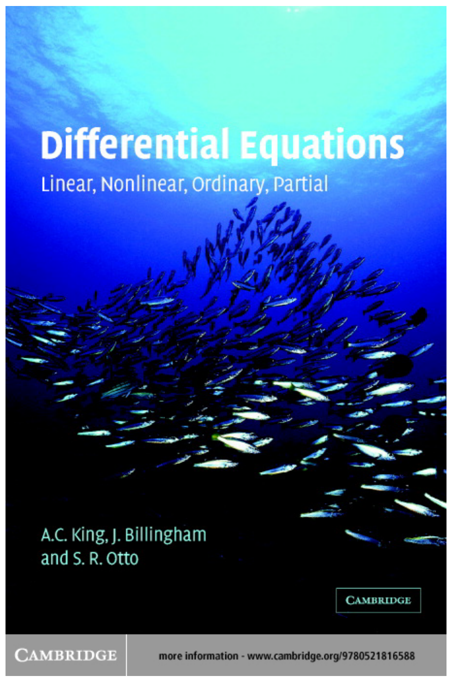
A Solution Manual For

Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003



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May 15, 2024

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1.1 problem Problem 1.1(a)

Internal problem ID [12394]
Internal file name [`OUTPUT/11046_Wednesday_October_04_2023_01_27_34_AM_22931473/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.1(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\, y''-y'x+y=0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\, y'+q(x)\, y=f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{x}{x-1}$$

Therefore

$$
\begin{aligned}
y_2(x) &= \mathrm{e}^{x}\left(\int \mathrm{e}^{-\left(\int -\frac{x}{x-1}dx\right)}\mathrm{e}^{-2x}dx\right) \\
y_2(x) &= \mathrm{e}^{x}\int \frac{\mathrm{e}^{x+\ln(x-1)}}{\mathrm{e}^{2x}}, dx \\
y_2(x) &= \mathrm{e}^{x}\left(\int (x-1)\,\mathrm{e}^{-x}dx\right) \\
y_2(x) &= -\mathrm{e}^{x} x\,\mathrm{e}^{-x}
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= c_1 y_1(x) + c_2 y_2(x) \\
&= \mathrm{e}^{x} c_1 - c_2 \mathrm{e}^{x} x\,\mathrm{e}^{-x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{x} c_1 - c_2 \mathrm{e}^{x} x\,\mathrm{e}^{-x} \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{x} c_1 - c_2 \mathrm{e}^{x} x\,\mathrm{e}^{-x}
$$

Verified OK.

1.1.1 Maple step by step solution

Let's solve

$(x-1)\,y'' - y'x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $$\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $$\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$$

- $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $(x-1)\, y'' - y'x + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = x-1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = x-1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k(x-1)^{k+2}\right), a_{1+k} = \frac{a_k}{1+k}, b_{1+k} = \frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 12

```
dsolve([(x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,exp(x)],singsol=all)
```

$$y(x) = c_2 \mathrm{e}^x + c_1 x$$

✓ Solution by Mathematica
Time used: 0.055 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 e^x - c_2 x$$

1.2 problem Problem 1.1(b)

Internal problem ID [12395]
Internal file name [`OUTPUT/11047_Wednesday_October_04_2023_01_27_35_AM_81430884/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.1(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 2y' + yx = 0}$$

Given that one solution of the ode is

$$y_1 = \frac{\sin(x)}{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{2}{x}$$

Therefore

$$y_2(x) = \frac{\sin(x)\left(\int \frac{e^{-\left(\int \frac{2}{x}dx\right)}x^2}{\sin(x)^2}dx\right)}{x}$$

$$y_2(x) = \frac{\sin(x)}{x}\int \frac{\frac{1}{x^2}}{\frac{\sin(x)^2}{x^2}}, dx$$

$$y_2(x) = \frac{\sin(x)\left(\int \csc(x)^2\,dx\right)}{x}$$

$$y_2(x) = -\frac{\sin(x)\cot(x)}{x}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{\sin(x)\,c_1}{x} - \frac{c_2\sin(x)\cot(x)}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(x)\,c_1}{x} - \frac{c_2\sin(x)\cot(x)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{\sin(x)\,c_1}{x} - \frac{c_2\sin(x)\cot(x)}{x}$$

Verified OK.

1.2.1 Maple step by step solution

Let's solve

$y''x + 2y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{2y'}{x} + y = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $y''x + 2y' + yx = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$
  - Shift index using $k- >k + 1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$a_1(1+r)(2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)(k+3+r) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2}=-\frac{a_k}{(1+k)(k+2)}, 0=0, b_{k+2}=-\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 17

```
dsolve([x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,sin(x)/x],singsol=all)
```

$$y(x)=\frac{c_1 \sin(x)+c_2 \cos(x)}{x}$$

✓ Solution by Mathematica
Time used: 0.046 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{2c_1 e^{-ix}-ic_2 e^{ix}}{2x}$$

1.3 problem Problem 1.3(a)

Internal problem ID [12396]
Internal file name [`OUTPUT/11048_Wednesday_October_04_2023_01_27_35_AM_74078627/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.3(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 2y' + y = x^{\frac{3}{2}} \mathrm{e}^{x}}$$

1.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = x^{\frac{3}{2}} \mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 x e^{x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{x} c_1 + c_2 x e^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{x} \\ y_2 &= x e^{x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{5}{2}} e^{2x}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int x^{\frac{5}{2}}\,dx$$

Hence

$$u_1 = -\frac{2x^{\frac{7}{2}}}{7}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} x^{\frac{3}{2}}}{e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int x^{\frac{3}{2}} dx$$

Hence

$$u_2 = \frac{2x^{\frac{5}{2}}}{5}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{4x^{\frac{7}{2}} e^x}{35}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 x\, e^x) + \left(\frac{4x^{\frac{7}{2}} e^x}{35}\right) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) + \frac{4x^{\frac{7}{2}} e^x}{35}$$

Summary

The solution(s) found are the following

$$y = e^x (c_2 x + c_1) + \frac{4x^{\frac{7}{2}} e^x}{35} \tag{1}$$

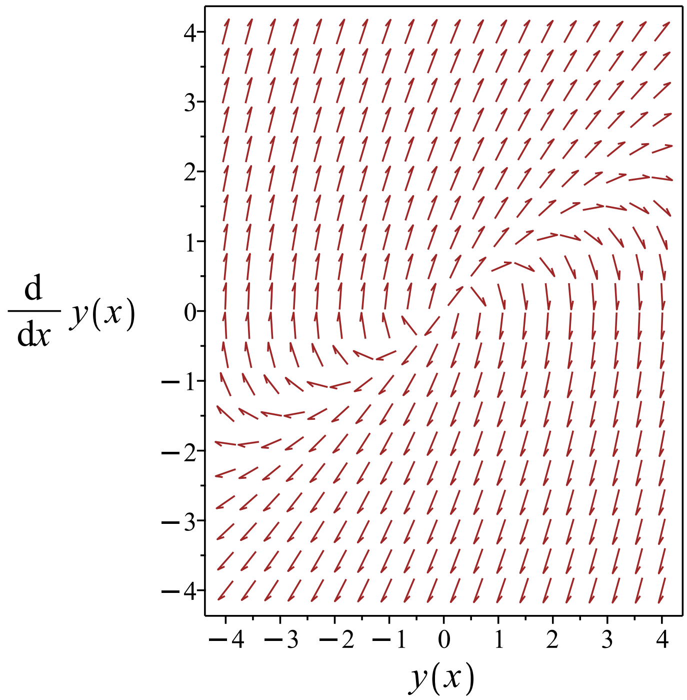

Figure 1: Slope field plot

Verification of solutions

$$y = e^x(c_2x + c_1) + \frac{4x^{\frac{7}{2}}e^x}{35}$$

Verified OK.

1.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \mathrm{e}^{-x} x^{\frac{3}{2}} \mathrm{e}^{x} \\
\left(\mathrm{e}^{-x} y\right)'' &= \mathrm{e}^{-x} x^{\frac{3}{2}} \mathrm{e}^{x}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-x} y\right)' = \frac{2x^{\frac{5}{2}}}{5} + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-x} y\right) = c_1 x + \frac{4x^{\frac{7}{2}}}{35} + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + \frac{4x^{\frac{7}{2}}}{35} + c_2}{\mathrm{e}^{-x}}
$$

Or

$$
y = \frac{4x^{\frac{7}{2}} \mathrm{e}^{x}}{35} + c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{4x^{\frac{7}{2}} \mathrm{e}^{x}}{35} + c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \tag{1}
$$

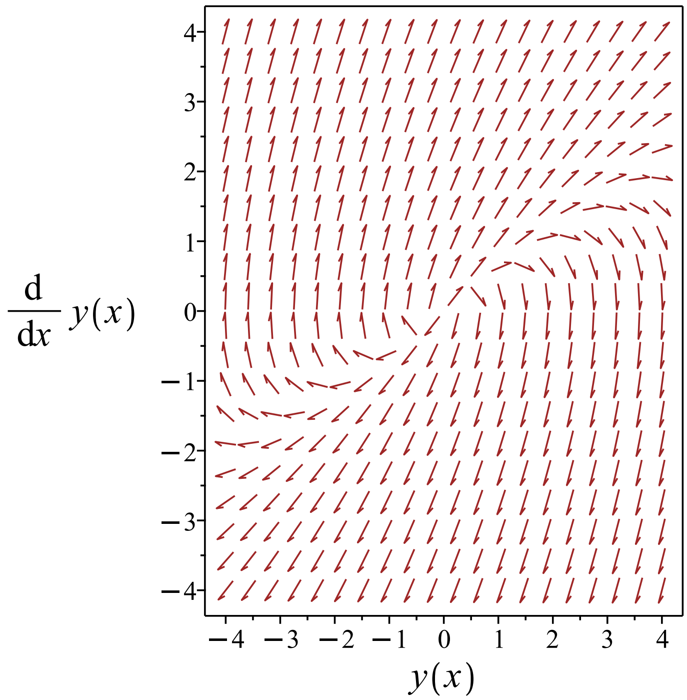

Figure 2: Slope field plot

Verification of solutions

$$
y = \frac{4x^{\frac{7}{2}} \mathrm{e}^{x}}{35} + c_1 x \,\mathrm{e}^{x} + c_2 \mathrm{e}^{x}
$$

Verified OK.

1.3.3 Solving using Kovacic algorithm

Writing the ode as

$$
y'' - 2y' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -2 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 3: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + c_2 x\,\mathrm{e}^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^x \\
y_2 &= x\,\mathrm{e}^x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{5}{2}} e^{2x}}{e^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int x^{\frac{5}{2}}\, dx$$

Hence

$$u_1 = -\frac{2x^{\frac{7}{2}}}{7}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} x^{\frac{3}{2}}}{e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int x^{\frac{3}{2}} dx$$

Hence

$$u_2 = \frac{2x^{\frac{5}{2}}}{5}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{4x^{\frac{7}{2}} \mathrm{e}^x}{35}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\mathrm{e}^x c_1 + c_2 x \,\mathrm{e}^x) + \left(\frac{4x^{\frac{7}{2}} \mathrm{e}^x}{35}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^x (c_2 x + c_1) + \frac{4x^{\frac{7}{2}} \mathrm{e}^x}{35}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x (c_2 x + c_1) + \frac{4x^{\frac{7}{2}} \mathrm{e}^x}{35} \tag{1}$$

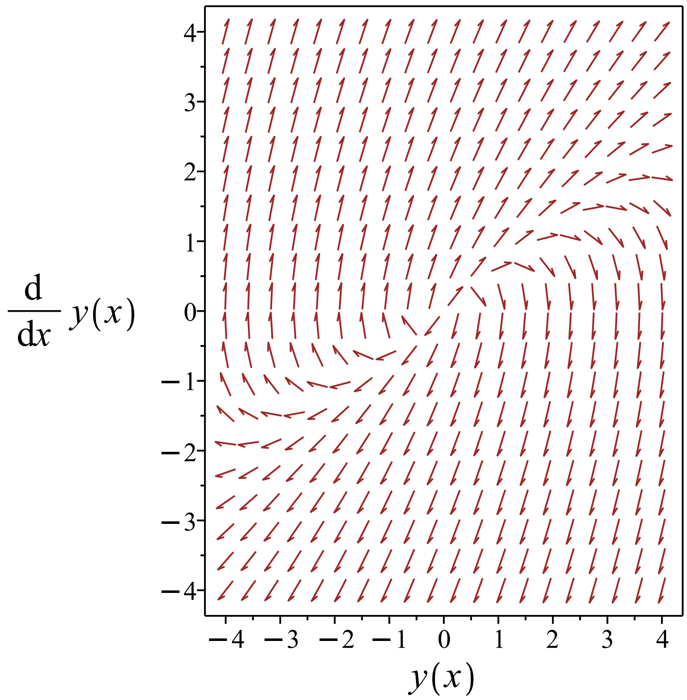

Figure 3: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + \frac{4x^{\frac{7}{2}} e^{x}}{35}$$

Verified OK.

1.3.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = x^{\frac{3}{2}} e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + c_2 x\, e^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^{\frac{3}{2}} e^x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x\, e^x \\ e^x & x\, e^x + e^x \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^x\left(-\left(\int x^{\frac{5}{2}} dx\right) + \left(\int x^{\frac{3}{2}} dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{4x^{\frac{7}{2}} e^x}{35}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\, e^x + e^x c_1 + \frac{4x^{\frac{7}{2}} e^x}{35}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=x^(3/2)*exp(x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x}\left(c_2 + c_1 x + \frac{4x^{\frac{7}{2}}}{35}\right)$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 29

```
DSolve[y''[x]-2*y'[x]+y[x]==x^(3/2)*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{35}e^{x}\left(4x^{7/2} + 35c_2 x + 35c_1\right)$$

1.4 problem Problem 1.3(b)

Internal problem ID [12397]
Internal file name [`OUTPUT/11049_Wednesday_October_04_2023_01_27_37_AM_60955606/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.3(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 2\sec(2x)}$$

1.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 2\sec(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(2x) \\ y_2 &= \sin(2x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}(\sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(2\cos(2x)) - (\sin(2x))(-2\sin(2x))$$

Which simplifies to

$$W = 2\cos(2x)^2 + 2\sin(2x)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\sin(2x)\sec(2x)}{2} dx$$

Which simplifies to

$$u_1 = -\int \tan(2x)\, dx$$

Hence

$$u_1 = -\frac{\ln\left(1+\tan(2x)^2\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(2x)\sec(2x)}{2} dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Which simplifies to

$$u_1 = -\frac{\ln\left(\sec(2x)^2\right)}{4}$$
$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln\left(\sec(2x)^2\right)\cos(2x)}{4} + \sin(2x)\, x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(-\frac{\ln(\sec(2x)^2)\cos(2x)}{4} + \sin(2x)\, x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{\ln(\sec(2x)^2)\cos(2x)}{4} + \sin(2x)\, x \tag{1}
$$

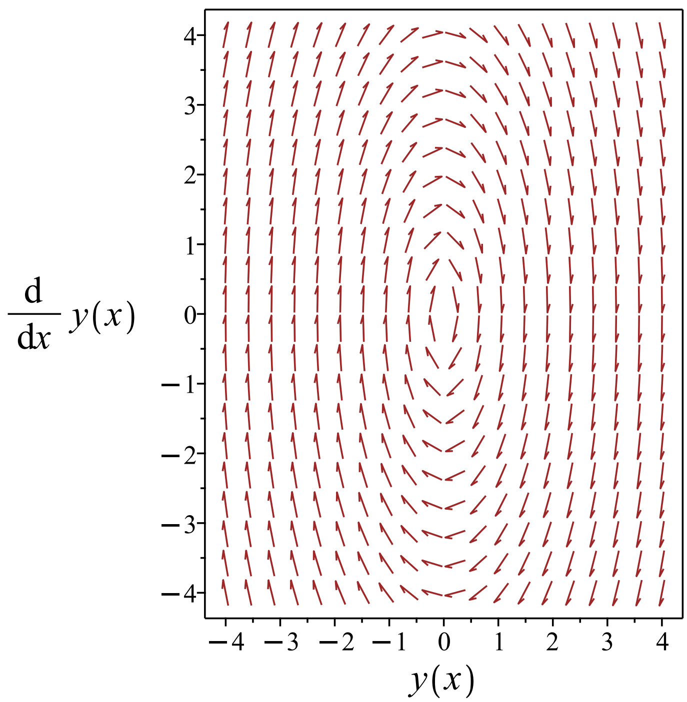

Figure 4: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{\ln(\sec(2x)^2)\cos(2x)}{4} + \sin(2x)\, x
$$

Verified OK.

1.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 5: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$

$$y_2 = \frac{\sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ -2\sin(2x) & \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(\cos(2x)) - \left(\frac{\sin(2x)}{2}\right)(-2\sin(2x))$$

Which simplifies to

$$W = \sin(2x)^2 + \cos(2x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(2x)\sec(2x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(2x)\,dx$$

Hence

$$u_1 = -\frac{\ln\left(1+\tan(2x)^2\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(2x)\sec(2x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 2dx$$

Hence

$$u_2 = 2x$$

Which simplifies to

$$u_1 = -\frac{\ln\left(\sec(2x)^2\right)}{4}$$
$$u_2 = 2x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln\left(\sec(2x)^2\right)\cos(2x)}{4} + \sin(2x)\,x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(-\frac{\ln\left(\sec(2x)^2\right)\cos(2x)}{4} + \sin(2x)\,x\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} - \frac{\ln\left(\sec(2x)^2\right)\cos(2x)}{4} + \sin(2x)\,x \tag{1}$$

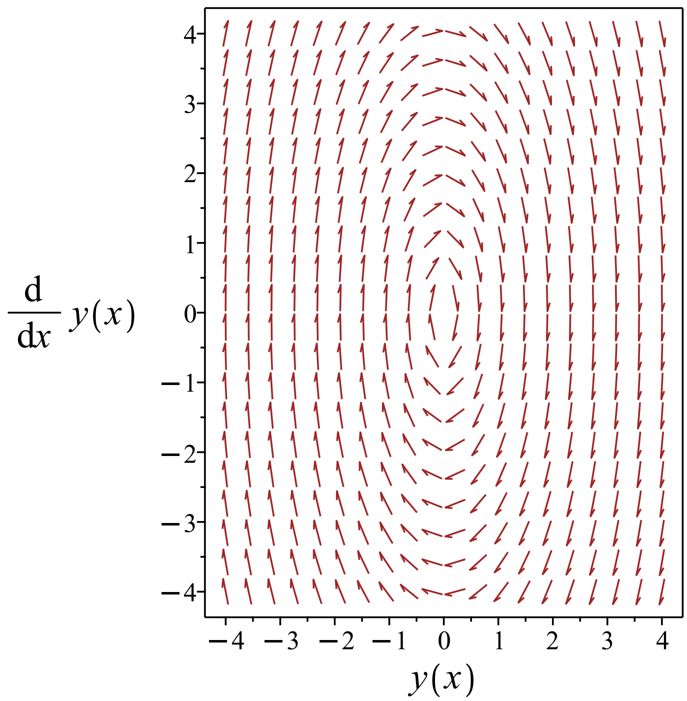

Figure 5: Slope field plot

<u>Verification of solutions</u>

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} - \frac{\ln\left(\sec(2x)^2\right)\cos(2x)}{4} + \sin(2x)\,x$$

Verified OK.

1.4.3 Maple step by step solution

Let's solve

$y'' + 4y = 2\sec(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = 2\sec(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(2x)\left(\int \tan(2x)\,dx\right) + \sin(2x)\left(\int 1dx\right)$$

- Compute integrals

$$y_p(x) = -\frac{\ln\left(\sec(2x)^2\right)\cos(2x)}{4} + \sin(2x)\,x$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(2x) + c_2\sin(2x) - \frac{\ln\left(\sec(2x)^2\right)\cos(2x)}{4} + \sin(2x)\,x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+4*y(x)=2*sec(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{\ln(\sec(2x))\cos(2x)}{2} + \cos(2x)\,c_1 + \sin(2x)(c_2 + x)$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 32

```
DSolve[y''[x]+4*y[x]==2*Sec[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x + c_2)\sin(2x) + \cos(2x)\left(\frac{1}{2}\log(\cos(2x)) + c_1\right)$$

1.5 problem Problem 1.3(c)

Internal problem ID [12398]
Internal file name [`OUTPUT/11050_Wednesday_October_04_2023_01_27_41_AM_79298836/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.3(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + \frac{y'}{x} + \left(1 - \frac{1}{4x^2}\right) y = x}$$

1.5.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + \frac{y'}{x} + \left(1 - \frac{1}{4x^2}\right) y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= 1 - \frac{1}{4x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 1 - \frac{1}{4x^2} - \frac{\left(\frac{1}{x}\right)'}{2} - \frac{\left(\frac{1}{x}\right)^2}{4} \\
&= 1 - \frac{1}{4x^2} - \frac{\left(-\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= 1 - \frac{1}{4x^2} - \left(-\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{1}{x}}{2}} \\
&= \frac{1}{\sqrt{x}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{\sqrt{x}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) + v(x) = x^{\frac{3}{2}}
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 1, B = 0, C = 1, f(x) = x^{\frac{3}{2}}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) + v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution v_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$v_p(x) = u_1 v_1 + u_2 v_2 \tag{1}$$

Where u_1, u_2 to be determined, and v_1, v_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$v_1 = \cos(x)$$
$$v_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{v_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{v_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of v'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\, x^{\frac{3}{2}}}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\, x^{\frac{3}{2}} dx$$

Hence

$$u_1 = \cos(x)\, x^{\frac{3}{2}} - \frac{3\sin(x)\sqrt{x}}{2} + \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\, x^{\frac{3}{2}}}{1}\, dx$$

Which simplifies to

$$u_2 = \int \cos(x)\, x^{\frac{3}{2}} dx$$

Hence

$$u_2 = \sin(x)\, x^{\frac{3}{2}} + \frac{3\cos(x)\sqrt{x}}{2} - \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}$$

Therefore the particular solution, from equation (1) is

$$v_p(x) = \left(\cos(x)\, x^{\frac{3}{2}} - \frac{3\sin(x)\sqrt{x}}{2} + \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}\right)\cos(x) + \left(\sin(x)\, x^{\frac{3}{2}} + \frac{3\cos(x)\sqrt{x}}{2} - \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}\right)\sin(x)$$

Which simplifies to

$$v_p(x) = -\frac{3\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + \frac{3\cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + x^{\frac{3}{2}}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= (c_1\cos(x) + c_2\sin(x)) \\ &\quad + \left(-\frac{3\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + \frac{3\cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + x^{\frac{3}{2}}\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1\cos(x) + c_2\sin(x) - \frac{3\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + \frac{3\cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + x\right. \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{\sqrt{x}}$$

Hence (7) becomes

$$y = \frac{c_1\cos(x) + c_2\sin(x) - \frac{3\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + \frac{3\cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + x^{\frac{3}{2}}}{\sqrt{x}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \cos(x) + c_2 \sin(x) - \frac{3\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + \frac{3\cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + x^{\frac{3}{2}}}{\sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

$$y_2 = \frac{\sin(x)}{\sqrt{x}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ \frac{d}{dx}\left(\frac{\cos(x)}{\sqrt{x}}\right) & \frac{d}{dx}\left(\frac{\sin(x)}{\sqrt{x}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ -\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}} & \frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(x)}{\sqrt{x}}\right)\left(\frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}}\right) - \left(\frac{\sin(x)}{\sqrt{x}}\right)\left(-\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}}\right)$$

Which simplifies to

$$W = \frac{\cos(x)^2 + \sin(x)^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sqrt{x}}{\frac{1}{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\, x^{\frac{3}{2}} dx$$

Hence

$$u_1 = \cos(x)\, x^{\frac{3}{2}} - \frac{3\sin(x)\sqrt{x}}{2} + \frac{3\sqrt{2}\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\sqrt{x}}{\sqrt{\pi}}\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sqrt{x}}{\frac{1}{x}}\, dx$$

Which simplifies to

$$u_2 = \int \cos(x)\, x^{\frac{3}{2}} dx$$

Hence

$$u_2 = \sin(x)\, x^{\frac{3}{2}} + \frac{3\cos(x)\sqrt{x}}{2} - \frac{3\sqrt{2}\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\sqrt{x}}{\sqrt{\pi}}\right)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\cos(x)\, x^{\frac{3}{2}} - \frac{3\sin(x)\sqrt{x}}{2} + \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}\right)\cos(x)}{\sqrt{x}} + \frac{\left(\sin(x)\, x^{\frac{3}{2}} + \frac{3\cos(x)\sqrt{x}}{2} - \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}\right)\sin(x)}{\sqrt{x}}$$

Which simplifies to

$$y_p(x) = -\frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\cos(x) + c_2\sin(x) - \frac{3\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + \frac{3\cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + x^{\frac{3}{2}}}{\sqrt{x}}\right) \\ &\quad + \left(-\frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\cos(x) + c_2\sin(x) - \frac{3\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + \frac{3\cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + x^{\frac{3}{2}}}{\sqrt{x}} - \frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos(x) + c_2 \sin(x) - \frac{3 \sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + \frac{3 \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4} + x^{\frac{3}{2}}}{\sqrt{x}}$$
$$- \frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}}$$

Verified OK.

1.5.2 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 + y'x + \left(x^2 - \frac{1}{4}\right)y = x^3 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$y''x^2 + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \,\mathrm{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \,\mathrm{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = 1$$
$$n = -\frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \cos(x)\sqrt{2}}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \cos(x)\sqrt{2}}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$
$$y_2 = \frac{\sin(x)}{\sqrt{x}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ \frac{d}{dx}\left(\frac{\cos(x)}{\sqrt{x}}\right) & \frac{d}{dx}\left(\frac{\sin(x)}{\sqrt{x}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ -\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}} & \frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(x)}{\sqrt{x}}\right)\left(\frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}}\right) - \left(\frac{\sin(x)}{\sqrt{x}}\right)\left(-\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}}\right)$$

Which simplifies to

$$W = \frac{\cos(x)^2 + \sin(x)^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\, x^{\frac{5}{2}}}{x}\, dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\, x^{\frac{3}{2}} dx$$

Hence

$$u_1 = \cos(x)\, x^{\frac{3}{2}} - \frac{3\sin(x)\sqrt{x}}{2} + \frac{3\sqrt{2}\,\sqrt{\pi}\ \text{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\, x^{\frac{5}{2}}}{x}\, dx$$

Which simplifies to

$$u_2 = \int \cos(x)\, x^{\frac{3}{2}} dx$$

Hence

$$u_2 = \sin(x)\, x^{\frac{3}{2}} + \frac{3\cos(x)\sqrt{x}}{2} - \frac{3\sqrt{2}\,\sqrt{\pi}\ \text{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\cos(x)\, x^{\frac{3}{2}} - \frac{3\sin(x)\sqrt{x}}{2} + \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}\right)\cos(x)}{\sqrt{x}} + \frac{\left(\sin(x)\, x^{\frac{3}{2}} + \frac{3\cos(x)\sqrt{x}}{2} - \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}\right)\sin(x)}{\sqrt{x}}$$

Which simplifies to

$$y_p(x) = -\frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\cos(x)\sqrt{2}}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}}\right) \\ &\quad + \left(-\frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \frac{c_1\cos(x)\sqrt{2}}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}} \\ & - \frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & \frac{c_1\cos(x)\sqrt{2}}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}} \\ & - \frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}} \end{aligned}$$

Verified OK.

1.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \frac{y'}{x} + \left(1 - \frac{1}{4x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{x} \\ C &= 1 - \frac{1}{4x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 7: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{1}{x}}{1}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(x)}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{1}{x}}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + \frac{y'}{x} + \left(1 - \frac{1}{4x^2}\right) y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{\cos(x)\, c_1}{\sqrt{x}} + \frac{\sin(x)\, c_2}{\sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

$$y_2 = \frac{\sin(x)}{\sqrt{x}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ \frac{d}{dx}\left(\frac{\cos(x)}{\sqrt{x}}\right) & \frac{d}{dx}\left(\frac{\sin(x)}{\sqrt{x}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ -\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}} & \frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(x)}{\sqrt{x}}\right)\left(\frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{\frac{3}{2}}}\right) - \left(\frac{\sin(x)}{\sqrt{x}}\right)\left(-\frac{\sin(x)}{\sqrt{x}} - \frac{\cos(x)}{2x^{\frac{3}{2}}}\right)$$

Which simplifies to

$$W = \frac{\cos(x)^2 + \sin(x)^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sqrt{x}}{\frac{1}{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\, x^{\frac{3}{2}} dx$$

Hence

$$u_1 = \cos(x)\, x^{\frac{3}{2}} - \frac{3\sin(x)\sqrt{x}}{2} + \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sqrt{x}}{\frac{1}{x}}\, dx$$

Which simplifies to

$$u_2 = \int \cos(x)\, x^{\frac{3}{2}} dx$$

Hence

$$u_2 = \sin(x)\, x^{\frac{3}{2}} + \frac{3\cos(x)\sqrt{x}}{2} - \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\cos(x)\, x^{\frac{3}{2}} - \frac{3\sin(x)\sqrt{x}}{2} + \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}\right)\cos(x)}{\sqrt{x}} + \frac{\left(\sin(x)\, x^{\frac{3}{2}} + \frac{3\cos(x)\sqrt{x}}{2} - \frac{3\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)}{4}\right)\sin(x)}{\sqrt{x}}$$

Which simplifies to

$$y_p(x) = -\frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\cos(x)\, c_1}{\sqrt{x}} + \frac{\sin(x)\, c_2}{\sqrt{x}}\right) \\ &\quad + \left(-\frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_1\cos(x) + c_2\sin(x)}{\sqrt{x}} - \frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(x) + c_2 \sin(x)}{\sqrt{x}} - \frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos(x) + c_2 \sin(x)}{\sqrt{x}} - \frac{3\left(\sin(x)\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \cos(x)\sqrt{2}\,\sqrt{\pi}\,\mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right) - \frac{4x^{\frac{3}{2}}}{3}\right)}{4\sqrt{x}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 62

```
dsolve(diff(y(x),x$2)+1/x*diff(y(x),x)+(1-1/(4*x^2))*y(x)=x,y(x), singsol=all)
```

$$y(x) = \frac{\sin(x)\, c_2 + c_1 \cos(x) + \frac{3\cos(x)\sqrt{\pi}\ \mathrm{FresnelS}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)\sqrt{2}}{4} - \frac{3\sin(x)\sqrt{\pi}\ \mathrm{FresnelC}\left(\frac{\sqrt{2}\,\sqrt{x}}{\sqrt{\pi}}\right)\sqrt{2}}{4} + x^{\frac{3}{2}}}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.443 (sec). Leaf size: 111

```
DSolve[y''[x]+1/x*y'[x]+(1-1/(4*x^2))*y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}\left(-\frac{e^{2ix}x^{3/2}\Gamma\left(\frac{5}{2},ix\right)}{\sqrt{-ix}} + \sqrt{x^2}\left(2c_1 - ic_2e^{2ix}\right) + \frac{(ix)^{3/2}\Gamma\left(\frac{5}{2},-ix\right)}{\sqrt{x}}\right)}{2\sqrt{x}\sqrt{x^2}}$$

1.6 problem Problem 1.3(d)

Internal problem ID [12399]
Internal file name [`OUTPUT/11051_Wednesday_October_04_2023_01_27_53_AM_92705771/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.3(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = f(x)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

1.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = f(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x) f(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int \sin(x) f(x) dx$$

Hence

$$u_1 = -\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x) f(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cos(x) f(x) dx$$

Hence

$$u_2 = \int_0^x \cos(\alpha) f(\alpha) d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) \\
&\quad + \left(-\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(x) + c_2 \sin(x) - \left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\sin(x) c_1 + c_2 \cos(x) + \left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \sin(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \cos(x)
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x)
$$

Summary

The solution(s) found are the following

$$
y = -\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x) \tag{1}
$$

Verification of solutions

$$y = -\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right)\cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right)\sin(x)$$

Verified OK.

1.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 8: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(x) \\ y_2 &= \sin(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x) f(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int \sin(x) f(x) dx$$

Hence

$$u_1 = -\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x) f(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cos(x) f(x) dx$$

Hence

$$u_2 = \int_0^x \cos(\alpha) f(\alpha) d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) \\
&\quad + \left(-\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(x) + c_2 \sin(x) - \left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\sin(x) c_1 + c_2 \cos(x) + \left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \sin(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \cos(x)
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x)
$$

Summary

The solution(s) found are the following

$$
y = -\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x) \tag{1}
$$

Verification of solutions

$$y = -\left(\int_0^x \sin(\alpha) f(\alpha) d\alpha\right) \cos(x) + \left(\int_0^x \cos(\alpha) f(\alpha) d\alpha\right) \sin(x)$$

Verified OK.

1.6.3 Maple step by step solution

Let's solve

$\left[y'' + y = f(x), y(0) = 0, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = f(x)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x) \left(\int \sin(x) f(x) dx\right) + \sin(x) \left(\int \cos(x) f(x) dx\right)$

- Compute integrals

  $y_p(x) = -\cos(x) \left(\int \sin(x) f(x) dx\right) + \sin(x) \left(\int \cos(x) f(x) dx\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) \left(\int \sin(x) f(x) dx\right) + \sin(x) \left(\int \cos(x) f(x) dx\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 34

```
dsolve([diff(y(x),x$2)+y(x)=f(x),y(0) = 0, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = \left(\int_0^x \cos(\_z1) f(\_z1) d\_z1\right) \sin(x) - \left(\int_0^x \sin(\_z1) f(\_z1) d\_z1\right) \cos(x)$$

✓ Solution by Mathematica

Time used: 0.105 (sec). Leaf size: 77

```
DSolve[{y''[x]+y[x]==f[x],{y[0]==0,y'[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) \to & -\sin(x)\int_1^0 \cos(K[2])f(K[2])dK[2] + \sin(x)\int_1^x \cos(K[2])f(K[2])dK[2] \\
& + \cos(x)\left(\int_1^x -f(K[1])\sin(K[1])dK[1] - \int_1^0 -f(K[1])\sin(K[1])dK[1]\right)
\end{aligned}
$$

1.7 problem Problem 1.6(a)

Internal problem ID [12400]
Internal file name [`OUTPUT/11052_Wednesday_October_04_2023_01_27_54_AM_9924872/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.6(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y''x^2 + x\left(x - \frac{1}{2}\right)y' + \frac{y}{2} = 0}$$

1.7.1 Solving using Kovacic algorithm

Writing the ode as

$$y''x^2 + \left(x^2 - \frac{1}{2}x\right)y' + \frac{y}{2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 - \frac{1}{2}x \\ C &= \frac{1}{2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 4x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 10: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{16x^2} - \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{4x} - \frac{1}{4x^2} - \frac{1}{8x^3} - \frac{1}{8x^4} - \frac{1}{8x^5} - \frac{9}{64x^6} - \frac{21}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^2 - 4x - 3}{16x^2} \\
&= Q + \frac{R}{16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x - 3}{16x^2}\right) \\
&= \frac{1}{4} + \frac{-4x - 3}{16x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 16 gives $-\frac{1}{4}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{4}\right) - (0) \\
&= -\frac{1}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{4}}{\frac{1}{2}} - 0\right) &= -\frac{1}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{4}}{\frac{1}{2}} - 0\right) &= \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 4x - 3}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{4x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{4x} - \frac{1}{2} \\ &= \frac{1}{4x} - \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{4x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{4x^2}\right) + \left(\frac{1}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2 - 4x - 3}{16x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} - \frac{1}{2}\right) dx} \\
&= x^{\frac{1}{4}} \mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 - \frac{1}{2}x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} + \frac{\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{1}{4}} \mathrm{e}^{-\frac{x}{2}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2 - \frac{1}{2}x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x + \frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\sqrt{\pi} \, \mathrm{erfi}\left(\sqrt{x}\right) \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x}\, \mathrm{e}^{-x} \right) + c_2 \left(\sqrt{x}\, \mathrm{e}^{-x} \left(\sqrt{\pi} \, \mathrm{erfi}\left(\sqrt{x}\right) \right) \right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x}\,\mathrm{e}^{-x} + c_2\sqrt{x}\,\mathrm{e}^{-x}\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\mathrm{e}^{-x} + c_2\sqrt{x}\,\mathrm{e}^{-x}\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)$$

Verified OK.

1.7.2 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$y''x^2 + \left(x^2 - \frac{1}{2}x\right)y' + \frac{y}{2} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2x-1}{2x} \\ q(x) &= \frac{1}{2x^2} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - \left(\frac{(2x-1)\,\xi(x)}{2x}\right)' + \left(\frac{\xi(x)}{2x^2}\right) &= 0 \\ \xi''(x) - \frac{(2x-1)\,\xi'(x)}{2x} + \left(-\frac{1}{x} + \frac{2x-1}{2x^2} + \frac{1}{2x^2}\right)\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order ode with missing dependent variable $\xi(x)$. Let

$$p(x) = \xi'(x)$$

Then

$$p'(x) = \xi''(x)$$

Hence the ode becomes

$$2p'(x)\,x+(-2x+1)\,p(x)=0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned}p' &= F(x,p)\\ &= f(x)g(p)\\ &= \frac{p(2x-1)}{2x}\end{aligned}$$

Where $f(x)=\frac{2x-1}{2x}$ and $g(p)=p$. Integrating both sides gives

$$\begin{aligned}\frac{1}{p}\,dp &= \frac{2x-1}{2x}\,dx\\ \int\frac{1}{p}\,dp &= \int\frac{2x-1}{2x}\,dx\\ \ln(p) &= x-\frac{\ln(x)}{2}+c_1\\ p &= \mathrm{e}^{x-\frac{\ln(x)}{2}+c_1}\\ &= c_1\mathrm{e}^{x-\frac{\ln(x)}{2}}\end{aligned}$$

Which simplifies to

$$p(x)=\frac{c_1\mathrm{e}^x}{\sqrt{x}}$$

Since $p=\xi'(x)$ then the new first order ode to solve is

$$\xi'(x)=\frac{c_1\mathrm{e}^x}{\sqrt{x}}$$

Integrating both sides gives

$$\begin{aligned}\xi(x) &= \int\frac{c_1\mathrm{e}^x}{\sqrt{x}}\,\mathrm{d}x\\ &= c_1\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\end{aligned}$$

The original ode (2) now reduces to first order ode

$$\begin{aligned}\xi(x)\,y'-y\xi'(x)+\xi(x)\,p(x)\,y &= \int\xi(x)\,r(x)\,dx\\ y'+y\left(p(x)-\frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int\xi(x)\,r(x)\,dx}{\xi(x)}\end{aligned}$$

$$y'+y\left(\frac{2x-1}{2x}-\frac{c_3\mathrm{e}^x}{\sqrt{x}\,\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}\right)=0$$

Which is now a first order ode. This is now solved for y. In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -\frac{y\left(2\,\operatorname{erfi}\left(\sqrt{x}\right) x^{\frac{3}{2}}\sqrt{\pi}\,c_3-\operatorname{erfi}\left(\sqrt{x}\right)\sqrt{x}\,\sqrt{\pi}\,c_3+2x^{\frac{3}{2}}c_2-2x\,\mathrm{e}^{x}c_3-\sqrt{x}\,c_2\right)}{2x^{\frac{3}{2}}\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}
\end{aligned}$$

Where $f(x)=-\frac{2\,\operatorname{erfi}\left(\sqrt{x}\right) x^{\frac{3}{2}}\sqrt{\pi}\,c_3-\operatorname{erfi}\left(\sqrt{x}\right)\sqrt{x}\,\sqrt{\pi}\,c_3+2x^{\frac{3}{2}}c_2-2x\,\mathrm{e}^{x}c_3-\sqrt{x}\,c_2}{2x^{\frac{3}{2}}\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}$ and $g(y)=y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -\frac{2\,\operatorname{erfi}\left(\sqrt{x}\right) x^{\frac{3}{2}}\sqrt{\pi}\,c_3-\operatorname{erfi}\left(\sqrt{x}\right)\sqrt{x}\,\sqrt{\pi}\,c_3+2x^{\frac{3}{2}}c_2-2x\,\mathrm{e}^{x}c_3-\sqrt{x}\,c_2}{2x^{\frac{3}{2}}\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}\,dx\\
\int\frac{1}{y}\,dy &= \int-\frac{2\,\operatorname{erfi}\left(\sqrt{x}\right) x^{\frac{3}{2}}\sqrt{\pi}\,c_3-\operatorname{erfi}\left(\sqrt{x}\right)\sqrt{x}\,\sqrt{\pi}\,c_3+2x^{\frac{3}{2}}c_2-2x\,\mathrm{e}^{x}c_3-\sqrt{x}\,c_2}{2x^{\frac{3}{2}}\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}\,dx\\
\ln(y) &= \int-\frac{2\,\operatorname{erfi}\left(\sqrt{x}\right) x^{\frac{3}{2}}\sqrt{\pi}\,c_3-\operatorname{erfi}\left(\sqrt{x}\right)\sqrt{x}\,\sqrt{\pi}\,c_3+2x^{\frac{3}{2}}c_2-2x\,\mathrm{e}^{x}c_3-\sqrt{x}\,c_2}{2x^{\frac{3}{2}}\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}dx+c_3\\
y &= \mathrm{e}^{\int-\frac{2\,\operatorname{erfi}\left(\sqrt{x}\right) x^{\frac{3}{2}}\sqrt{\pi}\,c_3-\operatorname{erfi}\left(\sqrt{x}\right)\sqrt{x}\,\sqrt{\pi}\,c_3+2x^{\frac{3}{2}}c_2-2x\,\mathrm{e}^{x}c_3-\sqrt{x}\,c_2}{2x^{\frac{3}{2}}\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}dx+c_3}\\
&= c_3\mathrm{e}^{\int-\frac{2\,\operatorname{erfi}\left(\sqrt{x}\right) x^{\frac{3}{2}}\sqrt{\pi}\,c_3-\operatorname{erfi}\left(\sqrt{x}\right)\sqrt{x}\,\sqrt{\pi}\,c_3+2x^{\frac{3}{2}}c_2-2x\,\mathrm{e}^{x}c_3-\sqrt{x}\,c_2}{2x^{\frac{3}{2}}\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}dx}
\end{aligned}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_3\mathrm{e}^{\int-\frac{2\,\operatorname{erfi}\left(\sqrt{x}\right) x^{\frac{3}{2}}\sqrt{\pi}\,c_3-\operatorname{erfi}\left(\sqrt{x}\right)\sqrt{x}\,\sqrt{\pi}\,c_3+2x^{\frac{3}{2}}c_2-2x\,\mathrm{e}^{x}c_3-\sqrt{x}\,c_2}{2x^{\frac{3}{2}}\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}dx}$$

Summary

The solution(s) found are the following

$$y = c_3\mathrm{e}^{\int-\frac{2\,\operatorname{erfi}\left(\sqrt{x}\right) x^{\frac{3}{2}}\sqrt{\pi}\,c_3-\operatorname{erfi}\left(\sqrt{x}\right)\sqrt{x}\,\sqrt{\pi}\,c_3+2x^{\frac{3}{2}}c_2-2x\,\mathrm{e}^{x}c_3-\sqrt{x}\,c_2}{2x^{\frac{3}{2}}\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}dx} \tag{1}$$

Verification of solutions

$$y = c_3\mathrm{e}^{\int-\frac{2\,\operatorname{erfi}\left(\sqrt{x}\right) x^{\frac{3}{2}}\sqrt{\pi}\,c_3-\operatorname{erfi}\left(\sqrt{x}\right)\sqrt{x}\,\sqrt{\pi}\,c_3+2x^{\frac{3}{2}}c_2-2x\,\mathrm{e}^{x}c_3-\sqrt{x}\,c_2}{2x^{\frac{3}{2}}\left(c_3\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)+c_2\right)}dx}$$

Verified OK.

1.7.3 Maple step by step solution

Let's solve

$y''x^2+\left(x^2-\frac{1}{2}x\right)y'+\frac{y}{2}=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{2x^2}-\frac{(2x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x-1)y'}{2x}+\frac{y}{2x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x-1}{2x}, P_3(x)=\frac{1}{2x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-\frac{1}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{1}{2}$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2y''x^2+x(2x-1)y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) + 2a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+r-1)\left(\left(k+r-\frac{1}{2}\right) a_k + a_{k-1}\right) = 0$$

- Shift index using $k- >k+1$

$$2(k+r)\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a_k}{2k+1+2r}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{2a_k}{2k+3}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{2a_k}{2k+3}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{2a_k}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{2a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{1+k} = -\frac{2a_k}{2k+3}, b_{1+k} = -\frac{2b_k}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 39

```
dsolve(x^2*diff(y(x),x$2)+x*(x-1/2)*diff(y(x),x)+1/2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{-x} \left(\operatorname{erf}\left(\sqrt{-x}\right) \sqrt{\pi}\, c_1 x + 2 c_2 \sqrt{x}\, \sqrt{-x}\right)}{2\sqrt{-x}}$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]+x*(x-1/2)*y'[x]+1/2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_2\sqrt{x} + c_1\sqrt{-x}\Gamma\left(\frac{1}{2}, -x\right)\right)$$

1.8 problem Problem 1.6(b)

Internal problem ID [12401]
Internal file name [`OUTPUT/11053_Wednesday_October_04_2023_01_27_55_AM_80091292/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.6(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y''x^2 + x(x+1)\,y' - y = 0}$$

1.8.1 Solving using Kovacic algorithm

Writing the ode as

$$y''x^2 + \left(x^2 + x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 2x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 12: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} + \frac{1}{2x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} + \frac{1}{2x^2} - \frac{1}{2x^3} + \frac{1}{4x^4} + \frac{1}{4x^5} - \frac{3}{4x^6} + \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{3 + 2x}{4x^2}\right) \\ &= \frac{1}{4} + \frac{3 + 2x}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x+1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+2x+3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2+x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^x (x-1)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{-x}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{-x}}{x} (\mathrm{e}^x (x-1))\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-x}}{x} + \frac{c_2(x-1)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-x}}{x} + \frac{c_2(x-1)}{x}$$

Verified OK.

1.8.2 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$y''x^2 + (x^2 + x) y' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{x+1}{x} \\ q(x) &= -\frac{1}{x^2} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(\frac{(x+1)\,\xi(x)}{x}\right)' + \left(-\frac{\xi(x)}{x^2}\right) &= 0 \\ \xi''(x) - \frac{(x+1)\,\xi'(x)}{x} + \left(-\frac{1}{x} + \frac{x+1}{x^2} - \frac{1}{x^2}\right)\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order ode with missing dependent variable $\xi(x)$. Let

$$p(x) = \xi'(x)$$

Then

$$p'(x) = \xi''(x)$$

Hence the ode becomes

$$p'(x)\, x + (-x - 1)\, p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x, p) \\ &= f(x) g(p) \\ &= \frac{p(x+1)}{x} \end{aligned}$$

Where $f(x) = \frac{x+1}{x}$ and $g(p) = p$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{p}\, dp &= \frac{x+1}{x}\, dx \\
\int \frac{1}{p}\, dp &= \int \frac{x+1}{x}\, dx \\
\ln(p) &= x + \ln(x) + c_1 \\
p &= \mathrm{e}^{x+\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{x+\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
p(x) = c_1 x\, \mathrm{e}^{x}
$$

Since $p = \xi'(x)$ then the new first order ode to solve is

$$
\xi'(x) = c_1 x\, \mathrm{e}^{x}
$$

Integrating both sides gives

$$
\begin{aligned}
\xi(x) &= \int c_1 x\, \mathrm{e}^{x}\, \mathrm{d}x \\
&= (x-1)\,\mathrm{e}^{x} c_1 + c_2
\end{aligned}
$$

The original ode (2) now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)} \\
y' + y\left(\frac{x+1}{x} - \frac{c_3 \mathrm{e}^{x}(x-1) + c_3 \mathrm{e}^{x}}{c_3 \mathrm{e}^{x}(x-1) + c_2}\right) &= 0
\end{aligned}
$$

Which is now a first order ode. This is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x) g(y) \\
&= \frac{y(c_3 \mathrm{e}^{x} - c_2 x - c_2)}{x\,(x\, \mathrm{e}^{x} c_3 - c_3 \mathrm{e}^{x} + c_2)}
\end{aligned}
$$

Where $f(x)=\frac{c_3 e^x - c_2 x - c_2}{x(x\,e^x c_3 - c_3 e^x + c_2)}$ and $g(y)=y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{c_3 e^x - c_2 x - c_2}{x\left(x\,e^x c_3 - c_3 e^x + c_2\right)}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{c_3 e^x - c_2 x - c_2}{x\left(x\,e^x c_3 - c_3 e^x + c_2\right)}\,dx \\
\ln(y) &= -x - \ln(x) + \ln\left(x\,e^x c_3 - c_3 e^x + c_2\right) + c_3 \\
y &= e^{-x-\ln(x)+\ln(x\,e^x c_3 - c_3 e^x + c_2)+c_3} \\
&= c_3 e^{-x-\ln(x)+\ln(x\,e^x c_3 - c_3 e^x + c_2)}
\end{aligned}
$$

Which simplifies to

$$
y = c_3\left(c_3 - \frac{c_3}{x} + \frac{e^{-x} c_2}{x}\right)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{c_3 e^{-x} c_2}{x} + c_3\left(c_3 - \frac{c_3}{x}\right)
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_3 e^{-x} c_2}{x} + c_3\left(c_3 - \frac{c_3}{x}\right) \tag{1}
$$

Verification of solutions

$$
y = \frac{c_3 e^{-x} c_2}{x} + c_3\left(c_3 - \frac{c_3}{x}\right)
$$

Verified OK.

1.8.3 Maple step by step solution

Let's solve

$y''x^2 + (x^2 + x)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x^2} - \frac{(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x+1)y'}{x}-\frac{y}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x+1}{x}, P_3(x)=-\frac{1}{x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-1$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $y''x^2+x(x+1)\,y'-y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y''=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(1+r)\,(-1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+1)\,(k+r-1)+a_{k-1}(k+r-1)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_k(k+r+1)+a_{k-1})=0$

- Shift index using $k- >k+1$

  $(k+r)(a_{k+1}(k+2+r)+a_k)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k+2+r}$

- Recursion relation for $r=-1$

  $a_{k+1} = -\frac{a_k}{k+1}$

- Solution for $r=-1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$

- Recursion relation for $r=1$

  $a_{k+1} = -\frac{a_k}{k+3}$

- Solution for $r=1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{1+k} = -\frac{a_k}{1+k}, b_{1+k} = -\frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 e^{-x} + c_1(-1 + x)}{x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]+x*(1+x)*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}(c_1 e^x (x - 1) + c_2)}{x}$$

1.9 problem Problem 1.7

Internal problem ID [12402]
Internal file name [`OUTPUT/11054_Wednesday_October_04_2023_01_27_56_AM_44044858/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{x(1-x)\,y''+(-5x+1)\,y'-4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^2+x\right)y''+(-5x+1)\,y'-4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{5x-1}{x\,(x-1)}$$
$$q(x)=\frac{4}{x\,(x-1)}$$

Table 14: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5x-1}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = \frac{4}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-y''x(x-1) + (-5x+1)\, y' - 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) + (-5x+1)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_{n} (n+r) (n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r-1} a_{n} (n+r) (n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-5 x^{n+r} a_{n} (n+r)\right)+\left(\sum_{n=0}^{\infty} (n+r) a_{n} x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-4 a_{n} x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_{n} (n+r) (n+r-1)\right)=\sum_{n=1}^{\infty}\left(-a_{n-1} (n+r-1) (n+r-2) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-5 x^{n+r} a_{n} (n+r)\right)=\sum_{n=1}^{\infty}\left(-5 a_{n-1} (n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-4 a_{n} x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-4 a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-a_{n-1} (n+r-1) (n+r-2) x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty} x^{n+r-1} a_{n} (n+r) (n+r-1)\right)+\sum_{n=1}^{\infty}\left(-5 a_{n-1} (n+r-1) x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty} (n+r) a_{n} x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-4 a_{n-1} x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_{n} (n+r) (n+r-1)+(n+r) a_{n} x^{n+r-1}=0$$

When $n=0$ the above becomes

$$x^{-1+r} a_{0} r (-1+r)+r a_{0} x^{-1+r}=0$$

Or

$$\left(x^{-1+r} r(-1+r)+r\, x^{-1+r}\right) a_{0}=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^{2}=0$$

Since the above is true for all x then the indicial equation becomes

$$r^{2}=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_{1} &= 0 \\ r_{2} &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^{2}=0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_{1}(x)=\sum_{n=0}^{\infty} a_{n} x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_{2}(x)=y_{1}(x) \ln (x)+\left(\sum_{n=1}^{\infty} b_{n} x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y=c_{1} y_{1}(x)+c_{2} y_{2}(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} & -a_{n-1}(n+r-1)(n+r-2)+a_{n}(n+r)(n+r-1) \\ & -5 a_{n-1}(n+r-1)+a_{n}(n+r)-4 a_{n-1}=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n^2 + 2nr + r^2 + 2n + 2r + 1)}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}(n+1)^2}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{(r+2)^2}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(r+2)^2}{(r+1)^2}$	4

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r+3)^2}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 9$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(r+2)^2}{(r+1)^2}$	4
a_2	$\frac{(r+3)^2}{(r+1)^2}$	9

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(r+4)^2}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = 16$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(r+2)^2}{(r+1)^2}$	4
a_2	$\frac{(r+3)^2}{(r+1)^2}$	9
a_3	$\frac{(r+4)^2}{(r+1)^2}$	16

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r+5)^2}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = 25$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(r+2)^2}{(r+1)^2}$	4
a_2	$\frac{(r+3)^2}{(r+1)^2}$	9
a_3	$\frac{(r+4)^2}{(r+1)^2}$	16
a_4	$\frac{(r+5)^2}{(r+1)^2}$	25

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(r+6)^2}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = 36$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(r+2)^2}{(r+1)^2}$	4
a_2	$\frac{(r+3)^2}{(r+1)^2}$	9
a_3	$\frac{(r+4)^2}{(r+1)^2}$	16
a_4	$\frac{(r+5)^2}{(r+1)^2}$	25
a_5	$\frac{(r+6)^2}{(r+1)^2}$	36

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 36x^5 + 25x^4 + 16x^3 + 9x^2 + 4x + 1 + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{(r+2)^2}{(r+1)^2}$	4	$\frac{-2r-4}{(r+1)^3}$	-4
b_2	$\frac{(r+3)^2}{(r+1)^2}$	9	$\frac{-4r-12}{(r+1)^3}$	-12
b_3	$\frac{(r+4)^2}{(r+1)^2}$	16	$\frac{-6r-24}{(r+1)^3}$	-24
b_4	$\frac{(r+5)^2}{(r+1)^2}$	25	$\frac{-8r-40}{(r+1)^3}$	-40
b_5	$\frac{(r+6)^2}{(r+1)^2}$	36	$\frac{-10r-60}{(r+1)^3}$	-60

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(36x^5 + 25x^4 + 16x^3 + 9x^2 + 4x + 1 + O(x^6)\right) \ln(x) \\
&\quad - 60x^5 - 40x^4 - 24x^3 - 12x^2 - 4x + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(36x^5 + 25x^4 + 16x^3 + 9x^2 + 4x + 1 + O(x^6)\right) \\
&\quad + c_2 \left(\left(36x^5 + 25x^4 + 16x^3 + 9x^2 + 4x + 1 + O(x^6)\right) \ln(x) - 60x^5 - 40x^4 - 24x^3 \right. \\
&\quad \left. - 12x^2 - 4x + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 \left(36x^5 + 25x^4 + 16x^3 + 9x^2 + 4x + 1 + O(x^6)\right) \\
&\quad + c_2 \left(\left(36x^5 + 25x^4 + 16x^3 + 9x^2 + 4x + 1 + O(x^6)\right) \ln(x) - 60x^5 - 40x^4 - 24x^3 \right. \\
&\quad \left. - 12x^2 - 4x + O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\left(36x^5+25x^4+16x^3+9x^2+4x+1+O\left(x^6\right)\right) \\
& + c_2\left(\left(36x^5+25x^4+16x^3+9x^2+4x+1+O\left(x^6\right)\right)\ln\left(x\right)-60x^5-40x^4 \right. \\
& \left. -24x^3-12x^2-4x+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1\left(36x^5+25x^4+16x^3+9x^2+4x+1+O\left(x^6\right)\right) \\
& + c_2\left(\left(36x^5+25x^4+16x^3+9x^2+4x+1+O\left(x^6\right)\right)\ln\left(x\right)-60x^5-40x^4-24x^3 \right. \\
& \left. -12x^2-4x+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

1.9.1 Maple step by step solution

Let's solve

$-y''x(x-1)+(-5x+1)\,y'-4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x(x-1)}-\frac{(5x-1)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(5x-1)y'}{x(x-1)}+\frac{4y}{x(x-1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{5x-1}{x(x-1)}, P_3(x)=\frac{4}{x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x(x-1) + (5x-1)\,y' + 4y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$$-a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)^2 + a_k (k+r+2)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

$-a_{k+1}(k+1)^2 + a_k (k+2)^2 = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k (k+2)^2}{(k+1)^2}$

- Recursion relation for $r = 0$

$a_{k+1} = \frac{a_k(k+2)^2}{(k+1)^2}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(k+2)^2}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*(1-x)*diff(y(x),x$2)+(1-5*x)*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_1 + c_2 \ln(x)) \left(1 + 4x + 9x^2 + 16x^3 + 25x^4 + 36x^5 + \mathrm{O}\left(x^6\right)\right) + \left((-4)x - 12x^2 - 24x^3 - 40x^4 - 60x^5 + \mathrm{O}\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica
Time used: 0.01 (sec). Leaf size: 87

```
AsymptoticDSolveValue[x*(1-x)*y''[x]+(1-5*x)*y'[x]-4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(36x^5 + 25x^4 + 16x^3 + 9x^2 + 4x + 1\right) + c_2\left(-60x^5 - 40x^4 - 24x^3 - 12x^2 + \left(36x^5 + 25x^4 + 16x^3 + 9x^2 + 4x + 1\right)\log(x) - 4x\right)$$

1.10 problem Problem 1.8(a)

Internal problem ID [12403]
Internal file name [`OUTPUT/11055_Wednesday_October_04_2023_01_27_57_AM_31548167/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.8(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{2}-1\right)^{2} y''+\left(x+1\right) y'-y=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f\big|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'\big|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (18) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (19) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n\big|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y'x + y' - y}{x^4 - 2x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{4\left(x - \frac{1}{2}\right)^2 \left(\left(x+1\right) y' - y\right)}{\left(x+1\right)^3 \left(x-1\right)^4} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\frac{\left(10x^3 - 15x^2 + 10x - 4\right) \left(2x - 1\right) \left(y'x + y' - y\right)}{\left(x-1\right)^6 \left(x+1\right)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{\left(120x^6 - 360x^5 + 540x^4 - 532x^3 + 321x^2 - 102x + 14\right) \left(y'x + y' - y\right)}{\left(x+1\right)^5 \left(x-1\right)^8} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -\frac{\left(840x^8 - 3360x^7 + 7140x^6 - 10360x^5 + 10135x^4 - 6484x^3 + 2696x^2 - 680x + 74\right) \left(y'x + y' - y\right)}{\left(x-1\right)^{10} \left(x+1\right)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) - y'(0) \\
F_1 &= y'(0) - y(0) \\
F_2 &= 4y(0) - 4y'(0) \\
F_3 &= -14y(0) + 14y'(0) \\
F_4 &= 74y(0) - 74y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$\begin{aligned}y &= \left(1+\frac{1}{2}x^{2}-\frac{1}{6}x^{3}+\frac{1}{6}x^{4}-\frac{7}{60}x^{5}+\frac{37}{360}x^{6}\right)y(0)\\&+\left(x-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{6}x^{4}+\frac{7}{60}x^{5}-\frac{37}{360}x^{6}\right)y'(0)+O(x^{6})\end{aligned}$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$y''(x^{4}-2x^{2}+1)+(x+1)\,y'-y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty}a_{n}x^{n}$$

Then

$$\begin{aligned}y' &= \sum_{n=1}^{\infty}na_{n}x^{n-1}\\y'' &= \sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\end{aligned}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)(x^{4}-2x^{2}+1)+(x+1)\left(\sum_{n=1}^{\infty}na_{n}x^{n-1}\right)-\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0 \tag{1}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=2}^{\infty}n\,x^{n+2}a_{n}(n-1)\right)+\sum_{n=2}^{\infty}\left(-2x^{n}a_{n}n(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)\\&+\left(\sum_{n=1}^{\infty}na_{n}x^{n}\right)+\left(\sum_{n=1}^{\infty}na_{n}x^{n-1}\right)+\sum_{n=0}^{\infty}\left(-a_{n}x^{n}\right)=0\end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^{n} and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n+2} a_n (n-1) = \sum_{n=4}^{\infty} (n-2)\, a_{n-2} (n-3)\, x^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, x^n$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}\left(\sum_{n=4}^{\infty} (n-2)\, a_{n-2}(n-3)\, x^n\right) + \sum_{n=2}^{\infty} \left(-2x^n a_n n(n-1)\right) \\ + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) \\ + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0\end{aligned} \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 - a_0 = 0$$

$$a_2 = \frac{a_0}{2} - \frac{a_1}{2}$$

$n = 1$ gives

$$6a_3 + 2a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$6a_3 + a_0 - a_1 = 0$$

Or

$$a_3 = -\frac{a_0}{6} + \frac{a_1}{6}$$

$n = 2$ gives

$$-3a_2 + 12a_4 + 3a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$-2a_0 + 2a_1 + 12a_4 = 0$$

Or

$$a_4 = \frac{a_0}{6} - \frac{a_1}{6}$$

$n = 3$ gives

$$-10a_3 + 20a_5 + 4a_4 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{7a_0}{3} - \frac{7a_1}{3} + 20a_5 = 0$$

Or

$$a_5 = -\frac{7a_0}{60} + \frac{7a_1}{60}$$

For $4 \leq n$, the recurrence equation is

$$(n-2)\, a_{n-2}(n-3) - 2na_n(n-1) + (n+2)\, a_{n+2}(n+1) + na_n + (n+1)\, a_{n+1} - a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{2n^2 a_n - n^2 a_{n-2} - 3na_n + 5na_{n-2} - na_{n+1} + a_n - 6a_{n-2} - a_{n+1}}{(n+2)\,(n+1)} \\ &= \frac{(2n^2 - 3n + 1)\, a_n}{(n+2)\,(n+1)} + \frac{(-n^2 + 5n - 6)\, a_{n-2}}{(n+2)\,(n+1)} + \frac{(-n-1)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \quad (5)$$

For $n = 4$ the recurrence equation gives

$$2a_2 - 21a_4 + 30a_6 + 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{37a_0}{360} - \frac{37a_1}{360}$$

For $n = 5$ the recurrence equation gives

$$6a_3 - 36a_5 + 42a_7 + 6a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{229a_0}{2520} + \frac{229a_1}{2520}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(\frac{a_0}{2} - \frac{a_1}{2}\right) x^2 + \left(-\frac{a_0}{6} + \frac{a_1}{6}\right) x^3 + \left(\frac{a_0}{6} - \frac{a_1}{6}\right) x^4 + \left(-\frac{7a_0}{60} + \frac{7a_1}{60}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{6}x^4 - \frac{7}{60}x^5\right) a_0 + \left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{7}{60}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{6}x^4 - \frac{7}{60}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{7}{60}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{6}x^4 - \frac{7}{60}x^5 + \frac{37}{360}x^6\right) y(0) \\ & + \left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{7}{60}x^5 - \frac{37}{360}x^6\right) y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{6}x^4 - \frac{7}{60}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{7}{60}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{6}x^4-\frac{7}{60}x^5+\frac{37}{360}x^6\right)y(0) \\ +\left(x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{6}x^4+\frac{7}{60}x^5-\frac{37}{360}x^6\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{6}x^4-\frac{7}{60}x^5\right)c_1+\left(x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{6}x^4+\frac{7}{60}x^5\right)c_2+O(x^6)$$

Verified OK.

1.10.1 Maple step by step solution

Let's solve

$y''(x^4-2x^2+1)+(x+1)\,y'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{x^4-2x^2+1}-\frac{y'}{x^3-x^2-x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x^3-x^2-x+1}-\frac{y}{x^4-2x^2+1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{x^3-x^2-x+1}, P_3(x)=-\frac{1}{x^4-2x^2+1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{1}{4}$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1}=-\frac{1}{4}$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $y''(x^3-x^2-x+1)(x^4-2x^2+1)+y'(x^4-2x^2+1)+(-x^3+x^2+x-1)y=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^7-8u^6+24u^5-32u^4+16u^3)\left(\frac{d^2}{du^2}y(u)\right)+(u^4-4u^3+4u^2)\left(\frac{d}{du}y(u)\right)+(-u^3+4u^2-4u)\,y($
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=1..3$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=2..4$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=3..7$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $4a_0(1+4r)(-1+r)\,u^{1+r}+(4a_1(5+4r)\,r-4a_0(1+8r)(-1+r))\,u^{2+r}+(4a_2(9+4r)(1+r)-$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(1+4r)(-1+r)=0$
- Values of r that satisfy the indicial equation

$r \in \left\{1, -\frac{1}{4}\right\}$

- The coefficients of each power of u must be 0

  $[4a_1(5+4r)r - 4a_0(1+8r)(-1+r) = 0, 4a_2(9+4r)(1+r) - 4a_1(9+8r)r + a_0(1+24r)(-1$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = \frac{a_0(8r^2-7r-1)}{r(5+4r)}, a_2 = \frac{a_0(160r^3+36r^2-165r-31)}{4(16r^3+72r^2+101r+45)}, a_3 = \frac{a_0(320r^4+736r^3-144r^2-761r-151)}{2(64r^4+560r^3+1772r^2+2401r+1170)}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(a_{k-5} - 8a_{k-4} + 24a_{k-3} - 32a_{k-2} + 16a_{k-1})k^2 + (2(a_{k-5} - 8a_{k-4} + 24a_{k-3} - 32a_{k-2} + 16a_{k-1})r$

- Shift index using $k- >k+5$

  $(a_k - 8a_{k+1} + 24a_{k+2} - 32a_{k+3} + 16a_{k+4})(k+5)^2 + (2(a_k - 8a_{k+1} + 24a_{k+2} - 32a_{k+3} + 16a_{k+4}$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{k^2a_k - 8k^2a_{k+1} + 24k^2a_{k+2} - 32k^2a_{k+3} + 2kra_k - 16kra_{k+1} + 48kra_{k+2} - 64kra_{k+3} + r^2a_k - 8r^2a_{k+1} + 24r^2a_{k+2} - 32r^2a_{k+}}{4(4k^2+8kr+4r^2+29k+29r+}$

- Recursion relation for $r = 1$

  $a_{k+4} = -\frac{k^2a_k - 8k^2a_{k+1} + 24k^2a_{k+2} - 32k^2a_{k+3} + ka_k - 24ka_{k+1} + 121ka_{k+2} - 228ka_{k+3} - 16a_{k+1} + 146a_{k+2} - 396a_{k+3}}{4(4k^2+37k+84)}$

- Solution for $r = 1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+4} = -\frac{k^2a_k - 8k^2a_{k+1} + 24k^2a_{k+2} - 32k^2a_{k+3} + ka_k - 24ka_{k+1} + 121ka_{k+2} - 228ka_{k+3} - 16a_{k+1} + 14}{4(4k^2+37k+84)}\right.$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^{k+1}, a_{k+4} = -\frac{k^2a_k - 8k^2a_{k+1} + 24k^2a_{k+2} - 32k^2a_{k+3} + ka_k - 24ka_{k+1} + 121ka_{k+2} - 228ka_{k+3} - 16a_{k+1}}{4(4k^2+37k+84)}\right.$

- Recursion relation for $r = -\frac{1}{4}$

  $a_{k+4} = -\frac{k^2a_k - 8k^2a_{k+1} + 24k^2a_{k+2} - 32k^2a_{k+3} - \frac{3}{2}ka_k - 4ka_{k+1} + 61ka_{k+2} - 148ka_{k+3} + \frac{5}{16}a_k + \frac{3}{2}a_{k+1} + \frac{129}{4}a_{k+2} - 161a_{k+3}}{4(4k^2+27k+44)}$

- Solution for $r = -\frac{1}{4}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{4}}, a_{k+4} = -\frac{k^2a_k - 8k^2a_{k+1} + 24k^2a_{k+2} - 32k^2a_{k+3} - \frac{3}{2}ka_k - 4ka_{k+1} + 61ka_{k+2} - 148ka_{k+3} + \frac{5}{16}a_k + \frac{3}{2}a_{k+}}{4(4k^2+27k+44)}\right.$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^{k-\frac{1}{4}}, a_{k+4} = -\frac{k^2a_k - 8k^2a_{k+1} + 24k^2a_{k+2} - 32k^2a_{k+3} - \frac{3}{2}ka_k - 4ka_{k+1} + 61ka_{k+2} - 148ka_{k+3} + \frac{5}{16}a_k + }{4(4k^2+27k+44)}\right.$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(x+1)^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k(x+1)^{k-\frac{1}{4}}\right), a_{k+4} = -\frac{k^2a_k - 8k^2a_{1+k} + 24k^2a_{k+2} - 32k^2a_{k+3} + ka_k - 24ka}{4(4k^2}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve((x^2-1)^2*diff(y(x),x$2)+(x+1)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{6}x^4-\frac{7}{60}x^5\right)y(0) + \left(x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{6}x^4+\frac{7}{60}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 70

```
AsymptoticDSolveValue[(x^2-1)^2*y''[x]+(x+1)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{7x^5}{60}+\frac{x^4}{6}-\frac{x^3}{6}+\frac{x^2}{2}+1\right)+c_2\left(\frac{7x^5}{60}-\frac{x^4}{6}+\frac{x^3}{6}-\frac{x^2}{2}+x\right)$$

1.11 problem Problem 1.8(b)

Internal problem ID [12404]
Internal file name [`OUTPUT/11056_Wednesday_October_04_2023_01_27_57_AM_74543273/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.8(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + 4y' - yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 4y' - yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = -1$$

Table 17: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 4y' - yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 4\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4(n+r) a_n x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 4(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 4r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 4r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(3+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -3$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^3}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-3} \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 4a_n(n+r) - a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{n^2 + 2nr + r^2 + 3n + 3r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-2}}{n(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r^2 + 7r + 10}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+7r+10}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+7r+10}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r + 5)(r + 2)(r + 7)(r + 4)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+7r+10}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{(r+5)(r+2)(r+7)(r+4)}$	$\frac{1}{280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{r^2+7r+10}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{(r+5)(r+2)(r+7)(r+4)}$	$\frac{1}{280}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + \frac{x^2}{10} + \frac{x^4}{280} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} 0 &= \lim_{r \to -3} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-3}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + 4(n+r) b_n - b_{n-2} = 0 \tag{4}
$$

Which for for the root $r = -3$ becomes

$$
b_n(n-3)(n-4) + 4(n-3) b_n - b_{n-2} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = \frac{b_{n-2}}{n^2 + 2nr + r^2 + 3n + 3r} \tag{5}
$$

Which for the root $r = -3$ becomes

$$
b_n = \frac{b_{n-2}}{n^2 - 3n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -3$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{r^2 + 7r + 10}$$

Which for the root $r = -3$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+7r+10}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+7r+10}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 7r + 10)(r^2 + 11r + 28)}$$

Which for the root $r = -3$ becomes

$$b_4 = -\frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+7r+10}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(r+5)(r+2)(r+7)(r+4)}$	$-\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{r^2+7r+10}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(r+5)(r+2)(r+7)(r+4)}$	$-\frac{1}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots\right) \\ &= \frac{1 - \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)}{x^3} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 + \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\right)}{x^3} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{x^2}{10}+\frac{x^4}{280}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}-\frac{x^4}{8}+O(x^6)\right)}{x^3}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1\left(1+\frac{x^2}{10}+\frac{x^4}{280}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}-\frac{x^4}{8}+O(x^6)\right)}{x^3} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1\left(1+\frac{x^2}{10}+\frac{x^4}{280}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}-\frac{x^4}{8}+O(x^6)\right)}{x^3}
$$

Verified OK.

1.11.1 Maple step by step solution

Let's solve

$y''x + 4y' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y'}{x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = -1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + 4y' - yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k- >k - 1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k + r + 1) x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + r + 1) (k + r) x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(3 + r) x^{-1+r} + a_1(1 + r) (4 + r) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k + r + 1) (k + 4 + r) - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$

- Each term must be 0

  $a_1(1+r)(4+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+4+r) - a_{k-1} = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)(k+5+r) - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+2+r)(k+5+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = \frac{a_k}{(k-1)(k+2)}$

- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = \frac{a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{(k+2)(k+5)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{-3+k}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{a_k}{(k-1)(k+2)}, -2a_1 = 0, b_{k+2} = \frac{b_k}{(k+2)(k+5)}, 4b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)+4*diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 + \frac{1}{10}x^2 + \frac{1}{280}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(12 - 6x^2 - \frac{3}{2}x^4 + \mathrm{O}\left(x^6\right)\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 42

```
AsymptoticDSolveValue[x*y''[x]+4*y'[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{x^3} - \frac{x}{8} - \frac{1}{2x}\right) + c_2\left(\frac{x^4}{280} + \frac{x^2}{10} + 1\right)$$

1.12 problem Problem 1.9

Internal problem ID [12405]
Internal file name [`OUTPUT/11057_Wednesday_October_04_2023_01_27_58_AM_74288418/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + (x+1)\,y' - yk = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + (x+1)\,y' - yk = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x+1}{2x}$$
$$q(x) = -\frac{k}{2x}$$

Table 19: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{k}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + (x+1)\, y' - yk = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + (x+1)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) k = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r}a_n(n+r)\right)$$
$$+\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-k a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r}a_n(n+r)=\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\,x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-k a_n x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-k a_{n-1}x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\,x^{n+r-1}\right)$$
$$+\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-k a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1}a_n(n+r)(n+r-1)+(n+r)a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$2x^{-1+r}a_0 r(-1+r)+r a_0 x^{-1+r}=0$$

Or

$$\left(2x^{-1+r}r(-1+r)+r\,x^{-1+r}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$r\,x^{-1+r}(-1+2r)=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_n(n+r) - ka_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(k-n-r+1)}{2n^2+4nr+2r^2-n-r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}(2k-2n+1)}{4n^2+2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{k-r}{2r^2+3r+1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{k}{3} - \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{k-r}{2r^2+3r+1}$	$\frac{k}{3} - \frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(k-1-r)(k-r)}{4r^4+20r^3+35r^2+25r+6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{30}k^2 - \frac{1}{15}k + \frac{1}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{k-r}{2r^2+3r+1}$	$\frac{k}{3} - \frac{1}{6}$
a_2	$\frac{(k-1-r)(k-r)}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}k^2 - \frac{1}{15}k + \frac{1}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(k-2-r)(k-1-r)(k-r)}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{(2k-5)(-3+2k)(2k-1)}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{k-r}{2r^2+3r+1}$	$\frac{k}{3} - \frac{1}{6}$
a_2	$\frac{(k-1-r)(k-r)}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}k^2 - \frac{1}{15}k + \frac{1}{40}$
a_3	$\frac{(k-2-r)(k-1-r)(k-r)}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{(2k-5)(-3+2k)(2k-1)}{5040}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(k-3-r)(k-2-r)(k-1-r)(k-r)}{(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(r+4)(2r+7)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{(2k-7)(2k-5)(-3+2k)(2k-1)}{362880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{k-r}{2r^2+3r+1}$	$\frac{k}{3} - \frac{1}{6}$
a_2	$\frac{(k-1-r)(k-r)}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}k^2 - \frac{1}{15}k + \frac{1}{40}$
a_3	$\frac{(k-2-r)(k-1-r)(k-r)}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{(2k-5)(-3+2k)(2k-1)}{5040}$
a_4	$\frac{(k-3-r)(k-2-r)(k-1-r)(k-r)}{(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(r+4)(2r+7)}$	$\frac{(2k-7)(2k-5)(-3+2k)(2k-1)}{362880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(k-4-r)(k-3-r)(k-2-r)(k-1-r)(k-r)}{(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(r+4)(2r+7)(2r+9)(r+5)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{(2k-9)(2k-7)(2k-5)(-3+2k)(2k-1)}{39916800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{k-r}{2r^2+3r+1}$	$\frac{k}{3}-\frac{1}{6}$
a_2	$\frac{(k-1-r)(k-r)}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}k^2-\frac{1}{15}k+\frac{1}{40}$
a_3	$\frac{(k-2-r)(k-1-r)(k-r)}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{(2k-5)(-3+2k)(2k-1)}{5040}$
a_4	$\frac{(k-3-r)(k-2-r)(k-1-r)(k-r)}{(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(r+4)(2r+7)}$	$\frac{(2k-7)(2k-5)(-3+2k)(2k-1)}{362880}$
a_5	$\frac{(k-4-r)(k-3-r)(k-2-r)(k-1-r)(k-r)}{(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(r+4)(2r+7)(2r+9)(r+5)}$	$\frac{(2k-9)(2k-7)(2k-5)(-3+2k)(2k-1)}{39916800}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \left(\frac{k}{3} - \frac{1}{6}\right) x + \left(\frac{1}{30}k^2 - \frac{1}{15}k + \frac{1}{40}\right) x^2 + \frac{(2k-5)(-3+2k)(2k-1)x^3}{5040} + \frac{(2k-7)(2k
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) + (n+r)b_n - kb_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}(k-n-r+1)}{2n^2+4nr+2r^2-n-r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{b_{n-1}(k-n+1)}{2n^2-n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{k - r}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = k$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{k-r}{2r^2+3r+1}$	k

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(k - 1 - r)(k - r)}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{(k - 1) k}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{k-r}{2r^2+3r+1}$	k
b_2	$\frac{(k-1-r)(k-r)}{4r^4+20r^3+35r^2+25r+6}$	$\frac{(k-1)k}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(k - 2 - r)(k - 1 - r)(k - r)}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{(k - 2)(k - 1) k}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{k-r}{2r^2+3r+1}$	k
b_2	$\frac{(k-1-r)(k-r)}{4r^4+20r^3+35r^2+25r+6}$	$\frac{(k-1)k}{6}$
b_3	$\frac{(k-2-r)(k-1-r)(k-r)}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{(k-2)(k-1)k}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(k-3-r)(k-2-r)(k-1-r)(k-r)}{(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(r+4)(2r+7)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{(-3+k)(k-2)(k-1)k}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{k-r}{2r^2+3r+1}$	k
b_2	$\frac{(k-1-r)(k-r)}{4r^4+20r^3+35r^2+25r+6}$	$\frac{(k-1)k}{6}$
b_3	$\frac{(k-2-r)(k-1-r)(k-r)}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{(k-2)(k-1)k}{90}$
b_4	$\frac{(k-3-r)(k-2-r)(k-1-r)(k-r)}{(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(r+4)(2r+7)}$	$\frac{(-3+k)(k-2)(k-1)k}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(k-4-r)(k-3-r)(k-2-r)(k-1-r)(k-r)}{(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(r+4)(2r+7)(2r+9)(r+5)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{(k-4)(-3+k)(k-2)(k-1)k}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{k-r}{2r^2+3r+1}$	k
b_2	$\frac{(k-1-r)(k-r)}{4r^4+20r^3+35r^2+25r+6}$	$\frac{(k-1)k}{6}$
b_3	$\frac{(k-2-r)(k-1-r)(k-r)}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{(k-2)(k-1)k}{90}$
b_4	$\frac{(k-3-r)(k-2-r)(k-1-r)(k-r)}{(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(r+4)(2r+7)}$	$\frac{(-3+k)(k-2)(k-1)k}{2520}$
b_5	$\frac{(k-4-r)(k-3-r)(k-2-r)(k-1-r)(k-r)}{(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(r+4)(2r+7)(2r+9)(r+5)}$	$\frac{(k-4)(-3+k)(k-2)(k-1)k}{113400}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\
&= 1 + kx + \frac{(k-1)\,k\,x^2}{6} + \frac{(k-2)\,(k-1)\,k\,x^3}{90} + \frac{(-3+k)\,(k-2)\,(k-1)\,k\,x^4}{2520} + \frac{(k-4)\,(-3+k)\,(}{113}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x}\left(1+\left(\frac{k}{3}-\frac{1}{6}\right)x+\left(\frac{1}{30}k^2-\frac{1}{15}k+\frac{1}{40}\right)x^2+\frac{(2k-5)\,(-3+2k)\,(2k-1)\,x^3}{5040} \right.\\
&\quad + \frac{(2k-7)\,(2k-5)\,(-3+2k)\,(2k-1)\,x^4}{362880} \\
&\quad \left. + \frac{(2k-9)\,(2k-7)\,(2k-5)\,(-3+2k)\,(2k-1)\,x^5}{39916800} + O\left(x^6\right)\right) \\
&\quad + c_2\left(1+kx+\frac{(k-1)\,k\,x^2}{6}+\frac{(k-2)\,(k-1)\,k\,x^3}{90}+\frac{(-3+k)\,(k-2)\,(k-1)\,k\,x^4}{2520} \right.\\
&\quad \left. + \frac{(k-4)\,(-3+k)\,(k-2)\,(k-1)\,k\,x^5}{113400} + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$
= c_1\sqrt{x}\left(1+\left(\frac{k}{3}-\frac{1}{6}\right)x+\left(\frac{1}{30}k^2-\frac{1}{15}k+\frac{1}{40}\right)x^2+\frac{(2k-5)(-3+2k)(2k-1)x^3}{5040} + \frac{(2k-7)(2k-5)(-3+2k)(2k-1)x^4}{362880} + \frac{(2k-9)(2k-7)(2k-5)(-3+2k)(2k-1)x^5}{39916800}+O\left(x^6\right)\right) + c_2\left(1+kx+\frac{(k-1)k\,x^2}{6}+\frac{(k-2)(k-1)k\,x^3}{90}+\frac{(-3+k)(k-2)(k-1)k\,x^4}{2520} + \frac{(k-4)(-3+k)(k-2)(k-1)k\,x^5}{113400}+O\left(x^6\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x}\left(1+\left(\frac{k}{3}-\frac{1}{6}\right)x+\left(\frac{1}{30}k^2-\frac{1}{15}k+\frac{1}{40}\right)x^2+\frac{(2k-5)(-3+2k)(2k-1)x^3}{5040} + \frac{(2k-7)(2k-5)(-3+2k)(2k-1)x^4}{362880} + \frac{(2k-9)(2k-7)(2k-5)(-3+2k)(2k-1)x^5}{39916800}+O\left(x^6\right)\right) + c_2\left(1+kx+\frac{(k-1)k\,x^2}{6}+\frac{(k-2)(k-1)k\,x^3}{90}+\frac{(-3+k)(k-2)(k-1)k\,x^4}{2520} + \frac{(k-4)(-3+k)(k-2)(k-1)k\,x^5}{113400}+O\left(x^6\right)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\left(1+\left(\frac{k}{3}-\frac{1}{6}\right)x+\left(\frac{1}{30}k^2-\frac{1}{15}k+\frac{1}{40}\right)x^2+\frac{(2k-5)(-3+2k)(2k-1)x^3}{5040} + \frac{(2k-7)(2k-5)(-3+2k)(2k-1)x^4}{362880} + \frac{(2k-9)(2k-7)(2k-5)(-3+2k)(2k-1)x^5}{39916800}+O\left(x^6\right)\right) + c_2\left(1+kx+\frac{(k-1)k\,x^2}{6}+\frac{(k-2)(k-1)k\,x^3}{90}+\frac{(-3+k)(k-2)(k-1)k\,x^4}{2520} + \frac{(k-4)(-3+k)(k-2)(k-1)k\,x^5}{113400}+O\left(x^6\right)\right)
$$

Verified OK.

1.12.1 Maple step by step solution

Let's solve

$2y''x + (x+1)\,y' - yk = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+1)y'}{2x} + \frac{yk}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+1)y'}{2x} - \frac{yk}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x+1}{2x}, P_3(x) = -\frac{k}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + (x+1)\,y' - yk = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) - a_k(k-k-r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} - a_k(k-k-r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k-k-r)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r=0$

  $$a_{k+1} = \frac{a_k(k-k)}{(2k+1)(k+1)}$$

- Solution for $r=0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(k-k)}{(2k+1)(k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+1} = \frac{a_k\left(k-k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r=\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k-k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{m=0}^{\infty} a_m x^m\right)+\left(\sum_{m=0}^{\infty} b_m x^{m+\frac{1}{2}}\right), a_{m+1}=\frac{a_m(-m+k)}{(2m+1)(m+1)}, b_{m+1}=\frac{b_m\left(k-m-\frac{1}{2}\right)}{(2m+2)\left(m+\frac{3}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 132

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+(1+x)*diff(y(x),x)-k*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\, c_1 \left(1 + \left(\frac{k}{3} - \frac{1}{6}\right) x + \left(\frac{1}{30}k^2 - \frac{1}{15}k + \frac{1}{40}\right) x^2 + \frac{1}{5040}(2k-5)(2k-3)(-1+2k)\, x^3 + \frac{1}{362880}(2k-7)(2k-5)(2k-3)(-1+2k)\, x^4 + \frac{1}{39916800}(2k-9)(2k-7)(2k-5)(2k-3)(-1+2k)\, x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(1 + kx + \frac{1}{6}(-1+k)\, kx^2 + \frac{1}{90}(-2+k)(-1+k)\, kx^3 + \frac{1}{2520}(k-3)(-2+k)(-1+k)\, kx^4 + \frac{1}{113400}(-4+k)(k-3)(-2+k)(-1+k)\, kx^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 304

```
AsymptoticDSolveValue[2*x*y''[x]+(1+x)*y'[x]-k*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{4\left(\frac{3}{4}-\frac{k}{2}\right)\left(\frac{5}{4}-\frac{k}{2}\right)\left(\frac{7}{4}-\frac{k}{2}\right)\left(\frac{9}{4}-\frac{k}{2}\right)\left(\frac{k}{2}-\frac{1}{4}\right)x^5}{155925} - \frac{2\left(\frac{3}{4}-\frac{k}{2}\right)\left(\frac{5}{4}-\frac{k}{2}\right)\left(\frac{7}{4}-\frac{k}{2}\right)\left(\frac{k}{2}-\frac{1}{4}\right)x^4}{2835} + \frac{4}{315}\left(\frac{3}{4}-\frac{k}{2}\right)\left(\frac{5}{4}-\frac{k}{2}\right)\left(\frac{k}{2}-\frac{1}{4}\right)x^3 - \frac{2}{15}\left(\frac{3}{4}-\frac{k}{2}\right)\left(\frac{k}{2}-\frac{1}{4}\right)x^2 + \frac{2}{3}\left(\frac{k}{2}-\frac{1}{4}\right)x + 1\right) + c_2\left(\frac{2\left(\frac{1}{2}-\frac{k}{2}\right)\left(1-\frac{k}{2}\right)\left(\frac{3}{2}-\frac{k}{2}\right)\left(2-\frac{k}{2}\right)kx^5}{14175} - \frac{1}{315}\left(\frac{1}{2}-\frac{k}{2}\right)\left(1-\frac{k}{2}\right)\left(\frac{3}{2}-\frac{k}{2}\right)kx^4 + \frac{2}{45}\left(\frac{1}{2}-\frac{k}{2}\right)\left(1-\frac{k}{2}\right)kx^3 - \frac{1}{3}\left(\frac{1}{2}-\frac{k}{2}\right)kx^2 + kx + 1\right)$$

1.13 problem Problem 1.11(a)

Internal problem ID [12406]
Internal file name [`OUTPUT/11058_Wednesday_October_04_2023_07_06_03_PM_64152595/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.11(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + x^2y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^3y'' + x^2y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{1}{x^3}$$

Table 21: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x^3}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^3*diff(y(x),x$2)+x^2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 222

```
AsymptoticDSolveValue[x^3*y''[x]+x^2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 e^{-\frac{2i}{\sqrt{x}}} \sqrt[4]{x} \left(\frac{418854310875 i x^{9/2}}{8796093022208} - \frac{57972915 i x^{7/2}}{4294967296} + \frac{59535 i x^{5/2}}{8388608} - \frac{75 i x^{3/2}}{8192} - \frac{30241281245175 x^5}{281474976710656} + \frac{13043905875 x^4}{549755813888} - \frac{2401245 x^3}{268435456} + \frac{3675 x^2}{524288} - \frac{9x}{512} + \frac{i\sqrt{x}}{16} + 1 \right) + c_2 e^{\frac{2i}{\sqrt{x}}} \sqrt[4]{x} \left(-\frac{418854310875 i x^{9/2}}{8796093022208} + \frac{57972915 i x^{7/2}}{4294967296} - \frac{59535 i x^{5/2}}{8388608} + \frac{75 i x^{3/2}}{8192} - \frac{30241281245175 x^5}{281474976710656} + \right.$$

1.14 problem Problem 1.11(b)

Internal problem ID [12407]
Internal file name [`OUTPUT/11059_Wednesday_October_04_2023_07_06_05_PM_68847962/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.11(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{y''x^2 + y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y''x^2 + y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x^2}$$
$$q(x) = -\frac{2}{x^2}$$

Table 22: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"irregular"

$q(x) = -\frac{2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 28

```
AsymptoticDSolveValue[x^2*y''[x]+y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 e^{\frac{1}{x}} x^2 + c_1 \left(2x^2 + 2x + 1\right)$$

1.15 problem Problem 1.12

Internal problem ID [12408]
Internal file name [`OUTPUT/11060_Wednesday_October_04_2023_07_06_06_PM_13727061/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2y''x^2 + x(1-x)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2y''x^2 + \left(-x^2 + x\right)y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{2x}$$
$$q(x) = -\frac{1}{2x^2}$$

Table 23: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2y''x^2 + \left(-x^2 + x\right) y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x^2 + \left(-x^2 + x\right) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= -\frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) - a_{n-1}(n + r - 1) + a_n(n + r) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n + 2r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}}{2n + 3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{3+2r}$$

Which for the root $r = 1$ becomes

$$a_1 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^2+16r+15}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8r^3+60r^2+142r+105}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$
a_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{315}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^4 + 192r^3 + 824r^2 + 1488r + 945}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{3465}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$
a_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{315}$
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{3465}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32r^5 + 560r^4 + 3760r^3 + 12040r^2 + 18258r + 10395}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1}{45045}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{3+2r}$	$\frac{1}{5}$
a_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{35}$
a_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{315}$
a_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{3465}$
a_5	$\frac{1}{32r^5+560r^4+3760r^3+12040r^2+18258r+10395}$	$\frac{1}{45045}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\ldots\right)\\ &= x\left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + b_n(n+r) - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{2n+2r+1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = \frac{b_{n-1}}{2n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{3+2r}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{8r^3 + 60r^2 + 142r + 105}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = \frac{1}{48}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^4 + 192r^3 + 824r^2 + 1488r + 945}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{48}$
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{32r^5 + 560r^4 + 3760r^3 + 12040r^2 + 18258r + 10395}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = \frac{1}{3840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{3+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4r^2+16r+15}$	$\frac{1}{8}$
b_3	$\frac{1}{8r^3+60r^2+142r+105}$	$\frac{1}{48}$
b_4	$\frac{1}{16r^4+192r^3+824r^2+1488r+945}$	$\frac{1}{384}$
b_5	$\frac{1}{32r^5+560r^4+3760r^3+12040r^2+18258r+10395}$	$\frac{1}{3840}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)}{\sqrt{x}}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right) \\ &\quad + \frac{c_2\left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1 x\left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right) \\ &\quad + \frac{c_2\left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}y &= c_1 x\left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right) \\ &\quad + \frac{c_2\left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}y &= c_1 x\left(1 + \frac{x}{5} + \frac{x^2}{35} + \frac{x^3}{315} + \frac{x^4}{3465} + \frac{x^5}{45045} + O(x^6)\right) \\ &\quad + \frac{c_2\left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)}{\sqrt{x}}\end{aligned}$$

Verified OK.

1.15.1 Maple step by step solution

Let's solve

$2y''x^2 + (-x^2 + x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{2x^2} + \frac{(x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{2x} - \frac{y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x-1}{2x}, P_3(x) = -\frac{1}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{2}$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2y''x^2 - x(x-1)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{1, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2(k+r-1)\left(\left(k+r+\frac{1}{2}\right) a_k - \frac{a_{k-1}}{2}\right) = 0$$

- Shift index using $k- >k+1$

  $$2(k+r)\left(\left(k+\frac{3}{2}+r\right) a_{k+1} - \frac{a_k}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k}{2k+3+2r}$$

- Recursion relation for $r = 1$

  $$a_{k+1} = \frac{a_k}{2k+5}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{2k+5}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+1} = \frac{a_k}{2k+2}$$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{1+k} = \frac{a_k}{2k+5}, b_{1+k} = \frac{b_k}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+x*(1-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1\left(1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\frac{1}{3840}x^5+O\left(x^6\right)\right)}{\sqrt{x}} + c_2 x\left(1+\frac{1}{5}x+\frac{1}{35}x^2+\frac{1}{315}x^3+\frac{1}{3465}x^4+\frac{1}{45045}x^5+O\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 86

```
AsymptoticDSolveValue[2*x^2*y''[x]+x*(1-x)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(\frac{x^5}{45045} + \frac{x^4}{3465} + \frac{x^3}{315} + \frac{x^2}{35} + \frac{x}{5} + 1\right) + \frac{c_2 \left(\frac{x^5}{3840} + \frac{x^4}{384} + \frac{x^3}{48} + \frac{x^2}{8} + \frac{x}{2} + 1\right)}{\sqrt{x}}$$

1.16 problem Problem 1.13

Internal problem ID [12409]
Internal file name [`OUTPUT/11061_Wednesday_October_04_2023_07_06_07_PM_97534733/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 1 VARIABLE COEFFICIENT, SECOND ORDER DIFFERENTIAL EQUATIONS. Problems page 28
Problem number: Problem 1.13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y''x(x-1)+3y'x+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2-x\right)y''+3y'x+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x-1}$$
$$q(x)=\frac{1}{x\,(x-1)}$$

Table 25: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x-1}$	
singularity	type
$x = 1$	"regular"

$q(x) = \frac{1}{x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x(x-1) + 3y'x + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) + 3\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$-x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$-x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$-x^{-1+r}r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$-r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$-x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) - a_n(n+r)(n+r-1) + 3a_{n-1}(n+r-1) + a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r)\, a_{n-1}}{n+r-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(n+1)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1+r}{r}$$

Which for the root $r = 1$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2+r}{r}$$

Which for the root $r = 1$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{3+r}{r}$$

Which for the root $r = 1$ becomes

$$a_3 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4+r}{r}$$

Which for the root $r = 1$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4
a_4	$\frac{4+r}{r}$	5

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{5+r}{r}$$

Which for the root $r = 1$ becomes

$$a_5 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4
a_4	$\frac{4+r}{r}$	5
a_5	$\frac{5+r}{r}$	6

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{1+r}{r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{1+r}{r} &= \lim_{r \to 0} \frac{1+r}{r} \\ &= \mathit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $y''x(x-1) + 3y'x + y = 0$ gives

$$\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x(x-1) \\
&+ 3\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) x \\
&+ Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(y_1''(x)\, x(x-1) + 3y_1'(x)\, x + y_1(x)\right) \ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x(x-1) + 3y_1(x)\right) C + \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x(x-1) + 3\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\, x(x-1) + 3y_1'(x)\, x + y_1(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x(x-1) + 3y_1(x)\right) C + \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x(x-1) + 3\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2x(x-1)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) + (2x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2(x-1) + 3\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2x(x-1)\left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) + (2x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x^2(x-1) + 3\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n}\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1)\right) + \sum_{n=0}^{\infty} \left(-2C\, x^n a_n (n+1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n\right) + \left(\sum_{n=0}^{\infty} C a_n x^n\right) + \left(\sum_{n=0}^{\infty} x^n b_n n(n-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} 3x^n b_n n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1) &= \sum_{n=2}^{\infty} 2C a_{-2+n} (n-1)\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-2C\, x^n a_n (n+1)\right) &= \sum_{n=1}^{\infty} \left(-2C a_{n-1} n\, x^{n-1}\right) \\
\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n &= \sum_{n=2}^{\infty} 2C a_{-2+n} x^{n-1} \\
\sum_{n=0}^{\infty} C a_n x^n &= \sum_{n=1}^{\infty} C a_{n-1} x^{n-1} \\
\sum_{n=0}^{\infty} x^n b_n n(n-1) &= \sum_{n=1}^{\infty} (n-1)\, b_{n-1} (-2+n)\, x^{n-1} \\
\sum_{n=0}^{\infty} 3x^n b_n n &= \sum_{n=1}^{\infty} 3(n-1)\, b_{n-1} x^{n-1} \\
\sum_{n=0}^{\infty} b_n x^n &= \sum_{n=1}^{\infty} b_{n-1} x^{n-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=2}^{\infty} 2Ca_{-2+n}(n-1)\,x^{n-1}\right)+\sum_{n=1}^{\infty}\left(-2Ca_{n-1}n\,x^{n-1}\right)$$
$$+\left(\sum_{n=2}^{\infty} 2Ca_{-2+n}x^{n-1}\right)+\left(\sum_{n=1}^{\infty} Ca_{n-1}x^{n-1}\right) \tag{2B}$$
$$+\left(\sum_{n=1}^{\infty}(n-1)\,b_{n-1}(-2+n)\,x^{n-1}\right)+\sum_{n=0}^{\infty}\left(-n\,x^{n-1}b_n(n-1)\right)$$
$$+\left(\sum_{n=1}^{\infty} 3(n-1)\,b_{n-1}x^{n-1}\right)+\left(\sum_{n=1}^{\infty} b_{n-1}x^{n-1}\right)=0$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$-C+1=0$$

Which is solved for C. Solving for C gives

$$C=1$$

For $n=2$, Eq (2B) gives

$$(4a_0-3a_1)\,C+4b_1-2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2-2b_2=0$$

Solving the above for b_2 gives

$$b_2=-1$$

For $n=3$, Eq (2B) gives

$$(6a_1-5a_2)\,C+9b_2-6b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-12-6b_3=0$$

Solving the above for b_3 gives

$$b_3=-2$$

For $n = 4$, Eq (2B) gives

$$(8a_2 - 7a_3)\, C + 16b_3 - 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-36 - 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -3$$

For $n = 5$, Eq (2B) gives

$$(10a_3 - 9a_4)\, C + 25b_4 - 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-80 - 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -4$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = {} & 1\left(x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\right)\right) \ln(x) \\ & + 1 - x^2 - 2x^3 - 3x^4 - 4x^5 + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\right) \\ &\quad + c_2\left(1\left(x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\right)\right) \ln(x) + 1 - x^2 - 2x^3 \right. \\ &\qquad \left. - 3x^4 - 4x^5 + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\right) \\ &\quad + c_2\left(x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\right) \ln(x) + 1 - x^2 - 2x^3 - 3x^4 - 4x^5 \right. \\ &\qquad \left. + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O\left(x^6\right)\right) + c_2\left(x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O\left(x^6\right)\right)\ln(x)+1-x^2-2x^3-3x^4-4x^5+O\left(x^6\right)\right) \quad (1)$$

Verification of solutions

$$y = c_1 x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O\left(x^6\right)\right) + c_2\left(x\left(1+2x+3x^2+4x^3+5x^4+6x^5+O\left(x^6\right)\right)\ln(x)+1-x^2-2x^3-3x^4-4x^5+O\left(x^6\right)\right)$$

Verified OK.

1.16.1 Maple step by step solution

Let's solve

$y''x(x-1)+3y'x+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x(x-1)}-\frac{3y'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x-1}+\frac{y}{x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3}{x-1}, P_3(x)=\frac{1}{x(x-1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  - $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x(x-1) + 3y'x + y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x \cdot y'$ to series expansion

$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- ○ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- ○ Shift index using $k- > k+2-m$

$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$-a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+r+1)\,(k+r) + a_k(k+r+1)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r+1)\,(-a_{k+1}(k+r) + a_k(k+r+1)) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k(k+r+1)}{k+r}$

- Recursion relation for $r = 0$

$a_{k+1} = \frac{a_k(k+1)}{k}$

- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{a_k(k+1)}{k}\right]$$

- Recursion relation for $r=1$

$$a_{k+1}=\frac{a_k(k+2)}{k+1}$$

- Solution for $r=1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=\frac{a_k(k+2)}{k+1}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{1+k}=\frac{a_k(1+k)}{k}, b_{1+k}=\frac{b_k(k+2)}{1+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 60

```
Order:=6;
dsolve(x*(x-1)*diff(y(x),x$2)+3*x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$\begin{aligned}y(x)=\ & c_1 x\left(1+2x+3x^2+4x^3+5x^4+6x^5+\mathrm{O}\left(x^6\right)\right)\\ &+\left(x+2x^2+3x^3+4x^4+5x^5+\mathrm{O}\left(x^6\right)\right)\ln(x)\, c_2\\ &+\left(1+3x+5x^2+7x^3+9x^4+11x^5+\mathrm{O}\left(x^6\right)\right) c_2\end{aligned}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 63

```
AsymptoticDSolveValue[x*(x-1)*y''[x]+3*x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(x^4 + x^3 + x^2 + \left(4x^3 + 3x^2 + 2x + 1\right)x\log(x) + x + 1\right) + c_2\left(5x^5 + 4x^4 + 3x^3 + 2x^2 + x\right)$$

2 Chapter 3 Bessel functions. Problems page 89

2.1 problem Problem 3.7(a)

Internal problem ID [12410]
Internal file name [`OUTPUT/11062_Wednesday_October_04_2023_07_06_08_PM_10656506/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 3 Bessel functions. Problems page 89
Problem number: Problem 3.7(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{y'' - x^2 y = 0}$$

2.1.1 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 - yx^4 = 0 \tag{1}$$

Bessel ode has the form

$$y''x^2 + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = \frac{i}{2}$$
$$n = \frac{1}{4}$$
$$\gamma = 2$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \sqrt{x}\, \text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2 \sqrt{x}\, \text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\, \text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2 \sqrt{x}\, \text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x}\, \text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2 \sqrt{x}\, \text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)$$

Verified OK.

2.1.2 Maple step by step solution

Let's solve

$y'' - x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y$ to series expansion

  $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

- Shift index using $k->k-2$

  $x^2\cdot y=\sum_{k=2}^{\infty} a_{k-2}x^k$
- Convert y'' to series expansion

  $y''=\sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
- Shift index using $k->k+2$

  $y''=\sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $6a_3x+2a_2+\left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1)-a_{k-2}\right) x^k\right)=0$

- The coefficients of each power of x must be 0

  $[2a_2=0, 6a_3=0]$
- Solve for the dependent coefficient(s)

  $\{a_2=0, a_3=0\}$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2}-a_{k-2}=0$
- Shift index using $k->k+2$

  $\left((k+2)^2+3k+8\right) a_{k+4}-a_k=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=\frac{a_k}{k^2+7k+12}, a_2=0, a_3=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\text{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_2 + \text{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1\right) \sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 37

```
DSolve[y''[x]-x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_2 \text{ParabolicCylinderD}\left(-\frac{1}{2}, i\sqrt{2}x\right) + c_1 \text{ParabolicCylinderD}\left(-\frac{1}{2}, \sqrt{2}x\right)$$

2.2 problem Problem 3.7(b)

Internal problem ID [12411]
Internal file name [OUTPUT/11063_Wednesday_October_04_2023_07_06_10_PM_71685414/index.tex]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 3 Bessel functions. Problems page 89
Problem number: Problem 3.7(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + y = 0}$$

2.2.1 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 + y'x + yx = 0 \tag{1}$$

Bessel ode has the form

$$y''x^2 + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = 2$$
$$n = 0$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right)$$

Verified OK.

2.2.2 Maple step by step solution

Let's solve

$y''x + y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{(k+1)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 31

```
DSolve[x*y''[x]+y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(0, 2\sqrt{x}\right) + 2c_2 \operatorname{BesselY}\left(0, 2\sqrt{x}\right)$$

2.3 problem Problem 3.7(c)

Internal problem ID [12412]
Internal file name [`OUTPUT/11064_Wednesday_October_04_2023_07_06_10_PM_70418792/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 3 Bessel functions. Problems page 89
Problem number: Problem 3.7(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (x+1)^2 y = 0}$$

2.3.1 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 + (x^3 + 2x^2 + x)\, y = 0 \tag{1}$$

Bessel ode has the form

$$y''x^2 + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1-2\alpha)\, xy' + (\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = 2$$
$$n = -1$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1\sqrt{x}\, \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\, \text{BesselY}\left(1, 2\sqrt{x}\right)$$

Summary
The solution(s) found are the following

$$y = -c_1\sqrt{x}\, \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\, \text{BesselY}\left(1, 2\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = -c_1\sqrt{x}\, \text{BesselJ}\left(1, 2\sqrt{x}\right) - c_2\sqrt{x}\, \text{BesselY}\left(1, 2\sqrt{x}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x$2)+(x+1)^2*y(x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+(x+1)^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

2.4 problem Problem 3.7(d)

Internal problem ID [12413]
Internal file name [`OUTPUT/11065_Wednesday_October_04_2023_07_06_10_PM_50793839/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 3 Bessel functions. Problems page 89
Problem number: Problem 3.7(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + \alpha^2 y = 0}$$

2.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = \alpha^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \alpha^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\alpha^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \alpha^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\alpha^2)} \\ &= \pm\sqrt{-\alpha^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-\alpha^2} \\ \lambda_2 &= -\sqrt{-\alpha^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-\alpha^2} \\ \lambda_2 &= -\sqrt{-\alpha^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\sqrt{-\alpha^2}\right)x} + c_2 e^{\left(-\sqrt{-\alpha^2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{-\alpha^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-\alpha^2}\,x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\sqrt{-\alpha^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-\alpha^2}\,x} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{\sqrt{-\alpha^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-\alpha^2}\,x}$$

Verified OK.

2.4.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + \alpha^2 y'y = 0$$

Integrating the above w.r.t x gives

$$\int \left(y'y'' + \alpha^2 y'y\right) dx = 0$$
$$\frac{y'^2}{2} + \frac{\alpha^2 y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-\alpha^2 y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-\alpha^2 y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{1}{\sqrt{-\alpha^2 y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{\sqrt{\alpha^2}\, y}{\sqrt{-\alpha^2 y^2 + 2c_1}}\right)}{\sqrt{\alpha^2}} = x + c_2$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-\alpha^2 y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{\sqrt{\alpha^2}\, y}{\sqrt{-\alpha^2 y^2 + 2c_1}}\right)}{\sqrt{\alpha^2}} = c_3 + x$$

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{\sqrt{\alpha^2}\, y}{\sqrt{-\alpha^2 y^2+2c_1}}\right)}{\sqrt{\alpha^2}} = x + c_2 \tag{1}$$

$$-\frac{\arctan\left(\frac{\sqrt{\alpha^2}\, y}{\sqrt{-\alpha^2 y^2+2c_1}}\right)}{\sqrt{\alpha^2}} = c_3 + x \tag{2}$$

Verification of solutions

$$\frac{\arctan\left(\frac{\sqrt{\alpha^2}\, y}{\sqrt{-\alpha^2 y^2+2c_1}}\right)}{\sqrt{\alpha^2}} = x + c_2$$

Verified OK.

$$-\frac{\arctan\left(\frac{\sqrt{\alpha^2}\, y}{\sqrt{-\alpha^2 y^2+2c_1}}\right)}{\sqrt{\alpha^2}} = c_3 + x$$

Verified OK.

2.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \alpha^2 y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \alpha^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\alpha^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\alpha^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (-\alpha^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 29: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\alpha^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{\sqrt{-\alpha^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\sqrt{-\alpha^2}\, x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{\sqrt{-\alpha^2}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= e^{\sqrt{-\alpha^2}\, x} \int \frac{1}{e^{2\sqrt{-\alpha^2}\, x}}\, dx \\ &= e^{\sqrt{-\alpha^2}\, x} \left(\frac{\sqrt{-\alpha^2}\, e^{-2\sqrt{-\alpha^2}\, x}}{2\alpha^2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\mathrm{e}^{\sqrt{-\alpha^2}\,x}\right) + c_2\left(\mathrm{e}^{\sqrt{-\alpha^2}\,x}\left(\frac{\sqrt{-\alpha^2}\,\mathrm{e}^{-2\sqrt{-\alpha^2}\,x}}{2\alpha^2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\mathrm{e}^{\sqrt{-\alpha^2}\,x} + \frac{c_2\sqrt{-\alpha^2}\,\mathrm{e}^{-\sqrt{-\alpha^2}\,x}}{2\alpha^2} \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{\sqrt{-\alpha^2}\,x} + \frac{c_2\sqrt{-\alpha^2}\,\mathrm{e}^{-\sqrt{-\alpha^2}\,x}}{2\alpha^2}
$$

Verified OK.

2.4.4 Maple step by step solution

Let's solve

$y'' + \alpha^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $\alpha^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm\left(\sqrt{-4\alpha^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-\alpha^2}, -\sqrt{-\alpha^2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\sqrt{-\alpha^2}\,x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\sqrt{-\alpha^2}\,x}$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\sqrt{-\alpha^2}\, x} + c_2 e^{-\sqrt{-\alpha^2}\, x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+alpha^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 20

```
DSolve[y''[x]+a^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(ax) + c_2 \sin(ax)$$

2.5 problem Problem 3.7(e)

Internal problem ID [12414]
Internal file name [`OUTPUT/11066_Wednesday_October_04_2023_07_06_12_PM_60725539/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 3 Bessel functions. Problems page 89
Problem number: Problem 3.7(e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - \alpha^2 y = 0}$$

2.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -\alpha^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \alpha^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$-\alpha^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -\alpha^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-\alpha^2)} \\ &= \pm\sqrt{\alpha^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{\alpha^2} \\ \lambda_2 &= -\sqrt{\alpha^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{\alpha^2} \\ \lambda_2 &= -\sqrt{\alpha^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\sqrt{\alpha^2}\right)x} + c_2 e^{\left(-\sqrt{\alpha^2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{\alpha^2}\, x} + c_2 \mathrm{e}^{-\sqrt{\alpha^2}\, x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\sqrt{\alpha^2}\, x} + c_2 \mathrm{e}^{-\sqrt{\alpha^2}\, x} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{\sqrt{\alpha^2}\, x} + c_2 \mathrm{e}^{-\sqrt{\alpha^2}\, x}$$

Verified OK.

2.5.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - \alpha^2 y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - \alpha^2 y'y)\, dx = 0$$
$$\frac{y'^2}{2} - \frac{\alpha^2 y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{\alpha^2 y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{\alpha^2 y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{\alpha^2 y^2 + 2c_1}} dy = \int dx$$
$$\frac{\ln\left(\frac{\alpha^2 y}{\sqrt{\alpha^2}} + \sqrt{\alpha^2 y^2 + 2c_1}\right)}{\sqrt{\alpha^2}} = x + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(\frac{\alpha^2 y}{\sqrt{\alpha^2}} + \sqrt{\alpha^2 y^2 + 2c_1}\right)}{\sqrt{\alpha^2}}} = e^{x + c_2}$$

Which simplifies to

$$\left(\alpha y \operatorname{csgn}(\alpha) + \sqrt{\alpha^2 y^2 + 2c_1}\right)^{\frac{1}{\sqrt{\alpha^2}}} = c_3 e^x$$

Simplifying the solution $y = \frac{\operatorname{csgn}(\alpha)\left((c_3 e^x)^{\operatorname{csgn}(\alpha)\alpha} - 2(c_3 e^x)^{-\operatorname{csgn}(\alpha)\alpha} c_1\right)}{2\alpha}$ to $y = \frac{(c_3 e^x)^{\alpha} - 2(c_3 e^x)^{-\alpha} c_1}{2\alpha}$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{\alpha^2 y^2 + 2c_1}} dy = \int dx$$

$$-\frac{\ln\left(\frac{\alpha^2 y}{\sqrt{\alpha^2}} + \sqrt{\alpha^2 y^2 + 2c_1}\right)}{\sqrt{\alpha^2}} = x + c_4$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln\left(\frac{\alpha^2 y}{\sqrt{\alpha^2}} + \sqrt{\alpha^2 y^2 + 2c_1}\right)}{\sqrt{\alpha^2}}} = \mathrm{e}^{x + c_4}$$

Which simplifies to

$$\left(\alpha y \,\mathrm{csgn}\,(\alpha) + \sqrt{\alpha^2 y^2 + 2c_1}\right)^{-\frac{\mathrm{csgn}(\alpha)}{\alpha}} = c_5 \mathrm{e}^{x}$$

Simplifying the solution $y = -\frac{\mathrm{csgn}(\alpha)\left(2(c_5 \mathrm{e}^{x})^{\mathrm{csgn}(\alpha)\alpha} c_1 - (c_5 \mathrm{e}^{x})^{-\mathrm{csgn}(\alpha)\alpha}\right)}{2\alpha}$ to $y = -\frac{2(c_5 \mathrm{e}^{x})^{\alpha} c_1 - (c_5 \mathrm{e}^{x})^{-\alpha}}{2\alpha}$
Summary
The solution(s) found are the following

$$y = \frac{(c_3 \mathrm{e}^{x})^{\alpha} - 2(c_3 \mathrm{e}^{x})^{-\alpha} c_1}{2\alpha} \tag{1}$$

$$y = -\frac{2(c_5 \mathrm{e}^{x})^{\alpha} c_1 - (c_5 \mathrm{e}^{x})^{-\alpha}}{2\alpha} \tag{2}$$

Verification of solutions

$$y = \frac{(c_3 \mathrm{e}^{x})^{\alpha} - 2(c_3 \mathrm{e}^{x})^{-\alpha} c_1}{2\alpha}$$

Verified OK.

$$y = -\frac{2(c_5 \mathrm{e}^{x})^{\alpha} c_1 - (c_5 \mathrm{e}^{x})^{-\alpha}}{2\alpha}$$

Verified OK.

2.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - \alpha^2 y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\alpha^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{\alpha^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= \alpha^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (\alpha^2) \, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 31: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \alpha^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{\alpha^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{\alpha^2}\, x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{\alpha^2}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{\sqrt{\alpha^2}\, x} \int \frac{1}{\mathrm{e}^{2\sqrt{\alpha^2}\, x}}\, dx \\ &= \mathrm{e}^{\sqrt{\alpha^2}\, x} \left(-\frac{\operatorname{csgn}(\alpha)\, \mathrm{e}^{-2 \operatorname{csgn}(\alpha)\alpha x}}{2\alpha} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\sqrt{\alpha^2}\, x} \right) + c_2 \left(\mathrm{e}^{\sqrt{\alpha^2}\, x} \left(-\frac{\operatorname{csgn}(\alpha)\, \mathrm{e}^{-2 \operatorname{csgn}(\alpha)\alpha x}}{2\alpha} \right) \right) \end{aligned}$$

Simplifying the solution $y = c_1 \mathrm{e}^{\sqrt{\alpha^2}\, x} - \frac{c_2 \operatorname{csgn}(\alpha) \mathrm{e}^{-\operatorname{csgn}(\alpha)\alpha x}}{2\alpha}$ to $y = c_1 \mathrm{e}^{\sqrt{\alpha^2}\, x} - \frac{c_2 \mathrm{e}^{-x\alpha}}{2\alpha}$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\sqrt{\alpha^2}\, x} - \frac{c_2 \mathrm{e}^{-x\alpha}}{2\alpha} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\sqrt{\alpha^2}\, x} - \frac{c_2 \mathrm{e}^{-x\alpha}}{2\alpha}$$

Verified OK.

2.5.4 Maple step by step solution

Let's solve

$y'' - \alpha^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $-\alpha^2 + r^2 = 0$

- Factor the characteristic polynomial

  $-(\alpha - r)(\alpha + r) = 0$

- Roots of the characteristic polynomial

  $r = (\alpha, -\alpha)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{x\alpha}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-x\alpha}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{x\alpha} + c_2 \mathrm{e}^{-x\alpha}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)-alpha^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\alpha x} + c_2 \mathrm{e}^{\alpha x}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 23

```
DSolve[y''[x]-a^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{ax} + c_2 e^{-ax}$$

2.6 problem Problem 3.7(f)

Internal problem ID [12415]
Internal file name [`OUTPUT/11067_Wednesday_October_04_2023_07_06_13_PM_11627360/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 3 Bessel functions. Problems page 89
Problem number: Problem 3.7(f).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + \beta y' + y\gamma = 0}$$

2.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = \beta, C = \gamma$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \beta\lambda\, \mathrm{e}^{\lambda x} + \gamma\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\beta\lambda + \lambda^2 + \gamma = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = \beta, C = \gamma$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-\beta}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{\beta^2 - (4)(1)(\gamma)} \\
&= -\frac{\beta}{2} \pm \frac{\sqrt{\beta^2 - 4\gamma}}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= -\frac{\beta}{2} + \frac{\sqrt{\beta^2 - 4\gamma}}{2} \\
\lambda_2 &= -\frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4\gamma}}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= -\frac{\beta}{2} + \frac{\sqrt{\beta^2 - 4\gamma}}{2} \\
\lambda_2 &= -\frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4\gamma}}{2}
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{\left(-\frac{\beta}{2} + \frac{\sqrt{\beta^2 - 4\gamma}}{2}\right)x} + c_2 e^{\left(-\frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4\gamma}}{2}\right)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{\left(-\frac{\beta}{2} + \frac{\sqrt{\beta^2 - 4\gamma}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4\gamma}}{2}\right)x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\left(-\frac{\beta}{2} + \frac{\sqrt{\beta^2 - 4\gamma}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4\gamma}}{2}\right)x} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\left(-\frac{\beta}{2} + \frac{\sqrt{\beta^2 - 4\gamma}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4\gamma}}{2}\right)x}
$$

Verified OK.

2.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \beta y' + y\gamma = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \beta \\ C &= \gamma \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{\beta^2 - 4\gamma}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= \beta^2 - 4\gamma \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{\beta^2}{4} - \gamma\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 33: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{\beta^2}{4} - \gamma$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\frac{x\sqrt{\beta^2-4\gamma}}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\beta}{1}\,dx} \\
&= z_1 e^{-\frac{\beta x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{\beta x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{\left(-\beta+\sqrt{\beta^2-4\gamma}\right)x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\beta}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\beta x}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-x\sqrt{\beta^2-4\gamma}}}{\sqrt{\beta^2-4\gamma}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{\left(-\beta+\sqrt{\beta^2-4\gamma}\right)x}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{\left(-\beta+\sqrt{\beta^2-4\gamma}\right)x}{2}} \left(-\frac{\mathrm{e}^{-x\sqrt{\beta^2-4\gamma}}}{\sqrt{\beta^2-4\gamma}}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{\left(-\beta+\sqrt{\beta^2-4\gamma}\right)x}{2}} - \frac{c_2 \mathrm{e}^{-\frac{\left(\beta+\sqrt{\beta^2-4\gamma}\right)x}{2}}}{\sqrt{\beta^2-4\gamma}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\frac{\left(-\beta+\sqrt{\beta^2-4\gamma}\right)x}{2}} - \frac{c_2 \mathrm{e}^{-\frac{\left(\beta+\sqrt{\beta^2-4\gamma}\right)x}{2}}}{\sqrt{\beta^2-4\gamma}}
$$

Verified OK.

2.6.3 Maple step by step solution

Let's solve

$y'' + \beta y' + y\gamma = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $\beta r + r^2 + \gamma = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-\beta)\pm\left(\sqrt{\beta^2-4\gamma}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{\beta}{2} - \frac{\sqrt{\beta^2-4\gamma}}{2}, -\frac{\beta}{2} + \frac{\sqrt{\beta^2-4\gamma}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\left(-\frac{\beta}{2}-\frac{\sqrt{\beta^2-4\gamma}}{2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\left(-\frac{\beta}{2}+\frac{\sqrt{\beta^2-4\gamma}}{2}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{\left(-\frac{\beta}{2}-\frac{\sqrt{\beta^2-4\gamma}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{\beta}{2}+\frac{\sqrt{\beta^2-4\gamma}}{2}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$2)+beta*diff(y(x),x)+gamma*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{\left(-\beta+\sqrt{\beta^2-4\gamma}\right)x}{2}} + c_2 e^{-\frac{\left(\beta+\sqrt{\beta^2-4\gamma}\right)x}{2}}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 47

```
DSolve[y''[x]+\[Beta]*y'[x]+\[Gamma]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{1}{2}x\left(\sqrt{\beta^2-4\gamma}+\beta\right)}\left(c_2 e^{x\sqrt{\beta^2-4\gamma}} + c_1\right)$$

2.7 problem Problem 3.7(g)

Internal problem ID [12416]
Internal file name [`OUTPUT/11068_Wednesday_October_04_2023_07_06_15_PM_52558194/index.tex`]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 3 Bessel functions. Problems page 89
Problem number: Problem 3.7(g).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_Gegenbauer]`

Unable to solve or complete the solution.

$$\boxed{(-x^2+1)\,y''-2y'x+n(n+1)\,y=0}$$

2.7.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-2y'x+(n^2+n)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{n(n+1)y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-1}-\frac{n(n+1)y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{n(n+1)}{x^2-1}\right]$

- $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $((x+1) \cdot P_2(x))\Big|_{x=-1} = 1$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $((x+1)^2 \cdot P_3(x))\Big|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''(x^2-1) + 2y'x - n(n+1)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) + (-n^2-n)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(r+1+n+k)(r-n+k)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(1+n+k)(-n+k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}\right]$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 15

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+n*(n+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \operatorname{LegendreP}(n, x) + c_2 \operatorname{LegendreQ}(n, x)$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 18

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+n*(n+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{LegendreP}(n, x) + c_2 \operatorname{LegendreQ}(n, x)$$

2.8 problem Problem 3.12

Internal problem ID [12417]
Internal file name [OUTPUT/11069_Wednesday_October_04_2023_07_06_15_PM_2375291/index.tex]

Book: Differential Equations, Linear, Nonlinear, Ordinary, Partial. A.C. King, J.Billingham, S.R.Otto. Cambridge Univ. Press 2003
Section: Chapter 3 Bessel functions. Problems page 89
Problem number: Problem 3.12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''x^2 + y'x + \left(-\nu^2 + x^2\right) y = \sin\left(x\right)}$$

2.8.1 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 + y'x + \left(-\nu^2 + x^2\right) y = \sin\left(x\right) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$y''x^2 + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \,\text{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \,\text{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= 1 \\
n &= \nu \\
\gamma &= 1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}(\nu, x) + c_2 \operatorname{BesselY}(\nu, x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \operatorname{BesselJ}(\nu, x) + c_2 \operatorname{BesselY}(\nu, x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \operatorname{BesselJ}(\nu, x) \\
y_2 &= \operatorname{BesselY}(\nu, x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \operatorname{BesselJ}(\nu, x) & \operatorname{BesselY}(\nu, x) \\ \frac{d}{dx}(\operatorname{BesselJ}(\nu, x)) & \frac{d}{dx}(\operatorname{BesselY}(\nu, x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \text{BesselJ}(\nu, x) & \text{BesselY}(\nu, x) \\ -\text{BesselJ}(\nu+1, x) + \frac{\nu \,\text{BesselJ}(\nu, x)}{x} & -\text{BesselY}(\nu+1, x) + \frac{\nu \,\text{BesselY}(\nu, x)}{x} \end{vmatrix}$$

Therefore

$$W = (\text{BesselJ}(\nu, x)) \left(-\text{BesselY}(\nu+1, x) + \frac{\nu \,\text{BesselY}(\nu, x)}{x}\right) - (\text{BesselY}(\nu, x)) \left(-\text{BesselJ}(\nu+1, x) + \frac{\nu \,\text{BesselJ}(\nu, x)}{x}\right)$$

Which simplifies to

$$W = -\text{BesselJ}(\nu, x)\, \text{BesselY}(\nu+1, x) + \text{BesselY}(\nu, x)\, \text{BesselJ}(\nu+1, x)$$

Which simplifies to

$$W = \frac{2}{\pi x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\text{BesselY}(\nu, x) \sin(x)}{\frac{2x}{\pi}} dx$$

Which simplifies to

$$u_1 = -\int \frac{\text{BesselY}(\nu, x) \sin(x)\, \pi}{2x} dx$$

Hence

$$u_1 = -\frac{x\Big(\text{hypergeom}\left(\left[\frac{1}{2}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2}\right], \left[\frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, \frac{3}{2}-\frac{\nu}{2}\right], -x^2\right) \Gamma(\nu+2)^2\, 2^{\nu} x^{-\nu} + \pi\, \text{hypergeom}\,(}{2\,(\nu-1)\,\nu\,(\nu+1)\,\Gamma\,(}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\text{BesselJ}(\nu, x) \sin(x)}{\frac{2x}{\pi}} dx$$

Which simplifies to

$$u_2 = \int \frac{\text{BesselJ}(\nu, x) \sin(x) \pi}{2x} dx$$

Hence

$$u_2 = \frac{2^{-1-\nu}\pi\, x^{\nu+1} \text{hypergeom}\left(\left[\frac{\nu}{2}+\frac{1}{2}, \frac{3}{4}+\frac{\nu}{2}, \frac{5}{4}+\frac{\nu}{2}\right], \left[\frac{3}{2}, \nu+1, \frac{3}{2}+\nu, \frac{3}{2}+\frac{\nu}{2}\right], -x^2\right)}{\Gamma(\nu+2)}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) =$$
$$-\frac{x\left(\text{hypergeom}\left(\left[\frac{1}{2}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2}\right], \left[\frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, \frac{3}{2}-\frac{\nu}{2}\right], -x^2\right)\Gamma(\nu+2)^2 2^{\nu}x^{-\nu} + \pi \,\text{hypergeom}\left(\right.}{2(\nu-1)\nu(\nu+}$$
$$+\frac{2^{-1-\nu}\pi\, x^{\nu+1} \text{hypergeom}\left(\left[\frac{\nu}{2}+\frac{1}{2}, \frac{3}{4}+\frac{\nu}{2}, \frac{5}{4}+\frac{\nu}{2}\right], \left[\frac{3}{2}, \nu+1, \frac{3}{2}+\nu, \frac{3}{2}+\frac{\nu}{2}\right], -x^2\right) \text{BesselY}(\nu, x)}{\Gamma(\nu+2)}$$

Which simplifies to

$$y_p(x) =$$
$$-\frac{\left(\text{BesselJ}(\nu, x) \text{hypergeom}\left(\left[\frac{1}{2}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2}\right], \left[\frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, \frac{3}{2}-\frac{\nu}{2}\right], -x^2\right)\Gamma(\nu+2)^2 2^{\nu}x^{-\nu} + \text{hy}\right.}{}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= (c_1 \text{BesselJ}(\nu, x) + c_2 \text{BesselY}(\nu, x))$$
$$+\left(-\frac{\left(\text{BesselJ}(\nu, x) \text{hypergeom}\left(\left[\frac{1}{2}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2}\right], \left[\frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, \frac{3}{2}-\frac{\nu}{2}\right], -x^2\right)\Gamma(\nu+2)^2 2^{\nu}x^{-\nu}\right.}{}\right.$$

Summary

The solution(s) found are the following

$$y = c_1 \text{BesselJ}(\nu, x) + c_2 \text{BesselY}(\nu, x) \tag{1}$$
$$-\frac{\left(\text{BesselJ}(\nu, x) \text{hypergeom}\left(\left[\frac{1}{2}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2}\right], \left[\frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, \frac{3}{2}-\frac{\nu}{2}\right], -x^2\right)\Gamma(\nu+2)^2 2^{\nu}x^{-\nu} + \text{hy}\right.}{}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}(\nu, x) + c_2 \operatorname{BesselY}(\nu, x) - \frac{\left(\operatorname{BesselJ}(\nu, x) \operatorname{hypergeom}\left(\left[\frac{1}{2} - \frac{\nu}{2}, \frac{5}{4} - \frac{\nu}{2}, \frac{3}{4} - \frac{\nu}{2}\right], \left[\frac{3}{2}, 1 - \nu, \frac{3}{2} - \nu, \frac{3}{2} - \frac{\nu}{2}\right], -x^2\right) \Gamma(\nu + 2)^2 2^\nu x^{-\nu} + \mathrm{hy}}{}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 158

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-nu^2)*y(x)=sin(x),y(x), singsol=all)
```

$$y(x) = -\frac{x^{1-\nu} 2^{\nu-1} \operatorname{BesselJ}(\nu, x) \Gamma(\nu + 2) \operatorname{hypergeom}\left(\left[\frac{1}{2} - \frac{\nu}{2}, \frac{5}{4} - \frac{\nu}{2}, \frac{3}{4} - \frac{\nu}{2}\right], \left[\frac{3}{2}, 1 - \nu, \frac{3}{2} - \nu, \frac{3}{2} - \frac{\nu}{2}\right], -x^2\right)}{\nu (\nu - 1) (\nu + 1)}$$
$$+ \operatorname{BesselJ}(\nu, x) c_2 + \operatorname{BesselY}(\nu, x) c_1$$
$$- \frac{\pi 2^{-1-\nu} x^{\nu+1} (\operatorname{BesselJ}(\nu, x) \cot(\pi\nu) - \operatorname{BesselY}(\nu, x)) \operatorname{hypergeom}\left(\left[\frac{\nu}{2} + \frac{1}{2}, \frac{5}{4} + \frac{\nu}{2}, \frac{3}{4} + \frac{\nu}{2}\right], \left[\frac{3}{2}, \nu + 1, \frac{3}{2} + \right.\right.}{\Gamma(\nu + 2)}$$

✓ Solution by Mathematica

Time used: 1.228 (sec). Leaf size: 205

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-\[Nu]^2)*y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True
```

$$
\begin{aligned}
y(x) \to & \\
& -\frac{\pi 2^{\nu-1}\csc(\pi\nu)x^{1-\nu}\,\text{BesselJ}(\nu,x)\,{}_3F_4\left(\frac{1}{2}-\frac{\nu}{2},\frac{3}{4}-\frac{\nu}{2},\frac{5}{4}-\frac{\nu}{2};\frac{3}{2},1-\nu,\frac{3}{2}-\nu,\frac{3}{2}-\frac{\nu}{2};-x^2\right)}{(\nu-1)\,\text{Gamma}(1-\nu)} \\
& +\frac{\pi 2^{-\nu-1}x^{\nu+1}(\text{BesselY}(\nu,x)-\cot(\pi\nu)\,\text{BesselJ}(\nu,x))\,{}_3F_4\left(\frac{\nu}{2}+\frac{1}{2},\frac{\nu}{2}+\frac{3}{4},\frac{\nu}{2}+\frac{5}{4};\frac{3}{2},\frac{\nu}{2}+\frac{3}{2},\nu+1,\nu+\frac{3}{2};-x\right.}{(\nu+1)\,\text{Gamma}(\nu+1)} \\
& +c_1\,\text{BesselJ}(\nu,x)+c_2\,\text{BesselY}(\nu,x)
\end{aligned}
$$