqrt{3}\,a_3 + 21870a_3 &= 0\\
-2916ia_2 + 972ib_3 - 972\sqrt{3}\,a_2 + 324\sqrt{3}\,b_3 &= 0\\
-486ia_2 + 162ib_3 + 162\sqrt{3}\,a_2 - 54\sqrt{3}\,b_3 &= 0\\
-8748i\sqrt{3}\,a_2 + 2916i\sqrt{3}\,b_3 - 8748a_2 + 2916b_3 &= 0\\
-1458i\sqrt{3}\,a_2 + 486i\sqrt{3}\,b_3 + 1458a_2 - 486b_3 &= 0\\
-243i\sqrt{3}\,a_2 + 81i\sqrt{3}\,b_3 - 243a_2 + 81b_3 &= 0\\
-81i\sqrt{3}\,a_2 + 27i\sqrt{3}\,b_3 + 81a_2 - 27b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{i\sqrt{3}\left(27y^2 + 3\sqrt{3x^3y^3 + 81y^4}\right)^{\frac{2}{3}} + 3i\sqrt{3}\,yx + \left(27y^2 + 3\sqrt{3x^3y^3 + 81y^4}\right)^{\frac{2}{3}} - 3xy}{6\left(27y^2 + 3\sqrt{3x^3y^3 + 81y^4}\right)^{\frac{1}{3}}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(i\sqrt{3}\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} + 3i\sqrt{3}\,yx + \left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} - 3xy\right)(b_3-a_2)}{6\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}} - \frac{\left(i\sqrt{3}\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} + 3i\sqrt{3}\,yx + \left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} - 3xy\right)^2 a_3}{36\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}}} - \left(-\frac{\frac{9i\sqrt{3}\,y^3x^2}{\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}\sqrt{3x^3y^3+81y^4}} + 3i\sqrt{3}\,y + \frac{9y^3x^2}{\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}\sqrt{3x^3y^3+81y^4}} - 3y}{6\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}} + \frac{3\left(i\sqrt{3}\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} + 3i\sqrt{3}\,yx + \left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} - 3xy\right)y^3x^2}{4\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{4}{3}}\sqrt{3x^3y^3+81y^4}}\right)(xa_2 + ya_3 + a_1) - \left(-\frac{\frac{2i\sqrt{3}\left(54y+\frac{\frac{27}{2}x^3y^2+486y^3}{\sqrt{3x^3y^3+81y^4}}\right)}{3\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}} + 3i\sqrt{3}\,x + \frac{36y+\frac{2\left(\frac{27}{2}x^3y^2+486y^3\right)}{3\sqrt{3x^3y^3+81y^4}}}{\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}} - 3x}{6\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{1}{3}}} + \frac{\left(i\sqrt{3}\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} + 3i\sqrt{3}\,yx + \left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{2}{3}} - 3xy\right)\left(54y+\frac{\frac{27}{2}x^3y^2+486y^3}{\sqrt{3x^3y^3+81y^4}}\right)}{18\left(27y^2+3\sqrt{3x^3y^3+81y^4}\right)^{\frac{4}{3}}}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{y^3 (x^3 + 27y)}, \left(27y^2+3\sqrt{3}\,\sqrt{y^3 (x^3 + 27y)}\right)^{\frac{1}{3}}, \left(27y^2+3\sqrt{3}\,\sqrt{y^3 (x^3 + 27y)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{y^3 (x^3 + 27y)} = v_3, \left(27y^2 + 3\sqrt{3}\,\sqrt{y^3 (x^3 + 27y)}\right)^{\frac{1}{3}} = v_4, \left(27y^2 + 3\sqrt{3}\,\sqrt{y^3 (x^3 + 27y)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&9v_2\Big(6v_3\sqrt{3}\,v_1^3v_2^2a_3+12v_3\sqrt{3}\,v_5v_1b_2+18v_3\sqrt{3}\,v_5v_2a_2\\
&-6v_3\sqrt{3}\,v_5v_2b_3-18v_3\sqrt{3}\,v_1^2v_2b_2-108v_3\sqrt{3}\,v_1v_2^2a_2\\
&+36v_3\sqrt{3}\,v_1v_2^2b_3+108v_3\sqrt{3}\,v_4v_2b_2-18v_3\sqrt{3}\,v_1v_2b_1\\
&+54iv_3v_5v_2a_2-18iv_3v_5v_2b_3+54iv_3v_1^2v_2b_2+324iv_3v_1v_2^2a_2\\
&-108iv_3v_1v_2^2b_3+54iv_3v_1v_2b_1+9i\sqrt{3}\,v_1^5v_2^2b_2+24v_4v_1^4v_2^3a_3\\
&+27i\sqrt{3}\,v_1^4v_2^3a_2-9i\sqrt{3}\,v_1^4v_2^3b_3-99i\sqrt{3}\,v_1^3v_2^4a_3\\
&+9i\sqrt{3}\,v_1^4v_2^2b_1+9i\sqrt{3}\,v_1^3v_2^3a_1-18iv_3v_1^3v_2^2a_3\\
&+162i\sqrt{3}\,v_5v_2^3a_2-54i\sqrt{3}\,v_5v_2^3b_3+162iv_2^3\sqrt{3}\,v_1^2b_2\\
&+972iv_2^4\sqrt{3}\,v_1a_2-324i\sqrt{3}\,v_1v_2^4b_3+108i\sqrt{3}\,v_5v_2^2b_1\\
&+162iv_2^3\sqrt{3}\,v_1b_1+36iv_3v_5v_1b_2+3i\sqrt{3}\,v_5v_1^3v_2b_1\\
&+3i\sqrt{3}\,v_5v_1^2v_2^2a_1+6iv_3v_5v_1^2v_2a_3+108i\sqrt{3}\,v_5v_1v_2^2b_2\\
&-2430iv_2^5\sqrt{3}\,a_3+486iv_2^4\sqrt{3}\,a_1-810iv_3v_2^3a_3+36iv_3v_5b_1\\
&+162iv_3v_2^2a_1+3v_5v_1^4v_2b_2+9v_5v_1^3v_2^2a_2-3v_5v_1^3v_2^2b_3\\
&+3v_5v_1^2v_2^3a_3+3v_5v_1^3v_2b_1+3v_5v_1^2v_2^2a_1+36v_4v_1^3v_2^2b_2\\
&+648v_4v_1v_2^4a_3+270v_3\sqrt{3}\,v_2^3a_3+108v_5v_1v_2^2b_2\\
&+12v_3\sqrt{3}\,v_5b_1-54v_3\sqrt{3}\,v_2^2a_1+2v_3\sqrt{3}\,v_5v_1^2v_2a_3\\
&+72v_3\sqrt{3}\,v_4v_1v_2^2a_3+3i\sqrt{3}\,v_5v_1^4v_2b_2+9i\sqrt{3}\,v_5v_1^3v_2^2a_2\\
&-3i\sqrt{3}\,v_5v_1^3v_2^2b_3+3i\sqrt{3}\,v_5v_1^2v_2^3a_3-54v_5v_2^3b_3+108v_5v_2^2b_1\\
&+972v_4v_2^3b_2+324v_1v_2^4b_3-27v_1^4v_2^3a_2+9v_1^4v_2^3b_3+99v_1^3v_2^4a_3\\
&-9v_1^4v_2^2b_1-9v_1^3v_2^3a_1+162v_5v_2^3a_2-162v_2^3v_1^2b_2-972v_2^4v_1a_2\\
&-162v_2^3v_1b_1-9v_1^5v_2^2b_2+2430v_2^5a_3-486v_2^4a_1\Big)=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3,v_4,v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&216a_3v_4v_1^4v_2^4+324b_2v_4v_1^3v_2^3+5832a_3v_4v_1v_2^5+972\sqrt{3}\,b_2v_3v_4v_2^2\\
&+\left(27i\sqrt{3}\,b_2+27b_2\right)v_5v_1^4v_2^2+\left(-162ia_3+54\sqrt{3}\,a_3\right)v_3v_1^3v_2^3\\
&+\left(81i\sqrt{3}\,a_2-27i\sqrt{3}\,b_3+81a_2-27b_3\right)v_5v_1^3v_2^3\\
&+\left(27i\sqrt{3}\,b_1+27b_1\right)v_5v_1^3v_2^2+\left(27i\sqrt{3}\,a_3+27a_3\right)v_5v_1^2v_2^4\\
&+\left(27i\sqrt{3}\,a_1+27a_1\right)v_5v_1^2v_2^3+\left(486ib_2-162\sqrt{3}\,b_2\right)v_3v_1^2v_2^2\\
&+\left(2916ia_2-972ib_3-972\sqrt{3}\,a_2+324\sqrt{3}\,b_3\right)v_3v_1v_2^3\\
&+\left(972i\sqrt{3}\,b_2+972b_2\right)v_5v_1v_2^3+\left(486ib_1-162\sqrt{3}\,b_1\right)v_3v_1v_2^2\\
&+\left(486ia_2-162ib_3+162\sqrt{3}\,a_2-54\sqrt{3}\,b_3\right)v_3v_5v_2^2\\
&+\left(324ib_1+108\sqrt{3}\,b_1\right)v_3v_5v_2\\
&+\left(324ib_2+108\sqrt{3}\,b_2\right)v_3v_5v_1v_2\\
&+\left(54ia_3+18\sqrt{3}\,a_3\right)v_3v_5v_1^2v_2^2+648\sqrt{3}\,a_3v_3v_4v_1v_2^3\\
&+\left(-891i\sqrt{3}\,a_3+891a_3\right)v_1^3v_2^5+\left(81i\sqrt{3}\,a_1-81a_1\right)v_1^3v_2^4\\
&+\left(243i\sqrt{3}\,a_2-81i\sqrt{3}\,b_3-243a_2+81b_3\right)v_1^4v_2^4\\
&+\left(81i\sqrt{3}\,b_1-81b_1\right)v_1^4v_2^3\\
&+\left(1458i\sqrt{3}\,a_2-486i\sqrt{3}\,b_3+1458a_2-486b_3\right)v_5v_2^4\\
&+\left(972i\sqrt{3}\,b_1+972b_1\right)v_5v_2^3+\left(-7290ia_3+2430\sqrt{3}\,a_3\right)v_3v_2^4\\
&+\left(81i\sqrt{3}\,b_2-81b_2\right)v_1^5v_2^3+\left(1458ia_1-486\sqrt{3}\,a_1\right)v_3v_2^3\\
&+\left(8748i\sqrt{3}\,a_2-2916i\sqrt{3}\,b_3-8748a_2+2916b_3\right)v_1v_2^5\\
&+\left(1458i\sqrt{3}\,b_1-1458b_1\right)v_1v_2^4+\left(1458i\sqrt{3}\,b_2-1458b_2\right)v_1^2v_2^4\\
&+8748b_2v_4v_2^4+\left(4374i\sqrt{3}\,a_1-4374a_1\right)v_2^5\\
&+\left(-21870i\sqrt{3}\,a_3+21870a_3\right)v_2^6=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$216a_3 = 0$$
$$5832a_3 = 0$$
$$324b_2 = 0$$
$$8748b_2 = 0$$
$$648\sqrt{3}\, a_3 = 0$$
$$972\sqrt{3}\, b_2 = 0$$
$$-7290ia_3 + 2430\sqrt{3}\, a_3 = 0$$
$$-162ia_3 + 54\sqrt{3}\, a_3 = 0$$
$$54ia_3 + 18\sqrt{3}\, a_3 = 0$$
$$324ib_1 + 108\sqrt{3}\, b_1 = 0$$
$$324ib_2 + 108\sqrt{3}\, b_2 = 0$$
$$486ib_1 - 162\sqrt{3}\, b_1 = 0$$
$$486ib_2 - 162\sqrt{3}\, b_2 = 0$$
$$1458ia_1 - 486\sqrt{3}\, a_1 = 0$$
$$-21870i\sqrt{3}\, a_3 + 21870a_3 = 0$$
$$-891i\sqrt{3}\, a_3 + 891a_3 = 0$$
$$27i\sqrt{3}\, a_1 + 27a_1 = 0$$
$$27i\sqrt{3}\, a_3 + 27a_3 = 0$$
$$27i\sqrt{3}\, b_1 + 27b_1 = 0$$
$$27i\sqrt{3}\, b_2 + 27b_2 = 0$$
$$81i\sqrt{3}\, a_1 - 81a_1 = 0$$
$$81i\sqrt{3}\, b_1 - 81b_1 = 0$$
$$81i\sqrt{3}\, b_2 - 81b_2 = 0$$
$$972i\sqrt{3}\, b_1 + 972b_1 = 0$$
$$972i\sqrt{3}\, b_2 + 972b_2 = 0$$
$$1458i\sqrt{3}\, b_1 - 1458b_1 = 0$$
$$1458i\sqrt{3}\, b_2 - 1458b_2 = 0$$
$$4374i\sqrt{3}\, a_1 - 4374a_1 = 0$$
$$486ia_2 - 162ib_3 + 162\sqrt{3}\, a_2 - 54\sqrt{3}\, b_3 = 0$$
$$2916ia_2 - 972ib_3 - 972\sqrt{3}\, a_2 + 324\sqrt{3}\, b_3 = 0$$
$$81i\sqrt{3}\, a_2 - 27i\sqrt{3}\, b_3 + 81a_2 - 27b_3 = 0$$
$$243i\sqrt{3}\, a_2 - 81i\sqrt{3}\, b_3 - 243a_2 + 81b_3 = 0$$
$$1458i\sqrt{3}\, a_2 - 486i\sqrt{3}\, b_3 + 1458a_2 - 486b_3 = 0$$
$$8748i\sqrt{3}\, a_2 - 2916i\sqrt{3}\, b_3 - 8748a_2 + 2916b_3 = 0$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   trying dAlembert
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-2*y(x)*x^3-2*y(x)^3)/(-x^4-x*y(x)^2),
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 28

```
dsolve(diff(y(x),x)^3+diff(y(x),x)*x*y(x)=2*y(x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{x^3}{27}$$
$$y(x) = 0$$
$$y(x) = \frac{(c_1x+1)^2}{4c_1^3}$$

✓ Solution by Mathematica

Time used: 141.328 (sec). Leaf size: 10666

```
DSolve[y'[x]^3+y'[x]*x*y[x]==2*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

20.13 problem 13

Internal problem ID [2350]
Internal file name [`OUTPUT/2350_Tuesday_February_27_2024_08_34_14_AM_76160943/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 13.
ODE order: 1.
ODE degree: 4.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{3y'^4x - y'^3y = 1}$$

20.13.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$3p^4x - p^3y = 1$$

Solving for y from the above results in

$$y = 3px - \frac{1}{p^3} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 3p$$
$$g = -\frac{1}{p^3}$$

Hence (2) becomes

$$-2p = \left(3x + \frac{3}{p^4}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-2p = 0$$

Solving for p from the above gives

$$p = 0$$

None of these values lead to defined solutions. Hence no singular solutions exist

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{2p(x)}{3x + \frac{3}{p(x)^4}} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{3x(p) + \frac{3}{p^4}}{2p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{3}{2p}$$
$$q(p) = -\frac{3}{2p^5}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{3x(p)}{2p} = -\frac{3}{2p^5}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{2p}dp} \\ &= p^{\frac{3}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{3}{2p^5}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(p^{\frac{3}{2}}x\right) &= \left(p^{\frac{3}{2}}\right)\left(-\frac{3}{2p^5}\right) \\ \mathrm{d}\left(p^{\frac{3}{2}}x\right) &= \left(-\frac{3}{2p^{\frac{7}{2}}}\right)\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^{\frac{3}{2}}x &= \int -\frac{3}{2p^{\frac{7}{2}}}\,\mathrm{d}p \\ p^{\frac{3}{2}}x &= \frac{3}{5p^{\frac{5}{2}}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^{\frac{3}{2}}$ results in

$$x(p) = \frac{3}{5p^4} + \frac{c_1}{p^{\frac{3}{2}}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \mathrm{RootOf}\left(3x\_Z^4 - y\_Z^3 - 1\right)$$

Substituting the above in the solution for x found above gives

$$x = \frac{5\,\mathrm{RootOf}\left(3x\_Z^4 - y\_Z^3 - 1\right)^3 c_1 y + 9\,\mathrm{RootOf}\left(3x\_Z^4 - y\_Z^3 - 1\right)^{\frac{3}{2}} x + 5c_1}{15x\,\mathrm{RootOf}\left(3x\_Z^4 - y\_Z^3 - 1\right)^{\frac{11}{2}}}$$

Summary

The solution(s) found are the following

$$x = \frac{5\,\mathrm{RootOf}\left(3x\_Z^4 - y\_Z^3 - 1\right)^3 c_1 y + 9\,\mathrm{RootOf}\left(3x\_Z^4 - y\_Z^3 - 1\right)^{\frac{3}{2}} x + 5c_1}{15x\,\mathrm{RootOf}\left(3x\_Z^4 - y\_Z^3 - 1\right)^{\frac{11}{2}}} \tag{1}$$

Verification of solutions

$$x = \frac{5 \operatorname{RootOf}\left(3x\_Z^4 - y\_Z^3 - 1\right)^3 c_1 y + 9 \operatorname{RootOf}\left(3x\_Z^4 - y\_Z^3 - 1\right)^{\frac{3}{2}} x + 5c_1}{15x \operatorname{RootOf}\left(3x\_Z^4 - y\_Z^3 - 1\right)^{\frac{11}{2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 34

```
dsolve(3*diff(y(x),x)^4*x=diff(y(x),x)^3*y(x)+1,y(x), singsol=all)
```

$$\left[x(\_T) = \frac{5c_1\_T^{\frac{5}{2}} + 3}{5\_T^4}, y(\_T) = \frac{15c_1\_T^{\frac{5}{2}} + 4}{5\_T^3}\right]$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[3*y'[x]^4*x==y'[x]^3*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

20.14 problem 14

Internal problem ID [2351]
Internal file name [`OUTPUT/2351_Tuesday_February_27_2024_08_34_16_AM_57941012/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 14.
ODE order: 1.
ODE degree: 5.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_dAlembert]`

$$\boxed{2y'^5 + 2xy' - y = 0}$$

20.14.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$2p^5 + 2xp - y = 0$$

Solving for y from the above results in

$$y = 2p^5 + 2xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 2p \\
g &= 2p^5
\end{aligned}
$$

Hence (2) becomes

$$
-p = \left(10p^4 + 2x\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = -\frac{p(x)}{10p\left(x\right)^4 + 2x} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = -\frac{10p^4 + 2x(p)}{p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{2}{p} \\
q(p) &= -10p^3
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p)+\frac{2x(p)}{p}=-10p^3$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{p}dp}\\ &= p^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-10p^3\right)\\ \frac{\mathrm{d}}{\mathrm{d}p}\left(p^2x\right) &= \left(p^2\right)\left(-10p^3\right)\\ \mathrm{d}\left(p^2x\right) &= \left(-10p^5\right)\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^2x &= \int -10p^5\,\mathrm{d}p\\ p^2x &= -\frac{5p^6}{3}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=p^2$ results in

$$x(p)=-\frac{5p^4}{3}+\frac{c_1}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p=\mathrm{RootOf}\left(2\_Z^5+2\_Zx-y\right)$$

Substituting the above in the solution for x found above gives

$$x=\frac{10x\,\mathrm{RootOf}\left(2\_Z^5+2\_Zx-y\right)^2-5y\,\mathrm{RootOf}\left(2\_Z^5+2\_Zx-y\right)+6c_1}{6\,\mathrm{RootOf}\left(2\_Z^5+2\_Zx-y\right)^2}$$

Summary

The solution(s) found are the following

$$y=0 \tag{1}$$

$$x=\frac{10x\,\mathrm{RootOf}\left(2\_Z^5+2\_Zx-y\right)^2-5y\,\mathrm{RootOf}\left(2\_Z^5+2\_Zx-y\right)+6c_1}{6\,\mathrm{RootOf}\left(2\_Z^5+2\_Zx-y\right)^2} \tag{2}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{10x \operatorname{RootOf}\left(2\_Z^5 + 2\_Zx - y\right)^2 - 5y \operatorname{RootOf}\left(2\_Z^5 + 2\_Zx - y\right) + 6c_1}{6 \operatorname{RootOf}\left(2\_Z^5 + 2\_Zx - y\right)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 1311

```
dsolve(2*diff(y(x),x)^5+2*diff(y(x),x)*x=y(x),y(x), singsol=all)
```

$$y(x) = \frac{20\sqrt{5}\sqrt{-\left(300c_1 + 20\sqrt{20x^3 + 225c_1^2}\right)^{\frac{1}{3}}\left(i\sqrt{3}\left(300c_1 + 20\sqrt{20x^3 + 225c_1^2}\right)^{\frac{2}{3}} + 20i\sqrt{3}\,x + \left(300c_1 + 2\right.\right.}}{\left(300c\right.}$$

$$y(x) = -\frac{20\sqrt{5}\sqrt{-\left(300c_1 + 20\sqrt{20x^3 + 225c_1^2}\right)^{\frac{1}{3}}\left(i\sqrt{3}\left(300c_1 + 20\sqrt{20x^3 + 225c_1^2}\right)^{\frac{2}{3}} + 20i\sqrt{3}\,x + \left(300c_1 +\right.\right.}}{\left(3\right.}$$

$$y(x) = -\frac{20\sqrt{5}\left(-\frac{3\left(c_1+\frac{\sqrt{20x^3+225c_1^2}}{15}\right)\left(1+i\sqrt{3}\right)\left(300c_1+20\sqrt{20x^3+225c_1^2}\right)^{\frac{2}{3}}}{4} + \left(\left(i\sqrt{3}-1\right)x\left(300c_1 + 20\sqrt{20x^3 + 225c_1^2}\right)\right.\right.}{\left(300\right.}$$

$$y(x) = \frac{20\sqrt{5}\left(-\frac{3\left(c_1+\frac{\sqrt{20x^3+225c_1^2}}{15}\right)\left(1+i\sqrt{3}\right)\left(300c_1+20\sqrt{20x^3+225c_1^2}\right)^{\frac{2}{3}}}{4} + \left(\left(i\sqrt{3}-1\right)x\left(300c_1 + 20\sqrt{20x^3 + 225c_1^2}\right)^{\frac{1}{3}}\right.\right.}{\left(300c_1\right.}$$

$$y(x) = -\frac{\left(\frac{\left(3c_1+\frac{\sqrt{20x^3+225c_1^2}}{5}\right)\left(300c_1+20\sqrt{20x^3+225c_1^2}\right)^{\frac{2}{3}}}{4} + x\left(x\left(300c_1 + 20\sqrt{20x^3 + 225c_1^2}\right)^{\frac{1}{3}} + 45c_1 + 3\sqrt{20x^3 + 2}\right.\right.}{25\left(300c_1 + 20\sqrt{20x^3 + 225c_1^2}\right)^{\frac{1}{3}}\left(15\right.}$$

$$y(x) = \frac{\left(\frac{\left(3c_1+\frac{\sqrt{20x^3+225c_1^2}}{5}\right)\left(300c_1+20\sqrt{20x^3+225c_1^2}\right)^{\frac{2}{3}}}{4} + x\left(x\left(300c_1 + 20\sqrt{20x^3 + 225c_1^2}\right)^{\frac{1}{3}} + 45c_1 + 3\sqrt{20x^3 + 225c}\right.\right.}{25\left(300c_1 + 20\sqrt{20x^3 + 225c_1^2}\right)^{\frac{1}{3}}\left(15c_1\right.}$$

✓ Solution by Mathematica

Time used: 2.303 (sec). Leaf size: 2226

```
DSolve[2*y'[x]^5+2*y'[x]*x==y[x],y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

20.15 problem 15

Internal problem ID [2352]
Internal file name [`OUTPUT/2352_Tuesday_February_27_2024_08_34_16_AM_70630014/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 15.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{\frac{1}{y'^2} + xy' - 2y = 0}$$

20.15.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$\frac{1}{p^2} + xp - 2y = 0$$

Solving for y from the above results in

$$y = \frac{xp}{2} + \frac{1}{2p^2} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p}{2}$$
$$g = \frac{1}{2p^2}$$

Hence (2) becomes

$$\frac{p}{2} = \left(\frac{x}{2} - \frac{1}{p^3}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$\frac{p}{2} = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = \infty$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x)}{x - \frac{2}{p(x)^3}} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p) - \frac{2}{p^3}}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{1}{p}$$
$$q(p) = -\frac{2}{p^4}$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)}{p} = -\frac{2}{p^4}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{p}dp} \\ &= \frac{1}{p}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{2}{p^4}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{x}{p}\right) &= \left(\frac{1}{p}\right)\left(-\frac{2}{p^4}\right) \\ \mathrm{d}\left(\frac{x}{p}\right) &= \left(-\frac{2}{p^5}\right)\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{x}{p} &= \int -\frac{2}{p^5}\,\mathrm{d}p \\ \frac{x}{p} &= \frac{1}{2p^4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{p}$ results in

$$x(p) = \frac{1}{2p^3} + c_1 p$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}}{6x} + \frac{8y^2}{3x\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}} + \frac{2y}{3x}$$

$$p = -\frac{\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}}{12x} - \frac{4y^2}{3x\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}} + \frac{2y}{3x}$$

$$p = -\frac{\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}}{12x} - \frac{4y^2}{3x\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}} + \frac{2y}{3x}$$

Substituting the above in the solution for x found above gives

$$x = \frac{108x^3}{\left(\frac{16y^2}{\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}} + 4y + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}\right)^3} + \frac{c_1\left(\frac{16y^2}{\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}} + 4y + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}\right)}{6x}$$

$$x = \frac{-10368x^4\sqrt{3}\,\sqrt{-32y^3+27x^2} - 55296y^3x^3 + \cdots}{\left(-i\sqrt{3}\,\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} + 16i\sqrt{3}\,y^2 + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-10\cdots\right.} + \frac{\left(\left(-1+i\sqrt{3}\right)\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} + 8y\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108\cdots\right.\right.}{12\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}x}$$

$$x = \frac{3456\left(-3\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x - 16y^3 + 27x\cdots\right.}{\left(i\sqrt{3}\,\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} - 16i\sqrt{3}\,y^2 + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108\cdots\right.} - \frac{\left(\left(1+i\sqrt{3}\right)\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} - 8y\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x\cdots\right.\right.}{12\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}x}$$

Summary

The solution(s) found are the following

$$y = \infty \tag{1}$$

$$x = \frac{108x^3}{\left(\frac{16y^2}{\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}} + 4y + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}\right)^3} + \frac{c_1\left(\frac{16y^2}{\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}} + 4y + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}\right)}{6x} \tag{2}$$

$$x = \frac{-10368x^4\sqrt{3}\,\sqrt{-32y^3+27x^2} - 55296y^3x^3 + \cdots}{\left(-i\sqrt{3}\,\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} + 16i\sqrt{3}\,y^2 + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-10\cdots\right.\right.} + \frac{\left(\left(-1+i\sqrt{3}\right)\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} + 8y\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108\cdots\right.\right.}{12\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}x} \tag{3}$$

$$x = \frac{3456\left(-3\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x - 16y^3 + 27x\cdots\right.}{\left(i\sqrt{3}\,\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} - 16i\sqrt{3}\,y^2 + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108\cdots\right.\right.} - \frac{\left(\left(1+i\sqrt{3}\right)\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} - 8y\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x\cdots\right.\right.}{12\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}x} \tag{4}$$

Verification of solutions

$$y = \infty$$

Warning, solution could not be verified

$$x = \frac{108x^3}{\left(\frac{16y^2}{\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}} + 4y + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}\right)^3} + \frac{c_1\left(\frac{16y^2}{\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}} + 4y + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}\right)}{6x}$$

Verified OK.

$$x = \frac{-10368x^4\sqrt{3}\,\sqrt{-32y^3+27x^2} - 55296y^3x^3 + 9}{\left(-i\sqrt{3}\,\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} + 16i\sqrt{3}\,y^2 + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-10\right.\right.} + \frac{\left(\left(-1+i\sqrt{3}\right)\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} + 8y\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108\right.\right.}{12\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}x}$$

Warning, solution could not be verified

$$x = \frac{3456\left(-3\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x-16y^3+27x\right.}{\left(i\sqrt{3}\,\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} - 16i\sqrt{3}\,y^2 + \left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108\right.\right.} - \frac{\left(\left(1+i\sqrt{3}\right)\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{2}{3}} - 8y\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x\right.\right.}{12\left(12\sqrt{3}\,\sqrt{-32y^3+27x^2}\,x+64y^3-108x^2\right)^{\frac{1}{3}}x}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 1441

```
dsolve(1/diff(y(x),x)^2+diff(y(x),x)*x=2*y(x),y(x), singsol=all)
```

$$\frac{279936\left(-\frac{x\left(-\frac{4x^2y(x)^2}{9}-\frac{16y(x)^3c_1}{3}+c_1x^2\right)\sqrt{-96y(x)^3+81x^2}}{108}-\frac{40y(x)^3c_1x^2}{81}-\frac{x^4y(x)^2}{27}+\frac{c_1x^4}{12}+\frac{32y(x)^6c_1}{81}+\frac{8x^2y(x)^5}{243}\right)\left(12x\cdots\right.}{\cdots}$$

$$= 0$$

$$\frac{1119744\left(\frac{\left(i-\frac{\sqrt{3}}{3}\right)x\left(-\frac{16x^2y(x)^2}{9}-\frac{16y(x)^3c_1}{3}+c_1x^2\right)\sqrt{-32y(x)^3+27x^2}}{144}-\frac{\left(i\sqrt{3}-1\right)\left(\frac{128x^2y(x)^5}{81}+\frac{128y(x)^6c_1}{27}-\frac{16x^4y(x)^2}{9}-\frac{160y(x)^3c_1x^2}{27}+\cdots\right.}{48}\cdots}{\left(12x\sqrt{\cdots}\right.\cdots}$$

$$= 0$$

$$\frac{1119744\left(\frac{\left(i+\frac{\sqrt{3}}{3}\right)x\left(-\frac{16x^2y(x)^2}{9}-\frac{16y(x)^3c_1}{3}+c_1x^2\right)\sqrt{-32y(x)^3+27x^2}}{144}-\frac{\left(\frac{128x^2y(x)^5}{81}+\frac{128y(x)^6c_1}{27}-\frac{16x^4y(x)^2}{9}-\frac{160y(x)^3c_1x^2}{27}+c_1x^4\right)\left(1+\cdots\right.}{48}\cdots}{\left(12x\sqrt{\cdots}\right.\cdots}$$

$$= 0$$

✓ Solution by Mathematica

Time used: 149.881 (sec). Leaf size: 10773

```
DSolve[1/(y'[x]^2)+y'[x]*x==2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

20.16 problem 16

Internal problem ID [2353]
Internal file name [`OUTPUT/2353_Tuesday_February_27_2024_08_36_02_AM_22694948/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 38, page 173
Problem number: 16.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{2y - 3xy' - 2\ln(y') = 4}$$

20.16.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$2y - 3xp - 2\ln(p) = 4$$

Solving for y from the above results in

$$y = \frac{3xp}{2} + \ln(p) + 2 \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{3p}{2} \\
g &= \ln(p) + 2
\end{aligned}
$$

Hence (2) becomes

$$
-\frac{p}{2} = \left(\frac{3x}{2} + \frac{1}{p}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-\frac{p}{2} = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = -\infty
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = -\frac{p(x)}{2\left(\frac{3x}{2} + \frac{1}{p(x)}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = -\frac{2\left(\frac{3x(p)}{2} + \frac{1}{p}\right)}{p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{3}{p} \\
q(p) &= -\frac{2}{p^2}
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{3x(p)}{p} = -\frac{2}{p^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{p}dp} \\ &= p^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dp}(\mu x) &= (\mu)\left(-\frac{2}{p^2}\right) \\ \frac{d}{dp}(p^3 x) &= (p^3)\left(-\frac{2}{p^2}\right) \\ d(p^3 x) &= (-2p)\, dp\end{aligned}$$

Integrating gives

$$\begin{aligned}p^3 x &= \int -2p\, dp \\ p^3 x &= -p^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^3$ results in

$$x(p) = -\frac{1}{p} + \frac{c_1}{p^3}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = e^{-\operatorname{LambertW}\left(\frac{3x\,e^{y-2}}{2}\right)+y-2}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{3x\left(-9c_1x^2 + 4\operatorname{LambertW}\left(\frac{3x\,e^{y-2}}{2}\right)^2\right)}{8\operatorname{LambertW}\left(\frac{3x\,e^{y-2}}{2}\right)^3}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\infty \tag{1}$$

$$x = -\frac{3x\left(-9c_1x^2 + 4\operatorname{LambertW}\left(\frac{3x\,e^{y-2}}{2}\right)^2\right)}{8\operatorname{LambertW}\left(\frac{3x\,e^{y-2}}{2}\right)^3} \tag{2}$$

Verification of solutions

$$y = -\infty$$

Warning, solution could not be verified

$$x = -\frac{3x\left(-9c_1x^2 + 4\,\mathrm{LambertW}\left(\frac{3x\,\mathrm{e}^{y-2}}{2}\right)^2\right)}{8\,\mathrm{LambertW}\left(\frac{3x\,\mathrm{e}^{y-2}}{2}\right)^3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 827

```
dsolve(2*y(x)=3*diff(y(x),x)*x+4+2*ln(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \frac{\ln\left(\frac{\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108c_1x^2-8\right)^{\frac{2}{3}}-2\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108c_1x^2-8\right)^{\frac{1}{3}}+4}{x\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108c_1x^2-8\right)^{\frac{1}{3}}}\right)\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108c_{1}\cdots\right.}{\left(12\sqrt{3}\,\sqrt{27c_1^2x^{2}\cdots}\right.}$$

$$y(x) = -\frac{-8\ln\left(-\frac{\left(\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108c_1x^2-8\right)^{\frac{1}{3}}+2\right)\left(2+i\left(\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108c_1x^2-8\right)^{\frac{1}{3}}-2\right)\sqrt{3}+\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-\cdots}\right.\right.}{x\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108c_1x^2-8\right)^{\frac{1}{3}}}\right.}{\cdots}$$

$$y(x) = \frac{8\ln\left(\frac{\left(-2+i\left(\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108c_1x^2-8\right)^{\frac{1}{3}}-2\right)\sqrt{3}-\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108c_1x^2-8\right)^{\frac{1}{3}}\right)\left(\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108\cdots\right.\right.}{x\left(12\sqrt{3}\,\sqrt{27c_1^2x^2-4c_1}\,x+108c_1x^2-8\right)^{\frac{1}{3}}}\right.}{\cdots}$$

✓ Solution by Mathematica

Time used: 0.929 (sec). Leaf size: 137

```
DSolve[2*y[x]==3*y'[x]*x+4+2*Log[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\left(2W\left(-\frac{3}{2}\sqrt{x^2e^{2y(x)-4}}\right)-\log\left(2W\left(-\frac{3}{2}\sqrt{x^2e^{2y(x)-4}}\right)+3\right)+3\right)-y(x)=c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{1}{2}\left(2W\left(\frac{3}{2}\sqrt{x^2e^{2y(x)-4}}\right)-\log\left(2W\left(\frac{3}{2}\sqrt{x^2e^{2y(x)-4}}\right)+3\right)+3\right)-y(x)=c_1,y(x)\right]$$

21 Exercise 39, page 179

21.1 problem 23

Internal problem ID [2354]
Internal file name [`OUTPUT/2354_Tuesday_February_27_2024_08_36_04_AM_36481757/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 39, page 179
Problem number: 23.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y - xy' - {y'}^2 = 0}$$

21.1.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$-p^2 - xp + y = 0$$

Solving for y from the above results in

$$y = p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= p^2 + xp \\ &= p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1^2 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = p^2$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x + 2p \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = -\frac{x}{2}$$

Substituting the above back in (1) results in

$$y_1 = -\frac{x^2}{4}$$

Summary

The solution(s) found are the following

$$y = c_1^2 + c_1 x \tag{1}$$

$$y = -\frac{x^2}{4} \tag{2}$$

Verification of solutions

$$y = c_1^2 + c_1 x$$

Verified OK.

$$y = -\frac{x^2}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 17

```
dsolve(y(x)=diff(y(x),x)*x+diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{4}$$
$$y(x) = c_1(c_1 + x)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 23

```
DSolve[y[x]==y'[x]*x+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x + c_1)$$
$$y(x) \to -\frac{x^2}{4}$$

21.2 problem 24

Internal problem ID [2355]
Internal file name [`OUTPUT/2355_Tuesday_February_27_2024_08_36_04_AM_11769021/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 39, page 179
Problem number: 24.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_homogeneous, `class G`], _rational, _Clairaut]`

$$\boxed{y - xy' - \frac{1}{y'} = 0}$$

21.2.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - xp - \frac{1}{p} = 0$$

Solving for y from the above results in

$$y = \frac{x\,p^2 + 1}{p} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \frac{1}{p} \\ &= xp + \frac{1}{p} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \frac{1}{p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{1}{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{1}{p}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{1}{p^2} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{1}{\sqrt{x}}$$
$$p_2 = -\frac{1}{\sqrt{x}}$$

Substituting the above back in (1) results in

$$y_1 = 2\sqrt{x}$$
$$y_2 = -2\sqrt{x}$$

Summary
The solution(s) found are the following

$$y = c_1 x + \frac{1}{c_1} \tag{1}$$
$$y = 2\sqrt{x} \tag{2}$$
$$y = -2\sqrt{x} \tag{3}$$

Verification of solutions

$$y = c_1 x + \frac{1}{c_1}$$

Verified OK.

$$y = 2\sqrt{x}$$

Verified OK.

$$y = -2\sqrt{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 27

```
dsolve(y(x)=diff(y(x),x)*x+1/diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -2\sqrt{x}$$
$$y(x) = 2\sqrt{x}$$
$$y(x) = c_1 x + \frac{1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 41

```
DSolve[y[x]==y'[x]*x+1/y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 x + \frac{1}{c_1}$$
$$y(x) \rightarrow \text{Indeterminate}$$
$$y(x) \rightarrow -2\sqrt{x}$$
$$y(x) \rightarrow 2\sqrt{x}$$

21.3 problem 25

Internal problem ID [2356]
Internal file name [`OUTPUT/2356_Tuesday_February_27_2024_08_36_05_AM_55688224/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 39, page 179
Problem number: 25.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _Clairaut]
```

$$\boxed{y - xy' + \sqrt{y'} = 0}$$

21.3.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - xp + \sqrt{p} = 0$$

Solving for y from the above results in

$$y = xp - \sqrt{p} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp - \sqrt{p} \\ &= xp - \sqrt{p} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -\sqrt{p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x - \sqrt{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\sqrt{p}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{1}{2\sqrt{p}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{1}{4x^2}$$

Substituting the above back in (1) results in

$$y_1 = \frac{\left(\operatorname{csgn}\left(\frac{1}{x}\right) - 2\right)\sqrt{\frac{1}{x^2}}}{4}$$

Simplifying the solution $y = \frac{\left(\operatorname{csgn}\left(\frac{1}{x}\right)-2\right)\sqrt{\frac{1}{x^2}}}{4}$ to $y = -\frac{\sqrt{\frac{1}{x^2}}}{4}$

Summary

The solution(s) found are the following

$$y = c_1 x - \sqrt{c_1}$$

$$y = -\frac{\sqrt{\frac{1}{x^2}}}{4}$$

Verification of solutions

$$y = c_1 x - \sqrt{c_1}$$

Verified OK.

$$y = -\frac{\sqrt{\frac{1}{x^2}}}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 21

```
dsolve(y(x)=diff(y(x),x)*x-sqrt(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = -\frac{1}{4x}$$
$$y(x) = c_1 x - \sqrt{c_1}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 23

```
DSolve[y[x]==y'[x]*x-Sqrt[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \sqrt{c_1}$$
$$y(x) \to 0$$

21.4 problem 26

Internal problem ID [2357]
Internal file name [`OUTPUT/2357_Tuesday_February_27_2024_08_36_05_AM_74284253/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 39, page 179
Problem number: 26.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y - xy' - \ln(y') = 0}$$

21.4.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - xp - \ln(p) = 0$$

Solving for y from the above results in

$$y = xp + \ln(p) \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \ln(p) \\ &= xp + \ln(p) \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \ln(p)$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \ln(c_1)$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \ln(p)$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{1}{p} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = -\frac{1}{x}$$

Substituting the above back in (1) results in

$$y_1 = \ln\left(-\frac{1}{x}\right) - 1$$

Summary

The solution(s) found are the following

$$y = c_1 x + \ln(c_1) \tag{1}$$

$$y = \ln\left(-\frac{1}{x}\right) - 1 \tag{2}$$

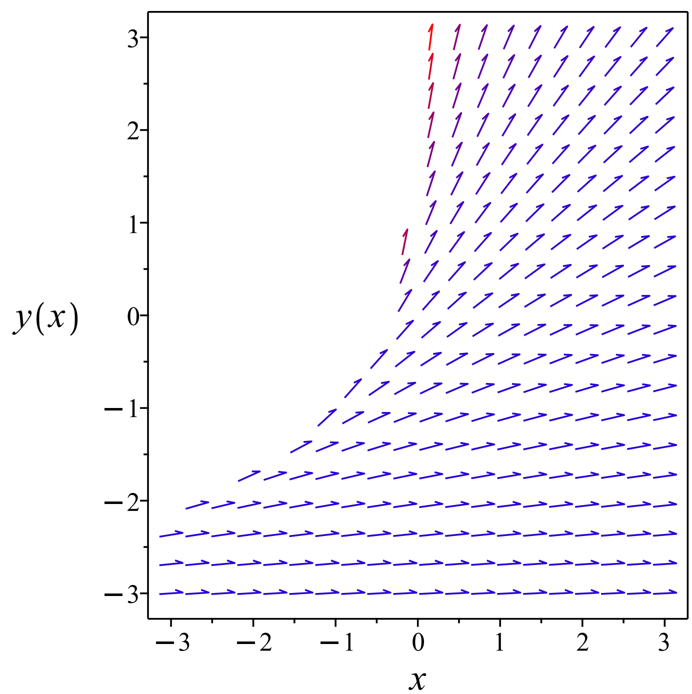


Figure 732: Slope field plot

Verification of solutions

$$y = c_1 x + \ln(c_1)$$

Verified OK.

$$y = \ln\left(-\frac{1}{x}\right) - 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- 1st order, parametric methods successful
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(y(x)=diff(y(x),x)*x+ln(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \ln\left(-\frac{1}{x}\right) - 1$$
$$y(x) = c_1 x + \ln(c_1)$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 25

```
DSolve[y[x]==y'[x]*x+Log[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x + \log(c_1)$$
$$y(x) \to \log\left(-\frac{1}{x}\right) - 1$$

21.5 problem 27

Internal problem ID [2358]
Internal file name [`OUTPUT/2358_Tuesday_February_27_2024_08_36_06_AM_26775764/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 39, page 179
Problem number: 27.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y - xy' - \frac{3}{y'^2} = 0}$$

21.5.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - xp - \frac{3}{p^2} = 0$$

Solving for y from the above results in

$$y = \frac{x\,p^3 + 3}{p^2} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \frac{3}{p^2} \\ &= xp + \frac{3}{p^2} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \frac{3}{p^2}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{3}{c_1^2}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{3}{p^2}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{6}{p^3} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{6^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{x}$$
$$p_2 = -\frac{6^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x} + \frac{i\sqrt{3}\, 6^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x}$$
$$p_3 = -\frac{6^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x} - \frac{i\sqrt{3}\, 6^{\frac{1}{3}} \left(x^2\right)^{\frac{1}{3}}}{2x}$$

Substituting the above back in (1) results in

$$y_1 = \frac{3x^2 6^{\frac{1}{3}}}{2 \left(x^2\right)^{\frac{2}{3}}}$$
$$y_2 = -\frac{9x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(3i3^{\frac{1}{6}} + 3^{\frac{2}{3}}\right)}$$
$$y_3 = \frac{9x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(3i3^{\frac{1}{6}} - 3^{\frac{2}{3}}\right)}$$

Summary

The solution(s) found are the following

$$y = c_1 x + \frac{3}{c_1^2} \tag{1}$$
$$y = \frac{3x^2 6^{\frac{1}{3}}}{2 \left(x^2\right)^{\frac{2}{3}}} \tag{2}$$
$$y = -\frac{9x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(3i3^{\frac{1}{6}} + 3^{\frac{2}{3}}\right)} \tag{3}$$
$$y = \frac{9x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(3i3^{\frac{1}{6}} - 3^{\frac{2}{3}}\right)} \tag{4}$$

Verification of solutions

$$y = c_1 x + \frac{3}{c_1^2}$$

Verified OK.

$$y = \frac{3x^2 6^{\frac{1}{3}}}{2 \left(x^2\right)^{\frac{2}{3}}}$$

Verified OK.

$$y = -\frac{9x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(3i 3^{\frac{1}{6}} + 3^{\frac{2}{3}}\right)}$$

Verified OK.

$$y = \frac{9x^2 2^{\frac{1}{3}}}{\left(x^2\right)^{\frac{2}{3}} \left(3i 3^{\frac{1}{6}} - 3^{\frac{2}{3}}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 74

```
dsolve(y(x)=diff(y(x),x)*x+3/diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = \frac{3\,6^{\frac{1}{3}} (x^2)^{\frac{1}{3}}}{2}$$
$$y(x) = -\frac{3\,2^{\frac{1}{3}} \left(i3^{\frac{5}{6}} + 3^{\frac{1}{3}}\right) (x^2)^{\frac{1}{3}}}{4}$$
$$y(x) = \frac{3(x^2)^{\frac{1}{3}}\, 2^{\frac{1}{3}} \left(i3^{\frac{5}{6}} - 3^{\frac{1}{3}}\right)}{4}$$
$$y(x) = c_1 x + \frac{3}{c_1^2}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 84

```
DSolve[y[x]==y'[x]*x+3/y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x + \frac{3}{c_1{}^2}$$
$$y(x) \to -\frac{3\sqrt[3]{-3} x^{2/3}}{2^{2/3}}$$
$$y(x) \to \frac{3\sqrt[3]{3} x^{2/3}}{2^{2/3}}$$
$$y(x) \to \frac{3(-1)^{2/3}\sqrt[3]{3} x^{2/3}}{2^{2/3}}$$

21.6 problem 28

Internal problem ID [2359]
Internal file name [OUTPUT/2359_Tuesday_February_27_2024_08_36_06_AM_99890233/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 39, page 179
Problem number: 28.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y - xy' + y'^{\frac{2}{3}} = 0}$$

21.6.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - xp + p^{\frac{2}{3}} = 0$$

Solving for y from the above results in

$$y = -p^{\frac{2}{3}} + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= -p^{\frac{2}{3}} + xp \\ &= -p^{\frac{2}{3}} + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -p^{\frac{2}{3}}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x - c_1^{\frac{2}{3}}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -p^{\frac{2}{3}}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{2}{3p^{\frac{1}{3}}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{8}{27x^3}$$

Substituting the above back in (1) results in

$$y_1 = \frac{-\frac{4\left(\frac{1}{x^3}\right)^{\frac{2}{3}} x^2}{9} + \frac{8}{27}}{x^2}$$

Summary

The solution(s) found are the following

$$y = c_1 x - c_1^{\frac{2}{3}} \tag{1}$$

$$y = \frac{-\frac{4\left(\frac{1}{x^3}\right)^{\frac{2}{3}} x^2}{9} + \frac{8}{27}}{x^2} \tag{2}$$

Verification of solutions

$$y = c_1 x - c_1^{\frac{2}{3}}$$

Verified OK.

$$y = \frac{-\frac{4\left(\frac{1}{x^3}\right)^{\frac{2}{3}} x^2}{9} + \frac{8}{27}}{x^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 25

```
dsolve(y(x)=diff(y(x),x)*x-diff(y(x),x)^(2/3),y(x), singsol=all)
```

$$y(x) = -\frac{4}{27x^2}$$
$$y(x) = 0$$
$$y(x) = c_1 x - c_1^{\frac{2}{3}}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 23

```
DSolve[y[x]==y'[x]*x-y'[x]^(2/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - c_1{}^{2/3}$$
$$y(x) \to 0$$

21.7 problem 29

Internal problem ID [2360]
Internal file name [`OUTPUT/2360_Tuesday_February_27_2024_08_36_07_AM_56273919/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 39, page 179
Problem number: 29.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y - xy' - e^{y'} = 0}$$

21.7.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - xp - e^{p} = 0$$

Solving for y from the above results in

$$y = xp + e^{p} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + e^{p} \\ &= xp + e^{p} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = e^p$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + e^{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = e^p$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + e^p \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \ln(-x)$$

Substituting the above back in (1) results in

$$y_1 = x(\ln(-x) - 1)$$

Summary

The solution(s) found are the following

$$y = c_1 x + e^{c_1} \tag{1}$$

$$y = x(\ln(-x) - 1) \tag{2}$$

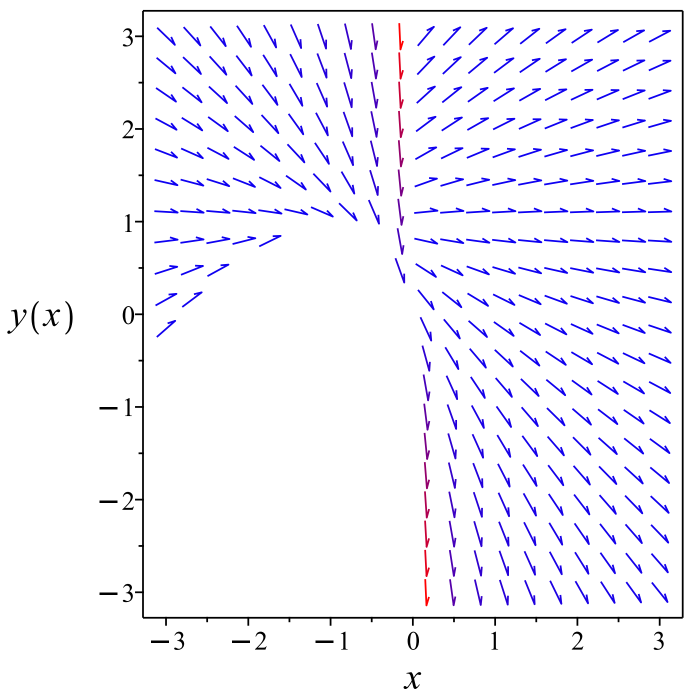


Figure 733: Slope field plot

Verification of solutions

$$y = c_1 x + e^{c_1}$$

Verified OK.

$$y = x(\ln(-x) - 1)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- 1st order, parametric methods successful
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(y(x)=diff(y(x),x)*x+exp(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = x(\ln(-x) - 1)$$
$$y(x) = c_1 x + e^{c_1}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 26

```
DSolve[y[x]==y'[x]*x+Exp[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x + e^{c_1}$$
$$y(x) \to x(\log(-x) - 1)$$

21.8 problem 30

Internal problem ID [2361]
Internal file name [`OUTPUT/2361_Tuesday_February_27_2024_08_36_08_AM_33399926/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 39, page 179
Problem number: 30.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _rational, _Clairaut]`

$$\boxed{(y - xy')^2 - y'^2 = 1}$$

21.8.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$(-xp + y)^2 - p^2 = 1$$

Solving for y from the above results in

$$y = xp + \sqrt{p^2 + 1} \tag{1A}$$

$$y = xp - \sqrt{p^2 + 1} \tag{2A}$$

Each of the above ode's is a Clairaut ode which is now solved. Solving ode 1A We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \sqrt{p^2 + 1} \\ &= xp + \sqrt{p^2 + 1} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \sqrt{p^2 + 1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \sqrt{c_1^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \sqrt{p^2 + 1}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{p}{\sqrt{p^2 + 1}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = -x\sqrt{-\frac{1}{x^2-1}}$$

Substituting the above back in (1) results in

$$y_1 = \left(-x^2+1\right)\sqrt{-\frac{1}{x^2-1}}$$

Solving ode 2A We start by replacing y' by p which gives

$$\begin{aligned} y &= xp - \sqrt{p^2+1} \\ &= xp - \sqrt{p^2+1} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -\sqrt{p^2+1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp+g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x+g')\frac{dp}{dx} \\ 0 &= (x+g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_2 x - \sqrt{c_2^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\sqrt{p^2 + 1}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - \frac{p}{\sqrt{p^2 + 1}} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = x\sqrt{-\frac{1}{x^2 - 1}}$$

Substituting the above back in (1) results in

$$y_1 = \sqrt{-\frac{1}{x^2 - 1}}\,(x^2 - 1)$$

Summary
The solution(s) found are the following

$$y = c_1 x + \sqrt{c_1^2 + 1} \tag{1}$$

$$y = (-x^2 + 1)\sqrt{-\frac{1}{x^2 - 1}} \tag{2}$$

$$y = c_2 x - \sqrt{c_2^2 + 1} \tag{3}$$

$$y = \sqrt{-\frac{1}{x^2 - 1}}\,(x^2 - 1) \tag{4}$$

Verification of solutions

$$y = c_1x + \sqrt{c_1^2 + 1}$$

Verified OK.

$$y = (-x^2 + 1)\sqrt{-\frac{1}{x^2 - 1}}$$

Verified OK.

$$y = c_2x - \sqrt{c_2^2 + 1}$$

Verified OK.

$$y = \sqrt{-\frac{1}{x^2 - 1}}\,(x^2 - 1)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 57

```
dsolve((y(x)-diff(y(x),x)*x)^2=diff(y(x),x)^2+1,y(x), singsol=all)
```

$$y(x) = \sqrt{-x^2+1}$$
$$y(x) = -\sqrt{-x^2+1}$$
$$y(x) = c_1 x - \sqrt{c_1^2+1}$$
$$y(x) = c_1 x + \sqrt{c_1^2+1}$$

✓ Solution by Mathematica

Time used: 0.141 (sec). Leaf size: 73

```
DSolve[(y[x]-y'[x]*x)^2==y'[x]^2+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \sqrt{1+c_1{}^2}$$
$$y(x) \to c_1 x + \sqrt{1+c_1{}^2}$$
$$y(x) \to -\sqrt{1-x^2}$$
$$y(x) \to \sqrt{1-x^2}$$

21.9 problem 31

Internal problem ID [2362]
Internal file name [`OUTPUT/2362_Tuesday_February_27_2024_08_36_08_AM_56211857/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 39, page 179
Problem number: 31.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Clairaut]
```

$$\boxed{{y'}^2 x - y y' = 2}$$

21.9.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = x y' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$p^2 x - y p = 2$$

Solving for y from the above results in

$$y = \frac{p^2 x - 2}{p} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= p x - \frac{2}{p} \\ &= p x - \frac{2}{p} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = -\frac{2}{p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x - \frac{2}{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\frac{2}{p}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{2}{p^2} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = -\frac{2}{\sqrt{-2x}}$$
$$p_2 = \frac{2}{\sqrt{-2x}}$$

Substituting the above back in (1) results in

$$y_1 = 2\sqrt{2}\,\sqrt{-x}$$
$$y_2 = -2\sqrt{2}\,\sqrt{-x}$$

Summary
The solution(s) found are the following

$$y = c_1 x - \frac{2}{c_1} \tag{1}$$
$$y = 2\sqrt{2}\,\sqrt{-x} \tag{2}$$
$$y = -2\sqrt{2}\,\sqrt{-x} \tag{3}$$

Verification of solutions

$$y = c_1 x - \frac{2}{c_1}$$

Verified OK.

$$y = 2\sqrt{2}\,\sqrt{-x}$$

Verified OK.

$$y = -2\sqrt{2}\,\sqrt{-x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 39

```
dsolve(diff(y(x),x)^2*x-diff(y(x),x)*y(x)-2=0,y(x), singsol=all)
```

$$y(x) = -2\sqrt{2}\,\sqrt{-x}$$
$$y(x) = 2\sqrt{2}\,\sqrt{-x}$$
$$y(x) = c_1 x - \frac{2}{c_1}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 57

```
DSolve[y'[x]^2*x-y'[x]*y[x]-2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{2}{c_1}$$
$$y(x) \to \text{Indeterminate}$$
$$y(x) \to -2i\sqrt{2}\sqrt{x}$$
$$y(x) \to 2i\sqrt{2}\sqrt{x}$$

21.10 problem 32

Internal problem ID [2363]
Internal file name [`OUTPUT/2363_Tuesday_February_27_2024_08_36_08_AM_76870095/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 39, page 179
Problem number: 32.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y^2 - 2xyy' + y'^2(x^2 - 1) = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y}{x+1} \tag{1}$$

$$y' = \frac{y}{x-1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x+1} \end{aligned}$$

Where $f(x) = \frac{1}{x+1}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x+1}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x+1}\,dx \\
\ln(y) &= \ln(x+1) + c_1 \\
y &= \mathrm{e}^{\ln(x+1)+c_1} \\
&= c_1(x+1)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(x+1) \tag{1}$$

Verification of solutions

$$y = c_1(x+1)$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{y}{x-1}
\end{aligned}$$

Where $f(x) = \frac{1}{x-1}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x-1}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x-1}\,dx \\
\ln(y) &= \ln(x-1) + c_2 \\
y &= \mathrm{e}^{\ln(x-1)+c_2} \\
&= c_2(x-1)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2(x-1) \tag{1}$$

Verification of solutions

$$y = c_2(x-1)$$

Verified OK.

21.10.1 Maple step by step solution

Let's solve

$y^2 - 2xyy' + y'^2(x^2-1) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x+1} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x+1) + c_1$

- Solve for y

  $y = e^{c_1}(x+1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(y(x)^2-2*diff(y(x),x)*x*y(x)+diff(y(x),x)^2*(x^2-1)=0,y(x), singsol=all)
```

$$y(x) = c_1(x-1)$$
$$y(x) = c_1(x+1)$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 26

```
DSolve[y[x]^2-2*y'[x]*x*y[x]+y'[x]^2*(x^2-1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x-1)$$
$$y(x) \to c_1(x+1)$$
$$y(x) \to 0$$

22 Exercise 40, page 186

22.1 problem 1

Internal problem ID [2364]
Internal file name [`OUTPUT/2364_Tuesday_February_27_2024_08_36_09_AM_55827639/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \sqrt{1-y} = 0}$$

With initial conditions

$$[y(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

22.1.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \sqrt{1-y} \end{aligned}$$

The y domain of $f(x,y)$ when $x = 0$ is

$$\{y \leq 1\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{1-y}\right) \\
&= -\frac{1}{2\sqrt{1-y}}
\end{aligned}
$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$
\{y < 1\}
$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

22.1.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$
y' = f(x, y)
$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y) \qquad (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)F_0 \qquad (5)
\end{aligned}
$$

For example, for $n=1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f
\end{aligned}
$$

Which is (1). And when $n=2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$
y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n\big|_{x_0,y_0} \qquad (6)
$$

Hence

$$
\begin{aligned}
F_0 &= \sqrt{1-y} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= -\frac{1}{2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= 0 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= -\frac{1}{2} \\
F_2 &= 0 \\
F_3 &= 0
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = x - \frac{x^2}{4} + O(x^5)
$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$
y(0) = y(0)
$$

Therefore the solution becomes

$$
y = x - \frac{1}{4}x^2
$$

Hence the solution can be written as

$$y = x - \frac{x^2}{4} + O(x^5)$$

which simplifies to

$$y = x - \frac{x^2}{4} + O(x^5)$$

Unable to also solve using normal power series since not linear ode. Not currently supported.

Summary

The solution(s) found are the following

$$y = x - \frac{x^2}{4} + O(x^5) \tag{1}$$

Verification of solutions

$$y = x - \frac{x^2}{4} + O(x^5)$$

Verified OK.

22.1.3 Maple step by step solution

Let's solve

$\left[y' = \sqrt{1-y}, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y}} dx = \int 1 dx + c_1$

- Evaluate integral

  $-2\sqrt{1-y} = x + c_1$

- Solve for y

  $y = -\frac{1}{4}c_1^2 - \frac{1}{2}c_1 x - \frac{1}{4}x^2 + 1$

- Use initial condition $y(0) = 0$

  $0 = -\frac{c_1^2}{4} + 1$

- Solve for c_1

  $c_1 = (2, -2)$
- Substitute $c_1 = (2, -2)$ into general solution and simplify

  $y = -\frac{(2+x)(x-2)}{4}$
- Solution to the IVP

  $y = -\frac{(2+x)(x-2)}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
Order:=5;
dsolve([diff(y(x),x)=(1-y(x))^(1/2),y(0) = 0],y(x),type='series',x=0);
```

$$y(x) = x - \frac{1}{4}x^2$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 12

```
AsymptoticDSolveValue[{y'[x]==(1-y[x])^(1/2),{y[0]==0}},y[x],{x,0,4}]
```

$$y(x) \to x - \frac{x^2}{4}$$

22.2 problem 2

Internal problem ID [2365]
Internal file name [OUTPUT/2365_Tuesday_February_27_2024_08_36_10_AM_12373942/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup", "first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - yx = -x^2}$$

With initial conditions

$$[y(0) = 2]$$

With the expansion point for the power series method at $x = 0$.

22.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -x$$
$$q(x) = -x^2$$

Hence the ode is

$$y' - yx = -x^2$$

The domain of $p(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

22.2.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{align*} F_0 &= f(x,y) \tag{4}\\ F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \tag{5} \end{align*}$$

For example, for $n=1$ we see that

$$\begin{align*} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{align*}$$

Which is (1). And when $n=2$

$$\begin{align*} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{align*}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$
\begin{aligned}
F_0 &= yx - x^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= x^2 y - x^3 + y - 2x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= \left(x^3 + 3x\right) y - x^4 - 4x^2 - 2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \left(x^4 + 6x^2 + 3\right) y - x^5 - 7x^3 - 8x
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 2$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 2 \\
F_2 &= -2 \\
F_3 &= 6
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = x^2 + 2 - \frac{x^3}{3} + \frac{x^4}{4} + O\left(x^5\right)
$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$
y(0) = y(0)
$$

Therefore the solution becomes

$$
y = x^2 + 2 - \frac{1}{3}x^3 + \frac{1}{4}x^4
$$

Hence the solution can be written as

$$y = x^2 + 2 - \frac{x^3}{3} + \frac{x^4}{4} + O(x^5)$$

which simplifies to

$$y = x^2 + 2 - \frac{x^3}{3} + \frac{x^4}{4} + O(x^5)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - yx &= -x^2 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -x \\ p(x) &= -x^2 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) x = -x^2 \tag{1}$$

Expanding $-x^2$ as Taylor series around $x = 0$ and keeping only the first 5 terms gives

$$\begin{aligned}-x^2 &= -x^2 + \dots \\ &= -x^2\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) x = -x^2 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = -x^2 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n \\ \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = -x^2 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$\left((1+n) a_{1+n} - a_{n-1}\right) x^n = -x^2 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned}(2a_2 - a_0) x &= 0 \\ 2a_2 - a_0 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(3a_3 - a_1) x^2 &= -x^2 \\ 3a_3 - a_1 &= -1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{1}{3}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(4a_4 - a_2) x^3 &= 0 \\ 4a_4 - a_2 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(5a_5 - a_3) x^4 &= 0 \\ 5a_5 - a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{1}{15}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + \frac{1}{2}a_0x^2 - \frac{1}{3}x^3 + \frac{1}{8}a_0x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right)a_0 - \frac{x^3}{3} + O(x^5) \tag{3}$$

At $x = 0$ the solution above becomes

$$y(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right)y(0) - \frac{x^3}{3} + O(x^5)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = 2$$

Therefore the solution becomes

$$y = x^2 + 2 - \frac{1}{3}x^3 + \frac{1}{4}x^4$$

Hence the solution can be written as

$$y = x^2 + 2 - \frac{x^3}{3} + \frac{x^4}{4} + O(x^5)$$

which simplifies to

$$y = x^2 + 2 - \frac{x^3}{3} + \frac{x^4}{4} + O(x^5)$$

Summary

The solution(s) found are the following

$$y = x^2 + 2 - \frac{x^3}{3} + \frac{x^4}{4} + O(x^5) \tag{1}$$

$$y = x^2 + 2 - \frac{x^3}{3} + \frac{x^4}{4} + O(x^5) \tag{2}$$

Verification of solutions

$$y = x^2 + 2 - \frac{x^3}{3} + \frac{x^4}{4} + O(x^5)$$

Verified OK.

$$y = x^2 + 2 - \frac{x^3}{3} + \frac{x^4}{4} + O(x^5)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=5;
dsolve([diff(y(x),x)=x*y(x)-x^2,y(0) = 2],y(x),type='series',x=0);
```

$$y(x) = 2 + x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 + \mathrm{O}\left(x^5\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 22

```
AsymptoticDSolveValue[{y'[x]==x*y[x]-x^2,{y[0]==2}},y[x],{x,0,4}]
```

$$y(x) \to \frac{x^4}{4} - \frac{x^3}{3} + x^2 + 2$$

22.3 problem 3

Internal problem ID [2366]
Internal file name [OUTPUT/2366_Tuesday_February_27_2024_08_36_12_AM_33596530/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y^2x^2 = 0}$$

With initial conditions

$$[y(1) = 0]$$

With the expansion point for the power series method at $x = 1$.

22.3.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^2x^2 \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 x^2) \\ &= 2x^2 y \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

22.3.2 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d}{dt}y(t) - y(t)^2 (t + 1)^2 = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 0$$

The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \qquad (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y) \qquad (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}$$

For example, for $n=1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f
\end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= y(t)^2 (t+1)^2 \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}F_0 \\
&= 2\left(1 + (t+1)^3 y(t)\right) y(t)^2 (t+1) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}F_1 \\
&= 6\left(\frac{1}{3} + (t+1)^6 y(t)^2 + 2(t+1)^3 y(t)\right) y(t)^2 \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}F_2 \\
&= 24y(t)^3 (t+1)^2 \left(\frac{5}{3} + (t+1)^6 y(t)^2 + 3(t+1)^3 y(t)\right)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t(0) = 0$ and $y(0) = 0$ gives

$$F_0 = 0$$
$$F_1 = 0$$
$$F_2 = 0$$
$$F_3 = 0$$

Substituting all the above in (6) and simplifying gives the solution as

$$y(t) = O(t^5)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y(t) = 0$$

Hence the solution can be written as

$$y(t) = O(t^5)$$

which simplifies to

$$y(t) = O(t^5)$$

Unable to also solve using normal power series since not linear ode. Not currently supported. Replacing t in the above with the original independent variable xs using $t = x - 1$ results in

$$y = O((x-1)^5)$$

Summary

The solution(s) found are the following

$$y = O((x-1)^5) \tag{1}$$

Verification of solutions

$$y = O((x-1)^5)$$

Verified OK.

22.3.3 Maple step by step solution

Let's solve

$[y' - y^2x^2 = 0, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2}dx = \int x^2dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = \frac{x^3}{3} + c_1$

- Solve for y

  $y = -\frac{3}{x^3+3c_1}$

- Use initial condition $y(1) = 0$

  $0 = -\frac{3}{3c_1+1}$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
Order:=5;
dsolve([diff(y(x),x)=x^2*y(x)^2,y(1) = 0],y(x),type='series',x=1);
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 4

```
AsymptoticDSolveValue[{y'[x]==x^2*y[x]^2,{y[1]==0}},y[x],{x,1,4}]
```

$$y(x) \to 0$$

22.4 problem 4

Internal problem ID [2367]
Internal file name [`OUTPUT/2367_Tuesday_February_27_2024_08_36_12_AM_97136872/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = 3x}$$

With initial conditions

$$[y(1) = 3]$$

With the expansion point for the power series method at $x = 1$.

22.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 3x$$

Hence the ode is

$$y' - \frac{y}{x} = 3x$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 3x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

22.4.2 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d}{dt}y(t) - \frac{y(t)}{t+1} = 3t + 3$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 3$$

The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2}y''(x_0) + \frac{(x-x_0)^3}{3!}y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2}\frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!}\frac{d^2f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f &(1)\\
\frac{d^2f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f &(2)\\
\frac{d^3f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2f}{dx^2}\right)f &(3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0=f(x,y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y) &(4)\\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)F_0 &(5)
\end{aligned}$$

For example, for $n=1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f
\end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}f\right)f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= \frac{3t^2+6t+y(t)+3}{t+1}\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}F_0\\
&= 6\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}F_1\\
&= 0\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}F_2\\
&= 0
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t(0)=0$ and $y(0)=3$ gives

$$\begin{aligned}
F_0 &= 6\\
F_1 &= 6\\
F_2 &= 0\\
F_3 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y(t) = 3t^2 + 6t + 3 + O(t^5)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y(t) = 3t^2 + 6t + 3$$

Hence the solution can be written as

$$y(t) = 3t^2 + 6t + 3 + O(t^5)$$

which simplifies to

$$y(t) = 3t^2 + 6t + 3 + O(t^5)$$

Since $t = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\frac{d}{dt}y(t) + q(t)y(t) = p(t)$$
$$\frac{d}{dt}y(t) - \frac{y(t)}{t+1} = 3t + 3$$

Where

$$q(t) = -\frac{1}{t+1}$$
$$p(t) = 3t + 3$$

Next, the type of the expansion point $t = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $t = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $t = 0$ is called an ordinary point $q(t)$ has a Taylor series expansion around the point $t = 0$. $t = 0$ is called a regular singular point if $q(t)$ is not not analytic at $t = 0$ but $tq(t)$ has Taylor series expansion. And finally, $t = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $t = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(t+1)\left(\frac{d}{dt}y(t)\right) - y(t) = 3(t+1)^2$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

Substituting the above back into the ode gives

$$(t+1)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n t^n\right) = 3(t+1)^2 \tag{1}$$

Expanding $3(t+1)^2$ as Taylor series around $t = 0$ and keeping only the first 5 terms gives

$$\begin{aligned} 3(t+1)^2 &= 3t^2 + 6t + 3 + \dots \\ &= 3t^2 + 6t + 3 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$(t+1)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n t^n\right) = 3t^2 + 6t + 3 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \sum_{n=0}^{\infty} (-a_n t^n) = 3t^2 + 6t + 3 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} n a_n t^n\right)+\left(\sum_{n=0}^{\infty}(n+1) a_{n+1} t^n\right)+\sum_{n=0}^{\infty}\left(-a_n t^n\right)=3t^2+6t+3 \tag{3}$$

$n=0$ gives

$$\begin{aligned}(a_1-a_0)\, t^0 &= 3\\ a_1-a_0 &= 3\\ a_1 &= a_0+3\end{aligned}$$

For $1 \leq n$, the recurrence equation is

$$\left(n a_n+(n+1) a_{n+1}-a_n\right) t^n=3t^2+6t+3 \tag{4}$$

For $n=1$ the recurrence equation gives

$$\begin{aligned}(2a_2)\, t &= 6t\\ 2a_2 &= 6\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2=3$$

For $n=2$ the recurrence equation gives

$$\begin{aligned}(a_2+3a_3)\, t^2 &= 3t^2\\ a_2+3a_3 &= 3\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3=0$$

For $n=3$ the recurrence equation gives

$$\begin{aligned}(2a_3+4a_4)\, t^3 &= 0\\ 2a_3+4a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(3a_4 + 5a_5)\, t^4 &= 0 \\ 3a_4 + 5a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

And so on. Therefore the solution is

$$\begin{aligned}y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + (a_0 + 3)\, t + 3t^2 + \dots$$

Collecting terms, the solution becomes

$$y(t) = (t + 1)\, a_0 + 3t^2 + O(t^5) + 3t \tag{3}$$

At $t = 0$ the solution above becomes

$$y(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

$$y(t) = (t + 1)\, y(0) + 3t^2 + O(t^5) + 3t$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = 3$$

Therefore the solution becomes

$$y(t) = 3t^2 + 6t + 3$$

Hence the solution can be written as

$$y(t) = 3t^2 + 6t + 3 + O(t^5)$$

which simplifies to

$$y(t) = 3t^2 + 6t + 3 + O(t^5)$$

Replacing t in the above with the original independent variable xs using $t = x - 1$ results in

$$y = 3(x-1)^2 + 6x - 3 + O\left((x-1)^5\right)$$

Summary

The solution(s) found are the following

$$y = 3(x-1)^2 + 6x - 3 + O\left((x-1)^5\right) \tag{1}$$

Verification of solutions

$$y = 3(x-1)^2 + 6x - 3 + O\left((x-1)^5\right)$$

Verified OK.

22.4.3 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = 3x, y(1) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 3x + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 3x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = 3\mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-\frac{y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x)=\frac{1}{x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int 3\mu(x)\,x dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int 3\mu(x)\,x dx+c_1$
- Solve for y

  $y=\frac{\int 3\mu(x)x dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{1}{x}$

  $y=x\left(\int 3dx+c_1\right)$
- Evaluate the integrals on the rhs

  $y=x(3x+c_1)$
- Use initial condition $y(1)=3$

  $3=3+c_1$
- Solve for c_1

  $c_1=0$
- Substitute $c_1=0$ into general solution and simplify

  $y=3x^2$
- Solution to the IVP

  $y=3x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=5;
dsolve([diff(y(x),x)=3*x+y(x)/x,y(1) = 3],y(x),type='series',x=1);
```

$$y(x) = 3x^2$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 17

```
AsymptoticDSolveValue[{y'[x]==3*x+y[x]/x,{y[1]==3}},y[x],{x,1,4}]
```

$$y(x) \to 3(x-1)^2 + 6(x-1) + 3$$

22.5 problem 5

Internal problem ID [2368]
Internal file name [`OUTPUT/2368_Tuesday_February_27_2024_08_36_13_AM_96813852/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{y' - \ln(yx) = 0}$$

With initial conditions

$$[y(1) = 1]$$

With the expansion point for the power series method at $x = 1$.

22.5.1 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d}{dt}y(t) - \ln(y(t)(t+1)) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$

The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\begin{aligned}
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned} \tag{2}$$

$$\begin{aligned}
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f
\end{aligned} \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$F_0 = f(x, y) \tag{4}$$

$$\begin{aligned}
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0
\end{aligned} \tag{5}$$

For example, for $n=1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f
\end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n\big|_{x_0,y_0} \tag{6}$$

Hence

$$
\begin{aligned}
F_0 &= \ln\left(y(t)\,(t+1)\right)\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} F_0\\
&= \frac{(t+1)\ln\left(y(t)\,(t+1)\right) + y(t)}{y\left(t\right)(t+1)}\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} F_1\\
&= \frac{-\ln\left(y(t)\,(t+1)\right)^2 (t+1)^2 + \ln\left(y(t)\,(t+1)\right)(t+1)^2 + y(t)\,(t-y(t)+1)}{y\left(t\right)^2 (t+1)^2}\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y} F_2\\
&= \frac{2\ln\left(y(t)\,(t+1)\right)^3 (t+1)^3 - 4(t+1)^3 \ln\left(y(t)\,(t+1)\right)^2 + (t+1)^2 (t-3y(t)+1)\ln\left(y(t)\,(t+1)\right) + (}{y\left(t\right)^3 (t+1)^3}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t(0)=0$ and $y(0)=1$ gives

$$
\begin{aligned}
F_0 &= 0\\
F_1 &= 1\\
F_2 &= 0\\
F_3 &= 2
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y(t) = 1 + \frac{t^2}{2} + \frac{t^4}{12} + O\left(t^5\right)
$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$
y(0) = y(0)
$$

Therefore the solution becomes

$$
y(t) = 1 + \frac{1}{2}t^2 + \frac{1}{12}t^4
$$

Hence the solution can be written as

$$y(t) = 1 + \frac{t^2}{2} + \frac{t^4}{12} + O(t^5)$$

which simplifies to

$$y(t) = 1 + \frac{t^2}{2} + \frac{t^4}{12} + O(t^5)$$

Unable to also solve using normal power series since not linear ode. Not currently supported. Replacing t in the above with the original independent variable xs using $t = x - 1$ results in

$$y = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{12} + O\left((x-1)^5\right)$$

Summary

The solution(s) found are the following

$$y = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{12} + O\left((x-1)^5\right) \tag{1}$$

Verification of solutions

$$y = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{12} + O\left((x-1)^5\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=5;
dsolve([diff(y(x),x)=ln(x*y(x)),y(1) = 1],y(x),type='series',x=1);
```

$$y(x) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-1)^4 + \mathrm{O}\left((x-1)^5\right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 23

```
AsymptoticDSolveValue[{y'[x]==Log[x*y[x]],{y[1]==1}},y[x],{x,1,4}]
```

$$y(x) \to \frac{1}{12}(x-1)^4 + \frac{1}{2}(x-1)^2 + 1$$

22.6 problem 6

Internal problem ID [2369]
Internal file name [`OUTPUT/2369_Tuesday_February_27_2024_08_36_15_AM_33641776/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = 1}$$

With initial conditions

$$[y(1) = -1]$$

With the expansion point for the power series method at $x = 1$.

22.6.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^2 + 1 \end{aligned}$$

The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2+1) \\ &= 2y\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

22.6.2 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d}{dt}y(t) - y(t)^2 = 1$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = -1$$

The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f \qquad (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) \qquad (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n=2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}f\right)f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= y(t)^2+1\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}F_0\\
&= 2y(t)\left(y(t)^2+1\right)\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}F_1\\
&= 6\left(y(t)^2+\frac{1}{3}\right)\left(y(t)^2+1\right)\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}F_2\\
&= 24y(t)^5+40y(t)^3+16y(t)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t(0)=0$ and $y(0)=-1$ gives

$$\begin{aligned}
F_0 &= 2\\
F_1 &= -4\\
F_2 &= 16\\
F_3 &= -80
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y(t) = -2t^2 + 2t - 1 + \frac{8t^3}{3} - \frac{10t^4}{3} + O(t^5)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y(t) = -2t^2 + 2t - 1 + \frac{8}{3}t^3 - \frac{10}{3}t^4$$

Hence the solution can be written as

$$y(t) = -2t^2 + 2t - 1 + \frac{8t^3}{3} - \frac{10t^4}{3} + O(t^5)$$

which simplifies to

$$y(t) = -2t^2 + 2t - 1 + \frac{8t^3}{3} - \frac{10t^4}{3} + O(t^5)$$

Unable to also solve using normal power series since not linear ode. Not currently supported. Replacing t in the above with the original independent variable xs using $t = x - 1$ results in

$$y = -2(x-1)^2 + 2x - 3 + \frac{8(x-1)^3}{3} - \frac{10(x-1)^4}{3} + O\left((x-1)^5\right)$$

Summary

The solution(s) found are the following

$$y = -2(x-1)^2 + 2x - 3 + \frac{8(x-1)^3}{3} - \frac{10(x-1)^4}{3} + O\left((x-1)^5\right) \tag{1}$$

Verification of solutions

$$y = -2(x-1)^2 + 2x - 3 + \frac{8(x-1)^3}{3} - \frac{10(x-1)^4}{3} + O\left((x-1)^5\right)$$

Verified OK.

22.6.3 Maple step by step solution

Let's solve

$[y' - y^2 = 1, y(1) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $\arctan(y) = x + c_1$

- Solve for y

  $y = \tan(x + c_1)$

- Use initial condition $y(1) = -1$

  $-1 = \tan(c_1 + 1)$

- Solve for c_1

  $c_1 = -1 - \frac{\pi}{4}$

- Substitute $c_1 = -1 - \frac{\pi}{4}$ into general solution and simplify

  $y = -\cot\left(x - 1 + \frac{\pi}{4}\right)$

- Solution to the IVP

  $y = -\cot\left(x - 1 + \frac{\pi}{4}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=5;
dsolve([diff(y(x),x)=1+y(x)^2,y(1) = -1],y(x),type='series',x=1);
```

$$y(x) = -1 + 2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3 - \frac{10}{3}(x-1)^4 + \mathrm{O}\left((x-1)^5\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 35

```
AsymptoticDSolveValue[{y'[x]==1+y[x]^2,{y[1]==-1}},y[x],{x,1,4}]
```

$$y(x) \to -\frac{10}{3}(x-1)^4 + \frac{8}{3}(x-1)^3 - 2(x-1)^2 + 2(x-1) - 1$$

22.7 problem 7

Internal problem ID [2370]
Internal file name [OUTPUT/2370_Tuesday_February_27_2024_08_36_16_AM_89319918/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Riccati, _special]]
```

$$\boxed{y' - y^2 = x^2}$$

With initial conditions

$$[y(2) = 0]$$

With the expansion point for the power series method at $x = 2$.

22.7.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x^2 + y^2 \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x, y)$ when $x = 2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2 + y^2) \\ &= 2y\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

22.7.2 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 2$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d}{dt}y(t) - y(t)^2 = (t + 2)^2$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 0$$

The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f \qquad (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y) \qquad (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= t^2 + y(t)^2 + 4t + 4 \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} F_0 \\
&= 2y(t)^3 + 2(t+2)^2\, y(t) + 2t + 4 \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} F_1 \\
&= 6y(t)^4 + 8(t+2)^2\, y(t)^2 + (4t+8)\, y(t) + 2t^4 + 16t^3 + 48t^2 + 64t + 34 \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y} F_2 \\
&= 24y(t)^5 + 40(t+2)^2\, y(t)^3 + 20(t+2)\, y(t)^2 + 4\left(4t^4 + 32t^3 + 96t^2 + 128t + 65\right) y(t) + 12(t+2)^3 \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y} F_3 \\
&= 120y(t)^6 + 240(t+2)^2\, y(t)^4 + 120(t+2)\, y(t)^3 + 8\left(17t^4 + 136t^3 + 408t^2 + 544t + 275\right) y(t)^2 + 104(t
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t(0) = 0$ and $y(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 4 \\
F_1 &= 4 \\
F_2 &= 34 \\
F_3 &= 96 \\
F_4 &= 1184
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y(t) = 2t^2 + 4t + \frac{17t^3}{3} + 4t^4 + \frac{148t^5}{15} + O(t^6)
$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$
y(0) = y(0)
$$

Therefore the solution becomes

$$
y(t) = 2t^2 + 4t + \frac{17}{3}t^3 + 4t^4 + \frac{148}{15}t^5
$$

Hence the solution can be written as

$$
y(t) = 2t^2 + 4t + \frac{17t^3}{3} + 4t^4 + \frac{148t^5}{15} + O(t^6)
$$

which simplifies to

$$
y(t) = 2t^2 + 4t + \frac{17t^3}{3} + 4t^4 + \frac{148t^5}{15} + O(t^6)
$$

Unable to also solve using normal power series since not linear ode. Not currently supported. Replacing t in the above with the original independent variable xs using $t = x - 2$ results in

$$
y = 2(x-2)^2 + 4x - 8 + \frac{17(x-2)^3}{3} + 4(x-2)^4 + \frac{148(x-2)^5}{15} + O\left((x-2)^6\right)
$$

Summary
The solution(s) found are the following

$$
y = 2(x-2)^2 + 4x - 8 + \frac{17(x-2)^3}{3} + 4(x-2)^4 + \frac{148(x-2)^5}{15} + O\left((x-2)^6\right) \quad (1)
$$

Verification of solutions

$$y = 2(x-2)^2 + 4x - 8 + \frac{17(x-2)^3}{3} + 4(x-2)^4 + \frac{148(x-2)^5}{15} + O\left((x-2)^6\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve([diff(y(x),x)=x^2+y(x)^2,y(2) = 0],y(x),type='series',x=2);
```

$$y(x) = 4(-2+x) + 2(-2+x)^2 + \frac{17}{3}(-2+x)^3 + 4(-2+x)^4 + \frac{148}{15}(-2+x)^5 + \mathrm{O}\left((-2+x)^6\right)$$

✓ Solution by Mathematica

Time used: 0.154 (sec). Leaf size: 9983

```
AsymptoticDSolveValue[{y'[x]==x^2+y[x]^2,{y[2]==0}},y[x],{x,2,5}]
```

Too large to display

22.8 problem 8

Internal problem ID [2371]
Internal file name [`OUTPUT/2371_Tuesday_February_27_2024_08_36_19_AM_27802779/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{y' - \sqrt{1 + yx} = 0}$$

With initial conditions

$$[y(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

22.8.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + x f + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f \qquad (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) \qquad (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= \sqrt{1 + yx} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{x\sqrt{1 + yx} + y}{2\sqrt{1 + yx}} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= \frac{3yx\sqrt{1 + yx} - y^2 + 4\sqrt{1 + yx}}{4(1 + yx)^{\frac{3}{2}}}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 1$ gives

$$\begin{aligned}
F_0 &= 1 \\
F_1 &= \frac{1}{2} \\
F_2 &= \frac{3}{4}
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = x + 1 + \frac{x^2}{4} + \frac{x^3}{8} + O(x^4)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y = x + 1 + \frac{1}{4}x^2 + \frac{1}{8}x^3$$

Hence the solution can be written as

$$y = x + 1 + \frac{x^2}{4} + \frac{x^3}{8} + O(x^4)$$

which simplifies to

$$y = x + 1 + \frac{x^2}{4} + \frac{x^3}{8} + O(x^4)$$

Unable to also solve using normal power series since not linear ode. Not currently supported.

Summary

The solution(s) found are the following

$$y = x + 1 + \frac{x^2}{4} + \frac{x^3}{8} + O(x^4) \quad (1)$$

Verification of solutions

$$y = x + 1 + \frac{x^2}{4} + \frac{x^3}{8} + O(x^4)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=4;
dsolve([diff(y(x),x)=sqrt(1+x*y(x)),y(0) = 1],y(x),type='series',x=0);
```

$$y(x) = 1 + x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \mathrm{O}\left(x^4\right)$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 20

```
AsymptoticDSolveValue[{y'[x]==Sqrt[1+x*y[x]],{y[0]==1}},y[x],{x,0,3}]
```

$$y(x) \to \frac{x^3}{8} + \frac{x^2}{4} + x + 1$$

22.9 problem 9

Internal problem ID [2372]
Internal file name [`OUTPUT/2372_Tuesday_February_27_2024_08_36_21_AM_53635095/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{y' - \sin(y) = \cos(x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\right]$$

With the expansion point for the power series method at $x = \frac{\pi}{2}$.

22.9.1 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - \frac{\pi}{2}$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d}{dt}y(t) - \sin(y(t)) = -\sin(t)$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = \frac{\pi}{2}$$

The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f &(1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f &(2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f &(3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y) &(4)\\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 &(5)
\end{aligned}$$

For example, for $n=1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= \sin(y(t)) - \sin(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} F_0 \\
&= (\sin(y(t)) - \sin(t)) \cos(y(t)) - \cos(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} F_1 \\
&= \cos(t)^2 \sin(y(t)) + 3\sin(y(t))^2 \sin(t) - 2\sin(y(t))^3 - \cos(y(t)) \cos(t)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t(0)=0$ and $y(0)=\frac{\pi}{2}$ gives

$$\begin{aligned}
F_0 &= 1 \\
F_1 &= -1 \\
F_2 &= -1
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y(t) = t + \frac{\pi}{2} - \frac{t^2}{2} - \frac{t^3}{6} + O(t^4)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y(t) = t + \frac{1}{2}\pi - \frac{1}{2}t^2 - \frac{1}{6}t^3$$

Hence the solution can be written as

$$y(t) = t + \frac{\pi}{2} - \frac{t^2}{2} - \frac{t^3}{6} + O(t^4)$$

which simplifies to

$$y(t) = t + \frac{\pi}{2} - \frac{t^2}{2} - \frac{t^3}{6} + O(t^4)$$

Unable to also solve using normal power series since not linear ode. Not currently supported. Replacing t in the above with the original independent variable xs using $t = x - \frac{\pi}{2}$ results in

$$y = x - \frac{\left(x - \frac{\pi}{2}\right)^2}{2} - \frac{\left(x - \frac{\pi}{2}\right)^3}{6} + O\left(\left(x - \frac{\pi}{2}\right)^4\right)$$

Summary

The solution(s) found are the following

$$y = x - \frac{\left(x - \frac{\pi}{2}\right)^2}{2} - \frac{\left(x - \frac{\pi}{2}\right)^3}{6} + O\left(\left(x - \frac{\pi}{2}\right)^4\right) \tag{1}$$

Verification of solutions

$$y = x - \frac{\left(x - \frac{\pi}{2}\right)^2}{2} - \frac{\left(x - \frac{\pi}{2}\right)^3}{6} + O\left(\left(x - \frac{\pi}{2}\right)^4\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=4;
dsolve([diff(y(x),x)=cos(x)+sin(y(x)),y(1/2*Pi) = 1/2*Pi],y(x),type='series',x=1/2*Pi);
```

$$y(x) = \frac{\pi}{2} + \left(-\frac{\pi}{2} + x\right) - \frac{1}{2}\left(-\frac{\pi}{2} + x\right)^2 - \frac{1}{6}\left(-\frac{\pi}{2} + x\right)^3 + \mathrm{O}\left(\left(-\frac{\pi}{2} + x\right)^4\right)$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 22

```
AsymptoticDSolveValue[{y'[x]==Cos[x]*Sin[y[x]],{y[Pi/2]==Pi/2}},y[x],{x,1/2*Pi,3}]
```

$$y(x) \to \frac{\pi}{2} - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$$

22.10 problem 10

Internal problem ID [2373]
Internal file name [`OUTPUT/2373_Tuesday_February_27_2024_08_36_24_AM_68353770/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \sin(x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

With the expansion point for the power series method at $x = 0$.

22.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -1 \\ F &= \sin(x) \end{aligned}$$

Hence the ode is

$$y'' - y = \sin(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (1084)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (1085)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= y+\sin(x) \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= y'+\cos(x) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= y+\sin(x) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x}+\frac{\partial F_4}{\partial y}y'+\frac{\partial F_4}{\partial y'}F_4 \\
&= y'+\cos(x)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=1$ and $y'(0)=2$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 3 \\
F_2 &= 1 \\
F_3 &= 2 \\
F_4 &= 1 \\
F_5 &= 3
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = 1 + 2x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{24} + \frac{x^5}{60} + \frac{x^6}{720} + \frac{x^7}{1680} + O(x^7)$$

$$y = 1 + 2x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{24} + \frac{x^5}{60} + \frac{x^6}{720} + \frac{x^7}{1680} + O(x^7)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \left(\sum_{n=0}^{\infty} a_n x^n\right) + \sin(x) \tag{1}$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 7 terms gives

$$\begin{aligned} \sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-a_n x^n) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$((n+2)\, a_{n+2}(n+1) - a_n)\, x^n = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \tag{4}$$

For $n = 0$ the recurrence equation gives

$$(2a_2 - a_0)\, 1 = 0$$
$$2a_2 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$(6a_3 - a_1)\, x = x$$
$$6a_3 - a_1 = 1$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{1}{6} + \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$(12a_4 - a_2) x^2 = 0$$
$$12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$(20a_5 - a_3) x^3 = -\frac{x^3}{6}$$
$$20a_5 - a_3 = -\frac{1}{6}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$(30a_6 - a_4) x^4 = 0$$
$$30a_6 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$(42a_7 - a_5) x^5 = \frac{x^5}{120}$$
$$42a_7 - a_5 = \frac{1}{120}$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{1}{5040} + \frac{a_1}{5040}$$

For $n = 6$ the recurrence equation gives

$$\begin{aligned}(56a_8 - a_6)\, x^6 &= 0\\ 56a_8 - a_6 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{a_0}{40320}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{a_0 x^2}{2} + \left(\frac{1}{6} + \frac{a_1}{6}\right) x^3 + \frac{a_0 x^4}{24} + \frac{a_1 x^5}{120} + \frac{a_0 x^6}{720} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) a_0 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) a_1 + \frac{x^3}{6} + O(x^7) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + \frac{x^3}{6} + O(x^7)$$

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + 2x + \frac{x^3}{2} + \frac{x^5}{60} + O(x^7)$$

Summary
The solution(s) found are the following

$$y = 1 + 2x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{24} + \frac{x^5}{60} + \frac{x^6}{720} + \frac{x^7}{1680} + O(x^7) \tag{1}$$

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + 2x + \frac{x^3}{2} + \frac{x^5}{60} + O(x^7) \tag{2}$$

Verification of solutions

$$y = 1 + 2x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{24} + \frac{x^5}{60} + \frac{x^6}{720} + \frac{x^7}{1680} + O(x^7)$$

Verified OK.

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + 2x + \frac{x^3}{2} + \frac{x^5}{60} + O(x^7)$$

Verified OK.

22.10.2 Maple step by step solution

Let's solve

$\left[y'' = y + \sin(x), y(0) = 1, y'\Big|_{\{x=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y = \sin(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$
- ☐ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{x} \sin(x) dx\right)}{2} + \frac{\mathrm{e}^{x}\left(\int \mathrm{e}^{-x} \sin(x) dx\right)}{2}$
  - Compute integrals

    $y_p(x) = -\frac{\sin(x)}{2}$
- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - \frac{\sin(x)}{2}$
- ☐ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - \frac{\sin(x)}{2}$
  - Use initial condition $y(0) = 1$

    $1 = c_1 + c_2$
  - Compute derivative of the solution

    $y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - \frac{\cos(x)}{2}$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 2$

    $2 = -c_1 + c_2 - \frac{1}{2}$
  - Solve for c_1 and c_2

    $\left\{c_1 = -\frac{3}{4}, c_2 = \frac{7}{4}\right\}$
  - Substitute constant values into general solution and simplify

$$y = -\frac{3\,e^{-x}}{4} + \frac{7\,e^{x}}{4} - \frac{\sin(x)}{2}$$

- Solution to the IVP

$$y = -\frac{3\,e^{-x}}{4} + \frac{7\,e^{x}}{4} - \frac{\sin(x)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
Order:=7;
dsolve([diff(y(x),x$2)-y(x)=sin(x),y(0) = 1, D(y)(0) = 2],y(x),type='series',x=0);
```

$$y(x) = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5 + \frac{1}{720}x^6 + O\left(x^7\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 43

```
AsymptoticDSolveValue[{y''[x]-y[x]==Sin[x],{y[0]==1,y'[0]==2}},y[x],{x,0,6}]
```

$$y(x) \to \frac{x^6}{720} + \frac{x^5}{60} + \frac{x^4}{24} + \frac{x^3}{2} + \frac{x^2}{2} + 2x + 1$$

22.11 problem 11

Internal problem ID [2374]
Internal file name [`OUTPUT/2374_Tuesday_February_27_2024_08_36_26_AM_23154222/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__linear__constant__coeff", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y = e^{2x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

22.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -2 \\ F &= e^{2x} \end{aligned}$$

Hence the ode is

$$y'' - 2y = e^{2x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = e^{2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (1087) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (1088) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$\begin{aligned}
F_0 &= 2y + \mathrm{e}^{2x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 2y' + 2\,\mathrm{e}^{2x} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 4y + 6\,\mathrm{e}^{2x} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 4y' + 12\,\mathrm{e}^{2x} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 8y + 28\,\mathrm{e}^{2x} \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= 8y' + 56\,\mathrm{e}^{2x}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=0$ and $y'(0)=0$ gives

$$\begin{aligned}
F_0 &= 1 \\
F_1 &= 2 \\
F_2 &= 6 \\
F_3 &= 12 \\
F_4 &= 28 \\
F_5 &= 56
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10} + \frac{7x^6}{180} + \frac{x^7}{90} + O(x^7)$$

$$y = \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10} + \frac{7x^6}{180} + \frac{x^7}{90} + O(x^7)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2\left(\sum_{n=0}^{\infty} a_n x^n\right) + e^{2x} \tag{1}$$

Expanding e^{2x} as Taylor series around $x = 0$ and keeping only the first 7 terms gives

$$\begin{aligned} e^{2x} &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \dots \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 \tag{3}$$

For $0 \le n$, the recurrence equation is

$$((n+2)\, a_{n+2}(n+1) - 2a_n)\, x^n = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 \tag{4}$$

For $n = 0$ the recurrence equation gives

$$(2a_2 - 2a_0)\, 1 = 1$$
$$2a_2 - 2a_0 = 1$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{1}{2} + a_0$$

For $n = 1$ the recurrence equation gives

$$(6a_3 - 2a_1)\, x = 2x$$
$$6a_3 - 2a_1 = 2$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{1}{3} + \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$(12a_4 - 2a_2)\, x^2 = 2x^2$$
$$12a_4 - 2a_2 = 2$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{4} + \frac{a_0}{6}$$

For $n = 3$ the recurrence equation gives

$$(20a_5 - 2a_3)\, x^3 = \frac{4x^3}{3}$$
$$20a_5 - 2a_3 = \frac{4}{3}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{10} + \frac{a_1}{30}$$

For $n = 4$ the recurrence equation gives

$$(30a_6 - 2a_4)\, x^4 = \frac{2x^4}{3}$$
$$30a_6 - 2a_4 = \frac{2}{3}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7}{180} + \frac{a_0}{90}$$

For $n = 5$ the recurrence equation gives

$$(42a_7 - 2a_5)\, x^5 = \frac{4x^5}{15}$$
$$42a_7 - 2a_5 = \frac{4}{15}$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{1}{90} + \frac{a_1}{630}$$

For $n = 6$ the recurrence equation gives

$$\left(56a_8 - 2a_6\right) x^6 = \frac{4x^6}{45}$$
$$56a_8 - 2a_6 = \frac{4}{45}$$

Which after substituting the earlier terms found becomes

$$a_8 = \frac{1}{336} + \frac{a_0}{2520}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(\frac{1}{2} + a_0\right) x^2 + \left(\frac{1}{3} + \frac{a_1}{3}\right) x^3 + \left(\frac{1}{4} + \frac{a_0}{6}\right) x^4 + \left(\frac{1}{10} + \frac{a_1}{30}\right) x^5 + \left(\frac{7}{180} + \frac{a_0}{90}\right) x^6 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x^2 + \frac{1}{6}x^4 + \frac{1}{90}x^6\right) a_0 + \left(x + \frac{1}{3}x^3 + \frac{1}{30}x^5\right) a_1 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10} + \frac{7x^6}{180} + O\left(x^7\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1+x^2+\frac{1}{6}x^4+\frac{1}{90}x^6\right) c_1 + \left(x+\frac{1}{3}x^3+\frac{1}{30}x^5\right) c_2 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10} + \frac{7x^6}{180} + O\left(x^7\right)$$

$$y=\frac{x^{2}}{2}+\frac{x^{3}}{3}+\frac{x^{4}}{4}+\frac{x^{5}}{10}+\frac{7x^{6}}{180}+O(x^{7})$$

Summary
The solution(s) found are the following

$$y=\frac{x^{2}}{2}+\frac{x^{3}}{3}+\frac{x^{4}}{4}+\frac{x^{5}}{10}+\frac{7x^{6}}{180}+\frac{x^{7}}{90}+O(x^{7}) \tag{1}$$
$$y=\frac{x^{2}}{2}+\frac{x^{3}}{3}+\frac{x^{4}}{4}+\frac{x^{5}}{10}+\frac{7x^{6}}{180}+O(x^{7}) \tag{2}$$

Verification of solutions

$$y=\frac{x^{2}}{2}+\frac{x^{3}}{3}+\frac{x^{4}}{4}+\frac{x^{5}}{10}+\frac{7x^{6}}{180}+\frac{x^{7}}{90}+O(x^{7})$$

Verified OK.

$$y=\frac{x^{2}}{2}+\frac{x^{3}}{3}+\frac{x^{4}}{4}+\frac{x^{5}}{10}+\frac{7x^{6}}{180}+O(x^{7})$$

Verified OK.

22.11.2 Maple step by step solution

Let's solve

$\left[y''=2y+e^{2x}, y(0)=0, y'\Big|_{\{x=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-2y=e^{2x}$

- Characteristic polynomial of homogeneous ODE

  $r^{2}-2=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{8})}{2}$

- Roots of the characteristic polynomial

  $r=(\sqrt{2}, -\sqrt{2})$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{\sqrt{2}\,x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\sqrt{2}\,x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\sqrt{2}\,x} + c_2 e^{-\sqrt{2}\,x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^{2x}\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\sqrt{2}\,x} & e^{-\sqrt{2}\,x} \\ \sqrt{2}\,e^{\sqrt{2}\,x} & -\sqrt{2}\,e^{-\sqrt{2}\,x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = -2\sqrt{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\sqrt{2}\left(e^{\sqrt{2}\,x}\left(\int e^{-x(\sqrt{2}-2)}dx\right) - e^{-\sqrt{2}\,x}\left(\int e^{x(2+\sqrt{2})}dx\right)\right)}{4}$

- Compute integrals

  $y_p(x) = \frac{e^{2x}}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{\sqrt{2}\,x} + c_2 e^{-\sqrt{2}\,x} + \frac{e^{2x}}{2}$

☐ Check validity of solution $y = c_1 e^{\sqrt{2}\,x} + c_2 e^{-\sqrt{2}\,x} + \frac{e^{2x}}{2}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 + \frac{1}{2}$

- Compute derivative of the solution

  $y' = c_1\sqrt{2}\,e^{\sqrt{2}\,x} - c_2\sqrt{2}\,e^{-\sqrt{2}\,x} + e^{2x}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = c_1\sqrt{2} - c_2\sqrt{2} + 1$

- Solve for c_1 and c_2

  $$\left\{c_1 = -\frac{1}{4} - \frac{\sqrt{2}}{4}, c_2 = -\frac{1}{4} + \frac{\sqrt{2}}{4}\right\}$$

- Substitute constant values into general solution and simplify

  $$y = \frac{\left(\sqrt{2}-1\right)e^{-\sqrt{2}\,x}}{4} + \frac{\left(-1-\sqrt{2}\right)e^{\sqrt{2}\,x}}{4} + \frac{e^{2x}}{2}$$

- Solution to the IVP

  $$y = \frac{\left(\sqrt{2}-1\right)e^{-\sqrt{2}\,x}}{4} + \frac{\left(-1-\sqrt{2}\right)e^{\sqrt{2}\,x}}{4} + \frac{e^{2x}}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=7;
dsolve([diff(y(x),x$2)-2*y(x)=exp(2*x),y(0) = 0, D(y)(0) = 0],y(x),type='series',x=0);
```

$$y(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{10}x^5 + \frac{7}{180}x^6 + O\left(x^7\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 39

```
AsymptoticDSolveValue[{y''[x]-2*y[x]==Exp[2*x],{y[0]==0,y'[0]==0}},y[x],{x,0,6}]
```

$$y(x) \to \frac{7x^6}{180} + \frac{x^5}{10} + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2}$$

22.12 problem 12

Internal problem ID [2375]
Internal file name [`OUTPUT/2375_Tuesday_February_27_2024_08_36_27_AM_54997599/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
   _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
    _reducible, _mu_xy]]
```

$$\boxed{y'' + 2yy' = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1090) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1091) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -2yy' \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= 4y^2y' - 2y'^2 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= -8yy'\left(y^2 - 2y'\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= 16\left(y^4 - \frac{11y^2y'}{2} + y'^2\right)y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= -32y'y\left(y^4 - 13y^2y' + \frac{17y'^2}{2}\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y}y' + \frac{\partial F_4}{\partial y'}F_4 \\
&= 64\left(y^6 - \frac{57y^4y'}{2} + 45y^2y'^2 - \frac{17y'^3}{4}\right)y'
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=0$ and

$y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -2 \\
F_2 &= 0 \\
F_3 &= 16 \\
F_4 &= 0 \\
F_5 &= -272
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + O(x^7)$$

$$y = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + O(x^7)$$

Unable to also solve using normal power series since not linear ode. Not currently supported.

Summary

The solution(s) found are the following

$$y = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + O(x^7) \tag{1}$$

Verification of solutions

$$y = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + O(x^7)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+2*_a*_b(_a) = 0, _b(_a), HINT =
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=7;
dsolve([diff(y(x),x$2)+2*y(x)*diff(y(x),x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',x=0);
```

$$y(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \mathrm{O}\left(x^7\right)$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]+2*y[x]*y'[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,6}]
```

$$y(x) \to \frac{2x^5}{15} - \frac{x^3}{3} + x$$

22.13 problem 13

Internal problem ID [2376]
Internal file name [`OUTPUT/2376_Tuesday_February_27_2024_08_36_27_AM_76446809/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y'' - \sin(y) = 0}$$

With initial conditions

$$\left[y(0) = \frac{\pi}{4}, y'(0) = 0\right]$$

With the expansion point for the power series method at $x = \frac{\pi}{4}$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - \frac{\pi}{4}$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) = \sin(y(t))$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$
\begin{aligned}
y(0) &= \frac{\pi}{4} \\
y'(0) &= 0
\end{aligned}
$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$
y'' = f(x, y, y')
$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1093) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1094) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \sin(y(t)) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \left(\frac{d}{dt}y(t)\right)\cos(y(t)) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \sin(y(t))\left(\cos(y(t)) - \left(\frac{d}{dt}y(t)\right)^2\right) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \left(4\cos(y(t))^2 - 3 - \left(\frac{d}{dt}y(t)\right)^2\cos(y(t))\right)\left(\frac{d}{dt}y(t)\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \sin(y(t))\left(\left(\frac{d}{dt}y(t)\right)^4 - 11\left(\frac{d}{dt}y(t)\right)^2\cos(y(t)) + 4\cos(y(t))^2 - 3\right) \\
F_5 &= \frac{dF_4}{dt} \\
&= \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_4}{\partial \frac{d}{dt}y(t)}F_4 \\
&= \left(\frac{d}{dt}y(t)\right)^5\cos(y(t)) + \left(-26\cos(y(t))^2 + 15\right)\left(\frac{d}{dt}y(t)\right)^3 + \left(34\cos(y(t))^3 - 33\cos(y(t))\right)\left(\frac{d}{dt}y(t)\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = \frac{\pi}{4}$ and

$y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= \frac{\sqrt{2}}{2} \\
F_1 &= 0 \\
F_2 &= \frac{1}{2} \\
F_3 &= 0 \\
F_4 &= -\frac{\sqrt{2}}{2} \\
F_5 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = \frac{\pi}{4} + \frac{\sqrt{2}\, t^2}{4} + \frac{t^4}{48} - \frac{\sqrt{2}\, t^6}{1440} + O(t^7)
$$

$$
y(t) = \frac{\pi}{4} + \frac{\sqrt{2}\, t^2}{4} + \frac{t^4}{48} - \frac{\sqrt{2}\, t^6}{1440} + O(t^7)
$$

Unable to also solve using normal power series since not linear ode. Not currently supported. Replacing t in the above with the original independent variable $x$using $t = x - \frac{\pi}{4}$ results in

$$
y = \frac{\pi}{4} + \frac{\sqrt{2}\left(x - \frac{\pi}{4}\right)^2}{4} + \frac{\left(x - \frac{\pi}{4}\right)^4}{48} - \frac{\sqrt{2}\left(x - \frac{\pi}{4}\right)^6}{1440} + O\left(\left(x - \frac{\pi}{4}\right)^7\right)
$$

Summary

The solution(s) found are the following

$$
y = \frac{\pi}{4} + \frac{\sqrt{2}\left(x - \frac{\pi}{4}\right)^2}{4} + \frac{\left(x - \frac{\pi}{4}\right)^4}{48} - \frac{\sqrt{2}\left(x - \frac{\pi}{4}\right)^6}{1440} + O\left(\left(x - \frac{\pi}{4}\right)^7\right) \tag{1}
$$

Verification of solutions

$$
y = \frac{\pi}{4} + \frac{\sqrt{2}\left(x - \frac{\pi}{4}\right)^2}{4} + \frac{\left(x - \frac{\pi}{4}\right)^4}{48} - \frac{\sqrt{2}\left(x - \frac{\pi}{4}\right)^6}{1440} + O\left(\left(x - \frac{\pi}{4}\right)^7\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-sin(_a) = 0, _b(_a)`   *** Suble
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✗ Solution by Maple

```
Order:=7;
dsolve([diff(y(x),x$2)=sin(y(x)),y(0) = 1/4*Pi, D(y)(0) = 0],y(x),type='series',x=1/4*Pi);
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
AsymptoticDSolveValue[{y''[x]==Sin[y[x]],{y[0]==Pi/4,y'[0]==0}},y[x],{x,1/4*Pi,6}]
```

Not solved

22.14 problem 14

Internal problem ID [2377]
Internal file name [`OUTPUT/2377_Tuesday_February_27_2024_08_36_31_AM_55098066/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + \frac{y'^2}{2} - y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1096) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1097) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y'^2}{2}+y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x}+\frac{\partial F_0}{\partial y}y'+\frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{y'^3}{2}+(1-y)\,y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x}+\frac{\partial F_1}{\partial y}y'+\frac{\partial F_1}{\partial y'}F_1 \\
&= -\frac{\left(-y'^2+2y-2\right)\left(-3y'^2+2y\right)}{4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x}+\frac{\partial F_2}{\partial y}y'+\frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{\left(-3y'^2+4y-1\right)y'\left(-y'^2+2y-2\right)}{2} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x}+\frac{\partial F_3}{\partial y}y'+\frac{\partial F_3}{\partial y'}F_3 \\
&= 4\left(\frac{15y'^4}{8}+\left(-\frac{15y}{4}+\frac{11}{8}\right)y'^2+y^2-\frac{y}{4}\right)\left(-\frac{y'^2}{2}+y-1\right) \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x}+\frac{\partial F_4}{\partial y}y'+\frac{\partial F_4}{\partial y'}F_4 \\
&= -34y'\left(\frac{1}{34}+\frac{45y'^4}{68}+\frac{(13-30y)\,y'^2}{17}+y^2-\frac{10y}{17}\right)\left(-\frac{y'^2}{2}+y-1\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=0$ and

$y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= -\frac{1}{2} \\
F_1 &= \frac{3}{2} \\
F_2 &= -\frac{9}{4} \\
F_3 &= 6 \\
F_4 &= -\frac{39}{2} \\
F_5 &= \frac{297}{4}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x - \frac{x^2}{4} + \frac{x^3}{4} - \frac{3x^4}{32} + \frac{x^5}{20} - \frac{13x^6}{480} + \frac{33x^7}{2240} + O(x^7)
$$

$$
y = x - \frac{x^2}{4} + \frac{x^3}{4} - \frac{3x^4}{32} + \frac{x^5}{20} - \frac{13x^6}{480} + \frac{33x^7}{2240} + O(x^7)
$$

Unable to also solve using normal power series since not linear ode. Not currently supported.

Summary

The solution(s) found are the following

$$
y = x - \frac{x^2}{4} + \frac{x^3}{4} - \frac{3x^4}{32} + \frac{x^5}{20} - \frac{13x^6}{480} + \frac{33x^7}{2240} + O(x^7) \tag{1}
$$

Verification of solutions

$$
y = x - \frac{x^2}{4} + \frac{x^3}{4} - \frac{3x^4}{32} + \frac{x^5}{20} - \frac{13x^6}{480} + \frac{33x^7}{2240} + O(x^7)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+(1/2)*_b(_a)^2-_a = 0, _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=7;
dsolve([diff(y(x),x$2)+1/2*diff(y(x),x)^2-y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',x
```

$$y(x) = x - \frac{1}{4}x^2 + \frac{1}{4}x^3 - \frac{3}{32}x^4 + \frac{1}{20}x^5 - \frac{13}{480}x^6 + \mathrm{O}\left(x^7\right)$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 40

```
AsymptoticDSolveValue[{y''[x]+1/2*y'[x]^2-y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,6}]
```

$$y(x) \to -\frac{13x^6}{480} + \frac{x^5}{20} - \frac{3x^4}{32} + \frac{x^3}{4} - \frac{x^2}{4} + x$$

22.15 problem 15

Internal problem ID [2378]
Internal file name [`OUTPUT/2378_Tuesday_February_27_2024_08_36_32_AM_36735985/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Taylor series method"**

Maple gives the following as the ode type

`[NONE]`

$$\boxed{y'' - \sin(yx) = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1, y'\left(\frac{\pi}{2}\right) = 1\right]$$

With the expansion point for the power series method at $x = \frac{\pi}{2}$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - \frac{\pi}{2}$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) = \sin\left(y(t)\left(t + \frac{\pi}{2}\right)\right)$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= 1
\end{aligned}$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1099) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1100) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = \sin\left(y(t)\left(t+\frac{\pi}{2}\right)\right)$$

$$\begin{aligned}
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \left(\left(\frac{d}{dt}y(t)\right)\left(t+\frac{\pi}{2}\right) + y(t)\right)\cos\left(y(t)\left(t+\frac{\pi}{2}\right)\right) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{\left((4t+2\pi)\cos\left(y(t)\left(t+\frac{\pi}{2}\right)\right) - 4\left(\left(\frac{d}{dt}y(t)\right)\left(t+\frac{\pi}{2}\right)+y(t)\right)^2\right)\sin\left(y(t)\left(t+\frac{\pi}{2}\right)\right)}{4} + 2\cos\left(y(t)\left(t+\frac{\pi}{2}\right.\right. \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{\left(24\sin\left(y(t)\left(t+\frac{\pi}{2}\right)\right) + (-\pi^3 - 6\pi^2 t - 12\pi t^2 - 8t^3)\left(\frac{d}{dt}y(t)\right)^3 + (-6\pi^2 - 24\pi t - 24t^2)\,y(t)\left(\frac{d}{dt}y(t)\right)^{\cdots}\right.}{8}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 1$ and $y'(0) = 1$ gives

$$\begin{aligned}
F_0 &= 1 \\
F_1 &= 0 \\
F_2 &= -\frac{1}{4}\pi^2 - \pi - 1 \\
F_3 &= -\frac{3}{4}\pi^2 - \frac{9}{2}\pi - 6
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = t + 1 + \frac{t^2}{2} - \frac{t^4\pi^2}{96} - \frac{t^4\pi}{24} - \frac{t^4}{24} - \frac{t^5\pi^2}{160} - \frac{3t^5\pi}{80} - \frac{t^5}{20} + O(t^5)$$

$$y(t) = t + 1 + \frac{t^2}{2} - \frac{t^4\pi^2}{96} - \frac{t^4\pi}{24} - \frac{t^4}{24} - \frac{t^5\pi^2}{160} - \frac{3t^5\pi}{80} - \frac{t^5}{20} + O(t^5)$$

Unable to also solve using normal power series since not linear ode. Not currently supported. Replacing t in the above with the original independent variable $x$using $t = x - \frac{\pi}{2}$ results in

$$y = x - \frac{\pi}{2} + 1 + \frac{\left(x - \frac{\pi}{2}\right)^2}{2} - \frac{\left(x - \frac{\pi}{2}\right)^4 \pi^2}{96} - \frac{\left(x - \frac{\pi}{2}\right)^4 \pi}{24} - \frac{\left(x - \frac{\pi}{2}\right)^4}{24} - \frac{\left(x - \frac{\pi}{2}\right)^5 \pi^2}{160} - \frac{3\left(x - \frac{\pi}{2}\right)^5 \pi}{80} - \frac{\left(x - \frac{\pi}{2}\right)^5}{20} + O\left(\left(x - \frac{\pi}{2}\right)^5\right)$$

Summary

The solution(s) found are the following

$$y = x - \frac{\pi}{2} + 1 + \frac{\left(x - \frac{\pi}{2}\right)^2}{2} - \frac{\left(x - \frac{\pi}{2}\right)^4 \pi^2}{96} - \frac{\left(x - \frac{\pi}{2}\right)^4 \pi}{24} - \frac{\left(x - \frac{\pi}{2}\right)^4}{24} - \frac{\left(x - \frac{\pi}{2}\right)^5 \pi^2}{160} - \frac{3\left(x - \frac{\pi}{2}\right)^5 \pi}{80} - \frac{\left(x - \frac{\pi}{2}\right)^5}{20} + O\left(\left(x - \frac{\pi}{2}\right)^5\right) \tag{1}$$

Verification of solutions

$$y = x - \frac{\pi}{2} + 1 + \frac{\left(x - \frac{\pi}{2}\right)^2}{2} - \frac{\left(x - \frac{\pi}{2}\right)^4 \pi^2}{96} - \frac{\left(x - \frac{\pi}{2}\right)^4 \pi}{24} - \frac{\left(x - \frac{\pi}{2}\right)^4}{24} - \frac{\left(x - \frac{\pi}{2}\right)^5 \pi^2}{160} - \frac{3\left(x - \frac{\pi}{2}\right)^5 \pi}{80} - \frac{\left(x - \frac{\pi}{2}\right)^5}{20} + O\left(\left(x - \frac{\pi}{2}\right)^5\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integrating f
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cases
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(y)
trying 2nd order, integrating factor of the form mu(x,y)
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
trying 2nd order, integrating factor of the form mu(y,y)
trying differential order: 2; mu polynomial in y
trying 2nd order, integrating factor of the form mu(x,y)
differential order: 2; looking for linear symmetries
-> trying 2nd order, the S-function method
   -> trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for the S-
   -> trying 2nd order, the S-function method
   -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
-> trying 2nd order, dynamical_symmetries, only a reduction of order through one integrating
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
`, `-> Computing symmetries using: way = patterns`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
Order:=5;
dsolve([diff(y(x),x$2)=sin(x*y(x)),y(1/2*Pi) = 1, D(y)(1/2*Pi) = 1],y(x),type='series',x=1/2
```

$$y(x) = 1 + \left(-\frac{\pi}{2} + x\right) + \frac{1}{2}\left(-\frac{\pi}{2} + x\right)^2 - \frac{1}{96}(\pi + 2)^2 \left(-\frac{\pi}{2} + x\right)^4 + \mathrm{O}\left(\left(-\frac{\pi}{2} + x\right)^5\right)$$

✓ Solution by Mathematica

Time used: 0.165 (sec). Leaf size: 47

```
AsymptoticDSolveValue[{y''[x]==Sin[x*y[x]],{y[Pi/2]==1,y'[Pi/2]==1}},y[x],{x,1/2*Pi,4}]
```

$$y(x) \to \frac{1}{96}(-4 - 4\pi - \pi^2) \left(x - \frac{\pi}{2}\right)^4 + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + x - \frac{\pi}{2} + 1$$

22.16 problem 16

Internal problem ID [2379]
Internal file name [`OUTPUT/2379_Tuesday_February_27_2024_08_36_33_AM_92924079/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 40, page 186
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Taylor series method"**

Maple gives the following as the ode type

`[NONE]`

$$\boxed{y'' - \cos(yx) = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1, y'\left(\frac{\pi}{2}\right) = 1\right]$$

With the expansion point for the power series method at $x = \frac{\pi}{2}$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - \frac{\pi}{2}$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) = \cos\left(y(t)\left(t + \frac{\pi}{2}\right)\right)$$

With its expansion point and initial conditions now at $t=0$. With initial conditions now becoming

$$
\begin{aligned}
y(0) &= 1 \\
y'(0) &= 1
\end{aligned}
$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$
y'' = f(x, y, y')
$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1102) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1103) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \cos\left(y(t)\left(t+\frac{\pi}{2}\right)\right)\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0\\
&= -\left(\left(\frac{d}{dt}y(t)\right)\left(t+\frac{\pi}{2}\right)+y(t)\right)\sin\left(y(t)\left(t+\frac{\pi}{2}\right)\right)\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1\\
&= \frac{\left((-4t-2\pi)\sin\left(y(t)\left(t+\frac{\pi}{2}\right)\right)-4\left(\left(\frac{d}{dt}y(t)\right)\left(t+\frac{\pi}{2}\right)+y(t)\right)^2\right)\cos\left(y(t)\left(t+\frac{\pi}{2}\right)\right)}{4} - 2\left(\frac{d}{dt}y(t)\right)\sin\left(y\right.\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2\\
&= \frac{\left(-24\cos\left(y(t)\left(t+\frac{\pi}{2}\right)\right)+(\pi^3+6\pi^2t+12\pi t^2+8t^3)\left(\frac{d}{dt}y(t)\right)^3+6(\pi^2+4\pi t+4t^2)\,y(t)\left(\frac{d}{dt}y(t)\right)^2+\right.}{8}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=1$ and $y'(0)=1$ gives

$$
\begin{aligned}
F_0 &= 0\\
F_1 &= -\frac{\pi}{2}-1\\
F_2 &= -2\\
F_3 &= \frac{1}{8}\pi^3+\pi^2+2\pi+1
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = t+1-\frac{t^3\pi}{12}-\frac{t^3}{6}-\frac{t^4}{12}+\frac{t^5\pi^3}{960}+\frac{t^5\pi^2}{120}+\frac{t^5\pi}{60}+\frac{t^5}{120}+O(t^5)
$$

$$
y(t) = t+1-\frac{t^3\pi}{12}-\frac{t^3}{6}-\frac{t^4}{12}+\frac{t^5\pi^3}{960}+\frac{t^5\pi^2}{120}+\frac{t^5\pi}{60}+\frac{t^5}{120}+O(t^5)
$$

Unable to also solve using normal power series since not linear ode. Not currently supported. Replacing t in the above with the original independent variable $x$susing $t = x - \frac{\pi}{2}$ results in

$$y = x - \frac{\pi}{2} + 1 - \frac{\left(x - \frac{\pi}{2}\right)^3 \pi}{12} - \frac{\left(x - \frac{\pi}{2}\right)^3}{6} - \frac{\left(x - \frac{\pi}{2}\right)^4}{12} + \frac{\left(x - \frac{\pi}{2}\right)^5 \pi^3}{960} + \frac{\left(x - \frac{\pi}{2}\right)^5 \pi^2}{120} + \frac{\left(x - \frac{\pi}{2}\right)^5 \pi}{60} + \frac{\left(x - \frac{\pi}{2}\right)^5}{120} + O\left(\left(x - \frac{\pi}{2}\right)^5\right)$$

Summary

The solution(s) found are the following

$$y = x - \frac{\pi}{2} + 1 - \frac{\left(x - \frac{\pi}{2}\right)^3 \pi}{12} - \frac{\left(x - \frac{\pi}{2}\right)^3}{6} - \frac{\left(x - \frac{\pi}{2}\right)^4}{12} + \frac{\left(x - \frac{\pi}{2}\right)^5 \pi^3}{960} + \frac{\left(x - \frac{\pi}{2}\right)^5 \pi^2}{120} + \frac{\left(x - \frac{\pi}{2}\right)^5 \pi}{60} + \frac{\left(x - \frac{\pi}{2}\right)^5}{120} + O\left(\left(x - \frac{\pi}{2}\right)^5\right) \tag{1}$$

Verification of solutions

$$y = x - \frac{\pi}{2} + 1 - \frac{\left(x - \frac{\pi}{2}\right)^3 \pi}{12} - \frac{\left(x - \frac{\pi}{2}\right)^3}{6} - \frac{\left(x - \frac{\pi}{2}\right)^4}{12} + \frac{\left(x - \frac{\pi}{2}\right)^5 \pi^3}{960} + \frac{\left(x - \frac{\pi}{2}\right)^5 \pi^2}{120} + \frac{\left(x - \frac{\pi}{2}\right)^5 \pi}{60} + \frac{\left(x - \frac{\pi}{2}\right)^5}{120} + O\left(\left(x - \frac{\pi}{2}\right)^5\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integrating f
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cases
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(y)
trying 2nd order, integrating factor of the form mu(x,y)
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
trying 2nd order, integrating factor of the form mu(y,y)
trying differential order: 2; mu polynomial in y
trying 2nd order, integrating factor of the form mu(x,y)
differential order: 2; looking for linear symmetries
-> trying 2nd order, the S-function method
   -> trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for the S-
   -> trying 2nd order, the S-function method
   -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
-> trying 2nd order, dynamical_symmetries, only a reduction of order through one integrating
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
`, `-> Computing symmetries using: way = patterns`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=5;
dsolve([diff(y(x),x$2)=cos(x*y(x)),y(1/2*Pi) = 1, D(y)(1/2*Pi) = 1],y(x),type='series',x=1/2
```

$$y(x) = 1 + \left(-\frac{\pi}{2} + x\right) + \left(-\frac{\pi}{12} - \frac{1}{6}\right)\left(-\frac{\pi}{2} + x\right)^3 - \frac{1}{12}\left(-\frac{\pi}{2} + x\right)^4 + \mathrm{O}\left(\left(-\frac{\pi}{2} + x\right)^5\right)$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 42

```
AsymptoticDSolveValue[{y''[x]==Cos[x*y[x]],{y[Pi/2]==1,y'[Pi/2]==1}},y[x],{x,1/2*Pi,4}]
```

$$y(x) \to -\frac{1}{12}\left(x - \frac{\pi}{2}\right)^4 + \frac{1}{12}(-2 - \pi)\left(x - \frac{\pi}{2}\right)^3 + x - \frac{\pi}{2} + 1$$

23 Exercise 41, page 195

23.1 problem 1

Internal problem ID [2380]
Internal file name [OUTPUT/2380_Tuesday_February_27_2024_08_36_35_AM_96855156/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + 5y' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + 5y' + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{5}{2x}$$
$$q(x) = \frac{1}{2}$$

Table 636: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2}$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + 5y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + 5\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + 5(n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + 5r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + 5r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(3+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -\frac{3}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(3+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 5a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{2n^2 + 4nr + 2r^2 + 3n + 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{2r^2 + 11r + 14}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 60r^3 + 325r^2 + 750r + 616}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{616}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{616}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{616}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n + r)(n + r - 1) + 5(n + r) b_n + b_{n-2} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{2n^2 + 4nr + 2r^2 + 3n + 3r} \tag{4}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_n = -\frac{b_{n-2}}{n (2n - 3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{2r^2 + 11r + 14}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 60r^3 + 325r^2 + 750r + 616}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$b_4 = \frac{1}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{40}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+11r+14}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4r^4+60r^3+325r^2+750r+616}$	$\frac{1}{40}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{x^{\frac{3}{2}}}
$$

Verified OK.

23.1.1 Maple step by step solution

Let's solve

$2y''x + 5y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{5y'}{2x} - \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{5y'}{2x} + \frac{y}{2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$\left[P_2(x) = \frac{5}{2x}, P_3(x) = \frac{1}{2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{5}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2y''x + 5y' + yx = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k - 1$

$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert y' to series expansion

$y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(3+2r)\, x^{-1+r} + a_1(1+r)\,(5+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(2k+5+2r) + a_{k-1}\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(1+r)\,(5+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+r+1)\left(k+\frac{5}{2}+r\right) a_{k+1} + a_{k-1} = 0$$

- Shift index using $k->k+1$

$$2(k+2+r)\left(k+\frac{7}{2}+r\right) a_{k+2} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+2+r)(2k+7+2r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k}{(k+2)(2k+7)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(2k+7)}, 5a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+4)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2}=-\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+4)}, -a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2}=-\frac{a_k}{(k+2)(2k+7)}, 5a_1=0, b_{k+2}=-\frac{b_k}{\left(k+\frac{1}{2}\right)(2k+4)}, -b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+5*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1\left(1-\frac{1}{2}x^2+\frac{1}{40}x^4+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{3}{2}}}+c_2\left(1-\frac{1}{14}x^2+\frac{1}{616}x^4+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 47

```
AsymptoticDSolveValue[2*x*y''[x]+5*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^4}{616}-\frac{x^2}{14}+1\right)+\frac{c_2\left(\frac{x^4}{40}-\frac{x^2}{2}+1\right)}{x^{3/2}}$$

23.2 problem 2

Internal problem ID [2381]
Internal file name [OUTPUT/2381_Tuesday_February_27_2024_08_36_35_AM_31564535/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x(2+3x)\, y''-4y'+4y=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(9x^2+6x\right) y''+4y-4y'=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x) = -\frac{4}{3x\,(2+3x)}$$
$$q(x) = \frac{4}{3x\,(2+3x)}$$

Table 638: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{4}{3x(2+3x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{2}{3}$	"regular"

$q(x) = \frac{4}{3x(2+3x)}$	
singularity	type
$x = 0$	"regular"
$x = -\frac{2}{3}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{2}{3}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x(2+3x)\, y'' - 4y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x(2+3x)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - 4\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-4(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 4a_n x^{n+r} = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} 9a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-4(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$6x^{n+r-1} a_n (n+r)(n+r-1) - 4(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$6x^{-1+r} a_0 r(-1+r) - 4r a_0 x^{-1+r} = 0$$

Or

$$\left(6x^{-1+r} r(-1+r) - 4r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(6r^2 - 10r\right) x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation becomes

$$6r^2 - 10r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{5}{3}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(6r^2 - 10r\right) x^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$9a_{n-1}(n + r - 1)(n + r - 2) + 6a_n(n + r)(n + r - 1) - 4a_n(n + r) + 4a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(9n^2 + 18nr + 9r^2 - 27n - 27r + 22)}{2\left(3n^2 + 6nr + 3r^2 - 5n - 5r\right)} \quad (4)$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_n = -\frac{a_{n-1}(9n^2 + 3n + 2)}{6n^2 + 10n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-9r^2 + 9r - 4}{6r^2 + 2r - 4}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_1 = -\frac{7}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-9r^2+9r-4}{6r^2+2r-4}$	$-\frac{7}{8}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{81r^4 - 9r^2 + 16}{36r^4 + 96r^3 + 28r^2 - 48r - 16}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_2 = \frac{7}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-9r^2+9r-4}{6r^2+2r-4}$	$-\frac{7}{8}$
a_2	$\frac{81r^4-9r^2+16}{36r^4+96r^3+28r^2-48r-16}$	$\frac{7}{8}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-729r^6 - 2187r^5 - 1701r^4 + 243r^3 + 54r^2 - 432r - 352}{216r^6 + 1512r^5 + 3528r^4 + 2744r^3 - 672r^2 - 1568r - 384}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_3 = -\frac{23}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-9r^2+9r-4}{6r^2+2r-4}$	$-\frac{7}{8}$
a_2	$\frac{81r^4-9r^2+16}{36r^4+96r^3+28r^2-48r-16}$	$\frac{7}{8}$
a_3	$\frac{-729r^6-2187r^5-1701r^4+243r^3+54r^2-432r-352}{216r^6+1512r^5+3528r^4+2744r^3-672r^2-1568r-384}$	$-\frac{23}{24}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(9r^2 - 9r + 4)(9r^2 + 9r + 4)(9r^2 + 45r + 58)(9r^2 + 27r + 22)}{16(27r^6 + 189r^5 + 441r^4 + 343r^3 - 84r^2 - 196r - 48)(3r^2 + 19r + 28)}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_4 = \frac{1817}{1632}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-9r^2+9r-4}{6r^2+2r-4}$	$-\frac{7}{8}$
a_2	$\frac{81r^4-9r^2+16}{36r^4+96r^3+28r^2-48r-16}$	$\frac{7}{8}$
a_3	$\frac{-729r^6-2187r^5-1701r^4+243r^3+54r^2-432r-352}{216r^6+1512r^5+3528r^4+2744r^3-672r^2-1568r-384}$	$-\frac{23}{24}$
a_4	$\frac{(9r^2-9r+4)(9r^2+9r+4)(9r^2+45r+58)(9r^2+27r+22)}{16(27r^6+189r^5+441r^4+343r^3-84r^2-196r-48)(3r^2+19r+28)}$	$\frac{1817}{1632}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(9r^2 - 9r + 4)(9r^2 + 9r + 4)(9r^2 + 45r + 58)(9r^2 + 27r + 22)(9r^2 + 63r + 112)}{32(27r^6 + 189r^5 + 441r^4 + 343r^3 - 84r^2 - 196r - 48)(3r^2 + 19r + 28)(3r^2 + 25r + 50)}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_5 = -\frac{219857}{163200}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-9r^2+9r-4}{6r^2+2r-4}$	$-\frac{7}{8}$
a_2	$\frac{81r^4-9r^2+16}{36r^4+96r^3+28r^2-48r-16}$	$\frac{7}{8}$
a_3	$\frac{-729r^6-2187r^5-1701r^4+243r^3+54r^2-432r-352}{216r^6+1512r^5+3528r^4+2744r^3-672r^2-1568r-384}$	$-\frac{23}{24}$
a_4	$\frac{(9r^2-9r+4)(9r^2+9r+4)(9r^2+45r+58)(9r^2+27r+22)}{16(27r^6+189r^5+441r^4+343r^3-84r^2-196r-48)(3r^2+19r+28)}$	$\frac{1817}{1632}$
a_5	$-\frac{(9r^2-9r+4)(9r^2+9r+4)(9r^2+45r+58)(9r^2+27r+22)(9r^2+63r+112)}{32(27r^6+189r^5+441r^4+343r^3-84r^2-196r-48)(3r^2+19r+28)(3r^2+25r+50)}$	$-\frac{219857}{163200}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{5}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{5}{3}}\left(1 - \frac{7x}{8} + \frac{7x^2}{8} - \frac{23x^3}{24} + \frac{1817x^4}{1632} - \frac{219857x^5}{163200} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$9b_{n-1}(n+r-1)(n+r-2) + 6b_n(n+r)(n+r-1) - 4(n+r)b_n + 4b_{n-1} = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(9n^2 + 18nr + 9r^2 - 27n - 27r + 22)}{2(3n^2 + 6nr + 3r^2 - 5n - 5r)} \quad (4)$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{9b_{n-1}\left(n^2 - 3n + \frac{22}{9}\right)}{6n^2 - 10n} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-9r^2 + 9r - 4}{6r^2 + 2r - 4}$$

Which for the root $r = 0$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-9r^2+9r-4}{6r^2+2r-4}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{81r^4 - 9r^2 + 16}{36r^4 + 96r^3 + 28r^2 - 48r - 16}$$

Which for the root $r = 0$ becomes

$$b_2 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-9r^2+9r-4}{6r^2+2r-4}$	1
b_2	$\frac{81r^4-9r^2+16}{36r^4+96r^3+28r^2-48r-16}$	-1

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-729r^6 - 2187r^5 - 1701r^4 + 243r^3 + 54r^2 - 432r - 352}{216r^6 + 1512r^5 + 3528r^4 + 2744r^3 - 672r^2 - 1568r - 384}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{11}{12}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-9r^2+9r-4}{6r^2+2r-4}$	1
b_2	$\frac{81r^4-9r^2+16}{36r^4+96r^3+28r^2-48r-16}$	-1
b_3	$\frac{-729r^6-2187r^5-1701r^4+243r^3+54r^2-432r-352}{216r^6+1512r^5+3528r^4+2744r^3-672r^2-1568r-384}$	$\frac{11}{12}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(9r^2-9r+4)(9r^2+9r+4)(9r^2+45r+58)(9r^2+27r+22)}{16(27r^6+189r^5+441r^4+343r^3-84r^2-196r-48)(3r^2+19r+28)}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{319}{336}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-9r^2+9r-4}{6r^2+2r-4}$	1
b_2	$\frac{81r^4-9r^2+16}{36r^4+96r^3+28r^2-48r-16}$	-1
b_3	$\frac{-729r^6-2187r^5-1701r^4+243r^3+54r^2-432r-352}{216r^6+1512r^5+3528r^4+2744r^3-672r^2-1568r-384}$	$\frac{11}{12}$
b_4	$\frac{(9r^2-9r+4)(9r^2+9r+4)(9r^2+45r+58)(9r^2+27r+22)}{16(27r^6+189r^5+441r^4+343r^3-84r^2-196r-48)(3r^2+19r+28)}$	$-\frac{319}{336}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(9r^2-9r+4)(9r^2+9r+4)(9r^2+45r+58)(9r^2+27r+22)(9r^2+63r+112)}{32(27r^6+189r^5+441r^4+343r^3-84r^2-196r-48)(3r^2+19r+28)(3r^2+25r+50)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{319}{300}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-9r^2+9r-4}{6r^2+2r-4}$	1
b_2	$\frac{81r^4-9r^2+16}{36r^4+96r^3+28r^2-48r-16}$	-1
b_3	$\frac{-729r^6-2187r^5-1701r^4+243r^3+54r^2-432r-352}{216r^6+1512r^5+3528r^4+2744r^3-672r^2-1568r-384}$	$\frac{11}{12}$
b_4	$\frac{(9r^2-9r+4)(9r^2+9r+4)(9r^2+45r+58)(9r^2+27r+22)}{16(27r^6+189r^5+441r^4+343r^3-84r^2-196r-48)(3r^2+19r+28)}$	$-\frac{319}{336}$
b_5	$-\frac{(9r^2-9r+4)(9r^2+9r+4)(9r^2+45r+58)(9r^2+27r+22)(9r^2+63r+112)}{32(27r^6+189r^5+441r^4+343r^3-84r^2-196r-48)(3r^2+19r+28)(3r^2+25r+50)}$	$\frac{319}{300}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 + x - x^2 + \frac{11x^3}{12} - \frac{319x^4}{336} + \frac{319x^5}{300} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{5}{3}} \left(1 - \frac{7x}{8} + \frac{7x^2}{8} - \frac{23x^3}{24} + \frac{1817x^4}{1632} - \frac{219857x^5}{163200} + O(x^6)\right) \\
&\quad + c_2 \left(1 + x - x^2 + \frac{11x^3}{12} - \frac{319x^4}{336} + \frac{319x^5}{300} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{5}{3}} \left(1 - \frac{7x}{8} + \frac{7x^2}{8} - \frac{23x^3}{24} + \frac{1817x^4}{1632} - \frac{219857x^5}{163200} + O(x^6)\right) \\
&\quad + c_2 \left(1 + x - x^2 + \frac{11x^3}{12} - \frac{319x^4}{336} + \frac{319x^5}{300} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{5}{3}} \left(1 - \frac{7x}{8} + \frac{7x^2}{8} - \frac{23x^3}{24} + \frac{1817x^4}{1632} - \frac{219857x^5}{163200} + O(x^6)\right) \\
+ c_2 \left(1 + x - x^2 + \frac{11x^3}{12} - \frac{319x^4}{336} + \frac{319x^5}{300} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{5}{3}} \left(1 - \frac{7x}{8} + \frac{7x^2}{8} - \frac{23x^3}{24} + \frac{1817x^4}{1632} - \frac{219857x^5}{163200} + O(x^6)\right) + c_2 \left(1 + x - x^2 + \frac{11x^3}{12} - \frac{319x^4}{336} + \frac{319x^5}{300} + O(x^6)\right)$$

Verified OK.

23.2.1 Maple step by step solution

Let's solve

$3x(2+3x)\,y'' - 4y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{3x(2+3x)} + \frac{4y'}{3x(2+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{3x(2+3x)} + \frac{4y}{3x(2+3x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{4}{3x(2+3x)}, P_3(x) = \frac{4}{3x(2+3x)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{2}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x(2+3x)\,y'' - 4y' + 4y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

  ◦ Shift index using $k->k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  ◦ Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$2a_0r(-5+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (2a_{k+1}(k+1+r)\,(3k-2+3r) + a_k(9k^2+18kr+9r^2-9k-9r+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-5+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{5}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$6(k+1+r)\left(k-\frac{2}{3}+r\right) a_{k+1} + 9\left(k^2+(2r-1)\,k+r^2-r+\frac{4}{9}\right) a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{(9k^2+18kr+9r^2-9k-9r+4)a_k}{2(k+1+r)(3k-2+3r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{(9k^2-9k+4)a_k}{2(k+1)(3k-2)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{(9k^2-9k+4)a_k}{2(k+1)(3k-2)}\right]$$

- Recursion relation for $r = \frac{5}{3}$

  $a_{k+1} = -\frac{(9k^2+21k+14)a_k}{2(k+\frac{8}{3})(3k+3)}$

- Solution for $r = \frac{5}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{3}}, a_{k+1} = -\frac{(9k^2+21k+14)a_k}{2(k+\frac{8}{3})(3k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{3}}\right), a_{k+1} = -\frac{(9k^2-9k+4)a_k}{2(k+1)(3k-2)}, b_{k+1} = -\frac{(9k^2+21k+14)b_k}{2(k+\frac{8}{3})(3k+3)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(3*x*(2+3*x)*diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{5}{3}} \left(1 - \frac{7}{8}x + \frac{7}{8}x^2 - \frac{23}{24}x^3 + \frac{1817}{1632}x^4 - \frac{219857}{163200}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(1 + x - x^2 + \frac{11}{12}x^3 - \frac{319}{336}x^4 + \frac{319}{300}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 79

```
AsymptoticDSolveValue[3*x*(2+3*x)*y''[x]-4*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{319x^5}{300} - \frac{319x^4}{336} + \frac{11x^3}{12} - x^2 + x + 1\right) + c_1 \left(-\frac{219857x^5}{163200} + \frac{1817x^4}{1632} - \frac{23x^3}{24} + \frac{7x^2}{8} - \frac{7x}{8} + 1\right) x^{5/3}$$

23.3 problem 3

Internal problem ID [2382]
Internal file name [`OUTPUT/2382_Tuesday_February_27_2024_08_36_39_AM_61879799/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+4)\,y''+7xy'-y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+4x^2\right)y''+7xy'-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{7}{x\,(x+4)}$$
$$q(x)=-\frac{1}{x^2\,(x+4)}$$

Table 640: Table $p(x), q(x)$ singularites.

$p(x) = \frac{7}{x(x+4)}$	
singularity	type
$x = -4$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{1}{x^2(x+4)}$	
singularity	type
$x = -4$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-4, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+4)\, y'' + 7xy' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+4)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 7x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 7x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) + 7x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + 7x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) + 7x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 + 3r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 + 3r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 + 3r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n + r - 1)(n + r - 2) + 4a_n(n + r)(n + r - 1) + 7a_n(n + r) - a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n + r - 1)(n + r - 2)}{4n^2 + 8nr + 4r^2 + 3n + 3r - 1} \quad (4)$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_n = -\frac{a_{n-1}(16n^2 - 40n + 21)}{64n^2 + 80n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r(-1+r)}{4r^2 + 11r + 6}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_1 = \frac{1}{48}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{4r^2+11r+6}$	$\frac{1}{48}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2(r^2 - 1)}{(4r^2 + 11r + 6)(4r^2 + 19r + 21)}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_2 = -\frac{5}{19968}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{4r^2+11r+6}$	$\frac{1}{48}$
a_2	$\frac{r^2(r^2-1)}{(4r^2+11r+6)(4r^2+19r+21)}$	$-\frac{5}{19968}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(1+r)^2 r^2(-1+r)}{64r^5 + 784r^4 + 3644r^3 + 7931r^2 + 7905r + 2772}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_3 = \frac{25}{1810432}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{4r^2+11r+6}$	$\frac{1}{48}$
a_2	$\frac{r^2(r^2-1)}{(4r^2+11r+6)(4r^2+19r+21)}$	$-\frac{5}{19968}$
a_3	$-\frac{(1+r)^2 r^2(-1+r)}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	$\frac{25}{1810432}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(2+r)(1+r)^2 r^2(-1+r)}{256r^6 + 4608r^5 + 32992r^4 + 119088r^3 + 225241r^2 + 206865r + 69300}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_4 = -\frac{75}{62390272}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{4r^2+11r+6}$	$\frac{1}{48}$
a_2	$\frac{r^2(r^2-1)}{(4r^2+11r+6)(4r^2+19r+21)}$	$-\frac{5}{19968}$
a_3	$-\frac{(1+r)^2 r^2(-1+r)}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	$\frac{25}{1810432}$
a_4	$\frac{(2+r)(1+r)^2 r^2(-1+r)}{256r^6+4608r^5+32992r^4+119088r^3+225241r^2+206865r+69300}$	$-\frac{75}{62390272}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(3+r)(2+r)(1+r)^2 r^2(-1+r)}{1024r^7 + 25344r^6 + 257920r^5 + 1388640r^4 + 4228276r^3 + 7175751r^2 + 6146865r + 1975050}$$

Which for the root $r = \frac{1}{4}$ becomes

$$a_5 = \frac{39}{293601280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{4r^2+11r+6}$	$\frac{1}{48}$
a_2	$\frac{r^2(r^2-1)}{(4r^2+11r+6)(4r^2+19r+21)}$	$-\frac{5}{19968}$
a_3	$-\frac{(1+r)^2 r^2(-1+r)}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	$\frac{25}{1810432}$
a_4	$\frac{(2+r)(1+r)^2 r^2(-1+r)}{256r^6+4608r^5+32992r^4+119088r^3+225241r^2+206865r+69300}$	$-\frac{75}{62390272}$
a_5	$-\frac{(3+r)(2+r)(1+r)^2 r^2(-1+r)}{1024r^7+25344r^6+257920r^5+1388640r^4+4228276r^3+7175751r^2+6146865r+1975050}$	$\frac{39}{293601280}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{4}}\left(1 + \frac{x}{48} - \frac{5x^2}{19968} + \frac{25x^3}{1810432} - \frac{75x^4}{62390272} + \frac{39x^5}{293601280} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2) + 4b_n(n+r)(n+r-1) + 7b_n(n+r) - b_n = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n+r-1)(n+r-2)}{4n^2+8nr+4r^2+3n+3r-1} \quad (4)$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-1}(n-2)(n-3)}{n(4n-5)} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r(-1+r)}{4r^2 + 11r + 6}$$

Which for the root $r = -1$ becomes

$$b_1 = 2$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r(-1+r)}{4r^2+11r+6}$	2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2(r^2-1)}{(4r^2+11r+6)(4r^2+19r+21)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r(-1+r)}{4r^2+11r+6}$	2
b_2	$\frac{r^2(r^2-1)}{(4r^2+11r+6)(4r^2+19r+21)}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(1+r)^2 r^2(-1+r)}{64r^5 + 784r^4 + 3644r^3 + 7931r^2 + 7905r + 2772}$$

Which for the root $r = -1$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r(-1+r)}{4r^2+11r+6}$	2
b_2	$\frac{r^2(r^2-1)}{(4r^2+11r+6)(4r^2+19r+21)}$	0
b_3	$-\frac{(1+r)^2 r^2(-1+r)}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(2+r)(1+r)^2 r^2(-1+r)}{256r^6 + 4608r^5 + 32992r^4 + 119088r^3 + 225241r^2 + 206865r + 69300}$$

Which for the root $r = -1$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r(-1+r)}{4r^2+11r+6}$	2
b_2	$\frac{r^2(r^2-1)}{(4r^2+11r+6)(4r^2+19r+21)}$	0
b_3	$-\frac{(1+r)^2 r^2(-1+r)}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	0
b_4	$\frac{(2+r)(1+r)^2 r^2(-1+r)}{256r^6+4608r^5+32992r^4+119088r^3+225241r^2+206865r+69300}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(3+r)(2+r)(1+r)^2 r^2(-1+r)}{1024r^7 + 25344r^6 + 257920r^5 + 1388640r^4 + 4228276r^3 + 7175751r^2 + 6146865r + 1975050}$$

Which for the root $r = -1$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r(-1+r)}{4r^2+11r+6}$	2
b_2	$\frac{r^2(r^2-1)}{(4r^2+11r+6)(4r^2+19r+21)}$	0
b_3	$-\frac{(1+r)^2 r^2(-1+r)}{64r^5+784r^4+3644r^3+7931r^2+7905r+2772}$	0
b_4	$\frac{(2+r)(1+r)^2 r^2(-1+r)}{256r^6+4608r^5+32992r^4+119088r^3+225241r^2+206865r+69300}$	0
b_5	$-\frac{(3+r)(2+r)(1+r)^2 r^2(-1+r)}{1024r^7+25344r^6+257920r^5+1388640r^4+4228276r^3+7175751r^2+6146865r+1975050}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{4}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right) \\
&= \frac{1+2x+O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1x^{\frac{1}{4}}\left(1+\frac{x}{48}-\frac{5x^2}{19968}+\frac{25x^3}{1810432}-\frac{75x^4}{62390272}+\frac{39x^5}{293601280}+O(x^6)\right) \\
&\quad+\frac{c_2(1+2x+O(x^6))}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x^{\frac{1}{4}}\left(1+\frac{x}{48}-\frac{5x^2}{19968}+\frac{25x^3}{1810432}-\frac{75x^4}{62390272}+\frac{39x^5}{293601280}+O(x^6)\right) \\
&\quad+\frac{c_2(1+2x+O(x^6))}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1x^{\frac{1}{4}}&\left(1+\frac{x}{48}-\frac{5x^2}{19968}+\frac{25x^3}{1810432}-\frac{75x^4}{62390272}+\frac{39x^5}{293601280}+O(x^6)\right) \\
&+\frac{c_2(1+2x+O(x^6))}{x}
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{1}{4}} \left(1 + \frac{x}{48} - \frac{5x^2}{19968} + \frac{25x^3}{1810432} - \frac{75x^4}{62390272} + \frac{39x^5}{293601280} + O(x^6)\right) + \frac{c_2(1 + 2x + O(x^6))}{x}$$

Verified OK.

23.3.1 Maple step by step solution

Let's solve

$x^2(x+4)\, y'' + 7xy' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x^2(x+4)} - \frac{7y'}{x(x+4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{7y'}{x(x+4)} - \frac{y}{x^2(x+4)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{7}{x(x+4)}, P_3(x) = -\frac{1}{x^2(x+4)}\right]$

  ◦ $(x+4) \cdot P_2(x)$ is analytic at $x = -4$

    $\left((x+4) \cdot P_2(x)\right)\Big|_{x=-4} = -\frac{7}{4}$

  ◦ $(x+4)^2 \cdot P_3(x)$ is analytic at $x = -4$

    $\left((x+4)^2 \cdot P_3(x)\right)\Big|_{x=-4} = 0$

  ◦ $x = -4$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -4$

- Multiply by denominators

  $x^2(x+4)\, y'' + 7xy' - y = 0$

- Change variables using $x = u - 4$ so that the regular singular point is at $u = 0$

$$\left(u^{3}-8u^{2}+16u\right)\left(\frac{d^{2}}{du^{2}}y(u)\right)+\left(7u-28\right)\left(\frac{d}{du}y(u)\right)-y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $u^{m}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..3$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_{0}r(-11+4r)\,u^{-1+r}+\left(4a_{1}(1+r)\,(-7+4r)-a_{0}(8r^{2}-15r+1)\right)u^{r}+\left(\sum_{k=1}^{\infty}(4a_{k+1}(k+1+r)\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$4r(-11+4r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{11}{4}\right\}$$

- Each term must be 0

$$4a_{1}(1+r)\,(-7+4r)-a_{0}(8r^{2}-15r+1)=0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(-8a_{k}+a_{k-1}+16a_{k+1}\right)k^{2}+\left(2(-8a_{k}+a_{k-1}+16a_{k+1})\,r+15a_{k}-3a_{k-1}-12a_{k+1}\right)k+(-8a_{k}+$$

- Shift index using $k->k+1$

$$\left(-8a_{k+1}+a_{k}+16a_{k+2}\right)(k+1)^{2}+\left(2(-8a_{k+1}+a_{k}+16a_{k+2})\,r+15a_{k+1}-3a_{k}-12a_{k+2}\right)(k+1$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2kra_k - 16kra_{k+1} + r^2 a_k - 8r^2 a_{k+1} - ka_k - ka_{k+1} - ra_k - ra_{k+1} + 6a_{k+1}}{4(4k^2 + 8kr + 4r^2 + 5k + 5r - 6)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - ka_k - ka_{k+1} + 6a_{k+1}}{4(4k^2 + 5k - 6)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - ka_k - ka_{k+1} + 6a_{k+1}}{4(4k^2 + 5k - 6)}, -28a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 4$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+4)^k, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - ka_k - ka_{k+1} + 6a_{k+1}}{4(4k^2 + 5k - 6)}, -28a_1 - a_0 = 0\right]$$

- Recursion relation for $r = \frac{11}{4}$

$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + \frac{9}{2}ka_k - 45ka_{k+1} + \frac{77}{16}a_k - \frac{229}{4}a_{k+1}}{4(4k^2 + 27k + 38)}$$

- Solution for $r = \frac{11}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{11}{4}}, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + \frac{9}{2}ka_k - 45ka_{k+1} + \frac{77}{16}a_k - \frac{229}{4}a_{k+1}}{4(4k^2 + 27k + 38)}, 60a_1 - \frac{81a_0}{4} = 0\right]$$

- Revert the change of variables $u = x + 4$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+4)^{k+\frac{11}{4}}, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + \frac{9}{2}ka_k - 45ka_{k+1} + \frac{77}{16}a_k - \frac{229}{4}a_{k+1}}{4(4k^2 + 27k + 38)}, 60a_1 - \frac{81a_0}{4} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+4)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+4)^{k+\frac{11}{4}}\right), a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - ka_k - ka_{k+1} + 6a_{k+1}}{4(4k^2 + 5k - 6)}, -28a_1 - a$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
Order:=6;
dsolve(x^2*(4+x)*diff(y(x),x$2)+7*x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{5}{4}} \left(1+\frac{1}{48}x-\frac{5}{19968}x^2+\frac{25}{1810432}x^3-\frac{75}{62390272}x^4+\frac{39}{293601280}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1+2x+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 58

```
AsymptoticDSolveValue[x^2*(4+x)*y''[x]+7*x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[4]{x} \left(\frac{39x^5}{293601280} - \frac{75x^4}{62390272} + \frac{25x^3}{1810432} - \frac{5x^2}{19968} + \frac{x}{48} + 1 \right) + \frac{c_2(2x+1)}{x}$$

23.4 problem 4

Internal problem ID [2383]
Internal file name [`OUTPUT/2383_Tuesday_February_27_2024_08_36_40_AM_3420785/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + \left(-x^2+x\right)y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + \left(-x^2+x\right)y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{2x}$$
$$q(x) = -\frac{1}{2x^2}$$

Table 642: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + (-x^2 + x)\, y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-x^2 + x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n+2r+1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}}{2n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{3 + 2r}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^2 + 16r + 15}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8r^3 + 60r^2 + 142r + 105}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$
a_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{315}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^4 + 192r^3 + 824r^2 + 1488r + 945}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{3465}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$
a_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{315}$
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{3465}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32r^5 + 560r^4 + 3760r^3 + 12040r^2 + 18258r + 10395}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1}{45045}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$
a_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{315}$
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{3465}$
a_5	$\frac{1}{32r^5+560r^4+3760r^3+12040r^2+18258r+10395}$	$\frac{1}{45045}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right)\\ &= x\left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + b_n(n+r) - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{2n+2r+1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = \frac{b_{n-1}}{2n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{3+2r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{8r^3 + 60r^2 + 142r + 105}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = \frac{1}{48}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^4 + 192r^3 + 824r^2 + 1488r + 945}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{48}$
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{32r^5 + 560r^4 + 3760r^3 + 12040r^2 + 18258r + 10395}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = \frac{1}{3840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{48}$
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$
b_5	$\frac{1}{32r^5+560r^4+3760r^3+12040r^2+18258r+10395}$	$\frac{1}{3840}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Verified OK.

23.4.1 Maple step by step solution

Let's solve

$2x^2y'' + (-x^2 + x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{2x^2} + \frac{(x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{2x} - \frac{y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-1}{2x}, P_3(x) = -\frac{1}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{2}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' - x(x-1)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(\left(k+r+\frac{1}{2}\right) a_k - \frac{a_{k-1}}{2}\right)(k+r-1) = 0$$

- Shift index using $k- >k+1$

$$2\left(\left(k+\frac{3}{2}+r\right) a_{k+1} - \frac{a_k}{2}\right)(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{2k+3+2r}$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{a_k}{2k+5}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{2k+5}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = \frac{a_k}{2k+2}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = \frac{a_k}{2k+5}, b_{k+1} = \frac{b_k}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+(x-x^2)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\frac{1}{3840}x^5+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}} + c_2 x\left(1+\frac{1}{5}x+\frac{1}{35}x^2+\frac{1}{315}x^3+\frac{1}{3465}x^4+\frac{1}{45045}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 86

```
AsymptoticDSolveValue[2*x^2*y''[x]+(x-x^2)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(\frac{x^5}{45045} + \frac{x^4}{3465} + \frac{x^3}{315} + \frac{x^2}{35} + \frac{x}{5} + 1 \right) + \frac{c_2 \left(\frac{x^5}{3840} + \frac{x^4}{384} + \frac{x^3}{48} + \frac{x^2}{8} + \frac{x}{2} + 1 \right)}{\sqrt{x}}$$

23.5 problem 5

Internal problem ID [2384]
Internal file name [OUTPUT/2384_Tuesday_February_27_2024_08_36_40_AM_45741848/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + 5xy' + y(x+1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 5xy' + y(x+1) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{5}{2x}$$
$$q(x) = \frac{x+1}{2x^2}$$

Table 644: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x+1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 5xy' + y(x + 1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2} \right) + 5x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) (x + 1) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) \tag{2A}
$$
$$
+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) \tag{2B}
$$
$$
+\left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n (n+r)(n+r-1)+5x^{n+r} a_n (n+r)+a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
2x^r a_0 r(-1+r)+5x^r a_0 r+a_0 x^r=0
$$

Or

$$
\left(2x^r r(-1+r)+5x^r r+x^r\right) a_0=0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(2r^2+3r+1\right) x^r=0
$$

Since the above is true for all x then the indicial equation becomes

$$
2r^2+3r+1=0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{1}{2}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 5a_n(n+r) + a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n^2 + 4nr + 2r^2 + 3n + 3r + 1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 7r + 6}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 36r^3 + 119r^2 + 171r + 90}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^6 + 132r^5 + 890r^4 + 3135r^3 + 6077r^2 + 6138r + 2520}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_3 = -\frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400)\,(2r^2 -}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_5 = -\frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{22680}$
a_5	$-\frac{1}{(16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400)(2r^2+23r+66)}$	$-\frac{1}{1247400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{\sqrt{x}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + 5b_n(n+r) + b_{n-1} + b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{2n^2 + 4nr + 2r^2 + 3n + 3r + 1} \tag{4}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{b_{n-1}}{n(2n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{2r^2 + 7r + 6}$$

Which for the root $r = -1$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 36r^3 + 119r^2 + 171r + 90}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{8r^6 + 132r^5 + 890r^4 + 3135r^3 + 6077r^2 + 6138r + 2520}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{(16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400)(2r^2 +}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{2520}$
b_5	$-\frac{1}{(16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400)(2r^2+23r+66)}$	$-\frac{1}{113400}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \frac{1}{\sqrt{x}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots\right) \\
&= \frac{1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1\left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + O(x^6)\right)}{\sqrt{x}} \\
&\quad + \frac{c_2\left(1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= \frac{c_1\left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + O(x^6)\right)}{\sqrt{x}} \\
&\quad + \frac{c_2\left(1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right)}{\sqrt{x}} + \frac{c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right)}{\sqrt{x}} + \frac{c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)}{x}$$

Verified OK.

23.5.1 Maple step by step solution

Let's solve

$2x^2y''+5xy'+y(x+1)=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{5y'}{2x}-\frac{(x+1)y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{5y'}{2x}+\frac{(x+1)y}{2x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

- ∘ Define functions

  $\left[P_2(x)=\frac{5}{2x}, P_3(x)=\frac{x+1}{2x^2}\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{5}{2}$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{2}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + 5xy' + y(x+1) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r+1) + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r+1)\left(k+r+\frac{1}{2}\right) a_k + a_{k-1} = 0$

- Shift index using $k->k+1$

  $2(k+2+r)\left(k+\frac{3}{2}+r\right)a_{k+1}+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{(k+2+r)(2k+3+2r)}$
- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{a_k}{(k+1)(2k+1)}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-1}, a_{k+1}=-\frac{a_k}{(k+1)(2k+1)}\right]$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+1}=-\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+1}=-\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{(k+1)(2k+1)}, b_{k+1}=-\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \frac{1}{2520}x^4 - \frac{1}{113400}x^5 + \mathrm{O}\left(x^6\right)\right)}{x} + \frac{c_2\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \frac{1}{22680}x^4 - \frac{1}{1247400}x^5 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 86

```
AsymptoticDSolveValue[2*x^2*y''[x]+5*x*y'[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(-\frac{x^5}{1247400} + \frac{x^4}{22680} - \frac{x^3}{630} + \frac{x^2}{30} - \frac{x}{3} + 1\right)}{\sqrt{x}} + \frac{c_2\left(-\frac{x^5}{113400} + \frac{x^4}{2520} - \frac{x^3}{90} + \frac{x^2}{6} - x + 1\right)}{x}$$

23.6 problem 6

Internal problem ID [2385]
Internal file name [`OUTPUT/2385_Tuesday_February_27_2024_08_36_41_AM_57152745/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2y'' + (2+3x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + (2+3x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{2+3x}{9x^2} \end{aligned}$$

Table 646: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{2+3x}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + (2 + 3x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (2+3x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) + 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 2a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 9r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 9r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 9r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) + 2a_n + 3a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_n = -\frac{a_{n-1}}{3n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3}{9r^2 + 9r + 2}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_1 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = \frac{1}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{4}$
a_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{56}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_3 = -\frac{1}{1680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{4}$
a_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{56}$
a_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{1680}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = \frac{1}{87360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{4}$
a_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{56}$
a_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{1680}$
a_4	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{1}{87360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_5 = -\frac{1}{6988800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{4}$
a_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{56}$
a_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{1680}$
a_4	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{1}{87360}$
a_5	$-\frac{243}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{1}{6988800}$

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = x^{\frac{2}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\dots\right)$$
$$= x^{\frac{2}{3}}\left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \frac{x^4}{87360} - \frac{x^5}{6988800} + O(x^6)\right)$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$9b_n(n+r)(n+r-1) + 2b_n + 3b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{3b_{n-1}}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_n = -\frac{b_{n-1}}{n(3n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{3}{9r^2 + 9r + 2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = \frac{1}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{20}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{27}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_3 = -\frac{1}{480}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{20}$
b_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{480}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{1}{21120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{20}$
b_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{480}$
b_4	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{1}{21120}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)(9r^2 + 81r + 182)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_5 = -\frac{1}{1478400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{20}$
b_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{480}$
b_4	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{1}{21120}$
b_5	$-\frac{243}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{1}{1478400}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{2}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \frac{x^4}{21120} - \frac{x^5}{1478400} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \frac{x^4}{87360} - \frac{x^5}{6988800} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \frac{x^4}{21120} - \frac{x^5}{1478400} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \frac{x^4}{87360} - \frac{x^5}{6988800} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \frac{x^4}{21120} - \frac{x^5}{1478400} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \frac{x^4}{87360} - \frac{x^5}{6988800} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \frac{x^4}{21120} - \frac{x^5}{1478400} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \frac{x^4}{87360} - \frac{x^5}{6988800} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \frac{x^4}{21120} - \frac{x^5}{1478400} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

23.6.1 Maple step by step solution

Let's solve

$9x^2y'' + (2+3x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2+3x)y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2+3x)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{2+3x}{9x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{2}{9}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2y'' + (2+3x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)(-2+3r) x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-1)(3k+3r-2) + 3a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+3r)(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$9\left(k+r-\frac{1}{3}\right)\left(k+r-\frac{2}{3}\right) a_k + 3a_{k-1} = 0$$

- Shift index using $k->k+1$

$$9\left(k+\frac{2}{3}+r\right)\left(k+\frac{1}{3}+r\right) a_{k+1} + 3a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{3a_k}{(3k+2+3r)(3k+1+3r)}$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = -\frac{3a_k}{(3k+3)(3k+2)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{3a_k}{(3k+3)(3k+2)}\right]$$

- Recursion relation for $r = \frac{2}{3}$

$$a_{k+1} = -\frac{3a_k}{(3k+4)(3k+3)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+1} = -\frac{3a_k}{(3k+4)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{k+1} = -\frac{3a_k}{(3k+3)(3k+2)}, b_{k+1} = -\frac{3b_k}{(3k+4)(3k+3)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)+(2+3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 - \frac{1}{2}x + \frac{1}{20}x^2 - \frac{1}{480}x^3 + \frac{1}{21120}x^4 - \frac{1}{1478400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{2}{3}} \left(1 - \frac{1}{4}x + \frac{1}{56}x^2 - \frac{1}{1680}x^3 + \frac{1}{87360}x^4 - \frac{1}{6988800}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 90

```
AsymptoticDSolveValue[9*x^2*y''[x]+(2+3*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \sqrt[3]{x} \left(-\frac{x^5}{1478400} + \frac{x^4}{21120} - \frac{x^3}{480} + \frac{x^2}{20} - \frac{x}{2} + 1\right) + c_1 x^{2/3} \left(-\frac{x^5}{6988800} + \frac{x^4}{87360} - \frac{x^3}{1680} + \frac{x^2}{56} - \frac{x}{4} + 1\right)$$

23.7 problem 7

Internal problem ID [2386]
Internal file name [`OUTPUT/2386_Tuesday_February_27_2024_08_36_42_AM_53682316/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{\left(x^3+2x^2\right)y''-xy'+(1-x)\,y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+2x^2\right)y''-xy'+(1-x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{1}{x\,(x+2)}$$
$$q(x)=-\frac{x-1}{x^2\,(x+2)}$$

Table 648: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{x-1}{x^2(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x^2(x+2) - xy' + (1-x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x^2(x+2) - x\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (1-x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) \tag{2A}$$
$$+ \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) \tag{2B}$$
$$+ \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n + r - 1)(n + r - 2) + 2a_n(n + r)(n + r - 1) - a_n(n + r) + a_n - a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 3n - 3r + 1)}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}(n^2 - n - 1)}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 + r + 1}{2r^2 + r}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^4 - 3r^2 + 1}{4r^4 + 12r^3 + 11r^2 + 3r}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{1}{3}$
a_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^6 - 3r^5 + 2r^4 + 9r^3 + 2r^2 - 3r - 1}{r\,(4r^3 + 12r^2 + 11r + 3)\,(2r^2 + 9r + 10)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{126}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{1}{3}$
a_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{1}{126}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^8 + 8r^7 + 18r^6 - 4r^5 - 57r^4 - 52r^3 + 6r^2 + 20r + 5}{r(4r^3 + 12r^2 + 11r + 3)(2r^2 + 9r + 10)(2r^2 + 13r + 21)}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{11}{4536}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{1}{3}$
a_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{1}{126}$
a_4	$\frac{r^8+8r^7+18r^6-4r^5-57r^4-52r^3+6r^2+20r+5}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$-\frac{11}{4536}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r^2 + 3r + 1)(r^2 + 5r + 5)(r^2 + 7r + 11)(r^2 + r - 1)(r^2 - r - 1)}{r(4r^3 + 12r^2 + 11r + 3)(2r^2 + 9r + 10)(2r^2 + 13r + 21)(2r^2 + 17r + 36)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{19}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{1}{3}$
a_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{1}{126}$
a_4	$\frac{r^8+8r^7+18r^6-4r^5-57r^4-52r^3+6r^2+20r+5}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$-\frac{11}{4536}$
a_5	$-\frac{(r^2+3r+1)(r^2+5r+5)(r^2+7r+11)(r^2+r-1)(r^2-r-1)}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)(2r^2+17r+36)}$	$\frac{19}{22680}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)\\ &= x\left(1 + \frac{x}{3} - \frac{x^2}{30} + \frac{x^3}{126} - \frac{11x^4}{4536} + \frac{19x^5}{22680} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) - b_n(n+r) + b_n - b_{n-1} = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n^2 + 2nr + r^2 - 3n - 3r + 1)}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \quad (4)$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -\frac{b_{n-1}(4n^2 - 8n - 1)}{8n^2 - 4n} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-r^2 + r + 1}{2r^2 + r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = \frac{5}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{5}{4}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^4 - 3r^2 + 1}{4r^4 + 12r^3 + 11r^2 + 3r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{5}{96}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{5}{4}$
b_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$\frac{5}{96}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-r^6 - 3r^5 + 2r^4 + 9r^3 + 2r^2 - 3r - 1}{r\left(4r^3 + 12r^2 + 11r + 3\right)\left(2r^2 + 9r + 10\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = -\frac{11}{1152}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{5}{4}$
b_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$\frac{5}{96}$
b_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$-\frac{11}{1152}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^8 + 8r^7 + 18r^6 - 4r^5 - 57r^4 - 52r^3 + 6r^2 + 20r + 5}{r(4r^3 + 12r^2 + 11r + 3)(2r^2 + 9r + 10)(2r^2 + 13r + 21)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{341}{129024}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{5}{4}$
b_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$\frac{5}{96}$
b_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$-\frac{11}{1152}$
b_4	$\frac{r^8+8r^7+18r^6-4r^5-57r^4-52r^3+6r^2+20r+5}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$\frac{341}{129024}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r^2 + 3r + 1)(r^2 + 5r + 5)(r^2 + 7r + 11)(r^2 + r - 1)(r^2 - r - 1)}{r(4r^3 + 12r^2 + 11r + 3)(2r^2 + 9r + 10)(2r^2 + 13r + 21)(2r^2 + 17r + 36)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = -\frac{20119}{23224320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{5}{4}$
b_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$\frac{5}{96}$
b_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$-\frac{11}{1152}$
b_4	$\frac{r^8+8r^7+18r^6-4r^5-57r^4-52r^3+6r^2+20r+5}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$\frac{341}{129024}$
b_5	$-\frac{(r^2+3r+1)(r^2+5r+5)(r^2+7r+11)(r^2+r-1)(r^2-r-1)}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)(2r^2+17r+36)}$	$-\frac{20119}{23224320}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{5x}{4} + \frac{5x^2}{96} - \frac{11x^3}{1152} + \frac{341x^4}{129024} - \frac{20119x^5}{23224320} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x}{3} - \frac{x^2}{30} + \frac{x^3}{126} - \frac{11x^4}{4536} + \frac{19x^5}{22680} + O(x^6)\right) \\ &\quad + c_2\sqrt{x}\left(1 + \frac{5x}{4} + \frac{5x^2}{96} - \frac{11x^3}{1152} + \frac{341x^4}{129024} - \frac{20119x^5}{23224320} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 + \frac{x}{3} - \frac{x^2}{30} + \frac{x^3}{126} - \frac{11x^4}{4536} + \frac{19x^5}{22680} + O(x^6)\right) \\ &\quad + c_2\sqrt{x}\left(1 + \frac{5x}{4} + \frac{5x^2}{96} - \frac{11x^3}{1152} + \frac{341x^4}{129024} - \frac{20119x^5}{23224320} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 x\left(1 + \frac{x}{3} - \frac{x^2}{30} + \frac{x^3}{126} - \frac{11x^4}{4536} + \frac{19x^5}{22680} + O(x^6)\right) \\ + c_2\sqrt{x}\left(1 + \frac{5x}{4} + \frac{5x^2}{96} - \frac{11x^3}{1152} + \frac{341x^4}{129024} - \frac{20119x^5}{23224320} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1+\frac{x}{3}-\frac{x^2}{30}+\frac{x^3}{126}-\frac{11x^4}{4536}+\frac{19x^5}{22680}+O(x^6)\right) + c_2\sqrt{x}\left(1+\frac{5x}{4}+\frac{5x^2}{96}-\frac{11x^3}{1152}+\frac{341x^4}{129024}-\frac{20119x^5}{23224320}+O(x^6)\right)$$

Verified OK.

23.7.1 Maple step by step solution

Let's solve

$y''x^2(x+2)-xy'+(1-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x-1)y}{x^2(x+2)}+\frac{y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{y'}{x(x+2)}-\frac{(x-1)y}{x^2(x+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{1}{x(x+2)}, P_3(x)=-\frac{x-1}{x^2(x+2)}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=\frac{1}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $y''x^2(x+2)-xy'+(1-x)\,y=0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

$$\left(u^{3}-4u^{2}+4u\right)\left(\frac{d^{2}}{d u^{2}}y(u)\right)+\left(-u+2\right)\left(\frac{d}{d u}y(u)\right)+\left(3-u\right)y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^{m}\cdot y(u)$ to series expansion for $m = 0..1$

$$u^{m}\cdot y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r+m}$$

- Shift index using $k->k-m$

$$u^{m}\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d}{d u}y(u)\right)$ to series expansion for $m = 0..1$

$$u^{m}\cdot\left(\frac{d}{d u}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{d u}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{d u^{2}}y(u)\right)$ to series expansion for $m = 1..3$

$$u^{m}\cdot\left(\frac{d^{2}}{d u^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d^{2}}{d u^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$2a_{0}r(2r-1)\,u^{-1+r}+\left(2a_{1}(1+r)\,(1+2r)-a_{0}(4r^{2}-3r-3)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(2a_{k+1}(k+1+r)\,(2k+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(2r-1)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{2}\right\}$$

- Each term must be 0

$2a_1(1+r)(1+2r) - a_0(4r^2-3r-3) = 0$

- Each term in the series must be 0, giving the recursion relation

$(-4a_k + a_{k-1} + 4a_{k+1}) k^2 + ((-8a_k + 2a_{k-1} + 8a_{k+1}) r + 3a_k - 3a_{k-1} + 6a_{k+1}) k + (-4a_k + a_{k-}$

- Shift index using $k->k+1$

$(-4a_{k+1} + a_k + 4a_{k+2}) (k+1)^2 + ((-8a_{k+1} + 2a_k + 8a_{k+2}) r + 3a_{k+1} - 3a_k + 6a_{k+2}) (k+1) + ($

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} + 2kra_k - 8kra_{k+1} + r^2 a_k - 4r^2 a_{k+1} - ka_k - 5ka_{k+1} - ra_k - 5ra_{k+1} - a_k + 2a_{k+1}}{2(2k^2 + 4kr + 2r^2 + 7k + 7r + 6)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - ka_k - 5ka_{k+1} - a_k + 2a_{k+1}}{2(2k^2 + 7k + 6)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - ka_k - 5ka_{k+1} - a_k + 2a_{k+1}}{2(2k^2 + 7k + 6)}, 2a_1 + 3a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - ka_k - 5ka_{k+1} - a_k + 2a_{k+1}}{2(2k^2 + 7k + 6)}, 2a_1 + 3a_0 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 9ka_{k+1} - \frac{5}{4}a_k - \frac{3}{2}a_{k+1}}{2(2k^2 + 9k + 10)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 9ka_{k+1} - \frac{5}{4}a_k - \frac{3}{2}a_{k+1}}{2(2k^2 + 9k + 10)}, 6a_1 + \frac{7a_0}{2} = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{1}{2}}, a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - 9ka_{k+1} - \frac{5}{4}a_k - \frac{3}{2}a_{k+1}}{2(2k^2 + 9k + 10)}, 6a_1 + \frac{7a_0}{2} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{k^2 a_k - 4k^2 a_{k+1} - ka_k - 5ka_{k+1} - a_k + 2a_{k+1}}{2(2k^2 + 7k + 6)}, 2a_1 + 3\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve((2*x^2+x^3)*diff(y(x),x$2)-x*diff(y(x),x)+(1-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{5}{4}x+\frac{5}{96}x^2-\frac{11}{1152}x^3+\frac{341}{129024}x^4-\frac{20119}{23224320}x^5+\mathrm{O}\left(x^6\right)\right) + c_2x\left(1+\frac{1}{3}x-\frac{1}{30}x^2+\frac{1}{126}x^3-\frac{11}{4536}x^4+\frac{19}{22680}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 86

```
AsymptoticDSolveValue[(2*x^2+x^3)*y''[x]-x*y'[x]+(1-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1x\left(\frac{19x^5}{22680}-\frac{11x^4}{4536}+\frac{x^3}{126}-\frac{x^2}{30}+\frac{x}{3}+1\right) + c_2\sqrt{x}\left(-\frac{20119x^5}{23224320}+\frac{341x^4}{129024}-\frac{11x^3}{1152}+\frac{5x^2}{96}+\frac{5x}{4}+1\right)$$

23.8 problem 8

Internal problem ID [2387]
Internal file name [OUTPUT/2387_Tuesday_February_27_2024_08_36_43_AM_24521173/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - 3(x^2 + x)\,y' + (2 + 3x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + (-3x^2 - 3x)\,y' + (2 + 3x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3(x + 1)}{2x}$$
$$q(x) = \frac{2 + 3x}{2x^2}$$

Table 650: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3(x+1)}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2+3x}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + (-3x^2 - 3x)\, y' + (2 + 3x)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-3x^2 - 3x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (2+3x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - 3x^r a_0 r + 2a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - 3x^r r + 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 5r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 5r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 5r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - 3a_{n-1}(n+r-1) - 3a_n(n+r) + 2a_n + 3a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{3a_{n-1}}{2n+2r-1} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{3a_{n-1}}{2n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{3}{1 + 2r}$$

Which for the root $r = 2$ becomes

$$a_1 = \frac{3}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3}{1+2r}$	$\frac{3}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{4r^2 + 8r + 3}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{9}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3}{1+2r}$	$\frac{3}{5}$
a_2	$\frac{9}{4r^2+8r+3}$	$\frac{9}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{27}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{3}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3}{1+2r}$	$\frac{3}{5}$
a_2	$\frac{9}{4r^2+8r+3}$	$\frac{9}{35}$
a_3	$\frac{27}{8r^3+36r^2+46r+15}$	$\frac{3}{35}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{9}{385}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3}{1+2r}$	$\frac{3}{5}$
a_2	$\frac{9}{4r^2+8r+3}$	$\frac{9}{35}$
a_3	$\frac{27}{8r^3+36r^2+46r+15}$	$\frac{3}{35}$
a_4	$\frac{81}{16r^4+128r^3+344r^2+352r+105}$	$\frac{9}{385}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{243}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{27}{5005}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3}{1+2r}$	$\frac{3}{5}$
a_2	$\frac{9}{4r^2+8r+3}$	$\frac{9}{35}$
a_3	$\frac{27}{8r^3+36r^2+46r+15}$	$\frac{3}{35}$
a_4	$\frac{81}{16r^4+128r^3+344r^2+352r+105}$	$\frac{9}{385}$
a_5	$\frac{243}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{27}{5005}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\
&= x^2\left(1 + \frac{3x}{5} + \frac{9x^2}{35} + \frac{3x^3}{35} + \frac{9x^4}{385} + \frac{27x^5}{5005} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) - 3b_{n-1}(n+r-1) - 3b_n(n+r) + 2b_n + 3b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{3b_{n-1}}{2n+2r-1} \tag{4}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
b_n = \frac{3b_{n-1}}{2n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = \frac{3}{1+2r}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = \frac{3}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3}{1+2r}$	$\frac{3}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{9}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3}{1+2r}$	$\frac{3}{2}$
b_2	$\frac{9}{4r^2+8r+3}$	$\frac{9}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{27}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = \frac{9}{16}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3}{1+2r}$	$\frac{3}{2}$
b_2	$\frac{9}{4r^2+8r+3}$	$\frac{9}{8}$
b_3	$\frac{27}{8r^3+36r^2+46r+15}$	$\frac{9}{16}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{27}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3}{1+2r}$	$\frac{3}{2}$
b_2	$\frac{9}{4r^2+8r+3}$	$\frac{9}{8}$
b_3	$\frac{27}{8r^3+36r^2+46r+15}$	$\frac{9}{16}$
b_4	$\frac{81}{16r^4+128r^3+344r^2+352r+105}$	$\frac{27}{128}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{243}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = \frac{81}{1280}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3}{1+2r}$	$\frac{3}{2}$
b_2	$\frac{9}{4r^2+8r+3}$	$\frac{9}{8}$
b_3	$\frac{27}{8r^3+36r^2+46r+15}$	$\frac{9}{16}$
b_4	$\frac{81}{16r^4+128r^3+344r^2+352r+105}$	$\frac{27}{128}$
b_5	$\frac{243}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$\frac{81}{1280}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^2\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{3x}{2} + \frac{9x^2}{8} + \frac{9x^3}{16} + \frac{27x^4}{128} + \frac{81x^5}{1280} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(1 + \frac{3x}{5} + \frac{9x^2}{35} + \frac{3x^3}{35} + \frac{9x^4}{385} + \frac{27x^5}{5005} + O(x^6)\right) \\
&\quad + c_2\sqrt{x}\left(1 + \frac{3x}{2} + \frac{9x^2}{8} + \frac{9x^3}{16} + \frac{27x^4}{128} + \frac{81x^5}{1280} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2\left(1 + \frac{3x}{5} + \frac{9x^2}{35} + \frac{3x^3}{35} + \frac{9x^4}{385} + \frac{27x^5}{5005} + O(x^6)\right) \\
&\quad + c_2\sqrt{x}\left(1 + \frac{3x}{2} + \frac{9x^2}{8} + \frac{9x^3}{16} + \frac{27x^4}{128} + \frac{81x^5}{1280} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2\left(1 + \frac{3x}{5} + \frac{9x^2}{35} + \frac{3x^3}{35} + \frac{9x^4}{385} + \frac{27x^5}{5005} + O(x^6)\right) \\
&\quad + c_2\sqrt{x}\left(1 + \frac{3x}{2} + \frac{9x^2}{8} + \frac{9x^3}{16} + \frac{27x^4}{128} + \frac{81x^5}{1280} + O(x^6)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^2\left(1 + \frac{3x}{5} + \frac{9x^2}{35} + \frac{3x^3}{35} + \frac{9x^4}{385} + \frac{27x^5}{5005} + O(x^6)\right) \\
&\quad + c_2\sqrt{x}\left(1 + \frac{3x}{2} + \frac{9x^2}{8} + \frac{9x^3}{16} + \frac{27x^4}{128} + \frac{81x^5}{1280} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

23.8.1 Maple step by step solution

Let's solve

$2x^2y'' + (-3x^2 - 3x)\, y' + (2 + 3x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3(x+1)y'}{2x} - \frac{(2+3x)y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3(x+1)y'}{2x} + \frac{(2+3x)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{3(x+1)}{2x}, P_3(x) = \frac{2+3x}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{3}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' - 3x(x + 1)\, y' + (2 + 3x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-2) - 3a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+2r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{2, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2(k+r-2)\left(\left(k+r-\frac{1}{2}\right) a_k - \frac{3a_{k-1}}{2}\right) = 0$$

- Shift index using $k->k+1$

  $$2(k+r-1)\left(\left(k+\frac{1}{2}+r\right) a_{k+1} - \frac{3a_k}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{3a_k}{2k+1+2r}$$

- Recursion relation for $r = 2$

  $$a_{k+1} = \frac{3a_k}{2k+5}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{3a_k}{2k+5}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = \frac{3a_k}{2k+2}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{3a_k}{2k+2}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{3a_k}{2k+5}, b_{k+1} = \frac{3b_k}{2k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)-3*(x+x^2)*diff(y(x),x)+(2+3*x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{3}{2}x+\frac{9}{8}x^2+\frac{9}{16}x^3+\frac{27}{128}x^4+\frac{81}{1280}x^5+\mathrm{O}\left(x^6\right)\right) + c_2x^2\left(1+\frac{3}{5}x+\frac{9}{35}x^2+\frac{3}{35}x^3+\frac{9}{385}x^4+\frac{27}{5005}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 88

```
AsymptoticDSolveValue[2*x^2*y''[x]-3*(x+x^2)*y'[x]+(2+3*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{27x^5}{5005}+\frac{9x^4}{385}+\frac{3x^3}{35}+\frac{9x^2}{35}+\frac{3x}{5}+1\right)x^2 + c_2\left(\frac{81x^5}{1280}+\frac{27x^4}{128}+\frac{9x^3}{16}+\frac{9x^2}{8}+\frac{3x}{2}+1\right)\sqrt{x}$$

23.9 problem 9

Internal problem ID [2388]
Internal file name [`OUTPUT/2388_Tuesday_February_27_2024_08_36_44_AM_17541881/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2y'' + \left(-x^2+5x\right)y' + \left(2x^2-1\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3x^2y'' + \left(-x^2+5x\right)y' + \left(2x^2-1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{x-5}{3x} \\ q(x) &= \frac{2x^2-1}{3x^2} \end{aligned}$$

Table 652: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-5}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2x^2-1}{3x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2y'' + \left(-x^2 + 5x\right) y' + \left(2x^2 - 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(-x^2 + 5x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(2x^2 - 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r) + 5x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(3x^r r(-1+r) + 5x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 + 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{r}{3r^2 + 8r + 4}$$

For $2 \leq n$ the recursive equation is

$$3a_n(n + r)(n + r - 1) - a_{n-1}(n + r - 1) + 5a_n(n + r) + 2a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n a_{n-1} + r a_{n-1} - 2a_{n-2} - a_{n-1}}{3n^2 + 6nr + 3r^2 + 2n + 2r - 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{3na_{n-1} - 6a_{n-2} - 2a_{n-1}}{9n^2 + 12n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{3r^2+8r+4}$	$\frac{1}{21}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-5r^2 - 15r - 8}{(3r^2 + 8r + 4)(3r^2 + 14r + 15)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = -\frac{61}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{3r^2+8r+4}$	$\frac{1}{21}$
a_2	$\frac{-5r^2-15r-8}{(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{61}{630}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-11r^3 - 53r^2 - 68r - 16}{(3r^2 + 20r + 32)(3r^2 + 8r + 4)(3r^2 + 14r + 15)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{607}{73710}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{3r^2+8r+4}$	$\frac{1}{21}$
a_2	$\frac{-5r^2-15r-8}{(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{61}{630}$
a_3	$\frac{-11r^3-53r^2-68r-16}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{607}{73710}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{19r^4 + 204r^3 + 741r^2 + 1060r + 464}{(3r^2 + 20r + 32)(3r^2 + 8r + 4)(3r^2 + 14r + 15)(3r^2 + 26r + 55)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{2297}{884520}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{3r^2+8r+4}$	$\frac{1}{21}$
a_2	$\frac{-5r^2-15r-8}{(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{61}{630}$
a_3	$\frac{-11r^3-53r^2-68r-16}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{607}{73710}$
a_4	$\frac{19r^4+204r^3+741r^2+1060r+464}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)(3r^2+26r+55)}$	$\frac{2297}{884520}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{85r^5 + 1170r^4 + 5931r^3 + 13486r^2 + 13016r + 3616}{(3r^2 + 20r + 32)(3r^2 + 8r + 4)(3r^2 + 14r + 15)(3r^2 + 26r + 55)(3r^2 + 32r + 84)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = \frac{14713}{50417640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{3r^2+8r+4}$	$\frac{1}{21}$
a_2	$\frac{-5r^2-15r-8}{(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{61}{630}$
a_3	$\frac{-11r^3-53r^2-68r-16}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{607}{73710}$
a_4	$\frac{19r^4+204r^3+741r^2+1060r+464}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)(3r^2+26r+55)}$	$\frac{2297}{884520}$
a_5	$\frac{85r^5+1170r^4+5931r^3+13486r^2+13016r+3616}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)(3r^2+26r+55)(3r^2+32r+84)}$	$\frac{14713}{50417640}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 + \frac{x}{21} - \frac{61x^2}{630} - \frac{607x^3}{73710} + \frac{2297x^4}{884520} + \frac{14713x^5}{50417640} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{r}{3r^2 + 8r + 4}$$

For $2 \leq n$ the recursive equation is

$$3b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + 5b_n(n+r) + 2b_{n-2} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{nb_{n-1} + rb_{n-1} - 2b_{n-2} - b_{n-1}}{3n^2 + 6nr + 3r^2 + 2n + 2r - 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{nb_{n-1} - 2b_{n-2} - 2b_{n-1}}{n(3n-4)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{3r^2+8r+4}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-5r^2 - 15r - 8}{(3r^2 + 8r + 4)(3r^2 + 14r + 15)}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{3r^2+8r+4}$	1
b_2	$\frac{-5r^2-15r-8}{(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-11r^3 - 53r^2 - 68r - 16}{(3r^2 + 20r + 32)(3r^2 + 8r + 4)(3r^2 + 14r + 15)}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{3r^2+8r+4}$	1
b_2	$\frac{-5r^2-15r-8}{(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{1}{2}$
b_3	$\frac{-11r^3-53r^2-68r-16}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{19r^4 + 204r^3 + 741r^2 + 1060r + 464}{(3r^2 + 20r + 32)(3r^2 + 8r + 4)(3r^2 + 14r + 15)(3r^2 + 26r + 55)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{48}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{3r^2+8r+4}$	1
b_2	$\frac{-5r^2-15r-8}{(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{1}{2}$
b_3	$\frac{-11r^3-53r^2-68r-16}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{1}{6}$
b_4	$\frac{19r^4+204r^3+741r^2+1060r+464}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)(3r^2+26r+55)}$	$\frac{1}{48}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{85r^5 + 1170r^4 + 5931r^3 + 13486r^2 + 13016r + 3616}{(3r^2 + 20r + 32)(3r^2 + 8r + 4)(3r^2 + 14r + 15)(3r^2 + 26r + 55)(3r^2 + 32r + 84)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{19}{2640}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{3r^2+8r+4}$	1
b_2	$\frac{-5r^2-15r-8}{(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{1}{2}$
b_3	$\frac{-11r^3-53r^2-68r-16}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{1}{6}$
b_4	$\frac{19r^4+204r^3+741r^2+1060r+464}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)(3r^2+26r+55)}$	$\frac{1}{48}$
b_5	$\frac{85r^5+1170r^4+5931r^3+13486r^2+13016r+3616}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)(3r^2+26r+55)(3r^2+32r+84)}$	$\frac{19}{2640}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{48} + \frac{19x^5}{2640} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}} \left(1 + \frac{x}{21} - \frac{61x^2}{630} - \frac{607x^3}{73710} + \frac{2297x^4}{884520} + \frac{14713x^5}{50417640} + O\left(x^6\right)\right) \\
&\quad + \frac{c_2 \left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{48} + \frac{19x^5}{2640} + O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}} \left(1 + \frac{x}{21} - \frac{61x^2}{630} - \frac{607x^3}{73710} + \frac{2297x^4}{884520} + \frac{14713x^5}{50417640} + O\left(x^6\right)\right) \\
&\quad + \frac{c_2 \left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{48} + \frac{19x^5}{2640} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{3}} \left(1 + \frac{x}{21} - \frac{61x^2}{630} - \frac{607x^3}{73710} + \frac{2297x^4}{884520} + \frac{14713x^5}{50417640} + O\left(x^6\right)\right) \\
&\quad + \frac{c_2 \left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{48} + \frac{19x^5}{2640} + O(x^6)\right)}{x}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{3}} \left(1 + \frac{x}{21} - \frac{61x^2}{630} - \frac{607x^3}{73710} + \frac{2297x^4}{884520} + \frac{14713x^5}{50417640} + O\left(x^6\right)\right) \\
&\quad + \frac{c_2 \left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{48} + \frac{19x^5}{2640} + O(x^6)\right)}{x}
\end{aligned}
$$

Verified OK.

23.9.1 Maple step by step solution

Let's solve

$3x^2y'' + (-x^2 + 5x)\, y' + (2x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-5)y'}{3x} - \frac{(2x^2-1)y}{3x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-5)y'}{3x} + \frac{(2x^2-1)y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{x-5}{3x}, P_3(x) = \frac{2x^2-1}{3x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{3}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' - x(x-5)\, y' + (2x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+r)\,(-1+3r)\,x^{r}+(a_{1}(2+r)\,(2+3r)-a_{0}r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_{k}(k+r+1)\,(3k+3r-1)-a\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+3r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-1,\frac{1}{3}\right\}$

- Each term must be 0

  $a_{1}(2+r)\,(2+3r)-a_{0}r=0$

- Solve for the dependent coefficient(s)

  $a_{1}=\frac{a_{0}r}{3r^{2}+8r+4}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(k+r-\frac{1}{3}\right)(k+r+1)\,a_{k}-a_{k-1}k-a_{k-1}r+2a_{k-2}+a_{k-1}=0$

- Shift index using $k->k+2$

  $3\left(k+\frac{5}{3}+r\right)(k+3+r)\,a_{k+2}-a_{k+1}(k+2)-a_{k+1}r+2a_{k}+a_{k+1}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{ka_{k+1}+a_{k+1}r-2a_{k}+a_{k+1}}{(3k+5+3r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=\frac{ka_{k+1}-2a_{k}}{(3k+2)(k+2)}$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{k a_{k+1} - 2a_k}{(3k+2)(k+2)}, a_1 = a_0\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = \frac{k a_{k+1} - 2a_k + \frac{4}{3} a_{k+1}}{(3k+6)\left(k+\frac{10}{3}\right)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{k a_{k+1} - 2a_k + \frac{4}{3} a_{k+1}}{(3k+6)\left(k+\frac{10}{3}\right)}, a_1 = \frac{a_0}{21}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{k a_{k+1} - 2a_k}{(3k+2)(k+2)}, a_1 = a_0, b_{k+2} = \frac{k b_{k+1} - 2b_k + \frac{4}{3} b_{k+1}}{(3k+6)\left(k+\frac{10}{3}\right)}, b_1 = \frac{b_0}{21}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(3*x^2*diff(y(x),x$2)+(5*x-x^2)*diff(y(x),x)+(2*x^2-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}}\left(1+\frac{1}{21}x-\frac{61}{630}x^2-\frac{607}{73710}x^3+\frac{2297}{884520}x^4+\frac{14713}{50417640}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1+x-\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{48}x^4+\frac{19}{2640}\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 84

```
AsymptoticDSolveValue[3*x^2*y''[x]+(5*x-x^2)*y'[x]+(2*x^2-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt[3]{x}\left(\frac{14713x^5}{50417640}+\frac{2297x^4}{884520}-\frac{607x^3}{73710}-\frac{61x^2}{630}+\frac{x}{21}+1\right) + \frac{c_2\left(\frac{19x^5}{2640}+\frac{x^4}{48}-\frac{x^3}{6}-\frac{x^2}{2}+x+1\right)}{x}$$

23.10 problem 10

Internal problem ID [2389]
Internal file name [`OUTPUT/2389_Tuesday_February_27_2024_08_36_45_AM_83292704/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + x(x^2-4)\,y' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + (x^3-4x)\,y' + 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2-4}{4x}$$
$$q(x) = \frac{3}{4x^2}$$

Table 654: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2-4}{4x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{3}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + (x^3 - 4x) y' + 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + (x^3 - 4x) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}\right) \\
&+ \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n (n+r)(n+r-1) - 4x^{n+r} a_n (n+r) + 3a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
4x^r a_0 r(-1+r) - 4x^r a_0 r + 3a_0 x^r = 0
$$

Or

$$
\left(4x^r r(-1+r) - 4x^r r + 3x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(4r^2 - 8r + 3\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
4r^2 - 8r + 3 = 0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{3}{2} \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(4r^2 - 8r + 3\right) x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^{\frac{3}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \sqrt{x} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
4a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) - 4a_n(n+r) + 3a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n+r-2)}{4n^2+8nr+4r^2-8n-8r+3} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{a_{n-2}(2n-1)}{8n\,(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{r}{4r^2+8r+3}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = -\frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{4r^2+8r+3}$	$-\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{4r^2+8r+3}$	$-\frac{1}{16}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(2+r)}{(4r^2+8r+3)(4r^2+24r+35)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{7}{2560}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{4r^2+8r+3}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{r(2+r)}{(4r^2+8r+3)(4r^2+24r+35)}$	$\frac{7}{2560}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{4r^2+8r+3}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{r(2+r)}{(4r^2+8r+3)(4r^2+24r+35)}$	$\frac{7}{2560}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{3}{2}}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\
&= x^{\frac{3}{2}}\left(1 - \frac{x^2}{16} + \frac{7x^4}{2560} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} 0 &= \lim_{r\to \frac{1}{2}} 0 \\
&= 0
\end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$4b_n(n + r)(n + r - 1) + b_{n-2}(n + r - 2) - 4b_n(n + r) + 3b_n = 0 \tag{4}$$

Which for for the root $r = \frac{1}{2}$ becomes

$$4b_n\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right) + b_{n-2}\left(n - \frac{3}{2}\right) - 4b_n\left(n + \frac{1}{2}\right) + 3b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}(n + r - 2)}{4n^2 + 8nr + 4r^2 - 8n - 8r + 3} \tag{5}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -\frac{b_{n-2}\left(n - \frac{3}{2}\right)}{4n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{r}{4r^2 + 8r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = -\frac{1}{16}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{4r^2+8r+3}$	$-\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{4r^2+8r+3}$	$-\frac{1}{16}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(2+r)}{(4r^2+8r+3)(4r^2+24r+35)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{5}{1536}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{4r^2+8r+3}$	$-\frac{1}{16}$
b_3	0	0
b_4	$\frac{r(2+r)}{(4r^2+8r+3)(4r^2+24r+35)}$	$\frac{5}{1536}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{r}{4r^2+8r+3}$	$-\frac{1}{16}$
b_3	0	0
b_4	$\frac{r(2+r)}{(4r^2+8r+3)(4r^2+24r+35)}$	$\frac{5}{1536}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x^2}{16} + \frac{5x^4}{1536} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{16} + \frac{7x^4}{2560} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{16} + \frac{5x^4}{1536} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{16} + \frac{7x^4}{2560} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{16} + \frac{5x^4}{1536} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{16} + \frac{7x^4}{2560} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{16} + \frac{5x^4}{1536} + O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{3}{2}}\left(1 - \frac{x^2}{16} + \frac{7x^4}{2560} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{16} + \frac{5x^4}{1536} + O(x^6)\right)
$$

Verified OK.

23.10.1 Maple step by step solution

Let's solve

$4x^2 y'' + (x^3 - 4x)\, y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{4x^2} - \frac{(x^2-4)y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y''+\frac{(x^2-4)y'}{4x}+\frac{3y}{4x^2}=0$$

☐ Check to see if $x_0=0$ is a regular singular point

- Define functions

$$\left[P_2(x)=\frac{x^2-4}{4x}, P_3(x)=\frac{3}{4x^2}\right]$$

- $x\cdot P_2(x)$ is analytic at $x=0$

$$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-1$$

- $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=\frac{3}{4}$$

- $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$$x_0=0$$

- Multiply by denominators

$$4x^2y''+x(x^2-4)\,y'+3y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty}a_kx^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y'$ to series expansion for $m=1..3$

$$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^2\cdot y''$ to series expansion

$$x^2\cdot y''=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-3+2r)\,x^r+a_1(1+2r)\,(-1+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(2k+2r-1)\,(2k+2r-3)+a\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$
- Each term must be 0

  $a_1(1+2r)(-1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right)a_k + a_{k-2}(k-2+r) = 0$
- Shift index using $k- >k+2$

  $4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right)a_{k+2} + a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)}{(2k+1+2r)(2k+3+2r)}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{(2k+2)(2k+4)}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{(2k+2)(2k+4)}, a_1 = 0\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{(2k+4)(2k+6)}$
- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{(2k+4)(2k+6)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{(2k+2)(2k+4)}, a_1 = 0, b_{k+2} = -\frac{b_k\left(k+\frac{3}{2}\right)}{(2k+4)(2k+6)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)+x*(x^2-4)*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(x\left(1-\frac{1}{16}x^2+\frac{7}{2560}x^4+\mathrm{O}\left(x^6\right)\right)c_1+\left(1-\frac{1}{16}x^2+\frac{5}{1536}x^4+\mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 58

```
AsymptoticDSolveValue[4*x^2*y''[x]+x*(x^2-4)*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{5x^{9/2}}{1536}-\frac{x^{5/2}}{16}+\sqrt{x}\right)+c_2\left(\frac{7x^{11/2}}{2560}-\frac{x^{7/2}}{16}+x^{3/2}\right)$$

23.11 problem 15

Internal problem ID [2390]
Internal file name [OUTPUT/2390_Tuesday_February_27_2024_08_36_45_AM_75593104/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 3(x^2 + x)\, y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' + (-3x^2 - 3x)\, y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3(x+1)}{4x}$$
$$q(x) = \frac{1}{2x^2}$$

Table 656: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3(x+1)}{4x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' + \left(-3x^2 - 3x\right) y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(-3x^2 - 3x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-3x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$4x^r a_0 r(-1+r) - 3x^r a_0 r + 2a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) - 3x^r r + 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 7r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 7r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{7}{8}+\frac{\sqrt{17}}{8} \\
r_2 &= \frac{7}{8}-\frac{\sqrt{17}}{8}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
\left(4r^2-7r+2\right)x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{\sqrt{17}}{4}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{7}{8}+\frac{\sqrt{17}}{8}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{7}{8}-\frac{\sqrt{17}}{8}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
4a_n(n+r)(n+r-1)-3a_{n-1}(n+r-1)-3a_n(n+r)+2a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{3a_{n-1}(n+r-1)}{4n^2+8nr+4r^2-7n-7r+2} \tag{4}
$$

Which for the root $r = \frac{7}{8}+\frac{\sqrt{17}}{8}$ becomes

$$
a_n = \frac{3a_{n-1}\left(8n-1+\sqrt{17}\right)}{8n\left(\sqrt{17}+4n\right)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{7}{8} + \frac{\sqrt{17}}{8}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{3r}{4r^2 + r - 1}$$

Which for the root $r = \frac{7}{8} + \frac{\sqrt{17}}{8}$ becomes

$$a_1 = \frac{21 + 3\sqrt{17}}{32 + 8\sqrt{17}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r}{4r^2+r-1}$	$\frac{21+3\sqrt{17}}{32+8\sqrt{17}}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9r(1 + r)}{16r^4 + 40r^3 + 21r^2 - 5r - 4}$$

Which for the root $r = \frac{7}{8} + \frac{\sqrt{17}}{8}$ becomes

$$a_2 = \frac{9(7 + \sqrt{17})\left(15 + \sqrt{17}\right)}{128\left(4 + \sqrt{17}\right)\left(8 + \sqrt{17}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r}{4r^2+r-1}$	$\frac{21+3\sqrt{17}}{32+8\sqrt{17}}$
a_2	$\frac{9r(1+r)}{16r^4+40r^3+21r^2-5r-4}$	$\frac{9\left(7+\sqrt{17}\right)\left(15+\sqrt{17}\right)}{128\left(4+\sqrt{17}\right)\left(8+\sqrt{17}\right)}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{27r(1+r)(2+r)}{64r^6+432r^5+1036r^4+1017r^3+256r^2-153r-68}$$

Which for the root $r = \frac{7}{8} + \frac{\sqrt{17}}{8}$ becomes

$$a_3 = \frac{9(7+\sqrt{17})(15+\sqrt{17})(23+\sqrt{17})}{1024(49+12\sqrt{17})(12+\sqrt{17})}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r}{4r^2+r-1}$	$\frac{21+3\sqrt{17}}{32+8\sqrt{17}}$
a_2	$\frac{9r(1+r)}{16r^4+40r^3+21r^2-5r-4}$	$\frac{9(7+\sqrt{17})(15+\sqrt{17})}{128(4+\sqrt{17})(8+\sqrt{17})}$
a_3	$\frac{27r(1+r)(2+r)}{64r^6+432r^5+1036r^4+1017r^3+256r^2-153r-68}$	$\frac{9(7+\sqrt{17})(15+\sqrt{17})(23+\sqrt{17})}{1024(49+12\sqrt{17})(12+\sqrt{17})}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r(1+r)(2+r)(3+r)}{256r^8+3328r^7+17376r^6+46384r^5+65817r^4+44434r^3+5631r^2-7514r-2584}$$

Which for the root $r = \frac{7}{8} + \frac{\sqrt{17}}{8}$ becomes

$$a_4 = \frac{27(7+\sqrt{17})(15+\sqrt{17})(23+\sqrt{17})(31+\sqrt{17})}{32768(792+193\sqrt{17})(16+\sqrt{17})}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r}{4r^2+r-1}$	$\frac{21+3\sqrt{17}}{32+8\sqrt{17}}$
a_2	$\frac{9r(1+r)}{16r^4+40r^3+21r^2-5r-4}$	$\frac{9(7+\sqrt{17})(15+\sqrt{17})}{128(4+\sqrt{17})(8+\sqrt{17})}$
a_3	$\frac{27r(1+r)(2+r)}{64r^6+432r^5+1036r^4+1017r^3+256r^2-153r-68}$	$\frac{9(7+\sqrt{17})(15+\sqrt{17})(23+\sqrt{17})}{1024(49+12\sqrt{17})(12+\sqrt{17})}$
a_4	$\frac{81r(1+r)(2+r)(3+r)}{256r^8+3328r^7+17376r^6+46384r^5+65817r^4+44434r^3+5631r^2-7514r-2584}$	$\frac{27(7+\sqrt{17})(15+\sqrt{17})(23+\sqrt{17})(31+\sqrt{17})}{32768(792+193\sqrt{17})(16+\sqrt{17})}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{243r(1+r)(2+r)(3+r)(4+r)}{(256r^8 + 3328r^7 + 17376r^6 + 46384r^5 + 65817r^4 + 44434r^3 + 5631r^2 - 7514r - 2584)(4r^2 + 33r +$$

Which for the root $r = \frac{7}{8} + \frac{\sqrt{17}}{8}$ becomes

$$a_5 = \frac{81(7+\sqrt{17})(15+\sqrt{17})(23+\sqrt{17})(31+\sqrt{17})(39+\sqrt{17})}{1310720(15953+3880\sqrt{17})(20+\sqrt{17})}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{3r}{4r^2+r-1}$	$\frac{21+3\sqrt{17}}{32+8\sqrt{17}}$
a_2	$\frac{9r(1+r)}{16r^4+40r^3+21r^2-5r-4}$	$\frac{9(7+\sqrt{17})(15+\sqrt{17})}{128(4+\sqrt{17})(8+\sqrt{17})}$
a_3	$\frac{27r(1+r)(2+r)}{64r^6+432r^5+1036r^4+1017r^3+256r^2-153r-68}$	$\frac{9(7+\sqrt{17})(15+\sqrt{17})(23+\sqrt{17})}{1024(49+12\sqrt{17})(12+\sqrt{17})}$
a_4	$\frac{81r(1+r)(2+r)(3+r)}{256r^8+3328r^7+17376r^6+46384r^5+65817r^4+44434r^3+5631r^2-7514r-2584}$	$\frac{27(7+\sqrt{17})(15+\sqrt{17})(23+\sqrt{17})(31}{32768(792+193\sqrt{17})(16+\sqrt{17})}$
a_5	$\frac{243r(1+r)(2+r)(3+r)(4+r)}{(256r^8+3328r^7+17376r^6+46384r^5+65817r^4+44434r^3+5631r^2-7514r-2584)(4r^2+33r+67)}$	$\frac{81(7+\sqrt{17})(15+\sqrt{17})(23+\sqrt{17})(31}{1310720(15953+3880\sqrt{17})(2}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{7}{8}+\frac{\sqrt{17}}{8}}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= x^{\frac{7}{8}+\frac{\sqrt{17}}{8}}\left(1 + \frac{(21+3\sqrt{17})x}{32+8\sqrt{17}} + \frac{9(7+\sqrt{17})(15+\sqrt{17})x^2}{128(4+\sqrt{17})(8+\sqrt{17})} + \frac{9(7+\sqrt{17})(15+\sqrt{17})(23+\sqrt{17})}{1024(49+12\sqrt{17})(12+\sqrt{17})}\right. \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) - 3b_{n-1}(n+r-1) - 3b_n(n+r) + 2b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{3b_{n-1}(n+r-1)}{4n^2 + 8nr + 4r^2 - 7n - 7r + 2} \tag{4}$$

Which for the root $r = \frac{7}{8} - \frac{\sqrt{17}}{8}$ becomes

$$b_n = \frac{3b_{n-1}\left(-8n+1+\sqrt{17}\right)}{8n\left(\sqrt{17}-4n\right)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{7}{8} - \frac{\sqrt{17}}{8}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{3r}{4r^2+r-1}$$

Which for the root $r = \frac{7}{8} - \frac{\sqrt{17}}{8}$ becomes

$$b_1 = \frac{-21+3\sqrt{17}}{-32+8\sqrt{17}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3r}{4r^2+r-1}$	$\frac{-21+3\sqrt{17}}{-32+8\sqrt{17}}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9r(1+r)}{16r^4+40r^3+21r^2-5r-4}$$

Which for the root $r = \frac{7}{8} - \frac{\sqrt{17}}{8}$ becomes

$$b_2 = \frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)}{128\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3r}{4r^2+r-1}$	$\frac{-21+3\sqrt{17}}{-32+8\sqrt{17}}$
b_2	$\frac{9r(1+r)}{16r^4+40r^3+21r^2-5r-4}$	$\frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)}{128\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{27r(1+r)(2+r)}{64r^6 + 432r^5 + 1036r^4 + 1017r^3 + 256r^2 - 153r - 68}$$

Which for the root $r = \frac{7}{8} - \frac{\sqrt{17}}{8}$ becomes

$$b_3 = -\frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)}{1024\left(-49+12\sqrt{17}\right)\left(-12+\sqrt{17}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3r}{4r^2+r-1}$	$\frac{-21+3\sqrt{17}}{-32+8\sqrt{17}}$
b_2	$\frac{9r(1+r)}{16r^4+40r^3+21r^2-5r-4}$	$\frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)}{128\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)}$
b_3	$\frac{27r(1+r)(2+r)}{64r^6+432r^5+1036r^4+1017r^3+256r^2-153r-68}$	$-\frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)}{1024\left(-49+12\sqrt{17}\right)\left(-12+\sqrt{17}\right)}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81r(1+r)(2+r)(3+r)}{256r^8 + 3328r^7 + 17376r^6 + 46384r^5 + 65817r^4 + 44434r^3 + 5631r^2 - 7514r - 2584}$$

Which for the root $r = \frac{7}{8} - \frac{\sqrt{17}}{8}$ becomes

$$b_4 = \frac{27\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)\left(-31+\sqrt{17}\right)}{32768\left(-792+193\sqrt{17}\right)\left(-16+\sqrt{17}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3r}{4r^2+r-1}$	$\frac{-21+3\sqrt{17}}{-32+8\sqrt{17}}$
b_2	$\frac{9r(1+r)}{16r^4+40r^3+21r^2-5r-4}$	$\frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)}{128\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)}$
b_3	$\frac{27r(1+r)(2+r)}{64r^6+432r^5+1036r^4+1017r^3+256r^2-153r-68}$	$-\frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)}{1024\left(-49+12\sqrt{17}\right)\left(-12+\sqrt{17}\right)}$
b_4	$\frac{81r(1+r)(2+r)(3+r)}{256r^8+3328r^7+17376r^6+46384r^5+65817r^4+44434r^3+5631r^2-7514r-2584}$	$\frac{27\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)\left(-31+\sqrt{17}\right)}{32768\left(-792+193\sqrt{17}\right)\left(-16+\sqrt{17}\right)}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{243r(1+r)(2+r)(3+r)(4+r)}{(256r^8+3328r^7+17376r^6+46384r^5+65817r^4+44434r^3+5631r^2-7514r-2584)(4r^2+33r+}$$

Which for the root $r = \frac{7}{8} - \frac{\sqrt{17}}{8}$ becomes

$$b_5 = -\frac{81\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)\left(-31+\sqrt{17}\right)\left(-39+\sqrt{17}\right)}{1310720\left(-15953+3880\sqrt{17}\right)\left(-20+\sqrt{17}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{3r}{4r^2+r-1}$	$\frac{-21+3\sqrt{17}}{-32+8\sqrt{17}}$
b_2	$\frac{9r(1+r)}{16r^4+40r^3+21r^2-5r-4}$	$\frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)}{128\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)}$
b_3	$\frac{27r(1+r)(2+r)}{64r^6+432r^5+1036r^4+1017r^3+256r^2-153r-68}$	$-\frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{1\right.}{1024\left(-49+12\sqrt{17}\right)\left(-12+\sqrt{17}\right)}$
b_4	$\frac{81r(1+r)(2+r)(3+r)}{256r^8+3328r^7+17376r^6+46384r^5+65817r^4+44434r^3+5631r^2-7514r-2584}$	$\frac{27\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right.}{32768\left(-792+193\sqrt{17}\right)\left(-16+\right.}$
b_5	$\frac{243r(1+r)(2+r)(3+r)(4+r)}{(256r^8+3328r^7+17376r^6+46384r^5+65817r^4+44434r^3+5631r^2-7514r-2584)(4r^2+33r+67)}$	$-\frac{81\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{\right.}{1310720\left(-15953+3880\sqrt{\right.}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{7}{8}+\frac{\sqrt{17}}{8}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{\frac{7}{8}-\frac{\sqrt{17}}{8}}\left(1 + \frac{\left(-21+3\sqrt{17}\right)x}{-32+8\sqrt{17}} + \frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)x^2}{128\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)} - \frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-\right.}{1024\left(-49+12\sqrt{17}\right)\left(-1\right.}\right.
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{7}{8}+\frac{\sqrt{17}}{8}}\left(1 + \frac{\left(21+3\sqrt{17}\right)x}{32+8\sqrt{17}} + \frac{9\left(7+\sqrt{17}\right)\left(15+\sqrt{17}\right)x^2}{128\left(4+\sqrt{17}\right)\left(8+\sqrt{17}\right)}\right. \\
&\quad + \frac{9\left(7+\sqrt{17}\right)\left(15+\sqrt{17}\right)\left(23+\sqrt{17}\right)x^3}{1024\left(49+12\sqrt{17}\right)\left(12+\sqrt{17}\right)} \\
&\quad + \frac{27\left(7+\sqrt{17}\right)\left(15+\sqrt{17}\right)\left(23+\sqrt{17}\right)\left(31+\sqrt{17}\right)x^4}{32768\left(792+193\sqrt{17}\right)\left(16+\sqrt{17}\right)} \\
&\quad \left. + \frac{81\left(7+\sqrt{17}\right)\left(15+\sqrt{17}\right)\left(23+\sqrt{17}\right)\left(31+\sqrt{17}\right)\left(39+\sqrt{17}\right)x^5}{1310720\left(15953+3880\sqrt{17}\right)\left(20+\sqrt{17}\right)} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{7}{8}-\frac{\sqrt{17}}{8}}\left(1 + \frac{\left(-21+3\sqrt{17}\right)x}{-32+8\sqrt{17}} + \frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)x^2}{128\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)}\right. \\
&\quad - \frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)x^3}{1024\left(-49+12\sqrt{17}\right)\left(-12+\sqrt{17}\right)} \\
&\quad + \frac{27\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)\left(-31+\sqrt{17}\right)x^4}{32768\left(-792+193\sqrt{17}\right)\left(-16+\sqrt{17}\right)} \\
&\quad - \frac{81\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)\left(-31+\sqrt{17}\right)\left(-39+\sqrt{17}\right)x^5}{1310720\left(-15953+3880\sqrt{17}\right)\left(-20+\sqrt{17}\right)} \\
&\quad \left. + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$
\begin{aligned}
&= c_1 x^{\frac{7}{8}+\frac{\sqrt{17}}{8}} \Bigg(1 + \frac{\left(21+3\sqrt{17}\right)x}{32+8\sqrt{17}} + \frac{9\left(7+\sqrt{17}\right)\left(15+\sqrt{17}\right)x^2}{128\left(4+\sqrt{17}\right)\left(8+\sqrt{17}\right)} \\
&\quad + \frac{9\left(7+\sqrt{17}\right)\left(15+\sqrt{17}\right)\left(23+\sqrt{17}\right)x^3}{1024\left(49+12\sqrt{17}\right)\left(12+\sqrt{17}\right)} \\
&\quad + \frac{27\left(7+\sqrt{17}\right)\left(15+\sqrt{17}\right)\left(23+\sqrt{17}\right)\left(31+\sqrt{17}\right)x^4}{32768\left(792+193\sqrt{17}\right)\left(16+\sqrt{17}\right)} \\
&\quad + \frac{81\left(7+\sqrt{17}\right)\left(15+\sqrt{17}\right)\left(23+\sqrt{17}\right)\left(31+\sqrt{17}\right)\left(39+\sqrt{17}\right)x^5}{1310720\left(15953+3880\sqrt{17}\right)\left(20+\sqrt{17}\right)} + O\left(x^6\right)\Bigg) \\
&\quad + c_2 x^{\frac{7}{8}-\frac{\sqrt{17}}{8}} \Bigg(1 + \frac{\left(-21+3\sqrt{17}\right)x}{-32+8\sqrt{17}} + \frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)x^2}{128\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)} \\
&\quad - \frac{9\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)x^3}{1024\left(-49+12\sqrt{17}\right)\left(-12+\sqrt{17}\right)} \\
&\quad + \frac{27\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)\left(-31+\sqrt{17}\right)x^4}{32768\left(-792+193\sqrt{17}\right)\left(-16+\sqrt{17}\right)} \\
&\quad - \frac{81\left(-7+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-23+\sqrt{17}\right)\left(-31+\sqrt{17}\right)\left(-39+\sqrt{17}\right)x^5}{1310720\left(-15953+3880\sqrt{17}\right)\left(-20+\sqrt{17}\right)} \\
&\quad + O\left(x^6\right)\Bigg)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^{\frac{7}{8}+\frac{\sqrt{17}}{8}} \left(1+\frac{\left(21+3\sqrt{17}\right) x}{32+8\sqrt{17}}+\frac{9\left(7+\sqrt{17}\right) \left(15+\sqrt{17}\right) x^{2}}{128 \left(4+\sqrt{17}\right) \left(8+\sqrt{17}\right)}\right. \\
& +\frac{9\left(7+\sqrt{17}\right) \left(15+\sqrt{17}\right) \left(23+\sqrt{17}\right) x^{3}}{1024 \left(49+12\sqrt{17}\right) \left(12+\sqrt{17}\right)} \\
& +\frac{27\left(7+\sqrt{17}\right) \left(15+\sqrt{17}\right) \left(23+\sqrt{17}\right) \left(31+\sqrt{17}\right) x^{4}}{32768 \left(792+193\sqrt{17}\right) \left(16+\sqrt{17}\right)} \\
& \left.+\frac{81\left(7+\sqrt{17}\right) \left(15+\sqrt{17}\right) \left(23+\sqrt{17}\right) \left(31+\sqrt{17}\right) \left(39+\sqrt{17}\right) x^{5}}{1310720 \left(15953+3880\sqrt{17}\right) \left(20+\sqrt{17}\right)}+O\left(x^{6}\right)\right) \\
& + c_2 x^{\frac{7}{8}-\frac{\sqrt{17}}{8}} \left(1+\frac{\left(-21+3\sqrt{17}\right) x}{-32+8\sqrt{17}}+\frac{9\left(-7+\sqrt{17}\right) \left(-15+\sqrt{17}\right) x^{2}}{128 \left(-4+\sqrt{17}\right) \left(-8+\sqrt{17}\right)}\right. \\
& -\frac{9\left(-7+\sqrt{17}\right) \left(-15+\sqrt{17}\right) \left(-23+\sqrt{17}\right) x^{3}}{1024 \left(-49+12\sqrt{17}\right) \left(-12+\sqrt{17}\right)} \\
& +\frac{27\left(-7+\sqrt{17}\right) \left(-15+\sqrt{17}\right) \left(-23+\sqrt{17}\right) \left(-31+\sqrt{17}\right) x^{4}}{32768 \left(-792+193\sqrt{17}\right) \left(-16+\sqrt{17}\right)} \\
& -\frac{81\left(-7+\sqrt{17}\right) \left(-15+\sqrt{17}\right) \left(-23+\sqrt{17}\right) \left(-31+\sqrt{17}\right) \left(-39+\sqrt{17}\right) x^{5}}{1310720 \left(-15953+3880\sqrt{17}\right) \left(-20+\sqrt{17}\right)} \\
& \left.+O\left(x^{6}\right)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{7}{8}+\frac{\sqrt{17}}{8}} \left(1+\frac{\left(21+3\sqrt{17}\right) x}{32+8\sqrt{17}}+\frac{9\left(7+\sqrt{17}\right) \left(15+\sqrt{17}\right) x^{2}}{128 \left(4+\sqrt{17}\right) \left(8+\sqrt{17}\right)} + \frac{9\left(7+\sqrt{17}\right) \left(15+\sqrt{17}\right) \left(23+\sqrt{17}\right) x^{3}}{1024 \left(49+12\sqrt{17}\right) \left(12+\sqrt{17}\right)} + \frac{27\left(7+\sqrt{17}\right) \left(15+\sqrt{17}\right) \left(23+\sqrt{17}\right) \left(31+\sqrt{17}\right) x^{4}}{32768 \left(792+193\sqrt{17}\right) \left(16+\sqrt{17}\right)} + \frac{81\left(7+\sqrt{17}\right) \left(15+\sqrt{17}\right) \left(23+\sqrt{17}\right) \left(31+\sqrt{17}\right) \left(39+\sqrt{17}\right) x^{5}}{1310720 \left(15953+3880\sqrt{17}\right) \left(20+\sqrt{17}\right)}+O\left(x^{6}\right)\right) + c_2 x^{\frac{7}{8}-\frac{\sqrt{17}}{8}} \left(1+\frac{\left(-21+3\sqrt{17}\right) x}{-32+8\sqrt{17}}+\frac{9\left(-7+\sqrt{17}\right) \left(-15+\sqrt{17}\right) x^{2}}{128 \left(-4+\sqrt{17}\right) \left(-8+\sqrt{17}\right)} - \frac{9\left(-7+\sqrt{17}\right) \left(-15+\sqrt{17}\right) \left(-23+\sqrt{17}\right) x^{3}}{1024 \left(-49+12\sqrt{17}\right) \left(-12+\sqrt{17}\right)} + \frac{27\left(-7+\sqrt{17}\right) \left(-15+\sqrt{17}\right) \left(-23+\sqrt{17}\right) \left(-31+\sqrt{17}\right) x^{4}}{32768 \left(-792+193\sqrt{17}\right) \left(-16+\sqrt{17}\right)} - \frac{81\left(-7+\sqrt{17}\right) \left(-15+\sqrt{17}\right) \left(-23+\sqrt{17}\right) \left(-31+\sqrt{17}\right) \left(-39+\sqrt{17}\right) x^{5}}{1310720 \left(-15953+3880\sqrt{17}\right) \left(-20+\sqrt{17}\right)} + O\left(x^{6}\right)\right)
$$

Verified OK.

23.11.1 Maple step by step solution

Let's solve

$4x^2y'' + (-3x^2 - 3x)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x^2} + \frac{3(x+1)y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3(x+1)y'}{4x} + \frac{y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{3(x+1)}{4x}, P_3(x) = \frac{1}{2x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{3}{4}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{2}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 3x(x+1)\,y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(4r^2 - 7r + 2)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k^2 + 8kr + 4r^2 - 7k - 7r + 2) - 3a_{k-1}(k+r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r^2 - 7r + 2 = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{7}{8}-\frac{\sqrt{17}}{8}, \frac{7}{8}+\frac{\sqrt{17}}{8}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(4k^2+(8r-7)k+4r^2-7r+2)a_k-3a_{k-1}(k+r-1)=0$

- Shift index using $k->k+1$

  $\left(4(k+1)^2+(8r-7)(k+1)+4r^2-7r+2\right)a_{k+1}-3a_k(k+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{3a_k(k+r)}{4k^2+8kr+4r^2+k+r-1}$

- Recursion relation for $r=\frac{7}{8}-\frac{\sqrt{17}}{8}$

  $a_{k+1}=\frac{3a_k\left(k+\frac{7}{8}-\frac{\sqrt{17}}{8}\right)}{4k^2+8k\left(\frac{7}{8}-\frac{\sqrt{17}}{8}\right)+4\left(\frac{7}{8}-\frac{\sqrt{17}}{8}\right)^2+k-\frac{1}{8}-\frac{\sqrt{17}}{8}}$

- Solution for $r=\frac{7}{8}-\frac{\sqrt{17}}{8}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{7}{8}-\frac{\sqrt{17}}{8}}, a_{k+1}=\frac{3a_k\left(k+\frac{7}{8}-\frac{\sqrt{17}}{8}\right)}{4k^2+8k\left(\frac{7}{8}-\frac{\sqrt{17}}{8}\right)+4\left(\frac{7}{8}-\frac{\sqrt{17}}{8}\right)^2+k-\frac{1}{8}-\frac{\sqrt{17}}{8}}\right]$

- Recursion relation for $r=\frac{7}{8}+\frac{\sqrt{17}}{8}$

  $a_{k+1}=\frac{3a_k\left(k+\frac{7}{8}+\frac{\sqrt{17}}{8}\right)}{4k^2+8k\left(\frac{7}{8}+\frac{\sqrt{17}}{8}\right)+4\left(\frac{7}{8}+\frac{\sqrt{17}}{8}\right)^2+k-\frac{1}{8}+\frac{\sqrt{17}}{8}}$

- Solution for $r=\frac{7}{8}+\frac{\sqrt{17}}{8}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{7}{8}+\frac{\sqrt{17}}{8}}, a_{k+1}=\frac{3a_k\left(k+\frac{7}{8}+\frac{\sqrt{17}}{8}\right)}{4k^2+8k\left(\frac{7}{8}+\frac{\sqrt{17}}{8}\right)+4\left(\frac{7}{8}+\frac{\sqrt{17}}{8}\right)^2+k-\frac{1}{8}+\frac{\sqrt{17}}{8}}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k+\frac{7}{8}-\frac{\sqrt{17}}{8}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{7}{8}+\frac{\sqrt{17}}{8}}\right), a_{k+1}=\frac{3a_k\left(k+\frac{7}{8}-\frac{\sqrt{17}}{8}\right)}{4k^2+8k\left(\frac{7}{8}-\frac{\sqrt{17}}{8}\right)+4\left(\frac{7}{8}-\frac{\sqrt{17}}{8}\right)^2+k-\frac{1}{8}-\frac{\sqrt{17}}{8}}, b_{k+1}=\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 879

```
Order:=6;
dsolve(4*x^2*diff(y(x),x$2)-3*(x+x^2)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^{\frac{7}{8}}\left(c_2 x^{\frac{\sqrt{17}}{8}}\left(1+\frac{21+3\sqrt{17}}{8\sqrt{17}+32}x+\frac{9}{128}\frac{\left(15+\sqrt{17}\right)\left(7+\sqrt{17}\right)}{\left(4+\sqrt{17}\right)\left(8+\sqrt{17}\right)}x^2 + \frac{9}{1024}\frac{\left(23+\sqrt{17}\right)\left(15+\sqrt{17}\right)\left(7+\sqrt{17}\right)}{\left(4+\sqrt{17}\right)\left(8+\sqrt{17}\right)\left(12+\sqrt{17}\right)}x^3 + \frac{27}{32768}\frac{\left(31+\sqrt{17}\right)\left(23+\sqrt{17}\right)\left(15+\sqrt{17}\right)\left(7+\sqrt{17}\right)}{\left(4+\sqrt{17}\right)\left(8+\sqrt{17}\right)\left(12+\sqrt{17}\right)\left(16+\sqrt{17}\right)}x^4 + \frac{81}{1310720}\frac{\left(39+\sqrt{17}\right)\left(31+\sqrt{17}\right)\left(23+\sqrt{17}\right)\left(15+\sqrt{17}\right)\left(7+\sqrt{17}\right)}{\left(4+\sqrt{17}\right)\left(8+\sqrt{17}\right)\left(12+\sqrt{17}\right)\left(16+\sqrt{17}\right)\left(20+\sqrt{17}\right)}x^5 + \mathrm{O}\left(x^6\right)\right) + c_1 x^{-\frac{\sqrt{17}}{8}}\left(1+\frac{-21+3\sqrt{17}}{8\sqrt{17}-32}x+\frac{9}{128}\frac{\left(-15+\sqrt{17}\right)\left(-7+\sqrt{17}\right)}{\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)}x^2 + \frac{9}{1024}\frac{\left(-23+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-7+\sqrt{17}\right)}{\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)\left(-12+\sqrt{17}\right)}x^3 + \frac{27}{32768}\frac{\left(-31+\sqrt{17}\right)\left(-23+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-7+\sqrt{17}\right)}{\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)\left(-12+\sqrt{17}\right)\left(-16+\sqrt{17}\right)}x^4 + \frac{81}{1310720}\frac{\left(-39+\sqrt{17}\right)\left(-31+\sqrt{17}\right)\left(-23+\sqrt{17}\right)\left(-15+\sqrt{17}\right)\left(-7+\sqrt{17}\right)}{\left(-4+\sqrt{17}\right)\left(-8+\sqrt{17}\right)\left(-12+\sqrt{17}\right)\left(-16+\sqrt{17}\right)\left(-20+\sqrt{17}\right)}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 2028

```
AsymptoticDSolveValue[4*x^2*y''[x]-3*(x+x^2)*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

Too large to display

23.12 problem 16

Internal problem ID [2391]
Internal file name [`OUTPUT/2391_Tuesday_February_27_2024_08_36_47_AM_32410806/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2y'' + 9\left(-x^2 + x\right)y' + y(x-1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + \left(-9x^2 + 9x\right)y' + y(x-1) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{x}$$
$$q(x) = \frac{x-1}{9x^2}$$

Table 658: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x-1}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + (-9x^2 + 9x)\, y' + y(x - 1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) + (-9x^2 + 9x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)(x - 1) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r) (n+r-1)\right) + \sum_{n=0}^{\infty} \left(-9x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-9x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-9a_{n-1} (n+r-1) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r) (n+r-1)\right) + \sum_{n=1}^{\infty} \left(-9a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r) (n+r-1) + 9x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 9x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(9x^r r(-1+r) + 9x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(9r^2 - 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) - 9a_{n-1}(n+r-1) + 9a_n(n+r) + a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(9n + 9r - 10)}{9n^2 + 18nr + 9r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{9na_{n-1} - 7a_{n-1}}{9n^2 + 6n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1 + 9r}{9r^2 + 18r + 8}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = \frac{2}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+9r}{9r^2+18r+8}$	$\frac{2}{15}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{81r^2 + 63r - 8}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{11}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+9r}{9r^2+18r+8}$	$\frac{2}{15}$
a_2	$\frac{81r^2+63r-8}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{11}{360}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{729r^3 + 1944r^2 + 999r - 136}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = \frac{1}{162}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+9r}{9r^2+18r+8}$	$\frac{2}{15}$
a_2	$\frac{81r^2+63r-8}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{11}{360}$
a_3	$\frac{729r^3+1944r^2+999r-136}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1}{162}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{6561r^4 + 36450r^3 + 59535r^2 + 24750r - 3536}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{29}{27216}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+9r}{9r^2+18r+8}$	$\frac{2}{15}$
a_2	$\frac{81r^2+63r-8}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{11}{360}$
a_3	$\frac{729r^3+1944r^2+999r-136}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1}{162}$
a_4	$\frac{6561r^4+36450r^3+59535r^2+24750r-3536}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{29}{27216}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{59049r^5 + 557685r^4 + 1811565r^3 + 2306475r^2 + 834426r - 123760}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)(9r^2 + 90r + 224)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = \frac{551}{3470040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+9r}{9r^2+18r+8}$	$\frac{2}{15}$
a_2	$\frac{81r^2+63r-8}{(9r^2+18r+8)(9r^2+36r+35)}$	$\frac{11}{360}$
a_3	$\frac{729r^3+1944r^2+999r-136}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$\frac{1}{162}$
a_4	$\frac{6561r^4+36450r^3+59535r^2+24750r-3536}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$\frac{29}{27216}$
a_5	$\frac{59049r^5+557685r^4+1811565r^3+2306475r^2+834426r-123760}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$	$\frac{551}{3470040}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 + \frac{2x}{15} + \frac{11x^2}{360} + \frac{x^3}{162} + \frac{29x^4}{27216} + \frac{551x^5}{3470040} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$9b_n(n+r)(n+r-1) - 9b_{n-1}(n+r-1) + 9b_n(n+r) + b_{n-1} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}(9n+9r-10)}{9n^2+18nr+9r^2-1} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = \frac{9nb_{n-1} - 13b_{n-1}}{9n^2-6n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-1 + 9r}{9r^2 + 18r + 8}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_1 = -\frac{4}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+9r}{9r^2+18r+8}$	$-\frac{4}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{81r^2 + 63r - 8}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_2 = -\frac{5}{18}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+9r}{9r^2+18r+8}$	$-\frac{4}{3}$
b_2	$\frac{81r^2+63r-8}{(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{5}{18}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{729r^3 + 1944r^2 + 999r - 136}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_3 = -\frac{5}{81}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+9r}{9r^2+18r+8}$	$-\frac{4}{3}$
b_2	$\frac{81r^2+63r-8}{(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{5}{18}$
b_3	$\frac{729r^3+1944r^2+999r-136}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$-\frac{5}{81}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{6561r^4 + 36450r^3 + 59535r^2 + 24750r - 3536}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = -\frac{23}{1944}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+9r}{9r^2+18r+8}$	$-\frac{4}{3}$
b_2	$\frac{81r^2+63r-8}{(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{5}{18}$
b_3	$\frac{729r^3+1944r^2+999r-136}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$-\frac{5}{81}$
b_4	$\frac{6561r^4+36450r^3+59535r^2+24750r-3536}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$-\frac{23}{1944}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{59049r^5 + 557685r^4 + 1811565r^3 + 2306475r^2 + 834426r - 123760}{(9r^2 + 18r + 8)(9r^2 + 36r + 35)(9r^2 + 54r + 80)(9r^2 + 72r + 143)(9r^2 + 90r + 224)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_5 = -\frac{92}{47385}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-1+9r}{9r^2+18r+8}$	$-\frac{4}{3}$
b_2	$\frac{81r^2+63r-8}{(9r^2+18r+8)(9r^2+36r+35)}$	$-\frac{5}{18}$
b_3	$\frac{729r^3+1944r^2+999r-136}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)}$	$-\frac{5}{81}$
b_4	$\frac{6561r^4+36450r^3+59535r^2+24750r-3536}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)}$	$-\frac{23}{1944}$
b_5	$\frac{59049r^5+557685r^4+1811565r^3+2306475r^2+834426r-123760}{(9r^2+18r+8)(9r^2+36r+35)(9r^2+54r+80)(9r^2+72r+143)(9r^2+90r+224)}$	$-\frac{92}{47385}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{4x}{3} - \frac{5x^2}{18} - \frac{5x^3}{81} - \frac{23x^4}{1944} - \frac{92x^5}{47385} + O(x^6)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}}\left(1 + \frac{2x}{15} + \frac{11x^2}{360} + \frac{x^3}{162} + \frac{29x^4}{27216} + \frac{551x^5}{3470040} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 - \frac{4x}{3} - \frac{5x^2}{18} - \frac{5x^3}{81} - \frac{23x^4}{1944} - \frac{92x^5}{47385} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{3}}\left(1 + \frac{2x}{15} + \frac{11x^2}{360} + \frac{x^3}{162} + \frac{29x^4}{27216} + \frac{551x^5}{3470040} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 - \frac{4x}{3} - \frac{5x^2}{18} - \frac{5x^3}{81} - \frac{23x^4}{1944} - \frac{92x^5}{47385} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{1}{3}}\left(1 + \frac{2x}{15} + \frac{11x^2}{360} + \frac{x^3}{162} + \frac{29x^4}{27216} + \frac{551x^5}{3470040} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 - \frac{4x}{3} - \frac{5x^2}{18} - \frac{5x^3}{81} - \frac{23x^4}{1944} - \frac{92x^5}{47385} + O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}}\left(1+\frac{2x}{15}+\frac{11x^2}{360}+\frac{x^3}{162}+\frac{29x^4}{27216}+\frac{551x^5}{3470040}+O(x^6)\right)$$
$$+\frac{c_2\left(1-\frac{4x}{3}-\frac{5x^2}{18}-\frac{5x^3}{81}-\frac{23x^4}{1944}-\frac{92x^5}{47385}+O(x^6)\right)}{x^{\frac{1}{3}}}$$

Verified OK.

23.12.1 Maple step by step solution

Let's solve

$9x^2y''+(-9x^2+9x)\,y'+y(x-1)=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x-1)y}{9x^2}+\frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x-1)y'}{x}+\frac{(x-1)y}{9x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x-1}{x}, P_3(x)=\frac{x-1}{9x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{1}{9}$

  ◦ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $9x^2y''-9x(x-1)\,y'+y(x-1)=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)\,(3k+3r-1) - a_{k-1}(9k-10+9r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(3k+3r+1)\,(3k+3r-1) - a_{k-1}(9k-10+9r) = 0$$

- Shift index using $k->k+1$

$$a_{k+1}(3k+4+3r)\,(3k+2+3r) - a_k(9k+9r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(9k+9r-1)}{(3k+4+3r)(3k+2+3r)}$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+1} = \frac{a_k(9k-4)}{(3k+3)(3k+1)}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = \frac{a_k(9k-4)}{(3k+3)(3k+1)}\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = \frac{a_k(9k+2)}{(3k+5)(3k+3)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{a_k(9k+2)}{(3k+5)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = \frac{a_k(9k-4)}{(3k+3)(3k+1)}, b_{k+1} = \frac{b_k(9k+2)}{(3k+5)(3k+3)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Whittaker successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)+9*(x-x^2)*diff(y(x),x)+(x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{2}{3}}\left(1+\frac{2}{15}x+\frac{11}{360}x^2+\frac{1}{162}x^3+\frac{29}{27216}x^4+\frac{551}{3470040}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1-\frac{4}{3}x-\frac{5}{18}x^2-\frac{5}{81}x^3-\frac{23}{1944}x^4-\frac{9}{47}\right.}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 90

```
AsymptoticDSolveValue[9*x^2*y''[x]+9*(x-x^2)*y'[x]+(x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt[3]{x}\left(\frac{551x^5}{3470040}+\frac{29x^4}{27216}+\frac{x^3}{162}+\frac{11x^2}{360}+\frac{2x}{15}+1\right)+\frac{c_2\left(-\frac{92x^5}{47385}-\frac{23x^4}{1944}-\frac{5x^3}{81}-\frac{5x^2}{18}-\frac{4x}{3}+1\right)}{\sqrt[3]{x}}$$

23.13 problem 17

Internal problem ID [2392]
Internal file name [OUTPUT/2392_Tuesday_February_27_2024_08_36_47_AM_67396664/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(1-x)\,y''+3x(1+2x)\,y'-3y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-4x^3+4x^2\right)y''+\left(6x^2+3x\right)y'-3y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{3(1+2x)}{4x\,(x-1)}$$
$$q(x)=\frac{3}{4x^2\,(x-1)}$$

Table 660: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3(1+2x)}{4x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{3}{4x^2(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-4y''x^2(x-1) + \left(6x^2 + 3x\right) y' - 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-4\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x-1) + \left(6x^2+3x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-4x^{1+n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}4x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}6x^{1+n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-3a_nx^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-4x^{1+n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-4a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}6x^{1+n+r}a_n(n+r)=\sum_{n=1}^{\infty}6a_{n-1}(n+r-1)x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty}\left(-4a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right)+\left(\sum_{n=0}^{\infty}4x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty}6a_{n-1}(n+r-1)x^{n+r}\right)+\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-3a_nx^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)(n+r-1)+3x^{n+r}a_n(n+r)-3a_nx^{n+r}=0$$

When $n=0$ the above becomes

$$4x^ra_0r(-1+r)+3x^ra_0r-3a_0x^r=0$$

Or

$$\left(4x^rr(-1+r)+3x^rr-3x^r\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - r - 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - r - 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -\frac{3}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - r - 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{7}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} -4a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) \\ + 6a_{n-1}(n+r-1) + 3a_n(n+r) - 3a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2(2n+2r-7)\, a_{n-1}}{4n+4r+3} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(4n - 10)\, a_{n-1}}{4n + 7} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{4r - 10}{7 + 4r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{6}{11}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r-10}{7+4r}$	$-\frac{6}{11}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16r^2 - 64r + 60}{16r^2 + 72r + 77}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{4}{55}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r-10}{7+4r}$	$-\frac{6}{11}$
a_2	$\frac{16r^2-64r+60}{16r^2+72r+77}$	$\frac{4}{55}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{64r^3 - 288r^2 + 368r - 120}{64r^3 + 528r^2 + 1388r + 1155}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{8}{1045}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r-10}{7+4r}$	$-\frac{6}{11}$
a_2	$\frac{16r^2-64r+60}{16r^2+72r+77}$	$\frac{4}{55}$
a_3	$\frac{64r^3-288r^2+368r-120}{64r^3+528r^2+1388r+1155}$	$\frac{8}{1045}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256r^4 - 1024r^3 + 896r^2 + 256r - 240}{256r^4 + 3328r^3 + 15584r^2 + 30992r + 21945}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{48}{24035}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r-10}{7+4r}$	$-\frac{6}{11}$
a_2	$\frac{16r^2-64r+60}{16r^2+72r+77}$	$\frac{4}{55}$
a_3	$\frac{64r^3-288r^2+368r-120}{64r^3+528r^2+1388r+1155}$	$\frac{8}{1045}$
a_4	$\frac{256r^4-1024r^3+896r^2+256r-240}{256r^4+3328r^3+15584r^2+30992r+21945}$	$\frac{48}{24035}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1024r^5 - 2560r^4 - 2560r^3 + 6400r^2 + 576r - 1440}{1024r^5 + 19200r^4 + 138880r^3 + 482400r^2 + 800596r + 504735}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{32}{43263}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{4r-10}{7+4r}$	$-\frac{6}{11}$
a_2	$\frac{16r^2-64r+60}{16r^2+72r+77}$	$\frac{4}{55}$
a_3	$\frac{64r^3-288r^2+368r-120}{64r^3+528r^2+1388r+1155}$	$\frac{8}{1045}$
a_4	$\frac{256r^4-1024r^3+896r^2+256r-240}{256r^4+3328r^3+15584r^2+30992r+21945}$	$\frac{48}{24035}$
a_5	$\frac{1024r^5-2560r^4-2560r^3+6400r^2+576r-1440}{1024r^5+19200r^4+138880r^3+482400r^2+800596r+504735}$	$\frac{32}{43263}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - \frac{6x}{11} + \frac{4x^2}{55} + \frac{8x^3}{1045} + \frac{48x^4}{24035} + \frac{32x^5}{43263} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}
&-4b_{n-1}(n+r-1)(n+r-2) + 4b_n(n+r)(n+r-1) \\
&+ 6b_{n-1}(n+r-1) + 3b_n(n+r) - 3b_n = 0
\end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{2(2n+2r-7)\, b_{n-1}}{4n+4r+3} \tag{4}$$

Which for the root $r = -\frac{3}{4}$ becomes

$$b_n = \frac{(4n-17)\, b_{n-1}}{4n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{3}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{4r - 10}{7 + 4r}$$

Which for the root $r = -\frac{3}{4}$ becomes

$$b_1 = -\frac{13}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r-10}{7+4r}$	$-\frac{13}{4}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{16r^2 - 64r + 60}{16r^2 + 72r + 77}$$

Which for the root $r = -\frac{3}{4}$ becomes

$$b_2 = \frac{117}{32}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r-10}{7+4r}$	$-\frac{13}{4}$
b_2	$\frac{16r^2-64r+60}{16r^2+72r+77}$	$\frac{117}{32}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{64r^3 - 288r^2 + 368r - 120}{64r^3 + 528r^2 + 1388r + 1155}$$

Which for the root $r = -\frac{3}{4}$ becomes

$$b_3 = -\frac{195}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r-10}{7+4r}$	$-\frac{13}{4}$
b_2	$\frac{16r^2-64r+60}{16r^2+72r+77}$	$\frac{117}{32}$
b_3	$\frac{64r^3-288r^2+368r-120}{64r^3+528r^2+1388r+1155}$	$-\frac{195}{128}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256r^4 - 1024r^3 + 896r^2 + 256r - 240}{256r^4 + 3328r^3 + 15584r^2 + 30992r + 21945}$$

Which for the root $r = -\frac{3}{4}$ becomes

$$b_4 = \frac{195}{2048}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r-10}{7+4r}$	$-\frac{13}{4}$
b_2	$\frac{16r^2-64r+60}{16r^2+72r+77}$	$\frac{117}{32}$
b_3	$\frac{64r^3-288r^2+368r-120}{64r^3+528r^2+1388r+1155}$	$-\frac{195}{128}$
b_4	$\frac{256r^4-1024r^3+896r^2+256r-240}{256r^4+3328r^3+15584r^2+30992r+21945}$	$\frac{195}{2048}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1024r^5 - 2560r^4 - 2560r^3 + 6400r^2 + 576r - 1440}{1024r^5 + 19200r^4 + 138880r^3 + 482400r^2 + 800596r + 504735}$$

Which for the root $r = -\frac{3}{4}$ becomes

$$b_5 = \frac{117}{8192}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{4r-10}{7+4r}$	$-\frac{13}{4}$
b_2	$\frac{16r^2-64r+60}{16r^2+72r+77}$	$\frac{117}{32}$
b_3	$\frac{64r^3-288r^2+368r-120}{64r^3+528r^2+1388r+1155}$	$-\frac{195}{128}$
b_4	$\frac{256r^4-1024r^3+896r^2+256r-240}{256r^4+3328r^3+15584r^2+30992r+21945}$	$\frac{195}{2048}$
b_5	$\frac{1024r^5-2560r^4-2560r^3+6400r^2+576r-1440}{1024r^5+19200r^4+138880r^3+482400r^2+800596r+504735}$	$\frac{117}{8192}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{13x}{4} + \frac{117x^2}{32} - \frac{195x^3}{128} + \frac{195x^4}{2048} + \frac{117x^5}{8192} + O(x^6)}{x^{\frac{3}{4}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{6x}{11} + \frac{4x^2}{55} + \frac{8x^3}{1045} + \frac{48x^4}{24035} + \frac{32x^5}{43263} + O(x^6)\right) \\
&+ \frac{c_2\left(1 - \frac{13x}{4} + \frac{117x^2}{32} - \frac{195x^3}{128} + \frac{195x^4}{2048} + \frac{117x^5}{8192} + O(x^6)\right)}{x^{\frac{3}{4}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - \frac{6x}{11} + \frac{4x^2}{55} + \frac{8x^3}{1045} + \frac{48x^4}{24035} + \frac{32x^5}{43263} + O(x^6)\right) \\
&+ \frac{c_2\left(1 - \frac{13x}{4} + \frac{117x^2}{32} - \frac{195x^3}{128} + \frac{195x^4}{2048} + \frac{117x^5}{8192} + O(x^6)\right)}{x^{\frac{3}{4}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x \left(1-\frac{6x}{11}+\frac{4x^2}{55}+\frac{8x^3}{1045}+\frac{48x^4}{24035}+\frac{32x^5}{43263}+O(x^6)\right) + \frac{c_2\left(1-\frac{13x}{4}+\frac{117x^2}{32}-\frac{195x^3}{128}+\frac{195x^4}{2048}+\frac{117x^5}{8192}+O(x^6)\right)}{x^{\frac{3}{4}}} \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1-\frac{6x}{11}+\frac{4x^2}{55}+\frac{8x^3}{1045}+\frac{48x^4}{24035}+\frac{32x^5}{43263}+O(x^6)\right) + \frac{c_2\left(1-\frac{13x}{4}+\frac{117x^2}{32}-\frac{195x^3}{128}+\frac{195x^4}{2048}+\frac{117x^5}{8192}+O(x^6)\right)}{x^{\frac{3}{4}}}$$

Verified OK.

23.13.1 Maple step by step solution

Let's solve

$-4y''x^2(x-1)+(6x^2+3x)\,y'-3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y}{4x^2(x-1)}+\frac{3(1+2x)y'}{4x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{3(1+2x)y'}{4x(x-1)}+\frac{3y}{4x^2(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{3(1+2x)}{4x(x-1)}, P_3(x)=\frac{3}{4x^2(x-1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=\frac{3}{4}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4y''x^2(x-1) - 3x(1+2x)\,y' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(3+4r)\,(-1+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(4k+4r+3)\,(k+r-1) + 2a_{k-1}(k+r-1)\,(2k-7+2r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(3+4r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, -\frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-4(k+r-1)\left(\left(-k-r+\frac{7}{2}\right)a_{k-1} + a_k\left(k+r+\frac{3}{4}\right)\right) = 0$

- Shift index using $k- >k+1$

  $-4(k+r)\left(\left(-k+\frac{5}{2}-r\right) a_{k}+a_{k+1}\left(k+\frac{7}{4}+r\right)\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2(2k+2r-5)a_{k}}{4k+7+4r}$
- Recursion relation for $r=1$

  $a_{k+1}=\frac{2(2k-3)a_{k}}{4k+11}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+1}, a_{k+1}=\frac{2(2k-3)a_{k}}{4k+11}\right]$
- Recursion relation for $r=-\frac{3}{4}$

  $a_{k+1}=\frac{2\left(2k-\frac{13}{2}\right)a_{k}}{4k+4}$
- Solution for $r=-\frac{3}{4}$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k-\frac{3}{4}}, a_{k+1}=\frac{2\left(2k-\frac{13}{2}\right)a_{k}}{4k+4}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_{k} x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_{k} x^{k-\frac{3}{4}}\right), a_{k+1}=\frac{2(2k-3)a_{k}}{4k+11}, b_{k+1}=\frac{2\left(2k-\frac{13}{2}\right)b_{k}}{4k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve(4*x^2*(1-x)*diff(y(x),x$2)+3*x*(1+2*x)*diff(y(x),x)-3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1-\frac{13}{4}x+\frac{117}{32}x^2-\frac{195}{128}x^3+\frac{195}{2048}x^4+\frac{117}{8192}x^5+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{3}{4}}} + c_2x\left(1-\frac{6}{11}x+\frac{4}{55}x^2+\frac{8}{1045}x^3+\frac{48}{24035}x^4+\frac{32}{43263}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 86

```
AsymptoticDSolveValue[4*x^2*(1-x)*y''[x]+3*x*(1+2*x)*y'[x]-3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(\frac{32x^5}{43263} + \frac{48x^4}{24035} + \frac{8x^3}{1045} + \frac{4x^2}{55} - \frac{6x}{11} + 1\right) + \frac{c_2 \left(\frac{117x^5}{8192} + \frac{195x^4}{2048} - \frac{195x^3}{128} + \frac{117x^2}{32} - \frac{13x}{4} + 1\right)}{x^{3/4}}$$

23.14 problem 18

Internal problem ID [2393]
Internal file name [`OUTPUT/2393_Tuesday_February_27_2024_08_36_49_AM_10627226/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(1-3x)\,y''+5xy'-2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-6x^3+2x^2\right)y''+5xy'-2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{5}{2x\,(3x-1)}$$
$$q(x)=\frac{1}{x^2\,(3x-1)}$$

Table 662: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5}{2x(3x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{3}$	"regular"

$q(x) = \frac{1}{x^2(3x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{3}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{3}, \infty\right]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-2y''x^2(3x-1) + 5xy' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2(3x-1) + 5x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-6x^{1+n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-6a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty} \left(-6a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + 5x^{n+r} a_n (n+r) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + 5x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + 5x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + 3r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + 3r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-2}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-6a_{n-1}(n + r - 1)(n + r - 2) + 2a_n(n + r)(n + r - 1) + 5a_n(n + r) - 2a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{6a_{n-1}(n + r - 1)(n + r - 2)}{2n^2 + 4nr + 2r^2 + 3n + 3r - 2} \quad (4)$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{3a_{n-1}(4n^2 - 8n + 3)}{4n^2 + 10n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{6r(-1+r)}{2r^2+7r+3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{3}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{6r(-1+r)}{2r^2+7r+3}$	$-\frac{3}{14}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{36r^4-36r^2}{4r^4+36r^3+107r^2+117r+36}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{3}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{6r(-1+r)}{2r^2+7r+3}$	$-\frac{3}{14}$
a_2	$\frac{36r^4-36r^2}{4r^4+36r^3+107r^2+117r+36}$	$-\frac{3}{56}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{216r^2(-1+r)(1+r)^2(2+r)}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{45}{1232}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{6r(-1+r)}{2r^2+7r+3}$	$-\frac{3}{14}$
a_2	$\frac{36r^4-36r^2}{4r^4+36r^3+107r^2+117r+36}$	$-\frac{3}{56}$
a_3	$\frac{216r^2(-1+r)(1+r)^2(2+r)}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	$-\frac{45}{1232}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1296r^2(-1+r)(1+r)^2(2+r)^2}{16r^7+368r^6+3448r^5+16904r^4+46201r^3+68903r^2+50010r+12600}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = -\frac{675}{18304}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{6r(-1+r)}{2r^2+7r+3}$	$-\frac{3}{14}$
a_2	$\frac{36r^4-36r^2}{4r^4+36r^3+107r^2+117r+36}$	$-\frac{3}{56}$
a_3	$\frac{216r^2(-1+r)(1+r)^2(2+r)}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	$-\frac{45}{1232}$
a_4	$\frac{1296r^2(-1+r)(1+r)^2(2+r)^2}{16r^7+368r^6+3448r^5+16904r^4+46201r^3+68903r^2+50010r+12600}$	$-\frac{675}{18304}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{7776(3+r)\,r^2(-1+r)(1+r)^2(2+r)^2}{32r^8+976r^7+12464r^6+86440r^5+352658r^4+854749r^3+1176456r^2+810495r+198450}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{1701}{36608}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{6r(-1+r)}{2r^2+7r+3}$	$-\frac{3}{14}$
a_2	$\frac{36r^4-36r^2}{4r^4+36r^3+107r^2+117r+36}$	$-\frac{3}{56}$
a_3	$\frac{216r^2(-1+r)(1+r)^2(2+r)}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	$-\frac{45}{1232}$
a_4	$\frac{1296r^2(-1+r)(1+r)^2(2+r)^2}{16r^7+368r^6+3448r^5+16904r^4+46201r^3+68903r^2+50010r+12600}$	$-\frac{675}{18304}$
a_5	$\frac{7776(3+r)r^2(-1+r)(1+r)^2(2+r)^2}{32r^8+976r^7+12464r^6+86440r^5+352658r^4+854749r^3+1176456r^2+810495r+198450}$	$-\frac{1701}{36608}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{3x}{14} - \frac{3x^2}{56} - \frac{45x^3}{1232} - \frac{675x^4}{18304} - \frac{1701x^5}{36608} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
-6b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) + 5b_n(n+r) - 2b_n = 0 \quad (3)
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{6b_{n-1}(n+r-1)(n+r-2)}{2n^2+4nr+2r^2+3n+3r-2} \tag{4}
$$

Which for the root $r = -2$ becomes

$$
b_n = \frac{6b_{n-1}(n-3)(n-4)}{n(2n-5)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{6r(-1+r)}{2r^2 + 7r + 3}$$

Which for the root $r = -2$ becomes

$$b_1 = -12$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{6r(-1+r)}{2r^2+7r+3}$	-12

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{36r^4 - 36r^2}{4r^4 + 36r^3 + 107r^2 + 117r + 36}$$

Which for the root $r = -2$ becomes

$$b_2 = 72$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{6r(-1+r)}{2r^2+7r+3}$	-12
b_2	$\frac{36r^4-36r^2}{4r^4+36r^3+107r^2+117r+36}$	72

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{216r^2(-1+r)(1+r)^2(2+r)}{8r^6 + 132r^5 + 854r^4 + 2739r^3 + 4502r^2 + 3465r + 900}$$

Which for the root $r = -2$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{6r(-1+r)}{2r^2+7r+3}$	-12
b_2	$\frac{36r^4-36r^2}{4r^4+36r^3+107r^2+117r+36}$	72
b_3	$\frac{216r^2(-1+r)(1+r)^2(2+r)}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1296r^2(-1+r)\,(1+r)^2\,(2+r)^2}{16r^7+368r^6+3448r^5+16904r^4+46201r^3+68903r^2+50010r+12600}$$

Which for the root $r = -2$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{6r(-1+r)}{2r^2+7r+3}$	-12
b_2	$\frac{36r^4-36r^2}{4r^4+36r^3+107r^2+117r+36}$	72
b_3	$\frac{216r^2(-1+r)(1+r)^2(2+r)}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	0
b_4	$\frac{1296r^2(-1+r)(1+r)^2(2+r)^2}{16r^7+368r^6+3448r^5+16904r^4+46201r^3+68903r^2+50010r+12600}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{7776(3+r)\,r^2(-1+r)\,(1+r)^2\,(2+r)^2}{32r^8+976r^7+12464r^6+86440r^5+352658r^4+854749r^3+1176456r^2+810495r+198450}$$

Which for the root $r = -2$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{6r(-1+r)}{2r^2+7r+3}$	-12
b_2	$\frac{36r^4-36r^2}{4r^4+36r^3+107r^2+117r+36}$	72
b_3	$\frac{216r^2(-1+r)(1+r)^2(2+r)}{8r^6+132r^5+854r^4+2739r^3+4502r^2+3465r+900}$	0
b_4	$\frac{1296r^2(-1+r)(1+r)^2(2+r)^2}{16r^7+368r^6+3448r^5+16904r^4+46201r^3+68903r^2+50010r+12600}$	0
b_5	$\frac{7776(3+r)r^2(-1+r)(1+r)^2(2+r)^2}{32r^8+976r^7+12464r^6+86440r^5+352658r^4+854749r^3+1176456r^2+810495r+198450}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - 12x + 72x^2 + O(x^6)}{x^2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1-\frac{3x}{14}-\frac{3x^2}{56}-\frac{45x^3}{1232}-\frac{675x^4}{18304}-\frac{1701x^5}{36608}+O\left(x^6\right)\right)+\frac{c_2(1 - 12x + 72x^2 + O(x^6))}{x^2}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1-\frac{3x}{14}-\frac{3x^2}{56}-\frac{45x^3}{1232}-\frac{675x^4}{18304}-\frac{1701x^5}{36608}+O\left(x^6\right)\right)+\frac{c_2(1 - 12x + 72x^2 + O(x^6))}{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(1-\frac{3x}{14}-\frac{3x^2}{56}-\frac{45x^3}{1232}-\frac{675x^4}{18304}-\frac{1701x^5}{36608}+O\left(x^6\right)\right) \\
& + \frac{c_2(1 - 12x + 72x^2 + O(x^6))}{x^2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{3x}{14}-\frac{3x^2}{56}-\frac{45x^3}{1232}-\frac{675x^4}{18304}-\frac{1701x^5}{36608}+O(x^6)\right) + \frac{c_2(1-12x+72x^2+O(x^6))}{x^2}$$

Verified OK.

23.14.1 Maple step by step solution

Let's solve

$-2y''x^2(3x-1)+5xy'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x^2(3x-1)}+\frac{5y'}{2x(3x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{5y'}{2x(3x-1)}+\frac{y}{x^2(3x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{5}{2x(3x-1)}, P_3(x)=\frac{1}{x^2(3x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{5}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=-1$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2y''x^2(3x-1)-5xy'+2y=0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(2+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+2)\,(2k+2r-1) + 6a_{k-1}(k+r-1)\,(k-2+r)\right) x\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(2+r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $6a_{k-1}(k+r-1)\,(k-2+r) - 2(k+r+2)\left(k+r-\frac{1}{2}\right) a_k = 0$

- Shift index using $k- >k+1$

  $6a_k(k+r)\,(k+r-1) - 2(k+3+r)\left(k+\frac{1}{2}+r\right) a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{6a_k(k+r)(k+r-1)}{(k+3+r)(2k+1+2r)}$

- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+1} = \frac{6a_k(k-2)(k-3)}{(k+1)(2k-3)}$

- Apply recursion relation for $k = 0$

  $a_1 = -12a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -6a_1$

- Express in terms of a_0

  $a_2 = 72a_0$

- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $y = a_0 \cdot (1 - 12x + 72x^2)$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{6a_k (k+\frac{1}{2})(k-\frac{1}{2})}{(k+\frac{7}{2})(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{6a_k (k+\frac{1}{2})(k-\frac{1}{2})}{(k+\frac{7}{2})(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot (1 - 12x + 72x^2) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{6b_k (k+\frac{1}{2})(k-\frac{1}{2})}{(k+\frac{7}{2})(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
Order:=6;
dsolve(2*x^2*(1-3*x)*diff(y(x),x$2)+5*x*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{5}{2}}\left(1-\frac{3}{14}x-\frac{3}{56}x^2-\frac{45}{1232}x^3-\frac{675}{18304}x^4-\frac{1701}{36608}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1-12x+72x^2+\mathrm{O}\left(x^6\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 63

```
AsymptoticDSolveValue[2*x^2*(1-3*x)*y''[x]+5*x*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_2(72x^2 - 12x + 1)}{x^2} + c_1\sqrt{x}\left(-\frac{1701x^5}{36608} - \frac{675x^4}{18304} - \frac{45x^3}{1232} - \frac{3x^2}{56} - \frac{3x}{14} + 1\right)$$

23.15 problem 19

Internal problem ID [2394]
Internal file name [`OUTPUT/2394_Tuesday_February_27_2024_08_36_50_AM_47576210/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x+1)\,y''-5xy'+2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(4x^3+4x^2\right)y''-5xy'+2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{5}{4x\,(x+1)}$$
$$q(x)=\frac{1}{2x^2\,(x+1)}$$

Table 664: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{5}{4x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1}{2x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2(x+1)\,y'' - 5xy' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x^2(x+1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - 5x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\
+ \sum_{n=0}^{\infty} \left(-5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} 4x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=1}^{\infty} 4a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) \\
+ \sum_{n=0}^{\infty} \left(-5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r} a_n (n+r)(n+r-1) - 5x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
4x^r a_0 r(-1+r) - 5x^r a_0 r + 2a_0 x^r = 0
$$

Or

$$
\left(4x^r r(-1+r) - 5x^r r + 2x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(4r^2 - 9r + 2\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
4r^2 - 9r + 2 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 9r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{7}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_{n-1}(n + r - 1)(n + r - 2) + 4a_n(n + r)(n + r - 1) - 5a_n(n + r) + 2a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4(n + r - 1) a_{n-1}}{4n + 4r - 1} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{4(1 + n) a_{n-1}}{4n + 7} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4r}{3+4r}$$

Which for the root $r = 2$ becomes

$$a_1 = -\frac{8}{11}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4r}{3+4r}$	$-\frac{8}{11}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16r(1+r)}{16r^2+40r+21}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{32}{55}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4r}{3+4r}$	$-\frac{8}{11}$
a_2	$\frac{16r(1+r)}{16r^2+40r+21}$	$\frac{32}{55}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64r(1+r)(2+r)}{64r^3+336r^2+524r+231}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{512}{1045}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4r}{3+4r}$	$-\frac{8}{11}$
a_2	$\frac{16r(1+r)}{16r^2+40r+21}$	$\frac{32}{55}$
a_3	$-\frac{64r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{512}{1045}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256r(1+r)(2+r)(3+r)}{256r^4 + 2304r^3 + 7136r^2 + 8784r + 3465}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{2048}{4807}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4r}{3+4r}$	$-\frac{8}{11}$
a_2	$\frac{16r(1+r)}{16r^2+40r+21}$	$\frac{32}{55}$
a_3	$-\frac{64r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{512}{1045}$
a_4	$\frac{256r(1+r)(2+r)(3+r)}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{2048}{4807}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024(3+r)\,r(1+r)(2+r)(4+r)}{1024r^5 + 14080r^4 + 72320r^3 + 170720r^2 + 180756r + 65835}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{16384}{43263}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4r}{3+4r}$	$-\frac{8}{11}$
a_2	$\frac{16r(1+r)}{16r^2+40r+21}$	$\frac{32}{55}$
a_3	$-\frac{64r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{512}{1045}$
a_4	$\frac{256r(1+r)(2+r)(3+r)}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{2048}{4807}$
a_5	$-\frac{1024(3+r)r(1+r)(2+r)(4+r)}{1024r^5+14080r^4+72320r^3+170720r^2+180756r+65835}$	$-\frac{16384}{43263}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x^2\left(1 - \frac{8x}{11} + \frac{32x^2}{55} - \frac{512x^3}{1045} + \frac{2048x^4}{4807} - \frac{16384x^5}{43263} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_{n-1}(n+r-1)(n+r-2) + 4b_n(n+r)(n+r-1) - 5b_n(n+r) + 2b_n = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{4(n+r-1)\,b_{n-1}}{4n+4r-1} \quad (4)$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_n = -\frac{(4n-3)\,b_{n-1}}{4n} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{4r}{3+4r}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_1 = -\frac{1}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4r}{3+4r}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{16r(1+r)}{16r^2 + 40r + 21}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_2 = \frac{5}{32}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4r}{3+4r}$	$-\frac{1}{4}$
b_2	$\frac{16r(1+r)}{16r^2+40r+21}$	$\frac{5}{32}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{64r(1+r)(2+r)}{64r^3 + 336r^2 + 524r + 231}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_3 = -\frac{15}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4r}{3+4r}$	$-\frac{1}{4}$
b_2	$\frac{16r(1+r)}{16r^2+40r+21}$	$\frac{5}{32}$
b_3	$-\frac{64r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{15}{128}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{256r(1+r)(2+r)(3+r)}{256r^4 + 2304r^3 + 7136r^2 + 8784r + 3465}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_4 = \frac{195}{2048}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4r}{3+4r}$	$-\frac{1}{4}$
b_2	$\frac{16r(1+r)}{16r^2+40r+21}$	$\frac{5}{32}$
b_3	$-\frac{64r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{15}{128}$
b_4	$\frac{256r(1+r)(2+r)(3+r)}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{195}{2048}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1024(3+r)r(1+r)(2+r)(4+r)}{1024r^5 + 14080r^4 + 72320r^3 + 170720r^2 + 180756r + 65835}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_5 = -\frac{663}{8192}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{4r}{3+4r}$	$-\frac{1}{4}$
b_2	$\frac{16r(1+r)}{16r^2+40r+21}$	$\frac{5}{32}$
b_3	$-\frac{64r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{15}{128}$
b_4	$\frac{256r(1+r)(2+r)(3+r)}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{195}{2048}$
b_5	$-\frac{1024(3+r)r(1+r)(2+r)(4+r)}{1024r^5+14080r^4+72320r^3+170720r^2+180756r+65835}$	$-\frac{663}{8192}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{\frac{1}{4}} \left(1 - \frac{x}{4} + \frac{5x^2}{32} - \frac{15x^3}{128} + \frac{195x^4}{2048} - \frac{663x^5}{8192} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 - \frac{8x}{11} + \frac{32x^2}{55} - \frac{512x^3}{1045} + \frac{2048x^4}{4807} - \frac{16384x^5}{43263} + O(x^6)\right) \\
&+ c_2 x^{\frac{1}{4}} \left(1 - \frac{x}{4} + \frac{5x^2}{32} - \frac{15x^3}{128} + \frac{195x^4}{2048} - \frac{663x^5}{8192} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 - \frac{8x}{11} + \frac{32x^2}{55} - \frac{512x^3}{1045} + \frac{2048x^4}{4807} - \frac{16384x^5}{43263} + O(x^6)\right) \\
&+ c_2 x^{\frac{1}{4}} \left(1 - \frac{x}{4} + \frac{5x^2}{32} - \frac{15x^3}{128} + \frac{195x^4}{2048} - \frac{663x^5}{8192} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^2 \left(1 - \frac{8x}{11} + \frac{32x^2}{55} - \frac{512x^3}{1045} + \frac{2048x^4}{4807} - \frac{16384x^5}{43263} + O(x^6)\right) \\
+ c_2 x^{\frac{1}{4}} \left(1 - \frac{x}{4} + \frac{5x^2}{32} - \frac{15x^3}{128} + \frac{195x^4}{2048} - \frac{663x^5}{8192} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^2 \left(1 - \frac{8x}{11} + \frac{32x^2}{55} - \frac{512x^3}{1045} + \frac{2048x^4}{4807} - \frac{16384x^5}{43263} + O(x^6)\right) \\
+ c_2 x^{\frac{1}{4}} \left(1 - \frac{x}{4} + \frac{5x^2}{32} - \frac{15x^3}{128} + \frac{195x^4}{2048} - \frac{663x^5}{8192} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

23.15.1 Maple step by step solution

Let's solve

$4x^2(x+1)\,y'' - 5xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{4x(x+1)} - \frac{y}{2x^2(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{4x(x+1)} + \frac{y}{2x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{5}{4x(x+1)}, P_3(x) = \frac{1}{2x^2(x+1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{5}{4}$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4x^2(x+1)\,y'' - 5xy' + 2y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (-5u+5)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+4r)\, u^{-1+r} + \left(a_1(1+r)\,(5+4r) - a_0(8r^2-3r-2)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(4k+5+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(1+4r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{1}{4}\right\}$$

- Each term must be 0

  $$a_1(1+r)\,(5+4r) - a_0(8r^2-3r-2) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$4(-2a_k + a_{k-1} + a_{k+1})\, k^2 + \left(8(-2a_k + a_{k-1} + a_{k+1})\, r + 3a_k - 12a_{k-1} + 9a_{k+1}\right) k + 4(-2a_k + a_k$$

- Shift index using $k- >k+1$

  $$4(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + \left(8(-2a_{k+1} + a_k + a_{k+2})\, r + 3a_{k+1} - 12a_k + 9a_{k+2}\right)(k+1) + 4$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} - 4ka_k - 13ka_{k+1} - 4ra_k - 13ra_{k+1} - 3a_{k+1}}{4k^2 + 8kr + 4r^2 + 17k + 17r + 18}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 13ka_{k+1} - 3a_{k+1}}{4k^2 + 17k + 18}$$

- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-4ka_k-13ka_{k+1}-3a_{k+1}}{4k^2+17k+18}, 5a_1+2a_0=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-4ka_k-13ka_{k+1}-3a_{k+1}}{4k^2+17k+18}, 5a_1+2a_0=0\right]$$

- Recursion relation for $r=-\frac{1}{4}$

$$a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-6ka_k-9ka_{k+1}+\frac{5}{4}a_k-\frac{1}{4}a_{k+1}}{4k^2+15k+14}$$

- Solution for $r=-\frac{1}{4}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{4}}, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-6ka_k-9ka_{k+1}+\frac{5}{4}a_k-\frac{1}{4}a_{k+1}}{4k^2+15k+14}, 3a_1+\frac{3a_0}{4}=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{1}{4}}, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-6ka_k-9ka_{k+1}+\frac{5}{4}a_k-\frac{1}{4}a_{k+1}}{4k^2+15k+14}, 3a_1+\frac{3a_0}{4}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x+1)^{k-\frac{1}{4}}\right), a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-4ka_k-13ka_{k+1}-3a_{k+1}}{4k^2+17k+18}, 5a_1+2a_0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(4*x^2*(1+x)*diff(y(x),x$2)-5*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{4}} \left(1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \frac{195}{2048}x^4 - \frac{663}{8192}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 x^2 \left(1 - \frac{8}{11}x + \frac{32}{55}x^2 - \frac{512}{1045}x^3 + \frac{2048}{4807}x^4 - \frac{16384}{43263}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 88

```
AsymptoticDSolveValue[4*x^2*(1+x)*y''[x]-5*x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{16384x^5}{43263} + \frac{2048x^4}{4807} - \frac{512x^3}{1045} + \frac{32x^2}{55} - \frac{8x}{11} + 1\right) x^2 + c_2 \left(-\frac{663x^5}{8192} + \frac{195x^4}{2048} - \frac{15x^3}{128} + \frac{5x^2}{32} - \frac{x}{4} + 1\right) \sqrt[4]{x}$$

23.16 problem 20

Internal problem ID [2395]
Internal file name [`OUTPUT/2395_Tuesday_February_27_2024_08_36_51_AM_79605024/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x+4)\, y''+x(x-1)\, y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+4x^2\right) y''+\left(x^2-x\right) y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x-1}{x\,(x+4)}$$
$$q(x)=\frac{1}{x^2\,(x+4)}$$

Table 666: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-1}{x(x+4)}$	
singularity	type
$x = -4$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1}{x^2(x+4)}$	
singularity	type
$x = -4$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-4, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+4)\, y'' + \left(x^2 - x\right) y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+4) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(x^2 - x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1) x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(4x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 5r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 5r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 5r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) \\ &+ a_{n-1}(n+r-1) - a_n(n+r) + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-1)}{4n+4r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}n}{4n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r}{3+4r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{3+4r}$	$-\frac{1}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(1+r)}{16r^2 + 40r + 21}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{2}{77}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{3+4r}$	$-\frac{1}{7}$
a_2	$\frac{r(1+r)}{16r^2+40r+21}$	$\frac{2}{77}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{r(1+r)(2+r)}{64r^3 + 336r^2 + 524r + 231}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{2}{385}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{3+4r}$	$-\frac{1}{7}$
a_2	$\frac{r(1+r)}{16r^2+40r+21}$	$\frac{2}{77}$
a_3	$-\frac{r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{2}{385}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(1+r)(2+r)(3+r)}{256r^4 + 2304r^3 + 7136r^2 + 8784r + 3465}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{8}{7315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{3+4r}$	$-\frac{1}{7}$
a_2	$\frac{r(1+r)}{16r^2+40r+21}$	$\frac{2}{77}$
a_3	$-\frac{r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{2}{385}$
a_4	$\frac{r(1+r)(2+r)(3+r)}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{8}{7315}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(3+r)\,r(1+r)(2+r)(4+r)}{1024r^5 + 14080r^4 + 72320r^3 + 170720r^2 + 180756r + 65835}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{8}{33649}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{3+4r}$	$-\frac{1}{7}$
a_2	$\frac{r(1+r)}{16r^2+40r+21}$	$\frac{2}{77}$
a_3	$-\frac{r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{2}{385}$
a_4	$\frac{r(1+r)(2+r)(3+r)}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{8}{7315}$
a_5	$-\frac{(3+r)r(1+r)(2+r)(4+r)}{1024r^5+14080r^4+72320r^3+170720r^2+180756r+65835}$	$-\frac{8}{33649}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - \frac{x}{7} + \frac{2x^2}{77} - \frac{2x^3}{385} + \frac{8x^4}{7315} - \frac{8x^5}{33649} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
&b_{n-1}(n+r-1)(n+r-2) + 4b_n(n+r)(n+r-1) \\
&+ b_{n-1}(n+r-1) - b_n(n+r) + b_n = 0
\end{aligned} \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}(n+r-1)}{4n+4r-1} \tag{4}
$$

Which for the root $r = \frac{1}{4}$ becomes

$$
b_n = -\frac{b_{n-1}(4n-3)}{16n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r}{3+4r}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_1 = -\frac{1}{16}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{3+4r}$	$-\frac{1}{16}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r(1+r)}{16r^2+40r+21}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_2 = \frac{5}{512}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{3+4r}$	$-\frac{1}{16}$
b_2	$\frac{r(1+r)}{16r^2+40r+21}$	$\frac{5}{512}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{r(1+r)(2+r)}{64r^3+336r^2+524r+231}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_3 = -\frac{15}{8192}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{3+4r}$	$-\frac{1}{16}$
b_2	$\frac{r(1+r)}{16r^2+40r+21}$	$\frac{5}{512}$
b_3	$-\frac{r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{15}{8192}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(1+r)(2+r)(3+r)}{256r^4 + 2304r^3 + 7136r^2 + 8784r + 3465}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_4 = \frac{195}{524288}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{3+4r}$	$-\frac{1}{16}$
b_2	$\frac{r(1+r)}{16r^2+40r+21}$	$\frac{5}{512}$
b_3	$-\frac{r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{15}{8192}$
b_4	$\frac{r(1+r)(2+r)(3+r)}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{195}{524288}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(3+r)\,r(1+r)(2+r)(4+r)}{1024r^5 + 14080r^4 + 72320r^3 + 170720r^2 + 180756r + 65835}$$

Which for the root $r = \frac{1}{4}$ becomes

$$b_5 = -\frac{663}{8388608}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{3+4r}$	$-\frac{1}{16}$
b_2	$\frac{r(1+r)}{16r^2+40r+21}$	$\frac{5}{512}$
b_3	$-\frac{r(1+r)(2+r)}{64r^3+336r^2+524r+231}$	$-\frac{15}{8192}$
b_4	$\frac{r(1+r)(2+r)(3+r)}{256r^4+2304r^3+7136r^2+8784r+3465}$	$\frac{195}{524288}$
b_5	$-\frac{(3+r)r(1+r)(2+r)(4+r)}{1024r^5+14080r^4+72320r^3+170720r^2+180756r+65835}$	$-\frac{663}{8388608}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots\right) \\
&= x^{\frac{1}{4}}\left(1 - \frac{x}{16} + \frac{5x^2}{512} - \frac{15x^3}{8192} + \frac{195x^4}{524288} - \frac{663x^5}{8388608} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x\left(1 - \frac{x}{7} + \frac{2x^2}{77} - \frac{2x^3}{385} + \frac{8x^4}{7315} - \frac{8x^5}{33649} + O(x^6)\right) \\
&\quad + c_2x^{\frac{1}{4}}\left(1 - \frac{x}{16} + \frac{5x^2}{512} - \frac{15x^3}{8192} + \frac{195x^4}{524288} - \frac{663x^5}{8388608} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x\left(1 - \frac{x}{7} + \frac{2x^2}{77} - \frac{2x^3}{385} + \frac{8x^4}{7315} - \frac{8x^5}{33649} + O(x^6)\right) \\
&\quad + c_2x^{\frac{1}{4}}\left(1 - \frac{x}{16} + \frac{5x^2}{512} - \frac{15x^3}{8192} + \frac{195x^4}{524288} - \frac{663x^5}{8388608} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1x\left(1 - \frac{x}{7} + \frac{2x^2}{77} - \frac{2x^3}{385} + \frac{8x^4}{7315} - \frac{8x^5}{33649} + O(x^6)\right) \\
+ c_2x^{\frac{1}{4}}\left(1 - \frac{x}{16} + \frac{5x^2}{512} - \frac{15x^3}{8192} + \frac{195x^4}{524288} - \frac{663x^5}{8388608} + O(x^6)\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = c_1 x \left(1-\frac{x}{7}+\frac{2x^2}{77}-\frac{2x^3}{385}+\frac{8x^4}{7315}-\frac{8x^5}{33649}+O\left(x^6\right)\right) + c_2 x^{\frac{1}{4}} \left(1-\frac{x}{16}+\frac{5x^2}{512}-\frac{15x^3}{8192}+\frac{195x^4}{524288}-\frac{663x^5}{8388608}+O\left(x^6\right)\right)$$

Verified OK.

23.16.1 Maple step by step solution

Let's solve

$x^2(x+4)\, y'' + (x^2 - x)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2(x+4)} - \frac{(x-1)y'}{x(x+4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-1)y'}{x(x+4)} + \frac{y}{x^2(x+4)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{x-1}{x(x+4)}, P_3(x) = \frac{1}{x^2(x+4)}\right]$

- ◦ $(x+4)\cdot P_2(x)$ is analytic at $x=-4$

  $\left((x+4)\cdot P_2(x)\right)\Big|_{x=-4} = \frac{5}{4}$

- ◦ $(x+4)^2\cdot P_3(x)$ is analytic at $x=-4$

  $\left((x+4)^2\cdot P_3(x)\right)\Big|_{x=-4} = 0$

- ◦ $x=-4$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -4$

- Multiply by denominators

  $x^2(x+4)\, y'' + x(x-1)\, y' + y = 0$

- Change variables using $x = u - 4$ so that the regular singular point is at $u = 0$

$$\left(u^{3}-8u^{2}+16u\right)\left(\frac{d^{2}}{du^{2}}y(u)\right)+\left(u^{2}-9u+20\right)\left(\frac{d}{du}y(u)\right)+y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(1+4r)\,u^{-1+r}+\left(4a_1(1+r)\,(5+4r)-a_0\left(8r^2+r-1\right)\right)u^r+\left(\sum_{k=1}^{\infty}\left(4a_{k+1}(k+1+r)\,(4k+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(1+4r)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{4}\right\}$$

- Each term must be 0

$$4a_1(1+r)\,(5+4r)-a_0\left(8r^2+r-1\right)=0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k-1}(k+r-1)^2+16\left(k+\frac{5}{4}+r\right)(k+1+r)\,a_{k+1}-a_k\left(8k^2+16kr+8r^2+k+r-1\right)=0$$

- Shift index using $k->k+1$

$$a_k(k+r)^2+16\left(k+\frac{9}{4}+r\right)(k+2+r)\,a_{k+2}-a_{k+1}\left(8(k+1)^2+16(k+1)\,r+8r^2+k+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} + 2kra_k - 16kra_{k+1} + r^2 a_k - 8r^2 a_{k+1} - 17ka_{k+1} - 17ra_{k+1} - 8a_{k+1}}{4(4k+9+4r)(k+2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - 17ka_{k+1} - 8a_{k+1}}{4(4k+9)(k+2)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - 17ka_{k+1} - 8a_{k+1}}{4(4k+9)(k+2)}, 20a_1 + a_0 = 0\right]$$

- Revert the change of variables $u = x + 4$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+4)^k, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - 17ka_{k+1} - 8a_{k+1}}{4(4k+9)(k+2)}, 20a_1 + a_0 = 0\right]$$

- Recursion relation for $r = -\frac{1}{4}$

$$a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - \frac{1}{2}ka_k - 13ka_{k+1} + \frac{1}{16}a_k - \frac{17}{4}a_{k+1}}{4(4k+8)\left(k+\frac{7}{4}\right)}$$

- Solution for $r = -\frac{1}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{4}}, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - \frac{1}{2}ka_k - 13ka_{k+1} + \frac{1}{16}a_k - \frac{17}{4}a_{k+1}}{4(4k+8)\left(k+\frac{7}{4}\right)}, 12a_1 + \frac{3a_0}{4} = 0\right]$$

- Revert the change of variables $u = x + 4$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+4)^{k-\frac{1}{4}}, a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - \frac{1}{2}ka_k - 13ka_{k+1} + \frac{1}{16}a_k - \frac{17}{4}a_{k+1}}{4(4k+8)\left(k+\frac{7}{4}\right)}, 12a_1 + \frac{3a_0}{4} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+4)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+4)^{k-\frac{1}{4}}\right), a_{k+2} = -\frac{k^2 a_k - 8k^2 a_{k+1} - 17ka_{k+1} - 8a_{k+1}}{4(4k+9)(k+2)}, 20a_1 + a_0 = 0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
Order:=6;
dsolve((4+x)*x^2*diff(y(x),x$2)+x*(x-1)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$
y(x) = c_1 x^{\frac{1}{4}} \left(1 - \frac{1}{16}x + \frac{5}{512}x^2 - \frac{15}{8192}x^3 + \frac{195}{524288}x^4 - \frac{663}{8388608}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 x \left(1 - \frac{1}{7}x + \frac{2}{77}x^2 - \frac{2}{385}x^3 + \frac{8}{7315}x^4 - \frac{8}{33649}x^5 + \mathrm{O}\left(x^6\right)\right)
$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 86

```
AsymptoticDSolveValue[(4+x)*x^2*y''[x]+x*(x-1)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(-\frac{8x^5}{33649} + \frac{8x^4}{7315} - \frac{2x^3}{385} + \frac{2x^2}{77} - \frac{x}{7} + 1\right) + c_2 \sqrt[4]{x} \left(-\frac{663x^5}{8388608} + \frac{195x^4}{524288} - \frac{15x^3}{8192} + \frac{5x^2}{512} - \frac{x}{16} + 1\right)$$

23.17 problem 21

Internal problem ID [2396]
Internal file name [`OUTPUT/2396_Tuesday_February_27_2024_08_36_51_AM_64556011/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(8-x)\,x^2y''+6xy'-y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^3+8x^2\right)y''+6xy'-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{6}{x\,(-8+x)}$$
$$q(x)=\frac{1}{(-8+x)\,x^2}$$

Table 668: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{6}{x(-8+x)}$	
singularity	type
$x = 0$	"regular"
$x = 8$	"regular"

$q(x) = \frac{1}{(-8+x)x^2}$	
singularity	type
$x = 0$	"regular"
$x = 8$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 8, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''(-8+x)\,x^2 + 6xy' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right)(-8+x)\,x^2 + 6x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$8x^{n+r} a_n (n+r)(n+r-1) + 6x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$8x^r a_0 r(-1+r) + 6x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(8x^r r(-1+r) + 6x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(8r^2 - 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$8r^2 - 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(8r^2 - 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{4}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2) + 8a_n(n+r)(n+r-1) + 6a_n(n+r) - a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-1)(n+r-2)}{8n^2 + 16nr + 8r^2 - 2n - 2r - 1} \quad (4)$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(2n-1)(2n-3)}{32n^2 + 24n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r(-1+r)}{8r^2+14r+5}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{1}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r(-1+r)}{8r^2+14r+5}$	$-\frac{1}{56}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2(r^2-1)}{(8r^2+14r+5)(8r^2+30r+27)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{3}{9856}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r(-1+r)}{8r^2+14r+5}$	$-\frac{1}{56}$
a_2	$\frac{r^2(r^2-1)}{(8r^2+14r+5)(8r^2+30r+27)}$	$-\frac{3}{9856}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^2(r^2-1)(r^2+3r+2)}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{78848}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r(-1+r)}{8r^2+14r+5}$	$-\frac{1}{56}$
a_2	$\frac{r^2(r^2-1)}{(8r^2+14r+5)(8r^2+30r+27)}$	$-\frac{3}{9856}$
a_3	$\frac{r^2(r^2-1)(r^2+3r+2)}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)}$	$-\frac{1}{78848}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(3+r)\,r^2(-1+r)\,(1+r)^2\,(2+r)^2}{(8r^2+14r+5)\,(8r^2+30r+27)\,(8r^2+46r+65)\,(8r^2+62r+119)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = -\frac{5}{6848512}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r(-1+r)}{8r^2+14r+5}$	$-\frac{1}{56}$
a_2	$\frac{r^2(r^2-1)}{(8r^2+14r+5)(8r^2+30r+27)}$	$-\frac{3}{9856}$
a_3	$\frac{r^2(r^2-1)(r^2+3r+2)}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)}$	$-\frac{1}{78848}$
a_4	$\frac{(3+r)r^2(-1+r)(1+r)^2(2+r)^2}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)(8r^2+62r+119)}$	$-\frac{5}{6848512}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(3+r)^2\,r^2(-1+r)\,(1+r)^2\,(2+r)^2\,(4+r)}{(8r^2+14r+5)\,(8r^2+30r+27)\,(8r^2+46r+65)\,(8r^2+62r+119)\,(8r^2+78r+189)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{63}{1260126208}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r(-1+r)}{8r^2+14r+5}$	$-\frac{1}{56}$
a_2	$\frac{r^2(r^2-1)}{(8r^2+14r+5)(8r^2+30r+27)}$	$-\frac{3}{9856}$
a_3	$\frac{r^2(r^2-1)(r^2+3r+2)}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)}$	$-\frac{1}{78848}$
a_4	$\frac{(3+r)r^2(-1+r)(1+r)^2(2+r)^2}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)(8r^2+62r+119)}$	$-\frac{5}{6848512}$
a_5	$\frac{(3+r)^2r^2(-1+r)(1+r)^2(2+r)^2(4+r)}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)(8r^2+62r+119)(8r^2+78r+189)}$	$-\frac{63}{1260126208}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x}{56} - \frac{3x^2}{9856} - \frac{x^3}{78848} - \frac{5x^4}{6848512} - \frac{63x^5}{1260126208} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$-b_{n-1}(n+r-1)(n+r-2) + 8b_n(n+r)(n+r-1) + 6b_n(n+r) - b_n = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}(n+r-1)(n+r-2)}{8n^2 + 16nr + 8r^2 - 2n - 2r - 1} \quad (4)$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_n = \frac{b_{n-1}(4n-5)(4n-9)}{128n^2 - 96n} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{r(-1+r)}{8r^2 + 14r + 5}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_1 = \frac{5}{32}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r(-1+r)}{8r^2+14r+5}$	$\frac{5}{32}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2(r^2-1)}{(8r^2+14r+5)(8r^2+30r+27)}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_2 = -\frac{3}{2048}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r(-1+r)}{8r^2+14r+5}$	$\frac{5}{32}$
b_2	$\frac{r^2(r^2-1)}{(8r^2+14r+5)(8r^2+30r+27)}$	$-\frac{3}{2048}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r^2(r^2-1)(r^2+3r+2)}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_3 = -\frac{7}{196608}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r(-1+r)}{8r^2+14r+5}$	$\frac{5}{32}$
b_2	$\frac{r^2(r^2-1)}{(8r^2+14r+5)(8r^2+30r+27)}$	$-\frac{3}{2048}$
b_3	$\frac{r^2(r^2-1)(r^2+3r+2)}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)}$	$-\frac{7}{196608}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(3+r)\, r^2(-1+r)\,(1+r)^2\,(2+r)^2}{(8r^2+14r+5)\,(8r^2+30r+27)\,(8r^2+46r+65)\,(8r^2+62r+119)}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_4 = -\frac{539}{327155712}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r(-1+r)}{8r^2+14r+5}$	$\frac{5}{32}$
b_2	$\frac{r^2(r^2-1)}{(8r^2+14r+5)(8r^2+30r+27)}$	$-\frac{3}{2048}$
b_3	$\frac{r^2(r^2-1)(r^2+3r+2)}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)}$	$-\frac{7}{196608}$
b_4	$\frac{(3+r)r^2(-1+r)(1+r)^2(2+r)^2}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)(8r^2+62r+119)}$	$-\frac{539}{327155712}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(3+r)^2\, r^2(-1+r)\,(1+r)^2\,(2+r)^2\,(4+r)}{(8r^2+14r+5)\,(8r^2+30r+27)\,(8r^2+46r+65)\,(8r^2+62r+119)\,(8r^2+78r+189)}$$

Which for the root $r = -\frac{1}{4}$ becomes

$$b_5 = -\frac{5929}{59324235776}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r(-1+r)}{8r^2+14r+5}$	$\frac{5}{32}$
b_2	$\frac{r^2(r^2-1)}{(8r^2+14r+5)(8r^2+30r+27)}$	$-\frac{3}{2048}$
b_3	$\frac{r^2(r^2-1)(r^2+3r+2)}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)}$	$-\frac{7}{196608}$
b_4	$\frac{(3+r)r^2(-1+r)(1+r)^2(2+r)^2}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)(8r^2+62r+119)}$	$-\frac{539}{327155712}$
b_5	$\frac{(3+r)^2r^2(-1+r)(1+r)^2(2+r)^2(4+r)}{(8r^2+14r+5)(8r^2+30r+27)(8r^2+46r+65)(8r^2+62r+119)(8r^2+78r+189)}$	$-\frac{5929}{59324235776}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= \sqrt{x}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\
&= \frac{1 + \frac{5x}{32} - \frac{3x^2}{2048} - \frac{7x^3}{196608} - \frac{539x^4}{327155712} - \frac{5929x^5}{59324235776} + O(x^6)}{x^{\frac{1}{4}}}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\sqrt{x}\left(1 - \frac{x}{56} - \frac{3x^2}{9856} - \frac{x^3}{78848} - \frac{5x^4}{6848512} - \frac{63x^5}{1260126208} + O(x^6)\right) \\
&+ \frac{c_2\left(1 + \frac{5x}{32} - \frac{3x^2}{2048} - \frac{7x^3}{196608} - \frac{539x^4}{327155712} - \frac{5929x^5}{59324235776} + O(x^6)\right)}{x^{\frac{1}{4}}}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1 - \frac{x}{56} - \frac{3x^2}{9856} - \frac{x^3}{78848} - \frac{5x^4}{6848512} - \frac{63x^5}{1260126208} + O(x^6)\right) \\
&+ \frac{c_2\left(1 + \frac{5x}{32} - \frac{3x^2}{2048} - \frac{7x^3}{196608} - \frac{539x^4}{327155712} - \frac{5929x^5}{59324235776} + O(x^6)\right)}{x^{\frac{1}{4}}}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x}\left(1-\frac{x}{56}-\frac{3x^2}{9856}-\frac{x^3}{78848}-\frac{5x^4}{6848512}-\frac{63x^5}{1260126208}+O(x^6)\right) + \frac{c_2\left(1+\frac{5x}{32}-\frac{3x^2}{2048}-\frac{7x^3}{196608}-\frac{539x^4}{327155712}-\frac{5929x^5}{59324235776}+O(x^6)\right)}{x^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1-\frac{x}{56}-\frac{3x^2}{9856}-\frac{x^3}{78848}-\frac{5x^4}{6848512}-\frac{63x^5}{1260126208}+O(x^6)\right) + \frac{c_2\left(1+\frac{5x}{32}-\frac{3x^2}{2048}-\frac{7x^3}{196608}-\frac{539x^4}{327155712}-\frac{5929x^5}{59324235776}+O(x^6)\right)}{x^{\frac{1}{4}}}$$

Verified OK.

23.17.1 Maple step by step solution

Let's solve

$-y''(-8+x)\,x^2+6xy'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{(-8+x)x^2}+\frac{6y'}{x(-8+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{6y'}{x(-8+x)}+\frac{y}{(-8+x)x^2}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{6}{x(-8+x)},P_3(x)=\frac{1}{(-8+x)x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{3}{4}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{8}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''(-8+x)\, x^2 - 6xy' + y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- >k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0(1+4r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(4k+4r+1)\,(2k+2r-1) + a_{k-1}(k+r-1)\,(k-2+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(1+4r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{4}, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(k+r-1)\,(k-2+r) - 8a_k\left(k+r+\frac{1}{4}\right)\left(k+r-\frac{1}{2}\right) = 0$

- Shift index using $k- >k+1$

  $a_k(k+r)\,(k+r-1) - 8a_{k+1}\left(k+\frac{5}{4}+r\right)\left(k+\frac{1}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r)(k+r-1)}{(4k+5+4r)(2k+1+2r)}$$

- Recursion relation for $r = -\frac{1}{4}$

$$a_{k+1} = \frac{a_k\left(k-\frac{1}{4}\right)\left(k-\frac{5}{4}\right)}{(4k+4)\left(2k+\frac{1}{2}\right)}$$

- Solution for $r = -\frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+1} = \frac{a_k\left(k-\frac{1}{4}\right)\left(k-\frac{5}{4}\right)}{(4k+4)\left(2k+\frac{1}{2}\right)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)}{(4k+7)(2k+2)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)}{(4k+7)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k\left(k-\frac{1}{4}\right)\left(k-\frac{5}{4}\right)}{(4k+4)\left(2k+\frac{1}{2}\right)}, b_{k+1} = \frac{b_k\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)}{(4k+7)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve((8-x)*x^2*diff(y(x),x$2)+6*x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{3}{4}}\left(1-\frac{1}{56}x-\frac{3}{9856}x^2-\frac{1}{78848}x^3-\frac{5}{6848512}x^4-\frac{63}{1260126208}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1+\frac{5}{32}x-\frac{3}{2048}x^2-\frac{7}{196608}x^3\right.}{x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 90

```
AsymptoticDSolveValue[(8-x)*x^2*y''[x]+6*x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{63x^5}{1260126208}-\frac{5x^4}{6848512}-\frac{x^3}{78848}-\frac{3x^2}{9856}-\frac{x}{56}+1\right)+\frac{c_2\left(-\frac{5929x^5}{59324235776}-\frac{539x^4}{327155712}-\frac{7x^3}{196608}-\frac{3x^2}{2048}+\frac{5x}{32}+1\right)}{\sqrt[4]{x}}$$

23.18 problem 22

Internal problem ID [2397]
Internal file name [OUTPUT/2397_Tuesday_February_27_2024_08_36_52_AM_49705858/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + x(x^2+1)\,y' - y(x+1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + (x^3+x)\,y' + (-x-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2+1}{2x}$$
$$q(x) = -\frac{x+1}{2x^2}$$

Table 670: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+1}{2x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{x+1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + \left(x^3 + x\right) y' + (-x - 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + \left(x^3 + x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (-x - 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{1}{r(2r+3)}$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) + a_n(n+r) - a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n a_{n-2} + r a_{n-2} - 2a_{n-2} - a_{n-1}}{2n^2 + 4nr + 2r^2 - n - r - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-n a_{n-2} + a_{n-2} + a_{n-1}}{n\,(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(2r+3)}$	$\frac{1}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-2r^2 - r + 1}{4r^3 + 16r^2 + 15r}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{2}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(2r+3)}$	$\frac{1}{5}$
a_2	$\frac{-2r^2-r+1}{4r^3+16r^2+15r}$	$-\frac{2}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{4(1+r)^2}{8r^5 + 76r^4 + 262r^3 + 389r^2 + 210r}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{16}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(2r+3)}$	$\frac{1}{5}$
a_2	$\frac{-2r^2-r+1}{4r^3+16r^2+15r}$	$-\frac{2}{35}$
a_3	$-\frac{4(1+r)^2}{8r^5+76r^4+262r^3+389r^2+210r}$	$-\frac{16}{945}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4r^5 + 32r^4 + 85r^3 + 73r^2 - 16r - 32}{(2r^2 + 15r + 27)\, r\, (8r^4 + 76r^3 + 262r^2 + 389r + 210)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{73}{20790}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(2r+3)}$	$\frac{1}{5}$
a_2	$\frac{-2r^2-r+1}{4r^3+16r^2+15r}$	$-\frac{2}{35}$
a_3	$-\frac{4(1+r)^2}{8r^5+76r^4+262r^3+389r^2+210r}$	$-\frac{16}{945}$
a_4	$\frac{4r^5+32r^4+85r^3+73r^2-16r-32}{(2r^2+15r+27)r(8r^4+76r^3+262r^2+389r+210)}$	$\frac{73}{20790}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{12r^5 + 132r^4 + 549r^3 + 1057r^2 + 920r + 292}{(2r^2 + 15r + 27)\, r\, (8r^4 + 76r^3 + 262r^2 + 389r + 210)\, (2r^2 + 19r + 44)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1481}{1351350}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(2r+3)}$	$\frac{1}{5}$
a_2	$\frac{-2r^2-r+1}{4r^3+16r^2+15r}$	$-\frac{2}{35}$
a_3	$-\frac{4(1+r)^2}{8r^5+76r^4+262r^3+389r^2+210r}$	$-\frac{16}{945}$
a_4	$\frac{4r^5+32r^4+85r^3+73r^2-16r-32}{(2r^2+15r+27)r(8r^4+76r^3+262r^2+389r+210)}$	$\frac{73}{20790}$
a_5	$\frac{12r^5+132r^4+549r^3+1057r^2+920r+292}{(2r^2+15r+27)r(8r^4+76r^3+262r^2+389r+210)(2r^2+19r+44)}$	$\frac{1481}{1351350}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)\\ &= x\left(1 + \frac{x}{5} - \frac{2x^2}{35} - \frac{16x^3}{945} + \frac{73x^4}{20790} + \frac{1481x^5}{1351350} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{1}{r(2r+3)}$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + b_{n-2}(n+r-2) + b_n(n+r) - b_{n-1} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{nb_{n-2} + rb_{n-2} - 2b_{n-2} - b_{n-1}}{2n^2 + 4nr + 2r^2 - n - r - 1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = \frac{-2nb_{n-2} + 5b_{n-2} + 2b_{n-1}}{4n^2 - 6n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(2r+3)}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-2r^2 - r + 1}{4r^3 + 16r^2 + 15r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(2r+3)}$	-1
b_2	$\frac{-2r^2-r+1}{4r^3+16r^2+15r}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{4(1+r)^2}{8r^5 + 76r^4 + 262r^3 + 389r^2 + 210r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = \frac{1}{36}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(2r+3)}$	-1
b_2	$\frac{-2r^2-r+1}{4r^3+16r^2+15r}$	$-\frac{1}{4}$
b_3	$-\frac{4(1+r)^2}{8r^5+76r^4+262r^3+389r^2+210r}$	$\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4r^5 + 32r^4 + 85r^3 + 73r^2 - 16r - 32}{(2r^2 + 15r + 27)\, r\, (8r^4 + 76r^3 + 262r^2 + 389r + 210)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{29}{1440}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(2r+3)}$	-1
b_2	$\frac{-2r^2-r+1}{4r^3+16r^2+15r}$	$-\frac{1}{4}$
b_3	$-\frac{4(1+r)^2}{8r^5+76r^4+262r^3+389r^2+210r}$	$\frac{1}{36}$
b_4	$\frac{4r^5+32r^4+85r^3+73r^2-16r-32}{(2r^2+15r+27)r(8r^4+76r^3+262r^2+389r+210)}$	$\frac{29}{1440}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{12r^5 + 132r^4 + 549r^3 + 1057r^2 + 920r + 292}{(2r^2 + 15r + 27)\, r\, (8r^4 + 76r^3 + 262r^2 + 389r + 210)\, (2r^2 + 19r + 44)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = -\frac{71}{50400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{r(2r+3)}$	-1
b_2	$\frac{-2r^2-r+1}{4r^3+16r^2+15r}$	$-\frac{1}{4}$
b_3	$-\frac{4(1+r)^2}{8r^5+76r^4+262r^3+389r^2+210r}$	$\frac{1}{36}$
b_4	$\frac{4r^5+32r^4+85r^3+73r^2-16r-32}{(2r^2+15r+27)r(8r^4+76r^3+262r^2+389r+210)}$	$\frac{29}{1440}$
b_5	$\frac{12r^5+132r^4+549r^3+1057r^2+920r+292}{(2r^2+15r+27)r(8r^4+76r^3+262r^2+389r+210)(2r^2+19r+44)}$	$-\frac{71}{50400}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - x - \frac{x^2}{4} + \frac{x^3}{36} + \frac{29x^4}{1440} - \frac{71x^5}{50400} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1+\frac{x}{5}-\frac{2x^2}{35}-\frac{16x^3}{945}+\frac{73x^4}{20790}+\frac{1481x^5}{1351350}+O(x^6)\right) \\
&+ \frac{c_2 \left(1-x-\frac{x^2}{4}+\frac{x^3}{36}+\frac{29x^4}{1440}-\frac{71x^5}{50400}+O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1+\frac{x}{5}-\frac{2x^2}{35}-\frac{16x^3}{945}+\frac{73x^4}{20790}+\frac{1481x^5}{1351350}+O(x^6)\right) \\
&+ \frac{c_2 \left(1-x-\frac{x^2}{4}+\frac{x^3}{36}+\frac{29x^4}{1440}-\frac{71x^5}{50400}+O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1+\frac{x}{5}-\frac{2x^2}{35}-\frac{16x^3}{945}+\frac{73x^4}{20790}+\frac{1481x^5}{1351350}+O(x^6)\right) \\
&+ \frac{c_2 \left(1-x-\frac{x^2}{4}+\frac{x^3}{36}+\frac{29x^4}{1440}-\frac{71x^5}{50400}+O(x^6)\right)}{\sqrt{x}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1+\frac{x}{5}-\frac{2x^2}{35}-\frac{16x^3}{945}+\frac{73x^4}{20790}+\frac{1481x^5}{1351350}+O(x^6)\right) \\
&+ \frac{c_2 \left(1-x-\frac{x^2}{4}+\frac{x^3}{36}+\frac{29x^4}{1440}-\frac{71x^5}{50400}+O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Verified OK.

23.18.1 Maple step by step solution

Let's solve

$2x^2y'' + (x^3 + x)\, y' + (-x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x+1)y}{2x^2} - \frac{(x^2+1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+1)y'}{2x} - \frac{(x+1)y}{2x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^2+1}{2x}, P_3(x) = -\frac{x+1}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{2}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + x(x^2 + 1)\, y' + (-x - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m=1..3$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+r)\, x^r+(a_1(3+2r)\, r-a_0)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(k+r-1)-a_{k-1}+a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, -\frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\, r-a_0=0$

- Solve for the dependent coefficient(s)

  $a_1=\frac{a_0}{r(3+2r)}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{1}{2}\right)\,(k+r-1)\, a_k+a_{k-2}k+a_{k-2}r-2a_{k-2}-a_{k-1}=0$

- Shift index using $k->k+2$

  $2\left(k+\frac{5}{2}+r\right)\,(k+1+r)\, a_{k+2}+a_k(k+2)+r a_k-2a_k-a_{k+1}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{k a_k+r a_k-a_{k+1}}{(2k+5+2r)(k+1+r)}$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{k a_k+a_k-a_{k+1}}{(2k+7)(k+2)}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k a_k + a_k - a_{k+1}}{(2k+7)(k+2)}, a_1 = \frac{a_0}{5}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+2} = -\frac{k a_k - \frac{1}{2} a_k - a_{k+1}}{(2k+4)\left(k+\frac{1}{2}\right)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{k a_k - \frac{1}{2} a_k - a_{k+1}}{(2k+4)\left(k+\frac{1}{2}\right)}, a_1 = -a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+2} = -\frac{k a_k + a_k - a_{k+1}}{(2k+7)(k+2)}, a_1 = \frac{a_0}{5}, b_{k+2} = -\frac{k b_k - \frac{1}{2} b_k - b_{k+1}}{(2k+4)\left(k+\frac{1}{2}\right)}, b_1 = -$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+x*(1+x^2)*diff(y(x),x)-(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1 - x - \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{29}{1440}x^4 - \frac{71}{50400}x^5 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}} + c_2 x\left(1 + \frac{1}{5}x - \frac{2}{35}x^2 - \frac{16}{945}x^3 + \frac{73}{20790}x^4 + \frac{1481}{1351350}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 84

```
AsymptoticDSolveValue[2*x^2*y''[x]+x*(1+x^2)*y'[x]-(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(\frac{1481x^5}{1351350} + \frac{73x^4}{20790} - \frac{16x^3}{945} - \frac{2x^2}{35} + \frac{x}{5} + 1 \right) + \frac{c_2 \left(-\frac{71x^5}{50400} + \frac{29x^4}{1440} + \frac{x^3}{36} - \frac{x^2}{4} - x + 1 \right)}{\sqrt{x}}$$

23.19 problem 23

Internal problem ID [2398]
Internal file name [OUTPUT/2398_Tuesday_February_27_2024_08_36_54_AM_43939945/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - xy' + (x^2 + 1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + (x^2 + 1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = \frac{x^2 + 1}{2x^2}$$

Table 672: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2+1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(x^2 + 1\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2} \right) - x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \right) + \left(x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0
$$

Or

$$
\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(2r^2 - 3r + 1\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
2r^2 - 3r + 1 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) + b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -\frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
$$

Verified OK.

23.19.1 Maple step by step solution

Let's solve

$2x^2y'' - xy' + (x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+1)y}{2x^2} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} + \frac{(x^2+1)y}{2x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{x^2+1}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{2}$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2y'' - xy' + (x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  ◦ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+2r)(-1+r)\,x^r + a_1(1+2r)\,r\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)(k+r-1) + a_{k-2}\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)\,r = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r-1)\left(k+r-\frac{1}{2}\right)a_k + a_{k-2} = 0$

- Shift index using $k->k+2$

  $2(k+1+r)\left(k+\frac{3}{2}+r\right)a_{k+2} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)(2k+3+2r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)(2k+5)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(2k+5)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+4)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+4)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k}{(k+2)(2k+5)}, a_1 = 0, b_{k+2} = -\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+4)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 33

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+(1+x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2x\left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 48

```
AsymptoticDSolveValue[2*x^2*y''[x]-x*y'[x]+(1+x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1x\left(\frac{x^4}{360} - \frac{x^2}{10} + 1\right) + c_2\sqrt{x}\left(\frac{x^4}{168} - \frac{x^2}{6} + 1\right)$$

23.20 problem 24

Internal problem ID [2399]
Internal file name [OUTPUT/2399_Tuesday_February_27_2024_08_36_54_AM_71638245/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2y'' + 2xy' + \left(x^2 - 2\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3x^2y'' + 2xy' + \left(x^2 - 2\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{3x}$$
$$q(x) = \frac{x^2 - 2}{3x^2}$$

Table 674: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-2}{3x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2y'' + 2xy' + \left(x^2 - 2\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 2x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^2 - 2\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n(n+r)(n+r-1) + 2x^{n+r} a_n(n+r) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r) + 2x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(3x^r r(-1+r) + 2x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 - r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -\frac{2}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 - r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{2}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$3a_n(n + r)(n + r - 1) + 2a_n(n + r) + a_{n-2} - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{3n^2 + 6nr + 3r^2 - n - r - 2} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n(3n + 5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{3r^2 + 11r + 8}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{22}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{3r^2+11r+8}$	$-\frac{1}{22}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{3r^2+11r+8}$	$-\frac{1}{22}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{9r^4 + 102r^3 + 403r^2 + 646r + 336}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{1496}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{3r^2+11r+8}$	$-\frac{1}{22}$
a_3	0	0
a_4	$\frac{1}{9r^4+102r^3+403r^2+646r+336}$	$\frac{1}{1496}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{3r^2+11r+8}$	$-\frac{1}{22}$
a_3	0	0
a_4	$\frac{1}{9r^4+102r^3+403r^2+646r+336}$	$\frac{1}{1496}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x^2}{22} + \frac{x^4}{1496} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$3b_n(n+r)(n+r-1) + 2b_n(n+r) + b_{n-2} - 2b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{3n^2 + 6nr + 3r^2 - n - r - 2} \tag{4}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_n = -\frac{b_{n-2}}{n(3n-5)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{3r^2 + 11r + 8}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{3r^2+11r+8}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{3r^2+11r+8}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{9r^4 + 102r^3 + 403r^2 + 646r + 336}$$

Which for the root $r = -\frac{2}{3}$ becomes

$$b_4 = \frac{1}{56}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{3r^2+11r+8}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{9r^4+102r^3+403r^2+646r+336}$	$\frac{1}{56}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{3r^2+11r+8}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{9r^4+102r^3+403r^2+646r+336}$	$\frac{1}{56}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x^2}{2} + \frac{x^4}{56} + O(x^6)}{x^{\frac{2}{3}}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{x^2}{22} + \frac{x^4}{1496} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{56} + O(x^6)\right)}{x^{\frac{2}{3}}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 - \frac{x^2}{22} + \frac{x^4}{1496} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{56} + O(x^6)\right)}{x^{\frac{2}{3}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\left(1 - \frac{x^2}{22} + \frac{x^4}{1496} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{56} + O(x^6)\right)}{x^{\frac{2}{3}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\left(1 - \frac{x^2}{22} + \frac{x^4}{1496} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{56} + O(x^6)\right)}{x^{\frac{2}{3}}}
$$

Verified OK.

23.20.1 Maple step by step solution

Let's solve

$3x^2y'' + 2xy' + (x^2 - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(x^2-2)y}{3x^2} - \frac{2y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{3x} + \frac{(x^2-2)y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{2}{3x}, P_3(x) = \frac{x^2-2}{3x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{2}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{2}{3}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' + 2xy' + (x^2 - 2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+3r)(-1+r)\, x^r + a_1(5+3r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+2)(k+r-1) + a_{k-2}\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+3r)(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, -\frac{2}{3}\right\}$$

- Each term must be 0

$$a_1(5+3r)\, r = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(k+r+\frac{2}{3}\right)(k+r-1)\, a_k + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$3\left(k+\frac{8}{3}+r\right)(k+1+r)\, a_{k+2} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(3k+8+3r)(k+1+r)}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{a_k}{(3k+11)(k+2)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(3k+11)(k+2)}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{2}{3}$

$$a_{k+2} = -\frac{a_k}{(3k+6)\left(k+\frac{1}{3}\right)}$$

- Solution for $r = -\frac{2}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{2}{3}}, a_{k+2} = -\frac{a_k}{(3k+6)\left(k+\frac{1}{3}\right)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k-\frac{2}{3}}\right), a_{k+2}=-\frac{a_k}{(3k+11)(k+2)}, a_1=0, b_{k+2}=-\frac{b_k}{(3k+6)\left(k+\frac{1}{3}\right)}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
Order:=6;
dsolve(3*x^2*diff(y(x),x$2)+2*x*diff(y(x),x)+(x^2-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1\left(1-\frac{1}{2}x^2+\frac{1}{56}x^4+\mathrm{O}\left(x^6\right)\right)}{x^{\frac{2}{3}}}+c_2x\left(1-\frac{1}{22}x^2+\frac{1}{1496}x^4+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 48

```
AsymptoticDSolveValue[3*x^2*y''[x]+2*x*y'[x]+(x^2-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1x\left(\frac{x^4}{1496}-\frac{x^2}{22}+1\right)+\frac{c_2\left(\frac{x^4}{56}-\frac{x^2}{2}+1\right)}{x^{2/3}}$$

23.21 problem 25

Internal problem ID [2400]
Internal file name [`OUTPUT/2400_Tuesday_February_27_2024_08_36_55_AM_96829618/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^3(x^2+3)\,y''+5xy'-y(x+1)=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^5+3x^3)\,y''+5xy'+(-x-1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{5}{x^2\,(x^2+3)}\\ q(x)&=-\frac{x+1}{x^3\,(x^2+3)}\end{aligned}$$

Table 676: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{x^2(x^2+3)}$	
singularity	type
$x = 0$	"irregular"
$x = -i\sqrt{3}$	"regular"
$x = i\sqrt{3}$	"regular"

$q(x) = -\frac{x+1}{x^3(x^2+3)}$	
singularity	type
$x = 0$	"irregular"
$x = -i\sqrt{3}$	"regular"
$x = i\sqrt{3}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-i\sqrt{3}, i\sqrt{3}, \infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c <
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^3*(3+x^2)*diff(y(x),x$2)+5*x*diff(y(x),x)-(1+x)*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 99

```
AsymptoticDSolveValue[x^3*(3+x^2)*y''[x]+5*x*y'[x]-(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{18968303719x^5}{1220703125000} - \frac{20383193x^4}{1953125000} + \frac{26731x^3}{3906250} + \frac{259x^2}{31250} + \frac{37x}{125} + 1\right) \sqrt[5]{x} + c_2 e^{\frac{5}{3}/x} \left(\frac{869909160612721304x^5}{27030487060546875} + \frac{46847788879262x^4}{4805419921875} + \frac{15542572604x^3}{4271484375} + \frac{2270672x^2}{1265625} + \frac{1372x}{1125} + 1\right) x^{9/5}$$

23.22 problem 26

Internal problem ID [2401]
Internal file name [OUTPUT/2401_Tuesday_February_27_2024_08_36_56_AM_15179823/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 41, page 195
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' - \left(x^3+1\right)y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + \left(-x^3-1\right)y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x^3+1}{2x}$$
$$q(x) = \frac{1}{2x}$$

Table 677: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^3+1}{2x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (-x^3 - 1)\, y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + (-x^3 - 1)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n (n+r)\right) = \sum_{n=3}^{\infty} \left(-a_{n-3}(n+r-3)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=3}^{\infty} \left(-a_{n-3}(n+r-3)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) - (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) - r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) - r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-3+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{2r^2 + r - 1}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{1}{4r^4 + 12r^3 + 7r^2 - 3r - 2}$$

For $3 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_{n-3}(n+r-3) - a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n a_{n-3} + r a_{n-3} - 3a_{n-3} - a_{n-1}}{2n^2 + 4nr + 2r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = \frac{2n a_{n-3} - 3a_{n-3} - 2a_{n-1}}{4n^2 + 6n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+r-1}$	$-\frac{1}{5}$
a_2	$\frac{1}{4r^4+12r^3+7r^2-3r-2}$	$\frac{1}{70}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4r^5 + 12r^4 + 7r^3 - 3r^2 - 2r - 1}{8r^6 + 60r^5 + 158r^4 + 165r^3 + 32r^2 - 45r - 18}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = \frac{52}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+r-1}$	$-\frac{1}{5}$
a_2	$\frac{1}{4r^4+12r^3+7r^2-3r-2}$	$\frac{1}{70}$
a_3	$\frac{4r^5+12r^4+7r^3-3r^2-2r-1}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{52}{945}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-8r^5 - 44r^4 - 102r^3 - 127r^2 - 79r - 17}{16r^8 + 224r^7 + 1256r^6 + 3584r^5 + 5369r^4 + 3626r^3 + 19r^2 - 1134r - 360}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = -\frac{1049}{83160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+r-1}$	$-\frac{1}{5}$
a_2	$\frac{1}{4r^4+12r^3+7r^2-3r-2}$	$\frac{1}{70}$
a_3	$\frac{4r^5+12r^4+7r^3-3r^2-2r-1}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{52}{945}$
a_4	$\frac{-8r^5-44r^4-102r^3-127r^2-79r-17}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$-\frac{1049}{83160}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{12r^5 + 96r^4 + 365r^3 + 774r^2 + 853r + 377}{32r^{10} + 720r^9 + 6880r^8 + 36360r^7 + 115626r^6 + 223965r^5 + 249595r^4 + 124965r^3 - 19333r^2 - 458}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = \frac{5207}{5405400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+r-1}$	$-\frac{1}{5}$
a_2	$\frac{1}{4r^4+12r^3+7r^2-3r-2}$	$\frac{1}{70}$
a_3	$\frac{4r^5+12r^4+7r^3-3r^2-2r-1}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{52}{945}$
a_4	$\frac{-8r^5-44r^4-102r^3-127r^2-79r-17}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$-\frac{1049}{83160}$
a_5	$\frac{12r^5+96r^4+365r^3+774r^2+853r+377}{32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600}$	$\frac{5207}{5405400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - \frac{x}{5} + \frac{x^2}{70} + \frac{52x^3}{945} - \frac{1049x^4}{83160} + \frac{5207x^5}{5405400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = -\frac{1}{2r^2 + r - 1}$$

Substituting $n = 2$ in Eq. (2B) gives

$$b_2 = \frac{1}{4r^4 + 12r^3 + 7r^2 - 3r - 2}$$

For $3 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_{n-3}(n+r-3) - (n+r)b_n + b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{nb_{n-3} + rb_{n-3} - 3b_{n-3} - b_{n-1}}{2n^2 + 4nr + 2r^2 - 3n - 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{(n-3)b_{n-3} - b_{n-1}}{n(2n-3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+r-1}$	1
b_2	$\frac{1}{4r^4+12r^3+7r^2-3r-2}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{4r^5 + 12r^4 + 7r^3 - 3r^2 - 2r - 1}{8r^6 + 60r^5 + 158r^4 + 165r^3 + 32r^2 - 45r - 18}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{18}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+r-1}$	1
b_2	$\frac{1}{4r^4+12r^3+7r^2-3r-2}$	$-\frac{1}{2}$
b_3	$\frac{4r^5+12r^4+7r^3-3r^2-2r-1}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{1}{18}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{-8r^5 - 44r^4 - 102r^3 - 127r^2 - 79r - 17}{16r^8 + 224r^7 + 1256r^6 + 3584r^5 + 5369r^4 + 3626r^3 + 19r^2 - 1134r - 360}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{17}{360}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+r-1}$	1
b_2	$\frac{1}{4r^4+12r^3+7r^2-3r-2}$	$-\frac{1}{2}$
b_3	$\frac{4r^5+12r^4+7r^3-3r^2-2r-1}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{1}{18}$
b_4	$\frac{-8r^5-44r^4-102r^3-127r^2-79r-17}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$\frac{17}{360}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{12r^5 + 96r^4 + 365r^3 + 774r^2 + 853r + 377}{32r^{10} + 720r^9 + 6880r^8 + 36360r^7 + 115626r^6 + 223965r^5 + 249595r^4 + 124965r^3 - 19333r^2 - 458}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{377}{12600}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+r-1}$	1
b_2	$\frac{1}{4r^4+12r^3+7r^2-3r-2}$	$-\frac{1}{2}$
b_3	$\frac{4r^5+12r^4+7r^3-3r^2-2r-1}{8r^6+60r^5+158r^4+165r^3+32r^2-45r-18}$	$\frac{1}{18}$
b_4	$\frac{-8r^5-44r^4-102r^3-127r^2-79r-17}{16r^8+224r^7+1256r^6+3584r^5+5369r^4+3626r^3+19r^2-1134r-360}$	$\frac{17}{360}$
b_5	$\frac{12r^5+96r^4+365r^3+774r^2+853r+377}{32r^{10}+720r^9+6880r^8+36360r^7+115626r^6+223965r^5+249595r^4+124965r^3-19333r^2-45810r-12600}$	$-\frac{377}{12600}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 + x - \frac{x^2}{2} + \frac{x^3}{18} + \frac{17x^4}{360} - \frac{377x^5}{12600} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}} \left(1 - \frac{x}{5} + \frac{x^2}{70} + \frac{52x^3}{945} - \frac{1049x^4}{83160} + \frac{5207x^5}{5405400} + O(x^6)\right) \\
&+ c_2 \left(1 + x - \frac{x^2}{2} + \frac{x^3}{18} + \frac{17x^4}{360} - \frac{377x^5}{12600} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}} \left(1 - \frac{x}{5} + \frac{x^2}{70} + \frac{52x^3}{945} - \frac{1049x^4}{83160} + \frac{5207x^5}{5405400} + O(x^6)\right) \\
&+ c_2 \left(1 + x - \frac{x^2}{2} + \frac{x^3}{18} + \frac{17x^4}{360} - \frac{377x^5}{12600} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^{\frac{3}{2}} \left(1 - \frac{x}{5} + \frac{x^2}{70} + \frac{52x^3}{945} - \frac{1049x^4}{83160} + \frac{5207x^5}{5405400} + O(x^6)\right) \\
+ c_2 \left(1 + x - \frac{x^2}{2} + \frac{x^3}{18} + \frac{17x^4}{360} - \frac{377x^5}{12600} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1 - \frac{x}{5} + \frac{x^2}{70} + \frac{52x^3}{945} - \frac{1049x^4}{83160} + \frac{5207x^5}{5405400} + O(x^6)\right) + c_2 \left(1 + x - \frac{x^2}{2} + \frac{x^3}{18} + \frac{17x^4}{360} - \frac{377x^5}{12600} + O(x^6)\right)$$

Verified OK.

23.22.1 Maple step by step solution

Let's solve

$2y''x + (-x^3 - 1) y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x} + \frac{(x^3+1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^3+1)y'}{2x} + \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x^3+1}{2x}, P_3(x) = \frac{1}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + (-x^3 - 1) y' + y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+2r)\, x^{-1+r} + (a_1(1+r)\,(-1+2r) + a_0)\, x^r + (a_2(2+r)\,(1+2r) + a_1)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_{k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{2}\right\}$$

- The coefficients of each power of x must be 0

$$[a_1(1+r)\,(-1+2r) + a_0 = 0, a_2(2+r)\,(1+2r) + a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = -\frac{a_0}{2r^2+r-1}, a_2 = \frac{a_0}{4r^4+12r^3+7r^2-3r-2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(2k-1+2r) + a_k - a_{k-2}(k-2+r) = 0$$

- Shift index using $k- >k+2$

$$a_{k+3}(k+3+r)\,(2k+3+2r) + a_{k+2} - a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = \frac{k a_k + r a_k - a_{k+2}}{(k+3+r)(2k+3+2r)}$$

- Recursion relation for $r = 0$

  $a_{k+3} = \frac{k a_k - a_{k+2}}{(k+3)(2k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_k - a_{k+2}}{(k+3)(2k+3)}, a_1 = a_0, a_2 = -\frac{a_0}{2}\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+3} = \frac{k a_k + \frac{3}{2} a_k - a_{k+2}}{\left(k+\frac{9}{2}\right)(2k+6)}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+3} = \frac{k a_k + \frac{3}{2} a_k - a_{k+2}}{\left(k+\frac{9}{2}\right)(2k+6)}, a_1 = -\frac{a_0}{5}, a_2 = \frac{a_0}{70}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+3} = \frac{k a_k - a_{k+2}}{(k+3)(2k+3)}, a_1 = a_0, a_2 = -\frac{a_0}{2}, b_{k+3} = \frac{k b_k + \frac{3}{2} b_k - b_{k+2}}{\left(k+\frac{9}{2}\right)(2k+6)}, b_1\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)-(1+x^3)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{3}{2}} \left(1 - \frac{1}{5}x + \frac{1}{70}x^2 + \frac{52}{945}x^3 - \frac{1049}{83160}x^4 + \frac{5207}{5405400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(1 + x - \frac{1}{2}x^2 + \frac{1}{18}x^3 + \frac{17}{360}x^4 - \frac{377}{12600}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 81

```
AsymptoticDSolveValue[2*x*y''[x]-(1+x^3)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(-\frac{377x^5}{12600} + \frac{17x^4}{360} + \frac{x^3}{18} - \frac{x^2}{2} + x + 1\right) + c_1 \left(\frac{5207x^5}{5405400} - \frac{1049x^4}{83160} + \frac{52x^3}{945} + \frac{x^2}{70} - \frac{x}{5} + 1\right) x^{3/2}$$

24 Exercise 42, page 206

24.1 problem 1

Internal problem ID [2402]
Internal file name [OUTPUT/2402_Tuesday_February_27_2024_08_36_56_AM_98626/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{2}{x}$$

Table 679: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{2a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{(r+1)^2 (2+r)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(2+r)^2}$	1

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{(r+1)^2 (2+r)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{2}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(2+r)^2}$	1
a_3	$-\frac{8}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{2}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(r+1)^2 (2+r)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(2+r)^2}$	1
a_3	$-\frac{8}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{2}{9}$
a_4	$\frac{16}{(r+1)^2(2+r)^2(r+3)^2(r+4)^2}$	$\frac{1}{36}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{(r+1)^2 (2+r)^2 (r+3)^2 (r+4)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{450}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(2+r)^2}$	1
a_3	$-\frac{8}{(r+1)^2(2+r)^2(r+3)^2}$	$-\frac{2}{9}$
a_4	$\frac{16}{(r+1)^2(2+r)^2(r+3)^2(r+4)^2}$	$\frac{1}{36}$
a_5	$-\frac{32}{(r+1)^2(2+r)^2(r+3)^2(r+4)^2(r+5)^2}$	$-\frac{1}{450}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{2}{(r+1)^2}$	-2	$\frac{4}{(r+1)^3}$	4
b_2	$\frac{4}{(r+1)^2 (2+r)^2}$	1	$\frac{-24-16r}{(r+1)^3 (2+r)^3}$	-3
b_3	$-\frac{8}{(r+1)^2 (2+r)^2 (r+3)^2}$	$-\frac{2}{9}$	$\frac{48r^2+192r+176}{(r+1)^3 (2+r)^3 (r+3)^3}$	$\frac{22}{27}$
b_4	$\frac{16}{(r+1)^2 (2+r)^2 (r+3)^2 (r+4)^2}$	$\frac{1}{36}$	$-\frac{64(2r^3+15r^2+35r+25)}{(r+1)^3 (2+r)^3 (r+3)^3 (r+4)^3}$	$-\frac{25}{216}$
b_5	$-\frac{32}{(r+1)^2 (2+r)^2 (r+3)^2 (r+4)^2 (r+5)^2}$	$-\frac{1}{450}$	$\frac{320r^4+3840r^3+16320r^2+28800r+17536}{(r+1)^3 (2+r)^3 (r+3)^3 (r+4)^3 (r+5)^3}$	$\frac{137}{13500}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\ &= \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 + 4x + \frac{22x^3}{27} - \frac{25x^4}{216} + \frac{137x^5}{13500} \\ &\quad + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \\
&\quad + c_2 \left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 + 4x + \frac{22x^3}{27} \right. \\
&\quad \left. - \frac{25x^4}{216} + \frac{137x^5}{13500} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \\
&\quad + c_2 \left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 + 4x + \frac{22x^3}{27} - \frac{25x^4}{216} \right. \\
&\quad \left. + \frac{137x^5}{13500} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \\
&\quad + c_2 \left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 + 4x + \frac{22x^3}{27} \right. \\
&\quad \left. - \frac{25x^4}{216} + \frac{137x^5}{13500} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \\
&\quad + c_2 \left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 + 4x + \frac{22x^3}{27} - \frac{25x^4}{216} \right. \\
&\quad \left. + \frac{137x^5}{13500} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

24.1.1 Maple step by step solution

Let's solve

$y''x + y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{2y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1}$

  ◦ Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- ○ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- ○ Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a_k}{(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{2a_k}{(k+1)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln (x) + c_1) \left(1 - 2x + x^2 - \frac{2}{9}x^3 + \frac{1}{36}x^4 - \frac{1}{450}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(4x - 3x^2 + \frac{22}{27}x^3 - \frac{25}{216}x^4 + \frac{137}{13500}x^5 + \mathrm{O}\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 101

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{x^5}{450} + \frac{x^4}{36} - \frac{2x^3}{9} + x^2 - 2x + 1\right) + c_2 \left(\frac{137x^5}{13500} - \frac{25x^4}{216} + \frac{22x^3}{27} - 3x^2 + \left(-\frac{x^5}{450} + \frac{x^4}{36} - \frac{2x^3}{9} + x^2 - 2x + 1\right) \log(x) + 4x\right)$$

24.2 problem 2

Internal problem ID [2403]
Internal file name [`OUTPUT/2403_Tuesday_February_27_2024_08_36_57_AM_34163828/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + y' + 2yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + 2yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = 2$$

Table 681: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 2$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + 2yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r}a_n = \sum_{n=2}^{\infty} 2a_{n-2}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2}x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)(n+r-1) + (n+r)a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r}r(-1+r) + r\,x^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r}r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r}r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + 2a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{2a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{2}{(2+r)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{(2+r)^2}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{(2+r)^2}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4}{(2+r)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{(2+r)^2}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{4}{(2+r)^2(r+4)^2}$	$\frac{1}{16}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{(2+r)^2}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{4}{(2+r)^2(r+4)^2}$	$\frac{1}{16}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned}
y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\
&= 1 - \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{2}{(2+r)^2}$	$-\frac{1}{2}$	$\frac{4}{(2+r)^3}$	$\frac{1}{2}$
b_3	0	0	0	0
b_4	$\frac{4}{(2+r)^2(r+4)^2}$	$\frac{1}{16}$	$\frac{-16r-48}{(2+r)^3(r+4)^3}$	$-\frac{3}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\ &= \left(1 - \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)\right)\ln(x) + \frac{x^2}{2} - \frac{3x^4}{32} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\left(1-\frac{x^2}{2}+\frac{x^4}{16}+O(x^6)\right)+c_2\left(\left(1-\frac{x^2}{2}+\frac{x^4}{16}+O(x^6)\right)\ln(x)+\frac{x^2}{2}-\frac{3x^4}{32}+O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\left(1-\frac{x^2}{2}+\frac{x^4}{16}+O(x^6)\right)+c_2\left(\left(1-\frac{x^2}{2}+\frac{x^4}{16}+O(x^6)\right)\ln(x)+\frac{x^2}{2}-\frac{3x^4}{32}+O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1&\left(1 - \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)\right) \\ &+ c_2\left(\left(1 - \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)\right)\ln(x) + \frac{x^2}{2} - \frac{3x^4}{32} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1\left(1-\frac{x^2}{2}+\frac{x^4}{16}+O(x^6)\right)+c_2\left(\left(1-\frac{x^2}{2}+\frac{x^4}{16}+O(x^6)\right)\ln(x)+\frac{x^2}{2}-\frac{3x^4}{32}+O(x^6)\right)$$

Verified OK.

24.2.1 Maple step by step solution

Let's solve

$y''x + y' + 2yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - 2y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + 2y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = 2\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y' + 2yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$
  - Shift index using $k->k-1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$
  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$
  - Shift index using $k->k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$
  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$
  - Shift index using $k->k+1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + 2a_{k-1}\right) x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$
- Values of r that satisfy the indicial equation

  $$r = 0$$
- Each term must be 0

  $$a_1(1+r)^2 = 0$$
- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^2 + 2a_{k-1} = 0$$
- Shift index using $k->k+1$

  $$a_{k+2}(k+2)^2 + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2a_k}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+2*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - \frac{1}{2}x^2 + \frac{1}{16}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{2}x^2 - \frac{3}{32}x^4 + \mathrm{O}\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+2*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{16} - \frac{x^2}{2} + 1\right) + c_2\left(-\frac{3x^4}{32} + \frac{x^2}{2} + \left(\frac{x^4}{16} - \frac{x^2}{2} + 1\right)\log(x)\right)$$

24.3 problem 3

Internal problem ID [2404]
Internal file name [`OUTPUT/2404_Tuesday_February_27_2024_08_36_57_AM_50708696/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3xy' + 4y(x+1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - 3xy' + (4x + 4)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4x + 4}{x^2}$$

Table 683: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x+4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - 3xy' + (4x + 4)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ & - 3x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (4x+4) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 3a_n(n+r) + 4a_{n-1} + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}}{n^2 + 2nr + r^2 - 4n - 4r + 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{4a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_1 = -4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{(-1+r)^2 r^2}$$

Which for the root $r = 2$ becomes

$$a_2 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{(-1+r)^2 r^2 (r+1)^2}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{16}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(-1+r)^2 r^2 (r+1)^2 (2+r)^2}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{4}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$
a_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (2+r)^2}$	$\frac{4}{9}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (2+r)^2 (r+3)^2}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{16}{225}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{(-1+r)^2}$	-4
a_2	$\frac{16}{(-1+r)^2 r^2}$	4
a_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$
a_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (2+r)^2}$	$\frac{4}{9}$
a_5	$-\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (2+r)^2 (r+3)^2}$	$-\frac{16}{225}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2 \left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = 2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{4}{(-1+r)^2}$	-4	$\frac{8}{(-1+r)^3}$	8
b_2	$\frac{16}{(-1+r)^2 r^2}$	4	$\frac{-64r+32}{(-1+r)^3 r^3}$	-12
b_3	$-\frac{64}{(-1+r)^2 r^2 (r+1)^2}$	$-\frac{16}{9}$	$\frac{384r^2-128}{(-1+r)^3 r^3 (r+1)^3}$	$\frac{176}{27}$
b_4	$\frac{256}{(-1+r)^2 r^2 (r+1)^2 (2+r)^2}$	$\frac{4}{9}$	$\frac{-2048r^3-3072r^2+1024r+1024}{(-1+r)^3 r^3 (r+1)^3 (2+r)^3}$	$-\frac{50}{27}$
b_5	$-\frac{1024}{(-1+r)^2 r^2 (r+1)^2 (2+r)^2 (r+3)^2}$	$-\frac{16}{225}$	$\frac{10240r^4+40960r^3+30720r^2-20480r-12288}{(-1+r)^3 r^3 (r+1)^3 (2+r)^3 (r+3)^3}$	$\frac{1096}{3375}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\
&= x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right)\ln(x) \\
&\quad + x^2\left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right)\ln(x)\right. \\
&\quad \left. + x^2\left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(4x^2 - 4x + 1 - \frac{16x^3}{9} + \frac{4x^4}{9} - \frac{16x^5}{225} + O(x^6)\right)\ln(x)\right. \\
&\quad \left. + x^2\left(-12x^2 + 8x + \frac{176x^3}{27} - \frac{50x^4}{27} + \frac{1096x^5}{3375} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^2 \left(4x^2-4x+1-\frac{16x^3}{9}+\frac{4x^4}{9}-\frac{16x^5}{225}+O\left(x^6\right)\right) \\
& + c_2 \left(x^2 \left(4x^2-4x+1-\frac{16x^3}{9}+\frac{4x^4}{9}-\frac{16x^5}{225}+O\left(x^6\right)\right) \ln\left(x\right)\right. \\
& \left. + x^2 \left(-12x^2+8x+\frac{176x^3}{27}-\frac{50x^4}{27}+\frac{1096x^5}{3375}+O\left(x^6\right)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x^2 \left(4x^2-4x+1-\frac{16x^3}{9}+\frac{4x^4}{9}-\frac{16x^5}{225}+O\left(x^6\right)\right) \\
& + c_2 \left(x^2 \left(4x^2-4x+1-\frac{16x^3}{9}+\frac{4x^4}{9}-\frac{16x^5}{225}+O\left(x^6\right)\right) \ln\left(x\right)\right. \\
& \left. + x^2 \left(-12x^2+8x+\frac{176x^3}{27}-\frac{50x^4}{27}+\frac{1096x^5}{3375}+O\left(x^6\right)\right)\right)
\end{aligned}
$$

Verified OK.

24.3.1 Maple step by step solution

Let's solve

$x^2y''-3xy'+(4x+4)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{3y'}{x}-\frac{4(x+1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{3y'}{x}+\frac{4(x+1)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{3}{x}, P_3(x)=\frac{4(x+1)}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 3xy' + (4x + 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (-2 + r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k + r - 2)^2 + 4a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2 + r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-2)^2+4a_{k-1}=0$$

- Shift index using $k->k+1$

$$a_{k+1}(k+r-1)^2+4a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{4a_k}{(k+r-1)^2}$$

- Recursion relation for $r=2$

$$a_{k+1}=-\frac{4a_k}{(k+1)^2}$$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{4a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 - 4x + 4x^2 - \frac{16}{9}x^3 + \frac{4}{9}x^4 - \frac{16}{225}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(8x - 12x^2 + \frac{176}{27}x^3 - \frac{50}{27}x^4 + \frac{1096}{3375}x^5 + \mathrm{O}\left(x^6\right)\right)c_2\right)x^2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 116

```
AsymptoticDSolveValue[x^2*y''[x]-3*x*y'[x]+4*(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{16x^5}{225} + \frac{4x^4}{9} - \frac{16x^3}{9} + 4x^2 - 4x + 1\right)x^2 + c_2\left(\left(\frac{1096x^5}{3375} - \frac{50x^4}{27} + \frac{176x^3}{27} - 12x^2 + 8x\right)x^2 + \left(-\frac{16x^5}{225} + \frac{4x^4}{9} - \frac{16x^3}{9} + 4x^2 - 4x + 1\right)x^2\log(x)\right)$$

24.4 problem 4

Internal problem ID [2405]
Internal file name [`OUTPUT/2405_Tuesday_February_27_2024_08_36_58_AM_40163664/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(x+1)\,y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-x^2 - x\right)y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x+1}{x}$$
$$q(x) = \frac{1}{x^2}$$

Table 685: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + \left(-x^2 - x\right)y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(-x^2 - x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{1+n}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - a_n(n+r) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{r}$$

Which for the root $r = 1$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r(1+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(2+r) r (1+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(2+r)r(1+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(3+r)(2+r) r (1+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(2+r)r(1+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(3+r)(2+r)r(1+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(4+r)(3+r)(2+r) r (1+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r}$	1
a_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(2+r)r(1+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(3+r)(2+r)r(1+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(4+r)(3+r)(2+r)r(1+r)}$	$\frac{1}{120}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(x + 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{r}$	1	$-\frac{1}{r^2}$	-1
b_2	$\frac{1}{r(1+r)}$	$\frac{1}{2}$	$\frac{-1-2r}{r^2(1+r)^2}$	$-\frac{3}{4}$
b_3	$\frac{1}{(2+r)r(1+r)}$	$\frac{1}{6}$	$\frac{-3r^2-6r-2}{(2+r)^2 r^2 (1+r)^2}$	$-\frac{11}{36}$
b_4	$\frac{1}{(3+r)(2+r)r(1+r)}$	$\frac{1}{24}$	$\frac{-4r^3-18r^2-22r-6}{(3+r)^2(2+r)^2 r^2 (1+r)^2}$	$-\frac{25}{288}$
b_5	$\frac{1}{(4+r)(3+r)(2+r)r(1+r)}$	$\frac{1}{120}$	$\frac{-5r^4-40r^3-105r^2-100r-24}{(4+r)^2(3+r)^2(2+r)^2 r^2 (1+r)^2}$	$-\frac{137}{7200}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= x\left(x + 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x) \\
&\quad + x\left(-x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x\left(x + 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(x\left(x + 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x\left(-x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1x\left(x + 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(x\left(x + 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x\left(-x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} + O(x^6)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1x\left(x + 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) \\
&\quad + c_2\left(x\left(x + 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x\left(-x - \frac{3x^2}{4} - \frac{11x^3}{36} - \frac{25x^4}{288} - \frac{137x^5}{7200} + O(x^6)\right)\right)
\end{aligned} \tag{1}$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x \left(x+1+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right) \\
+ c_2 \left(x \left(x+1+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right) \ln(x) \right. \\
\left. + x \left(-x-\frac{3x^2}{4}-\frac{11x^3}{36}-\frac{25x^4}{288}-\frac{137x^5}{7200}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

24.4.1 Maple step by step solution

Let's solve

$x^2y''+(-x^2-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x^2}+\frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x+1)y'}{x}+\frac{y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{x+1}{x}, P_3(x)=\frac{1}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=1$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

$$x^2 y'' - x(x+1)\, y' + y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_k(k+r-1) - a_{k-1}\right) = 0$$

- Shift index using $k->k+1$

$$(k+r)\left(a_{k+1}(k+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+r}$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 63

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(1+x)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(-x - \frac{3}{4}x^2 - \frac{11}{36}x^3 - \frac{25}{288}x^4 - \frac{137}{7200}x^5 + \mathrm{O}\left(x^6\right)\right)c_2\right)x$$

✓ Solution by Mathematica
Time used: 0.004 (sec). Leaf size: 112

```
AsymptoticDSolveValue[x^2*y''[x]-x*(1+x)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) + c_2\left(x\left(-\frac{137x^5}{7200} - \frac{25x^4}{288} - \frac{11x^3}{36} - \frac{3x^2}{4} - x\right) + x\left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right)\log(x)\right)$$

24.5 problem 5

Internal problem ID [2406]
Internal file name [`OUTPUT/2406_Tuesday_February_27_2024_08_36_59_AM_25213605/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(2x+3)\,y' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(-2x^2 - 3x\right) y' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2x+3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Table 687: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2x+3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (-2x^2 - 3x)\, y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (-2x^2 - 3x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) - 3a_n(n+r) + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - 4n - 4r + 4} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{2a_{n-1}(1+n)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2r}{(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_1 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r}{(-1+r)^2}$	4

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4+4r}{r(-1+r)^2}$$

Which for the root $r = 2$ becomes

$$a_2 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r}{(-1+r)^2}$	4
a_2	$\frac{4+4r}{r(-1+r)^2}$	6

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{16 + 8r}{(1 + r)\, r\, (-1 + r)^2}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{16}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r}{(-1+r)^2}$	4
a_2	$\frac{4+4r}{r(-1+r)^2}$	6
a_3	$\frac{16+8r}{(1+r)r(-1+r)^2}$	$\frac{16}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{48 + 16r}{(2 + r)\,(1 + r)\, r\, (-1 + r)^2}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{10}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r}{(-1+r)^2}$	4
a_2	$\frac{4+4r}{r(-1+r)^2}$	6
a_3	$\frac{16+8r}{(1+r)r(-1+r)^2}$	$\frac{16}{3}$
a_4	$\frac{48+16r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{10}{3}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{128 + 32r}{(3 + r)\,(2 + r)\,(1 + r)\, r\, (-1 + r)^2}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{8}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r}{(-1+r)^2}$	4
a_2	$\frac{4+4r}{r(-1+r)^2}$	6
a_3	$\frac{16+8r}{(1+r)r(-1+r)^2}$	$\frac{16}{3}$
a_4	$\frac{48+16r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{10}{3}$
a_5	$\frac{128+32r}{(3+r)(2+r)(1+r)r(-1+r)^2}$	$\frac{8}{5}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2 \left(6x^2 + 4x + 1 + \frac{16x^3}{3} + \frac{10x^4}{3} + \frac{8x^5}{5} + O(x^6)\right)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=2)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{2r}{(-1+r)^2}$	4	$\frac{-2r-2}{(-1+r)^3}$	-6
b_2	$\frac{4+4r}{r(-1+r)^2}$	6	$\frac{-8r^2-12r+4}{r^2(-1+r)^3}$	-13
b_3	$\frac{16+8r}{(1+r)r(-1+r)^2}$	$\frac{16}{3}$	$\frac{-24r^3-72r^2-16r+16}{(1+r)^2r^2(-1+r)^3}$	$-\frac{124}{9}$
b_4	$\frac{48+16r}{(2+r)(1+r)r(-1+r)^2}$	$\frac{10}{3}$	$\frac{-64r^4-352r^3-448r^2+96}{(2+r)^2(1+r)^2r^2(-1+r)^3}$	$-\frac{173}{18}$
b_5	$\frac{128+32r}{(3+r)(2+r)(1+r)r(-1+r)^2}$	$\frac{8}{5}$	$-\frac{32(5r^5+45r^4+125r^3+105r^2-16r-24)}{(3+r)^2(2+r)^2(1+r)^2r^2(-1+r)^3}$	$-\frac{374}{75}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\ldots \\
&= x^2\left(6x^2 + 4x + 1 + \frac{16x^3}{3} + \frac{10x^4}{3} + \frac{8x^5}{5} + O(x^6)\right)\ln(x) \\
&\quad + x^2\left(-13x^2 - 6x - \frac{124x^3}{9} - \frac{173x^4}{18} - \frac{374x^5}{75} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x^2\left(6x^2 + 4x + 1 + \frac{16x^3}{3} + \frac{10x^4}{3} + \frac{8x^5}{5} + O(x^6)\right) \\
&\quad + c_2\Bigg(x^2\left(6x^2 + 4x + 1 + \frac{16x^3}{3} + \frac{10x^4}{3} + \frac{8x^5}{5} + O(x^6)\right)\ln(x) \\
&\quad + x^2\left(-13x^2 - 6x - \frac{124x^3}{9} - \frac{173x^4}{18} - \frac{374x^5}{75} + O(x^6)\right)\Bigg)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1x^2\left(6x^2 + 4x + 1 + \frac{16x^3}{3} + \frac{10x^4}{3} + \frac{8x^5}{5} + O(x^6)\right) \\
&\quad + c_2\Bigg(x^2\left(6x^2 + 4x + 1 + \frac{16x^3}{3} + \frac{10x^4}{3} + \frac{8x^5}{5} + O(x^6)\right)\ln(x) \\
&\quad + x^2\left(-13x^2 - 6x - \frac{124x^3}{9} - \frac{173x^4}{18} - \frac{374x^5}{75} + O(x^6)\right)\Bigg)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^2 \left(6x^2+4x+1+\frac{16x^3}{3}+\frac{10x^4}{3}+\frac{8x^5}{5}+O(x^6)\right) \\
+ c_2 \left(x^2 \left(6x^2+4x+1+\frac{16x^3}{3}+\frac{10x^4}{3}+\frac{8x^5}{5}+O(x^6)\right) \ln(x) \right. \\
\left. + x^2 \left(-13x^2-6x-\frac{124x^3}{9}-\frac{173x^4}{18}-\frac{374x^5}{75}+O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^2 \left(6x^2+4x+1+\frac{16x^3}{3}+\frac{10x^4}{3}+\frac{8x^5}{5}+O(x^6)\right) \\
+ c_2 \left(x^2 \left(6x^2+4x+1+\frac{16x^3}{3}+\frac{10x^4}{3}+\frac{8x^5}{5}+O(x^6)\right) \ln(x) \right. \\
\left. + x^2 \left(-13x^2-6x-\frac{124x^3}{9}-\frac{173x^4}{18}-\frac{374x^5}{75}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

24.5.1 Maple step by step solution

Let's solve

$x^2 y'' + (-2x^2-3x)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2} + \frac{(2x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x+3)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x+3}{x}, P_3(x) = \frac{4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2 y'' - x(2x+3)\,y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - 2a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r=2$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 - 2a_{k-1}(k+r-1) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+r-1)^2 - 2a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r)}{(k+r-1)^2}$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{2a_k(k+2)}{(k+1)^2}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{2a_k(k+2)}{(k+1)^2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*(2*x+3)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1) \left(1 + 4x + 6x^2 + \frac{16}{3}x^3 + \frac{10}{3}x^4 + \frac{8}{5}x^5 + O(x^6)\right) + \left((-6)x - 13x^2 - \frac{124}{9}x^3 - \frac{173}{18}x^4 - \frac{374}{75}x^5 + O(x^6)\right) c_2 \right) x^2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 116

```
AsymptoticDSolveValue[x^2*y''[x]-x*(2*x+3)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{8x^5}{5}+\frac{10x^4}{3}+\frac{16x^3}{3}+6x^2+4x+1\right)x^2 + c_2\left(\left(-\frac{374x^5}{75}-\frac{173x^4}{18}-\frac{124x^3}{9}-13x^2-6x\right)x^2 + \left(\frac{8x^5}{5}+\frac{10x^4}{3}+\frac{16x^3}{3}+6x^2+4x+1\right)x^2\log(x)\right)$$

24.6 problem 6

Internal problem ID [2407]
Internal file name [OUTPUT/2407_Tuesday_February_27_2024_08_37_00_AM_51977979/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(-x^2+1)\,y''-5xy'+9y=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(-x^4+x^2)\,y''-5xy'+9y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{5}{x\,(x^2-1)}$$
$$q(x)=-\frac{9}{x^2\,(x^2-1)}$$

Table 689: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{x(x^2-1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{9}{x^2(x^2-1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(x^2-1) - 5xy' + 9y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2(x^2-1) - 5x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 9\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)$$
$$+\sum_{n=0}^{\infty}\left(-5x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}9a_nx^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n(n+r)(n+r-1)\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r)x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r)x^{n+r}\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)$$
$$+\sum_{n=0}^{\infty}\left(-5x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}9a_nx^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)(n+r-1)-5x^{n+r}a_n(n+r)+9a_nx^{n+r}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)-5x^r a_0 r+9a_0x^r=0$$

Or

$$\left(x^r r(-1+r)-5x^r r+9x^r\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$(r-3)^2 x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$(r-3)^2=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-3)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 3$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+3}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$-a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) - 5a_n(n+r) + 9a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n + r - 2)\, a_{n-2}}{n - 3 + r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = \frac{(n + 1)\, a_{n-2}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r}{-1 + r}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r}{-1+r}$	$\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r}{-1+r}$	$\frac{3}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(2+r)}{r^2 - 1}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{15}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r}{-1+r}$	$\frac{3}{2}$
a_3	0	0
a_4	$\frac{r(2+r)}{r^2-1}$	$\frac{15}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r}{-1+r}$	$\frac{3}{2}$
a_3	0	0
a_4	$\frac{r(2+r)}{r^2-1}$	$\frac{15}{8}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^3 \left(1 + \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr}a_{n,r}$$

And the above is then evaluated at $r = 3$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=3)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{r}{-1+r}$	$\frac{3}{2}$	$-\frac{1}{(-1+r)^2}$	$-\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{r(2+r)}{r^2-1}$	$\frac{15}{8}$	$\frac{-2r^2-2r-2}{(r^2-1)^2}$	$-\frac{13}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= x^3\left(1 + \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right)\ln(x) + x^3\left(-\frac{x^2}{4} - \frac{13x^4}{32} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3\left(1 + \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right) \\ &\quad + c_2\left(x^3\left(1 + \frac{3x^2}{2} + \frac{15x^4}{8} + O(x^6)\right)\ln(x) + x^3\left(-\frac{x^2}{4} - \frac{13x^4}{32} + O(x^6)\right)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3\left(1+\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right) \\
&\quad + c_2\left(x^3\left(1+\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)\ln(x)+x^3\left(-\frac{x^2}{4}-\frac{13x^4}{32}+O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^3\left(1+\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right) \\
+ c_2\left(x^3\left(1+\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)\ln(x)+x^3\left(-\frac{x^2}{4}-\frac{13x^4}{32}+O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^3\left(1+\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right) \\
+ c_2\left(x^3\left(1+\frac{3x^2}{2}+\frac{15x^4}{8}+O(x^6)\right)\ln(x)+x^3\left(-\frac{x^2}{4}-\frac{13x^4}{32}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

24.6.1 Maple step by step solution

Let's solve

$-y''x^2(x^2-1)-5xy'+9y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{9y}{x^2(x^2-1)}-\frac{5y'}{x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{5y'}{x(x^2-1)}-\frac{9y}{x^2(x^2-1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{5}{x(x^2-1)}, P_3(x) = -\frac{9}{x^2(x^2-1)}\right]$

- $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right)\bigg|_{x=-1} = \frac{5}{2}$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''x^2(x^2-1) + 5xy' - 9y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^4 - 4u^3 + 5u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (5u - 5)\left(\frac{d}{du}y(u)\right) - 9y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..4$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$-a_0 r(3+2r)\, u^{-1+r} + (-a_1(1+r)(5+2r) + a_0(5r^2-9))\, u^r + (-a_2(2+r)(7+2r) + a_1(5r^2+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, -\frac{3}{2}\}$
- The coefficients of each power of u must be 0

  $[-a_1(1+r)(5+2r) + a_0(5r^2-9) = 0, -a_2(2+r)(7+2r) + a_1(5r^2+10r-4) - 4a_0 r(-1+r)$
- Solve for the dependent coefficient(s)

  $\left\{a_1 = \frac{a_0(5r^2-9)}{2r^2+7r+5}, a_2 = \frac{a_0(17r^4+30r^3-57r^2-70r+36)}{4r^4+36r^3+115r^2+153r+70}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, k^2 + (2(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, r - 5a_{k-2} + 12a_{k-1} - 7a_{k+1})$
- Shift index using $k- >k+2$

  $(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\,(k+2)^2 + (2(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})\, r - 5a_k + 12a_{k+1} - 7a_k$
- Recursion relation that defines series solution to ODE

  $a_{k+3} = \frac{k^2 a_k - 4k^2 a_{k+1} + 5k^2 a_{k+2} + 2kr a_k - 8kr a_{k+1} + 10kr a_{k+2} + r^2 a_k - 4r^2 a_{k+1} + 5r^2 a_{k+2} - k a_k - 4k a_{k+1} + 20k a_{k+2} - r a_k - 4r a_{k+}}{2k^2+4kr+2r^2+15k+15r+27}$
- Recursion relation for $r = 0$

  $a_{k+3} = \frac{k^2 a_k - 4k^2 a_{k+1} + 5k^2 a_{k+2} - k a_k - 4k a_{k+1} + 20k a_{k+2} + 11a_{k+2}}{2k^2+15k+27}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = \frac{k^2 a_k - 4k^2 a_{k+1} + 5k^2 a_{k+2} - k a_k - 4k a_{k+1} + 20k a_{k+2} + 11a_{k+2}}{2k^2+15k+27}, a_1 = -\frac{9a_0}{5}, a_2 = \frac{18a_0}{35}\right]$
- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+3} = \frac{k^2 a_k - 4k^2 a_{k+1} + 5k^2 a_{k+2} - k a_k - 4k a_{k+1} + 20k a_{k+2} + 11a_{k+2}}{2k^2+15k+27}, a_1 = -\frac{9a_0}{5}, a_2 = \frac{18a_0}{35}\right]$
- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+3} = \frac{k^2 a_k - 4k^2 a_{k+1} + 5k^2 a_{k+2} - 4k a_k + 8k a_{k+1} + 5k a_{k+2} + \frac{15}{4} a_k - 3a_{k+1} - \frac{31}{4} a_{k+2}}{2k^2+9k+9}$
- Solution for $r = -\frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+3} = \frac{k^2 a_k - 4k^2 a_{k+1} + 5k^2 a_{k+2} - 4k a_k + 8k a_{k+1} + 5k a_{k+2} + \frac{15}{4} a_k - 3a_{k+1} - \frac{31}{4} a_{k+2}}{2k^2+9k+9}, a_1 = -\frac{9a_0}{4}, a\right.$
- Revert the change of variables $u = x+1$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+1)^{k-\frac{3}{2}}, a_{k+3}=\frac{k^2 a_k-4k^2 a_{k+1}+5k^2 a_{k+2}-4k a_k+8k a_{k+1}+5k a_{k+2}+\frac{15}{4} a_k-3a_{k+1}-\frac{31}{4} a_{k+2}}{2k^2+9k+9}, a_1=-\frac{9}{}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x+1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x+1)^{k-\frac{3}{2}}\right), a_{k+3}=\frac{k^2 a_k-4k^2 a_{k+1}+5k^2 a_{k+2}-k a_k-4k a_{k+1}+20k a_{k+2}+11a}{2k^2+15k+27}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 51

```
Order:=6;
dsolve(x^2*(1-x^2)*diff(y(x),x$2)-5*x*diff(y(x),x)+9*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left((c_2\ln(x)+c_1)\left(1+\frac{3}{2}x^2+\frac{15}{8}x^4+\mathrm{O}(x^6)\right)+\left(-\frac{1}{4}x^2-\frac{13}{32}x^4+\mathrm{O}(x^6)\right)c_2\right)x^3$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 71

```
AsymptoticDSolveValue[x^2*(1-x^2)*y''[x]-5*x*y'[x]+9*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{15x^4}{8}+\frac{3x^2}{2}+1\right)x^3+c_2\left(\left(-\frac{13x^4}{32}-\frac{x^2}{4}\right)x^3+\left(\frac{15x^4}{8}+\frac{3x^2}{2}+1\right)x^3\log(x)\right)$$

24.7 problem 7

Internal problem ID [2408]
Internal file name [OUTPUT/2408_Tuesday_February_27_2024_08_37_01_AM_97415678/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(x^2 - 1)\, y' + (-x^2 + 1)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (x^3 - x)\, y' + (-x^2 + 1)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2 - 1}{x}$$
$$q(x) = -\frac{x^2 - 1}{x^2}$$

Table 691: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2-1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{x^2-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^3 - x)\, y' + (-x^2 + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (x^3 - x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-x^2 + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n (n+r) = \sum_{n=2}^{\infty} a_{n-2} (n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} (n+r-2)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1)+a_{n-2}(n+r-2)-a_n(n+r)-a_{n-2}+a_n=0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n=-\frac{a_{n-2}(n+r-3)}{n^2+2nr+r^2-2n-2r+1} \tag{4}$$

Which for the root $r=1$ becomes

$$a_n=-\frac{a_{n-2}(n-2)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r=1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n=2$, using the above recursive equation gives

$$a_2=\frac{1-r}{(1+r)^2}$$

Which for the root $r=1$ becomes

$$a_2=0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-r}{(1+r)^2}$	0

For $n=3$, using the above recursive equation gives

$$a_3=0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-r}{(1+r)^2}$	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-1+r}{(1+r)(r+3)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-r}{(1+r)^2}$	0
a_3	0	0
a_4	$\frac{-1+r}{(1+r)(r+3)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1-r}{(1+r)^2}$	0
a_3	0	0
a_4	$\frac{-1+r}{(1+r)(r+3)^2}$	0
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x\left(1 + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{1-r}{(1+r)^2}$	0	$\frac{r-3}{(1+r)^3}$	$-\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{-1+r}{(1+r)(r+3)^2}$	0	$\frac{-2r^2+2r+8}{(1+r)^2(r+3)^3}$	$\frac{1}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= x\left(1 + O(x^6)\right)\ln(x) + x\left(-\frac{x^2}{4} + \frac{x^4}{32} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1x\left(1 + O(x^6)\right) + c_2\left(x\left(1 + O(x^6)\right)\ln(x) + x\left(-\frac{x^2}{4} + \frac{x^4}{32} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x(1+O(x^6)) + c_2\left(x(1+O(x^6))\ln(x) + x\left(-\frac{x^2}{4}+\frac{x^4}{32}+O(x^6)\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x(1+O(x^6)) + c_2\left(x(1+O(x^6))\ln(x) + x\left(-\frac{x^2}{4}+\frac{x^4}{32}+O(x^6)\right)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x(1+O(x^6)) + c_2\left(x(1+O(x^6))\ln(x) + x\left(-\frac{x^2}{4}+\frac{x^4}{32}+O(x^6)\right)\right)
$$

Verified OK.

24.7.1 Maple step by step solution

Let's solve
$x^2y'' + (x^3 - x)\,y' + (-x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative
  $y'' = \frac{(x^2-1)y}{x^2} - \frac{(x^2-1)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y'' + \frac{(x^2-1)y'}{x} - \frac{(x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = -\frac{x^2-1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x^2-1)\,y' + (-x^2+1)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2\,x^r + a_1r^2x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k-3+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1r^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k + r - 1)^2 + a_{k-2}(k - 3 + r) = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}(k + 1 + r)^2 + a_k(k + r - 1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r-1)}{(k+1+r)^2}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k k}{(k+2)^2}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k k}{(k+2)^2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(x^2-1)*diff(y(x),x)+(1-x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_2 \ln(x) + c_1)\left(1 + \mathrm{O}\left(x^6\right)\right) + \left(-\frac{1}{4}x^2 + \frac{1}{32}x^4 + \mathrm{O}\left(x^6\right)\right)c_2\right)x$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 33

```
AsymptoticDSolveValue[x^2*y''[x]+x*(x^2-1)*y'[x]+(1-x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x\left(\frac{x^4}{32} - \frac{x^2}{4}\right) + x\log(x)\right) + c_1 x$$

24.8 problem 8

Internal problem ID [2409]
Internal file name [`OUTPUT/2409_Tuesday_February_27_2024_08_37_01_AM_11847939/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y''+x(2x-1)\,y'+x(x-1)\,y=0}$$

With the expansion point for the power series method at $x = 0$.

The ODE is

$$x^2y''+\left(2x^2-x\right)y'+\left(x^2-x\right)y=0$$

Or

$$x(y''x+2xy'+yx-y'-y)=0$$

For $x \neq 0$ the above simplifies to

$$y''x+(2x-1)\,y'+y(x-1)=0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y''+\left(2x^2-x\right)y'+\left(x^2-x\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$
\begin{aligned}
p(x) &= \frac{2x-1}{x} \\
q(x) &= \frac{x-1}{x}
\end{aligned}
$$

Table 693: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x-1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x-1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$
x^2y'' + (2x^2 - x)\, y' + (x^2 - x)\, y = 0
$$

Let the solution be represented as Frobenius power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}\right) + (2x^2-x)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right) + (x^2-x)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty}2x^{1+n+r}a_n(n+r)\right) + \sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right) + \left(\sum_{n=0}^{\infty}x^{n+r+2}a_n\right) + \sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}2x^{1+n+r}a_n(n+r) = \sum_{n=1}^{\infty}2a_{n-1}(n+r-1)\,x^{n+r}$$

$$\sum_{n=0}^{\infty}x^{n+r+2}a_n = \sum_{n=2}^{\infty}a_{n-2}x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right) = \sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty}2a_{n-1}(n+r-1)\,x^{n+r}\right) + \sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right) + \left(\sum_{n=2}^{\infty}a_{n-2}x^{n+r}\right) + \sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r = 0$$

Or

$$(x^r r(-1+r) - x^r r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(-2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = \frac{-2r+1}{r^2-1}
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) - a_n(n+r) + a_{n-2} - a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{2na_{n-1} + 2ra_{n-1} + a_{n-2} - 3a_{n-1}}{n^2 + 2nr + r^2 - 2n - 2r} \tag{4}
$$

Which for the root $r = 2$ becomes

$$
a_n = \frac{-2na_{n-1} - a_{n-2} - a_{n-1}}{n(n+2)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{r^2-1}$	-1

For $n = 2$, using the above recursive equation gives

$$
a_2 = \frac{3r}{(r^2-1)(r+2)}
$$

Which for the root $r = 2$ becomes

$$
a_2 = \frac{1}{2}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{r^2-1}$	-1
a_2	$\frac{3r}{(r^2-1)(r+2)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-4r - 2}{(r + 3)(1 + r)(r + 2)(-1 + r)}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{r^2-1}$	-1
a_2	$\frac{3r}{(r^2-1)(r+2)}$	$\frac{1}{2}$
a_3	$\frac{-4r-2}{(r+3)(1+r)(r+2)(-1+r)}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5}{(r + 4)(r + 2)(-1 + r)(r + 3)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{r^2-1}$	-1
a_2	$\frac{3r}{(r^2-1)(r+2)}$	$\frac{1}{2}$
a_3	$\frac{-4r-2}{(r+3)(1+r)(r+2)(-1+r)}$	$-\frac{1}{6}$
a_4	$\frac{5}{(r+4)(r+2)(-1+r)(r+3)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-6r - 9}{(r^2 + 8r + 15)(r + 2)(r + 4)(r^2 - 1)}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2r+1}{r^2-1}$	-1
a_2	$\frac{3r}{(r^2-1)(r+2)}$	$\frac{1}{2}$
a_3	$\frac{-4r-2}{(r+3)(1+r)(r+2)(-1+r)}$	$-\frac{1}{6}$
a_4	$\frac{5}{(r+4)(r+2)(-1+r)(r+3)}$	$\frac{1}{24}$
a_5	$\frac{-6r-9}{(r^2+8r+15)(r+2)(r+4)(r^2-1)}$	$-\frac{1}{120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= x^2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{3r}{(r^2 - 1)(r + 2)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{3r}{(r^2 - 1)(r + 2)} &= \lim_{r \to 0} \frac{3r}{(r^2 - 1)(r + 2)} \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = -\frac{2r - 1}{r^2 - 1}$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) - b_n(n+r) + b_{n-2} - b_{n-1} = 0 \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_n n(n-1) + 2b_{n-1}(n-1) - b_n n + b_{n-2} - b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{2nb_{n-1} + 2rb_{n-1} + b_{n-2} - 3b_{n-1}}{n^2 + 2nr + r^2 - 2n - 2r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{2nb_{n-1} + b_{n-2} - 3b_{n-1}}{n^2 - 2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r+1}{r^2-1}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{3r}{(r^2 - 1)(r + 2)}$$

Which for the root $r = 0$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r+1}{r^2-1}$	-1
b_2	$\frac{3r}{(r^2-1)(r+2)}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{2(2r+1)}{(r^2+4r+3)(r+2)(-1+r)}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r+1}{r^2-1}$	-1
b_2	$\frac{3r}{(r^2-1)(r+2)}$	0
b_3	$\frac{-4r-2}{(r+3)(1+r)(r+2)(-1+r)}$	$\frac{1}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{5}{(r^2+6r+8)(-1+r)(r+3)}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{5}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r+1}{r^2-1}$	-1
b_2	$\frac{3r}{(r^2-1)(r+2)}$	0
b_3	$\frac{-4r-2}{(r+3)(1+r)(r+2)(-1+r)}$	$\frac{1}{3}$
b_4	$\frac{5}{(r+4)(r+2)(-1+r)(r+3)}$	$-\frac{5}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{3(2r+3)}{(r^2+8r+15)(1+r)(-1+r)(r+2)(r+4)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{3}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2r+1}{r^2-1}$	-1
b_2	$\frac{3r}{(r^2-1)(r+2)}$	0
b_3	$\frac{-4r-2}{(r+3)(1+r)(r+2)(-1+r)}$	$\frac{1}{3}$
b_4	$\frac{5}{(r+4)(r+2)(-1+r)(r+3)}$	$-\frac{5}{24}$
b_5	$\frac{-6r-9}{(r^2+8r+15)(r+2)(r+4)(r^2-1)}$	$\frac{3}{40}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - x + \frac{x^3}{3} - \frac{5x^4}{24} + \frac{3x^5}{40} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) + c_2 \left(1 - x + \frac{x^3}{3} - \frac{5x^4}{24} + \frac{3x^5}{40} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$y = y_h$$

$$= c_1 x^2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) + c_2 \left(1 - x + \frac{x^3}{3} - \frac{5x^4}{24} + \frac{3x^5}{40} + O(x^6)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) + c_2 \left(1 - x + \frac{x^3}{3} - \frac{5x^4}{24} + \frac{3x^5}{40} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) + c_2 \left(1 - x + \frac{x^3}{3} - \frac{5x^4}{24} + \frac{3x^5}{40} + O(x^6)\right)$$

Verified OK.

24.8.1 Maple step by step solution

Let's solve

$x^2 y'' + (2x^2 - x) y' + (x^2 - x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x} - \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (2x-1)\, y' + y(x-1) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, x^{-1+r} + \left(a_1(1+r)\,(-1+r) + a_0(-1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0

  $a_1(1+r)(-1+r) + a_0(-1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r-1) + a_k(2k+2r-1) + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r) + a_{k+1}(2k+1+2r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2ka_{k+1}+2ra_{k+1}+a_k+a_{k+1}}{(k+2+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k+a_{k+1}}{(k+2)k}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k+a_{k+1}}{(k+2)k}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k+5a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{2ka_{k+1}+a_k+5a_{k+1}}{(k+4)(k+2)}, 3a_1 + 3a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 42

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(2*x-1)*diff(y(x),x)+x*(x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(-2 + 2x - \frac{2}{3}x^3 + \frac{5}{12}x^4 - \frac{3}{20}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 59

```
AsymptoticDSolveValue[x^2*y''[x]+x*(2*x-1)*y'[x]+x*(x-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{5x^4}{24} + \frac{x^3}{3} - x + 1\right) + c_2 \left(\frac{x^6}{24} - \frac{x^5}{6} + \frac{x^4}{2} - x^3 + x^2\right)$$

24.9 problem 9

Internal problem ID [2410]
Internal file name [`OUTPUT/2410_Tuesday_February_27_2024_08_37_02_AM_3685757/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2 y'' - y' x^2 + \left(x^2 - 2\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2 y'' - y' x^2 + \left(x^2 - 2\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x) y' + q(x) y = 0$$

Where

$$\begin{aligned} p(x) &= -1 \\ q(x) &= \frac{x^2 - 2}{x^2} \end{aligned}$$

Table 695: Table $p(x), q(x)$ singularites.

$p(x) = -1$	
singularity	type

$q(x) = \frac{x^2-2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' - y' x^2 + \left(x^2 - 2\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^2 + \left(x^2 - 2\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-2 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} (n+r-1) x^{n+r}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 2 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 2 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - r - 2) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{r}{r^2 + r - 2}$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_{n-2} - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n a_{n-1} + r a_{n-1} - a_{n-2} - a_{n-1}}{n^2 + 2nr + r^2 - n - r - 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{n a_{n-1} - a_{n-2} + a_{n-1}}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2}{(r^2+r-2)\, r\, (r+3)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
a_2	$\frac{2}{(r^2+r-2)r(r+3)}$	$\frac{1}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^2 - 2r + 4}{(r+4)(r^2+r-2)\, r\, (r+3)}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{1}{60}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
a_2	$\frac{2}{(r^2+r-2)r(r+3)}$	$\frac{1}{20}$
a_3	$\frac{-r^2-2r+4}{(r+4)(r^2+r-2)r(r+3)}$	$-\frac{1}{60}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-r^2 - 3r + 2}{(r+5)(r+2)(r+3)r(-1+r)(r+4)}$$

Which for the root $r = 2$ becomes

$$a_4 = -\frac{1}{210}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
a_2	$\frac{2}{(r^2+r-2)r(r+3)}$	$\frac{1}{20}$
a_3	$\frac{-r^2-2r+4}{(r+4)(r^2+r-2)r(r+3)}$	$-\frac{1}{60}$
a_4	$\frac{-r^2-3r+2}{(r+5)(r+2)(r+3)r(-1+r)(r+4)}$	$-\frac{1}{210}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{4}{(r+6)(r+3)r(r+2)(-1+r)(r+4)(r+5)}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{1}{3360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
a_2	$\frac{2}{(r^2+r-2)r(r+3)}$	$\frac{1}{20}$
a_3	$\frac{-r^2-2r+4}{(r+4)(r^2+r-2)r(r+3)}$	$-\frac{1}{60}$
a_4	$\frac{-r^2-3r+2}{(r+5)(r+2)(r+3)r(-1+r)(r+4)}$	$-\frac{1}{210}$
a_5	$-\frac{4}{(r+6)(r+3)r(r+2)(-1+r)(r+4)(r+5)}$	$-\frac{1}{3360}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2\left(1 + \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{60} - \frac{x^4}{210} - \frac{x^5}{3360} + O\left(x^6\right)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= \frac{-r^2 - 2r + 4}{(r+4)\left(r^2 + r - 2\right) r \left(r+3\right)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{-r^2 - 2r + 4}{(r+4)\left(r^2 + r - 2\right) r \left(r+3\right)} &= \lim_{r\to -1} \frac{-r^2 - 2r + 4}{(r+4)\left(r^2 + r - 2\right) r \left(r+3\right)} \\
&= \frac{5}{12}
\end{aligned}$$

The limit is $\frac{5}{12}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = \frac{r}{r^2 + r - 2}$$

For $2 \leq n$ the recursive equation is

$$b_n(n + r)(n + r - 1) - b_{n-1}(n + r - 1) + b_{n-2} - 2b_n = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n - 1)(n - 2) - b_{n-1}(n - 2) + b_{n-2} - 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{nb_{n-1} + rb_{n-1} - b_{n-2} - b_{n-1}}{n^2 + 2nr + r^2 - n - r - 2} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{nb_{n-1} - b_{n-2} - 2b_{n-1}}{n^2 - 3n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{2}{(r^2 + r - 2) r (r + 3)}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{2}{(r^2+r-2)r(r+3)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{r^2 + 2r - 4}{(r+4)(r^2+r-2)\, r\, (r+3)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{5}{12}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{2}{(r^2+r-2)r(r+3)}$	$\frac{1}{2}$
b_3	$\frac{-r^2-2r+4}{(r+4)(r^2+r-2)r(r+3)}$	$\frac{5}{12}$

For $n = 4$, using the above recursive equation gives

$$b_4 = -\frac{r^2 + 3r - 2}{(r^2+7r+10)(r+3)\, r\, (-1+r)(r+4)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{12}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{2}{(r^2+r-2)r(r+3)}$	$\frac{1}{2}$
b_3	$\frac{-r^2-2r+4}{(r+4)(r^2+r-2)r(r+3)}$	$\frac{5}{12}$
b_4	$\frac{-r^2-3r+2}{(r+5)(r+2)(r+3)r(-1+r)(r+4)}$	$\frac{1}{12}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{4}{(r^2 + 9r + 18)\, r\, (r^2 + r - 2)\, (r + 4)\, (r + 5)}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{1}{60}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{r^2+r-2}$	$\frac{1}{2}$
b_2	$\frac{2}{(r^2+r-2)r(r+3)}$	$\frac{1}{2}$
b_3	$\frac{-r^2-2r+4}{(r+4)(r^2+r-2)r(r+3)}$	$\frac{5}{12}$
b_4	$\frac{-r^2-3r+2}{(r+5)(r+2)(r+3)r(-1+r)(r+4)}$	$\frac{1}{12}$
b_5	$-\frac{4}{(r+6)(r+3)r(r+2)(-1+r)(r+4)(r+5)}$	$-\frac{1}{60}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + \frac{x}{2} + \frac{x^2}{2} + \frac{5x^3}{12} + \frac{x^4}{12} - \frac{x^5}{60} + O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 \left(1 + \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{60} - \frac{x^4}{210} - \frac{x^5}{3360} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{2} + \frac{5x^3}{12} + \frac{x^4}{12} - \frac{x^5}{60} + O(x^6)\right)}{x}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^2 \left(1 + \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{60} - \frac{x^4}{210} - \frac{x^5}{3360} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{2} + \frac{5x^3}{12} + \frac{x^4}{12} - \frac{x^5}{60} + O(x^6)\right)}{x}
\end{aligned}$$

Summary
The solution(s) found are the following

$$
y = c_1 x^2 \left(1+\frac{x}{2}+\frac{x^2}{20}-\frac{x^3}{60}-\frac{x^4}{210}-\frac{x^5}{3360}+O(x^6)\right) + \frac{c_2\left(1+\frac{x}{2}+\frac{x^2}{2}+\frac{5x^3}{12}+\frac{x^4}{12}-\frac{x^5}{60}+O(x^6)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 \left(1+\frac{x}{2}+\frac{x^2}{20}-\frac{x^3}{60}-\frac{x^4}{210}-\frac{x^5}{3360}+O(x^6)\right) + \frac{c_2\left(1+\frac{x}{2}+\frac{x^2}{2}+\frac{5x^3}{12}+\frac{x^4}{12}-\frac{x^5}{60}+O(x^6)\right)}{x}
$$

Verified OK.

24.9.1 Maple step by step solution

Let's solve

$x^2 y'' - y' x^2 + (x^2-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' - \frac{(x^2-2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' + \frac{(x^2-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -1, P_3(x) = \frac{x^2-2}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''-y'x^2+(x^2-2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2\cdot y'$ to series expansion

  $x^2\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$

- Shift index using $k->k-1$

  $x^2\cdot y'=\sum_{k=1}^{\infty} a_{k-1}(k-1+r)\,x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\,x^r+(a_1(2+r)\,(-1+r)-a_0r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r+1)\,(k+r-2)-a_{k-1}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-1,2\}$

- Each term must be 0

$$a_1(2+r)(-1+r)-a_0r=0$$

- Solve for the dependent coefficient(s)

$$a_1=\frac{a_0r}{r^2+r-2}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)(k+r-2)-a_{k-1}k-a_{k-1}r+a_{k-2}+a_{k-1}=0$$

- Shift index using $k->k+2$

$$a_{k+2}(k+3+r)(k+r)-a_{k+1}(k+2)-a_{k+1}r+a_k+a_{k+1}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{ka_{k+1}+a_{k+1}r-a_k+a_{k+1}}{(k+3+r)(k+r)}$$

- Recursion relation for $r=-1$

$$a_{k+2}=\frac{ka_{k+1}-a_k}{(k+2)(k-1)}$$

- Series not valid for $r=-1$, division by 0 in the recursion relation at $k=1$

$$a_{k+2}=\frac{ka_{k+1}-a_k}{(k+2)(k-1)}$$

- Recursion relation for $r=2$

$$a_{k+2}=\frac{ka_{k+1}-a_k+3a_{k+1}}{(k+5)(k+2)}$$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+2},a_{k+2}=\frac{ka_{k+1}-a_k+3a_{k+1}}{(k+5)(k+2)},a_1=\frac{a_0}{2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x^2*diff(y(x),x)+(x^2-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 + \frac{1}{2}x + \frac{1}{20}x^2 - \frac{1}{60}x^3 - \frac{1}{210}x^4 - \frac{1}{3360}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(12 + 6x + 6x^2 + 5x^3 + x^4 - \frac{1}{5}x^5 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 68

```
AsymptoticDSolveValue[x^2*y''[x]-x^2*y'[x]+(x^2-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^3}{12} + \frac{5x^2}{12} + \frac{x}{2} + \frac{1}{x} + \frac{1}{2}\right) + c_2\left(-\frac{x^6}{210} - \frac{x^5}{60} + \frac{x^4}{20} + \frac{x^3}{2} + x^2\right)$$

24.10 problem 10

Internal problem ID [2411]
Internal file name [OUTPUT/2411_Tuesday_February_27_2024_08_37_03_AM_40584997/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2y'x^2 - (3x^2 + 2)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 2y'x^2 + (-3x^2 - 2)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 2$$
$$q(x) = -\frac{3x^2 + 2}{x^2}$$

Table 697: Table $p(x), q(x)$ singularites.

$p(x) = 2$	
singularity	type

$q(x) = -\frac{3x^2+2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 2y'x^2 + \left(-3x^2 - 2\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^2 + \left(-3x^2 - 2\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-3x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-3a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=2}^{\infty} \left(-3a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{2r}{r^2 + r - 2}$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_{n-1}(n+r-1) - 3a_{n-2} - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2na_{n-1} + 2ra_{n-1} - 3a_{n-2} - 2a_{n-1}}{n^2 + 2nr + r^2 - n - r - 2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{-2na_{n-1} + 3a_{n-2} - 2a_{n-1}}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{7r^2 + 7r - 6}{(r^2 + r - 2)\, r\, (r+3)}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{9}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1
a_2	$\frac{7r^2+7r-6}{(r^2+r-2)r(r+3)}$	$\frac{9}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-20r^2 - 40r + 24}{(r+4)(r^2 + r - 2)\, r\, (r+3)}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{17}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1
a_2	$\frac{7r^2+7r-6}{(r^2+r-2)r(r+3)}$	$\frac{9}{10}$
a_3	$\frac{-20r^2-40r+24}{(r+4)(r^2+r-2)r(r+3)}$	$-\frac{17}{30}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{61r^2 + 183r - 108}{(r+5)(r+2)(r+3)r(-1+r)(r+4)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{251}{840}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1
a_2	$\frac{7r^2+7r-6}{(r^2+r-2)r(r+3)}$	$\frac{9}{10}$
a_3	$\frac{-20r^2-40r+24}{(r+4)(r^2+r-2)r(r+3)}$	$-\frac{17}{30}$
a_4	$\frac{61r^2+183r-108}{(r+5)(r+2)(r+3)r(-1+r)(r+4)}$	$\frac{251}{840}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-182r^2 - 728r + 408}{(r+6)(r+3)r(r+2)(-1+r)(r+4)(r+5)}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{37}{280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2r}{r^2+r-2}$	-1
a_2	$\frac{7r^2+7r-6}{(r^2+r-2)r(r+3)}$	$\frac{9}{10}$
a_3	$\frac{-20r^2-40r+24}{(r+4)(r^2+r-2)r(r+3)}$	$-\frac{17}{30}$
a_4	$\frac{61r^2+183r-108}{(r+5)(r+2)(r+3)r(-1+r)(r+4)}$	$\frac{251}{840}$
a_5	$\frac{-182r^2-728r+408}{(r+6)(r+3)r(r+2)(-1+r)(r+4)(r+5)}$	$-\frac{37}{280}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2\left(1 - x + \frac{9x^2}{10} - \frac{17x^3}{30} + \frac{251x^4}{840} - \frac{37x^5}{280} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= \frac{-20r^2 - 40r + 24}{(r+4)(r^2+r-2)\, r\, (r+3)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{-20r^2 - 40r + 24}{(r+4)(r^2+r-2)\, r\, (r+3)} &= \lim_{r \to -1} \frac{-20r^2 - 40r + 24}{(r+4)(r^2+r-2)\, r\, (r+3)} \\
&= \frac{11}{3}
\end{aligned}
$$

The limit is $\frac{11}{3}$. Since the limit exists then the log term is not needed and we can set

$C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = -\frac{2r}{r^2 + r - 2}
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + 2b_{n-1}(n+r-1) - 3b_{n-2} - 2b_n = 0 \tag{4}
$$

Which for for the root $r = -1$ becomes

$$
b_n(n-1)(n-2) + 2b_{n-1}(n-2) - 3b_{n-2} - 2b_n = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{2nb_{n-1} + 2rb_{n-1} - 3b_{n-2} - 2b_{n-1}}{n^2 + 2nr + r^2 - n - r - 2} \tag{5}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{2nb_{n-1} - 3b_{n-2} - 4b_{n-1}}{n^2 - 3n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1

For $n = 2$, using the above recursive equation gives

$$
b_2 = \frac{7r^2 + 7r - 6}{(r^2 + r - 2)\, r\, (r+3)}
$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{3}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1
b_2	$\frac{7r^2+7r-6}{(r^2+r-2)r(r+3)}$	$-\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{4(5r^2 + 10r - 6)}{(r+4)(r^2+r-2)\,r\,(r+3)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{11}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1
b_2	$\frac{7r^2+7r-6}{(r^2+r-2)r(r+3)}$	$-\frac{3}{2}$
b_3	$\frac{-20r^2-40r+24}{(r+4)(r^2+r-2)r(r+3)}$	$\frac{11}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{61r^2 + 183r - 108}{(r^2+7r+10)(r+3)\,r\,(-1+r)(r+4)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{115}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1
b_2	$\frac{7r^2+7r-6}{(r^2+r-2)r(r+3)}$	$-\frac{3}{2}$
b_3	$\frac{-20r^2-40r+24}{(r+4)(r^2+r-2)r(r+3)}$	$\frac{11}{3}$
b_4	$\frac{61r^2+183r-108}{(r+5)(r+2)(r+3)r(-1+r)(r+4)}$	$-\frac{115}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{2(91r^2 + 364r - 204)}{(r^2 + 9r + 18)\, r\, (r^2 + r - 2)\, (r + 4)\, (r + 5)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{159}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2r}{r^2+r-2}$	-1
b_2	$\frac{7r^2+7r-6}{(r^2+r-2)r(r+3)}$	$-\frac{3}{2}$
b_3	$\frac{-20r^2-40r+24}{(r+4)(r^2+r-2)r(r+3)}$	$\frac{11}{3}$
b_4	$\frac{61r^2+183r-108}{(r+5)(r+2)(r+3)r(-1+r)(r+4)}$	$-\frac{115}{24}$
b_5	$\frac{-182r^2-728r+408}{(r+6)(r+3)r(r+2)(-1+r)(r+4)(r+5)}$	$\frac{159}{40}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^2 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - x - \frac{3x^2}{2} + \frac{11x^3}{3} - \frac{115x^4}{24} + \frac{159x^5}{40} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= c_1 x^2 \left(1 - x + \frac{9x^2}{10} - \frac{17x^3}{30} + \frac{251x^4}{840} - \frac{37x^5}{280} + O(x^6)\right)\\ &\quad + \frac{c_2 \left(1 - x - \frac{3x^2}{2} + \frac{11x^3}{3} - \frac{115x^4}{24} + \frac{159x^5}{40} + O(x^6)\right)}{x}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1 x^2 \left(1 - x + \frac{9x^2}{10} - \frac{17x^3}{30} + \frac{251x^4}{840} - \frac{37x^5}{280} + O(x^6)\right)\\ &\quad + \frac{c_2 \left(1 - x - \frac{3x^2}{2} + \frac{11x^3}{3} - \frac{115x^4}{24} + \frac{159x^5}{40} + O(x^6)\right)}{x}\end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned}y &= c_1 x^2 \left(1 - x + \frac{9x^2}{10} - \frac{17x^3}{30} + \frac{251x^4}{840} - \frac{37x^5}{280} + O(x^6)\right)\\ &\quad + \frac{c_2 \left(1 - x - \frac{3x^2}{2} + \frac{11x^3}{3} - \frac{115x^4}{24} + \frac{159x^5}{40} + O(x^6)\right)}{x}\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y &= c_1 x^2 \left(1 - x + \frac{9x^2}{10} - \frac{17x^3}{30} + \frac{251x^4}{840} - \frac{37x^5}{280} + O(x^6)\right)\\ &\quad + \frac{c_2 \left(1 - x - \frac{3x^2}{2} + \frac{11x^3}{3} - \frac{115x^4}{24} + \frac{159x^5}{40} + O(x^6)\right)}{x}\end{aligned}$$

Verified OK.

24.10.1 Maple step by step solution

Let's solve

$x^2y'' + 2y'x^2 + (-3x^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3x^2+2)y}{x^2} - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' - \frac{(3x^2+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 2, P_3(x) = -\frac{3x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + 2y'x^2 + (-3x^2 - 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$
- Shift index using $k->k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$
- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\, x^r + (a_1(2+r)\,(-1+r) + 2a_0 r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)\,(k+r-2) + 2a_k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term must be 0

  $a_1(2+r)\,(-1+r) + 2a_0 r = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -\frac{2a_0 r}{r^2+r-2}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) + 2a_{k-1}k + 2a_{k-1}r - 3a_{k-2} - 2a_{k-1} = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)\,(k+r) + 2a_{k+1}(k+2) + 2a_{k+1}r - 3a_k - 2a_{k+1} = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2ka_{k+1}+2a_{k+1}r-3a_k+2a_{k+1}}{(k+3+r)(k+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{2ka_{k+1}-3a_k}{(k+2)(k-1)}$

- Series not valid for $r=-1$, division by 0 in the recursion relation at $k=1$

$a_{k+2}=-\frac{2ka_{k+1}-3a_k}{(k+2)(k-1)}$

- Recursion relation for $r=2$

$a_{k+2}=-\frac{2ka_{k+1}-3a_k+6a_{k+1}}{(k+5)(k+2)}$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{2ka_{k+1}-3a_k+6a_{k+1}}{(k+5)(k+2)}, a_1=-a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+2*x^2*diff(y(x),x)-(3*x^2+2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1x^2\left(1-x+\frac{9}{10}x^2-\frac{17}{30}x^3+\frac{251}{840}x^4-\frac{37}{280}x^5+\mathrm{O}\left(x^6\right)\right)+\frac{c_2\left(12-12x-18x^2+44x^3-\frac{115}{2}x^4+\frac{477}{10}x^5+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 64

```
AsymptoticDSolveValue[x^2*y''[x]+2*x^2*y'[x]-(3*x^2+2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{115x^3}{24}+\frac{11x^2}{3}-\frac{3x}{2}+\frac{1}{x}-1\right)+c_2\left(\frac{251x^6}{840}-\frac{17x^5}{30}+\frac{9x^4}{10}-x^3+x^2\right)$$

24.11 problem 11

Internal problem ID [2412]
Internal file name [OUTPUT/2412_Tuesday_February_27_2024_08_37_03_AM_26828799/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-x)\,y''+x(x+1)\,y'-9y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^3+x^2\right)y''+\left(x^2+x\right)y'-9y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{x+1}{x\,(x-1)}$$
$$q(x)=\frac{9}{x^2\,(x-1)}$$

Table 699: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x+1}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{9}{x^2(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x^2(x-1) + (x^2 + x)\, y' - 9y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x-1) + (x^2 + x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 9\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-9 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-9 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 9 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - 9 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - 9 x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 9\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 9 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 3 \\ r_2 &= -3 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 9\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^3 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_n(n+r) - 9a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-1)}{n+r+3} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = \frac{a_{n-1}(n+2)}{n+6} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r}{4+r}$$

Which for the root $r = 3$ becomes

$$a_1 = \frac{3}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{4+r}$	$\frac{3}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(1+r)\,r}{(4+r)\,(5+r)}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{3}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{4+r}$	$\frac{3}{7}$
a_2	$\frac{(1+r)r}{(4+r)(5+r)}$	$\frac{3}{14}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(1+r)\, r(2+r)}{(4+r)\,(5+r)\,(6+r)}$$

Which for the root $r = 3$ becomes

$$a_3 = \frac{5}{42}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{4+r}$	$\frac{3}{7}$
a_2	$\frac{(1+r)r}{(4+r)(5+r)}$	$\frac{3}{14}$
a_3	$\frac{(1+r)r(2+r)}{(4+r)(5+r)(6+r)}$	$\frac{5}{42}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(1+r)\, r(2+r)\,(r+3)}{(4+r)\,(5+r)\,(6+r)\,(7+r)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{4+r}$	$\frac{3}{7}$
a_2	$\frac{(1+r)r}{(4+r)(5+r)}$	$\frac{3}{14}$
a_3	$\frac{(1+r)r(2+r)}{(4+r)(5+r)(6+r)}$	$\frac{5}{42}$
a_4	$\frac{(1+r)r(2+r)(r+3)}{(4+r)(5+r)(6+r)(7+r)}$	$\frac{1}{14}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(r+3)\, r(1+r)\,(2+r)}{(7+r)\,(6+r)\,(5+r)\,(8+r)}$$

Which for the root $r = 3$ becomes

$$a_5 = \frac{1}{22}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{4+r}$	$\frac{3}{7}$
a_2	$\frac{(1+r)r}{(4+r)(5+r)}$	$\frac{3}{14}$
a_3	$\frac{(1+r)r(2+r)}{(4+r)(5+r)(6+r)}$	$\frac{5}{42}$
a_4	$\frac{(1+r)r(2+r)(r+3)}{(4+r)(5+r)(6+r)(7+r)}$	$\frac{1}{14}$
a_5	$\frac{(r+3)r(1+r)(2+r)}{(7+r)(6+r)(5+r)(8+r)}$	$\frac{1}{22}$

For $n = 6$, using the above recursive equation gives

$$a_6 = \frac{(r+3)\,(1+r)\, r(2+r)}{(8+r)\,(7+r)\,(6+r)\,(9+r)}$$

Which for the root $r = 3$ becomes

$$a_6 = \frac{1}{33}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r}{4+r}$	$\frac{3}{7}$
a_2	$\frac{(1+r)r}{(4+r)(5+r)}$	$\frac{3}{14}$
a_3	$\frac{(1+r)r(2+r)}{(4+r)(5+r)(6+r)}$	$\frac{5}{42}$
a_4	$\frac{(1+r)r(2+r)(r+3)}{(4+r)(5+r)(6+r)(7+r)}$	$\frac{1}{14}$
a_5	$\frac{(r+3)r(1+r)(2+r)}{(7+r)(6+r)(5+r)(8+r)}$	$\frac{1}{22}$
a_6	$\frac{(r+3)(1+r)r(2+r)}{(8+r)(7+r)(6+r)(9+r)}$	$\frac{1}{33}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 \dots\right)\\
&= x^3\left(1 + \frac{3x}{7} + \frac{3x^2}{14} + \frac{5x^3}{42} + \frac{x^4}{14} + \frac{x^5}{22} + \frac{x^6}{33} + O(x^7)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_6\\
&= \frac{(r+3)(1+r)\,r(2+r)}{(8+r)(7+r)(6+r)(9+r)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{(r+3)(1+r)\,r(2+r)}{(8+r)(7+r)(6+r)(9+r)} &= \lim_{r\to -3} \frac{(r+3)(1+r)\,r(2+r)}{(8+r)(7+r)(6+r)(9+r)}\\
&= 0
\end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r}\\
&= \sum_{n=0}^{\infty} b_n x^{n-3}
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}
&-b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1)\\
&+ b_{n-1}(n+r-1) + b_n(n+r) - 9b_n = 0
\end{aligned} \tag{4}$$

Which for for the root $r = -3$ becomes

$$-b_{n-1}(n-4)(n-5) + b_n(n-3)(n-4) + b_{n-1}(n-4) + b_n(n-3) - 9b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-1}(n+r-1)}{n+r+3} \tag{5}$$

Which for the root $r = -3$ becomes

$$b_n = \frac{b_{n-1}(n-4)}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{r}{4+r}$$

Which for the root $r = -3$ becomes

$$b_1 = -3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{4+r}$	-3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(1+r)\,r}{(4+r)\,(5+r)}$$

Which for the root $r = -3$ becomes

$$b_2 = 3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{4+r}$	-3
b_2	$\frac{(1+r)r}{(4+r)(5+r)}$	3

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(1+r)\, r(2+r)}{(4+r)\,(5+r)\,(6+r)}$$

Which for the root $r = -3$ becomes

$$b_3 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{4+r}$	-3
b_2	$\frac{(1+r)r}{(4+r)(5+r)}$	3
b_3	$\frac{(1+r)r(2+r)}{(4+r)(5+r)(6+r)}$	-1

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(1+r)\, r(2+r)\,(r+3)}{(4+r)\,(5+r)\,(6+r)\,(7+r)}$$

Which for the root $r = -3$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{4+r}$	-3
b_2	$\frac{(1+r)r}{(4+r)(5+r)}$	3
b_3	$\frac{(1+r)r(2+r)}{(4+r)(5+r)(6+r)}$	-1
b_4	$\frac{(1+r)r(2+r)(r+3)}{(4+r)(5+r)(6+r)(7+r)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(r+3)\, r(1+r)\,(2+r)}{(7+r)\,(6+r)\,(5+r)\,(8+r)}$$

Which for the root $r = -3$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{4+r}$	-3
b_2	$\frac{(1+r)r}{(4+r)(5+r)}$	3
b_3	$\frac{(1+r)r(2+r)}{(4+r)(5+r)(6+r)}$	-1
b_4	$\frac{(1+r)r(2+r)(r+3)}{(4+r)(5+r)(6+r)(7+r)}$	0
b_5	$\frac{(r+3)r(1+r)(2+r)}{(7+r)(6+r)(5+r)(8+r)}$	0

For $n = 6$, using the above recursive equation gives

$$b_6 = \frac{(r+3)\,(1+r)\, r(2+r)}{(8+r)\,(7+r)\,(6+r)\,(9+r)}$$

Which for the root $r = -3$ becomes

$$b_6 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r}{4+r}$	-3
b_2	$\frac{(1+r)r}{(4+r)(5+r)}$	3
b_3	$\frac{(1+r)r(2+r)}{(4+r)(5+r)(6+r)}$	-1
b_4	$\frac{(1+r)r(2+r)(r+3)}{(4+r)(5+r)(6+r)(7+r)}$	0
b_5	$\frac{(r+3)r(1+r)(2+r)}{(7+r)(6+r)(5+r)(8+r)}$	0
b_6	$\frac{(r+3)(1+r)r(2+r)}{(8+r)(7+r)(6+r)(9+r)}$	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^3\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 \dots\right) \\
&= \frac{1 - 3x + 3x^2 - x^3 + O(x^7)}{x^3}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3\left(1+\frac{3x}{7}+\frac{3x^2}{14}+\frac{5x^3}{42}+\frac{x^4}{14}+\frac{x^5}{22}+\frac{x^6}{33}+O(x^7)\right)+\frac{c_2(1-3x+3x^2-x^3+O(x^7))}{x^3}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3\left(1+\frac{3x}{7}+\frac{3x^2}{14}+\frac{5x^3}{42}+\frac{x^4}{14}+\frac{x^5}{22}+\frac{x^6}{33}+O(x^7)\right)+\frac{c_2(1-3x+3x^2-x^3+O(x^7))}{x^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^3\left(1+\frac{3x}{7}+\frac{3x^2}{14}+\frac{5x^3}{42}+\frac{x^4}{14}+\frac{x^5}{22}+\frac{x^6}{33}+O(x^7)\right) \\
+\frac{c_2(1-3x+3x^2-x^3+O(x^7))}{x^3}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3\left(1+\frac{3x}{7}+\frac{3x^2}{14}+\frac{5x^3}{42}+\frac{x^4}{14}+\frac{x^5}{22}+\frac{x^6}{33}+O(x^7)\right)+\frac{c_2(1-3x+3x^2-x^3+O(x^7))}{x^3}
$$

Verified OK.

24.11.1 Maple step by step solution

Let's solve

$-y'' x^2(x-1) + (x^2+x)\, y' - 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{9y}{x^2(x-1)} + \frac{(x+1)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(x+1)y'}{x(x-1)} + \frac{9y}{x^2(x-1)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x) = -\frac{x+1}{x(x-1)}, P_3(x) = \frac{9}{x^2(x-1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -9$

- ○ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x^2(x-1) - x(x+1)\,y' + 9y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- ○ Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(3+r)(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+3)(k+r-3) + a_{k-1}(k+r-1)(k+r-3)\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(3+r)(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $-((-k-r+1)\, a_{k-1} + a_k(k+r+3))(k+r-3) = 0$
- Shift index using $k- >k+1$

  $-((-k-r)\, a_k + a_{k+1}(k+4+r))(k+r-2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k+r)a_k}{k+4+r}$
- Recursion relation for $r = -3$; series terminates at $k = 3$

  $a_{k+1} = \frac{(k-3)a_k}{k+1}$
- Apply recursion relation for $k = 0$

  $a_1 = -3a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -a_1$
- Express in terms of a_0

  $a_2 = 3a_0$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$
- Express in terms of a_0

  $a_3 = -a_0$
- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot (1 - 3x + 3x^2 - x^3)$
- Recursion relation for $r = 3$

$a_{k+1} = \frac{(k+3)a_k}{k+7}$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{(k+3)a_k}{k+7}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot (1 - 3x + 3x^2 - x^3) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), b_{k+1} = \frac{(k+3)b_k}{k+7}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 43

```
Order:=6;
dsolve(x^2*(1-x)*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-9*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 + \frac{3}{7}x + \frac{3}{14}x^2 + \frac{5}{42}x^3 + \frac{1}{14}x^4 + \frac{1}{22}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2(-86400 + 259200x - 259200x^2 + 86400x^3 + \mathrm{O}\left(x^6\right))}{x^3}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 57

```
AsymptoticDSolveValue[x^2*(1-x)*y''[x]+x*(1+x)*y'[x]-9*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{x^3} - \frac{3}{x^2} + \frac{3}{x} - 1\right) + c_2\left(\frac{x^7}{14} + \frac{5x^6}{42} + \frac{3x^5}{14} + \frac{3x^4}{7} + x^3\right)$$

24.12 problem 12

Internal problem ID [2413]
Internal file name [OUTPUT/2413_Tuesday_February_27_2024_08_37_05_AM_76053711/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{\left(-x^2+x\right) y''-3y'+2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^2+x\right) y''-3y'+2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x\left(x-1\right)} \\ q(x) &= -\frac{2}{x\left(x-1\right)} \end{aligned}$$

Table 701: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{2}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) - 3y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x(x-1) - 3\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)(n+r-2) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-4+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-4+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 4$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-4+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^4\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+4}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) - 3a_n(n+r) + 2a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r-3)\, a_{n-1}}{n-4+r} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = \frac{(n+1)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1+r}{r-3}$$

Which for the root $r = 4$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r}{r - 3}$$

Which for the root $r = 4$ becomes

$$a_3 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1 + r}{r - 3}$$

Which for the root $r = 4$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{2 + r}{r - 3}$$

Which for the root $r = 4$ becomes

$$a_5 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+r}{r-3}$	2
a_2	$\frac{-1+r}{r-3}$	3
a_3	$\frac{r}{r-3}$	4
a_4	$\frac{1+r}{r-3}$	5
a_5	$\frac{2+r}{r-3}$	6

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^4\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^4\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{1+r}{r-3} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{1+r}{r-3} &= \lim_{r\to 0} \frac{1+r}{r-3} \\ &= -\frac{1}{3} \end{aligned}$$

The limit is $-\frac{1}{3}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$-b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) - 3(n+r)b_n + 2b_{n-1} = 0 \quad (4)$$

Which for for the root $r = 0$ becomes

$$-b_{n-1}(n-1)(n-2) + b_n n(n-1) - 3nb_n + 2b_{n-1} = 0 \quad (4A)$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{(n+r-3)\,b_{n-1}}{n-4+r} \quad (5)$$

Which for the root $r = 0$ becomes

$$b_n = \frac{(n-3)\,b_{n-1}}{n-4} \quad (6)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-2+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{-1+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{2+r}{r-3}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+r}{r-3}$	$\frac{2}{3}$
b_2	$\frac{-1+r}{r-3}$	$\frac{1}{3}$
b_3	$\frac{r}{r-3}$	0
b_4	$\frac{1+r}{r-3}$	$-\frac{1}{3}$
b_5	$\frac{2+r}{r-3}$	$-\frac{2}{3}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + \frac{2x}{3} + \frac{x^2}{3} - \frac{x^4}{3} - \frac{2x^5}{3} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^4 \left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right)+c_2\left(1+\frac{2x}{3}+\frac{x^2}{3}-\frac{x^4}{3}-\frac{2x^5}{3}+O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^4 \left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right)+c_2\left(1+\frac{2x}{3}+\frac{x^2}{3}-\frac{x^4}{3}-\frac{2x^5}{3}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x^4 \left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right) \\
& + c_2\left(1+\frac{2x}{3}+\frac{x^2}{3}-\frac{x^4}{3}-\frac{2x^5}{3}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^4 \left(1+2x+3x^2+4x^3+5x^4+6x^5+O(x^6)\right)+c_2\left(1+\frac{2x}{3}+\frac{x^2}{3}-\frac{x^4}{3}-\frac{2x^5}{3}+O(x^6)\right)
$$

Verified OK.

24.12.1 Maple step by step solution

Let's solve

$-y''x(x-1)-3y'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2y}{x(x-1)}-\frac{3y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x(x-1)}-\frac{2y}{x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3}{x(x-1)}, P_3(x)=-\frac{2}{x(x-1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0}=-3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x(x-1) + 3y' - 2y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - Shift index using $k->k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-4+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(k-3+r) + a_k(k+1+r)(k+r-2)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-4+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 4\}$$

- Each term in the series must be 0, giving the recursion relation

$$((-k-r+3)\,a_{k+1}+a_k(k+r-2))\,(k+1+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+r-2)}{k-3+r}$$

- Recursion relation for $r=0$; series terminates at $k=2$

$$a_{k+1}=\frac{a_k(k-2)}{k-3}$$

- Apply recursion relation for $k=0$

$$a_1=\frac{2a_0}{3}$$

- Apply recursion relation for $k=1$

$$a_2=\frac{a_1}{2}$$

- Express in terms of a_0

$$a_2=\frac{a_0}{3}$$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

$$y=a_0\cdot\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)$$

- Recursion relation for $r=4$

$$a_{k+1}=\frac{a_k(k+2)}{k+1}$$

- Solution for $r=4$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+4}, a_{k+1}=\frac{a_k(k+2)}{k+1}\right]$$

- Combine solutions and rename parameters

$$\left[y=a_0\cdot\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+4}\right), b_{k+1}=\frac{b_k(k+2)}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 42

```
Order:=6;
dsolve((x-x^2)*diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^4 \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(-144 - 96x - 48x^2 + 48x^4 + 96x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 55

```
AsymptoticDSolveValue[(x-x^2)*y''[x]-3*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{x^4}{3} + \frac{x^2}{3} + \frac{2x}{3} + 1\right) + c_2 \left(5x^8 + 4x^7 + 3x^6 + 2x^5 + x^4\right)$$

24.13 problem 13

Internal problem ID [2414]
Internal file name [`OUTPUT/2414_Tuesday_February_27_2024_08_37_05_AM_38181934/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2 y'' + x(x-7)\,y' + (x+12)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2 y'' + \left(x^2 - 7x\right) y' + (x+12)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x-7}{x}$$
$$q(x) = \frac{x+12}{x^2}$$

Table 703: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-7}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x+12}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^2 - 7x)\, y' + (x + 12)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2} \right) + (x^2 - 7x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} \right) + (x + 12) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-7x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-7x^{n+r} a_n (n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 12 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 7x^{n+r} a_n (n+r) + 12 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 7x^r a_0 r + 12 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 7x^r r + 12x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)(r-6)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)(r-6)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 6 \\ r_2 &= 2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)(r-6)x^r=0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^{6}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + x^{2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+6} \\ y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+2}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - 7a_n(n+r) + a_{n-1} + 12a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r)}{n^2+2nr+r^2-8n-8r+12} \tag{4}$$

Which for the root $r = 6$ becomes

$$a_n = -\frac{a_{n-1}(n+6)}{n\,(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 6$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-r}{r^2-6r+5}$$

Which for the root $r = 6$ becomes

$$a_1 = -\frac{7}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2-6r+5}$	$-\frac{7}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2+3r+2}{(-1+r)\,(r-5)\,r\,(r-4)}$$

Which for the root $r = 6$ becomes

$$a_2 = \frac{14}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2-6r+5}$	$-\frac{7}{5}$
a_2	$\frac{r^2+3r+2}{(-1+r)(r-5)r(r-4)}$	$\frac{14}{15}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(2+r)(3+r)}{(r-3)(r-4)r(r-5)(-1+r)}$$

Which for the root $r = 6$ becomes

$$a_3 = -\frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2-6r+5}$	$-\frac{7}{5}$
a_2	$\frac{r^2+3r+2}{(-1+r)(r-5)r(r-4)}$	$\frac{14}{15}$
a_3	$-\frac{(2+r)(3+r)}{(r-3)(r-4)r(r-5)(-1+r)}$	$-\frac{2}{5}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(4+r)(3+r)}{(r-2)(r-3)(r-4)r(r-5)(-1+r)}$$

Which for the root $r = 6$ becomes

$$a_4 = \frac{1}{8}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2-6r+5}$	$-\frac{7}{5}$
a_2	$\frac{r^2+3r+2}{(-1+r)(r-5)r(r-4)}$	$\frac{14}{15}$
a_3	$-\frac{(2+r)(3+r)}{(r-3)(r-4)r(r-5)(-1+r)}$	$-\frac{2}{5}$
a_4	$\frac{(4+r)(3+r)}{(r-2)(r-3)(r-4)r(r-5)(-1+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(5+r)(4+r)}{(-1+r)^2(r-2)(r-3)(r-4)r(r-5)}$$

Which for the root $r = 6$ becomes

$$a_5 = -\frac{11}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{r^2-6r+5}$	$-\frac{7}{5}$
a_2	$\frac{r^2+3r+2}{(-1+r)(r-5)r(r-4)}$	$\frac{14}{15}$
a_3	$-\frac{(2+r)(3+r)}{(r-3)(r-4)r(r-5)(-1+r)}$	$-\frac{2}{5}$
a_4	$\frac{(4+r)(3+r)}{(r-2)(r-3)(r-4)r(r-5)(-1+r)}$	$\frac{1}{8}$
a_5	$-\frac{(5+r)(4+r)}{(-1+r)^2(r-2)(r-3)(r-4)r(r-5)}$	$-\frac{11}{360}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^6\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^6\left(1 - \frac{7x}{5} + \frac{14x^2}{15} - \frac{2x^3}{5} + \frac{x^4}{8} - \frac{11x^5}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{(4+r)(3+r)}{(r-2)(r-3)(r-4)r(r-5)(-1+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{(4+r)(3+r)}{(r-2)(r-3)(r-4)r(r-5)(-1+r)} &= \lim_{r\to 2} \frac{(4+r)(3+r)}{(r-2)(r-3)(r-4)r(r-5)(-1+r)} \\ &= \mathit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y'' + (x^2 - 7x)\,y' + (x+12)\,y = 0$ gives

$$\begin{aligned}
&x^2\Bigg(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) \\
&+ (x^2 - 7x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ (x+12)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(\left(x^{2} y_{1}''(x)+\left(x^{2}-7 x\right) y_{1}'(x)+(x+12) y_{1}(x)\right) \ln (x)+x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)\right.\right.$$
$$\left.+\frac{\left(x^{2}-7 x\right) y_{1}(x)}{x}\right) C+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right) \tag{7}$$
$$+\left(x^{2}-7 x\right)\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+(x+12)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

But since $y_1(x)$ is a solution to the ode, then

$$x^{2} y_{1}''(x)+\left(x^{2}-7 x\right) y_{1}'(x)+(x+12) y_{1}(x)=0$$

Eq (7) simplifes to

$$\left(x^{2}\left(\frac{2 y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)+\frac{\left(x^{2}-7 x\right) y_{1}(x)}{x}\right) C$$
$$+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)^{2}}{x^{2}}-\frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x^{2}}\right)\right) \tag{8}$$
$$+\left(x^{2}-7 x\right)\left(\sum_{n=0}^{\infty} \frac{b_{n} x^{n+r_{2}}\left(n+r_{2}\right)}{x}\right)+(x+12)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_{1}} a_{n}\left(n+r_{1}\right)\right) x+(-8+x)\left(\sum_{n=0}^{\infty} a_{n} x^{n+r_{1}}\right)\right) C$$
$$+\left(\sum_{n=0}^{\infty} x^{-2+n+r_{2}} b_{n}\left(n+r_{2}\right)\left(-1+n+r_{2}\right)\right) x^{2} \tag{9}$$
$$+\left(x^{2}-7 x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_{2}} b_{n}\left(n+r_{2}\right)\right)+(x+12)\left(\sum_{n=0}^{\infty} b_{n} x^{n+r_{2}}\right)=0$$

Since $r_1 = 6$ and $r_2 = 2$ then the above becomes

$$\left(2\left(\sum_{n=0}^{\infty} x^{5+n} a_n (n+6)\right) x + (-8+x)\left(\sum_{n=0}^{\infty} a_n x^{n+6}\right)\right) C + \left(\sum_{n=0}^{\infty} x^n b_n (n+2)(1+n)\right) x^2 + (x^2 - 7x)\left(\sum_{n=0}^{\infty} x^{1+n} b_n (n+2)\right) + (x+12)\left(\sum_{n=0}^{\infty} b_n x^{n+2}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{n+6} a_n (n+6)\right) + \sum_{n=0}^{\infty} \left(-8C\, x^{n+6} a_n\right) + \left(\sum_{n=0}^{\infty} C\, x^{n+7} a_n\right) + \left(\sum_{n=0}^{\infty} x^{n+2} b_n (n^2+3n+2)\right) + \left(\sum_{n=0}^{\infty} x^{n+3} b_n (n+2)\right) + \sum_{n=0}^{\infty} \left(-7x^{n+2} b_n (n+2)\right) + \left(\sum_{n=0}^{\infty} x^{n+3} b_n\right) + \left(\sum_{n=0}^{\infty} 12 b_n x^{n+2}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+2} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+6} a_n (n+6) = \sum_{n=4}^{\infty} 2C a_{n-4} (n+2)\, x^{n+2}$$

$$\sum_{n=0}^{\infty} \left(-8C\, x^{n+6} a_n\right) = \sum_{n=4}^{\infty} \left(-8C a_{n-4} x^{n+2}\right)$$

$$\sum_{n=0}^{\infty} C\, x^{n+7} a_n = \sum_{n=5}^{\infty} C a_{n-5} x^{n+2}$$

$$\sum_{n=0}^{\infty} x^{n+3} b_n (n+2) = \sum_{n=1}^{\infty} b_{n-1} (1+n)\, x^{n+2}$$

$$\sum_{n=0}^{\infty} x^{n+3} b_n = \sum_{n=1}^{\infty} b_{n-1} x^{n+2}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+2$.

$$
\begin{aligned}
&\left(\sum_{n=4}^{\infty} 2 C a_{n-4}(n+2)\, x^{n+2}\right)+\sum_{n=4}^{\infty}\left(-8 C a_{n-4} x^{n+2}\right)+\left(\sum_{n=5}^{\infty} C a_{n-5} x^{n+2}\right) \\
&+\left(\sum_{n=0}^{\infty} x^{n+2} b_{n}\left(n^{2}+3 n+2\right)\right)+\left(\sum_{n=1}^{\infty} b_{n-1}(1+n)\, x^{n+2}\right) \\
&+\sum_{n=0}^{\infty}\left(-7 x^{n+2} b_{n}(n+2)\right)+\left(\sum_{n=1}^{\infty} b_{n-1} x^{n+2}\right)+\left(\sum_{n=0}^{\infty} 12 b_{n} x^{n+2}\right)=0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-3b_1+3b_0=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1+3=0$$

Solving the above for b_1 gives

$$b_1=1$$

For $n=2$, Eq (2B) gives

$$-4b_2+4b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_2+4=0$$

Solving the above for b_2 gives

$$b_2=1$$

For $n=3$, Eq (2B) gives

$$-3b_3+5b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3+5=0$$

Solving the above for b_3 gives

$$b_3=\frac{5}{3}$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + 10 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{5}{2}$$

For $n = 5$, Eq (2B) gives

$$(a_0 + 6a_1) C + 7b_4 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{37}{2} + 5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{37}{10}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{5}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{5}{2}\left(x^6\left(1 - \frac{7x}{5} + \frac{14x^2}{15} - \frac{2x^3}{5} + \frac{x^4}{8} - \frac{11x^5}{360} + O(x^6)\right)\right)\ln(x) + x^2\left(1 + x + x^2 + \frac{5x^3}{3} - \frac{37x^5}{10} + O(x^6)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^6\left(1 - \frac{7x}{5} + \frac{14x^2}{15} - \frac{2x^3}{5} + \frac{x^4}{8} - \frac{11x^5}{360} + O(x^6)\right) \\ &+ c_2\left(-\frac{5}{2}\left(x^6\left(1 - \frac{7x}{5} + \frac{14x^2}{15} - \frac{2x^3}{5} + \frac{x^4}{8} - \frac{11x^5}{360} + O(x^6)\right)\right)\ln(x) \right. \\ &\left. + x^2\left(1 + x + x^2 + \frac{5x^3}{3} - \frac{37x^5}{10} + O(x^6)\right)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^6 \left(1 - \frac{7x}{5} + \frac{14x^2}{15} - \frac{2x^3}{5} + \frac{x^4}{8} - \frac{11x^5}{360} + O(x^6)\right) \\
&+ c_2 \left(-\frac{5x^6 \left(1 - \frac{7x}{5} + \frac{14x^2}{15} - \frac{2x^3}{5} + \frac{x^4}{8} - \frac{11x^5}{360} + O(x^6)\right) \ln(x)}{2} \right. \\
&\left. + x^2 \left(1 + x + x^2 + \frac{5x^3}{3} - \frac{37x^5}{10} + O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^6 \left(1 - \frac{7x}{5} + \frac{14x^2}{15} - \frac{2x^3}{5} + \frac{x^4}{8} - \frac{11x^5}{360} + O(x^6)\right) \\
&+ c_2 \left(-\frac{5x^6 \left(1 - \frac{7x}{5} + \frac{14x^2}{15} - \frac{2x^3}{5} + \frac{x^4}{8} - \frac{11x^5}{360} + O(x^6)\right) \ln(x)}{2} \right. \\
&\left. + x^2 \left(1 + x + x^2 + \frac{5x^3}{3} - \frac{37x^5}{10} + O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^6 \left(1 - \frac{7x}{5} + \frac{14x^2}{15} - \frac{2x^3}{5} + \frac{x^4}{8} - \frac{11x^5}{360} + O(x^6)\right) \\
&+ c_2 \left(-\frac{5x^6 \left(1 - \frac{7x}{5} + \frac{14x^2}{15} - \frac{2x^3}{5} + \frac{x^4}{8} - \frac{11x^5}{360} + O(x^6)\right) \ln(x)}{2} \right. \\
&\left. + x^2 \left(1 + x + x^2 + \frac{5x^3}{3} - \frac{37x^5}{10} + O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

24.13.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 - 7x)\,y' + (x + 12)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+12)y}{x^2} - \frac{(x-7)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-7)y'}{x} + \frac{(x+12)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x-7}{x}, P_3(x) = \frac{x+12}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -7$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 12$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x - 7)\,y' + (x + 12)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)\,(-6+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)\,(k+r-6) + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)\,(-6+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{2, 6\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-2)\,(k+r-6) + a_{k-1}(k+r) = 0$$

- Shift index using $k->k+1$

$$a_{k+1}(k+r-1)\,(k-5+r) + a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+1)}{(k+r-1)(k-5+r)}$$

- Recursion relation for $r = 2$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-3)}$$

- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 3$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-3)}$$

- Recursion relation for $r = 6$

$$a_{k+1} = -\frac{a_k(k+7)}{(k+5)(k+1)}$$

- Solution for $r = 6$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+1}=-\frac{a_k(k+7)}{(k+5)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 63

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(x-7)*diff(y(x),x)+(x+12)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(c_1x^4\left(1-\frac{7}{5}x+\frac{14}{15}x^2-\frac{2}{5}x^3+\frac{1}{8}x^4-\frac{11}{360}x^5+\mathrm{O}\left(x^6\right)\right)\right.$$
$$+c_2\left(\ln\left(x\right)\left(360x^4-504x^5+\mathrm{O}\left(x^6\right)\right)\right.$$
$$\left.\left.+\left(-144-144x-144x^2-240x^3+342x^4+54x^5+\mathrm{O}\left(x^6\right)\right)\right)\right)x^2$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 79

```
AsymptoticDSolveValue[x^2*y''[x]+x*(x-7)*y'[x]+(x+12)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(-\frac{5}{2}x^6\log(x)-\frac{1}{12}\left(21x^4-20x^3-12x^2-12x-12\right)x^2\right)$$
$$+c_2\left(\frac{x^{10}}{8}-\frac{2x^9}{5}+\frac{14x^8}{15}-\frac{7x^7}{5}+x^6\right)$$

24.14 problem 14

Internal problem ID [2415]
Internal file name [`OUTPUT/2415_Tuesday_February_27_2024_08_37_07_AM_35431443/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x+1)\,y''+x(x-4)\,y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+\left(x^2-4x\right)y'+4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x-4}{x\,(x+1)}$$
$$q(x)=\frac{4}{x^2\,(x+1)}$$

Table 705: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-4}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{4}{x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\, y'' + \left(x^2 - 4x\right) y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2(x+1) \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(x^2 - 4x\right) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \tag{2A}$$
$$+ \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)(n+r-2) x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \tag{2B}$$
$$+ \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 4x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 4x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 4x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 5r + 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 5r + 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 4 \\ r_2 &= 1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 5r + 4\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x^4 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+4} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{1+n}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-1}(n + r - 1)(n + r - 2) + a_n(n + r)(n + r - 1) \\ &+ a_{n-1}(n + r - 1) - 4a_n(n + r) + 4a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-1)}{n+r-4} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = -\frac{a_{n-1}(n+3)}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r}{-3+r}$$

Which for the root $r = 4$ becomes

$$a_1 = -4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{-3+r}$	-4

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(1+r)}{(-3+r)(-2+r)}$$

Which for the root $r = 4$ becomes

$$a_2 = 10$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{-3+r}$	-4
a_2	$\frac{r(1+r)}{(-3+r)(-2+r)}$	10

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(2+r)\, r(1+r)}{(-3+r)\,(-2+r)\,(-1+r)}$$

Which for the root $r = 4$ becomes

$$a_3 = -20$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{-3+r}$	-4
a_2	$\frac{r(1+r)}{(-3+r)(-2+r)}$	10
a_3	$-\frac{(2+r)r(1+r)}{(-3+r)(-2+r)(-1+r)}$	-20

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(3+r)\,(1+r)\,(2+r)}{(-3+r)\,(-2+r)\,(-1+r)}$$

Which for the root $r = 4$ becomes

$$a_4 = 35$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{-3+r}$	-4
a_2	$\frac{r(1+r)}{(-3+r)(-2+r)}$	10
a_3	$-\frac{(2+r)r(1+r)}{(-3+r)(-2+r)(-1+r)}$	-20
a_4	$\frac{(3+r)(1+r)(2+r)}{(-3+r)(-2+r)(-1+r)}$	35

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(3+r)\,(2+r)\,(4+r)}{(-3+r)\,(-2+r)\,(-1+r)}$$

Which for the root $r = 4$ becomes

$$a_5 = -56$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{-3+r}$	-4
a_2	$\frac{r(1+r)}{(-3+r)(-2+r)}$	10
a_3	$-\frac{(2+r)r(1+r)}{(-3+r)(-2+r)(-1+r)}$	-20
a_4	$\frac{(3+r)(1+r)(2+r)}{(-3+r)(-2+r)(-1+r)}$	35
a_5	$-\frac{(3+r)(2+r)(4+r)}{(-3+r)(-2+r)(-1+r)}$	-56

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^4\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^4\left(1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= -\frac{(2+r)\, r(1+r)}{(-3+r)\,(-2+r)\,(-1+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} -\frac{(2+r)\, r(1+r)}{(-3+r)\,(-2+r)\,(-1+r)} &= \lim_{r \to 1} -\frac{(2+r)\, r(1+r)}{(-3+r)\,(-2+r)\,(-1+r)} \\ &= \mathit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)\\ \frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\\ &\quad+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)\end{aligned}$$

Substituting these back into the given ode $x^2(x+1)\,y'' + (x^2-4x)\,y' + 4y = 0$ gives

$$\begin{aligned}&x^2(x+1)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right.\\ &\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\ &+ (x^2-4x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)\\ &+ 4Cy_1(x)\ln(x) + 4\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0\end{aligned}$$

Which can be written as

$$\begin{aligned}&\left(\left(x^2(x+1)\,y_1''(x)+\left(x^2-4x\right)y_1'(x)+4y_1(x)\right)\ln\left(x\right)\right.\\&+x^2(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(x^2-4x\right)y_1(x)}{x}\Bigg)\,C\\&+x^2(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\&+\left(x^2-4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+4\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2(x+1)\,y_1''(x)+\left(x^2-4x\right)y_1'(x)+4y_1(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(x^2(x+1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{\left(x^2-4x\right)y_1(x)}{x}\right)C\\&+x^2(x+1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\&+\left(x^2-4x\right)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+4\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\left(2x(x+1)\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)-5\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C\\&+x^2(x+1)\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)\,(-1+n+r_2)\right)\\&+\left(x^2-4x\right)\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)+4\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0\end{aligned}\tag{9}$$

Since $r_1 = 4$ and $r_2 = 1$ then the above becomes

$$\begin{aligned} &\left(2x(x+1)\left(\sum_{n=0}^{\infty} x^{n+3} a_n (n+4)\right) - 5\left(\sum_{n=0}^{\infty} a_n x^{n+4}\right)\right) C \\ &+ x^2(x+1)\left(\sum_{n=0}^{\infty} x^{n-1} b_n (1+n)\, n\right) \\ &+ \left(x^2-4x\right)\left(\sum_{n=0}^{\infty} x^n b_n (1+n)\right) + 4\left(\sum_{n=0}^{\infty} b_n x^{1+n}\right) = 0 \end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n (n+4)\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4)\right) + \sum_{n=0}^{\infty} \left(-5C a_n x^{n+4}\right) \\ &+ \left(\sum_{n=0}^{\infty} n\, x^{n+2} b_n (1+n)\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (1+n)\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n+2} b_n (1+n)\right) + \sum_{n=0}^{\infty} \left(-4x^{1+n} b_n (1+n)\right) + \left(\sum_{n=0}^{\infty} 4 b_n x^{1+n}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $1+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{1+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+5} a_n (n+4) = \sum_{n=4}^{\infty} 2C a_{-4+n} n\, x^{1+n}$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4) = \sum_{n=3}^{\infty} 2C a_{n-3} (1+n)\, x^{1+n}$$

$$\sum_{n=0}^{\infty} \left(-5C a_n x^{n+4}\right) = \sum_{n=3}^{\infty} \left(-5C a_{n-3} x^{1+n}\right)$$

$$\sum_{n=0}^{\infty} n\, x^{n+2} b_n (1+n) = \sum_{n=1}^{\infty} (n-1)\, b_{n-1} n\, x^{1+n}$$

$$\sum_{n=0}^{\infty} x^{n+2} b_n (1+n) = \sum_{n=1}^{\infty} b_{n-1} n\, x^{1+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $1+n$.

$$\left(\sum_{n=4}^{\infty} 2C a_{-4+n} n\, x^{1+n}\right) + \left(\sum_{n=3}^{\infty} 2C a_{n-3}(1+n)\, x^{1+n}\right) + \sum_{n=3}^{\infty} \left(-5C a_{n-3} x^{1+n}\right) + \left(\sum_{n=1}^{\infty} (n-1)\, b_{n-1} n\, x^{1+n}\right) + \left(\sum_{n=0}^{\infty} n\, x^{1+n} b_n (1+n)\right) + \left(\sum_{n=1}^{\infty} b_{n-1} n\, x^{1+n}\right) + \sum_{n=0}^{\infty} \left(-4x^{1+n} b_n (1+n)\right) + \left(\sum_{n=0}^{\infty} 4 b_n x^{1+n}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-2b_1 + b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1 + 1 = 0$$

Solving the above for b_1 gives

$$b_1 = \frac{1}{2}$$

For $n=2$, Eq (2B) gives

$$4b_1 - 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2 - 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = 1$$

For $n=N$, where $N=3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n=3$, Eq (2B) gives

$$3C + 9 = 0$$

Which is solved for C. Solving for C gives

$$C = -3$$

For $n=4$, Eq (2B) gives

$$(8a_0 + 5a_1)\, C + 16b_3 + 4b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$36+4b_4=0$$

Solving the above for b_4 gives

$$b_4=-9$$

For $n=5$, Eq (2B) gives

$$(10a_1+7a_2)\,C+25b_4+10b_5=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-315+10b_5=0$$

Solving the above for b_5 gives

$$b_5=\frac{63}{2}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)$$

Using the above value found for $C=-3$ and all b_n, then the second solution becomes

$$\begin{aligned}y_2(x)&=(-3)\left(x^4\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right)\right)\ln(x)\\&\quad+x\left(1+\frac{x}{2}+x^2-9x^4+\frac{63x^5}{2}+O\left(x^6\right)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x)&=c_1y_1(x)+c_2y_2(x)\\&=c_1x^4\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right)\\&\quad+c_2\Bigg((-3)\left(x^4\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right)\right)\ln(x)\\&\quad+x\left(1+\frac{x}{2}+x^2-9x^4+\frac{63x^5}{2}+O\left(x^6\right)\right)\Bigg)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y&=y_h\\&=c_1x^4\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right)\\&\quad+c_2\Bigg(-3x^4\left(1-4x+10x^2-20x^3+35x^4-56x^5+O\left(x^6\right)\right)\ln(x)\\&\quad+x\left(1+\frac{x}{2}+x^2-9x^4+\frac{63x^5}{2}+O\left(x^6\right)\right)\Bigg)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^4 \left(1-4x+10x^2-20x^3+35x^4-56x^5+O(x^6)\right) + c_2 \left(-3x^4 \left(1-4x+10x^2-20x^3+35x^4-56x^5+O(x^6)\right) \ln(x) + x\left(1+\frac{x}{2}+x^2-9x^4+\frac{63x^5}{2}+O(x^6)\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^4 \left(1-4x+10x^2-20x^3+35x^4-56x^5+O(x^6)\right) + c_2 \left(-3x^4 \left(1-4x+10x^2-20x^3+35x^4-56x^5+O(x^6)\right) \ln(x) + x\left(1+\frac{x}{2}+x^2-9x^4+\frac{63x^5}{2}+O(x^6)\right)\right)$$

Verified OK.

24.14.1 Maple step by step solution

Let's solve

$x^2(x+1)\,y''+(x^2-4x)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x^2(x+1)}-\frac{(x-4)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x-4)y'}{x(x+1)}+\frac{4y}{x^2(x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x-4}{x(x+1)}, P_3(x)=\frac{4}{x^2(x+1)}\right]$

  ◦ $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.\left((x+1)\cdot P_2(x)\right)\right|_{x=-1}=5$

  ◦ $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(x+1)\, y'' + x(x-4)\, y' + 4y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 6u + 5)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(4+r)\, u^{-1+r} + \left(a_1(1+r)\,(5+r) - 2a_0(r^2+2r-2)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+5+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(4+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-4, 0\}$

- Each term must be 0

  $a_1(1+r)(5+r) - 2a_0(r^2+2r-2) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(k+r-1)^2 + a_{k+1}(k+1+r)(k+5+r) - 2(k^2+(2r+2)k+r^2+2r-2)a_k = 0$

- Shift index using $k- >k+1$

  $a_k(k+r)^2 + a_{k+2}(k+2+r)(k+6+r) - 2\left((k+1)^2+(2r+2)(k+1)+r^2+2r-2\right)a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2kra_k-4kra_{k+1}+r^2a_k-2r^2a_{k+1}-8ka_{k+1}-8ra_{k+1}-2a_{k+1}}{(k+2+r)(k+6+r)}$

- Recursion relation for $r = -4$

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-8ka_k+8ka_{k+1}+16a_k-2a_{k+1}}{(k-2)(k+2)}$

- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-8ka_k+8ka_{k+1}+16a_k-2a_{k+1}}{(k-2)(k+2)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-8ka_{k+1}-2a_{k+1}}{(k+2)(k+6)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-8ka_{k+1}-2a_{k+1}}{(k+2)(k+6)}, 5a_1+4a_0 = 0\right]$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-8ka_{k+1}-2a_{k+1}}{(k+2)(k+6)}, 5a_1+4a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 61

```
Order:=6;
dsolve(x^2*(x+1)*diff(y(x),x$2)+x*(x-4)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(c_1 x^3 \left(1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(\left(-36\right) x^3 + 144x^4 - 360x^5 + \mathrm{O}\left(x^6\right)\right) + \left(12 + 6x + 12x^2 - 240x^3 + 852x^4 - 2022x^5 + \mathrm{O}\left(x^6\right)\right)\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 70

```
AsymptoticDSolveValue[x^2*(x+1)*y''[x]+x*(x-4)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(3(4x-1)x^4 \log(x) + \frac{1}{2}\left(62x^4 - 20x^3 + 2x^2 + x + 2\right) x\right) + c_2 \left(35x^8 - 20x^7 + 10x^6 - 4x^5 + x^4\right)$$

24.15 problem 15

Internal problem ID [2416]
Internal file name [`OUTPUT/2416_Tuesday_February_27_2024_08_37_09_AM_91422767/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 42, page 206
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(-x^2+3)\,y' - 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + (-x^3+3x)\,y' - 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x^2-3}{x}$$
$$q(x) = -\frac{3}{x^2}$$

Table 707: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2-3}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{3}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + \left(-x^3 + 3x\right) y' - 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \left(-x^3 + 3x\right) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n(n+r)\right) = \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-3a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) + 3x^{n+r} a_n(n+r) - 3a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r - 3a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 3x^r r - 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 2r - 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 2r - 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 2r - 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) - a_{n-2}(n + r - 2) + 3a_n(n + r) - 3a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}(n+r-2)}{n^2+2nr+r^2+2n+2r-3} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}(n-1)}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r}{r^2+6r+5}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r}{r^2+6r+5}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r}{r^2+6r+5}$	$\frac{1}{12}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(2+r)}{(r^2+6r+5)(r^2+10r+21)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r}{r^2+6r+5}$	$\frac{1}{12}$
a_3	0	0
a_4	$\frac{r(2+r)}{(r^2+6r+5)(r^2+10r+21)}$	$\frac{1}{128}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r}{r^2+6r+5}$	$\frac{1}{12}$
a_3	0	0
a_4	$\frac{r(2+r)}{(r^2+6r+5)(r^2+10r+21)}$	$\frac{1}{128}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{12} + \frac{x^4}{128} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{r(2+r)}{(r^2+6r+5)(r^2+10r+21)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{r(2+r)}{(r^2+6r+5)(r^2+10r+21)} &= \lim_{r \to -3} \frac{r(2+r)}{(r^2+6r+5)(r^2+10r+21)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x^2y''+(-x^3+3x)\,y'-3y=0$ gives

$$x^2\left(Cy_1''(x)\ln(x)+\frac{2Cy_1'(x)}{x}-\frac{Cy_1(x)}{x^2}+\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(-x^3+3x)\left(Cy_1'(x)\ln(x)+\frac{Cy_1(x)}{x}+\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)-3Cy_1(x)\ln(x)-3\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0$$

Which can be written as

$$\left(\left(x^2y_1''(x)+(-x^3+3x)\,y_1'(x)-3y_1(x)\right)\ln(x)+x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(-x^3+3x)\,y_1(x)}{x}\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(-x^3+3x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-3\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2y_1''(x)+(-x^3+3x)\,y_1'(x)-3y_1(x)=0$$

Eq (7) simplifes to

$$\left(x^2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+\frac{(-x^3+3x)\,y_1(x)}{x}\right)C+x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)+(-x^3+3x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-3\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x + \left(-x^2+2\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\ + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\left(-1+n+r_2\right)\right) x^2 \\ + \left(-x^3+3x\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) - 3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = -3$ then the above becomes

$$\left(2\left(\sum_{n=0}^{\infty} x^{n} a_n(n+1)\right) x + \left(-x^2+2\right)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C \\ + \left(\sum_{n=0}^{\infty} x^{-5+n} b_n(n-3)\left(-4+n\right)\right) x^2 \\ + \left(-x^3+3x\right)\left(\sum_{n=0}^{\infty} x^{-4+n} b_n(n-3)\right) - 3\left(\sum_{n=0}^{\infty} b_n x^{n-3}\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n(n+1)\right) + \sum_{n=0}^{\infty}\left(-C\, x^{n+3} a_n\right) + \left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n\right) \\ + \left(\sum_{n=0}^{\infty} x^{n-3} b_n(-4+n)\left(n-3\right)\right) + \sum_{n=0}^{\infty}\left(-x^{n-1} b_n(n-3)\right) \\ + \left(\sum_{n=0}^{\infty} 3x^{n-3} b_n(n-3)\right) + \sum_{n=0}^{\infty}\left(-3b_n x^{n-3}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-3$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-3} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n(n+1) = \sum_{n=4}^{\infty} 2C a_{-4+n}(n-3)\, x^{n-3}$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n+3} a_n\right) = \sum_{n=6}^{\infty} \left(-C a_{n-6} x^{n-3}\right)$$

$$\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n = \sum_{n=4}^{\infty} 2C a_{-4+n} x^{n-3}$$

$$\sum_{n=0}^{\infty} \left(-x^{n-1} b_n(n-3)\right) = \sum_{n=2}^{\infty} \left(-b_{n-2}(-5+n)\, x^{n-3}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-3$.

$$\begin{aligned}&\left(\sum_{n=4}^{\infty} 2C a_{-4+n}(n-3)\, x^{n-3}\right) + \sum_{n=6}^{\infty} \left(-C a_{n-6} x^{n-3}\right) + \left(\sum_{n=4}^{\infty} 2C a_{-4+n} x^{n-3}\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{n-3} b_n(-4+n)\,(n-3)\right) + \sum_{n=2}^{\infty} \left(-b_{n-2}(-5+n)\, x^{n-3}\right) \\ &+ \left(\sum_{n=0}^{\infty} 3x^{n-3} b_n(n-3)\right) + \sum_{n=0}^{\infty} \left(-3b_n x^{n-3}\right) = 0\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n=2$, Eq (2B) gives

$$-4b_2 + 3b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-4b_2 + 3 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$2b_1 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + \frac{3}{4} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{3}{16}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{3}{16}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{3}{16}\left(x\left(1 + \frac{x^2}{12} + \frac{x^4}{128} + O(x^6)\right)\right)\ln(x) + \frac{1 + \frac{3x^2}{4} + O(x^6)}{x^3}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 + \frac{x^2}{12} + \frac{x^4}{128} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{3}{16}\left(x\left(1 + \frac{x^2}{12} + \frac{x^4}{128} + O\left(x^6\right)\right)\right)\ln\left(x\right) + \frac{1 + \frac{3x^2}{4} + O(x^6)}{x^3}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 + \frac{x^2}{12} + \frac{x^4}{128} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{3x\left(1 + \frac{x^2}{12} + \frac{x^4}{128} + O(x^6)\right)\ln\left(x\right)}{16} + \frac{1 + \frac{3x^2}{4} + O(x^6)}{x^3}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1 + \frac{x^2}{12} + \frac{x^4}{128} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{3x\left(1 + \frac{x^2}{12} + \frac{x^4}{128} + O(x^6)\right)\ln\left(x\right)}{16} + \frac{1 + \frac{3x^2}{4} + O(x^6)}{x^3}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1 + \frac{x^2}{12} + \frac{x^4}{128} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{3x\left(1 + \frac{x^2}{12} + \frac{x^4}{128} + O(x^6)\right)\ln\left(x\right)}{16} + \frac{1 + \frac{3x^2}{4} + O(x^6)}{x^3}\right)
\end{aligned}
$$

Verified OK.

24.15.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^3 + 3x)\, y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y}{x^2} + \frac{(x^2-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-3)y'}{x} - \frac{3y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x^2-3}{x}, P_3(x) = -\frac{3}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -3$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(x^2 - 3)\, y' - 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(3+r)\,(-1+r)\, x^{r} + a_{1}(4+r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_{k}(k+r+3)\,(k+r-1) - a_{k-2}(k-2+r)\right) x^{k-}\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 1\}$

- Each term must be 0

  $a_{1}(4+r)\, r = 0$

- Solve for the dependent coefficient(s)

  $a_{1} = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+3)\,(k+r-1) - a_{k-2}(k-2+r) = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+5+r)\,(k+1+r) - a_{k}(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_{k}(k+r)}{(k+5+r)(k+1+r)}$

- Recursion relation for $r=-3$

  $a_{k+2} = \frac{a_{k}(k-3)}{(k+2)(k-2)}$

- Series not valid for $r=-3$, division by 0 in the recursion relation at $k=2$

  $a_{k+2} = \frac{a_{k}(k-3)}{(k+2)(k-2)}$

- Recursion relation for $r=1$

  $a_{k+2} = \frac{a_{k}(k+1)}{(k+6)(k+2)}$

- Solution for $r=1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=\frac{a_k(k+1)}{(k+6)(k+2)}, a_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*(3-x^2)*diff(y(x),x)-3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^4\left(1+\frac{1}{12}x^2+\frac{1}{128}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(\ln\left(x\right)\left(27x^4+\mathrm{O}\left(x^6\right)\right)+\left(-144-108x^2-36x^4+\mathrm{O}\left(x^6\right)\right)\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 53

```
AsymptoticDSolveValue[x^2*y''[x]+x*(3-x^2)*y'[x]-3*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{x^5}{128}+\frac{x^3}{12}+x\right)+c_1\left(\frac{19x^4+48x^2+64}{64x^3}-\frac{3}{16}x\log(x)\right)$$

25 Exercise 43, page 209

25.1 problem 1

Internal problem ID [2417]
Internal file name [OUTPUT/2417_Tuesday_February_27_2024_08_37_10_AM_51190382/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''x + 3y' - y = x}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y''x + 3y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 709: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x + 3y' - y = x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y''x+3y'-y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(x^{-1+m} m(-1+m) + 3m\, x^{-1+m}\right) c_0 = x$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n\,(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{r^2 + 4r + 3}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+4r+3}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r^4 + 10r^3 + 35r^2 + 50r + 24}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+4r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{r^4+10r^3+35r^2+50r+24}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{\left(r+3\right)^2 \left(r+1\right) \left(r+4\right) \left(2+r\right) \left(r+5\right)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+4r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{r^4+10r^3+35r^2+50r+24}$	$\frac{1}{24}$
a_3	$\frac{1}{(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$\frac{1}{360}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{\left(r+3\right)^2 \left(r+1\right) \left(r+4\right)^2 \left(2+r\right) \left(r+5\right) \left(r+6\right)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{8640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+4r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{r^4+10r^3+35r^2+50r+24}$	$\frac{1}{24}$
a_3	$\frac{1}{(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$\frac{1}{360}$
a_4	$\frac{1}{(r+3)^2(r+1)(r+4)^2(2+r)(r+5)(r+6)}$	$\frac{1}{8640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{\left(r+3\right)^2 \left(r+1\right) \left(r+4\right)^2 \left(2+r\right) \left(r+5\right)^2 \left(r+6\right) \left(7+r\right)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{302400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+4r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{r^4+10r^3+35r^2+50r+24}$	$\frac{1}{24}$
a_3	$\frac{1}{(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$\frac{1}{360}$
a_4	$\frac{1}{(r+3)^2(r+1)(r+4)^2(2+r)(r+5)(r+6)}$	$\frac{1}{8640}$
a_5	$\frac{1}{(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)(7+r)}$	$\frac{1}{302400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots \\ &= 1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O\left(x^6\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_2 \\ &= \frac{1}{r^4 + 10r^3 + 35r^2 + 50r + 24}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r \to r_2} \frac{1}{r^4 + 10r^3 + 35r^2 + 50r + 24} &= \lim_{r \to -2} \frac{1}{r^4 + 10r^3 + 35r^2 + 50r + 24} \\ &= \mathit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $y''x + 3y' - y = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + 3Cy_1'(x)\ln(x) \\
&+ \frac{3Cy_1(x)}{x} + 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - Cy_1(x)\ln(x) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left((y_1''(x)\,x - y_1(x) + 3y_1'(x))\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{3y_1(x)}{x}\right)C \\
&+ \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&+ 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
y_1''(x)\,x - y_1(x) + 3y_1'(x) = 0
$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x+\frac{3y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + 3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x+2\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} + \frac{\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x+3\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x}{x} = 0 \tag{9}$$

Since $r_1=0$ and $r_2=-2$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{n-1}a_n n\right)x+2\left(\sum_{n=0}^{\infty}a_n x^{n}\right)\right)C}{x} + \frac{\left(\sum_{n=0}^{\infty}x^{-4+n}b_n(n-2)(-3+n)\right)x^2-\left(\sum_{n=0}^{\infty}b_n x^{n-2}\right)x+3\left(\sum_{n=0}^{\infty}x^{-3+n}b_n(n-2)\right)x}{x}=0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}2C\,x^{n-1}a_n n\right)+\left(\sum_{n=0}^{\infty}2C\,x^{n-1}a_n\right)+\left(\sum_{n=0}^{\infty}x^{-3+n}b_n\left(n^2-5n+6\right)\right) + \sum_{n=0}^{\infty}\left(-b_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty}3x^{-3+n}b_n(n-2)\right)=0 \tag{2A}$$

The next step is to make all powers of x be $-3+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-3+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n = \sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n = \sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} \left(-b_n x^{n-2}\right) = \sum_{n=1}^{\infty} \left(-b_{n-1} x^{-3+n}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-3+n$.

$$\left(\sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}\right) + \left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}\right) + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n \left(n^2-5n+6\right)\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} x^{-3+n}\right) + \left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n (n-2)\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1 - b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 - 1 = 0$$

Solving the above for b_1 gives

$$b_1 = -1$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 - b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 - \frac{2}{3} = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{2}{9}$$

For $n = 4$, Eq (2B) gives

$$6Ca_2 - b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 - \frac{25}{72} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{576}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 - b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 - \frac{157}{2880} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{157}{43200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O(x^6)\right)\ln(x) + \frac{1 - x + \frac{2x^3}{9} + \frac{25x^4}{576} + \frac{157x^5}{43200} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O(x^6)\right) \\
&+ c_2 \left(-\frac{1}{2}\left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O(x^6)\right) \ln(x) \right. \\
&\left. + \frac{1 - x + \frac{2x^3}{9} + \frac{25x^4}{576} + \frac{157x^5}{43200} + O(x^6)}{x^2}\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(x^{-1+m} m(-1+m) + 3m\, x^{-1+m}\right) c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\boxed{
\begin{aligned}
a_1 &= \frac{a_0}{r^2+4r+3} \\
a_2 &= \frac{a_0}{(r^2+4r+3)(r^2+6r+8)} \\
a_3 &= \frac{a_0}{(r+3)^2 (r+1)(r+4)(2+r)(r+5)} \\
a_4 &= \frac{a_0}{(r+3)^2 (r+1)(r+4)^2 (2+r)(r+5)(r+6)} \\
a_5 &= \frac{a_0}{(r+3)^2 (r+1)(r+4)^2 (2+r)(r+5)^2 (r+6)(7+r)}
\end{aligned}
}
$$

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$
\left(x^{-1+m} m(-1+m) + 3m\, x^{-1+m}\right) c_0 = x
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{8} \\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \end{aligned}$$

Where in the above $c_0 = \frac{1}{8}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{8}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\boxed{\begin{aligned} c_0 &= \frac{1}{8} \\ c_1 &= \frac{1}{120} \\ c_2 &= \frac{1}{2880} \\ c_3 &= \frac{1}{100800} \\ c_4 &= \frac{1}{4838400} \\ c_5 &= \frac{1}{304819200} \end{aligned}}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^2 \left(\frac{1}{8} + \frac{1}{120} x + \frac{1}{2880} x^2 + \frac{1}{100800} x^3 + \frac{1}{4838400} x^4 + \frac{1}{304819200} x^5 \right) \\ &= \frac{1}{8} x^2 + \frac{1}{120} x^3 + \frac{1}{2880} x^4 + \frac{1}{100800} x^5 + \frac{1}{4838400} x^6 + \frac{1}{304819200} x^7 \end{aligned}$$

Adding all the above particular solution(s) gives

$$y_p = \frac{x^2}{8}+\frac{x^3}{120}+\frac{x^4}{2880}+\frac{x^5}{100800}+\frac{x^6}{4838400}+\frac{x^7}{304819200}+O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{x^2}{8}+\frac{x^3}{120}+\frac{x^4}{2880}+\frac{x^5}{100800}+O(x^6)$$

Hence the final solution is

$y = y_h + y_p$

$$= \frac{x^2}{8}+\frac{x^3}{120}+\frac{x^4}{2880}+\frac{x^5}{100800}+O(x^6)+c_1\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right) + c_2\left(\left(-\frac{1}{2}-\frac{x}{6}-\frac{x^2}{48}-\frac{x^3}{720}-\frac{x^4}{17280}-\frac{x^5}{604800}-\frac{O(x^6)}{2}\right)\ln(x) + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{x^2}\right)$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{8}+\frac{x^3}{120}+\frac{x^4}{2880}+\frac{x^5}{100800}+O(x^6) + c_1\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right) + c_2\left(\left(-\frac{1}{2}-\frac{x}{6}-\frac{x^2}{48}-\frac{x^3}{720}-\frac{x^4}{17280}-\frac{x^5}{604800}-\frac{O(x^6)}{2}\right)\ln(x) + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{x^2}\right) \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{8}+\frac{x^3}{120}+\frac{x^4}{2880}+\frac{x^5}{100800}+O(x^6)+c_1\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right) + c_2\left(\left(-\frac{1}{2}-\frac{x}{6}-\frac{x^2}{48}-\frac{x^3}{720}-\frac{x^4}{17280}-\frac{x^5}{604800}-\frac{O(x^6)}{2}\right)\ln(x) + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{x^2}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 78

```
Order:=6;
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)-y(x)=x,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1\left(1+\frac{1}{3}x+\frac{1}{24}x^2+\frac{1}{360}x^3+\frac{1}{8640}x^4+\frac{1}{302400}x^5+\mathrm{O}\left(x^6\right)\right) \\
& + \frac{c_2\left(\ln\left(x\right)\left(x^2+\frac{1}{3}x^3+\frac{1}{24}x^4+\frac{1}{360}x^5+\mathrm{O}\left(x^6\right)\right)+\left(-2+2x-\frac{4}{9}x^3-\frac{25}{288}x^4-\frac{157}{21600}x^5+\mathrm{O}\left(x^6\right)\right)\right)}{x^2} \\
& + x^2\left(\frac{1}{8}+\frac{1}{120}x+\frac{1}{2880}x^2+\frac{1}{100800}x^3+\mathrm{O}\left(x^4\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 248

```
AsymptoticDSolveValue[x*y''[x]+3*y'[x]-y[x]==x,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_2\left(x^4\left(\frac{25}{576}-\frac{\log(x)}{48}\right)+x^3\left(\frac{2}{9}-\frac{\log(x)}{6}\right)-\frac{1}{2}x^2\log(x)-x+1\right)}{x^2} + c_1\left(\frac{x^5}{302400}+\frac{x^4}{8640}+\frac{x^3}{360}+\frac{x^2}{24}+\frac{x}{3}+1\right)+\left(\frac{x^5}{302400}+\frac{x^4}{8640}+\frac{x^3}{360}+\frac{x^2}{24}+\frac{x}{3}+1\right)\left(\frac{x^6(9-4\log(x))}{2304}+\frac{1}{900}x^5(23-15\log(x))+\frac{1}{64}x^4(1-4\log(x))-\frac{x^3}{6}+\frac{x^2}{4}\right) + \frac{\left(-\frac{x^6}{288}-\frac{x^5}{30}-\frac{x^4}{8}\right)\left(x^4\left(\frac{25}{576}-\frac{\log(x)}{48}\right)+x^3\left(\frac{2}{9}-\frac{\log(x)}{6}\right)-\frac{1}{2}x^2\log(x)-x+1\right)}{x^2}$$

25.2 problem 1

Internal problem ID [2418]
Internal file name [OUTPUT/2418_Tuesday_February_27_2024_08_37_12_AM_81087829/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y''x + 3y' - y = x}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y''x + 3y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 710: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x + 3y' - y = x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y''x+3y'-y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + 3\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 3r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 3r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(x^{-1+m} m(-1+m) + 3m\, x^{-1+m}\right) c_0 = x$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{r^2 + 4r + 3}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+4r+3}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r^4 + 10r^3 + 35r^2 + 50r + 24}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+4r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{r^4+10r^3+35r^2+50r+24}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{\left(r+3\right)^2 \left(r+1\right) \left(r+4\right) \left(2+r\right) \left(r+5\right)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+4r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{r^4+10r^3+35r^2+50r+24}$	$\frac{1}{24}$
a_3	$\frac{1}{(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$\frac{1}{360}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+3)^2 (r+1) (r+4)^2 (2+r) (r+5) (r+6)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{8640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+4r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{r^4+10r^3+35r^2+50r+24}$	$\frac{1}{24}$
a_3	$\frac{1}{(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$\frac{1}{360}$
a_4	$\frac{1}{(r+3)^2(r+1)(r+4)^2(2+r)(r+5)(r+6)}$	$\frac{1}{8640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(r+3)^2 (r+1) (r+4)^2 (2+r) (r+5)^2 (r+6) (7+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{302400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r^2+4r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{r^4+10r^3+35r^2+50r+24}$	$\frac{1}{24}$
a_3	$\frac{1}{(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$\frac{1}{360}$
a_4	$\frac{1}{(r+3)^2(r+1)(r+4)^2(2+r)(r+5)(r+6)}$	$\frac{1}{8640}$
a_5	$\frac{1}{(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)(7+r)}$	$\frac{1}{302400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O(x^6)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \rightarrow r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= \frac{1}{r^4 + 10r^3 + 35r^2 + 50r + 24}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \rightarrow r_2} \frac{1}{r^4 + 10r^3 + 35r^2 + 50r + 24} &= \lim_{r \rightarrow -2} \frac{1}{r^4 + 10r^3 + 35r^2 + 50r + 24} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $y''x + 3y' - y = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x + 3Cy_1'(x)\ln(x) \\
&+ \frac{3Cy_1(x)}{x} + 3\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - Cy_1(x)\ln(x) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(y_1''(x)\,x - y_1(x) + 3y_1'(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{3y_1(x)}{x}\right)C \\
&+ \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&+ 3\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
y_1''(x)\,x - y_1(x) + 3y_1'(x) = 0
$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x+\frac{3y_1(x)}{x}\right)C \\ +\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\ +3\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x+2\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} \\ +\frac{\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2+3\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x-\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x} \\ =0 \tag{9}$$

Since $r_1=0$ and $r_2=-2$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty}x^{n-1}a_n n\right)x+2\left(\sum_{n=0}^{\infty}a_n x^{n}\right)\right)C}{x} \\ +\frac{\left(\sum_{n=0}^{\infty}x^{-4+n}b_n(n-2)(-3+n)\right)x^2+3\left(\sum_{n=0}^{\infty}x^{-3+n}b_n(n-2)\right)x-\left(\sum_{n=0}^{\infty}b_n x^{n-2}\right)x}{x}=0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}2C\,x^{n-1}a_n n\right)+\left(\sum_{n=0}^{\infty}2C\,x^{n-1}a_n\right)+\left(\sum_{n=0}^{\infty}x^{-3+n}b_n\left(n^2-5n+6\right)\right) \\ +\left(\sum_{n=0}^{\infty}3x^{-3+n}b_n(n-2)\right)+\sum_{n=0}^{\infty}\left(-b_n x^{n-2}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $-3+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-3+n} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n n &= \sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n} \\
\sum_{n=0}^{\infty} 2C\, x^{n-1} a_n &= \sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n} \\
\sum_{n=0}^{\infty} \left(-b_n x^{n-2}\right) &= \sum_{n=1}^{\infty} \left(-b_{n-1} x^{-3+n}\right)
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-3+n$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2C(n-2)\, a_{n-2} x^{-3+n}\right) + \left(\sum_{n=2}^{\infty} 2C a_{n-2} x^{-3+n}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n \left(n^2-5n+6\right)\right) \\
&+ \left(\sum_{n=0}^{\infty} 3x^{-3+n} b_n (n-2)\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} x^{-3+n}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1 - b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 - 1 = 0$$

Solving the above for b_1 gives

$$b_1 = -1$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 - b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 - \frac{2}{3} = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{2}{9}$$

For $n = 4$, Eq (2B) gives

$$6Ca_2 - b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 - \frac{25}{72} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{576}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 - b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 - \frac{157}{2880} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{157}{43200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(1 + \frac{x}{3} + \frac{x^2}{24} + \frac{x^3}{360} + \frac{x^4}{8640} + \frac{x^5}{302400} + O(x^6)\right)\ln(x) + \frac{1 - x + \frac{2x^3}{9} + \frac{25x^4}{576} + \frac{157x^5}{43200} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{2}\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{x^2}\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(x^{-1+m} m(-1+m) + 3m\, x^{-1+m}\right) c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\boxed{
\begin{aligned}
a_1 &= \frac{a_0}{r^2+4r+3} \\
a_2 &= \frac{a_0}{(r^2+4r+3)(r^2+6r+8)} \\
a_3 &= \frac{a_0}{(r+3)^2 (r+1)(r+4)(2+r)(r+5)} \\
a_4 &= \frac{a_0}{(r+3)^2 (r+1)(r+4)^2 (2+r)(r+5)(r+6)} \\
a_5 &= \frac{a_0}{(r+3)^2 (r+1)(r+4)^2 (2+r)(r+5)^2 (r+6)(7+r)}
\end{aligned}
}
$$

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$
\left(x^{-1+m} m(-1+m) + 3m\, x^{-1+m}\right) c_0 = x
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{8} \\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+2}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{8}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{8}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$
\boxed{
\begin{aligned}
c_0 &= \frac{1}{8} \\
c_1 &= \frac{1}{120} \\
c_2 &= \frac{1}{2880} \\
c_3 &= \frac{1}{100800} \\
c_4 &= \frac{1}{4838400} \\
c_5 &= \frac{1}{304819200}
\end{aligned}
}
$$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^2 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^2 \left(\frac{1}{8} + \frac{1}{120} x + \frac{1}{2880} x^2 + \frac{1}{100800} x^3 + \frac{1}{4838400} x^4 + \frac{1}{304819200} x^5\right) \\
&= \frac{1}{8} x^2 + \frac{1}{120} x^3 + \frac{1}{2880} x^4 + \frac{1}{100800} x^5 + \frac{1}{4838400} x^6 + \frac{1}{304819200} x^7
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$y_p = \frac{x^2}{8}+\frac{x^3}{120}+\frac{x^4}{2880}+\frac{x^5}{100800}+\frac{x^6}{4838400}+\frac{x^7}{304819200}+O\left(x^6\right)$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{x^2}{8}+\frac{x^3}{120}+\frac{x^4}{2880}+\frac{x^5}{100800}+O\left(x^6\right)$$

Hence the final solution is

$y = y_h + y_p$

$$= \frac{x^2}{8}+\frac{x^3}{120}+\frac{x^4}{2880}+\frac{x^5}{100800}+O\left(x^6\right)+c_1\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O\left(x^6\right)\right) + c_2\left(\left(-\frac{1}{2}-\frac{x}{6}-\frac{x^2}{48}-\frac{x^3}{720}-\frac{x^4}{17280}-\frac{x^5}{604800}-\frac{O(x^6)}{2}\right)\ln\left(x\right) + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{x^2}\right)$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{8}+\frac{x^3}{120}+\frac{x^4}{2880}+\frac{x^5}{100800}+O\left(x^6\right) + c_1\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O\left(x^6\right)\right) + c_2\left(\left(-\frac{1}{2}-\frac{x}{6}-\frac{x^2}{48}-\frac{x^3}{720}-\frac{x^4}{17280}-\frac{x^5}{604800}-\frac{O(x^6)}{2}\right)\ln\left(x\right) + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{x^2}\right) \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{8}+\frac{x^3}{120}+\frac{x^4}{2880}+\frac{x^5}{100800}+O\left(x^6\right)+c_1\left(1+\frac{x}{3}+\frac{x^2}{24}+\frac{x^3}{360}+\frac{x^4}{8640}+\frac{x^5}{302400}+O\left(x^6\right)\right) + c_2\left(\left(-\frac{1}{2}-\frac{x}{6}-\frac{x^2}{48}-\frac{x^3}{720}-\frac{x^4}{17280}-\frac{x^5}{604800}-\frac{O(x^6)}{2}\right)\ln\left(x\right) + \frac{1-x+\frac{2x^3}{9}+\frac{25x^4}{576}+\frac{157x^5}{43200}+O(x^6)}{x^2}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 78

```
Order:=6;
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)-y(x)=x,y(x),type='series',x=0);
```

$$y(x) = c_1 \left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \frac{1}{8640}x^4 + \frac{1}{302400}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2 \left(\ln(x) \left(x^2 + \frac{1}{3}x^3 + \frac{1}{24}x^4 + \frac{1}{360}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(-2 + 2x - \frac{4}{9}x^3 - \frac{25}{288}x^4 - \frac{157}{21600}x^5 + \mathrm{O}\left(x^6\right)\right)\right)}{x^2} + x^2 \left(\frac{1}{8} + \frac{1}{120}x + \frac{1}{2880}x^2 + \frac{1}{100800}x^3 + \mathrm{O}\left(x^4\right)\right)$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 248

```
AsymptoticDSolveValue[x*y''[x]+3*y'[x]-y[x]==x,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_2\left(x^4\left(\frac{25}{576}-\frac{\log(x)}{48}\right)+x^3\left(\frac{2}{9}-\frac{\log(x)}{6}\right)-\frac{1}{2}x^2\log(x)-x+1\right)}{x^2} + c_1\left(\frac{x^5}{302400}+\frac{x^4}{8640}+\frac{x^3}{360}+\frac{x^2}{24}+\frac{x}{3}+1\right)+\left(\frac{x^5}{302400}+\frac{x^4}{8640}+\frac{x^3}{360}+\frac{x^2}{24}+\frac{x}{3}+1\right)\left(\frac{x^6(9-4\log(x))}{2304}+\frac{1}{900}x^5(23-15\log(x))+\frac{1}{64}x^4(1-4\log(x))-\frac{x^3}{6}+\frac{x^2}{4}\right) + \frac{\left(-\frac{x^6}{288}-\frac{x^5}{30}-\frac{x^4}{8}\right)\left(x^4\left(\frac{25}{576}-\frac{\log(x)}{48}\right)+x^3\left(\frac{2}{9}-\frac{\log(x)}{6}\right)-\frac{1}{2}x^2\log(x)-x+1\right)}{x^2}$$

25.3 problem 2

Internal problem ID [2419]
Internal file name [`OUTPUT/2419_Tuesday_February_27_2024_08_37_14_AM_73832806/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''x + y' - 2yx = x^2}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y''x + y' - 2yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -2$$

Table 711: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -2$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x + y' - 2yx = x^2$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y''x + y' - 2yx = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-2a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r) (n+r-1) + (n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(x^{-1+m} m(-1+m) + m\, x^{-1+m}\right) c_0 = x^2$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - 2a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{2a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2}{(2+r)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{(2+r)^2}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{(2+r)^2}$	$\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4}{(2+r)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{(2+r)^2}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{4}{(2+r)^2(r+4)^2}$	$\frac{1}{16}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{2}{(2+r)^2}$	$\frac{1}{2}$
a_3	0	0
a_4	$\frac{4}{(2+r)^2(r+4)^2}$	$\frac{1}{16}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + \frac{x^2}{2} + \frac{x^4}{16} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$\frac{2}{(2+r)^2}$	$\frac{1}{2}$	$-\frac{4}{(2+r)^3}$	$-\frac{1}{2}$
b_3	0	0	0	0
b_4	$\frac{4}{(2+r)^2 (r+4)^2}$	$\frac{1}{16}$	$\frac{-16r-48}{(2+r)^3 (r+4)^3}$	$-\frac{3}{32}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 + \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)\right) \ln(x) - \frac{x^2}{2} - \frac{3x^4}{32} + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 + \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)\right) + c_2 \left(\left(1 + \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)\right) \ln(x) - \frac{x^2}{2} - \frac{3x^4}{32} + O(x^6)\right)
\end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(x^{-1+m} m(-1+m) + m\, x^{-1+m}\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{array}{|l|}
\hline
a_1 = 0 \\
a_2 = \frac{2a_0}{(2+r)^2} \\
a_3 = 0 \\
a_4 = \frac{4a_0}{(2+r)^2 (r+4)^2} \\
a_5 = 0 \\
\hline
\end{array}
$$

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$
\left(x^{-1+m} m(-1+m) + m\, x^{-1+m}\right) c_0 = x^2
$$

For c_0 and x. This results in

$$
c_0 = \frac{1}{9}
$$
$$
m = 3
$$

The particular solution is therefore

$$
y_p = \sum_{n=0}^{\infty} c_n x^{n+m}
$$
$$
= \sum_{n=0}^{\infty} c_n x^{n+3}
$$

Where in the above $c_0 = \frac{1}{9}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{9}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$
\begin{array}{|l|}
\hline
c_0 = \frac{1}{9} \\
c_1 = 0 \\
c_2 = \frac{2}{225} \\
c_3 = 0 \\
c_4 = \frac{4}{11025} \\
c_5 = 0 \\
\hline
\end{array}
$$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3 \left(\frac{1}{9} + \frac{2}{225} x^2 + \frac{4}{11025} x^4\right) \\
&= \frac{1}{9} x^3 + \frac{2}{225} x^5 + \frac{4}{11025} x^7
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = \frac{x^3}{9} + \frac{2x^5}{225} + \frac{4x^7}{11025} + O\left(x^6\right)
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = \frac{x^3}{9} + \frac{2x^5}{225} + O\left(x^6\right)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x^3}{9} + \frac{2x^5}{225} + O\left(x^6\right) + c_1 \left(1 + \frac{x^2}{2} + \frac{x^4}{16} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(\left(1 + \frac{x^2}{2} + \frac{x^4}{16} + O\left(x^6\right)\right) \ln\left(x\right) - \frac{x^2}{2} - \frac{3x^4}{32} + O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = \frac{x^3}{9} + \frac{2x^5}{225} + O\left(x^6\right) + c_1 \left(1 + \frac{x^2}{2} + \frac{x^4}{16} + O\left(x^6\right)\right) \\
+ c_2 \left(\left(1 + \frac{x^2}{2} + \frac{x^4}{16} + O\left(x^6\right)\right) \ln\left(x\right) - \frac{x^2}{2} - \frac{3x^4}{32} + O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = \frac{x^3}{9} + \frac{2x^5}{225} + O(x^6) + c_1\left(1 + \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)\right) + c_2\left(\left(1 + \frac{x^2}{2} + \frac{x^4}{16} + O(x^6)\right)\ln(x) - \frac{x^2}{2} - \frac{3x^4}{32} + O(x^6)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 53

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)-2*x*y(x)=x^2,y(x),type='series',x=0);
```

$$y(x) = (c_2\ln(x) + c_1)\left(1 + \frac{1}{2}x^2 + \frac{1}{16}x^4 + \mathrm{O}(x^6)\right) + x^3\left(\frac{1}{9} + \frac{2}{225}x^2 + \mathrm{O}(x^3)\right) + \left(-\frac{1}{2}x^2 - \frac{3}{32}x^4 + \mathrm{O}(x^6)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 188

```
AsymptoticDSolveValue[x*y''[x]+y'[x]-2*x*y[x]==x^2,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^6}{288}+\frac{x^4}{16}+\frac{x^2}{2}+1\right) + c_1\left(x^6\left(\frac{\log(x)}{144}-\frac{1}{108}\right)+x^4\left(\frac{\log(x)}{8}-\frac{1}{8}\right)+x^2\left(\log(x)-\frac{1}{2}\right)+2\log(x)+1\right) + \left(\frac{x^6}{288}+\frac{x^4}{16}+\frac{x^2}{2}+1\right)\left(\frac{1}{100}x^5(7-10\log(x))+\frac{1}{18}x^3(-6\log(x)-1)\right) + \left(\frac{x^5}{20} + \frac{x^3}{6}\right)\left(x^6\left(\frac{\log(x)}{144}-\frac{1}{108}\right)+x^4\left(\frac{\log(x)}{8}-\frac{1}{8}\right)+x^2\left(\log(x)-\frac{1}{2}\right)+2\log(x)+1\right)$$

25.4 problem 3

Internal problem ID [2420]
Internal file name [`OUTPUT/2420_Tuesday_February_27_2024_08_37_14_AM_21863710/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y''x - xy' + y = x^3}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y''x - xy' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -1$$
$$q(x) = \frac{1}{x}$$

Table 712: Table $p(x), q(x)$ singularites.

$p(x) = -1$	
singularity	type

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x - xy' + y = x^3$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y''x - xy' + y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x - x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)\,(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)\,(n+r-1) = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$x^{-1+m} c_0 m(-1+m) = x^3$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n+r-2)}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(n-1)}{(n+1)\, n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1+r}{(1+r)\, r}$$

Which for the root $r = 1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-1+r}{(1+r)^2\, (2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-1+r}{(1+r)(2+r)^2(3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0
a_4	$\frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-1+r}{(1+r)(2+r)(3+r)(4+r)^2(5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1+r}{(1+r)r}$	0
a_2	$\frac{-1+r}{(1+r)^2(2+r)}$	0
a_3	$\frac{-1+r}{(1+r)(2+r)^2(3+r)}$	0
a_4	$\frac{-1+r}{(1+r)(2+r)(3+r)^2(4+r)}$	0
a_5	$\frac{-1+r}{(1+r)(2+r)(3+r)(4+r)^2(5+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-1 + r}{(1 + r) r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{-1 + r}{(1 + r) r} &= \lim_{r \to 0} \frac{-1 + r}{(1 + r) r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n + r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $y''x - xy' + y = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x \\
&- x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left((-y_1'(x)\,x + y_1''(x)\,x + y_1(x))\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - y_1(x)\right)C \\
&+ \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&- x\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$-y_1'(x)\,x + y_1''(x)\,x + y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - y_1(x)\right)C \\
&+ \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&- x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1}a_n(n+r_1)\right)x - (x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C}{x} \\
&+ \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2 - \left(\sum_{n=0}^{\infty} x^{-1+n+r_2}b_n(n+r_2)\right)x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x}{x} \\
&= 0
\end{aligned} \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\begin{aligned}
&\frac{\left(2\left(\sum_{n=0}^{\infty} x^n a_n(n+1)\right)x - (x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right)C}{x} \\
&+ \frac{\left(\sum_{n=0}^{\infty} x^{-2+n}b_n n(n-1)\right)x^2 - \left(\sum_{n=0}^{\infty} x^{n-1}b_n n\right)x^2 + \left(\sum_{n=0}^{\infty} b_n x^n\right)x}{x} = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\,x^n a_n(n+1)\right) + \sum_{n=0}^{\infty}\left(-C\,x^{n+1}a_n\right) + \sum_{n=0}^{\infty}\left(-Ca_n x^n\right) \\
&+ \left(\sum_{n=0}^{\infty} n\,x^{n-1}b_n(n-1)\right) + \sum_{n=0}^{\infty}\left(-x^n b_n n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n+1} a_n\right) = \sum_{n=2}^{\infty} \left(-C a_{-2+n} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-C a_n x^n\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^n b_n n\right) = \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right) + \sum_{n=2}^{\infty} \left(-C a_{-2+n} x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1} x^{n-1}\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 2$, Eq (2B) gives

$$(-a_0 + 3a_1)\, C + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 + 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$(-a_1 + 5a_2)\, C - b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{2} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{12}$$

For $n = 4$, Eq (2B) gives

$$(-a_2 + 7a_3)\, C - 2b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{6} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{1}{72}$$

For $n = 5$, Eq (2B) gives

$$(-a_3 + 9a_4)\, C - 3b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{24} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{1}{480}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1)\left(x\left(1+O\left(x^6\right)\right)\right)\ln(x) + 1 - \frac{x^2}{2} - \frac{x^3}{12} - \frac{x^4}{72} - \frac{x^5}{480} + O\left(x^6\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1+O\left(x^6\right)\right) + c_2\left((-1)\left(x\left(1+O\left(x^6\right)\right)\right)\ln(x) + 1 - \frac{x^2}{2} - \frac{x^3}{12} - \frac{x^4}{72} - \frac{x^5}{480} + O\left(x^6\right)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$x^{-1+m} c_0 m(-1+m) = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\begin{aligned} a_1 &= \frac{a_0(-1+r)}{(1+r)r} \\ a_2 &= \frac{a_0(-1+r)}{(1+r)^2(2+r)} \\ a_3 &= \frac{a_0(-1+r)}{(1+r)(2+r)^2(3+r)} \\ a_4 &= \frac{a_0(-1+r)}{(1+r)(2+r)(3+r)^2(4+r)} \\ a_5 &= \frac{a_0(-1+r)}{(1+r)(2+r)(3+r)(4+r)^2(5+r)} \end{aligned}$$

Now we determine the particular solution y_p associated with $F = x^3$ by solving the balance equation

$$x^{-1+m} c_0 m(-1+m) = x^3$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{12} \\ m &= 4 \end{aligned}$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+4}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{12}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{12}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{12}$
$c_1 = \frac{1}{80}$
$c_2 = \frac{1}{600}$
$c_3 = \frac{1}{5040}$
$c_4 = \frac{1}{47040}$
$c_5 = \frac{1}{483840}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^4 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^4 \left(\frac{1}{12} + \frac{1}{80}x + \frac{1}{600}x^2 + \frac{1}{5040}x^3 + \frac{1}{47040}x^4 + \frac{1}{483840}x^5\right) \\
&= \frac{1}{12}x^4 + \frac{1}{80}x^5 + \frac{1}{600}x^6 + \frac{1}{5040}x^7 + \frac{1}{47040}x^8 + \frac{1}{483840}x^9
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$y_p = \frac{x^4}{12} + \frac{x^5}{80} + \frac{x^6}{600} + \frac{x^7}{5040} + \frac{x^8}{47040} + \frac{x^9}{483840} + O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{x^4}{12} + \frac{x^5}{80} + O(x^6)$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{x^4}{12} + \frac{x^5}{80} + O(x^6) + c_1 x(1 + O(x^6)) \\ &\quad + c_2\left(-x(1 + O(x^6))\ln(x) + 1 - \frac{x^2}{2} - \frac{x^3}{12} - \frac{x^4}{72} - \frac{x^5}{480} + O(x^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned} y &= \frac{x^4}{12} + \frac{x^5}{80} + O(x^6) + c_1 x(1 + O(x^6)) \\ &\quad + c_2\left(-x(1 + O(x^6))\ln(x) + 1 - \frac{x^2}{2} - \frac{x^3}{12} - \frac{x^4}{72} - \frac{x^5}{480} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= \frac{x^4}{12} + \frac{x^5}{80} + O(x^6) + c_1 x(1 + O(x^6)) \\ &\quad + c_2\left(-x(1 + O(x^6))\ln(x) + 1 - \frac{x^2}{2} - \frac{x^3}{12} - \frac{x^4}{72} - \frac{x^5}{480} + O(x^6)\right) \end{aligned}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
Order:=6;
dsolve(x*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=x^3,y(x),type='series',x=0);
```

$$y(x) = x^4\left(\frac{1}{12} + \frac{1}{80}x + \mathrm{O}\left(x^2\right)\right) + \ln\left(x\right)\left(-x + \mathrm{O}\left(x^6\right)\right)c_2 + c_1x\left(1 + \mathrm{O}\left(x^6\right)\right) + \left(1 + x - \frac{1}{2}x^2 - \frac{1}{12}x^3 - \frac{1}{72}x^4 - \frac{1}{480}x^5 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 139

```
AsymptoticDSolveValue[x*y''[x]-x*y'[x]+y[x]==x^3,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^4}{72} - \frac{x^3}{12} - \frac{x^2}{2} - x\log(x) + 1\right) + x\left(\frac{1}{36}x^6(2 - 3\log(x)) + \frac{1}{25}x^5(5\log(x) - 1) + \frac{1}{16}x^4(-4\log(x) - 3) + \frac{x^3}{3}\right) + \left(-\frac{x^6}{12} + \frac{x^5}{5} - \frac{x^4}{4}\right)\left(-\frac{x^4}{72} - \frac{x^3}{12} - \frac{x^2}{2} - x\log(x) + 1\right) + c_2x$$

25.5 problem 4

Internal problem ID [2421]
Internal file name [OUTPUT/2421_Tuesday_February_27_2024_08_37_17_AM_31975214/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-2x)\,y''+4xy'-4y=x^2-x}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy_0'+\frac{x^2}{2}\,f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y_0'}+\cdots \\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1145}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1146}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= \frac{4xy' - x^2 - 4y + x}{2x - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{8xy' - 2x^2 - 8y + 1}{2x - 1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{16xy' - 4x^2 - 16y + 2}{2x - 1} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{32xy' - 8x^2 - 32y + 4}{2x - 1} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{64xy' - 16x^2 - 64y + 8}{2x - 1}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= 4y(0) \\
F_1 &= 8y(0) - 1 \\
F_2 &= 16y(0) - 2 \\
F_3 &= 32y(0) - 4 \\
F_4 &= 64y(0) - 8
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6\right) y(0) + xy'(0) - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{30} - \frac{x^6}{90} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(1-2x)\,y'' + 4xy' - 4y = x^2 - x$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(1-2x)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + 4x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = x^2 - x \quad (1)$$

Expanding $x^2 - x$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} x^2 - x &= x^2 - x + \dots \\ &= x^2 - x \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$(1-2x)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + 4x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^n\right) = x^2 - x$$

Which simplifies to

$$\sum_{n=2}^{\infty} \left(-2n\, x^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 4n a_n x^n\right) + \sum_{n=0}^{\infty} \left(-4a_n x^n\right) \quad (2)$$
$$= x^2 - x$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty}\left(-2n\,x^{n-1}a_n(n-1)\right) &= \sum_{n=1}^{\infty}\left(-2(n+1)\,a_{n+1}n\,x^n\right) \\
\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} &= \sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left(-2(n+1)\,a_{n+1}n\,x^n\right) + \left(\sum_{n=0}^{\infty}(n+2)\,a_{n+2}(n+1)\,x^n\right) \\
+ \left(\sum_{n=1}^{\infty}4na_n x^n\right) + \sum_{n=0}^{\infty}\left(-4a_n x^n\right) = x^2 - x
\end{aligned} \tag{3}
$$

$n = 0$ gives

$$
2a_2 - 4a_0 = 0
$$

$$
a_2 = 2a_0
$$

For $1 \leq n$, the recurrence equation is

$$
\left(-2(n+1)\,a_{n+1}n + (n+2)\,a_{n+2}(n+1) + 4na_n - 4a_n\right)x^n = x^2 - x \tag{4}
$$

For $n = 1$ the recurrence equation gives

$$
\begin{aligned}
(-4a_2 + 6a_3)\,x &= -x \\
-4a_2 + 6a_3 &= -1
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_3 = -\frac{1}{6} + \frac{4a_0}{3}
$$

For $n = 2$ the recurrence equation gives

$$(-12a_3 + 12a_4 + 4a_2) x^2 = x^2$$
$$-12a_3 + 12a_4 + 4a_2 = 1$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{1}{12} + \frac{2a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$(-24a_4 + 20a_5 + 8a_3) x^3 = 0$$
$$-24a_4 + 20a_5 + 8a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{1}{30} + \frac{4a_0}{15}$$

For $n = 4$ the recurrence equation gives

$$(-40a_5 + 30a_6 + 12a_4) x^4 = 0$$
$$-40a_5 + 30a_6 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{90} + \frac{4a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$(-60a_6 + 42a_7 + 16a_5) x^5 = 0$$
$$-60a_6 + 42a_7 + 16a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{315} + \frac{8a_0}{315}$$

And so on. Therefore the solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + 2a_0x^2 + \left(-\frac{1}{6} + \frac{4a_0}{3}\right)x^3 + \left(-\frac{1}{12} + \frac{2a_0}{3}\right)x^4 + \left(-\frac{1}{30} + \frac{4a_0}{15}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5\right)a_0 + a_1x - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{30} + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5\right)c_1 + c_2x - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{30} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6\right)y(0) + xy'(0) - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{30} - \frac{x^6}{90} + O(x^6) \tag{1}$$

$$y = \left(1 + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5\right)c_1 + c_2x - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{30} + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6\right)y(0) + xy'(0) - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{30} - \frac{x^6}{90} + O(x^6)$$

Verified OK.

$$y = \left(1 + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5\right)c_1 + c_2x - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{30} + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
Order:=6;
dsolve((1-2*x)*diff(y(x),x$2)+4*x*diff(y(x),x)-4*y(x)=x^2-x,y(x),type='series',x=0);
```

$$y(x) = \left(1+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4+\frac{4}{15}x^5\right)y(0)+D(y)(0)\,x-\frac{x^3}{6}-\frac{x^4}{12}-\frac{x^5}{30}+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 60

```
AsymptoticDSolveValue[(1-2*x)*y''[x]+4*x*y'[x]-4*y[x]==x^2-x,y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^5}{30}-\frac{x^4}{12}-\frac{x^3}{6}+c_1\left(\frac{4x^5}{15}+\frac{2x^4}{3}+\frac{4x^3}{3}+2x^2+1\right)+c_2x$$

25.6 problem 5

Internal problem ID [2422]
Internal file name [`OUTPUT/2422_Tuesday_February_27_2024_08_37_17_AM_39828123/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x^2y'' + xy' + (x+12)\,y = x^2 + x}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (x+12)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x+12}{x^2}$$

Table 713: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x+12}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + (x + 12)\,y = x^2 + x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $x^2y'' + xy' + (x + 12)\,y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&x^2\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}\right) \\
&+x\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+(x+12)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n\right)+\left(\sum_{n=0}^{\infty}12a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty}x^{1+n+r}a_n=\sum_{n=1}^{\infty}a_{n-1}x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\right) \\
&+\left(\sum_{n=1}^{\infty}a_{n-1}x^{n+r}\right)+\left(\sum_{n=0}^{\infty}12a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r}a_n(n+r)(n+r-1)+x^{n+r}a_n(n+r)+12a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r)+x^r a_0 r+12a_0 x^r=0
$$

Or

$$\left(x^{r} r(-1+r)+x^{r} r+12 x^{r}\right) a_{0}=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^{2}+12\right) x^{r}=0$$

Since the above is true for all x then the indicial equation becomes

$$r^{2}+12=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_{1} &= 2 i \sqrt{3} \\ r_{2} &= -2 i \sqrt{3} \end{aligned}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(x^{m} m(-1+m)+x^{m} m+12 x^{m}\right) c_{0}=x^{2}+x$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^{2}+12\right) x^{r}=0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$\begin{aligned} y_{1}(x) &= x^{r_{1}}\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) \\ y_{2}(x) &= x^{r_{2}}\left(\sum_{n=0}^{\infty} b_{n} x^{n}\right) \end{aligned}$$

Or

$$\begin{aligned} y_{1}(x) &= \sum_{n=0}^{\infty} a_{n} x^{n+2 i \sqrt{3}} \\ y_{2}(x) &= \sum_{n=0}^{\infty} b_{n} x^{n-2 i \sqrt{3}} \end{aligned}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-1} + 12a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n^2 + 2nr + r^2 + 12} \tag{4}$$

Which for the root $r = 2i\sqrt{3}$ becomes

$$a_n = -\frac{a_{n-1}}{n^2 + 4in\sqrt{3}} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2i\sqrt{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{r^2 + 2r + 13}$$

Which for the root $r = 2i\sqrt{3}$ becomes

$$a_1 = \frac{1}{-1 - 4i\sqrt{3}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+2r+13}$	$\frac{1}{-1-4i\sqrt{3}}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r^2 + 2r + 13)(r^2 + 4r + 16)}$$

Which for the root $r = 2i\sqrt{3}$ becomes

$$a_2 = -\frac{1}{4\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+2r+13}$	$\frac{1}{-1-4i\sqrt{3}}$
a_2	$\frac{1}{(r^2+2r+13)(r^2+4r+16)}$	$-\frac{1}{4\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)}$$

Which for the root $r = 2i\sqrt{3}$ becomes

$$a_3 = \frac{1}{48\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i\sqrt{3}+\frac{3}{4}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+2r+13}$	$\frac{1}{-1-4i\sqrt{3}}$
a_2	$\frac{1}{(r^2+2r+13)(r^2+4r+16)}$	$-\frac{1}{4\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)}$
a_3	$-\frac{1}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)}$	$\frac{1}{48\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i\sqrt{3}+\frac{3}{4}\right)}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)(r^2+8r+28)}$$

Which for the root $r = 2i\sqrt{3}$ becomes

$$a_4 = \frac{1}{192\left(-4\sqrt{3}+3i\right)\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i-\sqrt{3}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+2r+13}$	$\frac{1}{-1-4i\sqrt{3}}$
a_2	$\frac{1}{(r^2+2r+13)(r^2+4r+16)}$	$-\frac{1}{4\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)}$
a_3	$-\frac{1}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)}$	$\frac{1}{48\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i\sqrt{3}+\frac{3}{4}\right)}$
a_4	$\frac{1}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)(r^2+8r+28)}$	$\frac{1}{192\left(-4\sqrt{3}+3i\right)\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i-\sqrt{3}\right)}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)(r^2+8r+28)(r^2+10r+37)}$$

Which for the root $r = 2i\sqrt{3}$ becomes

$$a_5 = \frac{1}{15360\left(\sqrt{3}-\frac{3i}{4}\right)\left(-i+4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+\frac{5}{4}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+2r+13}$	$\frac{1}{-1-4i\sqrt{3}}$
a_2	$\frac{1}{(r^2+2r+13)(r^2+4r+16)}$	$-\frac{1}{4\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)}$
a_3	$-\frac{1}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)}$	$\frac{1}{48\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i\sqrt{3}+\frac{3}{4}\right)}$
a_4	$\frac{1}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)(r^2+8r+28)}$	$\frac{1}{192\left(-4\sqrt{3}+3i\right)\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i-\sqrt{3}\right)}$
a_5	$-\frac{1}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)(r^2+8r+28)(r^2+10r+37)}$	$\frac{1}{15360\left(\sqrt{3}-\frac{3i}{4}\right)\left(-i+4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+\frac{5}{4}\right)}$

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = x^{2i\sqrt{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right)$$

$$= x^{2i\sqrt{3}} \left(1 + \frac{x}{-1 - 4i\sqrt{3}} - \frac{x^2}{4\left(i - 4\sqrt{3}\right)\left(-2\sqrt{3} + i\right)} + \frac{x^3}{48\left(i - 4\sqrt{3}\right)\left(-2\sqrt{3} + i\right)\left(i\sqrt{3} + \frac{3}{4}\right)} + \frac{x^4}{192\left(-4\sqrt{3} + 3i\right)\left(i - 4\sqrt{3}\right)\left(-2\sqrt{3} + i\right)\left(i - \sqrt{3}\right)} + \frac{x^5}{15360\left(\sqrt{3} - \frac{3i}{4}\right)\left(-i + 4\sqrt{3}\right)\left(-2\sqrt{3} + i\right)\left(-i + \sqrt{3}\right)\left(i\sqrt{3} + \frac{5}{4}\right)} + O(x^6)\right)$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$y_2(x) = x^{-2i\sqrt{3}} \left(1 + \frac{x}{-1 + 4i\sqrt{3}} - \frac{x^2}{4\left(-i - 4\sqrt{3}\right)\left(-2\sqrt{3} - i\right)} + \frac{x^3}{48\left(-i - 4\sqrt{3}\right)\left(-2\sqrt{3} - i\right)\left(-i\sqrt{3} + \frac{3}{4}\right)} + \frac{x^4}{192\left(-4\sqrt{3} - 3i\right)\left(-i - 4\sqrt{3}\right)\left(-2\sqrt{3} - i\right)\left(-i - \sqrt{3}\right)} + \frac{x^5}{15360\left(\sqrt{3} + \frac{3i}{4}\right)\left(i + 4\sqrt{3}\right)\left(-2\sqrt{3} - i\right)\left(\sqrt{3} + i\right)\left(-i\sqrt{3} + \frac{5}{4}\right)} + O(x^6)\right)$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$
\begin{aligned}
&= c_1 x^{2i\sqrt{3}}\left(1+\frac{x}{-1-4i\sqrt{3}}-\frac{x^2}{4\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)}\right.\\
&\quad+\frac{x^3}{48\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i\sqrt{3}+\frac{3}{4}\right)}\\
&\quad+\frac{x^4}{192\left(-4\sqrt{3}+3i\right)\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i-\sqrt{3}\right)}\\
&\quad\left.+\frac{x^5}{15360\left(\sqrt{3}-\frac{3i}{4}\right)\left(-i+4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+\frac{5}{4}\right)}+O\left(x^6\right)\right)\\
&\quad+c_2 x^{-2i\sqrt{3}}\left(1+\frac{x}{-1+4i\sqrt{3}}-\frac{x^2}{4\left(-i-4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)}\right.\\
&\quad+\frac{x^3}{48\left(-i-4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)\left(-i\sqrt{3}+\frac{3}{4}\right)}\\
&\quad+\frac{x^4}{192\left(-4\sqrt{3}-3i\right)\left(-i-4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)\left(-i-\sqrt{3}\right)}\\
&\quad\left.+\frac{x^5}{15360\left(\sqrt{3}+\frac{3i}{4}\right)\left(i+4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)\left(\sqrt{3}+i\right)\left(-i\sqrt{3}+\frac{5}{4}\right)}+O\left(x^6\right)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(x^m m(-1+m)+x^m m+12x^m\right)c_0=F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{aligned}
a_1 &= -\frac{a_0}{r^2+2r+13}\\
a_2 &= \frac{a_0}{(r^2+2r+13)(r^2+4r+16)}\\
a_3 &= -\frac{a_0}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)}\\
a_4 &= \frac{a_0}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)(r^2+8r+28)}\\
a_5 &= -\frac{a_0}{(r^2+2r+13)(r^2+4r+16)(r^2+6r+21)(r^2+8r+28)(r^2+10r+37)}
\end{aligned}
$$

Since the $F = x^2 + x$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$\left(x^m m(-1+m) + x^m m + 12x^m\right) c_0 = x^2$$

For c_0 and x. This results in

$$c_0 = \frac{1}{16}$$
$$m = 2$$

The particular solution is therefore

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+m}$$
$$= \sum_{n=0}^{\infty} c_n x^{n+2}$$

Where in the above $c_0 = \frac{1}{16}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{16}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{16}$
$c_1 = -\frac{1}{336}$
$c_2 = \frac{1}{9408}$
$c_3 = -\frac{1}{348096}$
$c_4 = \frac{1}{16708608}$
$c_5 = -\frac{1}{1019225088}$

The particular solution is now found using

$$y_p = x^m \sum_{n=0}^{\infty} c_n x^n$$

$$= x^2 \sum_{n=0}^{\infty} c_n x^n$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^2\left(\frac{1}{16}-\frac{1}{336}x+\frac{1}{9408}x^2-\frac{1}{348096}x^3+\frac{1}{16708608}x^4-\frac{1}{1019225088}x^5\right) \\ &= \frac{1}{16}x^2-\frac{1}{336}x^3+\frac{1}{9408}x^4-\frac{1}{348096}x^5+\frac{1}{16708608}x^6-\frac{1}{1019225088}x^7 \end{aligned}$$

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$(x^m m(-1+m)+x^m m+12x^m)\, c_0 = x$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{13} \\ m &= 1 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+1} \end{aligned}$$

Where in the above $c_0 = \frac{1}{13}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{13}$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{13}$
$c_1 = -\frac{1}{208}$
$c_2 = \frac{1}{4368}$
$c_3 = -\frac{1}{122304}$
$c_4 = \frac{1}{4525248}$
$c_5 = -\frac{1}{217211904}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x \left(\frac{1}{13} - \frac{1}{208}x + \frac{1}{4368}x^2 - \frac{1}{122304}x^3 + \frac{1}{4525248}x^4 - \frac{1}{217211904}x^5\right) \\
&= \frac{1}{13}x - \frac{1}{208}x^2 + \frac{1}{4368}x^3 - \frac{1}{122304}x^4 + \frac{1}{4525248}x^5 - \frac{1}{217211904}x^6
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = \frac{x}{13} + \frac{3x^2}{52} - \frac{x^3}{364} + \frac{x^4}{10192} - \frac{x^5}{377104} + \frac{x^6}{18100992} - \frac{x^7}{1019225088} + O\left(x^6\right)
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = \frac{x}{13} + \frac{3x^2}{52} - \frac{x^3}{364} + \frac{x^4}{10192} - \frac{x^5}{377104} + O\left(x^6\right)
$$

Hence the final solution is

$$
y = y_h + y_p
$$

$$
\begin{aligned}
&= \frac{x}{13} + \frac{3x^2}{52} - \frac{x^3}{364} + \frac{x^4}{10192} - \frac{x^5}{377104} + O\left(x^6\right) + c_1 x^{2i\sqrt{3}} \left(1 + \frac{x}{-1 - 4i\sqrt{3}}\right. \\
&\quad - \frac{x^2}{4\left(i - 4\sqrt{3}\right)\left(-2\sqrt{3} + i\right)} + \frac{x^3}{48\left(i - 4\sqrt{3}\right)\left(-2\sqrt{3} + i\right)\left(i\sqrt{3} + \frac{3}{4}\right)} \\
&\quad + \frac{x^4}{192\left(-4\sqrt{3} + 3i\right)\left(i - 4\sqrt{3}\right)\left(-2\sqrt{3} + i\right)\left(i - \sqrt{3}\right)} \\
&\quad \left. + \frac{x^5}{15360\left(\sqrt{3} - \frac{3i}{4}\right)\left(-i + 4\sqrt{3}\right)\left(-2\sqrt{3} + i\right)\left(-i + \sqrt{3}\right)\left(i\sqrt{3} + \frac{5}{4}\right)} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{-2i\sqrt{3}} \left(1 + \frac{x}{-1 + 4i\sqrt{3}} - \frac{x^2}{4\left(-i - 4\sqrt{3}\right)\left(-2\sqrt{3} - i\right)}\right. \\
&\quad + \frac{x^3}{48\left(-i - 4\sqrt{3}\right)\left(-2\sqrt{3} - i\right)\left(-i\sqrt{3} + \frac{3}{4}\right)} \\
&\quad + \frac{x^4}{192\left(-4\sqrt{3} - 3i\right)\left(-i - 4\sqrt{3}\right)\left(-2\sqrt{3} - i\right)\left(-i - \sqrt{3}\right)} \\
&\quad \left. + \frac{x^5}{15360\left(\sqrt{3} + \frac{3i}{4}\right)\left(i + 4\sqrt{3}\right)\left(-2\sqrt{3} - i\right)\left(\sqrt{3} + i\right)\left(-i\sqrt{3} + \frac{5}{4}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{x}{13}+\frac{3x^{2}}{52}-\frac{x^{3}}{364}+\frac{x^{4}}{10192}-\frac{x^{5}}{377104}+O\left(x^{6}\right)+c_{1} x^{2 i \sqrt{3}} \left(1+\frac{x}{-1-4 i \sqrt{3}} \right. \\
& -\frac{x^{2}}{4\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)}+\frac{x^{3}}{48\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i\sqrt{3}+\frac{3}{4}\right)} \\
& +\frac{x^{4}}{192\left(-4\sqrt{3}+3i\right)\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i-\sqrt{3}\right)} \\
& \left. +\frac{x^{5}}{15360\left(\sqrt{3}-\frac{3i}{4}\right)\left(-i+4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+\frac{5}{4}\right)}+O\left(x^{6}\right)\right) \\
& +c_{2} x^{-2 i \sqrt{3}} \left(1+\frac{x}{-1+4 i \sqrt{3}}-\frac{x^{2}}{4\left(-i-4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)} \right. \\
& +\frac{x^{3}}{48\left(-i-4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)\left(-i\sqrt{3}+\frac{3}{4}\right)} \\
& +\frac{x^{4}}{192\left(-4\sqrt{3}-3i\right)\left(-i-4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)\left(-i-\sqrt{3}\right)} \\
& \left. +\frac{x^{5}}{15360\left(\sqrt{3}+\frac{3i}{4}\right)\left(i+4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)\left(\sqrt{3}+i\right)\left(-i\sqrt{3}+\frac{5}{4}\right)}+O\left(x^{6}\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{x}{13}+\frac{3x^2}{52}-\frac{x^3}{364}+\frac{x^4}{10192}-\frac{x^5}{377104}+O(x^6)+c_1 x^{2i\sqrt{3}}\left(1+\frac{x}{-1-4i\sqrt{3}}\right. \\
& -\frac{x^2}{4\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)}+\frac{x^3}{48\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i\sqrt{3}+\frac{3}{4}\right)} \\
& +\frac{x^4}{192\left(-4\sqrt{3}+3i\right)\left(i-4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(i-\sqrt{3}\right)} \\
& \left.+\frac{x^5}{15360\left(\sqrt{3}-\frac{3i}{4}\right)\left(-i+4\sqrt{3}\right)\left(-2\sqrt{3}+i\right)\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+\frac{5}{4}\right)}+O(x^6)\right) \\
& +c_2 x^{-2i\sqrt{3}}\left(1+\frac{x}{-1+4i\sqrt{3}}-\frac{x^2}{4\left(-i-4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)}\right. \\
& +\frac{x^3}{48\left(-i-4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)\left(-i\sqrt{3}+\frac{3}{4}\right)} \\
& +\frac{x^4}{192\left(-4\sqrt{3}-3i\right)\left(-i-4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)\left(-i-\sqrt{3}\right)} \\
& \left.+\frac{x^5}{15360\left(\sqrt{3}+\frac{3i}{4}\right)\left(i+4\sqrt{3}\right)\left(-2\sqrt{3}-i\right)\left(\sqrt{3}+i\right)\left(-i\sqrt{3}+\frac{5}{4}\right)}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 379

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x+12)*y(x)=x^2+x,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_2 x^{2i\sqrt{3}} \Bigg(1 + \frac{1}{-4i\sqrt{3}-1}x - \frac{1}{4}\frac{1}{(i-2\sqrt{3})(-4\sqrt{3}+i)}x^2 \\
& + \frac{1}{48}\frac{1}{(i-2\sqrt{3})(-4\sqrt{3}+i)(i\sqrt{3}+\frac{3}{4})}x^3 \\
& + \frac{1}{768}\frac{1}{(-\sqrt{3}+\frac{3i}{4})(-i+2\sqrt{3})(-i+\sqrt{3})(-4\sqrt{3}+i)}x^4 \\
& + \frac{1}{15360}\frac{1}{(\sqrt{3}-\frac{3i}{4})(-i+2\sqrt{3})(-i+\sqrt{3})(-4\sqrt{3}+i)(i\sqrt{3}+\frac{5}{4})}x^5 + \mathrm{O}(x^6)\Bigg) \\
& + c_1 x^{-2i\sqrt{3}} \Bigg(1 + \frac{1}{4i\sqrt{3}-1}x - \frac{1}{4}\frac{1}{(2\sqrt{3}+i)(4\sqrt{3}+i)}x^2 \\
& - \frac{1}{48}\frac{1}{(2\sqrt{3}+i)(4\sqrt{3}+i)(i\sqrt{3}-\frac{3}{4})}x^3 \\
& + \frac{1}{192}\frac{1}{(2\sqrt{3}+i)(3i+4\sqrt{3})(\sqrt{3}+i)(4\sqrt{3}+i)}x^4 \\
& + \frac{1}{960}\frac{1}{(4i\sqrt{3}-3)(2\sqrt{3}+i)(\sqrt{3}+i)(5i+4\sqrt{3})(4\sqrt{3}+i)}x^5 + \mathrm{O}(x^6)\Bigg) \\
& + x\left(\frac{1}{13} + \frac{3}{52}x - \frac{1}{364}x^2 + \frac{1}{10192}x^3 - \frac{1}{377104}x^4 + \mathrm{O}(x^5)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 1.224 (sec). Leaf size: 704

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x+12)*y[x]==x^2+x,y[x],{x,0,5}]
```

$$y(x) \to$$

$$-\frac{\left(518\left(139\sqrt{3}-100i\right)x^5-2555\left(929\sqrt{3}+1053i\right)x^4-46720\left(121\sqrt{3}-2726i\right)x^3+9320640\left(125\sqrt{3}-\right.\right.}{}$$

$$+\frac{\left(141+74i\sqrt{3}\right)\left(4\sqrt{3}-i\right)\left(6\sqrt{3}+23i\right)\left(215\sqrt{3}+81i\right)\left(751\sqrt{3}+2985i\right)\left(-\frac{ix^5}{720960\sqrt{3}+2865600i}+\frac{ix}{41280\sqrt{3}}\right.}{}$$

$$+c_2\left(-\frac{ix^5}{720960\sqrt{3}+2865600i}+\frac{ix^4}{41280\sqrt{3}+15552i}-\frac{ix^3}{888\sqrt{3}-1692i}-\frac{ix^2}{24\sqrt{3}+92i}\right.$$
$$\left.-\frac{x}{1+4i\sqrt{3}}+1\right)x^{2i\sqrt{3}}+c_1\left(\frac{ix^5}{720960\sqrt{3}-2865600i}-\frac{ix^4}{41280\sqrt{3}-15552i}\right.$$
$$\left.+\frac{ix^3}{888\sqrt{3}+1692i}+\frac{ix^2}{24\sqrt{3}-92i}-\frac{x}{1-4i\sqrt{3}}+1\right)x^{-2i\sqrt{3}}$$

25.7 problem 6

Internal problem ID [2423]
Internal file name [`OUTPUT/2423_Tuesday_February_27_2024_08_37_18_AM_52728926/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x^2(x+1)\,y''+x\left(x^2+3\right)y'+y=-2x^2+x}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+\left(x^3+3x\right)y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x^2+3}{x\,(x+1)}$$
$$q(x)=\frac{1}{x^2\,(x+1)}$$

Table 714: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2+3}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1}{x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\, y'' + \left(x^3 + 3x\right) y' + y = -2x^2 + x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $x^2(x+1)\, y'' + \left(x^3 + 3x\right) y' + y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&x^2(x+1)\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}\right) \\
&+ \left(x^3+3x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=0}^{\infty}x^{n+r+2}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}x^{1+n+r}a_n(n+r)(n+r-1) &= \sum_{n=1}^{\infty}a_{n-1}(n+r-1)(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty}x^{n+r+2}a_n(n+r) &= \sum_{n=2}^{\infty}a_{n-2}(n+r-2)\,x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=1}^{\infty}a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right) \\
&+\left(\sum_{n=2}^{\infty}a_{n-2}(n+r-2)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r}a_n(n+r)(n+r-1)+3x^{n+r}a_n(n+r)+a_n x^{n+r}=0
$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 3x^r r + x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r+1)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r+1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -1$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(x^m m(-1+m) + 3x^m m + x^m) c_0 = -2x^2 + x$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = -1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n-1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{r(-1+r)}{(2+r)^2}$$

For $2 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) + 3a_n(n+r) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-1} + 2nr a_{n-1} + r^2 a_{n-1} + n a_{n-2} - 3n a_{n-1} + r a_{n-2} - 3r a_{n-1} - 2a_{n-2} + 2a_{n-1}}{n^2 + 2nr + r^2 + 2n + 2r + 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = -\frac{(n-3)\left((n-2)\, a_{n-1} + a_{n-2}\right)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{(2+r)^2}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r(r^3 - r^2 - 5r - 4)}{(2 + r)^2 (r + 3)^2}$$

Which for the root $r = -1$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{(2+r)^2}$	-2
a_2	$\frac{r(r^3-r^2-5r-4)}{(2+r)^2(r+3)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{r(r + 1)(r^4 - 12r^2 - 17r + 1)}{(2 + r)^2 (r + 3)^2 (r + 4)^2}$$

Which for the root $r = -1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{(2+r)^2}$	-2
a_2	$\frac{r(r^3-r^2-5r-4)}{(2+r)^2(r+3)^2}$	$\frac{1}{4}$
a_3	$-\frac{r(r+1)(r^4-12r^2-17r+1)}{(2+r)^2(r+3)^2(r+4)^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r^6 + 3r^5 - 16r^4 - 68r^3 - 43r^2 + 65r + 67)\, r}{(r + 5)^2 (r + 4)^2 (r + 3)^2 (2 + r)}$$

Which for the root $r = -1$ becomes

$$a_4 = -\frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{(2+r)^2}$	-2
a_2	$\frac{r(r^3-r^2-5r-4)}{(2+r)^2(r+3)^2}$	$\frac{1}{4}$
a_3	$-\frac{r(r+1)(r^4-12r^2-17r+1)}{(2+r)^2(r+3)^2(r+4)^2}$	0
a_4	$\frac{(r^6+3r^5-16r^4-68r^3-43r^2+65r+67)r}{(r+5)^2(r+4)^2(r+3)^2(2+r)}$	$-\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r^8 + 8r^7 - r^6 - 163r^5 - 455r^4 - 131r^3 + 997r^2 + 1312r + 511)\,r}{(r+6)^2\,(2+r)^2\,(r+3)\,(r+4)^2\,(r+5)^2}$$

Which for the root $r = -1$ becomes

$$a_5 = \frac{3}{800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r(-1+r)}{(2+r)^2}$	-2
a_2	$\frac{r(r^3-r^2-5r-4)}{(2+r)^2(r+3)^2}$	$\frac{1}{4}$
a_3	$-\frac{r(r+1)(r^4-12r^2-17r+1)}{(2+r)^2(r+3)^2(r+4)^2}$	0
a_4	$\frac{(r^6+3r^5-16r^4-68r^3-43r^2+65r+67)r}{(r+5)^2(r+4)^2(r+3)^2(2+r)}$	$-\frac{1}{64}$
a_5	$-\frac{(r^8+8r^7-r^6-163r^5-455r^4-131r^3+997r^2+1312r+511)r}{(r+6)^2(2+r)^2(r+3)(r+4)^2(r+5)^2}$	$\frac{3}{800}$

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \frac{1 - 2x + \frac{x^2}{4} - \frac{x^4}{64} + \frac{3x^5}{800} + O(x^6)}{x}
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = -1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$-\frac{r(-1+r)}{(2+r)^2}$	-2	$\frac{-5r+2}{(2+r)^3}$
b_2	$\frac{r(r^3-r^2-5r-4)}{(2+r)^2(r+3)^2}$	$\frac{1}{4}$	$\frac{11r^4+29r^3-6r^2-40r-24}{(2+r)^3(r+3)^3}$
b_3	$-\frac{r(r+1)(r^4-12r^2-17r+1)}{(2+r)^2(r+3)^2(r+4)^2}$	0	$\frac{-17r^7-137r^6-309r^5+179r^4+1623r^3+2115r^2+794r-24}{(2+r)^3(r+3)^3(r+4)^3}$
b_4	$\frac{(r^6+3r^5-16r^4-68r^3-43r^2+65r+67)r}{(r+5)^2(r+4)^2(r+3)^2(2+r)}$	$-\frac{1}{64}$	$\frac{23r^9+328r^8+1633r^7+2240r^6-8423r^5-36854r^4-51903r^3-227\cdots}{(r+5)^3(r+4)^3(r+3)^3(2+r)^2}$
b_5	$-\frac{(r^8+8r^7-r^6-163r^5-455r^4-131r^3+997r^2+1312r+511)r}{(r+6)^2(2+r)^2(r+3)(r+4)^2(r+5)^2}$	$\frac{3}{800}$	$\frac{-29r^{12}-707r^{11}-6972r^{10}-33562r^9-59772r^8+166668r^7+1175\cdots}{(r+6)^3(2+r\cdots}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \frac{\left(1 - 2x + \frac{x^2}{4} - \frac{x^4}{64} + \frac{3x^5}{800} + O(x^6)\right)\ln(x)}{x} + \frac{-x^2 + 7x + \frac{7x^3}{36} + \frac{35x^4}{1152} - \frac{191x^5}{9000} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= \frac{c_1 \left(1 - 2x + \frac{x^2}{4} - \frac{x^4}{64} + \frac{3x^5}{800} + O(x^6)\right)}{x} \\
&\quad + c_2 \left(\frac{\left(1 - 2x + \frac{x^2}{4} - \frac{x^4}{64} + \frac{3x^5}{800} + O(x^6)\right) \ln(x)}{x} \right. \\
&\quad \left. + \frac{-x^2 + 7x + \frac{7x^3}{36} + \frac{35x^4}{1152} - \frac{191x^5}{9000} + O(x^6)}{x}\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
(x^m m(-1+m) + 3x^m m + x^m)\, c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{aligned}
a_1 &= -\frac{a_0 r(-1+r)}{(2+r)^2} \\
a_2 &= \frac{a_0 r(r^3 - r^2 - 5r - 4)}{(2+r)^2 (r+3)^2} \\
a_3 &= -\frac{a_0 r(r+1)(r^4 - 12r^2 - 17r + 1)}{(2+r)^2 (r+3)^2 (r+4)^2} \\
a_4 &= \frac{(r^6 + 3r^5 - 16r^4 - 68r^3 - 43r^2 + 65r + 67) a_0 r}{(r+5)^2 (r+4)^2 (r+3)^2 (2+r)} \\
a_5 &= -\frac{(r^8 + 8r^7 - r^6 - 163r^5 - 455r^4 - 131r^3 + 997r^2 + 1312r + 511) a_0 r}{(r+6)^2 (2+r)^2 (r+3)(r+4)^2 (r+5)^2}
\end{aligned}
$$

Since the $F = -2x^2 + x$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = -2x^2$ by solving the balance equation

$$
(x^m m(-1+m) + 3x^m m + x^m)\, c_0 = -2x^2
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -\frac{2}{9} \\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+2}
\end{aligned}
$$

Where in the above $c_0 = -\frac{2}{9}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -\frac{2}{9}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = -\frac{2}{9}$
$c_1 = \frac{1}{36}$
$c_2 = \frac{1}{90}$
$c_3 = -\frac{13}{2160}$
$c_4 = \frac{41}{26460}$
$c_5 = -\frac{347}{1354752}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^2 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
y_p = x^2 \left(-\frac{2}{9} + \frac{1}{36} x + \frac{1}{90} x^2 - \frac{13}{2160} x^3 + \frac{41}{26460} x^4 - \frac{347}{1354752} x^5 \right)
$$

$$= -\frac{2}{9}x^2 + \frac{1}{36}x^3 + \frac{1}{90}x^4 - \frac{13}{2160}x^5 + \frac{41}{26460}x^6 - \frac{347}{1354752}x^7$$

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$(x^m m(-1+m) + 3x^m m + x^m)\, c_0 = x$$

For c_0 and x. This results in

$$c_0 = \frac{1}{4}$$
$$m = 1$$

The particular solution is therefore

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+m}$$
$$= \sum_{n=0}^{\infty} c_n x^{n+1}$$

Where in the above $c_0 = \frac{1}{4}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{4}$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{4}$
$c_1 = 0$
$c_2 = -\frac{1}{64}$
$c_3 = \frac{3}{800}$
$c_4 = \frac{1}{19200}$
$c_5 = -\frac{11}{33600}$

The particular solution is now found using

$$y_p = x^m \sum_{n=0}^{\infty} c_n x^n$$
$$= x \sum_{n=0}^{\infty} c_n x^n$$

Using the values found above for c_n into the above sum gives

$$y_p = x\left(\frac{1}{4} - \frac{1}{64}x^2 + \frac{3}{800}x^3 + \frac{1}{19200}x^4 - \frac{11}{33600}x^5\right)$$
$$= \frac{1}{4}x - \frac{1}{64}x^3 + \frac{3}{800}x^4 + \frac{1}{19200}x^5 - \frac{11}{33600}x^6$$

Adding all the above particular solution(s) gives

$$y_p = \frac{x}{4} - \frac{2x^2}{9} + \frac{7x^3}{576} + \frac{107x^4}{7200} - \frac{1031x^5}{172800} + \frac{2587x^6}{2116800} - \frac{347x^7}{1354752} + O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{x}{4} - \frac{2x^2}{9} + \frac{7x^3}{576} + \frac{107x^4}{7200} - \frac{1031x^5}{172800} + O(x^6)$$

Hence the final solution is

$y = y_h + y_p$

$$= \frac{x}{4} - \frac{2x^2}{9} + \frac{7x^3}{576} + \frac{107x^4}{7200} - \frac{1031x^5}{172800} + O(x^6) + \frac{c_1\left(1 - 2x + \frac{x^2}{4} - \frac{x^4}{64} + \frac{3x^5}{800} + O(x^6)\right)}{x} + c_2\left(\frac{\left(1 - 2x + \frac{x^2}{4} - \frac{x^4}{64} + \frac{3x^5}{800} + O(x^6)\right)\ln(x)}{x} + \frac{-x^2 + 7x + \frac{7x^3}{36} + \frac{35x^4}{1152} - \frac{191x^5}{9000} + O(x^6)}{x}\right)$$

Summary

The solution(s) found are the following

$$y = \frac{x}{4} - \frac{2x^2}{9} + \frac{7x^3}{576} + \frac{107x^4}{7200} - \frac{1031x^5}{172800} + O(x^6) + \frac{c_1\left(1 - 2x + \frac{x^2}{4} - \frac{x^4}{64} + \frac{3x^5}{800} + O(x^6)\right)}{x} + c_2\left(\frac{\left(1 - 2x + \frac{x^2}{4} - \frac{x^4}{64} + \frac{3x^5}{800} + O(x^6)\right)\ln(x)}{x} + \frac{-x^2 + 7x + \frac{7x^3}{36} + \frac{35x^4}{1152} - \frac{191x^5}{9000} + O(x^6)}{x}\right) \tag{1}$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{x}{4}-\frac{2x^{2}}{9}+\frac{7x^{3}}{576}+\frac{107x^{4}}{7200}-\frac{1031x^{5}}{172800}+O\left(x^{6}\right)+\frac{c_{1}\left(1-2x+\frac{x^{2}}{4}-\frac{x^{4}}{64}+\frac{3x^{5}}{800}+O(x^{6})\right)}{x} \\
& + c_{2}\left(\frac{\left(1-2x+\frac{x^{2}}{4}-\frac{x^{4}}{64}+\frac{3x^{5}}{800}+O(x^{6})\right)\ln\left(x\right)}{x}\right. \\
& \left. + \frac{-x^{2}+7x+\frac{7x^{3}}{36}+\frac{35x^{4}}{1152}-\frac{191x^{5}}{9000}+O(x^{6})}{x}\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0,
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 81

```
Order:=6;
dsolve(x^2*(x+1)*diff(y(x),x$2)+x*(x^2+3)*diff(y(x),x)+y(x)=x-2*x^2,y(x),type='series',x=0);
```

$$y(x) = \frac{x^2\left(\frac{1}{4}-\frac{2}{9}x+\frac{7}{576}x^2+\frac{107}{7200}x^3-\frac{1031}{172800}x^4+\mathrm{O}\left(x^5\right)\right)+\left(c_2\ln\left(x\right)+c_1\right)\left(1-2x+\frac{1}{4}x^2-\frac{1}{64}x^4+\frac{3}{800}x^5+\mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 300

```
AsymptoticDSolveValue[x^2*(x+1)*y''[x]+x*(x^2+3)*y'[x]+y[x]==x-2*x^2,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_2\left(\frac{3x^5}{800} - \frac{x^4}{64} + \frac{x^2}{4} - 2x + 1\right)}{x} + \frac{c_1\left(x^5\left(\frac{3\log(x)}{800} - \frac{629}{36000}\right) + x^4\left(\frac{17}{1152} - \frac{\log(x)}{64}\right) + \frac{7x^3}{36} + x^2\left(\frac{\log(x)}{4} - \frac{3}{4}\right) + x(5 - 2\log(x)) + \log(x) + 1\right)}{x} + \frac{\left(\frac{34109x^6}{1152} - \frac{491x^5}{30} + \frac{123x^4}{16} - \frac{8x^3}{3} + \frac{x^2}{2}\right)\left(x^5\left(\frac{3\log(x)}{800} - \frac{629}{36000}\right) + x^4\left(\frac{17}{1152} - \frac{\log(x)}{64}\right) + \frac{7x^3}{36} + x^2\left(\frac{\log(x)}{4} - \frac{3}{4}\right) + \right.}{x} + \frac{\left(\frac{3x^5}{800} - \frac{x^4}{64} + \frac{x^2}{4} - 2x + 1\right)\left(\frac{x^6(33586 - 34109\log(x))}{1152} + \frac{1}{900}x^5(14730\log(x) - 12641) + \frac{1}{64}x^4(319 - 492\log(x\right.}{x}$$

25.8 problem 7

Internal problem ID [2424]
Internal file name [`OUTPUT/2424_Tuesday_February_27_2024_08_37_20_AM_71160138/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{3x^2(x+1)\,y''+x(5-x)\,y'+\left(2x^2-1\right)y=-x^3+x}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(3x^3+3x^2\right)y''+\left(-x^2+5x\right)y'+\left(2x^2-1\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned} p(x) &= -\frac{x-5}{3x\,(x+1)} \\ q(x) &= \frac{2x^2-1}{3x^2\,(x+1)} \end{aligned}$$

Table 715: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-5}{3x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{2x^2-1}{3x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3x^2(x+1)\,y'' + (-x^2+5x)\,y' + (2x^2-1)\,y = -x^3 + x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $3x^2(x+1)\,y'' + (-x^2+5x)\,y' + (2x^2-1)\,y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&3x^2(x+1)\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\
&+\left(-x^2+5x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(2x^2-1\right)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned}
\tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}3x^{1+n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}3x^{n+r}a_n(n+r)\,(n+r-1)\right) \\
&+\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}5x^{n+r}a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty}2x^{n+r+2}a_n\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}3x^{1+n+r}a_n(n+r)\,(n+r-1) &= \sum_{n=1}^{\infty}3a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r} \\
\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n(n+r)\right) &= \sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,x^{n+r}\right) \\
\sum_{n=0}^{\infty}2x^{n+r+2}a_n &= \sum_{n=2}^{\infty}2a_{n-2}x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r}a_n(n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r}a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2}x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r}a_n(n+r)(n+r-1) + 5x^{n+r}a_n(n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r) + 5x^r a_0 r - a_0 x^r = 0$$

Or

$$(3x^r r(-1+r) + 5x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 + 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -1$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(3x^m m(-1+m) + 5x^m m - x^m)\, c_0 = -x^3 + x$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-3r^2 + 4r}{3r^2 + 8r + 4}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} 3a_{n-1}(n + r - 1) (n + r - 2) + 3a_n(n + r) (n + r - 1) \\ - a_{n-1}(n + r - 1) + 5a_n(n + r) + 2a_{n-2} - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3n^2 a_{n-1} + 6nr a_{n-1} + 3r^2 a_{n-1} - 10n a_{n-1} - 10r a_{n-1} + 2a_{n-2} + 7a_{n-1}}{3n^2 + 6nr + 3r^2 + 2n + 2r - 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{-3n^2 a_{n-1} + 8n a_{n-1} - 2a_{n-2} - 4a_{n-1}}{n\,(3n + 4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+4r}{3r^2+8r+4}$	$\frac{1}{7}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9r^4 - 6r^3 - 17r^2 - 12r - 8}{9r^4 + 66r^3 + 169r^2 + 176r + 60}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = -\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+4r}{3r^2+8r+4}$	$\frac{1}{7}$
a_2	$\frac{9r^4-6r^3-17r^2-12r-8}{9r^4+66r^3+169r^2+176r+60}$	$-\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-27r^6 - 54r^5 + 81r^4 + 256r^3 + 166r^2 - 8r + 32}{(3r^2 + 20r + 32)(3r^2 + 8r + 4)(3r^2 + 14r + 15)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = \frac{29}{2730}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+4r}{3r^2+8r+4}$	$\frac{1}{7}$
a_2	$\frac{9r^4-6r^3-17r^2-12r-8}{9r^4+66r^3+169r^2+176r+60}$	$-\frac{1}{10}$
a_3	$\frac{-27r^6-54r^5+81r^4+256r^3+166r^2-8r+32}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{29}{2730}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r^8 + 540r^7 + 864r^6 - 1416r^5 - 5531r^4 - 5004r^3 - 858r^2 + 760r + 32}{(3r^2 + 26r + 55)(9r^4 + 66r^3 + 169r^2 + 176r + 60)(3r^2 + 20r + 32)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = -\frac{17}{87360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+4r}{3r^2+8r+4}$	$\frac{1}{7}$
a_2	$\frac{9r^4-6r^3-17r^2-12r-8}{9r^4+66r^3+169r^2+176r+60}$	$-\frac{1}{10}$
a_3	$\frac{-27r^6-54r^5+81r^4+256r^3+166r^2-8r+32}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{29}{2730}$
a_4	$\frac{81r^8+540r^7+864r^6-1416r^5-5531r^4-5004r^3-858r^2+760r+32}{(3r^2+26r+55)(9r^4+66r^3+169r^2+176r+60)(3r^2+20r+32)}$	$-\frac{17}{87360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-243r^{10} - 3240r^9 - 15822r^8 - 28584r^7 + 22557r^6 + 171136r^5 + 256428r^4 + 138264r^3 - 5876r^2 - 2}{(3r^2 + 32r + 84)(3r^2 + 26r + 55)(9r^4 + 66r^3 + 169r^2 + 176r + 60)(3r^2 + 20r + 32)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{1193}{8299200}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-3r^2+4r}{3r^2+8r+4}$	$\frac{1}{7}$
a_2	$\frac{9r^4-6r^3-17r^2-12r-8}{9r^4+66r^3+169r^2+176r+60}$	$-\frac{1}{10}$
a_3	$\frac{-27r^6-54r^5+81r^4+256r^3+166r^2-8r+32}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$\frac{29}{2730}$
a_4	$\frac{81r^8+540r^7+864r^6-1416r^5-5531r^4-5004r^3-858r^2+760r+32}{(3r^2+26r+55)(9r^4+66r^3+169r^2+176r+60)(3r^2+20r+32)}$	$-\frac{17}{87360}$
a_5	$\frac{-243r^{10}-3240r^9-15822r^8-28584r^7+22557r^6+171136r^5+256428r^4+138264r^3-5876r^2-25744r-4544}{(3r^2+32r+84)(3r^2+26r+55)(9r^4+66r^3+169r^2+176r+60)(3r^2+20r+32)}$	$-\frac{1193}{8299200}$

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = x^{\frac{1}{3}}(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots)$$
$$= x^{\frac{1}{3}}\left(1 + \frac{x}{7} - \frac{x^2}{10} + \frac{29x^3}{2730} - \frac{17x^4}{87360} - \frac{1193x^5}{8299200} + O(x^6)\right)$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{-3r^2 + 4r}{3r^2 + 8r + 4}$$

For $2 \leq n$ the recursive equation is

$$3b_{n-1}(n + r - 1)(n + r - 2) + 3b_n(n + r)(n + r - 1) - b_{n-1}(n + r - 1) + 5b_n(n + r) + 2b_{n-2} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{3n^2b_{n-1} + 6nrb_{n-1} + 3r^2b_{n-1} - 10nb_{n-1} - 10rb_{n-1} + 2b_{n-2} + 7b_{n-1}}{3n^2 + 6nr + 3r^2 + 2n + 2r - 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{-3n^2b_{n-1} + 16nb_{n-1} - 2b_{n-2} - 20b_{n-1}}{n(3n - 4)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3r^2+4r}{3r^2+8r+4}$	7

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9r^4 - 6r^3 - 17r^2 - 12r - 8}{9r^4 + 66r^3 + 169r^2 + 176r + 60}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3r^2+4r}{3r^2+8r+4}$	7
b_2	$\frac{9r^4-6r^3-17r^2-12r-8}{9r^4+66r^3+169r^2+176r+60}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-27r^6 - 54r^5 + 81r^4 + 256r^3 + 166r^2 - 8r + 32}{(3r^2 + 20r + 32)(3r^2 + 8r + 4)(3r^2 + 14r + 15)}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{29}{30}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3r^2+4r}{3r^2+8r+4}$	7
b_2	$\frac{9r^4-6r^3-17r^2-12r-8}{9r^4+66r^3+169r^2+176r+60}$	$-\frac{1}{2}$
b_3	$\frac{-27r^6-54r^5+81r^4+256r^3+166r^2-8r+32}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{29}{30}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81r^8 + 540r^7 + 864r^6 - 1416r^5 - 5531r^4 - 5004r^3 - 858r^2 + 760r + 32}{(3r^2 + 26r + 55)(9r^4 + 66r^3 + 169r^2 + 176r + 60)(3r^2 + 20r + 32)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{73}{480}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3r^2+4r}{3r^2+8r+4}$	7
b_2	$\frac{9r^4-6r^3-17r^2-12r-8}{9r^4+66r^3+169r^2+176r+60}$	$-\frac{1}{2}$
b_3	$\frac{-27r^6-54r^5+81r^4+256r^3+166r^2-8r+32}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{29}{30}$
b_4	$\frac{81r^8+540r^7+864r^6-1416r^5-5531r^4-5004r^3-858r^2+760r+32}{(3r^2+26r+55)(9r^4+66r^3+169r^2+176r+60)(3r^2+20r+32)}$	$\frac{73}{480}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-243r^{10} - 3240r^9 - 15822r^8 - 28584r^7 + 22557r^6 + 171136r^5 + 256428r^4 + 138264r^3 - 5876r^2 - 2}{(3r^2 + 32r + 84)(3r^2 + 26r + 55)(9r^4 + 66r^3 + 169r^2 + 176r + 60)(3r^2 + 20r + 32)}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{167}{26400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-3r^2+4r}{3r^2+8r+4}$	7
b_2	$\frac{9r^4-6r^3-17r^2-12r-8}{9r^4+66r^3+169r^2+176r+60}$	$-\frac{1}{2}$
b_3	$\frac{-27r^6-54r^5+81r^4+256r^3+166r^2-8r+32}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)}$	$-\frac{29}{30}$
b_4	$\frac{81r^8+540r^7+864r^6-1416r^5-5531r^4-5004r^3-858r^2+760r+32}{(3r^2+26r+55)(9r^4+66r^3+169r^2+176r+60)(3r^2+20r+32)}$	$\frac{73}{480}$
b_5	$\frac{-243r^{10}-3240r^9-15822r^8-28584r^7+22557r^6+171136r^5+256428r^4+138264r^3-5876r^2-25744r-4544}{(3r^2+32r+84)(3r^2+26r+55)(9r^4+66r^3+169r^2+176r+60)(3r^2+20r+32)}$	$-\frac{167}{26400}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{3}}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\ &= \frac{1 + 7x - \frac{x^2}{2} - \frac{29x^3}{30} + \frac{73x^4}{480} - \frac{167x^5}{26400} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{3}} \left(1 + \frac{x}{7} - \frac{x^2}{10} + \frac{29x^3}{2730} - \frac{17x^4}{87360} - \frac{1193x^5}{8299200} + O(x^6)\right) \\
&\quad + \frac{c_2 \left(1 + 7x - \frac{x^2}{2} - \frac{29x^3}{30} + \frac{73x^4}{480} - \frac{167x^5}{26400} + O(x^6)\right)}{x}
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
(3x^m m(-1+m) + 5x^m m - x^m)\, c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{aligned}
a_1 &= \frac{a_0(-3r^2+4r)}{3r^2+8r+4} \\
a_2 &= \frac{a_0(9r^4-6r^3-17r^2-12r-8)}{9r^4+66r^3+169r^2+176r+60} \\
a_3 &= -\frac{(27r^6+54r^5-81r^4-256r^3-166r^2+8r-32)a_0}{(3r^2+20r+32)(3r^2+8r+4)(3r^2+14r+15)} \\
a_4 &= \frac{a_0(81r^8+540r^7+864r^6-1416r^5-5531r^4-5004r^3-858r^2+760r+32)}{(3r^2+26r+55)(9r^4+66r^3+169r^2+176r+60)(3r^2+20r+32)} \\
a_5 &= -\frac{a_0(243r^{10}+3240r^9+15822r^8+28584r^7-22557r^6-171136r^5-256428r^4-138264r^3+5876r^2+25744r+4544)}{(3r^2+32r+84)(3r^2+26r+55)(9r^4+66r^3+169r^2+176r+60)(3r^2+20r+32)}
\end{aligned}
$$

Since the $F = -x^3 + x$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = -x^3$ by solving the balance equation

$$
(3x^m m(-1+m) + 5x^m m - x^m)\, c_0 = -x^3
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -\frac{1}{32} \\
m &= 3
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+3}
\end{aligned}
$$

Where in the above $c_0 = -\frac{1}{32}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -\frac{1}{32}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = -\frac{1}{32}$
$c_1 = \frac{3}{352}$
$c_2 = -\frac{37}{14784}$
$c_3 = \frac{1783}{1759296}$
$c_4 = -\frac{10069}{20106240}$
$c_5 = \frac{8102197}{29133941760}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3 \left(-\frac{1}{32} + \frac{3}{352} x - \frac{37}{14784} x^2 + \frac{1783}{1759296} x^3 - \frac{10069}{20106240} x^4 + \frac{8102197}{29133941760} x^5 \right) \\
&= -\frac{1}{32} x^3 + \frac{3}{352} x^4 - \frac{37}{14784} x^5 + \frac{1783}{1759296} x^6 - \frac{10069}{20106240} x^7 + \frac{8102197}{29133941760} x^8
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$\left(3x^m m(-1+m)+5x^m m - x^m\right) c_0 = x$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{4} \\ m &= 1 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+1} \end{aligned}$$

Where in the above $c_0 = \frac{1}{4}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{4}$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{4}$
$c_1 = \frac{1}{60}$
$c_2 = -\frac{17}{960}$
$c_3 = \frac{223}{52800}$
$c_4 = -\frac{2633}{2217600}$
$c_5 = \frac{126083}{263894400}$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x\left(\frac{1}{4}+\frac{1}{60}x-\frac{17}{960}x^2+\frac{223}{52800}x^3-\frac{2633}{2217600}x^4+\frac{126083}{263894400}x^5\right) \\
&= \frac{1}{4}x+\frac{1}{60}x^2-\frac{17}{960}x^3+\frac{223}{52800}x^4-\frac{2633}{2217600}x^5+\frac{126083}{263894400}x^6
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = \frac{x}{4}+\frac{x^2}{60}-\frac{47x^3}{960}+\frac{673x^4}{52800}-\frac{1169x^5}{316800}+\frac{3307x^6}{2217600}-\frac{10069x^7}{20106240}+\frac{8102197x^8}{29133941760}+O\left(x^6\right)
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = \frac{x}{4}+\frac{x^2}{60}-\frac{47x^3}{960}+\frac{673x^4}{52800}-\frac{1169x^5}{316800}+O\left(x^6\right)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x}{4}+\frac{x^2}{60}-\frac{47x^3}{960}+\frac{673x^4}{52800}-\frac{1169x^5}{316800}+O\left(x^6\right) \\
&\quad + c_1 x^{\frac{1}{3}}\left(1+\frac{x}{7}-\frac{x^2}{10}+\frac{29x^3}{2730}-\frac{17x^4}{87360}-\frac{1193x^5}{8299200}+O\left(x^6\right)\right) \\
&\quad + \frac{c_2\left(1+7x-\frac{x^2}{2}-\frac{29x^3}{30}+\frac{73x^4}{480}-\frac{167x^5}{26400}+O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{x}{4}+\frac{x^2}{60}-\frac{47x^3}{960}+\frac{673x^4}{52800}-\frac{1169x^5}{316800}+O\left(x^6\right) \\
&\quad + c_1 x^{\frac{1}{3}}\left(1+\frac{x}{7}-\frac{x^2}{10}+\frac{29x^3}{2730}-\frac{17x^4}{87360}-\frac{1193x^5}{8299200}+O\left(x^6\right)\right) \\
&\quad + \frac{c_2\left(1+7x-\frac{x^2}{2}-\frac{29x^3}{30}+\frac{73x^4}{480}-\frac{167x^5}{26400}+O(x^6)\right)}{x}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$y = \frac{x}{4}+\frac{x^2}{60}-\frac{47x^3}{960}+\frac{673x^4}{52800}-\frac{1169x^5}{316800}+O(x^6) + c_1 x^{\frac{1}{3}}\left(1+\frac{x}{7}-\frac{x^2}{10}+\frac{29x^3}{2730}-\frac{17x^4}{87360}-\frac{1193x^5}{8299200}+O(x^6)\right) + \frac{c_2\left(1+7x-\frac{x^2}{2}-\frac{29x^3}{30}+\frac{73x^4}{480}-\frac{167x^5}{26400}+O(x^6)\right)}{x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0,
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 63

```
Order:=6;
dsolve(3*x^2*(x+1)*diff(y(x),x$2)+x*(5-x)*diff(y(x),x)+(2*x^2-1)*y(x)=x-x^3,y(x),type='serie
```

$$y(x) = \frac{c_2 x^{\frac{4}{3}} \left(1+\frac{1}{7}x-\frac{1}{10}x^2+\frac{29}{2730}x^3-\frac{17}{87360}x^4-\frac{1193}{8299200}x^5+\mathrm{O}\left(x^6\right)\right)+x^2\left(\frac{1}{4}+\frac{1}{60}x-\frac{47}{960}x^2+\frac{673}{52800}x^3-\frac{1169}{316800}x^{\cdots}\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 255

```
AsymptoticDSolveValue[3*x^2*(x+1)*y''[x]+x*(5-x)*y'[x]+(2*x^2-1)*y[x]==x-x^3,y[x],{x,0,5}]
```

$$\begin{aligned} y(x) \to\ & \frac{c_1\left(-\frac{167x^5}{26400}+\frac{73x^4}{480}-\frac{29x^3}{30}-\frac{x^2}{2}+7x+1\right)}{x} \\ & + c_2\sqrt[3]{x}\left(-\frac{1193x^5}{8299200}-\frac{17x^4}{87360}+\frac{29x^3}{2730}-\frac{x^2}{10}+\frac{x}{7}+1\right)+\sqrt[3]{x}\left(-\frac{1193x^5}{8299200}-\frac{17x^4}{87360}\right. \\ & \left.+\frac{29x^3}{2730}-\frac{x^2}{10}+\frac{x}{7}+1\right)\left(\frac{19491x^{17/3}}{8800}-\frac{541x^{14/3}}{256}+\frac{107x^{11/3}}{55}-\frac{99x^{8/3}}{64}+\frac{3x^{5/3}}{5}\right. \\ & \left.+\frac{3x^{2/3}}{8}\right)+\frac{\left(-\frac{167x^5}{26400}+\frac{73x^4}{480}-\frac{29x^3}{30}-\frac{x^2}{2}+7x+1\right)\left(-\frac{652399x^6}{2096640}+\frac{2039x^5}{6825}-\frac{313x^4}{1120}+\frac{5x^3}{21}-\frac{x^2}{8}\right)}{x} \end{aligned}$$

25.9 problem 8

Internal problem ID [2425]
Internal file name [`OUTPUT/2425_Tuesday_February_27_2024_08_37_21_AM_15207984/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{9x^2y'' + (2+3x)\,y = x^4 + x^2}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + (2+3x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{2+3x}{9x^2}$$

Table 716: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{2+3x}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + (2 + 3x)\, y = x^4 + x^2$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $9x^2y'' + (2 + 3x)\, y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) + (2 + 3x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n + r)\, (n + r - 1)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r)(n+r-1) + 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$9x^r a_0 r(-1+r) + 2a_0 x^r = 0$$

Or

$$(9x^r r(-1+r) + 2x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(9r^2 - 9r + 2)\, x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 9r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{2}{3}$$
$$r_2 = \frac{1}{3}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(9x^m m(-1+m) + 2x^m)\, c_0 = x^4 + x^2$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 9r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$9a_n(n + r)(n + r - 1) + 2a_n + 3a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_n = -\frac{a_{n-1}}{3n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3}{9r^2 + 9r + 2}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_1 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_2 = \frac{1}{56}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{4}$
a_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{56}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_3 = -\frac{1}{1680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{4}$
a_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{56}$
a_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{1680}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_4 = \frac{1}{87360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{4}$
a_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{56}$
a_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{1680}$
a_4	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{1}{87360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)(9r^2 + 81r + 182)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$a_5 = -\frac{1}{6988800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{4}$
a_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{56}$
a_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{1680}$
a_4	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{1}{87360}$
a_5	$-\frac{243}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{1}{6988800}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{2}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{2}{3}}\left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \frac{x^4}{87360} - \frac{x^5}{6988800} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$9b_n(n+r)(n+r-1) + 2b_n + 3b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{3b_{n-1}}{9n^2 + 18nr + 9r^2 - 9n - 9r + 2} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_n = -\frac{b_{n-1}}{n(3n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{3}{9r^2 + 9r + 2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = \frac{1}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{20}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{27}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_3 = -\frac{1}{480}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{20}$
b_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{480}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{1}{21120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{20}$
b_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{480}$
b_4	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{1}{21120}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243}{(9r^2 + 9r + 2)(9r^2 + 27r + 20)(9r^2 + 45r + 56)(9r^2 + 63r + 110)(9r^2 + 81r + 182)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_5 = -\frac{1}{1478400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{9r^2+9r+2}$	$-\frac{1}{2}$
b_2	$\frac{9}{(9r^2+9r+2)(9r^2+27r+20)}$	$\frac{1}{20}$
b_3	$-\frac{27}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)}$	$-\frac{1}{480}$
b_4	$\frac{81}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)}$	$\frac{1}{21120}$
b_5	$-\frac{243}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}$	$-\frac{1}{1478400}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{2}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \frac{x^4}{21120} - \frac{x^5}{1478400} + O\left(x^6\right)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{2}{3}}\left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \frac{x^4}{87360} - \frac{x^5}{6988800} + O\left(x^6\right)\right) \\
&\quad + c_2 x^{\frac{1}{3}}\left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \frac{x^4}{21120} - \frac{x^5}{1478400} + O\left(x^6\right)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(9x^m m(-1+m) + 2x^m\right) c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{aligned}
a_1 &= -\frac{3a_0}{9r^2+9r+2} \\
a_2 &= \frac{9a_0}{(9r^2+9r+2)(9r^2+27r+20)} \\
a_3 &= -\frac{27a_0}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)} \\
a_4 &= \frac{81a_0}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)} \\
a_5 &= -\frac{243a_0}{(9r^2+9r+2)(9r^2+27r+20)(9r^2+45r+56)(9r^2+63r+110)(9r^2+81r+182)}
\end{aligned}
$$

Since the $F = x^4 + x^2$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^4$ by solving the balance equation

$$
\left(9x^m m(-1+m) + 2x^m\right) c_0 = x^4
$$

For c_0 and x. This results in

$$c_0 = \frac{1}{110}$$
$$m = 4$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+4} \end{aligned}$$

Where in the above $c_0 = \frac{1}{110}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{110}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{110}$
$c_1 = -\frac{3}{20020}$
$c_2 = \frac{9}{5445440}$
$c_3 = -\frac{27}{2069267200}$
$c_4 = \frac{81}{1047049203200}$
$c_5 = -\frac{243}{680581982080000}$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^4 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^4\left(\frac{1}{110} - \frac{3}{20020}x + \frac{9}{5445440}x^2 - \frac{27}{2069267200}x^3 + \frac{81}{1047049203200}x^4 \right.\\
&\quad \left. - \frac{243}{680581982080000}x^5\right) \\
&= \frac{1}{110}x^4 - \frac{3}{20020}x^5 + \frac{9}{5445440}x^6 - \frac{27}{2069267200}x^7 \\
&\quad + \frac{81}{1047049203200}x^8 - \frac{243}{680581982080000}x^9
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$
\left(9x^m m(-1+m) + 2x^m\right) c_0 = x^2
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{20} \\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+2}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{20}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{20}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$
\begin{aligned}
c_0 &= \frac{1}{20}\\
c_1 &= -\frac{3}{1120}\\
c_2 &= \frac{9}{123200}\\
c_3 &= -\frac{27}{22422400}\\
c_4 &= \frac{81}{6098892800}\\
c_5 &= -\frac{243}{2317579264000}
\end{aligned}
$$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n\\
&= x^2 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^2 \left(\frac{1}{20} - \frac{3}{1120}x + \frac{9}{123200}x^2 - \frac{27}{22422400}x^3 + \frac{81}{6098892800}x^4 - \frac{243}{2317579264000}x^5\right)\\
&= \frac{1}{20}x^2 - \frac{3}{1120}x^3 + \frac{9}{123200}x^4 - \frac{27}{22422400}x^5 + \frac{81}{6098892800}x^6 - \frac{243}{2317579264000}x^7
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
\begin{aligned}
y_p = {} & \frac{x^2}{20} - \frac{3x^3}{1120} + \frac{1129x^4}{123200} - \frac{3387x^5}{22422400} + \frac{10161x^6}{6098892800}\\
& - \frac{30483x^7}{2317579264000} + \frac{81x^8}{1047049203200} - \frac{243x^9}{680581982080000} + O\left(x^6\right)
\end{aligned}
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = \frac{x^2}{20} - \frac{3x^3}{1120} + \frac{1129x^4}{123200} - \frac{3387x^5}{22422400} + O\left(x^6\right)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x^2}{20} - \frac{3x^3}{1120} + \frac{1129x^4}{123200} - \frac{3387x^5}{22422400} + O(x^6) \\
&\quad + c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \frac{x^4}{87360} - \frac{x^5}{6988800} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \frac{x^4}{21120} - \frac{x^5}{1478400} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{x^2}{20} - \frac{3x^3}{1120} + \frac{1129x^4}{123200} - \frac{3387x^5}{22422400} + O(x^6) \\
&\quad + c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \frac{x^4}{87360} - \frac{x^5}{6988800} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \frac{x^4}{21120} - \frac{x^5}{1478400} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{x^2}{20} - \frac{3x^3}{1120} + \frac{1129x^4}{123200} - \frac{3387x^5}{22422400} + O(x^6) \\
&\quad + c_1 x^{\frac{2}{3}} \left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \frac{x^4}{87360} - \frac{x^5}{6988800} + O(x^6)\right) \\
&\quad + c_2 x^{\frac{1}{3}} \left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \frac{x^4}{21120} - \frac{x^5}{1478400} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 63

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)+(2+3*x)*y(x)=x^2+x^4,y(x),type='series',x=0);
```

$$\begin{aligned}y(x) &= c_1 x^{\frac{1}{3}}\left(1-\frac{1}{2}x+\frac{1}{20}x^2-\frac{1}{480}x^3+\frac{1}{21120}x^4-\frac{1}{1478400}x^5+\mathrm{O}\left(x^6\right)\right)\\&\quad+c_2 x^{\frac{2}{3}}\left(1-\frac{1}{4}x+\frac{1}{56}x^2-\frac{1}{1680}x^3+\frac{1}{87360}x^4-\frac{1}{6988800}x^5+\mathrm{O}\left(x^6\right)\right)\\&\quad+x^2\left(\frac{1}{20}-\frac{3}{1120}x+\frac{1129}{123200}x^2-\frac{3387}{22422400}x^3+\mathrm{O}\left(x^4\right)\right)\end{aligned}$$

✓ Solution by Mathematica

Time used: 0.182 (sec). Leaf size: 264

```
AsymptoticDSolveValue[9*x^2*y''[x]+(2+3*x)*y[x]==x^2+x^4,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt[3]{x}\left(-\frac{x^5}{1478400}+\frac{x^4}{21120}-\frac{x^3}{480}+\frac{x^2}{20}-\frac{x}{2}+1\right)+x^{2/3}\left(-\frac{x^5}{6988800}+\frac{x^4}{87360}-\frac{x^3}{1680}+\frac{x^2}{56}-\frac{x}{4}+1\right)\left(\frac{1057x^{16/3}}{337920}-\frac{241x^{13/3}}{6240}+\frac{21x^{10/3}}{200}-\frac{x^{7/3}}{14}+\frac{x^{4/3}}{4}\right)+\sqrt[3]{x}\left(-\frac{x^5}{1478400}+\frac{x^4}{21120}-\frac{x^3}{480}+\frac{x^2}{20}-\frac{x}{2}+1\right)\left(-\frac{223x^{17/3}}{212160}+\frac{421x^{14/3}}{23520}-\frac{57x^{11/3}}{616}+\frac{x^{8/3}}{32}-\frac{x^{5/3}}{5}\right)$$

25.10 problem 9

Internal problem ID [2426]
Internal file name [`OUTPUT/2426_Tuesday_February_27_2024_08_37_22_AM_87092516/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 10xy' + y = x - 1}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$9x^2y'' + 10xy' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{10}{9x}$$
$$q(x) = \frac{1}{9x^2}$$

Table 717: Table $p(x), q(x)$ singularites.

$p(x) = \frac{10}{9x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$9x^2y'' + 10xy' + y = x - 1$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $9x^2y'' + 10xy' + y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$9x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 10x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 9x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 10x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$9x^{n+r} a_n (n+r) (n+r-1) + 10x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$9x^r a_0 r(-1+r) + 10x^r a_0 r + a_0 x^r = 0$$

Or

$$(9x^r r(-1+r) + 10x^r r + x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 + r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 + r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{1}{18} + \frac{i\sqrt{35}}{18}$$
$$r_2 = -\frac{1}{18} - \frac{i\sqrt{35}}{18}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(9x^m m(-1+m) + 10x^m m + x^m) c_0 = x - 1$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2+r+1\right)x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{18}+\frac{i\sqrt{35}}{18}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{18}-\frac{i\sqrt{35}}{18}}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$9a_n(n+r)(n+r-1) + 10a_n(n+r) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = -\frac{1}{18} + \frac{i\sqrt{35}}{18}$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{18} + \frac{i\sqrt{35}}{18}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{-\frac{1}{18}+\frac{i\sqrt{35}}{18}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{-\frac{1}{18}+\frac{i\sqrt{35}}{18}}\left(1 + O(x^6)\right) \end{aligned}$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$y_2(x) = x^{-\frac{1}{18}-\frac{i\sqrt{35}}{18}}\left(1 + O(x^6)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{-\frac{1}{18}+\frac{i\sqrt{35}}{18}}\left(1 + O(x^6)\right) + c_2 x^{-\frac{1}{18}-\frac{i\sqrt{35}}{18}}\left(1 + O(x^6)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(9x^m m(-1+m) + 10x^m m + x^m\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\boxed{\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0 \\ a_4 &= 0 \\ a_5 &= 0 \end{aligned}}$$

Since the $F = x - 1$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$\left(9x^m m(-1+m) + 10x^m m + x^m\right) c_0 = x$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{11} \\ m &= 1 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+1} \end{aligned}$$

Where in the above $c_0 = \frac{1}{11}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{11}$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{11}$
$c_1 = 0$
$c_2 = 0$
$c_3 = 0$
$c_4 = 0$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x \left(\frac{1}{11}\right) \\
&= \frac{x}{11}
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = -1$ by solving the balance equation

$$
(9x^m m(-1+m) + 10x^m m + x^m)\, c_0 = -1
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -1 \\
m &= 0
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+0}
\end{aligned}
$$

Where in the above $c_0 = -1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = -1$
$c_1 = 0$
$c_2 = 0$
$c_3 = 0$
$c_4 = 0$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= 1 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= 1(-1) \\
&= -1
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = -1 + \frac{x}{11} + O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= -1 + \frac{x}{11} + O(x^6) + c_1 x^{-\frac{1}{18}+\frac{i\sqrt{35}}{18}} \left(1 + O(x^6)\right) + c_2 x^{-\frac{1}{18}-\frac{i\sqrt{35}}{18}} \left(1 + O(x^6)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = -1+\frac{x}{11}+O\left(x^{6}\right)+c_{1} x^{-\frac{1}{18}+\frac{i \sqrt{35}}{18}} \left(1+O\left(x^{6}\right)\right)+c_{2} x^{-\frac{1}{18}-\frac{i \sqrt{35}}{18}} \left(1+O\left(x^{6}\right)\right) \tag{1}$$

Verification of solutions

$$y = -1+\frac{x}{11}+O\left(x^{6}\right)+c_{1} x^{-\frac{1}{18}+\frac{i \sqrt{35}}{18}} \left(1+O\left(x^{6}\right)\right)+c_{2} x^{-\frac{1}{18}-\frac{i \sqrt{35}}{18}} \left(1+O\left(x^{6}\right)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 49

```
Order:=6;
dsolve(9*x^2*diff(y(x),x$2)+10*x*diff(y(x),x)+y(x)=x-1,y(x),type='series',x=0);
```

$$y(x) = c_{1} x^{-\frac{1}{18}-\frac{i \sqrt{35}}{18}} \left(1+\mathrm{O}\left(x^{6}\right)\right)+c_{2} x^{-\frac{1}{18}+\frac{i \sqrt{35}}{18}} \left(1+\mathrm{O}\left(x^{6}\right)\right)+\left(-1+\frac{1}{11} x+\mathrm{O}\left(x^{6}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.294 (sec). Leaf size: 198

```
AsymptoticDSolveValue[9*x^2*y''[x]+10*x*y'[x]+y[x]==x-1,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \rightarrow & \frac{9i\left(\left(\sqrt{35}-i\right)x-\sqrt{35}+19i\right)x^{\frac{1}{18}\left(-1-i\sqrt{35}\right)+\frac{1}{18}\left(1+i\sqrt{35}\right)}}{\sqrt{35}\left(10\sqrt{35}+8i\right)} \\
& -\frac{9i\left(\left(\sqrt{35}+i\right)x-\sqrt{35}-19i\right)x^{\frac{1}{18}\left(1-i\sqrt{35}\right)+\frac{1}{18}\left(-1+i\sqrt{35}\right)}}{\sqrt{35}\left(10\sqrt{35}-8i\right)} \\
& +c_2x^{\frac{1}{18}\left(-1+i\sqrt{35}\right)}+c_1x^{\frac{1}{18}\left(-1-i\sqrt{35}\right)}
\end{aligned}
$$

25.11 problem 10

Internal problem ID [2427]
Internal file name [OUTPUT/2427_Tuesday_February_27_2024_08_37_22_AM_79242121/index.tex]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{2x^2y'' + \left(-x^2+x\right)y' - y = x^3+1}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + \left(-x^2+x\right)y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{x-1}{2x} \\ q(x) &= -\frac{1}{2x^2} \end{aligned}$$

Table 718: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + (-x^2 + x)\,y' - y = x^3 + 1$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' + (-x^2 + x)\,y' - y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
2x^2\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right)& \\
+\left(-x^2+x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)-\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)&=0
\end{aligned} \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)\,(n+r-1)\right)+\sum_{n\,=0}^{\infty}\left(-x^{1+n+r}a_n(n+r)\right)& \\
+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\right)+\sum_{n\,=0}^{\infty}\left(-a_n x^{n+r}\right)&=0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n\,=0}^{\infty}\left(-x^{1+n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)\,(n+r-1)\right)+\sum_{n\,=1}^{\infty}\left(-a_{n-1}(n+r-1)\,x^{n+r}\right)& \\
+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\right)+\sum_{n\,=0}^{\infty}\left(-a_n x^{n+r}\right)&=0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r}a_n(n+r)\,(n+r-1)+x^{n+r}a_n(n+r)-a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
2x^r a_0 r(-1+r)+x^r a_0 r-a_0 x^r=0
$$

Or

$$\left(2x^r r(-1+r)+x^r r-x^r\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2-r-1\right)x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2-r-1=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=1$$
$$r_2=-\frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(2x^m m(-1+m)+x^m m-x^m\right)c_0=x^3+1$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2-r-1\right)x^r=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=\frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x)=x^{r_1}\left(\sum_{n=0}^{\infty}a_n x^n\right)$$
$$y_2(x)=x^{r_2}\left(\sum_{n=0}^{\infty}b_n x^n\right)$$

Or

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^{1+n}$$
$$y_2(x)=\sum_{n=0}^{\infty}b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n+2r+1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}}{2n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{3+2r}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^2+16r+15}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8r^3 + 60r^2 + 142r + 105}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$
a_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{315}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^4 + 192r^3 + 824r^2 + 1488r + 945}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{3465}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$
a_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{315}$
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{3465}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32r^5 + 560r^4 + 3760r^3 + 12040r^2 + 18258r + 10395}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1}{45045}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$
a_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{315}$
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{3465}$
a_5	$\frac{1}{32r^5+560r^4+3760r^3+12040r^2+18258r+10395}$	$\frac{1}{45045}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + b_n(n+r) - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{2n + 2r + 1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = \frac{b_{n-1}}{2n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{3 + 2r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{8r^3 + 60r^2 + 142r + 105}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = \frac{1}{48}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^4 + 192r^3 + 824r^2 + 1488r + 945}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{48}$
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{32r^5 + 560r^4 + 3760r^3 + 12040r^2 + 18258r + 10395}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = \frac{1}{3840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{48}$
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$
b_5	$\frac{1}{32r^5+560r^4+3760r^3+12040r^2+18258r+10395}$	$\frac{1}{3840}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)}{\sqrt{x}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(2x^m m(-1+m) + x^m m - x^m\right) c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{aligned}
a_1 &= \frac{a_0}{3+2r} \\
a_2 &= \frac{a_0}{4r^2+16r+15} \\
a_3 &= \frac{a_0}{8r^3+60r^2+142r+105} \\
a_4 &= \frac{a_0}{16r^4+192r^3+824r^2+1488r+945} \\
a_5 &= \frac{a_0}{32r^5+560r^4+3760r^3+12040r^2+18258r+10395}
\end{aligned}
$$

Since the $F = x^3 + 1$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^3$ by solving the balance equation

$$
(2x^m m(-1+m) + x^m m - x^m) c_0 = x^3
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{14} \\
m &= 3
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+3}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{14}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{14}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$
\boxed{
\begin{aligned}
c_0 &= \frac{1}{14} \\
c_1 &= \frac{1}{126} \\
c_2 &= \frac{1}{1386} \\
c_3 &= \frac{1}{18018} \\
c_4 &= \frac{1}{270270} \\
c_5 &= \frac{1}{4594590}
\end{aligned}
}
$$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3 \left(\frac{1}{14} + \frac{1}{126}x + \frac{1}{1386}x^2 + \frac{1}{18018}x^3 + \frac{1}{270270}x^4 + \frac{1}{4594590}x^5\right) \\
&= \frac{1}{14}x^3 + \frac{1}{126}x^4 + \frac{1}{1386}x^5 + \frac{1}{18018}x^6 + \frac{1}{270270}x^7 + \frac{1}{4594590}x^8
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$
\left(2x^m m(-1+m) + x^m m - x^m\right) c_0 = 1
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -1 \\
m &= 0
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+0}
\end{aligned}
$$

Where in the above $c_0 = -1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = -1$
$c_1 = -\frac{1}{3}$
$c_2 = -\frac{1}{15}$
$c_3 = -\frac{1}{105}$
$c_4 = -\frac{1}{945}$
$c_5 = -\frac{1}{10395}$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= 1 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= 1\left(-1 - \frac{1}{3}x - \frac{1}{15}x^2 - \frac{1}{105}x^3 - \frac{1}{945}x^4 - \frac{1}{10395}x^5\right) \\ &= -1 - \frac{1}{3}x - \frac{1}{15}x^2 - \frac{1}{105}x^3 - \frac{1}{945}x^4 - \frac{1}{10395}x^5 \end{aligned}$$

Adding all the above particular solution(s) gives

$$y_p = -1 - \frac{x}{3} - \frac{x^2}{15} + \frac{13x^3}{210} + \frac{13x^4}{1890} + \frac{13x^5}{20790} + \frac{x^6}{18018} + \frac{x^7}{270270} + \frac{x^8}{4594590} + O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = -1 - \frac{x}{3} - \frac{x^2}{15} + \frac{13x^3}{210} + \frac{13x^4}{1890} + \frac{13x^5}{20790} + O(x^6)$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= -1 - \frac{x}{3} - \frac{x^2}{15} + \frac{13x^3}{210} + \frac{13x^4}{1890} + \frac{13x^5}{20790} + O\left(x^6\right) \\
&\quad + c_1 x \left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O\left(x^6\right)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= -1 - \frac{x}{3} - \frac{x^2}{15} + \frac{13x^3}{210} + \frac{13x^4}{1890} + \frac{13x^5}{20790} + O\left(x^6\right) \\
&\quad + c_1 x \left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O\left(x^6\right)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= -1 - \frac{x}{3} - \frac{x^2}{15} + \frac{13x^3}{210} + \frac{13x^4}{1890} + \frac{13x^5}{20790} + O\left(x^6\right) \\
&\quad + c_1 x \left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O\left(x^6\right)\right) \\
&\quad + \frac{c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
      Solution has integrals. Trying a special function solution free of integrals...
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         <- Kummer successful
      <- special function solution successful
         -> Trying to convert hypergeometric functions to elementary form...
         <- elementary form is not straightforward to achieve - returning special function s
      <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 57

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+(x-x^2)*diff(y(x),x)-y(x)=1+x^3,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\frac{1}{3840}x^5+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}} + c_2 x\left(1+\frac{1}{5}x+\frac{1}{35}x^2+\frac{1}{315}x^3+\frac{1}{3465}x^4+\frac{1}{45045}x^5+\mathrm{O}\left(x^6\right)\right) + \left(-1+\frac{1}{14}x^3+\frac{1}{126}x^4+\frac{1}{1386}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 248

```
AsymptoticDSolveValue[2*x^2*y''[x]+(x-x^2)*y'[x]-y[x]==1+x^3,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(\frac{x^5}{3840}+\frac{x^4}{384}+\frac{x^3}{48}+\frac{x^2}{8}+\frac{x}{2}+1\right)}{\sqrt{x}} + c_2 x\left(\frac{x^5}{45045}+\frac{x^4}{3465}+\frac{x^3}{315}+\frac{x^2}{35}+\frac{x}{5}+1\right) + \frac{\left(\frac{x^5}{3840}+\frac{x^4}{384}+\frac{x^3}{48}+\frac{x^2}{8}+\frac{x}{2}+1\right)\left(-\frac{6233x^{11/2}}{1921920}+\frac{2107x^{9/2}}{95040}-\frac{143x^{7/2}}{1512}-\frac{x^{5/2}}{140}+\frac{x^{3/2}}{15}-\frac{2\sqrt{x}}{3}\right)}{\sqrt{x}} + x\left(\frac{x^5}{45045}+\frac{x^4}{3465}+\frac{x^3}{315}+\frac{x^2}{35}+\frac{x}{5}+1\right)\left(\frac{x^6}{691200}-\frac{2747x^5}{518918400}+\frac{x^2}{6}-\frac{1}{3x}\right)$$

25.12 problem 11

Internal problem ID [2428]
Internal file name [`OUTPUT/2428_Tuesday_February_27_2024_08_37_23_AM_7201147/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(-x^{2}+1\right) y''+2xy'-2y=6\left(-x^{2}+1\right)^{2}}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1154}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1155}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{-6x^4 + 2xy' + 12x^2 - 2y - 6}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -24x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -24 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) + 6 \\
F_1 &= 0 \\
F_2 &= -24 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(x^2 + 1\right) y(0) - x^4 + xy'(0) + 3x^2 + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(-x^2 + 1\right) y'' + 2xy' - 2y = 6x^4 - 12x^2 + 6
$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-x^2+1)\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 6x^4 - 12x^2 + 6 \tag{1}$$

Expanding $6x^4 - 12x^2 + 6$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} 6x^4 - 12x^2 + 6 &= 6x^4 - 12x^2 + 6 + \dots \\ &= 6x^4 - 12x^2 + 6 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$(-x^2+1)\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 6x^4 - 12x^2 + 6$$

Which simplifies to

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) \tag{2}$$
$$= 6x^4 - 12x^2 + 6$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2n a_n x^n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 6x^4 - 12x^2 + 6 \tag{3}$$

$n = 0$ gives

$$(2a_2 - 2a_0)\, x^0 = 6$$
$$2a_2 - 2a_0 = 6$$

$$a_2 = a_0 + 3$$

For $2 \leq n$, the recurrence equation is

$$(-n a_n(n-1) + (n+2)\, a_{n+2}(n+1) + 2n a_n - 2a_n)\, x^n = 6x^4 - 12x^2 + 6 \tag{4}$$

For $n = 2$ the recurrence equation gives

$$(12a_4)\, x^2 = -12x^2$$
$$12a_4 = -12$$

Which after substituting the earlier terms found becomes

$$a_4 = -1$$

For $n = 3$ the recurrence equation gives

$$(-2a_3 + 20a_5)\, x^3 = 0$$
$$-2a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$(-6a_4 + 30a_6)\, x^4 = 6x^4$$
$$-6a_4 + 30a_6 = 6$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$(-12a_5 + 42a_7)\, x^5 = 0$$
$$-12a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + (a_0 + 3)\, x^2 - x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(x^2 + 1\right) a_0 - x^4 + 3x^2 + a_1 x + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(x^2 + 1\right) c_1 - x^4 + 3x^2 + c_2 x + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(x^2 + 1\right) y(0) - x^4 + xy'(0) + 3x^2 + O\left(x^6\right) \tag{1}$$
$$y = \left(x^2 + 1\right) c_1 - x^4 + 3x^2 + c_2 x + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = (x^2 + 1)\, y(0) - x^4 + xy'(0) + 3x^2 + O(x^6)$$

Verified OK.

$$y = (x^2 + 1)\, c_1 - x^4 + 3x^2 + c_2 x + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=6*(1-x^2)^2,y(x),type='series',x=0);
```

$$y(x) = (x^2 + 1)\, y(0) - x^4 + D(y)(0)\, x + 3x^2 + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 26

```
AsymptoticDSolveValue[(1-x^2)*y''[x]+2*x*y'[x]-2*y[x]==6*(1-x^2)^2,y[x],{x,0,5}]
```

$$y(x) \to -x^4 + 3x^2 + c_1(x^2 + 1) + c_2 x$$

25.13 problem 12

Internal problem ID [2429]
Internal file name [`OUTPUT/2429_Tuesday_February_27_2024_08_37_24_AM_94257474/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^2+2x\right)y''-(2+2x)\,y'+2y=x^2(x+2)^2}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2+2x\right)y''+(-2-2x)\,y'+2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{2(x+1)}{x\,(x+2)}$$
$$q(x)=\frac{2}{x\,(x+2)}$$

Table 719: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2(x+1)}{x(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

$q(x) = \frac{2}{x(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x(x+2) + (-2-2x)\,y' + 2y = x^2(x+2)^2$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y''x(x+2) + (-2-2x)\,y' + 2y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}\right)x(x+2) \\ +(-2-2x)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+2\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2x^{n+r-1}a_n(n+r)(n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-2(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}2a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)=\sum_{n=1}^{\infty}a_{n-1}(n+r-1)(n+r-2)\,x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}2a_n x^{n+r}=\sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty}a_{n-1}(n+r-1)(n+r-2)\,x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty}2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,x^{n+r-1}\right) \\ +\sum_{n=0}^{\infty}\left(-2(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r-1}a_n(n+r)\,(n+r-1) - 2(n+r)\,a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$2x^{-1+r}a_0 r(-1+r) - 2r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r}r(-1+r) - 2r\,x^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2r\,x^{-1+r}(-2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(2x^{-1+m}m(-1+m) - 2m\,x^{-1+m}\right)c_0 = x^2(x+2)^2$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$2r\,x^{-1+r}(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$
\begin{aligned}
y_1(x) &= x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
\begin{aligned}
& a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\
& - 2a_{n-1}(n+r-1) - 2a_n(n+r) + 2a_{n-1} = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{(n+r-3)\, a_{n-1}}{2(n+r)} \tag{4}
$$

Which for the root $r = 2$ becomes

$$
a_n = -\frac{(n-1)\, a_{n-1}}{2n+4} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = \frac{2-r}{2+2r}
$$

Which for the root $r = 2$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{2+2r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 - 3r + 2}{4(1+r)(2+r)}$$

Which for the root $r = 2$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{2+2r}$	0
a_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r^2 - 3r + 2)\, r}{8(2+r)(3+r)(1+r)}$$

Which for the root $r = 2$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{2+2r}$	0
a_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	0
a_3	$-\frac{(r^2-3r+2)r}{8(2+r)(3+r)(1+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r^2 - 3r + 2) r}{16 (3 + r) (4 + r) (2 + r)}$$

Which for the root $r = 2$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{2+2r}$	0
a_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	0
a_3	$-\frac{(r^2-3r+2)r}{8(2+r)(3+r)(1+r)}$	0
a_4	$\frac{(r^2-3r+2)r}{16(3+r)(4+r)(2+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r^2 - 3r + 2) r}{32 (4 + r) (3 + r) (5 + r)}$$

Which for the root $r = 2$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{2+2r}$	0
a_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	0
a_3	$-\frac{(r^2-3r+2)r}{8(2+r)(3+r)(1+r)}$	0
a_4	$\frac{(r^2-3r+2)r}{16(3+r)(4+r)(2+r)}$	0
a_5	$-\frac{(r^2-3r+2)r}{32(4+r)(3+r)(5+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2 \left(1 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{r^2 - 3r + 2}{4(1+r)(2+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{r^2 - 3r + 2}{4(1+r)(2+r)} &= \lim_{r \to 0} \frac{r^2 - 3r + 2}{4(1+r)(2+r)} \\ &= \frac{1}{4} \end{aligned}$$

The limit is $\frac{1}{4}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} & b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) \\ & - 2b_{n-1}(n+r-1) - 2(n+r) b_n + 2b_{n-1} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_{n-1}(n-1)(n-2) + 2b_n n(n-1) - 2b_{n-1}(n-1) - 2nb_n + 2b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(n+r-3)\, b_{n-1}}{2\,(n+r)} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{(n-3)\, b_{n-1}}{2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{-2+r}{2\,(1+r)}$$

Which for the root $r = 0$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{2+2r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 - 3r + 2}{4\,(1+r)\,(2+r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{2+2r}$	1
b_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(r^2 - 3r + 2)\, r}{8\,(2 + r)\,(3 + r)\,(1 + r)}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{2+2r}$	1
b_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	$\frac{1}{4}$
b_3	$-\frac{(r^2-3r+2)r}{8(2+r)(3+r)(1+r)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r^2 - 3r + 2)\, r}{16\,(3 + r)\,(4 + r)\,(2 + r)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{2+2r}$	1
b_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	$\frac{1}{4}$
b_3	$-\frac{(r^2-3r+2)r}{8(2+r)(3+r)(1+r)}$	0
b_4	$\frac{(r^2-3r+2)r}{16(3+r)(4+r)(2+r)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r^2 - 3r + 2)\, r}{32\,(4 + r)\,(3 + r)\,(5 + r)}$$

Which for the root $r = 0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{2+2r}$	1
b_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	$\frac{1}{4}$
b_3	$-\frac{(r^2-3r+2)r}{8(2+r)(3+r)(1+r)}$	0
b_4	$\frac{(r^2-3r+2)r}{16(3+r)(4+r)(2+r)}$	0
b_5	$-\frac{(r^2-3r+2)r}{32(4+r)(3+r)(5+r)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + x + \frac{x^2}{4} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2 (1 + O(x^6)) + c_2 \left(1 + x + \frac{x^2}{4} + O(x^6)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(2x^{-1+m} m(-1+m) - 2m\, x^{-1+m}\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{array}{|l|}
\hline
a_1 = -\frac{(-2+r)a_0}{2+2r} \\
a_2 = \frac{(-1+r)(-2+r)a_0}{4(1+r)(2+r)} \\
a_3 = -\frac{r(-1+r)(-2+r)a_0}{8(1+r)(2+r)(3+r)} \\
a_4 = \frac{r(-1+r)(-2+r)a_0}{16(2+r)(3+r)(4+r)} \\
a_5 = -\frac{r(-1+r)(-2+r)a_0}{32(3+r)(4+r)(5+r)} \\
\hline
\end{array}
$$

Expanding the rhs of the ode $x^2(x+2)^2$ in series gives

$$
x^2(x+2)^2 = x^4 + 4x^3 + 4x^2
$$

Since the $F = x^4 + 4x^3 + 4x^2$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^4$ by solving the balance equation

$$
\left(2x^{-1+m}m(-1+m) - 2m\, x^{-1+m}\right) c_0 = x^4
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{30} \\
m &= 5
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+5}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{30}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 5$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{30}$ and $r = m$ or $r = 5$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$
\boxed{
\begin{aligned}
c_0 &= \tfrac{1}{30} \\
c_1 &= -\tfrac{1}{120} \\
c_2 &= \tfrac{1}{420} \\
c_3 &= -\tfrac{1}{1344} \\
c_4 &= \tfrac{1}{4032} \\
c_5 &= -\tfrac{1}{11520}
\end{aligned}
}
$$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^5 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^5 \left(\frac{1}{30} - \frac{1}{120}x + \frac{1}{420}x^2 - \frac{1}{1344}x^3 + \frac{1}{4032}x^4 - \frac{1}{11520}x^5\right) \\
&= \frac{1}{30}x^5 - \frac{1}{120}x^6 + \frac{1}{420}x^7 - \frac{1}{1344}x^8 + \frac{1}{4032}x^9 - \frac{1}{11520}x^{10}
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = 4x^3$ by solving the balance equation

$$
\left(2x^{-1+m} m(-1+m) - 2m\, x^{-1+m}\right) c_0 = 4x^3
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{4} \\
m &= 4
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+4}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{4}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{4}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{4}$
$c_1 = -\frac{1}{20}$
$c_2 = \frac{1}{80}$
$c_3 = -\frac{1}{280}$
$c_4 = \frac{1}{896}$
$c_5 = -\frac{1}{2688}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^4 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^4 \left(\frac{1}{4} - \frac{1}{20}x + \frac{1}{80}x^2 - \frac{1}{280}x^3 + \frac{1}{896}x^4 - \frac{1}{2688}x^5\right) \\
&= \frac{1}{4}x^4 - \frac{1}{20}x^5 + \frac{1}{80}x^6 - \frac{1}{280}x^7 + \frac{1}{896}x^8 - \frac{1}{2688}x^9
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = 4x^2$ by solving the balance equation

$$
\left(2x^{-1+m} m(-1+m) - 2m\, x^{-1+m}\right) c_0 = 4x^2
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{2}{3} \\
m &= 3
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+3}
\end{aligned}
$$

Where in the above $c_0 = \frac{2}{3}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{2}{3}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{2}{3}$
$c_1 = -\frac{1}{12}$
$c_2 = \frac{1}{60}$
$c_3 = -\frac{1}{240}$
$c_4 = \frac{1}{840}$
$c_5 = -\frac{1}{2688}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3 \left(\frac{2}{3} - \frac{1}{12}x + \frac{1}{60}x^2 - \frac{1}{240}x^3 + \frac{1}{840}x^4 - \frac{1}{2688}x^5\right) \\
&= \frac{2}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5 - \frac{1}{240}x^6 + \frac{1}{840}x^7 - \frac{1}{2688}x^8
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$y_p = \frac{2x^3}{3} + \frac{x^4}{6} - \frac{x^9}{8064} - \frac{x^{10}}{11520} + O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{2x^3}{3} + \frac{x^4}{6} + O(x^6)$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{2x^3}{3} + \frac{x^4}{6} + O(x^6) + c_1 x^2 (1 + O(x^6)) + c_2 \left(1 + x + \frac{x^2}{4} + O(x^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{2x^3}{3} + \frac{x^4}{6} + O(x^6) + c_1 x^2 (1 + O(x^6)) + c_2 \left(1 + x + \frac{x^2}{4} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = \frac{2x^3}{3} + \frac{x^4}{6} + O(x^6) + c_1 x^2 (1 + O(x^6)) + c_2 \left(1 + x + \frac{x^2}{4} + O(x^6)\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
Order:=6;
dsolve((x^2+2*x)*diff(y(x),x$2)-(2+2*x)*diff(y(x),x)+2*y(x)=x^2*(x+2)^2,y(x),type='series',x
```

$$y(x) = c_1 x^2 \left(1 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(-2 - 2x - \frac{1}{2}x^2 + \mathrm{O}\left(x^6\right)\right) + x^3 \left(\frac{2}{3} + \frac{1}{6}x + \mathrm{O}\left(x^3\right)\right)$$

✓ Solution by Mathematica

Time used: 0.317 (sec). Leaf size: 39

```
AsymptoticDSolveValue[(x^2+2*x)*y''[x]-(2+2*x)*y'[x]+2*y[x]==x^2*(x+2)^2,y[x],{x,0,5}]
```

$$y(x) \to -\frac{1}{3}(x+1)x^3 + \left(\frac{x^2}{2} + x\right)x^2 + c_2 x^2 + c_1(x+1)$$

25.14 problem 13

Internal problem ID [2430]
Internal file name [`OUTPUT/2430_Tuesday_February_27_2024_08_37_25_AM_43501069/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{2x^2y'' + 5xy' + y(x+1) = x(x^2+x+1)}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 5xy' + y(x+1) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{5}{2x}$$
$$q(x) = \frac{x+1}{2x^2}$$

Table 720: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x+1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 5xy' + y(x + 1) = x^3 + x^2 + x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' + 5xy' + y(x + 1) = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}\right)\\ +5x\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)(x+1)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}5x^{n+r}a_n(n+r)\right)\\ +\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}x^{1+n+r}a_n=\sum_{n=1}^{\infty}a_{n-1}x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}5x^{n+r}a_n(n+r)\right)\\ +\left(\sum_{n=1}^{\infty}a_{n-1}x^{n+r}\right)+\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)(n+r-1)+5x^{n+r}a_n(n+r)+a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)+5x^r a_0 r+a_0 x^r=0$$

Or

$$\left(2x^r r(-1+r)+5x^r r+x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2+3r+1\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2+3r+1=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= -\frac{1}{2} \\ r_2 &= -1 \end{aligned}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(2x^m m(-1+m)+5x^m m+x^m\right) c_0=x^3+x^2+x$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2+3r+1\right) x^r=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=\frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 5a_n(n+r) + a_{n-1} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n^2 + 4nr + 2r^2 + 3n + 3r + 1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 7r + 6}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 36r^3 + 119r^2 + 171r + 90}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^6 + 132r^5 + 890r^4 + 3135r^3 + 6077r^2 + 6138r + 2520}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_3 = -\frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400)\,(2r^2 +$$

Which for the root $r = -\frac{1}{2}$ becomes

$$a_5 = -\frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+7r+6}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{22680}$
a_5	$-\frac{1}{(16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400)(2r^2+23r+66)}$	$-\frac{1}{1247400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{\sqrt{x}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 5b_n(n+r) + b_{n-1} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}}{2n^2 + 4nr + 2r^2 + 3n + 3r + 1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-1}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{2r^2 + 7r + 6}$$

Which for the root $r = -1$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 36r^3 + 119r^2 + 171r + 90}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{8r^6 + 132r^5 + 890r^4 + 3135r^3 + 6077r^2 + 6138r + 2520}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{(16r^8 + 416r^7 + 4648r^6 + 29120r^5 + 111769r^4 + 268814r^3 + 395127r^2 + 324090r + 113400)\,(2r^2 +}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+7r+6}$	-1
b_2	$\frac{1}{4r^4+36r^3+119r^2+171r+90}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+132r^5+890r^4+3135r^3+6077r^2+6138r+2520}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400}$	$\frac{1}{2520}$
b_5	$-\frac{1}{(16r^8+416r^7+4648r^6+29120r^5+111769r^4+268814r^3+395127r^2+324090r+113400)(2r^2+23r+66)}$	$-\frac{1}{113400}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \frac{1}{\sqrt{x}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1\left(1 - \frac{x}{3} + \frac{x^2}{30} - \frac{x^3}{630} + \frac{x^4}{22680} - \frac{x^5}{1247400} + O(x^6)\right)}{\sqrt{x}} \\ &\quad + \frac{c_2\left(1 - x + \frac{x^2}{6} - \frac{x^3}{90} + \frac{x^4}{2520} - \frac{x^5}{113400} + O(x^6)\right)}{x} \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(2x^m m(-1+m) + 5x^m m + x^m\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{aligned}
a_1 &= -\frac{a_0}{2r^2+7r+6} \\
a_2 &= \frac{a_0}{(2r^2+7r+6)(2r^2+11r+15)} \\
a_3 &= -\frac{a_0}{(2r^2+7r+6)(2r^2+11r+15)(2r^2+15r+28)} \\
a_4 &= \frac{a_0}{(2r^2+7r+6)(2r^2+11r+15)(2r^2+15r+28)(2r^2+19r+45)} \\
a_5 &= -\frac{a_0}{(2r^2+7r+6)(2r^2+11r+15)(2r^2+15r+28)(2r^2+19r+45)(2r^2+23r+66)}
\end{aligned}
$$

Since the $F = x^3 + x^2 + x$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^3$ by solving the balance equation

$$
(2x^m m(-1+m) + 5x^m m + x^m) c_0 = x^3
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{28} \\
m &= 3
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+3}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{28}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{28}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$
\boxed{\begin{aligned}
c_0 &= \tfrac{1}{28}\\
c_1 &= -\tfrac{1}{1260}\\
c_2 &= \tfrac{1}{83160}\\
c_3 &= -\tfrac{1}{7567560}\\
c_4 &= \tfrac{1}{908107200}\\
c_5 &= -\tfrac{1}{138940401600}
\end{aligned}}
$$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n\\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3 \left(\frac{1}{28} - \frac{1}{1260}x + \frac{1}{83160}x^2 - \frac{1}{7567560}x^3 + \frac{1}{908107200}x^4 - \frac{1}{138940401600}x^5\right)\\
&= \frac{1}{28}x^3 - \frac{1}{1260}x^4 + \frac{1}{83160}x^5 - \frac{1}{7567560}x^6 + \frac{1}{908107200}x^7 - \frac{1}{138940401600}x^8
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$
(2x^m m(-1+m) + 5x^m m + x^m)\, c_0 = x^2
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{15}\\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m}\\
&= \sum_{n=0}^{\infty} c_n x^{n+2}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{15}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{15}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{15}$
$c_1 = -\frac{1}{420}$
$c_2 = \frac{1}{18900}$
$c_3 = -\frac{1}{1247400}$
$c_4 = \frac{1}{113513400}$
$c_5 = -\frac{1}{13621608000}$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^2 \left(\frac{1}{15} - \frac{1}{420}x + \frac{1}{18900}x^2 - \frac{1}{1247400}x^3 + \frac{1}{113513400}x^4 - \frac{1}{13621608000}x^5 \right) \\ &= \frac{1}{15}x^2 - \frac{1}{420}x^3 + \frac{1}{18900}x^4 - \frac{1}{1247400}x^5 + \frac{1}{113513400}x^6 - \frac{1}{13621608000}x^7 \end{aligned}$$

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$(2x^m m(-1+m) + 5x^m m + x^m)\, c_0 = x$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{6} \\ m &= 1 \end{aligned}$$

The particular solution is therefore

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+m}$$
$$= \sum_{n=0}^{\infty} c_n x^{n+1}$$

Where in the above $c_0 = \frac{1}{6}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{6}$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{6}$
$c_1 = -\frac{1}{90}$
$c_2 = \frac{1}{2520}$
$c_3 = -\frac{1}{113400}$
$c_4 = \frac{1}{7484400}$
$c_5 = -\frac{1}{681080400}$

The particular solution is now found using

$$y_p = x^m \sum_{n=0}^{\infty} c_n x^n$$
$$= x \sum_{n=0}^{\infty} c_n x^n$$

Using the values found above for c_n into the above sum gives

$$y_p = x\left(\frac{1}{6} - \frac{1}{90}x + \frac{1}{2520}x^2 - \frac{1}{113400}x^3 + \frac{1}{7484400}x^4 - \frac{1}{681080400}x^5\right)$$
$$= \frac{1}{6}x - \frac{1}{90}x^2 + \frac{1}{2520}x^3 - \frac{1}{113400}x^4 + \frac{1}{7484400}x^5 - \frac{1}{681080400}x^6$$

Adding all the above particular solution(s) gives

$$y_p = \frac{x}{6}+\frac{x^2}{18}+\frac{17x^3}{504}-\frac{17x^4}{22680}+\frac{17x^5}{1496880}-\frac{17x^6}{136216080} + \frac{x^7}{972972000}-\frac{x^8}{138940401600}+O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{x}{6}+\frac{x^2}{18}+\frac{17x^3}{504}-\frac{17x^4}{22680}+\frac{17x^5}{1496880}+O(x^6)$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{x}{6}+\frac{x^2}{18}+\frac{17x^3}{504}-\frac{17x^4}{22680}+\frac{17x^5}{1496880}+O(x^6) \\ &+ \frac{c_1\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right)}{\sqrt{x}} \\ &+ \frac{c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \frac{x}{6}+\frac{x^2}{18}+\frac{17x^3}{504}-\frac{17x^4}{22680}+\frac{17x^5}{1496880}+O(x^6) \\ &+ \frac{c_1\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right)}{\sqrt{x}} \\ &+ \frac{c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)}{x} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= \frac{x}{6}+\frac{x^2}{18}+\frac{17x^3}{504}-\frac{17x^4}{22680}+\frac{17x^5}{1496880}+O(x^6) \\ &+ \frac{c_1\left(1-\frac{x}{3}+\frac{x^2}{30}-\frac{x^3}{630}+\frac{x^4}{22680}-\frac{x^5}{1247400}+O(x^6)\right)}{\sqrt{x}} \\ &+ \frac{c_2\left(1-x+\frac{x^2}{6}-\frac{x^3}{90}+\frac{x^4}{2520}-\frac{x^5}{113400}+O(x^6)\right)}{x} \end{aligned}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 63

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+(1+x)*y(x)=x*(1+x+x^2),y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \frac{1}{2520}x^4 - \frac{1}{113400}x^5 + \mathrm{O}\left(x^6\right)\right)}{x} + \frac{c_2\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \frac{1}{22680}x^4 - \frac{1}{1247400}x^5 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}} + x\left(\frac{1}{6} + \frac{1}{18}x + \frac{17}{504}x^2 - \frac{17}{22680}x^3 + \frac{17}{1496880}x^4 + \mathrm{O}\left(x^5\right)\right)$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 237

```
AsymptoticDSolveValue[2*x^2*y''[x]+5*x*y'[x]+(1+x)*y[x]==x*(1+x+x^2),y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1\left(-\frac{x^5}{113400}+\frac{x^4}{2520}-\frac{x^3}{90}+\frac{x^2}{6}-x+1\right)}{x} + \frac{c_2\left(-\frac{x^5}{1247400}+\frac{x^4}{22680}-\frac{x^3}{630}+\frac{x^2}{30}-\frac{x}{3}+1\right)}{\sqrt{x}} + \frac{\left(-\frac{x^5}{1247400}+\frac{x^4}{22680}-\frac{x^3}{630}+\frac{x^2}{30}-\frac{x}{3}+1\right)\left(\frac{131x^{11/2}}{4620}-\frac{76x^{9/2}}{405}+\frac{x^{7/2}}{21}+\frac{2x^{3/2}}{3}\right)}{\sqrt{x}} + \frac{\left(-\frac{x^5}{113400}+\frac{x^4}{2520}-\frac{x^3}{90}+\frac{x^2}{6}-x+1\right)\left(-\frac{103x^6}{19440}+\frac{19x^5}{315}-\frac{7x^4}{40}-\frac{2x^3}{9}-\frac{x^2}{2}\right)}{x}$$

25.15 problem 14

Internal problem ID [2431]
Internal file name [`OUTPUT/2431_Tuesday_February_27_2024_08_37_26_AM_1954633/index.tex`]

Book: Differential Equations by Alfred L. Nelson, Karl W. Folley, Max Coral. 3rd ed. DC heath. Boston. 1964
Section: Exercise 43, page 209
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{(x^3+2x^2)\,y''-xy'+(1-x)\,y=x^2(x+1)^2}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^3+2x^2)\,y''-xy'+(1-x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{1}{x\,(x+2)}$$
$$q(x)=-\frac{x-1}{x^2\,(x+2)}$$

Table 721: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

$q(x) = -\frac{x-1}{x^2(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x^2(x+2) - xy' + (1-x)\,y = x^2(x+1)^2$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y''x^2(x+2) - xy' + (1-x)\,y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}\right)x^2(x+2) - x\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}\right) + (1-x)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right) + \left(\sum_{n=0}^{\infty}a_n x^{n+r}\right) + \sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}x^{1+n+r}a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty}a_{n-1}(n+r-1)(n+r-2)x^{n+r}$$

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right) = \sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty}a_{n-1}(n+r-1)(n+r-2)x^{n+r}\right) + \left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right) + \left(\sum_{n=0}^{\infty}a_n x^{n+r}\right) + \sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)(n+r-1) - x^{n+r}a_n(n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) - x^r r + x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^2(x+1)^2$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) - a_n(n+r) + a_n - a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 3n - 3r + 1)}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}(n^2 - n - 1)}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-r^2 + r + 1}{2r^2 + r}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^4 - 3r^2 + 1}{4r^4 + 12r^3 + 11r^2 + 3r}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{1}{3}$
a_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^6 - 3r^5 + 2r^4 + 9r^3 + 2r^2 - 3r - 1}{r (4r^3 + 12r^2 + 11r + 3) (2r^2 + 9r + 10)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{126}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{1}{3}$
a_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{1}{126}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^8 + 8r^7 + 18r^6 - 4r^5 - 57r^4 - 52r^3 + 6r^2 + 20r + 5}{r (4r^3 + 12r^2 + 11r + 3) (2r^2 + 9r + 10) (2r^2 + 13r + 21)}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{11}{4536}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{1}{3}$
a_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{1}{126}$
a_4	$\frac{r^8+8r^7+18r^6-4r^5-57r^4-52r^3+6r^2+20r+5}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$-\frac{11}{4536}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r^2+3r+1)(r^2+5r+5)(r^2+7r+11)(r^2+r-1)(r^2-r-1)}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)(2r^2+17r+36)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{19}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{1}{3}$
a_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$-\frac{1}{30}$
a_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$\frac{1}{126}$
a_4	$\frac{r^8+8r^7+18r^6-4r^5-57r^4-52r^3+6r^2+20r+5}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$-\frac{11}{4536}$
a_5	$-\frac{(r^2+3r+1)(r^2+5r+5)(r^2+7r+11)(r^2+r-1)(r^2-r-1)}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)(2r^2+17r+36)}$	$\frac{19}{22680}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + \frac{x}{3} - \frac{x^2}{30} + \frac{x^3}{126} - \frac{11x^4}{4536} + \frac{19x^5}{22680} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) - b_n(n+r) + b_n - b_{n-1} = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n^2+2nr+r^2-3n-3r+1)}{2n^2+4nr+2r^2-3n-3r+1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -\frac{b_{n-1}(4n^2-8n-1)}{8n^2-4n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-r^2+r+1}{2r^2+r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = \frac{5}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{5}{4}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{5}{96}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{5}{4}$
b_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$\frac{5}{96}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-r^6 - 3r^5 + 2r^4 + 9r^3 + 2r^2 - 3r - 1}{r\left(4r^3 + 12r^2 + 11r + 3\right)\left(2r^2 + 9r + 10\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = -\frac{11}{1152}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{5}{4}$
b_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$\frac{5}{96}$
b_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$-\frac{11}{1152}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^8 + 8r^7 + 18r^6 - 4r^5 - 57r^4 - 52r^3 + 6r^2 + 20r + 5}{r\left(4r^3 + 12r^2 + 11r + 3\right)\left(2r^2 + 9r + 10\right)\left(2r^2 + 13r + 21\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{341}{129024}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{5}{4}$
b_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$\frac{5}{96}$
b_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$-\frac{11}{1152}$
b_4	$\frac{r^8+8r^7+18r^6-4r^5-57r^4-52r^3+6r^2+20r+5}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$\frac{341}{129024}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r^2+3r+1)(r^2+5r+5)(r^2+7r+11)(r^2+r-1)(r^2-r-1)}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)(2r^2+17r+36)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = -\frac{20119}{23224320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r+1}{2r^2+r}$	$\frac{5}{4}$
b_2	$\frac{r^4-3r^2+1}{4r^4+12r^3+11r^2+3r}$	$\frac{5}{96}$
b_3	$\frac{-r^6-3r^5+2r^4+9r^3+2r^2-3r-1}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)}$	$-\frac{11}{1152}$
b_4	$\frac{r^8+8r^7+18r^6-4r^5-57r^4-52r^3+6r^2+20r+5}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)}$	$\frac{341}{129024}$
b_5	$-\frac{(r^2+3r+1)(r^2+5r+5)(r^2+7r+11)(r^2+r-1)(r^2-r-1)}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)(2r^2+17r+36)}$	$-\frac{20119}{23224320}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{5x}{4} + \frac{5x^2}{96} - \frac{11x^3}{1152} + \frac{341x^4}{129024} - \frac{20119x^5}{23224320} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1+\frac{x}{3}-\frac{x^2}{30}+\frac{x^3}{126}-\frac{11x^4}{4536}+\frac{19x^5}{22680}+O(x^6)\right) \\
&\quad + c_2\sqrt{x}\left(1+\frac{5x}{4}+\frac{5x^2}{96}-\frac{11x^3}{1152}+\frac{341x^4}{129024}-\frac{20119x^5}{23224320}+O(x^6)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
(2x^m m(-1+m) - x^m m + x^m)\, c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{aligned}
a_1 &= -\frac{a_0(r^2-r-1)}{2r^2+r} \\
a_2 &= \frac{a_0(r^2-r-1)(r^2+r-1)}{4r^4+12r^3+11r^2+3r} \\
a_3 &= -\frac{a_0(r^2-r-1)(r^2+r-1)(r^2+3r+1)}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)} \\
a_4 &= \frac{a_0(r^2-r-1)(r^2+r-1)(r^2+3r+1)(r^2+5r+5)}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)} \\
a_5 &= -\frac{a_0(r^2-r-1)(r^2+r-1)(r^2+3r+1)(r^2+5r+5)(r^2+7r+11)}{r(4r^3+12r^2+11r+3)(2r^2+9r+10)(2r^2+13r+21)(2r^2+17r+36)}
\end{aligned}
$$

Expanding the rhs of the ode $x^2(x+1)^2$ in series gives

$$
x^2(x+1)^2 = x^4 + 2x^3 + x^2
$$

Since the $F = x^4 + 2x^3 + x^2$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^4$ by solving the balance equation

$$
(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^4
$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{21} \\ m &= 4 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+4} \end{aligned}$$

Where in the above $c_0 = \frac{1}{21}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{21}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\begin{aligned} c_0 &= \frac{1}{21} \\ c_1 &= -\frac{11}{756} \\ c_2 &= \frac{19}{3780} \\ c_3 &= -\frac{551}{294840} \\ c_4 &= \frac{22591}{30958200} \\ c_5 &= -\frac{248501}{842063040} \end{aligned}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^4 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$y_p = x^4 \left(\frac{1}{21} - \frac{11}{756} x + \frac{19}{3780} x^2 - \frac{551}{294840} x^3 + \frac{22591}{30958200} x^4 - \frac{248501}{842063040} x^5 \right)$$

$$
= \frac{1}{21}x^4 - \frac{11}{756}x^5 + \frac{19}{3780}x^6 - \frac{551}{294840}x^7 + \frac{22591}{30958200}x^8 - \frac{248501}{842063040}x^9
$$

Now we determine the particular solution y_p associated with $F = 2x^3$ by solving the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = 2x^3
$$

For c_0 and x. This results in

$$
c_0 = \frac{1}{5}
$$
$$
m = 3
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+3}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{5}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{5}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{5}$
$c_1 = -\frac{1}{21}$
$c_2 = \frac{11}{756}$
$c_3 = -\frac{19}{3780}$
$c_4 = \frac{551}{294840}$
$c_5 = -\frac{22591}{30958200}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3\left(\frac{1}{5}-\frac{1}{21}x+\frac{11}{756}x^2-\frac{19}{3780}x^3+\frac{551}{294840}x^4-\frac{22591}{30958200}x^5\right)\\
&= \frac{1}{5}x^3-\frac{1}{21}x^4+\frac{11}{756}x^5-\frac{19}{3780}x^6+\frac{551}{294840}x^7-\frac{22591}{30958200}x^8
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$
(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^2
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{3}\\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m}\\
&= \sum_{n=0}^{\infty} c_n x^{n+2}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{3}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{3}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{3}$
$c_1 = -\frac{1}{30}$
$c_2 = \frac{1}{126}$
$c_3 = -\frac{11}{4536}$
$c_4 = \frac{19}{22680}$
$c_5 = -\frac{551}{1769040}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^2 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^2 \left(\frac{1}{3} - \frac{1}{30}x + \frac{1}{126}x^2 - \frac{11}{4536}x^3 + \frac{19}{22680}x^4 - \frac{551}{1769040}x^5\right) \\
&= \frac{1}{3}x^2 - \frac{1}{30}x^3 + \frac{1}{126}x^4 - \frac{11}{4536}x^5 + \frac{19}{22680}x^6 - \frac{551}{1769040}x^7
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = \frac{x^2}{3} + \frac{x^3}{6} + \frac{x^4}{126} - \frac{11x^5}{4536} + \frac{19x^6}{22680} - \frac{551x^7}{1769040} - \frac{248501x^9}{842063040} + O(x^6)
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = \frac{x^2}{3} + \frac{x^3}{6} + \frac{x^4}{126} - \frac{11x^5}{4536} + O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x^2}{3} + \frac{x^3}{6} + \frac{x^4}{126} - \frac{11x^5}{4536} + O(x^6) + c_1 x \left(1 + \frac{x}{3} - \frac{x^2}{30} + \frac{x^3}{126} - \frac{11x^4}{4536} + \frac{19x^5}{22680} + O(x^6)\right) \\
&\quad + c_2 \sqrt{x} \left(1 + \frac{5x}{4} + \frac{5x^2}{96} - \frac{11x^3}{1152} + \frac{341x^4}{129024} - \frac{20119x^5}{23224320} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{x^2}{3} + \frac{x^3}{6} + \frac{x^4}{126} - \frac{11x^5}{4536} + O(x^6) + c_1 x \left(1 + \frac{x}{3} - \frac{x^2}{30} + \frac{x^3}{126} - \frac{11x^4}{4536} + \frac{19x^5}{22680} + O(x^6)\right) \\
& + c_2 \sqrt{x} \left(1 + \frac{5x}{4} + \frac{5x^2}{96} - \frac{11x^3}{1152} + \frac{341x^4}{129024} - \frac{20119x^5}{23224320} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = \frac{x^2}{3}+\frac{x^3}{6}+\frac{x^4}{126}-\frac{11x^5}{4536}+O(x^6)+c_1x\left(1+\frac{x}{3}-\frac{x^2}{30}+\frac{x^3}{126}-\frac{11x^4}{4536}+\frac{19x^5}{22680}+O(x^6)\right)$$
$$+c_2\sqrt{x}\left(1+\frac{5x}{4}+\frac{5x^2}{96}-\frac{11x^3}{1152}+\frac{341x^4}{129024}-\frac{20119x^5}{23224320}+O(x^6)\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
Order:=6;
dsolve((2*x^2+x^3)*diff(y(x),x$2)-x*diff(y(x),x)+(1-x)*y(x)=x^2*(1+x)^2,y(x),type='series',x
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{5}{4}x+\frac{5}{96}x^2-\frac{11}{1152}x^3+\frac{341}{129024}x^4-\frac{20119}{23224320}x^5+\mathrm{O}\left(x^6\right)\right)$$
$$+c_2x\left(1+\frac{1}{3}x-\frac{1}{30}x^2+\frac{1}{126}x^3-\frac{11}{4536}x^4+\frac{19}{22680}x^5+\mathrm{O}\left(x^6\right)\right)$$
$$+x^2\left(\frac{1}{3}+\frac{1}{6}x+\frac{1}{126}x^2-\frac{11}{4536}x^3+\mathrm{O}\left(x^4\right)\right)$$

✓ Solution by Mathematica

Time used: 0.273 (sec). Leaf size: 247

```
AsymptoticDSolveValue[(2*x^2+x^3)*y''[x]-x*y'[x]+(1-x)*y[x]==x^2*(1+x)^2,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_1\sqrt{x}\left(-\frac{20119x^5}{23224320}+\frac{341x^4}{129024}-\frac{11x^3}{1152}+\frac{5x^2}{96}+\frac{5x}{4}+1\right) \\
& + c_2x\left(\frac{19x^5}{22680}-\frac{11x^4}{4536}+\frac{x^3}{126}-\frac{x^2}{30}+\frac{x}{3}+1\right)+\sqrt{x}\left(-\frac{20119x^5}{23224320}+\frac{341x^4}{129024}-\frac{11x^3}{1152}\right. \\
& \left.+\frac{5x^2}{96}+\frac{5x}{4}+1\right)\left(\frac{4997x^{11/2}}{2903040}-\frac{1853x^{9/2}}{181440}-\frac{183x^{7/2}}{560}-\frac{5x^{5/2}}{6}\right. \\
& \left.-\frac{2x^{3/2}}{3}\right)+x\left(\frac{19x^5}{22680}-\frac{11x^4}{4536}+\frac{x^3}{126}-\frac{x^2}{30}+\frac{x}{3}+1\right)\left(\frac{479x^6}{136080}-\frac{13x^5}{840}+\frac{13x^4}{72}+\frac{17x^3}{18}+\frac{3x^2}{2}+x\right)
\end{aligned}
$$