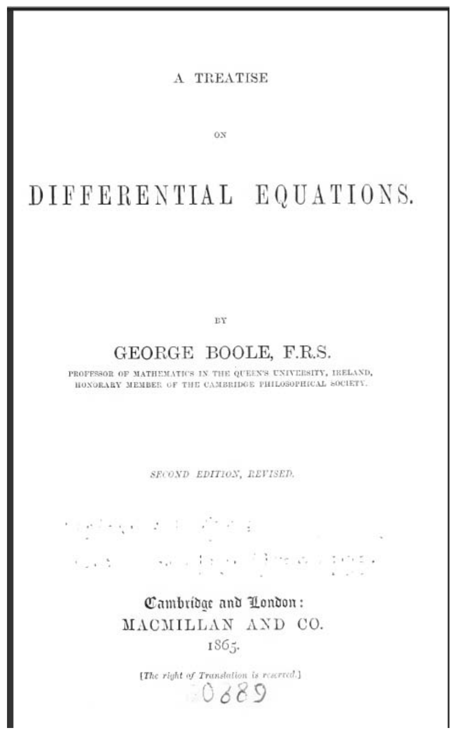
A Solution Manual For

Differential Equations, By George Boole F.R.S. 1865



A TREATISE

ON

DIFFERENTIAL EQUATIONS.

BY

GEORGE BOOLE, F.R.S.

PROFESSOR OF MATHEMATICS IN THE QUEEN'S UNIVERSITY, IRELAND,
HONORARY MEMBER OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

SECOND EDITION, REVISED.

Cambridge and London:
MACMILLAN AND CO.
1865.

[*The right of Translation is reserved.*]

0689

Nasser M. Abbasi

May 15, 2024

Contents

1 Chapter 2

1.1 problem 1.1

Internal problem ID [4355]
Internal file name [`OUTPUT/3848_Sunday_June_05_2022_11_28_25_AM_59433606/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 1.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x+1)\,y+(1-y)\,xy'=0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{(x+1)\,y}{(y-1)\,x} \end{aligned}$$

Where $f(x)=\frac{x+1}{x}$ and $g(y)=\frac{y}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y-1}}\,dy = \frac{x+1}{x}\,dx$$

$$\int \frac{1}{\frac{y}{y-1}}\,dy = \int \frac{x+1}{x}\,dx$$

$$y-\ln(y) = x+\ln(x)+c_1$$

Which results in

$$y = -\operatorname{LambertW}\left(-\frac{e^{-c_1-x}}{x}\right)$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$y = -\operatorname{LambertW}\left(-\frac{e^{-c_1-x}}{x}\right)$$

gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right)$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right) \tag{1}$$

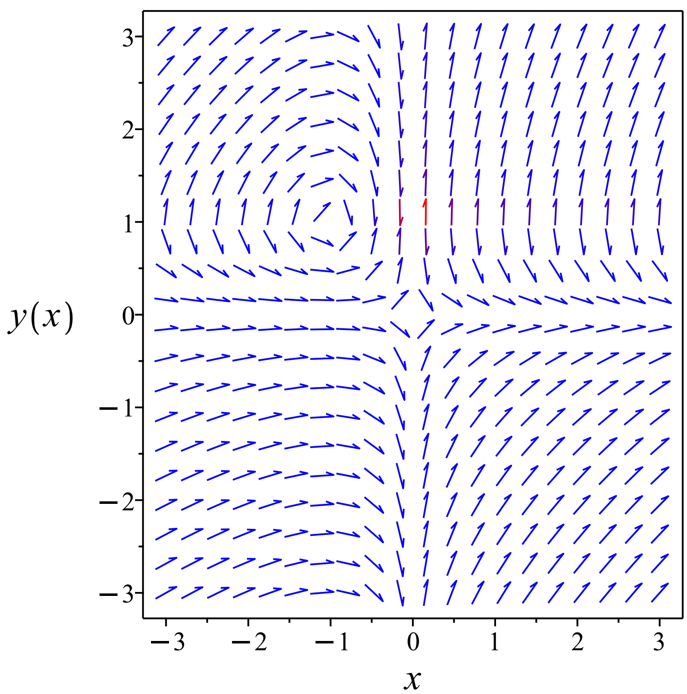


Figure 1: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right)$$

Verified OK.

1.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{(x+1)\,y}{(y-1)\,x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= \frac{x}{x+1} \\ \eta(x, y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x}{x+1}} dx \end{aligned}$$

Which results in

$$S = x + \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{(x + 1) y}{(y - 1) x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 1 + \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y - 1}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R - 1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x + \ln(x) = y - \ln(y) + c_1$$

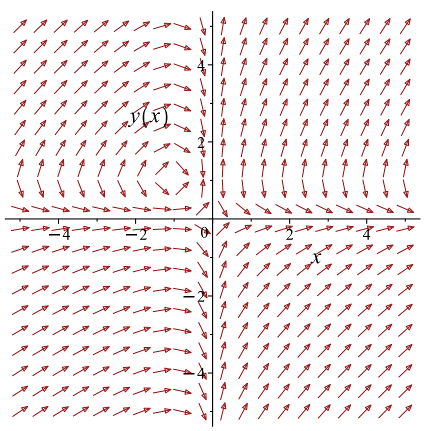
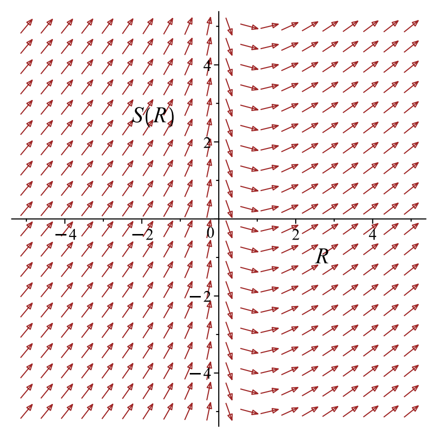
Which simplifies to

$$x + \ln(x) = y - \ln(y) + c_1$$

Which gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(x+1)y}{(y-1)x}$ 	$R = y$ $S = x + \ln(x)$	$\frac{dS}{dR} = \frac{R-1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right) \tag{1}$$

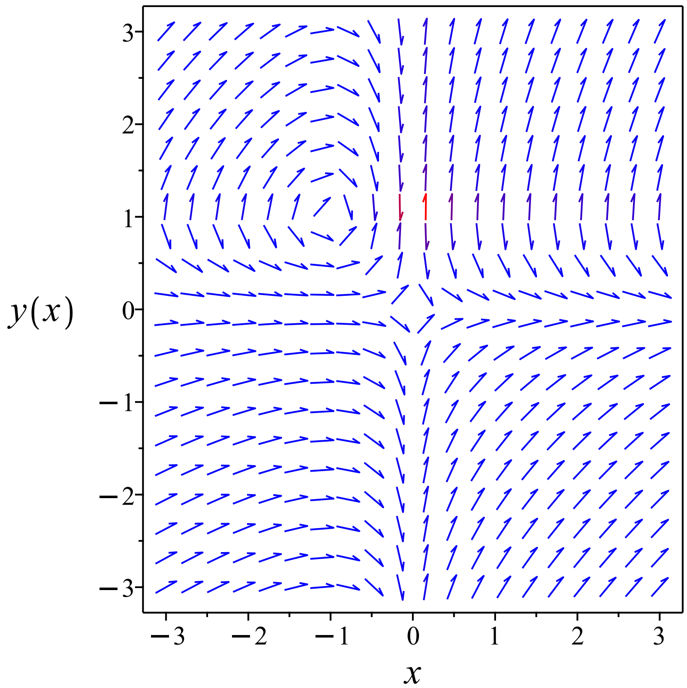


Figure 2: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right)$$

Verified OK.

1.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y-1}{y}\right) \mathrm{d}y = \left(\frac{x+1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{x+1}{x}\right) \mathrm{d}x + \left(\frac{y-1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x+1}{x}$$
$$N(x, y) = \frac{y-1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+1}{x}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y-1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x+1}{x}\,\mathrm{d}x\end{aligned}$$
$$\phi = -x - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-1}{y}$. Therefore equation (4) becomes

$$\frac{y-1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y-1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{y-1}{y}\right)\mathrm{d}y\\ f(y) &= y - \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \ln(x) + y - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \ln(x) + y - \ln(y)$$

The solution becomes

$$y = -\mathrm{LambertW}\left(-\frac{e^{-c_1-x}}{x}\right)$$

Summary

The solution(s) found are the following

$$y = -\mathrm{LambertW}\left(-\frac{e^{-c_1-x}}{x}\right) \tag{1}$$

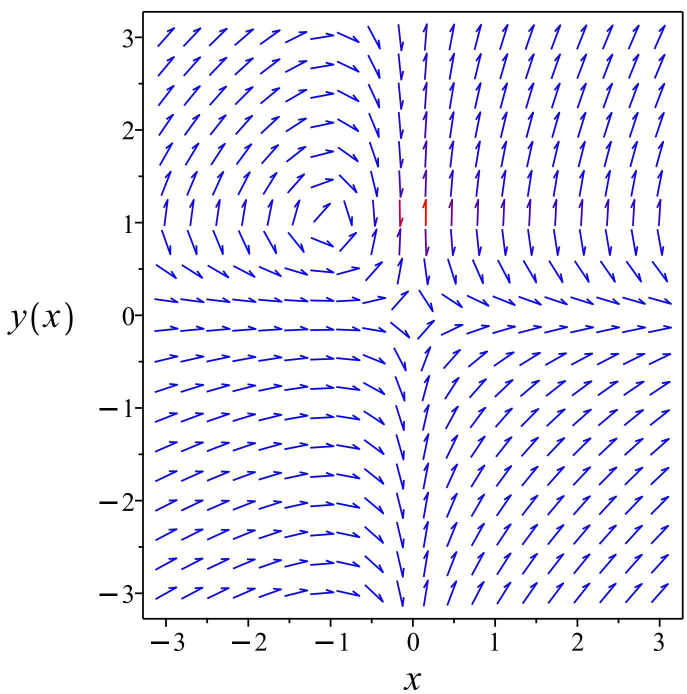


Figure 3: Slope field plot

Verification of solutions

$$y = -\text{LambertW}\left(-\frac{e^{-c_1-x}}{x}\right)$$

Verified OK.

1.1.4 Maple step by step solution

Let's solve

$(x+1)\, y + (1-y)\, x y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1-y)}{y} = -\frac{x+1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'(1-y)}{y} dx = \int -\frac{x+1}{x} dx + c_1$

- Evaluate integral

  $-y + \ln(y) = -x - \ln(x) + c_1$

- Solve for y

  $y = -\mathit{LambertW}\left(-\frac{e^{-x+c_1}}{x}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve((1+x)*y(x)+(1-y(x))*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-\frac{e^{-x}}{c_1 x}\right)$$

✓ Solution by Mathematica

Time used: 3.139 (sec). Leaf size: 28

```
DSolve[(1+x)*y[x]+(1-y[x])*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -W\left(-\frac{e^{-x-c_1}}{x}\right)$$
$$y(x) \to 0$$

1.2 problem 1.2

Internal problem ID [4356]
Internal file name [`OUTPUT/3849_Sunday_June_05_2022_11_28_32_AM_88189904/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 1.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 + x y^2 + \left(x^2 - y x^2\right) y' = 0}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2(x+1)}{x^2\,(y-1)} \end{aligned}$$

Where $f(x) = \frac{x+1}{x^2}$ and $g(y) = \frac{y^2}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{y^2}{y-1}}\,dy = \frac{x+1}{x^2}\,dx$$

$$\int \frac{1}{\frac{y^2}{y-1}}\,dy = \int \frac{x+1}{x^2}\,dx$$

$$
\ln \left(y\right)+\frac{1}{y} = \ln \left(x\right)-\frac{1}{x}+c_{1}
$$

Which results in

$$
y = {\mathrm e}^{\frac{\ln \left(x\right) x +\operatorname{LambertW}\left(-{\mathrm e}^{-\frac{\ln \left(x\right) x +c_{1} x -1}{x}}\right) x +c_{1} x -1}{x}}
$$

Which simplifies to

$$
y = x \,{\mathrm e}^{\operatorname{LambertW}\left(-\frac{{\mathrm e}^{\frac{1}{x}} {\mathrm e}^{-c_{1}}}{x}\right)} {\mathrm e}^{c_{1}} {\mathrm e}^{-\frac{1}{x}}
$$

Summary

The solution(s) found are the following

$$
y = x \,{\mathrm e}^{\operatorname{LambertW}\left(-\frac{{\mathrm e}^{\frac{1}{x}} {\mathrm e}^{-c_{1}}}{x}\right)} {\mathrm e}^{c_{1}} {\mathrm e}^{-\frac{1}{x}} \tag{1}
$$

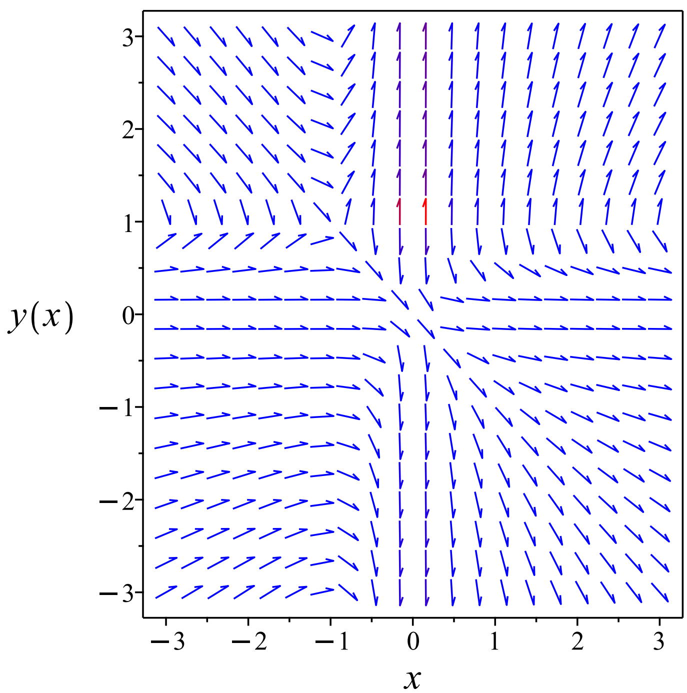


Figure 4: Slope field plot

Verification of solutions

$$
y = x \,{\mathrm e}^{\operatorname{LambertW}\left(-\frac{{\mathrm e}^{\frac{1}{x}} {\mathrm e}^{-c_{1}}}{x}\right)} {\mathrm e}^{c_{1}} {\mathrm e}^{-\frac{1}{x}}
$$

Verified OK.

1.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2(x+1)}{x^2(y-1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}
\xi(x, y) &= \frac{x^2}{x+1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x^2}{x+1}} dx
\end{aligned}$$

Which results in

$$S = \ln(x) - \frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2(x+1)}{x^2 (y-1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x+1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y-1}{y^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R-1}{R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + \frac{1}{R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(x)\, x - 1}{x} = \ln(y) + \frac{1}{y} + c_1
$$

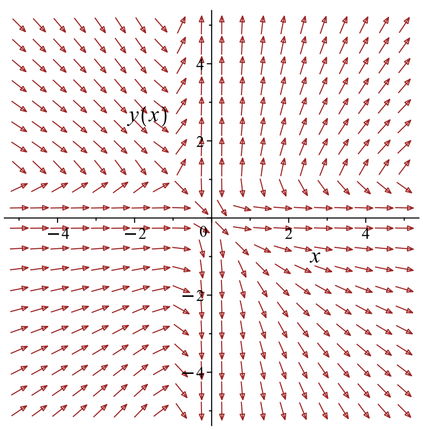
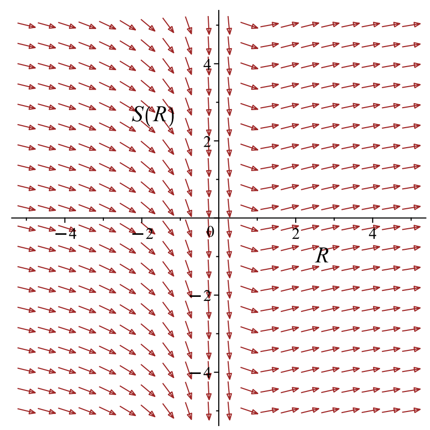
Which simplifies to

$$
\frac{\ln(x)\, x - 1}{x} = \ln(y) + \frac{1}{y} + c_1
$$

Which gives

$$
y = \mathrm{e}^{\frac{\ln(x)x + \operatorname{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x - c_1 x - 1}{x}}\right)x - c_1 x - 1}{x}}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2(x+1)}{x^2(y-1)}$ 	$R = y$ $S = \frac{\ln(x)\, x - 1}{x}$	$\frac{dS}{dR} = \frac{R-1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = e^{\frac{\ln(x)x + \mathrm{LambertW}\left(-e^{-\frac{\ln(x)x - c_1 x - 1}{x}}\right)x - c_1 x - 1}{x}} \tag{1}$$

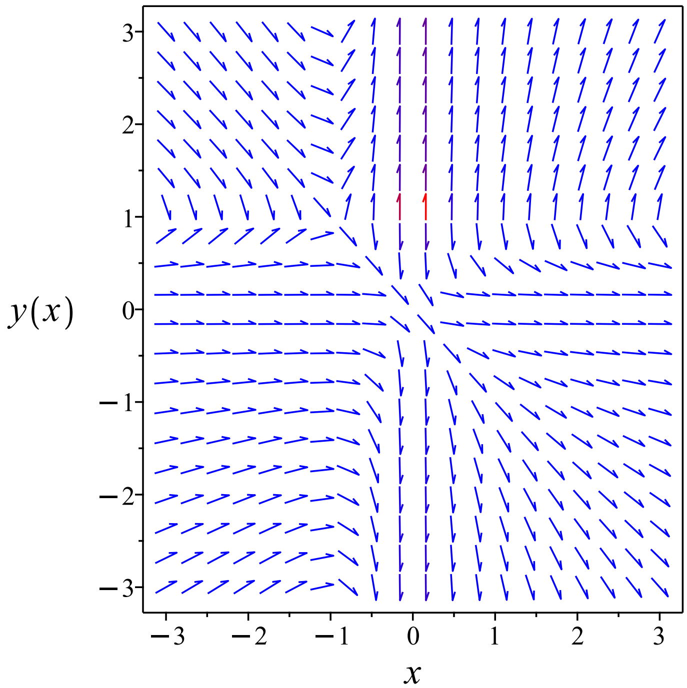


Figure 5: Slope field plot

Verification of solutions

$$y = e^{\frac{\ln(x)x + \mathrm{LambertW}\left(-e^{-\frac{\ln(x)x - c_1 x - 1}{x}}\right)x - c_1 x - 1}{x}}$$

Verified OK.

1.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{y-1}{y^2}\right) \mathrm{d}y &= \left(\frac{x+1}{x^2}\right) \mathrm{d}x \\
\left(-\frac{x+1}{x^2}\right) \mathrm{d}x + \left(\frac{y-1}{y^2}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x+1}{x^2} \\
N(x, y) &= \frac{y-1}{y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+1}{x^2}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y-1}{y^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x+1}{x^2}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\ln(x) + \frac{1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-1}{y^2}$. Therefore equation (4) becomes

$$\frac{y-1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y-1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y-1}{y^2}\right) \mathrm{d}y$$

$$f(y) = \ln(y) + \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{1}{x} + \ln(y) + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{1}{x} + \ln(y) + \frac{1}{y}$$

The solution becomes

$$y = \mathrm{e}^{\frac{\ln(x)x+\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x+c_1x-1}{x}}\right)x+c_1x-1}{x}}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\ln(x)x+\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x+c_1x-1}{x}}\right)x+c_1x-1}{x}} \tag{1}$$

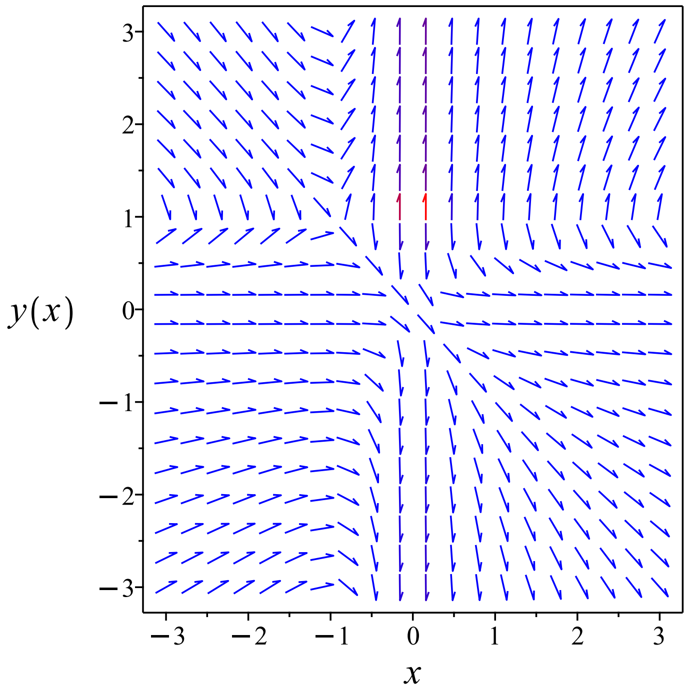


Figure 6: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{\ln(x)x+\mathrm{LambertW}\left(-\mathrm{e}^{-\frac{\ln(x)x+c_1x-1}{x}}\right)x+c_1x-1}{x}}$$

Verified OK.

1.2.4 Maple step by step solution

Let's solve

$y^2 + xy^2 + (x^2 - yx^2)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y-1)}{y^2} = \frac{x+1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'(y-1)}{y^2}dx = \int \frac{x+1}{x^2}dx + c_1$

- Evaluate integral

  $\ln(y) + \frac{1}{y} = \ln(x) - \frac{1}{x} + c_1$

- Solve for y

  $y = e^{\frac{\ln(x)x + \mathit{LambertW}\left(-e^{-\frac{\ln(x)x + c_1 x - 1}{x}}\right)x + c_1 x - 1}{x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve((y(x)^2+x*y(x)^2)+(x^2-y(x)*x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x\,e^{\frac{\mathrm{LambertW}\left(-\frac{e^{\frac{-c_1 x+1}{x}}}{x}\right)x + c_1 x - 1}{x}}$$

✓ Solution by Mathematica

Time used: 5.328 (sec). Leaf size: 30

```
DSolve[(y[x]^2+x*y[x]^2)+(x^2-y[x]*x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{1}{W\left(-\frac{e^{\frac{1}{x}-c_1}}{x}\right)}$$
$$y(x) \rightarrow 0$$

1.3 problem 1.3

Internal problem ID [4357]
Internal file name [`OUTPUT/3850_Sunday_June_05_2022_11_28_38_AM_52286439/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 1.3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{xy(x^2+1)\,y'-y^2=1}$$

1.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2+1}{xy\,(x^2+1)} \end{aligned}$$

Where $f(x)=\frac{1}{x(x^2+1)}$ and $g(y)=\frac{y^2+1}{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{y^2+1}{y}}\,dy &= \frac{1}{x\,(x^2+1)}\,dx \\ \int \frac{1}{\frac{y^2+1}{y}}\,dy &= \int \frac{1}{x\,(x^2+1)}\,dx \end{aligned}$$

$$
\frac{\ln \left(y^{2}+1\right)}{2} = -\frac{\ln \left(x^{2}+1\right)}{2}+\ln \left(x \right)+c_{1}
$$

Raising both side to exponential gives

$$
\sqrt{y^{2}+1} = {\mathrm e}^{-\frac{\ln \left(x^{2}+1\right)}{2}+\ln \left(x \right)+c_{1}}
$$

Which simplifies to

$$
\sqrt{y^{2}+1} = c_{2} {\mathrm e}^{-\frac{\ln \left(x^{2}+1\right)}{2}+\ln \left(x \right)}
$$

Which simplifies to

$$
\sqrt{1+y^{2}} = \frac{c_{2} x \,{\mathrm e}^{c_{1}}}{\sqrt{x^{2}+1}}
$$

The solution is

$$
\sqrt{1+y^{2}} = \frac{c_{2} x \,{\mathrm e}^{c_{1}}}{\sqrt{x^{2}+1}}
$$

Summary

The solution(s) found are the following

$$
\sqrt{1+y^{2}} = \frac{c_{2} x \,{\mathrm e}^{c_{1}}}{\sqrt{x^{2}+1}} \tag{1}
$$

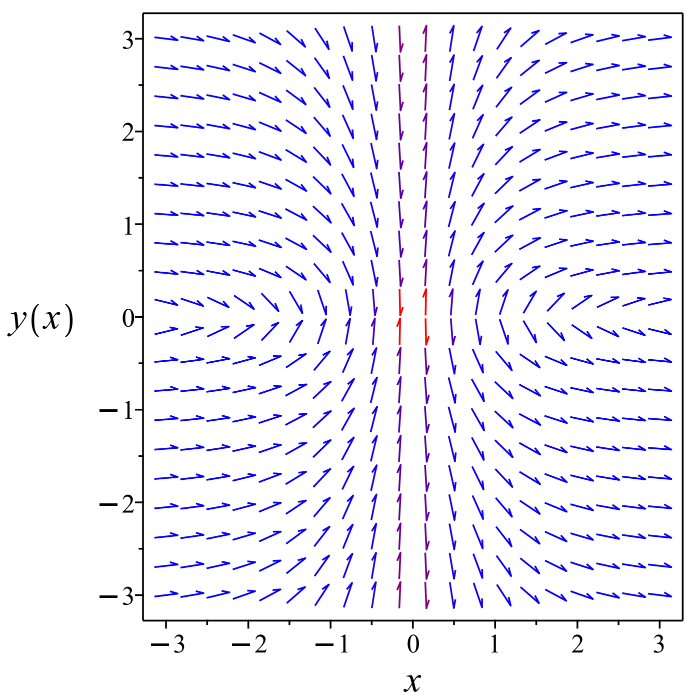


Figure 7: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} = \frac{c_2 x \,\mathrm{e}^{c_1}}{\sqrt{x^2+1}}$$

Verified OK.

1.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y^2+1}{xy\,(x^2+1)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= x\left(x^2 + 1\right) \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x(x^2+1)} dx \end{aligned}$$

Which results in

$$S = -\frac{\ln(x^2+1)}{2} + \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2+1}{xy(x^2+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x(x^2+1)} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

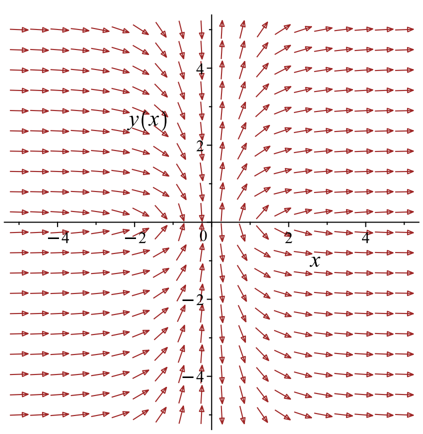
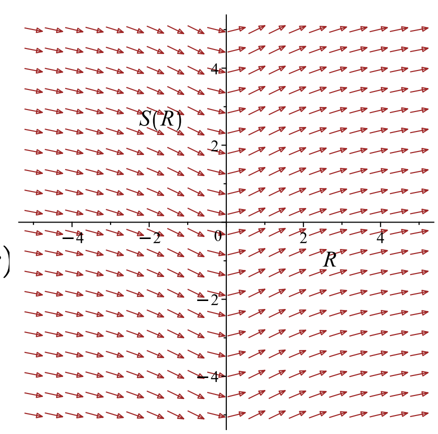
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(x^2+1)}{2} + \ln(x) = \frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$-\frac{\ln(x^2+1)}{2} + \ln(x) = \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+1}{xy(x^2+1)}$ 	$R = y$ $S = -\frac{\ln(x^2+1)}{2} + \ln(x)$	$\frac{dS}{dR} = \frac{R}{R^2+1}$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(x^2+1)}{2} + \ln(x) = \frac{\ln(1+y^2)}{2} + c_1 \tag{1}$$

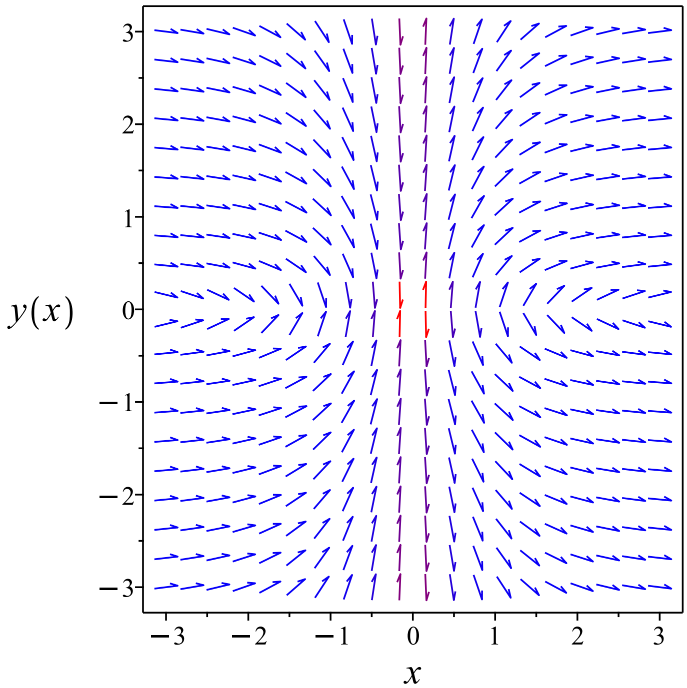


Figure 8: Slope field plot

Verification of solutions

$$-\frac{\ln\left(x^{2}+1\right)}{2}+\ln\left(x\right)=\frac{\ln\left(1+y^{2}\right)}{2}+c_{1}$$

Verified OK.

1.3.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^{2}+1}{xy\left(x^{2}+1\right)} \end{aligned}$$

This is a Bernoulli ODE.

$$y'=\frac{1}{x\left(x^{2}+1\right)}y+\frac{1}{x\left(x^{2}+1\right)}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y'=f_{0}(x)y+f_{1}(x)y^{n} \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x\,(x^2+1)} \\ f_1(x) &= \frac{1}{x\,(x^2+1)} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{x\,(x^2+1)} + \frac{1}{x\,(x^2+1)} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{x\,(x^2+1)} + \frac{1}{x\,(x^2+1)} \\ w' &= \frac{2w}{x\,(x^2+1)} + \frac{2}{x\,(x^2+1)} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x\left(x^{2}+1\right)} \\
q(x) &= \frac{2}{x\left(x^{2}+1\right)}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{2w(x)}{x\left(x^{2}+1\right)} = \frac{2}{x\left(x^{2}+1\right)}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{x\left(x^{2}+1\right)}dx} \\
&= \mathrm{e}^{\ln\left(x^{2}+1\right)-2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{x^{2}+1}{x^{2}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{2}{x\left(x^{2}+1\right)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\left(x^{2}+1\right)w}{x^{2}}\right) &= \left(\frac{x^{2}+1}{x^{2}}\right)\left(\frac{2}{x\left(x^{2}+1\right)}\right) \\
\mathrm{d}\left(\frac{\left(x^{2}+1\right)w}{x^{2}}\right) &= \left(\frac{2}{x^{3}}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\left(x^{2}+1\right)w}{x^{2}} &= \int \frac{2}{x^{3}}\,\mathrm{d}x \\
\frac{\left(x^{2}+1\right)w}{x^{2}} &= -\frac{1}{x^{2}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{x^{2}+1}{x^{2}}$ results in

$$
w(x) = -\frac{1}{x^{2}+1} + \frac{c_1 x^{2}}{x^{2}+1}
$$

which simplifies to

$$
w(x) = \frac{c_1 x^{2}-1}{x^{2}+1}
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{c_1 x^2 - 1}{x^2 + 1}$$

Solving for y gives

$$y(x) = \frac{\sqrt{(x^2+1)(c_1 x^2 - 1)}}{x^2+1}$$
$$y(x) = -\frac{\sqrt{(x^2+1)(c_1 x^2 - 1)}}{x^2+1}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{(x^2+1)(c_1 x^2 - 1)}}{x^2+1} \tag{1}$$
$$y = -\frac{\sqrt{(x^2+1)(c_1 x^2 - 1)}}{x^2+1} \tag{2}$$

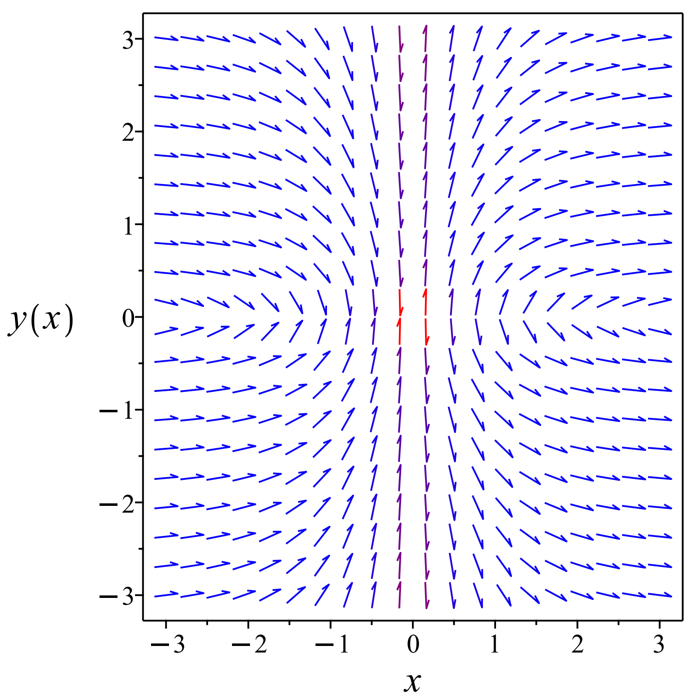


Figure 9: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{(x^2+1)(c_1x^2-1)}}{x^2+1}$$

Verified OK.

$$y = -\frac{\sqrt{(x^2+1)(c_1x^2-1)}}{x^2+1}$$

Verified OK.

1.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{y^2+1}\right)\mathrm{d}y &= \left(\frac{1}{x\,(x^2+1)}\right)\mathrm{d}x\\ \left(-\frac{1}{x\,(x^2+1)}\right)\mathrm{d}x + \left(\frac{y}{y^2+1}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x\,(x^2+1)}\\ N(x,y) &= \frac{y}{y^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x\,(x^2+1)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{y^2+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x\left(x^2+1\right)} \, \mathrm{d}x$$
$$\phi = \frac{\ln\left(x^2+1\right)}{2} - \ln\left(x\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{y^2+1}$. Therefore equation (4) becomes

$$\frac{y}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{y}{y^2+1}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln\left(y^2+1\right)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln\left(x^2+1\right)}{2} - \ln\left(x\right) + \frac{\ln\left(y^2+1\right)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x^2+1)}{2} - \ln(x) + \frac{\ln(y^2+1)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+1)}{2} - \ln(x) + \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

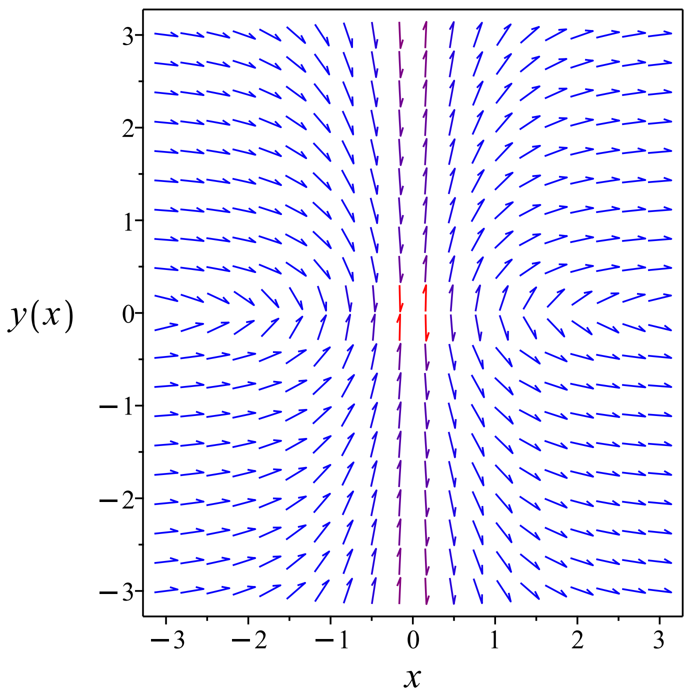


Figure 10: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+1)}{2} - \ln(x) + \frac{\ln(1+y^2)}{2} = c_1$$

Verified OK.

1.3.5 Maple step by step solution

Let's solve

$xy(x^2+1)\,y' - y^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{1+y^2} = \frac{1}{x(x^2+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{1+y^2}dx = \int \frac{1}{x(x^2+1)}dx + c_1$

- Evaluate integral

  $\frac{\ln(1+y^2)}{2} = -\frac{\ln(x^2+1)}{2} + \ln(x) + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{(x^2+1)\left((e^{c_1})^2x^2-x^2-1\right)}}{x^2+1}, y = -\frac{\sqrt{(x^2+1)\left((e^{c_1})^2x^2-x^2-1\right)}}{x^2+1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
dsolve(x*y(x)*(1+x^2)*diff(y(x),x)-(1+y(x)^2)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(x^2+1)(c_1x^2-1)}}{x^2+1}$$
$$y(x) = -\frac{\sqrt{(x^2+1)(c_1x^2-1)}}{x^2+1}$$

✓ Solution by Mathematica

Time used: 1.211 (sec). Leaf size: 131

```
DSolve[x*y[x]*(1+x^2)*y'[x]-(1+y[x]^2)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-1+(-1+e^{2c_1})\,x^2}}{\sqrt{x^2+1}}$$
$$y(x) \to \frac{\sqrt{-1+(-1+e^{2c_1})\,x^2}}{\sqrt{x^2+1}}$$
$$y(x) \to -i$$
$$y(x) \to i$$
$$y(x) \to -\frac{\sqrt{-x^2-1}}{\sqrt{x^2+1}}$$
$$y(x) \to \frac{\sqrt{-x^2-1}}{\sqrt{x^2+1}}$$

1.4 problem 1.4

Internal problem ID [4358]
Internal file name [`OUTPUT/3851_Sunday_June_05_2022_11_28_45_AM_11589406/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 1.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 - \left(y + \sqrt{1+y^2}\right)\left(x^2+1\right)^{\frac{3}{2}} y' = -1}$$

1.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2+1}{\left(y+\sqrt{y^2+1}\right)\left(x^2+1\right)^{\frac{3}{2}}} \end{aligned}$$

Where $f(x) = \frac{1}{(x^2+1)^{\frac{3}{2}}}$ and $g(y) = \frac{y^2+1}{y+\sqrt{y^2+1}}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{y+\sqrt{y^2+1}}}\, dy = \frac{1}{\left(x^2+1\right)^{\frac{3}{2}}}\, dx$$

$$\int \frac{1}{\frac{y^2+1}{y+\sqrt{y^2+1}}}\,dy = \int \frac{1}{(x^2+1)^{\frac{3}{2}}}\,dx$$

$$\operatorname{arcsinh}(y) + \frac{\ln(y^2+1)}{2} = \frac{x}{\sqrt{x^2+1}} + c_1$$

Which results in

$$y = \operatorname{RootOf}\left(\_Z^2 - e^{\operatorname{RootOf}\left(-\sinh\left(\frac{2c_1x^2-x^2\_Z+2\sqrt{x^2+1}\,x+2c_1-\_Z}{2x^2+2}\right)^2+e^{\_Z}-1\right)}+1\right)$$

Summary

The solution(s) found are the following

$$y = \operatorname{RootOf}\left(\_Z^2 - e^{\operatorname{RootOf}\left(-\sinh\left(\frac{2c_1x^2-x^2\_Z+2\sqrt{x^2+1}\,x+2c_1-\_Z}{2x^2+2}\right)^2+e^{\_Z}-1\right)}+1\right) \tag{1}$$

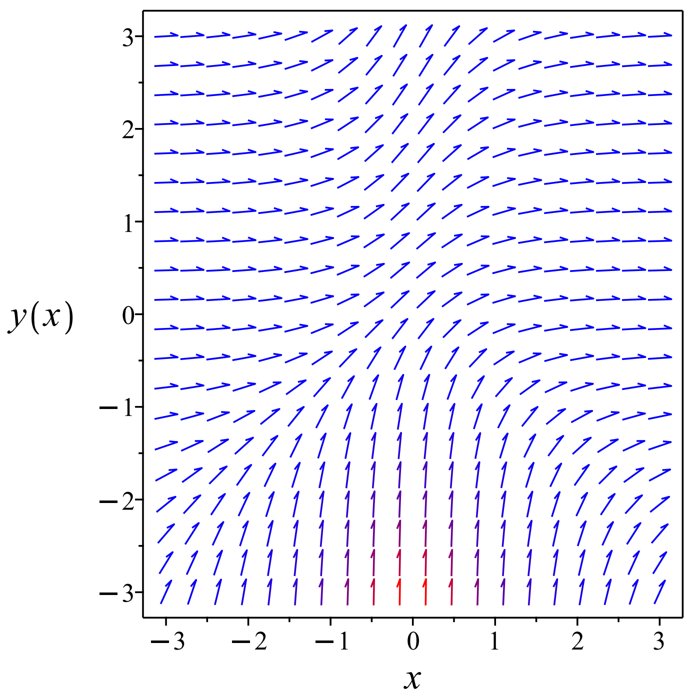


Figure 11: Slope field plot

Verification of solutions

$$y = \text{RootOf}\left(\_Z^2 - e^{\text{RootOf}\left(-\sinh\left(\frac{2c_1x^2 - x^2\_Z + 2\sqrt{x^2+1}\,x + 2c_1 - \_Z}{2x^2+2}\right)^2 + e^{\_Z} - 1\right)} + 1\right)$$

Warning, solution could not be verified

1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2 + 1}{\left(y + \sqrt{y^2 + 1}\right)(x^2 + 1)^{\frac{3}{2}}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 10: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \left(x^2+1\right)^{\frac{3}{2}} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{(x^2+1)^{\frac{3}{2}}} dx \end{aligned}$$

Which results in

$$S = \frac{x}{\sqrt{x^2+1}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2+1}{\left(y+\sqrt{y^2+1}\right)(x^2+1)^{\frac{3}{2}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{(x^2+1)^{\frac{3}{2}}} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y+\sqrt{y^2+1}}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R+\sqrt{R^2+1}}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \operatorname{arcsinh}(R) + \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

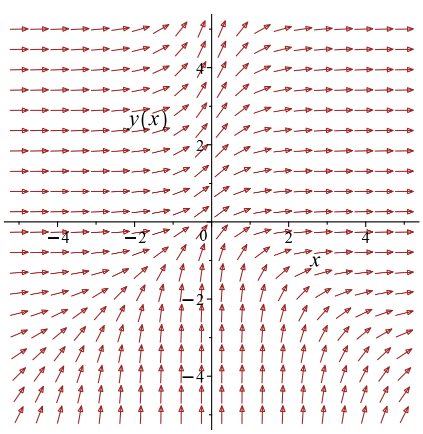
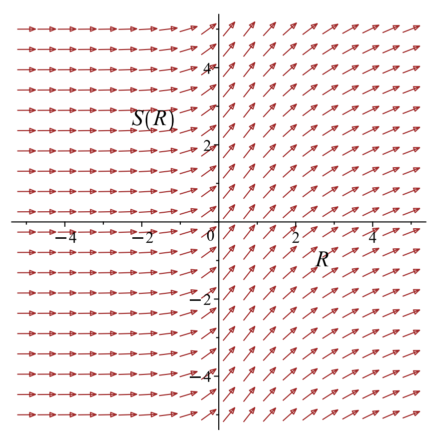
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x}{\sqrt{x^2+1}} = \operatorname{arcsinh}(y) + \frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$\frac{x}{\sqrt{x^2+1}} = \operatorname{arcsinh}(y) + \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+1}{\left(y+\sqrt{y^2+1}\right)(x^2+1)^{\frac{3}{2}}}$ 	$R = y$ $S = \frac{x}{\sqrt{x^2+1}}$	$\frac{dS}{dR} = \frac{R+\sqrt{R^2+1}}{R^2+1}$ 

Summary

The solution(s) found are the following

$$\frac{x}{\sqrt{x^2+1}} = \operatorname{arcsinh}(y) + \frac{\ln(1+y^2)}{2} + c_1 \tag{1}$$

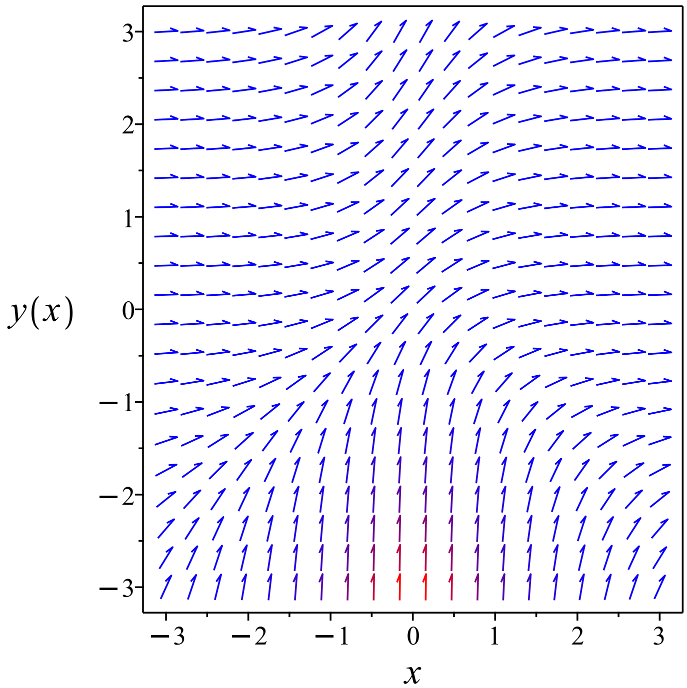


Figure 12: Slope field plot

Verification of solutions

$$\frac{x}{\sqrt{x^2+1}} = \operatorname{arcsinh}(y) + \frac{\ln(1+y^2)}{2} + c_1$$

Verified OK.

1.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y + \sqrt{y^2 + 1}}{y^2 + 1}\right) \mathrm{d}y = \left(\frac{1}{(x^2 + 1)^{\frac{3}{2}}}\right) \mathrm{d}x$$
$$\left(-\frac{1}{(x^2 + 1)^{\frac{3}{2}}}\right) \mathrm{d}x + \left(\frac{y + \sqrt{y^2 + 1}}{y^2 + 1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{(x^2 + 1)^{\frac{3}{2}}}$$
$$N(x, y) = \frac{y + \sqrt{y^2 + 1}}{y^2 + 1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{(x^2 + 1)^{\frac{3}{2}}}\right)$$
$$= 0$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y+\sqrt{y^2+1}}{y^2+1}\right)\\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{(x^2+1)^{\frac{3}{2}}} \,\mathrm{d}x\\
\phi &= -\frac{x}{\sqrt{x^2+1}} + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y+\sqrt{y^2+1}}{y^2+1}$. Therefore equation (4) becomes

$$\frac{y+\sqrt{y^2+1}}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y+\sqrt{y^2+1}}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y+\sqrt{y^2+1}}{y^2+1}\right)\mathrm{d}y$$

$$f(y) = \operatorname{arcsinh}(y) + \frac{\ln(y^2+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x}{\sqrt{x^2+1}} + \operatorname{arcsinh}(y) + \frac{\ln(y^2+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x}{\sqrt{x^2+1}} + \operatorname{arcsinh}(y) + \frac{\ln(y^2+1)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x}{\sqrt{x^2+1}} + \operatorname{arcsinh}(y) + \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

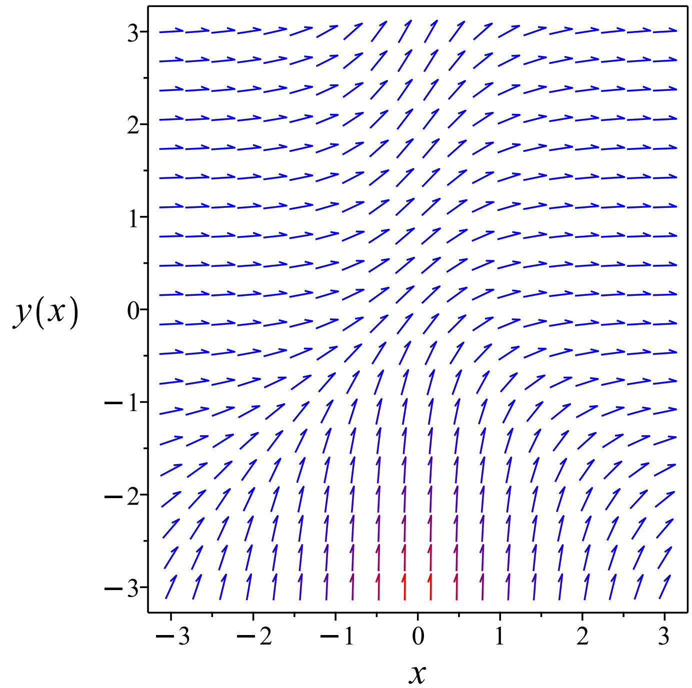


Figure 13: Slope field plot

Verification of solutions

$$-\frac{x}{\sqrt{x^2+1}} + \operatorname{arcsinh}(y) + \frac{\ln(1+y^2)}{2} = c_1$$

Verified OK.

1.4.4 Maple step by step solution

Let's solve

$y^2 - \left(y + \sqrt{1+y^2}\right)(x^2+1)^{\frac{3}{2}} y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\left(y+\sqrt{1+y^2}\right)}{-1-y^2} = -\frac{1}{(x^2+1)^{\frac{3}{2}}}$

- Integrate both sides with respect to x

  $\int \frac{y'\left(y+\sqrt{1+y^2}\right)}{-1-y^2} dx = \int -\frac{1}{(x^2+1)^{\frac{3}{2}}} dx + c_1$

- Evaluate integral

  $-\mathrm{arcsinh}(y) - \frac{\ln(1+y^2)}{2} = -\frac{x}{\sqrt{x^2+1}} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 28

```
dsolve((1+y(x)^2)-(y(x)+sqrt(1+y(x)^2))*(1+x^2)^(3/2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{x}{\sqrt{x^2+1}} - \mathrm{arcsinh}\left(y(x)\right) - \frac{\ln\left(1+y(x)^2\right)}{2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 15.063 (sec). Leaf size: 115

```
DSolve[(1+y[x]^2)-(y[x]+Sqrt[1+y[x]^2])*(1+x^2)^(3/2)*y'[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to -\frac{i\left(1+e^{\frac{x}{\sqrt{x^2+1}}+c_1}\right)}{\sqrt{1+2e^{\frac{x}{\sqrt{x^2+1}}+c_1}}}$$
$$y(x) \to \frac{i\left(1+e^{\frac{x}{\sqrt{x^2+1}}+c_1}\right)}{\sqrt{1+2e^{\frac{x}{\sqrt{x^2+1}}+c_1}}}$$
$$y(x) \to -i$$
$$y(x) \to i$$

1.5 problem 1.5

Internal problem ID [4359]
Internal file name [`OUTPUT/3852_Sunday_June_05_2022_11_28_55_AM_73803550/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 1.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sin (x) \cos (y)-\cos (x) \sin (y) y'=0}$$

1.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{\sin (x) \cot (y)}{\cos (x)} \end{aligned}$$

Where $f(x)=\frac{\sin(x)}{\cos(x)}$ and $g(y)=\cot (y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cot (y)}\, dy &= \frac{\sin (x)}{\cos (x)}\, dx\\ \int \frac{1}{\cot (y)}\, dy &= \int \frac{\sin (x)}{\cos (x)}\, dx\\ -\ln (\cos (y)) &= -\ln (\cos (x))+c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = e^{-\ln(\cos(x))+c_1}$$

Which simplifies to

$$\sec(y) = \frac{c_2}{\cos(x)}$$

Summary

The solution(s) found are the following

$$y = \operatorname{arcsec}\left(\frac{c_2 e^{c_1}}{\cos(x)}\right) \tag{1}$$

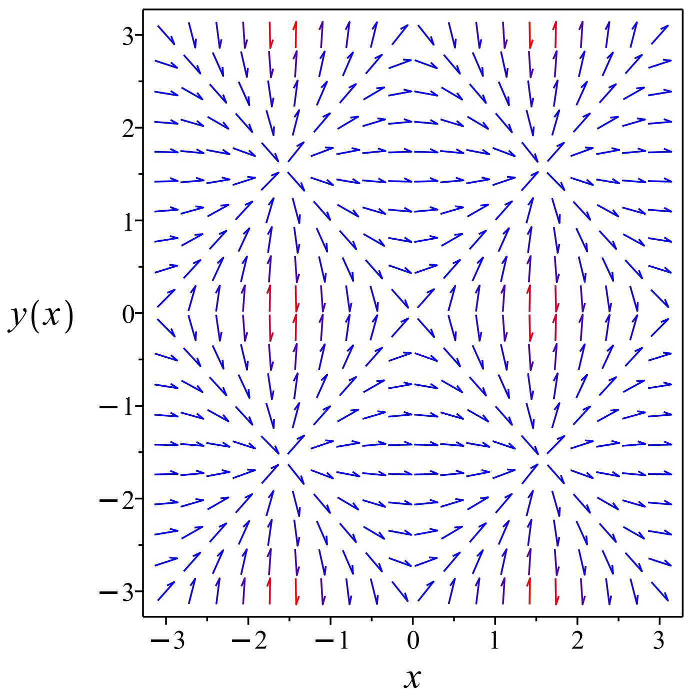


Figure 14: Slope field plot

Verification of solutions

$$y = \operatorname{arcsec}\left(\frac{c_2 e^{c_1}}{\cos(x)}\right)$$

Verified OK.

1.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sin(x)\cos(y)}{\cos(x)\sin(y)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 13: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \frac{\cos(x)}{\sin(x)} \\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{\cos(x)}{\sin(x)}} dx\end{aligned}$$

Which results in

$$S = -\ln(\cos(x))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sin(x)\cos(y)}{\cos(x)\sin(y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \tan(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \tan(y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \tan(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(\cos(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(\cos(x)) = -\ln(\cos(y)) + c_1
$$

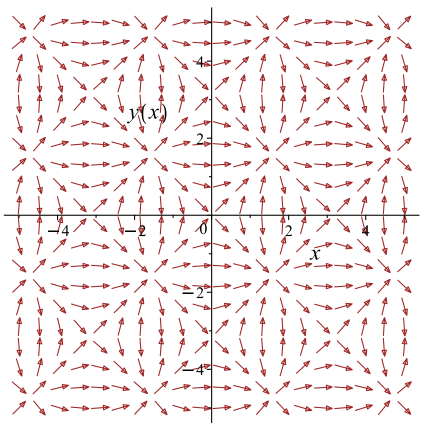
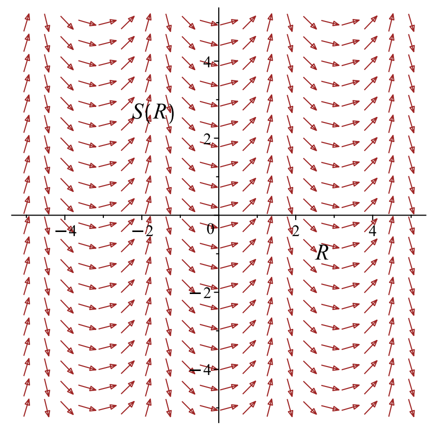
Which simplifies to

$$
-\ln(\cos(x)) = -\ln(\cos(y)) + c_1
$$

Which gives

$$
y = \arccos\left(\cos(x)\,\mathrm{e}^{c_1}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sin(x)\cos(y)}{\cos(x)\sin(y)}$ 	$R = y$ $S = -\ln(\cos(x))$	$\frac{dS}{dR} = \tan(R)$ 

Summary

The solution(s) found are the following

$$y = \arccos\left(\cos(x)\,\mathrm{e}^{c_1}\right) \tag{1}$$

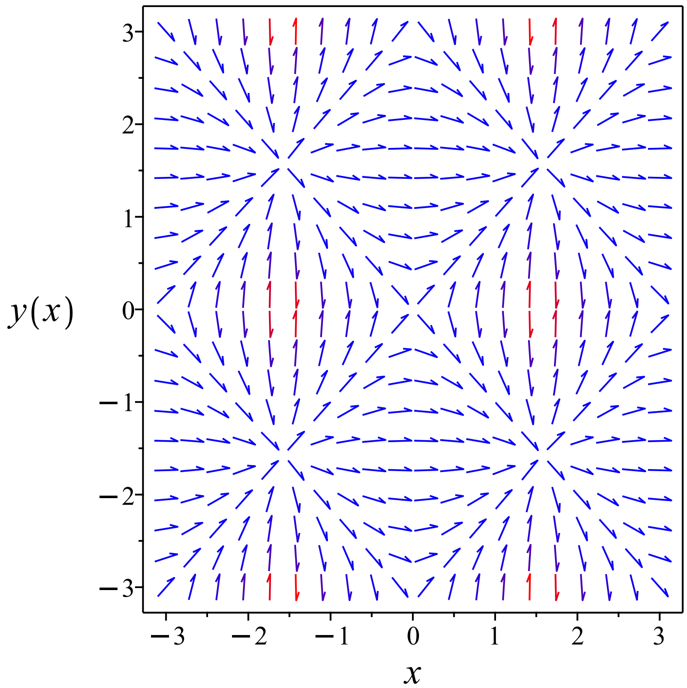


Figure 15: Slope field plot

Verification of solutions

$$y = \arccos\left(\cos\left(x\right) e^{c_1}\right)$$

Verified OK.

1.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y = \left(\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x$$
$$\left(-\frac{\sin(x)}{\cos(x)}\right)\mathrm{d}x + \left(\frac{\sin(y)}{\cos(y)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\sin(x)}{\cos(x)}$$
$$N(x, y) = \frac{\sin(y)}{\cos(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{\cos(x)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\sin(y)}{\cos(y)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\sin(x)}{\cos(x)}\,\mathrm{d}x\end{aligned}$$

$$\phi = \ln(\cos(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sin(y)}{\cos(y)}$. Therefore equation (4) becomes

$$\frac{\sin(y)}{\cos(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= \frac{\sin(y)}{\cos(y)}\\ &= \tan(y)\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\, \mathrm{d}y = \int (\tan (y))\, \mathrm{d}y$$
$$f(y) = -\ln (\cos (y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln (\cos (x)) - \ln (\cos (y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln (\cos (x)) - \ln (\cos (y))$$

Summary

The solution(s) found are the following

$$\ln (\cos (x)) - \ln (\cos (y)) = c_1 \tag{1}$$

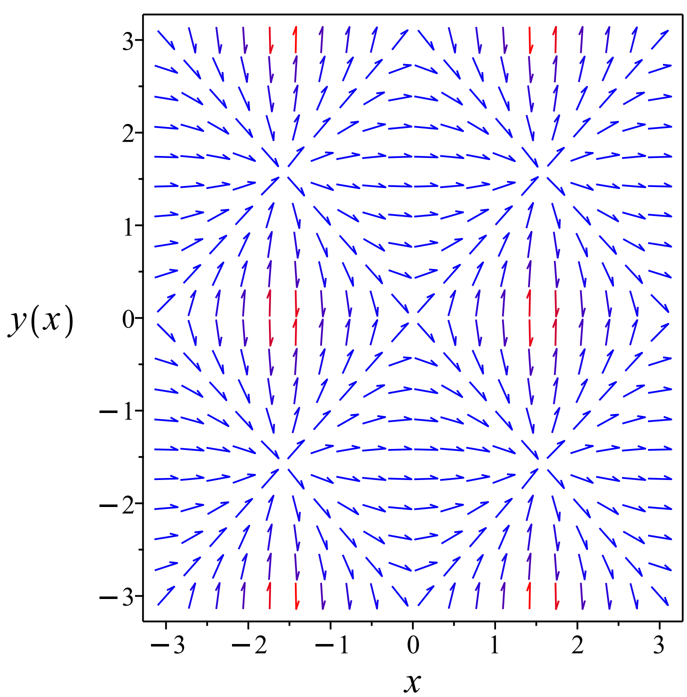


Figure 16: Slope field plot

Verification of solutions

$$\ln(\cos(x)) - \ln(\cos(y)) = c_1$$

Verified OK.

1.5.4 Maple step by step solution

Let's solve

$\sin(x)\cos(y) - \cos(x)\sin(y)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\sin(y)}{\cos(y)} = \frac{\sin(x)}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'\sin(y)}{\cos(y)}dx = \int \frac{\sin(x)}{\cos(x)}dx + c_1$

- Evaluate integral

  $-\ln(\cos(y)) = -\ln(\cos(x)) + c_1$

- Solve for y

  $y = \arccos\left(\frac{\cos(x)}{e^{c_1}}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 11

```
dsolve(sin(x)*cos(y(x))-cos(x)*sin(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{\cos(x)}{c_1}\right)$$

✓ Solution by Mathematica

Time used: 5.43 (sec). Leaf size: 47

```
DSolve[Sin[x]*Cos[y[x]]-Cos[x]*Sin[y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arccos\left(\frac{1}{2}c_1\cos(x)\right)$$
$$y(x) \to \arccos\left(\frac{1}{2}c_1\cos(x)\right)$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

1.6 problem 1.6

Internal problem ID [4360]
Internal file name [`OUTPUT/3853_Sunday_June_05_2022_11_29_03_AM_79424893/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 1.6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\sec(x)^2 \tan(y) + \sec(y)^2 \tan(x)\, y' = 0}$$

1.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\sec(x)^2 \sin(2y)}{2\tan(x)} \end{aligned}$$

Where $f(x) = -\frac{\sec(x)^2}{\tan(x)}$ and $g(y) = \frac{\sin(2y)}{2}$. Integrating both sides gives

$$\frac{1}{\frac{\sin(2y)}{2}}\, dy = -\frac{\sec(x)^2}{\tan(x)}\, dx$$

$$\int \frac{1}{\frac{\sin(2y)}{2}}\, dy = \int -\frac{\sec(x)^2}{\tan(x)}\, dx$$

$$\ln(\csc(2y) - \cot(2y)) = -\ln(\tan(x)) + c_1$$

Raising both side to exponential gives

$$\csc(2y) - \cot(2y) = e^{-\ln(\tan(x))+c_1}$$

Which simplifies to

$$\csc(2y) - \cot(2y) = \frac{c_2}{\tan(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\arctan\left(\frac{2c_2\tan(x)e^{c_1}}{e^{2c_1}c_2^2+\tan(x)^2}, -\frac{e^{2c_1}c_2^2-\tan(x)^2}{e^{2c_1}c_2^2+\tan(x)^2}\right)}{2} \tag{1}$$

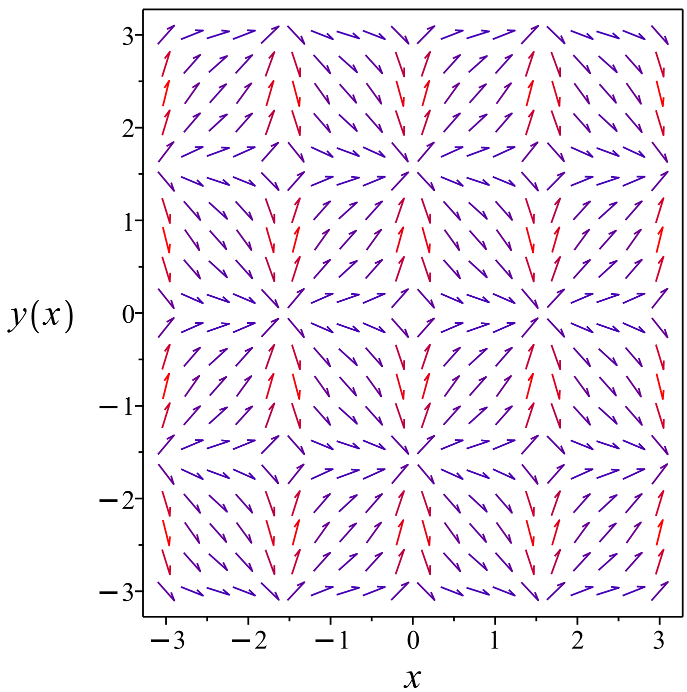


Figure 17: Slope field plot

Verification of solutions

$$y = \frac{\arctan\left(\frac{2c_2\tan(x)e^{c_1}}{e^{2c_1}c_2^2+\tan(x)^2}, -\frac{e^{2c_1}c_2^2-\tan(x)^2}{e^{2c_1}c_2^2+\tan(x)^2}\right)}{2}$$

Verified OK.

1.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\sec(x)^2 \tan(y)}{\sec(y)^2 \tan(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 16: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{\tan(x)}{\sec(x)^2}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{\tan(x)}{\sec(x)^2}} dx \end{aligned}$$

Which results in

$$S = -\ln(\tan(x))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sec(x)^2 \tan(y)}{\sec(y)^2 \tan(x)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\cot(x) - \tan(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \sec(y)\csc(y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \sec(R)\csc(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(\tan(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(\tan(x)) = \ln(\tan(y)) + c_1
$$

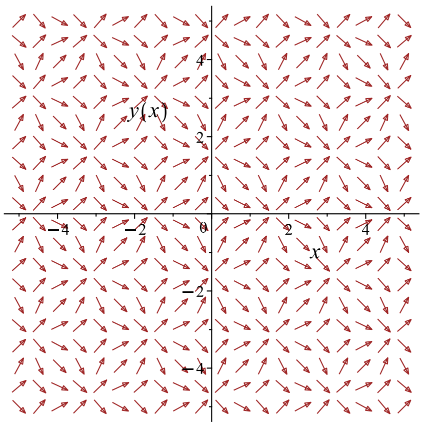
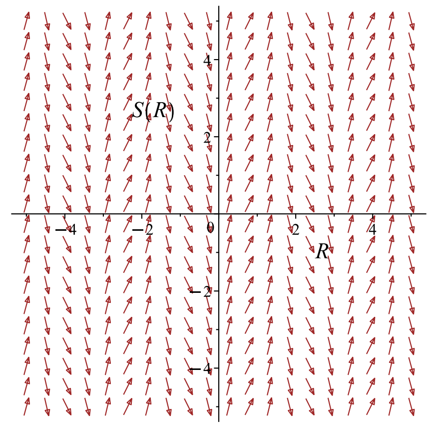
Which simplifies to

$$
-\ln(\tan(x)) = \ln(\tan(y)) + c_1
$$

Which gives

$$
y = \arctan\left(\frac{e^{-c_1}}{\tan(x)}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sec(x)^2 \tan(y)}{\sec(y)^2 \tan(x)}$ 	$R = y$ $S = -\ln(\tan(x))$	$\frac{dS}{dR} = \sec(R)\csc(R)$ 

Summary
The solution(s) found are the following

$$y = \arctan\left(\frac{e^{-c_1}}{\tan(x)}\right) \tag{1}$$

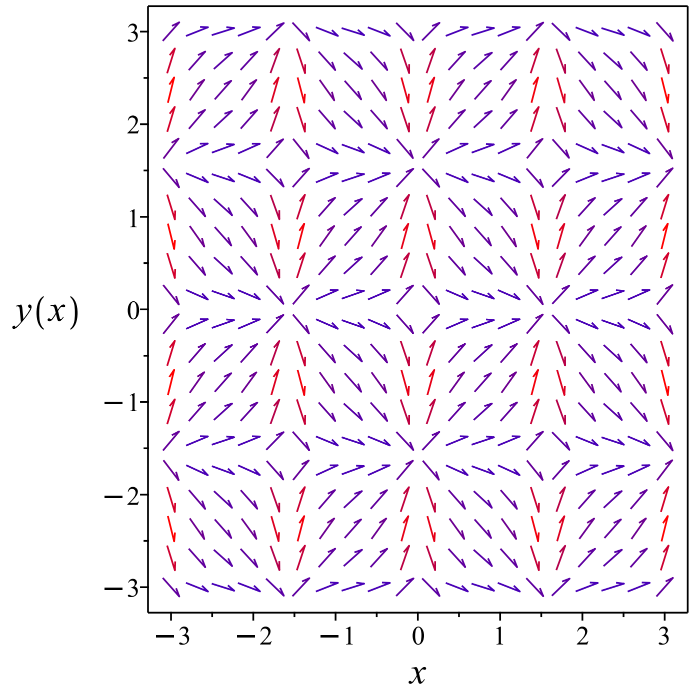


Figure 18: Slope field plot

Verification of solutions

$$y = \arctan\left(\frac{e^{-c_1}}{\tan(x)}\right)$$

Verified OK.

1.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{\sec(y)^2}{\tan(y)}\right)\mathrm{d}y = \left(\frac{\sec(x)^2}{\tan(x)}\right)\mathrm{d}x$$
$$\left(-\frac{\sec(x)^2}{\tan(x)}\right)\mathrm{d}x + \left(-\frac{\sec(y)^2}{\tan(y)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\sec(x)^2}{\tan(x)}$$
$$N(x, y) = -\frac{\sec(y)^2}{\tan(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\sec(x)^2}{\tan(x)}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\sec(y)^2}{\tan(y)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{\sec(x)^2}{\tan(x)} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(\tan(x)) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\sec(y)^2}{\tan(y)}$. Therefore equation (4) becomes

$$
-\frac{\sec(y)^2}{\tan(y)} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -\frac{\sec(y)^2}{\tan(y)} \\
&= -\sec(y)\csc(y)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\, dy = \int (-\sec(y)\csc(y))\, dy$$
$$f(y) = -\ln(\tan(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\tan(x)) - \ln(\tan(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\tan(x)) - \ln(\tan(y))$$

Summary

The solution(s) found are the following

$$-\ln(\tan(x)) - \ln(\tan(y)) = c_1 \tag{1}$$

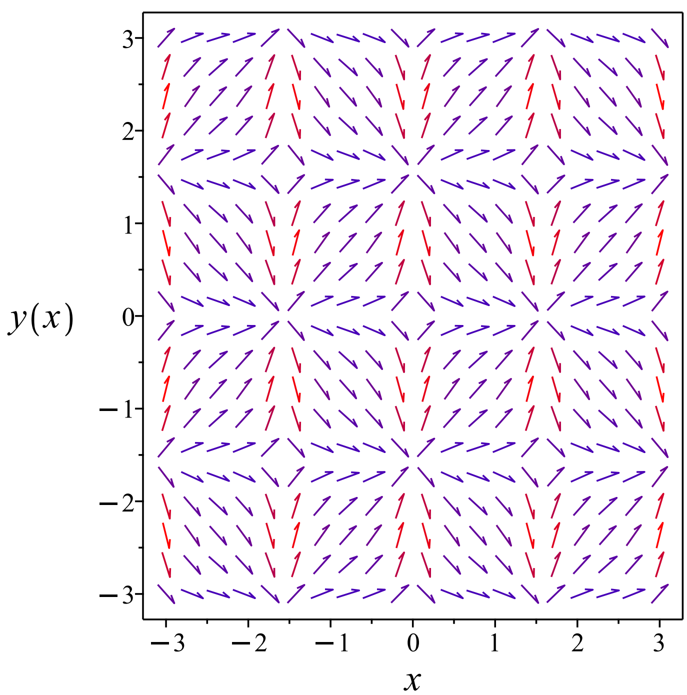


Figure 19: Slope field plot

Verification of solutions

$$-\ln(\tan(x)) - \ln(\tan(y)) = c_1$$

Verified OK.

1.6.4 Maple step by step solution

Let's solve

$\sec(x)^2 \tan(y) + \sec(y)^2 \tan(x) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(\sec(x)^2 \tan(y) + \sec(y)^2 \tan(x) y'\right) dx = \int 0 dx + c_1$

- Evaluate integral

  $\tan(y)\tan(x) = c_1$

- Solve for y

  $y = \arctan\left(\frac{c_1}{\tan(x)}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 47

```
dsolve(sec(x)^2*tan(y(x))+sec(y(x))^2*tan(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\arctan\left(\frac{2\tan(x)c_1}{c_1^2\tan(x)^2+1}, \frac{c_1^2\tan(x)^2-1}{c_1^2\tan(x)^2+1}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.457 (sec). Leaf size: 68

```
DSolve[Sec[x]^2*Tan[y[x]]+Sec[y[x]]^2*Tan[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -\frac{1}{2}\arccos(-\tanh(\operatorname{arctanh}(\cos(2x)) + 2c_1))$$
$$y(x) \to \frac{1}{2}\arccos(-\tanh(\operatorname{arctanh}(\cos(2x)) + 2c_1))$$
$$y(x) \to 0$$
$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{\pi}{2}$$

1.7 problem 3.1

Internal problem ID [4361]
Internal file name [`OUTPUT/3854_Sunday_June_05_2022_11_29_12_AM_27037630/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 3.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(-x+y)\,y'+y=0}$$

1.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(-x+u(x)\,x)\,(u'(x)\,x+u(x))+u(x)\,x=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2}{x\,(u-1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u-1}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u^2}{u-1}}\,du = \int -\frac{1}{x}\,dx$$
$$\ln(u) + \frac{1}{u} = -\ln(x) + c_2$$

The solution is

$$\ln(u(x)) + \frac{1}{u(x)} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{y}{x}\right) + \frac{x}{y} + \ln(x) - c_2 = 0$$
$$\ln\left(\frac{y}{x}\right) + \frac{x}{y} + \ln(x) - c_2 = 0$$

Summary
The solution(s) found are the following

$$\ln\left(\frac{y}{x}\right) + \frac{x}{y} + \ln(x) - c_2 = 0 \tag{1}$$

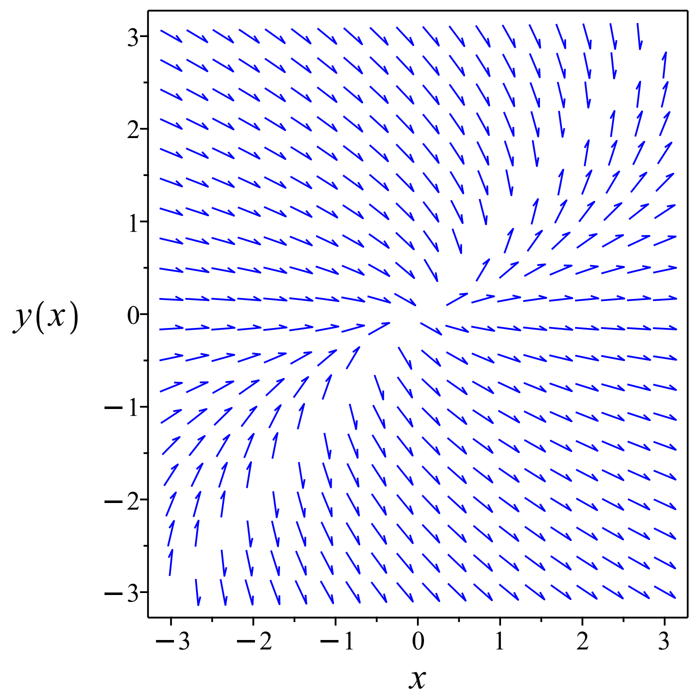


Figure 20: Slope field plot

Verification of solutions

$$\ln\left(\frac{y}{x}\right)+\frac{x}{y}+\ln\left(x\right)-c_2=0$$

Verified OK.

1.7.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y}{-x+y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(b_3 - a_2)}{-x + y} - \frac{y^2 a_3}{(-x + y)^2} + \frac{y(x a_2 + y a_3 + a_1)}{(-x + y)^2} - \left(-\frac{1}{-x + y} + \frac{y}{(-x + y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2xyb_2 - y^2 a_2 - y^2 b_2 + y^2 b_3 + x b_1 - y a_1}{(x - y)^2} = 0$$

Setting the numerator to zero gives

$$-2xyb_2 + y^2 a_2 + y^2 b_2 - y^2 b_3 - x b_1 + y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$a_2 v_2^2 - 2b_2 v_1 v_2 + b_2 v_2^2 - b_3 v_2^2 + a_1 v_2 - b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-2b_2 v_1 v_2 - b_1 v_1 + (a_2 + b_2 - b_3) v_2^2 + a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
-b_1 &= 0 \\
-2b_2 &= 0 \\
a_2 + b_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{y}{-x + y}\right)(x) \\
&= -\frac{y^2}{x - y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{y^2}{x-y}} dy \end{aligned}$$

Which results in

$$S = \ln(y) + \frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y}{-x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y} \\ S_y &= \frac{-x+y}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y \ln (y) + x}{y} = c_1$$

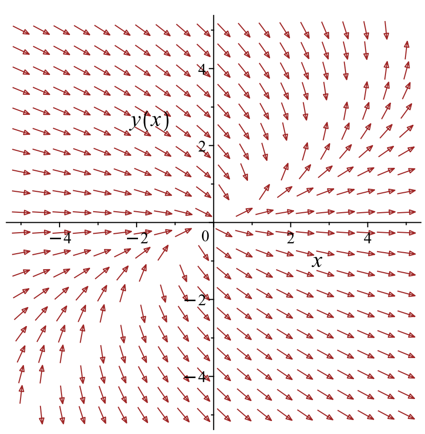
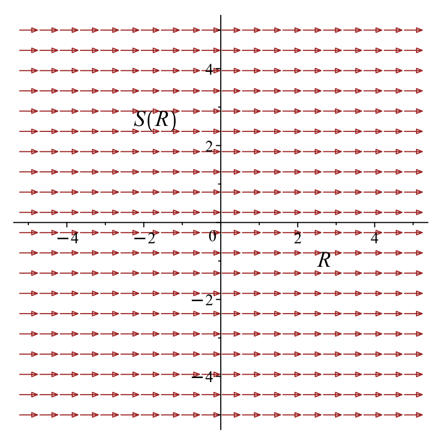
Which simplifies to

$$\frac{y \ln (y) + x}{y} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{-x+y}$ 	$R = x$ $S = \frac{\ln (y)\, y + x}{y}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1} \tag{1}$$

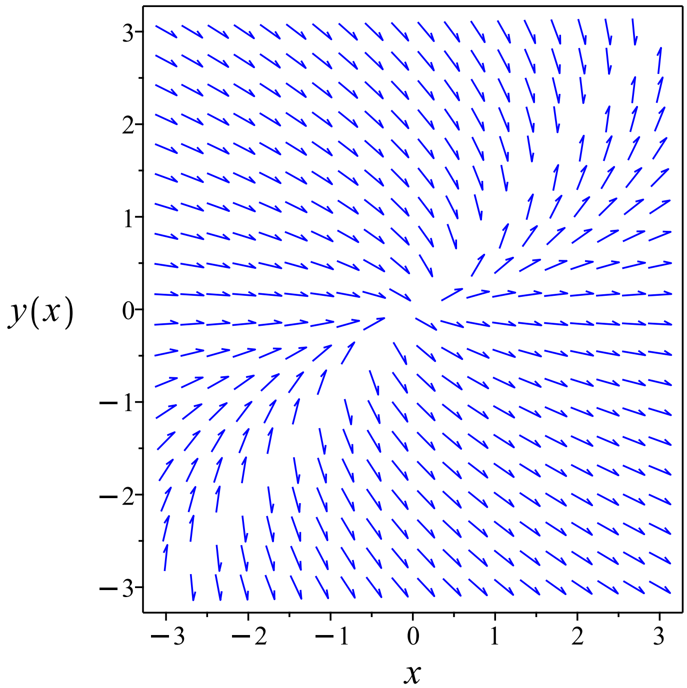


Figure 21: Slope field plot

Verification of solutions

$$y = e^{\text{LambertW}\left(-x\,e^{-c_1}\right)+c_1}$$

Verified OK.

1.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x+y)\,\mathrm{d}y &= (-y)\,\mathrm{d}x\\ (y)\,\mathrm{d}x + (-x+y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\\ N(x,y) &= -x+y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x+y)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{-x+y}((1) - (-1)) \\ &= -\frac{2}{x-y} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y}((-1) - (1)) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2}(y) \\ &= \frac{1}{y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^2}(-x+y) \\ &= \frac{-x+y}{y^2} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{1}{y}\right) + \left(\frac{-x+y}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{1}{y}\,\mathrm{d}x$$
$$\phi = \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+y}{y^2}$. Therefore equation (4) becomes

$$\frac{-x+y}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(y) + \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(y) + \frac{x}{y}$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)+c_1} \tag{1}$$

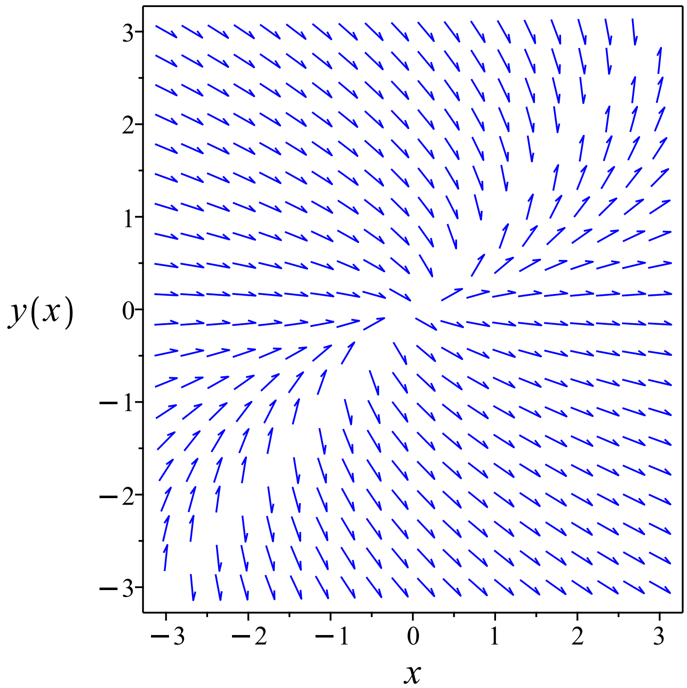


Figure 22: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}\left(-x\,e^{-c_1}\right)+c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve((y(x)-x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{\text{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)}$$

✓ Solution by Mathematica

Time used: 3.943 (sec). Leaf size: 25

```
DSolve[(y[x]-x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{W\left(-e^{-c_1}x\right)}$$
$$y(x) \to 0$$

1.8 problem 3.2

Internal problem ID [4362]
Internal file name [`OUTPUT/3855_Sunday_June_05_2022_11_29_20_AM_68775741/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 3.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{(2\sqrt{xy}-x)\,y'+y=0}$$

1.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y}{2\sqrt{xy}-x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(b_3 - a_2)}{2\sqrt{xy} - x} - \frac{y^2 a_3}{\left(2\sqrt{xy} - x\right)^2} - \frac{y\left(\frac{y}{\sqrt{xy}} - 1\right)(x a_2 + y a_3 + a_1)}{\left(2\sqrt{xy} - x\right)^2} - \left(-\frac{1}{2\sqrt{xy} - x} + \frac{yx}{\left(2\sqrt{xy} - x\right)^2 \sqrt{xy}}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{4(xy)^{\frac{3}{2}} b_2 - 3x^2 y b_2 + x\,y^2 a_2 - x\,y^2 b_3 - y^3 a_3 + xyb_1 - \sqrt{xy}\,x b_1 + \sqrt{xy}\,y a_1 - y^2 a_1}{\left(2\sqrt{xy} - x\right)^2 \sqrt{xy}} = 0$$

Setting the numerator to zero gives

$$4(xy)^{\frac{3}{2}} b_2 - 3x^2 y b_2 + x\,y^2 a_2 - x\,y^2 b_3 - y^3 a_3 + xyb_1 - \sqrt{xy}\,x b_1 + \sqrt{xy}\,y a_1 - y^2 a_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-3x^2 y b_2 + 4xy\sqrt{xy}\,b_2 + x\,y^2 a_2 - x\,y^2 b_3 - y^3 a_3 - \sqrt{xy}\,x b_1 + xyb_1 + \sqrt{xy}\,y a_1 - y^2 a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{xy}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{xy} = v_3\}$$

The above PDE (6E) now becomes

$$v_1 v_2^2 a_2 - v_2^3 a_3 - 3v_1^2 v_2 b_2 + 4v_1 v_2 v_3 b_2 - v_1 v_2^2 b_3 - v_2^2 a_1 + v_3 v_2 a_1 + v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-3v_1^2v_2b_2+(-b_3+a_2)\,v_1v_2^2+4v_1v_2v_3b_2+v_1v_2b_1-v_3v_1b_1-v_2^3a_3-v_2^2a_1+v_3v_2a_1=0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -3b_2 &= 0 \\ 4b_2 &= 0 \\ -b_3+a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y}{2\sqrt{xy}-x}\right)(x) \\ &= \frac{2y\sqrt{xy}}{2\sqrt{xy}-x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{\frac{2y\sqrt{xy}}{2\sqrt{xy}-x}}dy\end{aligned}$$

Which results in

$$S=\ln(y)+\frac{x}{\sqrt{xy}}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=-\frac{y}{2\sqrt{xy}-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=\frac{1}{2\sqrt{x}\,\sqrt{y}}\\S_y&=-\frac{-2\sqrt{y}+\sqrt{x}}{2y^{\frac{3}{2}}}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sqrt{x}\,\sqrt{y} - \sqrt{xy}}{\sqrt{x}\,\sqrt{y}\,\left(-2\sqrt{xy} + x\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

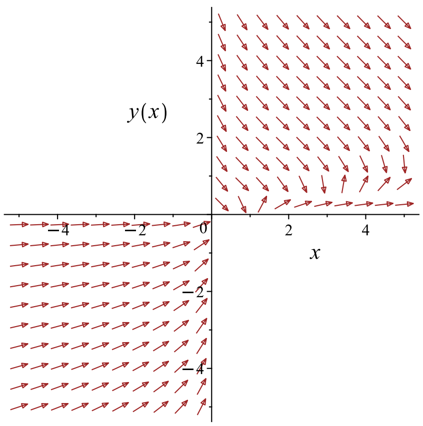
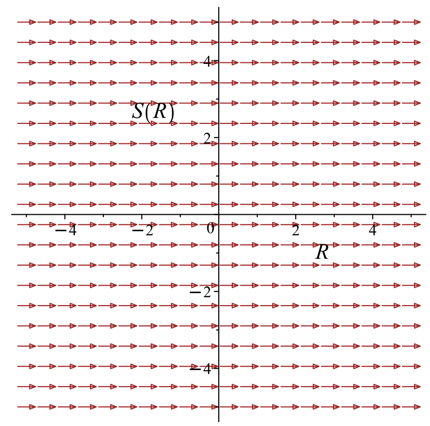
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(y\right)\sqrt{y} + \sqrt{x}}{\sqrt{y}} = c_1$$

Which simplifies to

$$\frac{\ln\left(y\right)\sqrt{y} + \sqrt{x}}{\sqrt{y}} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{2\sqrt{xy}-x}$ 	$R = x$ $S = \frac{\ln(y)\sqrt{y}+\sqrt{x}}{\sqrt{y}}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(y)\sqrt{y}+\sqrt{x}}{\sqrt{y}} = c_1 \tag{1}$$

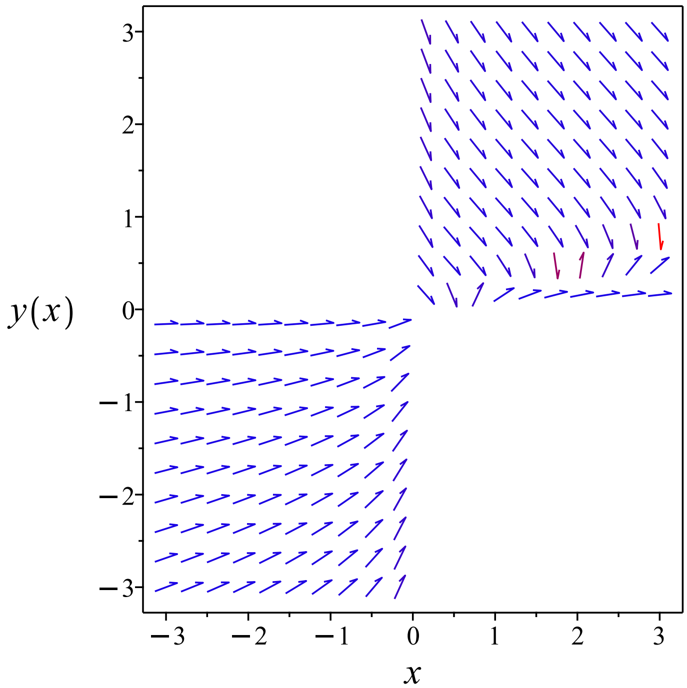


Figure 23: Slope field plot

Verification of solutions

$$\frac{\ln(y)\sqrt{y}+\sqrt{x}}{\sqrt{y}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve((2*sqrt(x*y(x))-x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$\ln(y(x)) + \frac{x}{\sqrt{xy(x)}} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.244 (sec). Leaf size: 33

```
DSolve[(2*Sqrt[x*y[x]]-x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2}{\sqrt{\frac{y(x)}{x}}} + 2\log\left(\frac{y(x)}{x}\right) = -2\log(x) + c_1, y(x)\right]$$

1.9 problem 3.3

Internal problem ID [4363]
Internal file name [`OUTPUT/3856_Sunday_June_05_2022_11_29_30_AM_82276429/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 3.3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy' - y - \sqrt{x^2 + y^2} = 0}$$

1.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y + \sqrt{x^2 + y^2}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{\left(y+\sqrt{x^2+y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{x^2+y^2}\right)^2 a_3}{x^2} \\
& - \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{y+\sqrt{x^2+y^2}}{x^2}\right)(x a_2 + y a_3 + a_1) \\
& - \frac{\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)(x b_2 + y b_3 + b_1)}{x} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{\left(x^2+y^2\right)^{\frac{3}{2}} a_3 + x^3 a_2 - x^3 b_3 + 2x^2 y a_3 + x^2 y b_2 + y^3 a_3 + \sqrt{x^2+y^2}\, x b_1 - \sqrt{x^2+y^2}\, y a_1 + x y b_1 - y^2 a_1}{\sqrt{x^2+y^2}\, x^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -\left(x^2+y^2\right)^{\frac{3}{2}} a_3 - x^3 a_2 + x^3 b_3 - 2x^2 y a_3 - x^2 y b_2 - y^3 a_3 \\
& - \sqrt{x^2+y^2}\, x b_1 + \sqrt{x^2+y^2}\, y a_1 - x y b_1 + y^2 a_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
& -\left(x^2+y^2\right)^{\frac{3}{2}} a_3 + \left(x^2+y^2\right) x b_3 - \left(x^2+y^2\right) y a_3 - x^3 a_2 - x^2 y a_3 - x^2 y b_2 \\
& - x y^2 b_3 + \left(x^2+y^2\right) a_1 - \sqrt{x^2+y^2}\, x b_1 + \sqrt{x^2+y^2}\, y a_1 - x^2 a_1 - x y b_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
& -x^3 a_2 + x^3 b_3 - x^2 \sqrt{x^2+y^2}\, a_3 - 2x^2 y a_3 - x^2 y b_2 - \sqrt{x^2+y^2}\, y^2 a_3 \\
& - y^3 a_3 - \sqrt{x^2+y^2}\, x b_1 - x y b_1 + \sqrt{x^2+y^2}\, y a_1 + y^2 a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^2+y^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-v_1^3 a_2 - 2v_1^2 v_2 a_3 - v_1^2 v_3 a_3 - v_2^3 a_3 - v_3 v_2^2 a_3 - v_1^2 v_2 b_2 \\ &+ v_1^3 b_3 + v_2^2 a_1 + v_3 v_2 a_1 - v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &(b_3 - a_2) v_1^3 + (-2a_3 - b_2) v_1^2 v_2 - v_1^2 v_3 a_3 - v_1 v_2 b_1 \\ &- v_3 v_1 b_1 - v_2^3 a_3 - v_3 v_2^2 a_3 + v_2^2 a_1 + v_3 v_2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2a_3 - b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(\frac{y + \sqrt{x^2 + y^2}}{x}\right)(x) \\ &= -\sqrt{x^2 + y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\sqrt{x^2 + y^2}} dy\end{aligned}$$

Which results in

$$S = -\ln\left(y + \sqrt{x^2 + y^2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + \sqrt{x^2 + y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x}{\sqrt{x^2 + y^2}\,\left(y + \sqrt{x^2 + y^2}\right)} \\
S_y &= -\frac{1}{\sqrt{x^2 + y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2\left(\sqrt{x^2 + y^2}\, y + x^2 + y^2\right)}{x\sqrt{x^2 + y^2}\,\left(y + \sqrt{x^2 + y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y + \sqrt{x^2 + y^2}\right) = -2\ln(x) + c_1$$

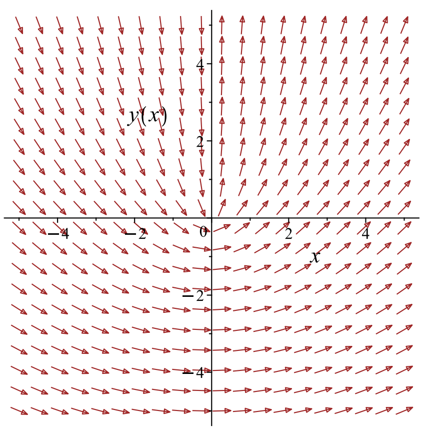
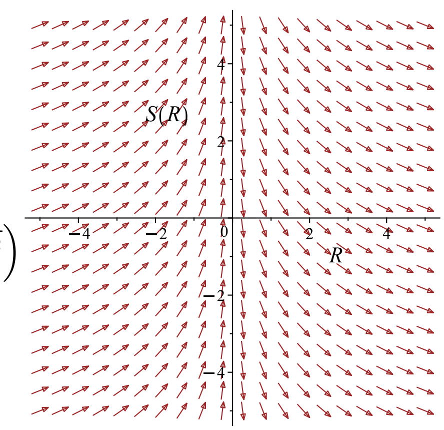
Which simplifies to

$$-\ln\left(y + \sqrt{x^2 + y^2}\right) = -2\ln(x) + c_1$$

Which gives

$$y = -\frac{\mathrm{e}^{-c_1}\left(\mathrm{e}^{2c_1} - x^2\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ 	$R = x$ $S = -\ln\left(y + \sqrt{x^2 + y^2}\right)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2} \tag{1}$$

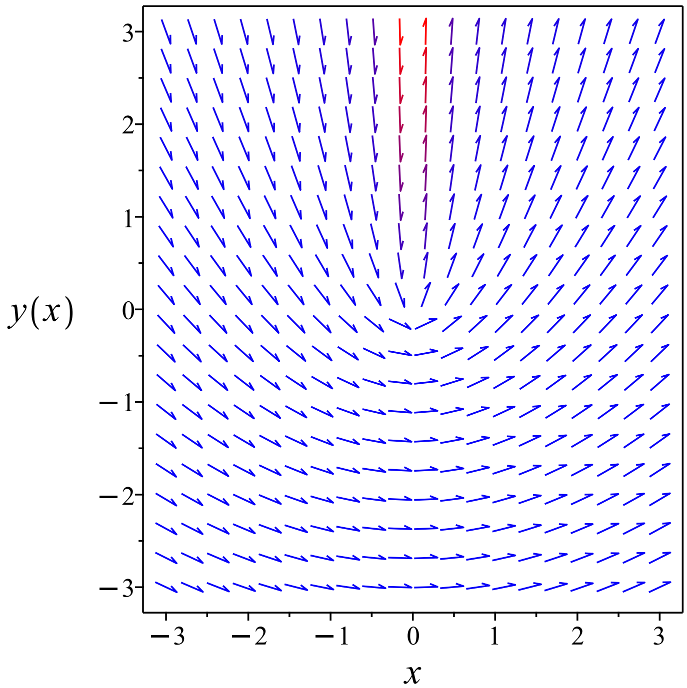


Figure 24: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_1}(e^{2c_1} - x^2)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x*diff(y(x),x)-y(x)-sqrt(x^2+y(x)^2)=0,y(x), singsol=all)
```

$$\frac{-c_1 x^2 + \sqrt{x^2 + y(x)^2} + y(x)}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.327 (sec). Leaf size: 27

```
DSolve[x*y'[x]-y[x]-Sqrt[x^2+y[x]^2]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-c_1} \left(-1 + e^{2c_1} x^2\right)$$

1.10 problem 3.4

Internal problem ID [4364]
Internal file name [`OUTPUT/3857_Sunday_June_05_2022_11_29_40_AM_80945842/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 3.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{-y\cos\left(\frac{y}{x}\right)+x\cos\left(\frac{y}{x}\right)y'=-x}$$

1.10.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y'=\frac{y}{x}-\frac{1}{\cos\left(\frac{y}{x}\right)} \tag{A}$$

The given ode has the form

$$y'=\frac{y}{x}+g(x)\,f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x)=u(x)\,x$ then

$$\frac{dy}{dx}=\frac{du}{dx}x+u$$

Hence the given ode becomes

$$\begin{aligned}\frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\ u' &= \frac{1}{x} g(x)\, f(bu)^{\frac{n}{m}} \qquad (2)\end{aligned}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$\begin{aligned}g(x) &= -1 \\ b &= 1 \\ f\left(\frac{bx}{y}\right) &= \cos\left(\frac{y}{x}\right)\end{aligned}$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x) = -\frac{1}{x \cos(u(x))}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{\sec(u)}{x}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \sec(u)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sec(u)}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{\sec(u)}\, du &= \int -\frac{1}{x}\, dx \\ \sin(u) &= -\ln(x) + c_1\end{aligned}$$

The solution is

$$\sin(u(x)) + \ln(x) - c_1 = 0$$

Therefore the solution is found using $y = ux$. Hence

$$\sin\left(\frac{y}{x}\right) + \ln(x) - c_1 = 0$$

Summary
The solution(s) found are the following

$$\sin\left(\frac{y}{x}\right)+\ln\left(x\right)-c_1=0 \tag{1}$$

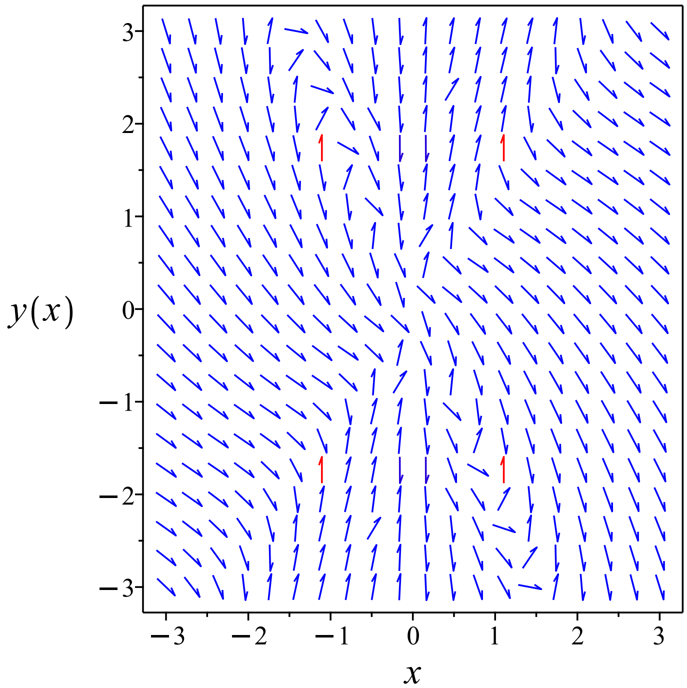


Figure 25: Slope field plot

Verification of solutions

$$\sin\left(\frac{y}{x}\right)+\ln\left(x\right)-c_1=0$$

Verified OK.

1.10.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)\,x\cos\left(u(x)\right)+x\cos\left(u(x)\right)\left(u'(x)\,x+u(x)\right)=-x$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{\sec\left(u\right)}{x}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \sec(u)$. Integrating both sides gives

$$\frac{1}{\sec(u)}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\sec(u)}\, du = \int -\frac{1}{x}\, dx$$

$$\sin(u) = -\ln(x) + c_2$$

The solution is

$$\sin(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sin\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\sin\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

<u>Summary</u>

The solution(s) found are the following

$$\sin\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

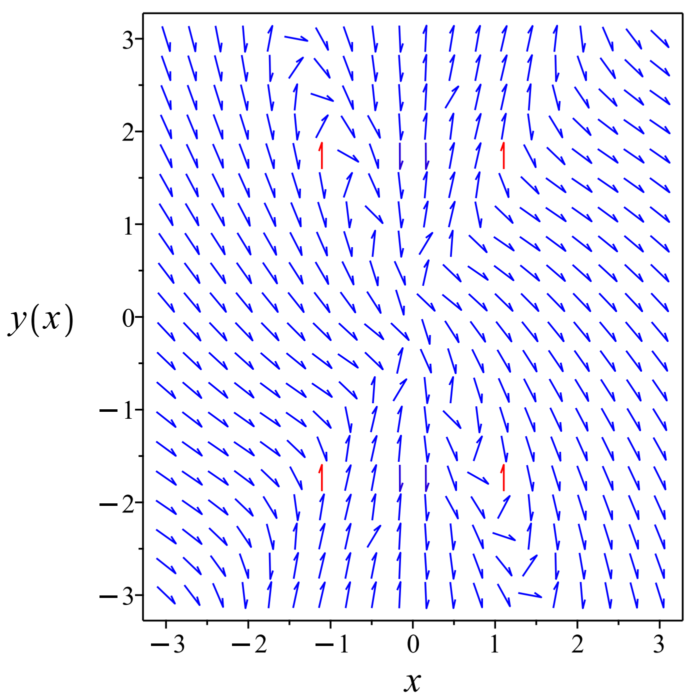


Figure 26: Slope field plot

Verification of solutions

$$\sin\left(\frac{y}{x}\right)+\ln\left(x\right)-c_2=0$$

Verified OK.

1.10.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=\frac{-x+y\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 19: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= x^2 \\ \eta(x,y) &= xy \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{xy}{x^2} \\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x + y\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x^2} \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\cos\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)\, S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{\sin(R)} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{\sin\left(\frac{y}{x}\right)}$$

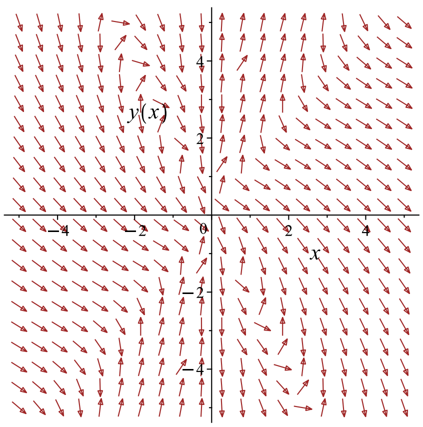
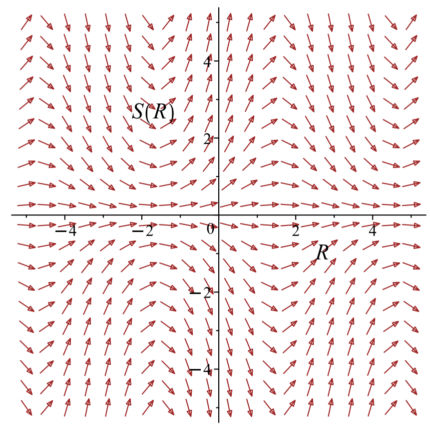
Which simplifies to

$$-\frac{1}{x} = c_1 e^{\sin\left(\frac{y}{x}\right)}$$

Which gives

$$y = \arcsin\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+y\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = \cos(R)\, S(R)$ 

Summary

The solution(s) found are the following

$$y = \arcsin\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x \tag{1}$$

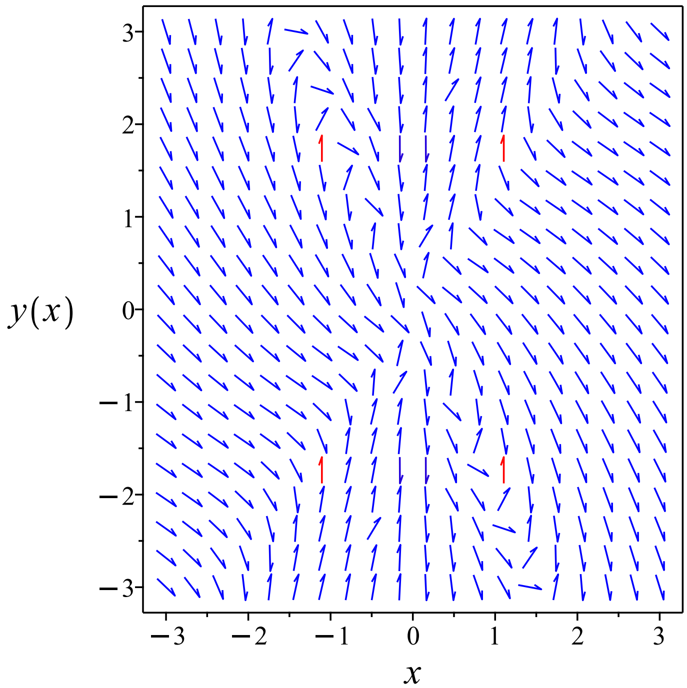


Figure 27: Slope field plot

Verification of solutions

$$y = \arcsin\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

Verified OK.

1.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x \cos\left(\frac{y}{x}\right)\right) \mathrm{d}y = \left(-x + y \cos\left(\frac{y}{x}\right)\right) \mathrm{d}x$$
$$\left(x - y \cos\left(\frac{y}{x}\right)\right) \mathrm{d}x + \left(x \cos\left(\frac{y}{x}\right)\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x - y \cos\left(\frac{y}{x}\right)$$
$$N(x, y) = x \cos\left(\frac{y}{x}\right)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x - y \cos\left(\frac{y}{x}\right)\right) \\ &= -\cos\left(\frac{y}{x}\right) + \frac{y \sin\left(\frac{y}{x}\right)}{x} \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\cos\left(\frac{y}{x}\right)\right)\\ &= \cos\left(\frac{y}{x}\right) + \frac{y\sin\left(\frac{y}{x}\right)}{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{\sec\left(\frac{y}{x}\right)}{x}\left(\left(-\cos\left(\frac{y}{x}\right) + \frac{y\sin\left(\frac{y}{x}\right)}{x}\right) - \left(\cos\left(\frac{y}{x}\right) + \frac{y\sin\left(\frac{y}{x}\right)}{x}\right)\right)\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(x - y\cos\left(\frac{y}{x}\right)\right)\\ &= \frac{x - y\cos\left(\frac{y}{x}\right)}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}\left(x\cos\left(\frac{y}{x}\right)\right)\\ &= \frac{\cos\left(\frac{y}{x}\right)}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{x - y\cos\left(\frac{y}{x}\right)}{x^2}\right) + \left(\frac{\cos\left(\frac{y}{x}\right)}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{x - y\cos\left(\frac{y}{x}\right)}{x^2}\,\mathrm{d}x$$
$$\phi = \sin\left(\frac{y}{x}\right) - \ln\left(\frac{1}{x}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\cos\left(\frac{y}{x}\right)}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\cos\left(\frac{y}{x}\right)}{x}$. Therefore equation (4) becomes

$$\frac{\cos\left(\frac{y}{x}\right)}{x} = \frac{\cos\left(\frac{y}{x}\right)}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin\left(\frac{y}{x}\right) - \ln\left(\frac{1}{x}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin\left(\frac{y}{x}\right) - \ln\left(\frac{1}{x}\right)$$

Summary

The solution(s) found are the following

$$\sin\left(\frac{y}{x}\right) - \ln\left(\frac{1}{x}\right) = c_1 \tag{1}$$

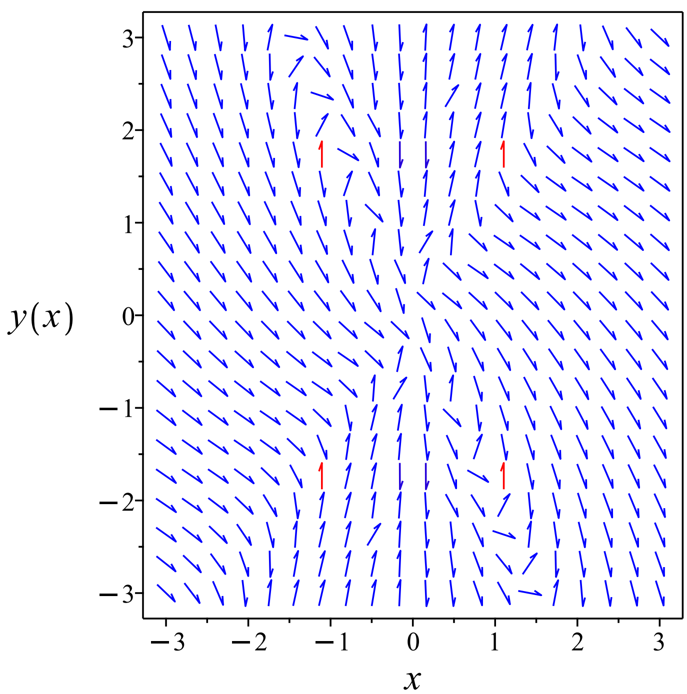


Figure 28: Slope field plot

Verification of solutions

$$\sin\left(\frac{y}{x}\right) - \ln\left(\frac{1}{x}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-y(x)*cos(y(x)/x))+x*cos(y(x)/x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\arcsin\left(\ln\left(x\right) + c_1\right) x$$

✓ Solution by Mathematica

Time used: 0.359 (sec). Leaf size: 15

```
DSolve[(x-y[x]*Cos[y[x]/x])+x*Cos[y[x]/x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin(-\log(x) + c_1)$$

1.11 problem 3.5

Internal problem ID [4365]
Internal file name [`OUTPUT/3858_Sunday_June_05_2022_11_29_49_AM_24603019/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 3.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{8y + (5y + 7x)\, y' = -10x}$$

1.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$8u(x)\, x + (5u(x)\, x + 7x)\, (u'(x)\, x + u(x)) = -10x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{5(u^2 + 3u + 2)}{x\,(5u + 7)} \end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = \frac{u^2+3u+2}{5u+7}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+3u+2}{5u+7}}\, du = -\frac{5}{x}\, dx$$

$$\int \frac{1}{\frac{u^2+3u+2}{5u+7}} du = \int -\frac{5}{x} dx$$

$$2\ln(u+1) + 3\ln(u+2) = -5\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u+1)+3\ln(u+2)} = e^{-5\ln(x)+c_2}$$

Which simplifies to

$$(u+1)^2 (u+2)^3 = \frac{c_3}{x^5}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= \mathrm{RootOf}\left(\_Z^5 + 8x\_Z^4 + 25x^2\_Z^3 + 38x^3\_Z^2 + 28x^4\_Z + 8x^5 - c_3\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{RootOf}\left(\_Z^5 + 8x\_Z^4 + 25x^2\_Z^3 + 38x^3\_Z^2 + 28x^4\_Z + 8x^5 - c_3\right) \tag{1}$$

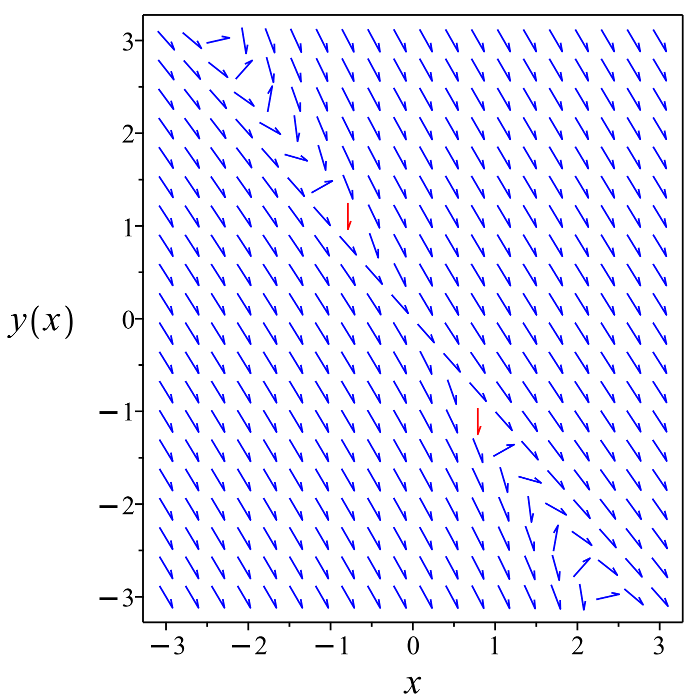


Figure 29: Slope field plot

Verification of solutions

$$y = \operatorname{RootOf}\left(\_Z^5 + 8x\_Z^4 + 25x^2\_Z^3 + 38x^3\_Z^2 + 28x^4\_Z + 8x^5 - c_3\right)$$

Verified OK.

1.11.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2(4y+5x)}{5y+7x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{2(4y+5x)(b_3-a_2)}{5y+7x} - \frac{4(4y+5x)^2 a_3}{(5y+7x)^2} \\ &- \left(-\frac{10}{5y+7x} + \frac{56y+70x}{(5y+7x)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(-\frac{8}{5y+7x} + \frac{40y+50x}{(5y+7x)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{70x^2a_2 - 100x^2a_3 + 55x^2b_2 - 70x^2b_3 + 100xya_2 - 160xya_3 + 70xyb_2 - 100xyb_3 + 40y^2a_2 - 70y^2a_3 + 25y}{(5y+7x)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &70x^2a_2 - 100x^2a_3 + 55x^2b_2 - 70x^2b_3 + 100xya_2 - 160xya_3 + 70xyb_2 \\ &- 100xyb_3 + 40y^2a_2 - 70y^2a_3 + 25y^2b_2 - 40y^2b_3 + 6xb_1 - 6ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &70a_2v_1^2 + 100a_2v_1v_2 + 40a_2v_2^2 - 100a_3v_1^2 - 160a_3v_1v_2 - 70a_3v_2^2 + 55b_2v_1^2 \\ &+ 70b_2v_1v_2 + 25b_2v_2^2 - 70b_3v_1^2 - 100b_3v_1v_2 - 40b_3v_2^2 - 6a_1v_2 + 6b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(70a_2 - 100a_3 + 55b_2 - 70b_3)\, v_1^2 + (100a_2 - 160a_3 + 70b_2 - 100b_3)\, v_1v_2 \\ &+ 6b_1v_1 + (40a_2 - 70a_3 + 25b_2 - 40b_3)\, v_2^2 - 6a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -6a_1 &= 0 \\ 6b_1 &= 0 \\ 40a_2 - 70a_3 + 25b_2 - 40b_3 &= 0 \\ 70a_2 - 100a_3 + 55b_2 - 70b_3 &= 0 \\ 100a_2 - 160a_3 + 70b_2 - 100b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 3a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= -2a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{2(4y + 5x)}{5y + 7x}\right)(x) \\
&= \frac{10x^2 + 15xy + 5y^2}{5y + 7x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{10x^2 + 15xy + 5y^2}{5y + 7x}} dy
\end{aligned}
$$

Which results in

$$S = \frac{2 \ln (x + y)}{5} + \frac{3 \ln (2x + y)}{5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2(4y + 5x)}{5y + 7x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x + \frac{8y}{5}}{(x + y)(2x + y)} \\
S_y &= \frac{5y + 7x}{5(x + y)(2x + y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2 \ln (x + y)}{5} + \frac{3 \ln (2x + y)}{5} = c_1$$

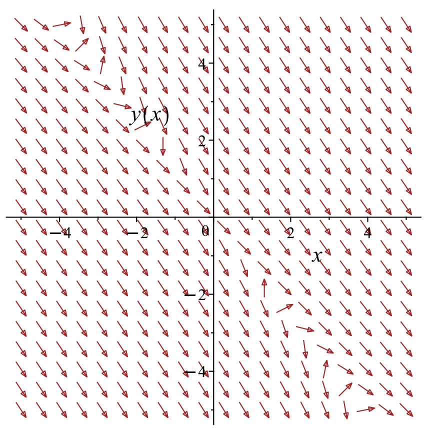
Which simplifies to

$$\frac{2 \ln (x+y)}{5}+\frac{3 \ln (2 x+y)}{5}=c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{2(4y+5x)}{5y+7x}$



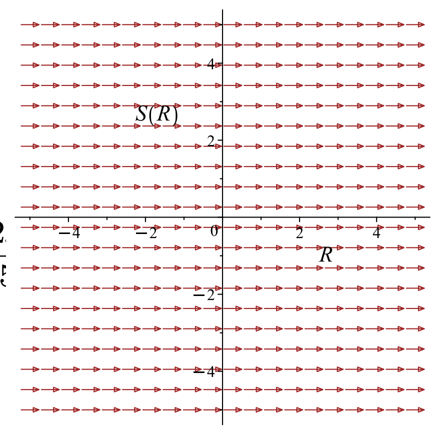
</td><td>

$R = x$

$S = \frac{2 \ln (x+y)}{5} + \frac{3 \ln (2}{5}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{2 \ln (x+y)}{5}+\frac{3 \ln (2 x+y)}{5}=c_1 \tag{1}$$

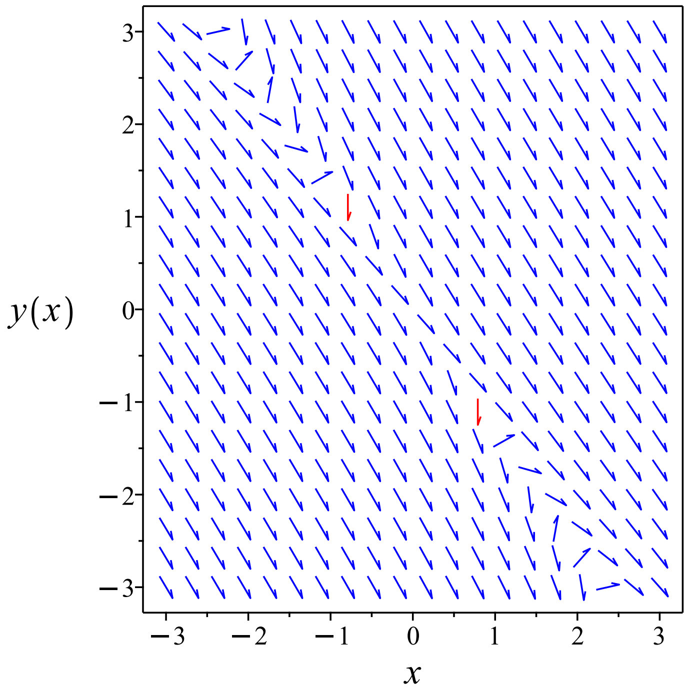


Figure 30: Slope field plot

Verification of solutions

$$\frac{2\ln(x+y)}{5}+\frac{3\ln(2x+y)}{5}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 38

```
dsolve((8*y(x)+10*x)+(5*y(x)+7*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x\left(\operatorname{RootOf}\left(\_Z^{25}c_1x^5 - 2\_Z^{20}c_1x^5 + \_Z^{15}c_1x^5 - 1\right)^5 - 2\right)$$

✓ Solution by Mathematica

Time used: 2.163 (sec). Leaf size: 276

```
DSolve[(8*y[x]+10*x)+(5*y[x]+7*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \operatorname{Root}\left[\#1^5 + 8\#1^4x + 25\#1^3x^2 + 38\#1^2x^3 + 28\#1x^4 + 8x^5 - e^{c_1}\&, 1\right]$$
$$y(x) \to \operatorname{Root}\left[\#1^5 + 8\#1^4x + 25\#1^3x^2 + 38\#1^2x^3 + 28\#1x^4 + 8x^5 - e^{c_1}\&, 2\right]$$
$$y(x) \to \operatorname{Root}\left[\#1^5 + 8\#1^4x + 25\#1^3x^2 + 38\#1^2x^3 + 28\#1x^4 + 8x^5 - e^{c_1}\&, 3\right]$$
$$y(x) \to \operatorname{Root}\left[\#1^5 + 8\#1^4x + 25\#1^3x^2 + 38\#1^2x^3 + 28\#1x^4 + 8x^5 - e^{c_1}\&, 4\right]$$
$$y(x) \to \operatorname{Root}\left[\#1^5 + 8\#1^4x + 25\#1^3x^2 + 38\#1^2x^3 + 28\#1x^4 + 8x^5 - e^{c_1}\&, 5\right]$$

1.12 problem 4.1

Internal problem ID [4366]
Internal file name [`OUTPUT/3859_Sunday_June_05_2022_11_30_00_AM_7213669/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 4.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y + (2y - 1)\, y' = -2x - 1}$$

1.12.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-2X - 2x_0 + Y(X) + y_0 - 1}{2Y(X) + 2y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -\frac{1}{4}$$
$$y_0 = \frac{1}{2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-2X + Y(X)}{2Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{-2X + Y}{2Y} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -2X+Y$ and $N = 2Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= -\frac{1}{u} + \frac{1}{2} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-\frac{1}{u(X)} + \frac{1}{2} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{-\frac{1}{u(X)} + \frac{1}{2} - u(X)}{X} = 0
$$

Or

$$
2\left(\frac{d}{dX}u(X)\right)u(X)\,X + 2u(X)^2 - u(X) + 2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X)g(u) \\
&= -\frac{2u^2 - u + 2}{2uX}
\end{aligned}
$$

Where $f(X) = -\frac{1}{2X}$ and $g(u) = \frac{2u^2-u+2}{u}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2-u+2}{u}}\, du = -\frac{1}{2X}\, dX$$
$$\int \frac{1}{\frac{2u^2-u+2}{u}}\, du = \int -\frac{1}{2X}\, dX$$
$$\frac{\ln\left(2u^2-u+2\right)}{4} + \frac{\sqrt{15}\, \arctan\left(\frac{(4u-1)\sqrt{15}}{15}\right)}{30} = -\frac{\ln\left(X\right)}{2} + c_2$$

The solution is

$$\frac{\ln\left(2u(X)^2-u(X)+2\right)}{4} + \frac{\sqrt{15}\, \arctan\left(\frac{(4u(X)-1)\sqrt{15}}{15}\right)}{30} + \frac{\ln\left(X\right)}{2} - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{2Y(X)^2}{X^2} - \frac{Y(X)}{X} + 2\right)}{4} + \frac{\sqrt{15}\, \arctan\left(\frac{\left(\frac{4Y(X)}{X}-1\right)\sqrt{15}}{15}\right)}{30} + \frac{\ln\left(X\right)}{2} - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{2Y(X)^2}{X^2} - \frac{Y(X)}{X} + 2\right)}{4} - \frac{\sqrt{15}\, \arctan\left(\frac{(-4Y(X)+X)\sqrt{15}}{15X}\right)}{30} + \frac{\ln\left(X\right)}{2} - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + \frac{1}{2}$$
$$X = x - \frac{1}{4}$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{2\left(y-\frac{1}{2}\right)^2}{\left(x+\frac{1}{4}\right)^2} - \frac{y-\frac{1}{2}}{x+\frac{1}{4}} + 2\right)}{4} - \frac{\sqrt{15}\, \arctan\left(\frac{\left(-4y+\frac{9}{4}+x\right)\sqrt{15}}{15x+\frac{15}{4}}\right)}{30} + \frac{\ln\left(x+\frac{1}{4}\right)}{2} - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{2\left(y-\frac{1}{2}\right)^2}{\left(x+\frac{1}{4}\right)^2}-\frac{y-\frac{1}{2}}{x+\frac{1}{4}}+2\right)}{4}-\frac{\sqrt{15}\,\arctan\left(\frac{\left(-4y+\frac{9}{4}+x\right)\sqrt{15}}{15x+\frac{15}{4}}\right)}{30}+\frac{\ln\left(x+\frac{1}{4}\right)}{2}-c_2=0 \quad (1)$$

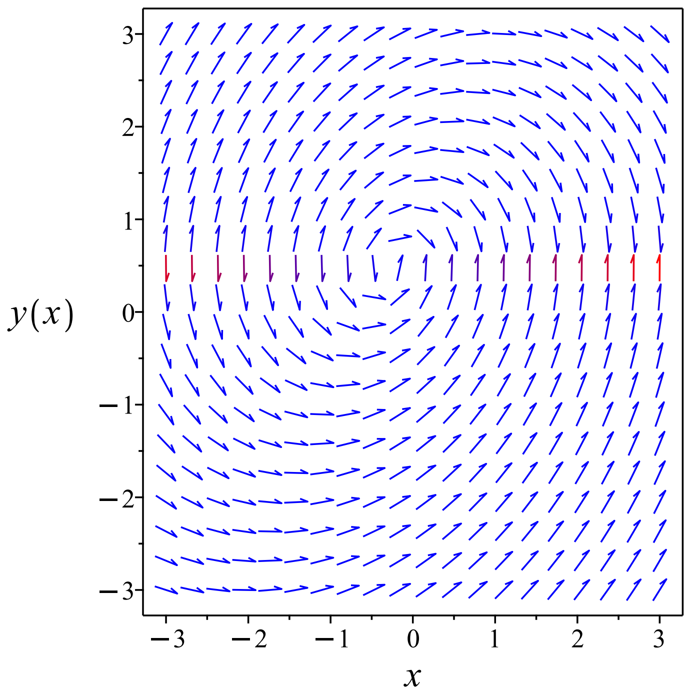


Figure 31: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{2\left(y-\frac{1}{2}\right)^2}{\left(x+\frac{1}{4}\right)^2}-\frac{y-\frac{1}{2}}{x+\frac{1}{4}}+2\right)}{4}-\frac{\sqrt{15}\,\arctan\left(\frac{\left(-4y+\frac{9}{4}+x\right)\sqrt{15}}{15x+\frac{15}{4}}\right)}{30}+\frac{\ln\left(x+\frac{1}{4}\right)}{2}-c_2=0$$

Verified OK.

1.12.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-2x + y - 1}{2y - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-2x + y - 1)(b_3 - a_2)}{2y - 1} - \frac{(-2x + y - 1)^2 a_3}{(2y - 1)^2} + \frac{2x a_2 + 2y a_3 + 2a_1}{2y - 1} - \left(\frac{1}{2y - 1} - \frac{2(-2x + y - 1)}{(2y - 1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4x^2 a_3 + 4x^2 b_2 - 8xy a_2 - 4xy a_3 + 8xy b_3 + 2y^2 a_2 - 3y^2 a_3 - 4y^2 b_2 - 2y^2 b_3 + 4x a_2 + 4x a_3 + 4x b_1 + x b_2 +}{(2y - 1)^2} = 0$$

Setting the numerator to zero gives

$$-4x^2 a_3 - 4x^2 b_2 + 8xy a_2 + 4xy a_3 - 8xy b_3 - 2y^2 a_2 + 3y^2 a_3 + 4y^2 b_2 + 2y^2 b_3 - 4x a_2 - 4x a_3 - 4x b_1 - x b_2 + 2x b_3 + 4y a_1 + 3y a_2 - 4y b_2 - 4y b_3 - 2a_1 - a_2 - a_3 - b_1 + b_2 + b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &8a_2v_1v_2 - 2a_2v_2^2 - 4a_3v_1^2 + 4a_3v_1v_2 + 3a_3v_2^2 - 4b_2v_1^2 + 4b_2v_2^2 \\ &- 8b_3v_1v_2 + 2b_3v_2^2 + 4a_1v_2 - 4a_2v_1 + 3a_2v_2 - 4a_3v_1 - 4b_1v_1 - b_2v_1 \\ &- 4b_2v_2 + 2b_3v_1 - 4b_3v_2 - 2a_1 - a_2 - a_3 - b_1 + b_2 + b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-4a_3 - 4b_2)\, v_1^2 + (8a_2 + 4a_3 - 8b_3)\, v_1v_2 \\ &+ (-4a_2 - 4a_3 - 4b_1 - b_2 + 2b_3)\, v_1 + (-2a_2 + 3a_3 + 4b_2 + 2b_3)\, v_2^2 \\ &+ (4a_1 + 3a_2 - 4b_2 - 4b_3)\, v_2 - 2a_1 - a_2 - a_3 - b_1 + b_2 + b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -4a_3 - 4b_2 &= 0 \\ 8a_2 + 4a_3 - 8b_3 &= 0 \\ 4a_1 + 3a_2 - 4b_2 - 4b_3 &= 0 \\ -2a_2 + 3a_3 + 4b_2 + 2b_3 &= 0 \\ -4a_2 - 4a_3 - 4b_1 - b_2 + 2b_3 &= 0 \\ -2a_1 - a_2 - a_3 - b_1 + b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= \frac{3b_3}{2} + \frac{5b_1}{2} \\
a_2 &= 2b_3 + 2b_1 \\
a_3 &= -2b_3 - 4b_1 \\
b_1 &= b_1 \\
b_2 &= 2b_3 + 4b_1 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{3}{2} + 2x - 2y \\
\eta &= 2x + y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 2x + y - \left(\frac{-2x + y - 1}{2y - 1}\right)\left(\frac{3}{2} + 2x - 2y\right) \\
&= \frac{8x^2 - 4xy + 8y^2 + 6x - 9y + 3}{4y - 2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{8x^2-4xy+8y^2+6x-9y+3}{4y-2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln \left(8x^2 - 4xy + 8y^2 + 6x - 9y + 3\right)}{4} + \frac{4\left(\frac{x}{2} + \frac{1}{8}\right)\sqrt{15}\, \arctan\left(\frac{(16y-4x-9)\sqrt{15}}{60x+15}\right)}{15\,(4x+1)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-2x + y - 1}{2y - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{4x - 2y + 2}{8x^2 + (-4y + 6)\,x + 8y^2 - 9y + 3} \\
S_y &= \frac{4y - 2}{8y^2 + (-4x - 9)\,y + 8x^2 + 6x + 3}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

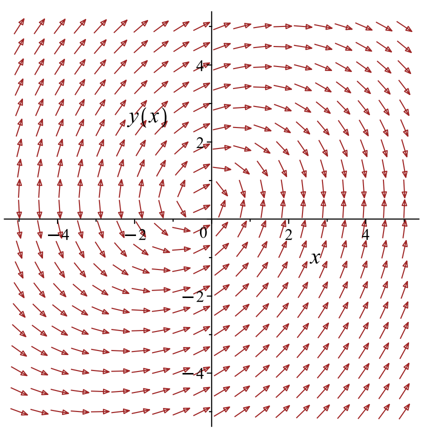
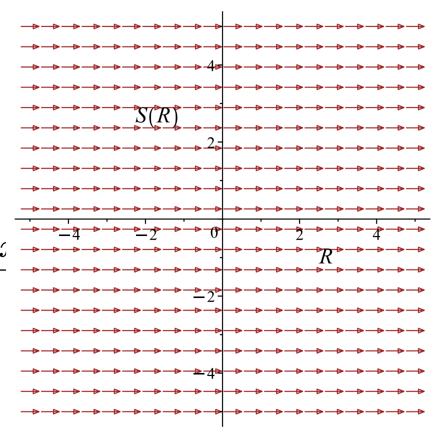
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(8x^2 + (-4y+6)x + 8y^2 - 9y + 3\right)}{4} - \frac{\sqrt{15}\,\arctan\left(\frac{(-16y+4x+9)\sqrt{15}}{60x+15}\right)}{30} = c_1$$

Which simplifies to

$$\frac{\ln\left(8x^2 + (-4y+6)x + 8y^2 - 9y + 3\right)}{4} - \frac{\sqrt{15}\,\arctan\left(\frac{(-16y+4x+9)\sqrt{15}}{60x+15}\right)}{30} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-2x+y-1}{2y-1}$ 	$R = x$ $S = \frac{\ln\left(8x^2 + (-4y+6)x \ldots\right.}{4} \ldots \frac{\left((-16y \ldots\right.}{30}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln\left(8x^2 + (-4y+6)x + 8y^2 - 9y + 3\right)}{4} - \frac{\sqrt{15}\,\arctan\left(\frac{(-16y+4x+9)\sqrt{15}}{60x+15}\right)}{30} = c_1 \tag{1}$$

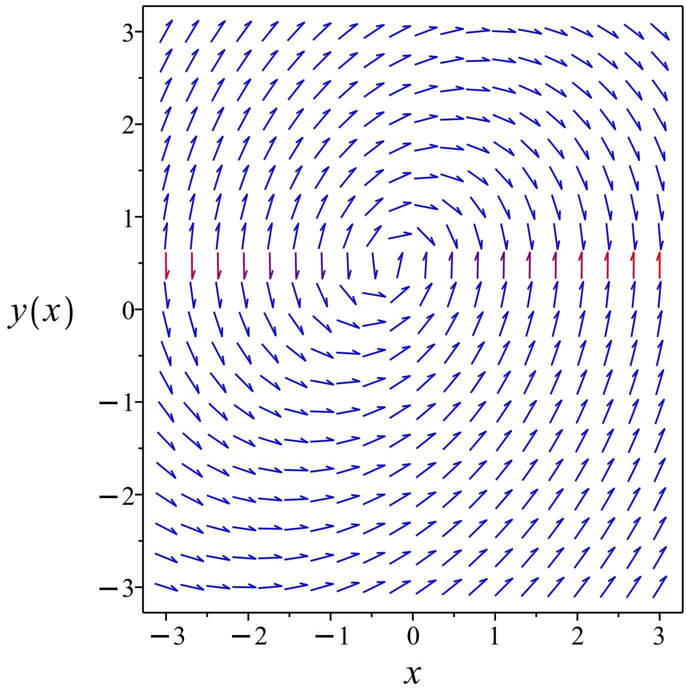


Figure 32: Slope field plot

Verification of solutions

$$\frac{\ln\left(8x^2+(-4y+6)x+8y^2-9y+3\right)}{4}-\frac{\sqrt{15}\,\arctan\left(\frac{(-16y+4x+9)\sqrt{15}}{60x+15}\right)}{30}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 67

```
dsolve((2*x-y(x)+1)+(2*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{15}\,\tan\left(\mathrm{RootOf}\left(\sqrt{15}\,\ln\left((1+4x)^2\sec\left(\_Z\right)^2\right) - 3\sqrt{15}\,\ln(2) + \sqrt{15}\,\ln(3) + \sqrt{15}\,\ln(5) + 2\sqrt{15}\,c_1 - \right.\right.}{16} + \frac{x}{4} + \frac{9}{16}$$

✓ Solution by Mathematica

Time used: 0.139 (sec). Leaf size: 85

```
DSolve[(2*x-y[x]+1)+(2*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\sqrt{15}\arctan\left(\frac{-2y(x)+8x+3}{\sqrt{15}(2y(x)-1)}\right) = 15\left(\log\left(\frac{2(8x^2+8y(x)^2-(4x+9)y(x)+6x+3)}{(4x+1)^2}\right) + 2\log(4x+1) + 8c_1\right), y(x)\right]$$

1.13 problem 4.2

Internal problem ID [4367]
Internal file name [`OUTPUT/3860_Sunday_June_05_2022_11_31_32_AM_81731100/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 4.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y + (7y - 3x + 3)\, y' = 7x - 7}$$

1.13.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) + 3y_0 - 7X - 7x_0 + 7}{7Y(X) + 7y_0 - 3X - 3x_0 + 3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) - 7X}{7Y(X) - 3X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{3Y - 7X}{7Y - 3X}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 3Y - 7X$ and $N = -7Y + 3X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-3u + 7}{7u - 3} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X} = 0
$$

Or

$$
7\left(\frac{d}{dX}u(X)\right) Xu(X) - 3\left(\frac{d}{dX}u(X)\right) X + 7u(X)^2 - 7 = 0
$$

Or

$$
-7 + X(7u(X) - 3)\left(\frac{d}{dX}u(X)\right) + 7u(X)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X)g(u) \\
&= -\frac{7(u^2 - 1)}{X(7u - 3)}
\end{aligned}
$$

Where $f(X) = -\frac{7}{X}$ and $g(u) = \frac{u^2-1}{7u-3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{7u-3}}\,du = -\frac{7}{X}\,dX$$

$$\int \frac{1}{\frac{u^2-1}{7u-3}}\,du = \int -\frac{7}{X}\,dX$$

$$2\ln(u-1) + 5\ln(u+1) = -7\ln(X) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u-1)+5\ln(u+1)} = e^{-7\ln(X)+c_2}$$

Which simplifies to

$$(u-1)^2 (u+1)^5 = \frac{c_3}{X^7}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \mathrm{RootOf}\left(X^7 + 3X^6\_Z + X^5\_Z^2 - 5X^4\_Z^3 - 5X^3\_Z^4 + X^2\_Z^5 + 3X\_Z^6 + \_Z^7 - c_3\right)$$

Using the solution for $Y(X)$

$$Y(X) = \mathrm{RootOf}\left(X^7 + 3X^6\_Z + X^5\_Z^2 - 5X^4\_Z^3 - 5X^3\_Z^4 + X^2\_Z^5 + 3X\_Z^6 + \_Z^7 - c_3\right)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x + 1$$

Then the solution in y becomes

$$y = \mathrm{RootOf}\left(\_Z^7 + (-3+3x)\_Z^6 + \left(x^2 - 2x + 1\right)\_Z^5 + \left(-5x^3 + 15x^2 - 15x + 5\right)\_Z^4 + \left(-5x^4 + 20x\right.\right.$$

Summary
The solution(s) found are the following

$$y = \text{RootOf}\left(\_Z^7 + (-3+3x)\_Z^6 + \left(x^2-2x+1\right)\_Z^5 + \left(-5x^3+15x^2-15x+5\right)\_Z^4 + \left(-5x^4+20x^3-30x^2+20x-5\right)\_Z^3 + \left(x^5-5x^4+10x^3-10x^2+5x-1\right)\_Z^2 + \left(3x^6-18x^5+45x^4-60x^3+45x^2-18x+3\right)\_Z + x^7-7x^6+21x^5-35x^4 + 35x^3-21x^2-c_3+7x-1\right) \tag{1}$$

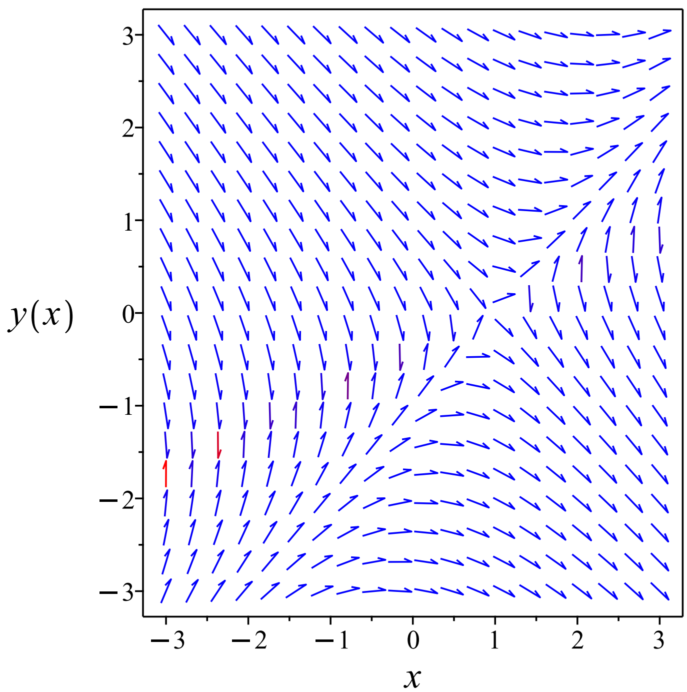


Figure 33: Slope field plot

Verification of solutions

$$y = \text{RootOf}\left(\_Z^7 + (-3+3x)\_Z^6 + \left(x^2-2x+1\right)\_Z^5 + \left(-5x^3+15x^2-15x+5\right)\_Z^4 + \left(-5x^4+20x^3-30x^2+20x-5\right)\_Z^3 + \left(x^5-5x^4+10x^3-10x^2+5x-1\right)\_Z^2 + \left(3x^6-18x^5+45x^4-60x^3+45x^2-18x+3\right)\_Z + x^7-7x^6+21x^5-35x^4 + 35x^3-21x^2-c_3+7x-1\right)$$

Verified OK.

1.13.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3y-7x+7}{7y-3x+3} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(3y-7x+7)(b_3-a_2)}{7y-3x+3} - \frac{(3y-7x+7)^2 a_3}{(7y-3x+3)^2} \\
&- \left(\frac{7}{7y-3x+3} - \frac{3(3y-7x+7)}{(7y-3x+3)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{3}{7y-3x+3} + \frac{21y-49x+49}{(7y-3x+3)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{21x^2 a_2 - 49x^2 a_3 + 49x^2 b_2 - 21x^2 b_3 - 98xy a_2 + 42xy a_3 - 42xy b_2 + 98xy b_3 + 21y^2 a_2 - 49y^2 a_3 + 49y^2 b_2 -}{}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&21x^2 a_2 - 49x^2 a_3 + 49x^2 b_2 - 21x^2 b_3 - 98xy a_2 + 42xy a_3 - 42xy b_2 + 98xy b_3 \\
&+ 21y^2 a_2 - 49y^2 a_3 + 49y^2 b_2 - 21y^2 b_3 - 42x a_2 + 98x a_3 + 40x b_1 - 58x b_2 + 42x b_3 \\
&- 40y a_1 + 58y a_2 - 42y a_3 + 42y b_2 - 98y b_3 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&21a_2v_1^2 - 98a_2v_1v_2 + 21a_2v_2^2 - 49a_3v_1^2 + 42a_3v_1v_2 - 49a_3v_2^2 + 49b_2v_1^2 \\
&- 42b_2v_1v_2 + 49b_2v_2^2 - 21b_3v_1^2 + 98b_3v_1v_2 - 21b_3v_2^2 - 40a_1v_2 \\
&- 42a_2v_1 + 58a_2v_2 + 98a_3v_1 - 42a_3v_2 + 40b_1v_1 - 58b_2v_1 + 42b_2v_2 \\
&+ 42b_3v_1 - 98b_3v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(21a_2 - 49a_3 + 49b_2 - 21b_3)\, v_1^2 + (-98a_2 + 42a_3 - 42b_2 + 98b_3)\, v_1v_2 \\
&+ (-42a_2 + 98a_3 + 40b_1 - 58b_2 + 42b_3)\, v_1 + (21a_2 - 49a_3 + 49b_2 - 21b_3)\, v_2^2 \\
&+ (-40a_1 + 58a_2 - 42a_3 + 42b_2 - 98b_3)\, v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-98a_2 + 42a_3 - 42b_2 + 98b_3 &= 0 \\
21a_2 - 49a_3 + 49b_2 - 21b_3 &= 0 \\
-40a_1 + 58a_2 - 42a_3 + 42b_2 - 98b_3 &= 0 \\
-42a_2 + 98a_3 + 40b_1 - 58b_2 + 42b_3 &= 0 \\
21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -b_3 \\
a_2 &= b_3 \\
a_3 &= b_2 \\
b_1 &= -b_2 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= x - 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - 1 - \left(-\frac{3y - 7x + 7}{7y - 3x + 3}\right)(y) \\
&= \frac{3x^2 - 3y^2 - 6x + 3}{-7y + 3x - 3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2 - 3y^2 - 6x + 3}{-7y + 3x - 3}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{5 \ln (x - 1 + y)}{3} + \frac{2 \ln (-x + y + 1)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y - 7x + 7}{7y - 3x + 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{5}{3x - 3 + 3y} + \frac{2}{3x - 3 - 3y} \\
S_y &= \frac{5}{3x - 3 + 3y} - \frac{2}{3x - 3 - 3y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

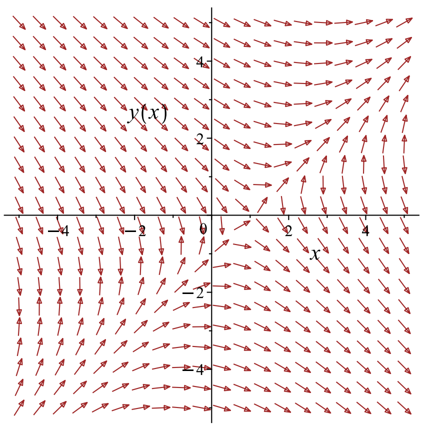
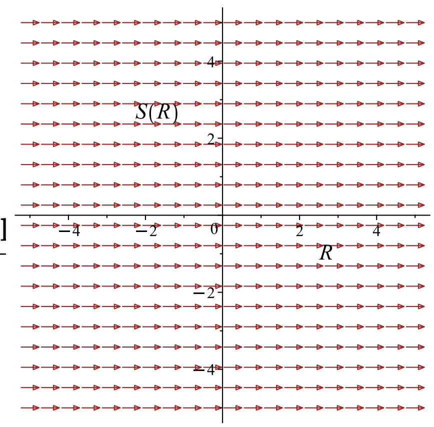
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{5\ln(x - 1 + y)}{3} + \frac{2\ln(-x + y + 1)}{3} = c_1$$

Which simplifies to

$$\frac{5\ln(x - 1 + y)}{3} + \frac{2\ln(-x + y + 1)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y-7x+7}{7y-3x+3}$ 	$R = x$ $S = \frac{5 \ln (x-1+y)}{3} + 2\mathrm{l}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{5 \ln (x-1+y)}{3} + \frac{2 \ln (-x+y+1)}{3} = c_1 \tag{1}$$

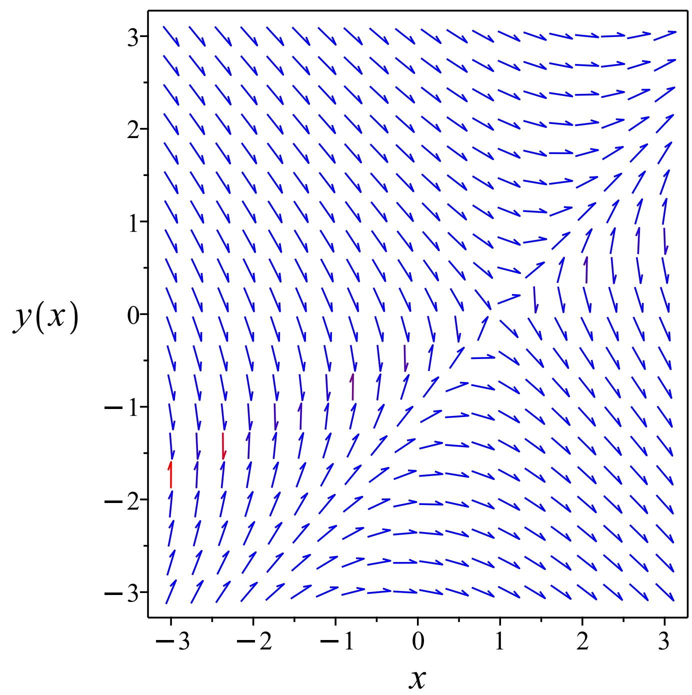


Figure 34: Slope field plot

<u>Verification of solutions</u>

$$\frac{5\ln(x-1+y)}{3}+\frac{2\ln(-x+y+1)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 1814

```
dsolve((3*y(x)-7*x+7)+(7*y(x)-3*x+3)*diff(y(x),x)=0,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 60.698 (sec). Leaf size: 7785

```
DSolve[(3*y[x]-7*x+7)+(7*y[x]-3*x+3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

1.14 problem 6.1

Internal problem ID [4368]
Internal file name [`OUTPUT/3861_Sunday_June_05_2022_11_31_42_AM_28711573/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 6.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{xy}{x^2+1} = \frac{1}{2x\left(x^2+1\right)}}$$

1.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{x}{x^2+1}$$
$$q(x) = \frac{1}{2x\left(x^2+1\right)}$$

Hence the ode is

$$y' + \frac{xy}{x^2+1} = \frac{1}{2x\left(x^2+1\right)}$$

The integrating factor μ is

$$\mu = e^{\int \frac{x}{x^2+1}dx}$$
$$= \sqrt{x^2+1}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{2x(x^2+1)}\right)$$
$$\frac{d}{dx}\left(\sqrt{x^2+1}\,y\right) = \left(\sqrt{x^2+1}\right)\left(\frac{1}{2x(x^2+1)}\right)$$
$$d\left(\sqrt{x^2+1}\,y\right) = \left(\frac{1}{2x\sqrt{x^2+1}}\right) dx$$

Integrating gives

$$\sqrt{x^2+1}\,y = \int \frac{1}{2x\sqrt{x^2+1}}\, dx$$
$$\sqrt{x^2+1}\,y = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{x^2+1}$ results in

$$y = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{2\sqrt{x^2+1}} + \frac{c_1}{\sqrt{x^2+1}}$$

which simplifies to

$$y = \frac{-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) + 2c_1}{2\sqrt{x^2+1}}$$

Summary
The solution(s) found are the following

$$y = \frac{-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) + 2c_1}{2\sqrt{x^2+1}} \tag{1}$$

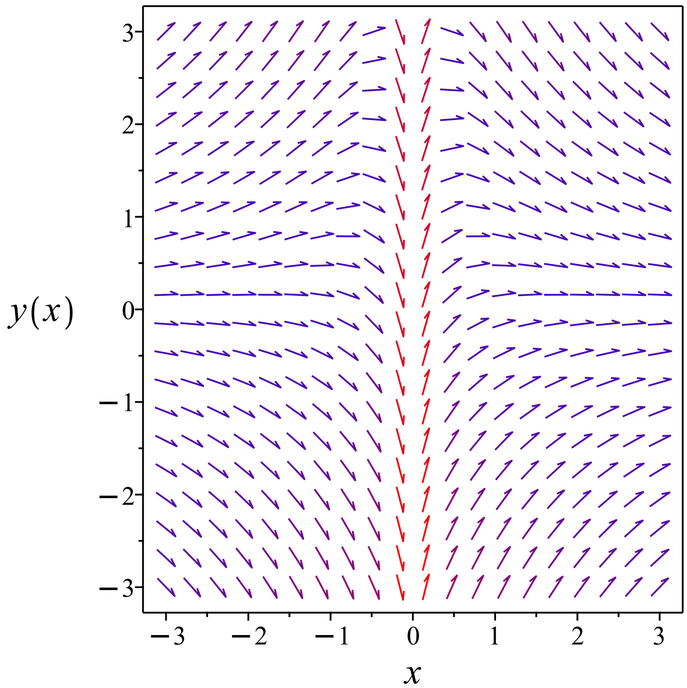


Figure 35: Slope field plot

Verification of solutions

$$y = \frac{-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) + 2c_1}{2\sqrt{x^2+1}}$$

Verified OK.

1.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y\,x^2 - 1}{2x\,(x^2+1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 21: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{\sqrt{x^2+1}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x^2+1}}} dy \end{aligned}$$

Which results in

$$S = \sqrt{x^2 + 1}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2y\, x^2 - 1}{2x\,(x^2 + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{yx}{\sqrt{x^2 + 1}} \\ S_y &= \sqrt{x^2 + 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x\sqrt{x^2 + 1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R\sqrt{R^2 + 1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{R^2+1}}\right)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x^2+1}\, y = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{2} + c_1$$

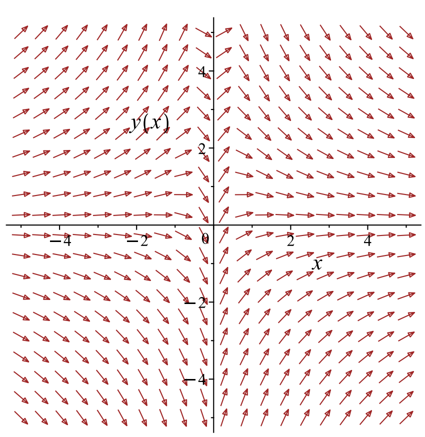
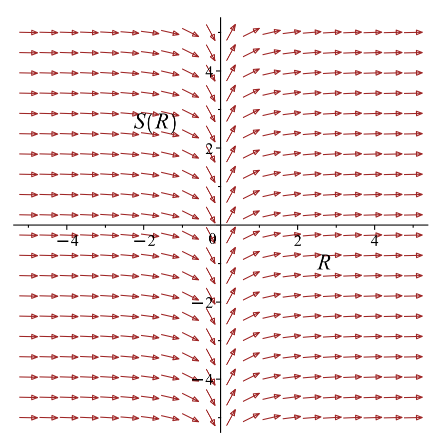
Which simplifies to

$$\sqrt{x^2+1}\, y = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{2} + c_1$$

Which gives

$$y = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) - 2c_1}{2\sqrt{x^2+1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y\,x^2-1}{2x(x^2+1)}$ 	$R = x$ $S = \sqrt{x^2+1}\, y$	$\frac{dS}{dR} = \frac{1}{2R\sqrt{R^2+1}}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) - 2c_1}{2\sqrt{x^2+1}} \tag{1}$$

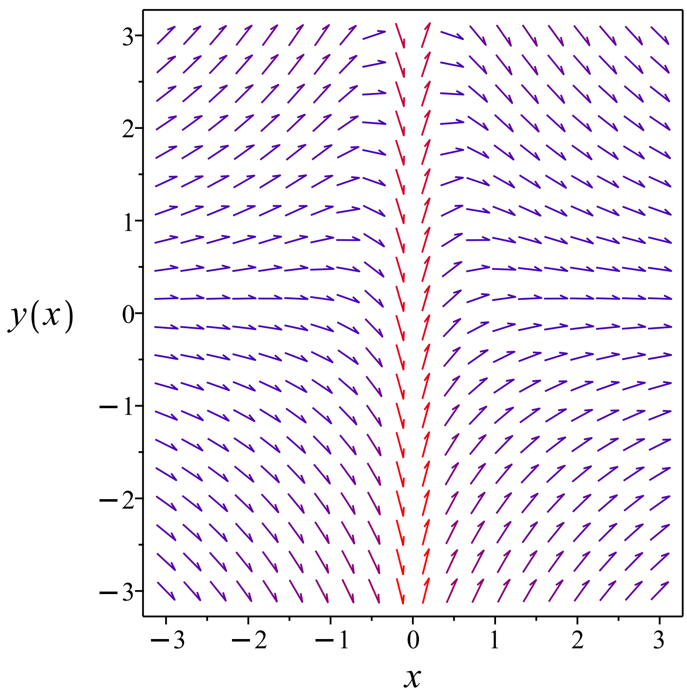


Figure 36: Slope field plot

Verification of solutions

$$y = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) - 2c_1}{2\sqrt{x^2+1}}$$

Verified OK.

1.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{xy}{x^2+1} + \frac{1}{2x\,(x^2+1)}\right)\mathrm{d}x$$

$$\left(\frac{xy}{x^2+1} - \frac{1}{2x\,(x^2+1)}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \frac{xy}{x^2+1} - \frac{1}{2x\,(x^2+1)} \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{xy}{x^2+1} - \frac{1}{2x\,(x^2+1)}\right) \\ &= \frac{x}{x^2+1}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(\frac{x}{x^2+1}\right) - (0)\right) \\ &= \frac{x}{x^2+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int \frac{x}{x^2+1} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{\ln\left(x^2+1\right)}{2}} \\ &= \sqrt{x^2+1}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \sqrt{x^2+1}\left(\frac{xy}{x^2+1} - \frac{1}{2x\,(x^2+1)}\right) \\ &= \frac{2y\,x^2-1}{2\sqrt{x^2+1}\,x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \sqrt{x^2+1}(1)\\ &= \sqrt{x^2+1}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(\frac{2y\,x^2-1}{2\sqrt{x^2+1}\,x}\right) + \left(\sqrt{x^2+1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{2y\,x^2-1}{2\sqrt{x^2+1}\,x}\,\mathrm{d}x$$

$$\phi = \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{2} + \sqrt{x^2+1}\,y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \sqrt{x^2+1} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \sqrt{x^2+1}$. Therefore equation (4) becomes

$$\sqrt{x^2+1} = \sqrt{x^2+1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{2} + \sqrt{x^2+1}\, y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{2} + \sqrt{x^2+1}\, y$$

The solution becomes

$$y = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) - 2c_1}{2\sqrt{x^2+1}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) - 2c_1}{2\sqrt{x^2+1}} \tag{1}$$

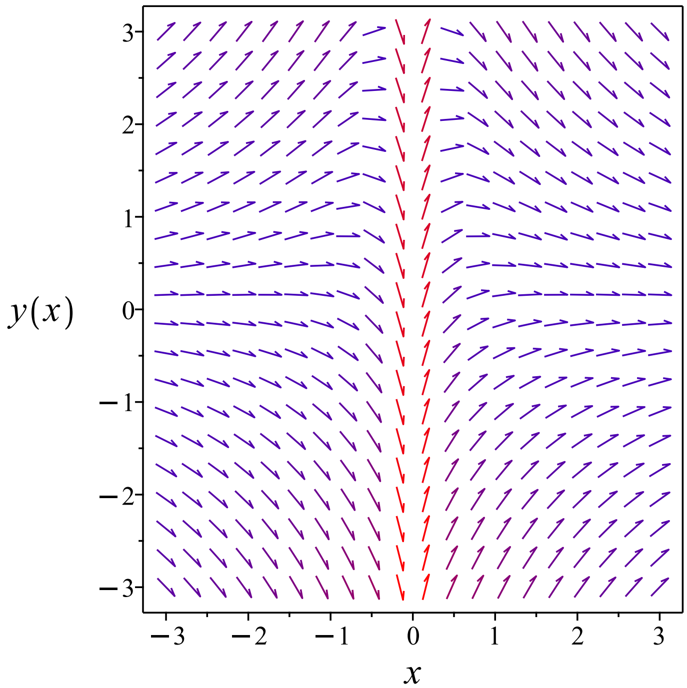


Figure 37: Slope field plot

Verification of solutions

$$y = -\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) - 2c_1}{2\sqrt{x^2+1}}$$

Verified OK.

1.14.4 Maple step by step solution

Let's solve

$y' + \frac{xy}{x^2+1} = \frac{1}{2x(x^2+1)}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{xy}{x^2+1} + \frac{1}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{xy}{x^2+1} = \frac{1}{2x(x^2+1)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{xy}{x^2+1}\right)=\frac{\mu(x)}{2x(x^2+1)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{xy}{x^2+1}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)x}{x^2+1}$

- Solve to find the integrating factor

  $\mu(x)=\sqrt{x^2+1}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)}{2x(x^2+1)}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)}{2x(x^2+1)}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(x)}{2x\left(x^2+1\right)}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\sqrt{x^2+1}$

  $y=\frac{\int\frac{1}{2x\sqrt{x^2+1}}dx+c_1}{\sqrt{x^2+1}}$

- Evaluate the integrals on the rhs

  $y=\frac{-\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{2}+c_1}{\sqrt{x^2+1}}$

- Simplify

  $y=\frac{-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)+2c_1}{2\sqrt{x^2+1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)+x/(1+x^2)*y(x)=1/(2*x*(1+x^2)),y(x), singsol=all)
```

$$y(x) = \frac{-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) + 2c_1}{2\sqrt{x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 33

```
DSolve[y'[x]+x/(1+x^2)*y[x]==1/(2*x*(1+x^2)),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\operatorname{arctanh}\left(\sqrt{x^2+1}\right) - 2c_1}{2\sqrt{x^2+1}}$$

1.15 problem 6.2

Internal problem ID [4369]
Internal file name [`OUTPUT/3862_Sunday_June_05_2022_11_31_50_AM_33158875/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 6.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x\left(-x^{2}+1\right) y' + \left(2x^{2}-1\right) y = a\,x^{3}}$$

1.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2x^{2}-1}{x^{3}-x}$$
$$q(x) = -\frac{a\,x^{2}}{x^{2}-1}$$

Hence the ode is

$$y' - \frac{(2x^{2}-1)\,y}{x^{3}-x} = -\frac{a\,x^{2}}{x^{2}-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2x^2-1}{x^3-x}dx} \\ &= \mathrm{e}^{-\frac{\ln(x+1)}{2}-\frac{\ln(x-1)}{2}-\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x+1}\,\sqrt{x-1}\,x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{a\,x^2}{x^2-1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x+1}\,\sqrt{x-1}\,x}\right) &= \left(\frac{1}{\sqrt{x+1}\,\sqrt{x-1}\,x}\right)\left(-\frac{a\,x^2}{x^2-1}\right) \\ \mathrm{d}\left(\frac{y}{\sqrt{x+1}\,\sqrt{x-1}\,x}\right) &= \left(-\frac{ax}{(x^2-1)\,\sqrt{x+1}\,\sqrt{x-1}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{\sqrt{x+1}\,\sqrt{x-1}\,x} &= \int -\frac{ax}{(x^2-1)\,\sqrt{x+1}\,\sqrt{x-1}}\,\mathrm{d}x \\ \frac{y}{\sqrt{x+1}\,\sqrt{x-1}\,x} &= \frac{\sqrt{x-1}\,\sqrt{x+1}\,a}{x^2-1} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x+1}\,\sqrt{x-1}\,x}$ results in

$$y = \frac{(x+1)\,(x-1)\,xa}{x^2-1} + c_1\sqrt{x+1}\,\sqrt{x-1}\,x$$

which simplifies to

$$y = x\left(a + c_1\sqrt{x+1}\,\sqrt{x-1}\right)$$

Summary

The solution(s) found are the following

$$y = x\left(a + c_1\sqrt{x+1}\,\sqrt{x-1}\right) \tag{1}$$

Verification of solutions

$$y = x\left(a + c_1\sqrt{x+1}\,\sqrt{x-1}\right)$$

Verified OK.

1.15.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x\left(-x^{2}+1\right)\left(u'(x)\,x+u(x)\right)+\left(2x^{2}-1\right)u(x)\,x = a\,x^{3}$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{x(u-a)}{x^{2}-1} \end{aligned}$$

Where $f(x) = \frac{x}{x^{2}-1}$ and $g(u) = u-a$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u-a}\,du &= \frac{x}{x^{2}-1}\,dx\\ \int \frac{1}{u-a}\,du &= \int \frac{x}{x^{2}-1}\,dx\\ \ln\left(u-a\right) &= \frac{\ln\left(x-1\right)}{2}+\frac{\ln\left(x+1\right)}{2}+c_{2} \end{aligned}$$

Raising both side to exponential gives

$$u-a = \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}+c_{2}}$$

Which simplifies to

$$u-a = c_{3}\mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}$$

Which simplifies to

$$u(x) = c_{3}\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_{2}}+a$$

Therefore the solution y is

$$\begin{aligned} y &= xu\\ &= x\left(c_{3}\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_{2}}+a\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\left(c_{3}\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_{2}}+a\right) \tag{1}$$

Verification of solutions

$$y = x\left(c_3\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_2}+a\right)$$

Verified OK.

1.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \frac{-a\,x^3+2y\,x^2-y}{x\,(x^2-1)}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 24: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{\ln(x+1)}{2} + \frac{\ln(x-1)}{2} + \ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{\ln(x+1)}{2} + \frac{\ln(x-1)}{2} + \ln(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{\ln\left(\frac{1}{\sqrt{x-1}}\right) + \ln\left(\frac{1}{\sqrt{x+1}}\right)} y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-a\,x^3 + 2y\,x^2 - y}{x\,(x^2-1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2\left(x^2 - \frac{1}{2}\right) y}{(x-1)^{\frac{3}{2}} (x+1)^{\frac{3}{2}} x^2} \\ S_y &= \frac{1}{\sqrt{x+1}\,\sqrt{x-1}\,x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{ax}{(x+1)^{\frac{3}{2}} (x-1)^{\frac{3}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{aR}{(R+1)^{\frac{3}{2}} (R-1)^{\frac{3}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{a}{\sqrt{R+1}\,\sqrt{R-1}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\sqrt{x+1}\,\sqrt{x-1}\,x} = \frac{a}{\sqrt{x+1}\,\sqrt{x-1}} + c_1$$

Which simplifies to

$$\frac{y}{\sqrt{x+1}\,\sqrt{x-1}\,x} = \frac{a}{\sqrt{x+1}\,\sqrt{x-1}} + c_1$$

Which gives

$$y = ax + c_1\sqrt{x+1}\,\sqrt{x-1}\,x$$

Summary
The solution(s) found are the following

$$y = ax + c_1\sqrt{x+1}\,\sqrt{x-1}\,x \tag{1}$$

Verification of solutions

$$y = ax + c_1\sqrt{x+1}\,\sqrt{x-1}\,x$$

Verified OK.

1.15.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\left(-x^2+1\right)\right)\mathrm{d}y&=\left(-\left(2x^2-1\right)y+a\,x^3\right)\mathrm{d}x\\ \left(\left(2x^2-1\right)y-a\,x^3\right)\mathrm{d}x+\left(x\left(-x^2+1\right)\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\left(2x^2-1\right)y-a\,x^3\\ N(x,y)&=x\left(-x^2+1\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(\left(2x^2-1\right)y-a\,x^3\right)\\ &=2x^2-1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(-x^2+1\right)\right) \\
&= -3x^2+1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x^3-x}\left(\left(2x^2-1\right) - \left(-3x^2+1\right)\right) \\
&= \frac{-5x^2+2}{x^3-x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \frac{-5x^2+2}{x^3-x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{3\ln(x+1)}{2} - \frac{3\ln(x-1)}{2} - 2\ln(x)} \\
&= \frac{1}{(x+1)^{\frac{3}{2}}\,(x-1)^{\frac{3}{2}}\,x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(x+1)^{\frac{3}{2}}\,(x-1)^{\frac{3}{2}}\,x^2}\left(\left(2x^2-1\right)y - a\,x^3\right) \\
&= -\frac{a\,x^3 - 2y\,x^2 + y}{(x+1)^{\frac{3}{2}}\,(x-1)^{\frac{3}{2}}\,x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(x+1)^{\frac{3}{2}}\,(x-1)^{\frac{3}{2}}\,x^2}\left(x\left(-x^2+1\right)\right) \\
&= -\frac{1}{\sqrt{x+1}\,\sqrt{x-1}\,x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(-\frac{a\,x^3 - 2y\,x^2 + y}{(x+1)^{\frac{3}{2}}\,(x-1)^{\frac{3}{2}}\,x^2}\right) + \left(-\frac{1}{\sqrt{x+1}\,\sqrt{x-1}\,x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{a\,x^3 - 2y\,x^2 + y}{(x+1)^{\frac{3}{2}}\,(x-1)^{\frac{3}{2}}\,x^2}\,\mathrm{d}x$$

$$\phi = \frac{ax - y}{\sqrt{x+1}\,\sqrt{x-1}\,x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{\sqrt{x+1}\,\sqrt{x-1}\,x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{\sqrt{x+1}\,\sqrt{x-1}\,x}$. Therefore equation (4) becomes

$$-\frac{1}{\sqrt{x+1}\,\sqrt{x-1}\,x} = -\frac{1}{\sqrt{x+1}\,\sqrt{x-1}\,x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{ax - y}{\sqrt{x+1}\,\sqrt{x-1}\,x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{ax - y}{\sqrt{x+1}\,\sqrt{x-1}\,x}$$

The solution becomes

$$y = -c_1\sqrt{x+1}\,\sqrt{x-1}\,x + ax$$

Summary

The solution(s) found are the following

$$y = -c_1\sqrt{x+1}\,\sqrt{x-1}\,x + ax \tag{1}$$

Verification of solutions

$$y = -c_1\sqrt{x+1}\,\sqrt{x-1}\,x + ax$$

Verified OK.

1.15.5 Maple step by step solution

Let's solve

$x(-x^2+1)\,y' + (2x^2-1)\,y = a\,x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{(2x^2-1)y}{x(x^2-1)} - \frac{a\,x^2}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{(2x^2-1)y}{x(x^2-1)} = -\frac{a\,x^2}{x^2-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{(2x^2-1)y}{x(x^2-1)}\right) = -\frac{\mu(x)a\,x^2}{x^2-1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y'-\frac{(2x^2-1)y}{x(x^2-1)}\right)=\mu'(x)\, y+\mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)(2x^2-1)}{x(x^2-1)}$
- Solve to find the integrating factor

  $\mu(x)=\frac{1}{\sqrt{x+1}\,\sqrt{x-1}\, x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx=\int -\frac{\mu(x) a\, x^2}{x^2-1}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y=\int -\frac{\mu(x) a\, x^2}{x^2-1}dx+c_1$
- Solve for y

  $y=\frac{\int -\frac{\mu(x) a\, x^2}{x^2-1}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{1}{\sqrt{x+1}\,\sqrt{x-1}\, x}$

  $y=\sqrt{x+1}\,\sqrt{x-1}\, x\left(\int -\frac{ax}{(x^2-1)\sqrt{x+1}\,\sqrt{x-1}}dx+c_1\right)$
- Evaluate the integrals on the rhs

  $y=\sqrt{x+1}\,\sqrt{x-1}\, x\left(\frac{\sqrt{x-1}\,\sqrt{x+1}\, a}{x^2-1}+c_1\right)$
- Simplify

  $y=\frac{\sqrt{x-1}\, x(\sqrt{x-1}\,\sqrt{x+1}\, a+c_1(x^2-1))\sqrt{x+1}}{x^2-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x*(1-x^2)*diff(y(x),x)+(2*x^2-1)*y(x)=a*x^3,y(x), singsol=all)
```

$$y(x) = x\left(\sqrt{x-1}\,\sqrt{1+x}\,c_1 + a\right)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 23

```
DSolve[x*(1-x^2)*y'[x]+(2*x^2-1)*y[x]==a*x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(a + c_1\sqrt{1-x^2}\right)$$

1.16 problem 6.3

Internal problem ID [4370]
Internal file name [`OUTPUT/3863_Sunday_June_05_2022_11_31_58_AM_23803442/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 6.3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{\left(-x^2+1\right)^{\frac{3}{2}}} = \frac{x+\sqrt{-x^2+1}}{\left(-x^2+1\right)^2}}$$

1.16.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{\left(-x^2+1\right)^{\frac{3}{2}}}$$
$$q(x) = -\frac{x^2-\sqrt{-x^2+1}\,x-1}{\left(-x^2+1\right)^{\frac{5}{2}}}$$

Hence the ode is

$$y' + \frac{y}{\left(-x^2+1\right)^{\frac{3}{2}}} = -\frac{x^2-\sqrt{-x^2+1}\,x-1}{\left(-x^2+1\right)^{\frac{5}{2}}}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{\left(-x^{2}+1\right)^{\frac{3}{2}}}d x} \\
&= \mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{x^{2}-\sqrt{-x^{2}+1}\,x-1}{\left(-x^{2}+1\right)^{\frac{5}{2}}}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}} y\right) &= \left(\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}\right)\left(-\frac{x^{2}-\sqrt{-x^{2}+1}\,x-1}{\left(-x^{2}+1\right)^{\frac{5}{2}}}\right) \\
\mathrm{d}\left(\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}} y\right) &= \left(-\frac{\left(x^{2}-\sqrt{-x^{2}+1}\,x-1\right)\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}}{\left(-x^{2}+1\right)^{\frac{5}{2}}}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}} y &= \int -\frac{\left(x^{2}-\sqrt{-x^{2}+1}\,x-1\right)\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}}{\left(-x^{2}+1\right)^{\frac{5}{2}}}\,\mathrm{d}x \\
\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}} y &= \int -\frac{\left(x^{2}-\sqrt{-x^{2}+1}\,x-1\right)\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}}{\left(-x^{2}+1\right)^{\frac{5}{2}}}dx + c_{1}
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}$ results in

$$
y = \mathrm{e}^{-\frac{x}{\sqrt{-x^{2}+1}}}\left(\int -\frac{\left(x^{2}-\sqrt{-x^{2}+1}\,x-1\right)\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}}{\left(-x^{2}+1\right)^{\frac{5}{2}}}dx\right) + c_{1}\mathrm{e}^{-\frac{x}{\sqrt{-x^{2}+1}}}
$$

which simplifies to

$$
y = \mathrm{e}^{-\frac{x}{\sqrt{-x^{2}+1}}}\left(\int -\frac{\left(x^{2}-\sqrt{-x^{2}+1}\,x-1\right)\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}}{\left(-x^{2}+1\right)^{\frac{5}{2}}}dx + c_{1}\right)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-\frac{x}{\sqrt{-x^{2}+1}}}\left(\int -\frac{\left(x^{2}-\sqrt{-x^{2}+1}\,x-1\right)\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}}{\left(-x^{2}+1\right)^{\frac{5}{2}}}dx + c_{1}\right) \tag{1}
$$

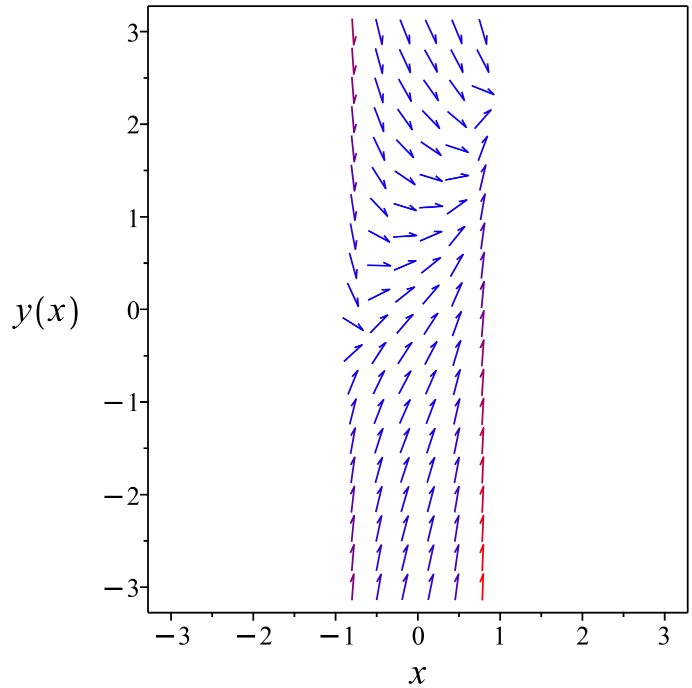


Figure 38: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{\sqrt{-x^2+1}}} \left(\int -\frac{\left(x^2 - \sqrt{-x^2+1}\, x - 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{\left(-x^2+1\right)^{\frac{5}{2}}} dx + c_1 \right)$$

Verified OK.

1.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y\,x^4 + \left(-x^2+1\right)^{\frac{3}{2}} x + x^4 + 2y\,x^2 - 2x^2 - y + 1}{\left(-x^2+1\right)^{\frac{3}{2}} \left(x^2-1\right)^2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 27: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{\frac{(x-1)(x+1)x}{(-x^2+1)^{\frac{3}{2}}}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{(x-1)(x+1)x}{(-x^2+1)^{\frac{3}{2}}}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\frac{(x-1)(x+1)x}{(-x^2+1)^{\frac{3}{2}}}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-y\,x^4 + (-x^2+1)^{\frac{3}{2}} x + x^4 + 2y\,x^2 - 2x^2 - y + 1}{(-x^2+1)^{\frac{3}{2}} (x^2-1)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\mathrm{e}^{\frac{-x^3+x}{(-x^2+1)^{\frac{3}{2}}}} y}{(-x^2+1)^{\frac{3}{2}}} \\ S_y &= \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\left(x^2 - \sqrt{-x^2+1}\,x - 1\right) \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{\left(R^2 - \sqrt{-R^2+1}\, R - 1\right) \mathrm{e}^{\frac{R}{\sqrt{-R^2+1}}}}{\left(-R^2+1\right)^{\frac{5}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{\left(-R^2 + \sqrt{-R^2+1}\, R + 1\right) \mathrm{e}^{\frac{R}{\sqrt{-R^2+1}}}}{\left(-R^2+1\right)^{\frac{5}{2}}} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}} y = \int \frac{\left(-x^2 + \sqrt{-x^2+1}\, x + 1\right) \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}}{\left(-x^2+1\right)^{\frac{5}{2}}} dx + c_1$$

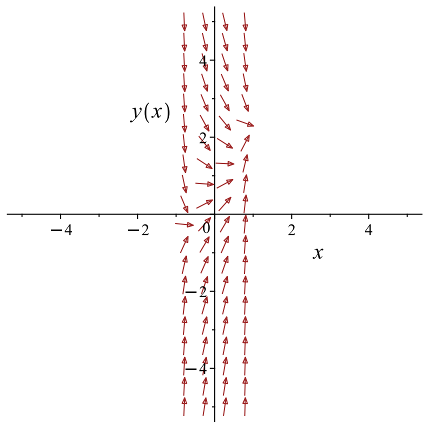
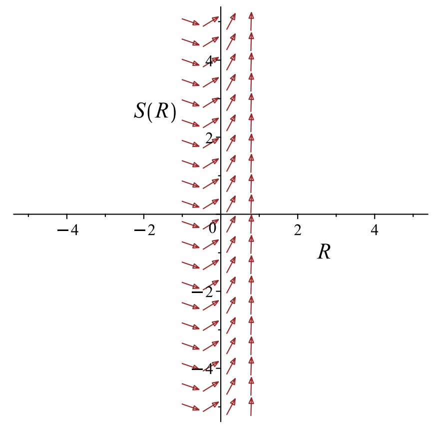
Which simplifies to

$$\mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}} y = \int \frac{\left(-x^2 + \sqrt{-x^2+1}\, x + 1\right) \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}}{\left(-x^2+1\right)^{\frac{5}{2}}} dx + c_1$$

Which gives

$$y = \left(\int \frac{\left(-x^2 + \sqrt{-x^2+1}\, x + 1\right) \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}}{\left(-x^2+1\right)^{\frac{5}{2}}} dx + c_1\right) \mathrm{e}^{-\frac{x}{\sqrt{-x^2+1}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y\,x^4+(-x^2+1)^{\frac{3}{2}}x+x^4+2y\,x^2-2x^2-y+1}{(-x^2+1)^{\frac{3}{2}}(x^2-1)^2}$ 	$R = x$ $S = \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}y$	$\frac{dS}{dR} = -\frac{\left(R^2-\sqrt{-R^2+1}\,R-1\right)\mathrm{e}^{\frac{R}{\sqrt{-R^2+1}}}}{(-R^2+1)^{\frac{5}{2}}}$ 

Summary

The solution(s) found are the following

$$y = \left(\int \frac{\left(-x^2+\sqrt{-x^2+1}\,x+1\right)\mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}}dx + c_1\right)\mathrm{e}^{-\frac{x}{\sqrt{-x^2+1}}} \tag{1}$$

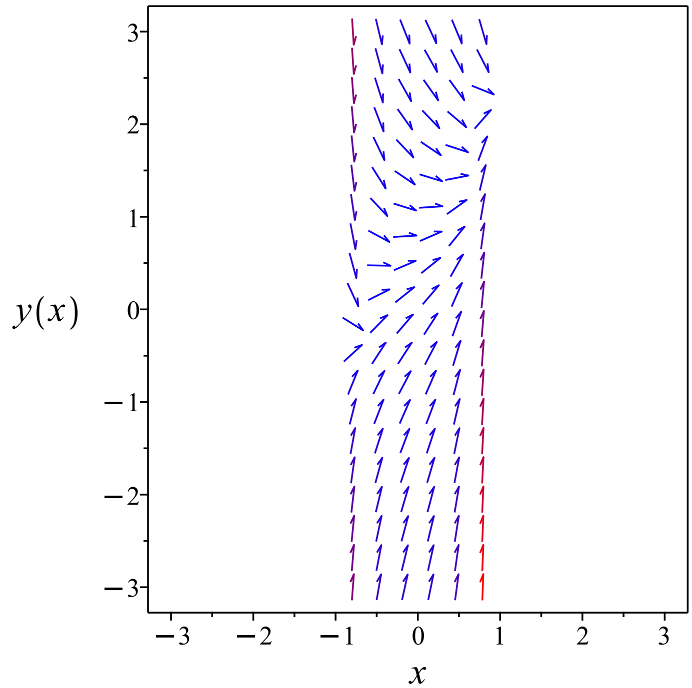


Figure 39: Slope field plot

Verification of solutions

$$y = \left(\int \frac{\left(-x^2 + \sqrt{-x^2+1}\, x + 1\right) e^{\frac{x}{\sqrt{-x^2+1}}}}{\left(-x^2+1\right)^{\frac{5}{2}}} dx + c_1\right) e^{-\frac{x}{\sqrt{-x^2+1}}}$$

Verified OK.

1.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-\frac{y}{(-x^2+1)^{\frac{3}{2}}} + \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}\right)\mathrm{d}x$$

$$\left(\frac{y}{(-x^2+1)^{\frac{3}{2}}} - \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{y}{(-x^2+1)^{\frac{3}{2}}} - \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{(-x^2+1)^{\frac{3}{2}}} - \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}\right)\\ &= \frac{1}{(-x^2+1)^{\frac{3}{2}}}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{1}{(-x^2+1)^{\frac{3}{2}}}\right) - (0)\right) \\
&= \frac{1}{(-x^2+1)^{\frac{3}{2}}}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{(-x^2+1)^{\frac{3}{2}}} \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{(x-1)(x+1)x}{(-x^2+1)^{\frac{3}{2}}}} \\
&= \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}\left(\frac{y}{(-x^2+1)^{\frac{3}{2}}} - \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}\right) \\
&= -\frac{\left(\sqrt{-x^2+1}\, x + (y-1)\, x^2 - y + 1\right) \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}(1) \\
&= \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(-\frac{\left(\sqrt{-x^2+1}\,x + (y-1)\,x^2 - y + 1\right)\mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}}\right) + \left(\mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{\left(\sqrt{-x^2+1}\,x + (y-1)\,x^2 - y + 1\right)\mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}}{(-x^2+1)^{\frac{5}{2}}}\,\mathrm{d}x$$

$$\phi = \int^x -\frac{\left(\sqrt{-\_a^2+1}\,\_a + (y-1)\,\_a^2 - y + 1\right)\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{5}{2}}} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -\left(\int^x \frac{(\_a^2-1)\,\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{5}{2}}} d\_a\right) + f'(y) \\ &= \int^x \frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{3}{2}}} d\_a + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}}$. Therefore equation (4) becomes

$$\mathrm{e}^{\frac{x}{\sqrt{-x^2+1}}} = \int^x \frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{3}{2}}} d\_a + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\left(\int^{x} \frac{e^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{3}{2}}} d\_a\right) + e^{\frac{x}{\sqrt{-x^2+1}}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\left(\int^{x} \frac{e^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{3}{2}}} d\_a\right) + e^{\frac{x}{\sqrt{-x^2+1}}}\right) \mathrm{d}y$$

$$f(y) = \left(-\left(\int^{x} \frac{e^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{3}{2}}} d\_a\right) + e^{\frac{x}{\sqrt{-x^2+1}}}\right) y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^{x} -\frac{\left(\sqrt{-\_a^2+1}\,\_a + (y-1)\_a^2 - y + 1\right) e^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{5}{2}}} d\_a + \left(-\left(\int^{x} \frac{e^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{3}{2}}} d\_a\right) + e^{\frac{x}{\sqrt{-x^2+1}}}\right) y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^{x} -\frac{\left(\sqrt{-\_a^2+1}\,\_a + (y-1)\_a^2 - y + 1\right) e^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{5}{2}}} d\_a + \left(-\left(\int^{x} \frac{e^{\frac{\_a}{\sqrt{-\_a^2+1}}}}{(-\_a^2+1)^{\frac{3}{2}}} d\_a\right) + e^{\frac{x}{\sqrt{-x^2+1}}}\right) y$$

Summary
The solution(s) found are the following

$$\int^{x} -\frac{\left(\sqrt{-\_a^{2}+1}\,\_a+(y-1)\,\_a^{2}-y+1\right) \mathrm{e}^{\frac{\_a}{\sqrt{-\_a^{2}+1}}}}{\left(-\_a^{2}+1\right)^{\frac{5}{2}}}d\_a + \left(-\left(\int^{x} \frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^{2}+1}}}}{\left(-\_a^{2}+1\right)^{\frac{3}{2}}}d\_a\right)+\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}\right) y = c_{1} \tag{1}$$

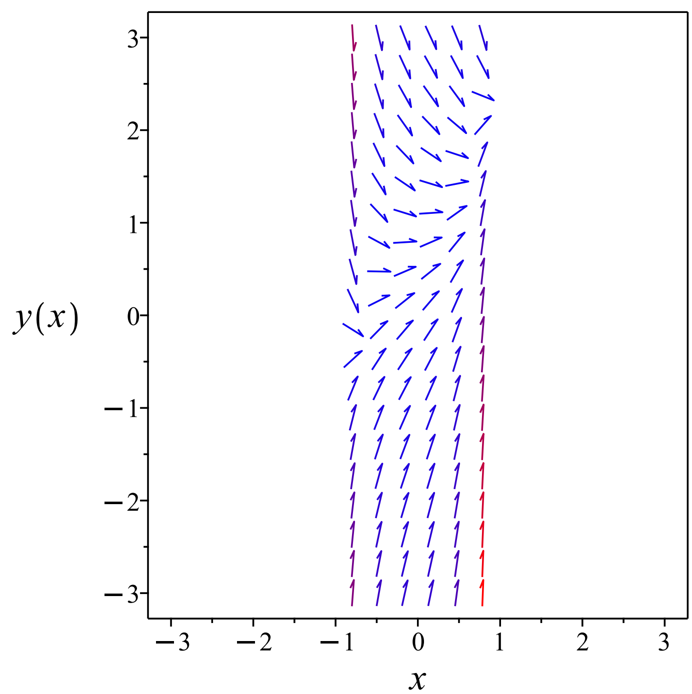


Figure 40: Slope field plot

Verification of solutions

$$\int^{x} -\frac{\left(\sqrt{-\_a^{2}+1}\,\_a+(y-1)\,\_a^{2}-y+1\right) \mathrm{e}^{\frac{\_a}{\sqrt{-\_a^{2}+1}}}}{\left(-\_a^{2}+1\right)^{\frac{5}{2}}}d\_a + \left(-\left(\int^{x} \frac{\mathrm{e}^{\frac{\_a}{\sqrt{-\_a^{2}+1}}}}{\left(-\_a^{2}+1\right)^{\frac{3}{2}}}d\_a\right)+\mathrm{e}^{\frac{x}{\sqrt{-x^{2}+1}}}\right) y = c_{1}$$

Verified OK.

1.16.4 Maple step by step solution

Let's solve

$y' + \frac{y}{(-x^2+1)^{\frac{3}{2}}} = \frac{x+\sqrt{-x^2+1}}{(-x^2+1)^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{(-x^2+1)^{\frac{3}{2}}} + \frac{x+\sqrt{-x^2+1}}{(x^2-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{(-x^2+1)^{\frac{3}{2}}} = \frac{x+\sqrt{-x^2+1}}{(x^2-1)^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{(-x^2+1)^{\frac{3}{2}}}\right) = \frac{\mu(x)\left(x+\sqrt{-x^2+1}\right)}{(x^2-1)^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{(-x^2+1)^{\frac{3}{2}}}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{(-x^2+1)^{\frac{3}{2}}}$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\frac{x}{\sqrt{-(x-1)(x+1)}}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)\left(x+\sqrt{-x^2+1}\right)}{(x^2-1)^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)\left(x+\sqrt{-x^2+1}\right)}{(x^2-1)^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\left(x+\sqrt{-x^2+1}\right)}{\left(x^2-1\right)^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\frac{x}{\sqrt{-(x-1)(x+1)}}}$

$$y = \frac{\int \frac{e^{\frac{x}{\sqrt{-(x-1)(x+1)}}}\left(x+\sqrt{-x^2+1}\right)}{\left(x^2-1\right)^2}dx + c_1}{e^{\frac{x}{\sqrt{-(x-1)(x+1)}}}}$$

- Simplify

$$y = \left(\int \frac{e^{\frac{x}{\sqrt{-x^2+1}}}\left(x+\sqrt{-x^2+1}\right)}{(x^2-1)^2}dx + c_1\right) e^{-\frac{x}{\sqrt{-x^2+1}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve(diff(y(x),x)+y(x)/(1-x^2)^(3/2)=(x+sqrt(1-x^2))/(1-x^2)^2,y(x), singsol=all)
```

$$y(x) = \left(\int \frac{e^{\frac{x}{\sqrt{-x^2+1}}}\left(x+\sqrt{-x^2+1}\right)}{(x-1)^2(1+x)^2}dx + c_1\right) e^{-\frac{x}{\sqrt{-x^2+1}}}$$

✓ Solution by Mathematica

Time used: 0.169 (sec). Leaf size: 38

```
DSolve[y'[x]+y[x]/(1-x^2)^(3/2)==(x+Sqrt[1-x^2])/(1-x^2)^2,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{x}{\sqrt{1-x^2}} + c_1 e^{-\frac{x}{\sqrt{1-x^2}}}$$

1.17 problem 6.4

Internal problem ID [4371]
Internal file name [`OUTPUT/3864_Sunday_June_05_2022_11_32_09_AM_66022866/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 6.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cos(x) = \frac{\sin(2x)}{2}}$$

1.17.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \cos(x) \\ q(x) &= \frac{\sin(2x)}{2} \end{aligned}$$

Hence the ode is

$$y' + y\cos(x) = \frac{\sin(2x)}{2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \cos(x)dx} \\
&= \mathrm{e}^{\sin(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\sin(2x)}{2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sin(x)} y\right) &= \left(\mathrm{e}^{\sin(x)}\right)\left(\frac{\sin(2x)}{2}\right) \\
\mathrm{d}\left(\mathrm{e}^{\sin(x)} y\right) &= \left(\frac{\sin(2x)\,\mathrm{e}^{\sin(x)}}{2}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\sin(x)} y &= \int \frac{\sin(2x)\,\mathrm{e}^{\sin(x)}}{2}\,\mathrm{d}x \\
\mathrm{e}^{\sin(x)} y &= \sin(x)\,\mathrm{e}^{\sin(x)} - \mathrm{e}^{\sin(x)} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\sin(x)}$ results in

$$
y = \mathrm{e}^{-\sin(x)}\left(\sin(x)\,\mathrm{e}^{\sin(x)} - \mathrm{e}^{\sin(x)}\right) + c_1 \mathrm{e}^{-\sin(x)}
$$

which simplifies to

$$
y = \sin(x) - 1 + c_1 \mathrm{e}^{-\sin(x)}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \sin(x) - 1 + c_1 \mathrm{e}^{-\sin(x)} \tag{1}
$$

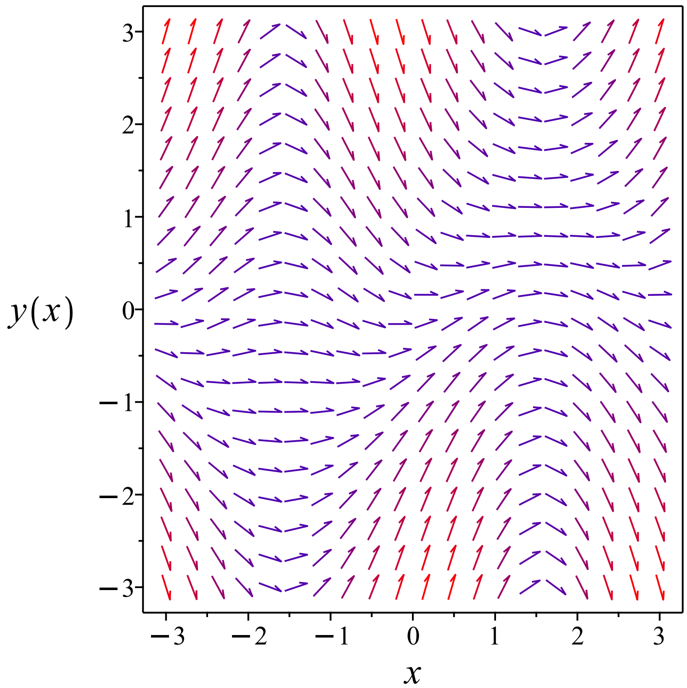


Figure 41: Slope field plot

Verification of solutions

$$y = \sin(x) - 1 + c_1 e^{-\sin(x)}$$

Verified OK.

1.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y\cos(x) + \frac{\sin(2x)}{2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= e^{-\sin(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\sin(x)}} dy \end{aligned}$$

Which results in

$$S = e^{\sin(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y\cos(x) + \frac{\sin(2x)}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\, e^{\sin(x)} y \\ S_y &= e^{\sin(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(2x)\, e^{\sin(x)}}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\sin(2R)\, e^{\sin(R)}}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + e^{\sin(R)}(-1 + \sin(R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\sin(x)}y = e^{\sin(x)}(-1 + \sin(x)) + c_1$$

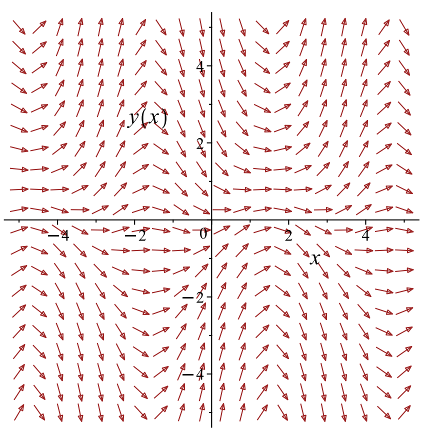
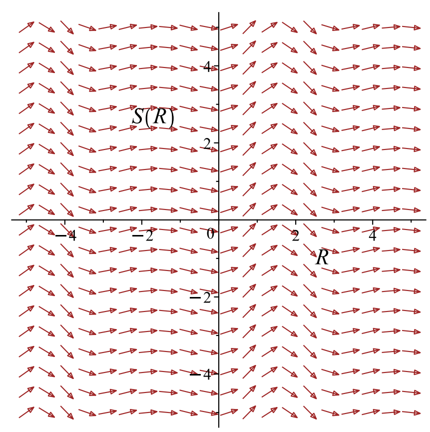
Which simplifies to

$$e^{\sin(x)}y = e^{\sin(x)}(-1 + \sin(x)) + c_1$$

Which gives

$$y = e^{-\sin(x)}\left(\sin(x)\, e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\cos(x) + \frac{\sin(2x)}{2}$ 	$R = x$ $S = e^{\sin(x)}y$	$\frac{dS}{dR} = \frac{\sin(2R)e^{\sin(R)}}{2}$ 

Summary

The solution(s) found are the following

$$y = e^{-\sin(x)}\left(\sin(x)\, e^{\sin(x)} - e^{\sin(x)} + c_1\right) \tag{1}$$

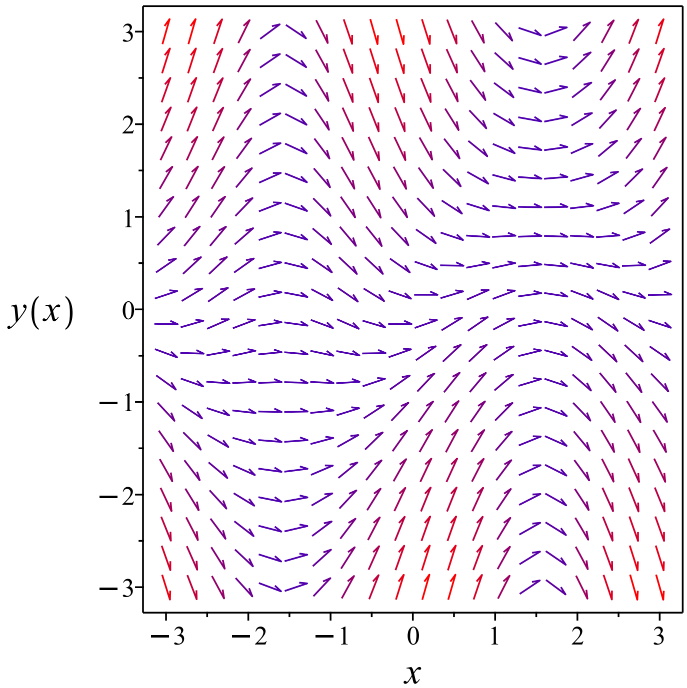


Figure 42: Slope field plot

Verification of solutions

$$y = e^{-\sin(x)}\left(\sin(x)\, e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

Verified OK.

1.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-y \cos(x) + \frac{\sin(2x)}{2}\right) \mathrm{d}x$$

$$\left(y \cos(x) - \frac{\sin(2x)}{2}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y \cos(x) - \frac{\sin(2x)}{2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y \cos(x) - \frac{\sin(2x)}{2}\right) \\ &= \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((\cos(x)) - (0)) \\ &= \cos(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cos(x) \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\sin(x)} \\ &= \mathrm{e}^{\sin(x)}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{\sin(x)}\left(y\cos(x) - \frac{\sin(2x)}{2}\right) \\ &= \cos(x)(-\sin(x) + y)\,\mathrm{e}^{\sin(x)}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{\sin(x)}(1) \\ &= \mathrm{e}^{\sin(x)}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\cos(x)(-\sin(x) + y)\,\mathrm{e}^{\sin(x)}\right) + \left(\mathrm{e}^{\sin(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \cos(x) (-\sin(x) + y) \mathrm{e}^{\sin(x)} \mathrm{d}x$$
$$\phi = (y - \sin(x) + 1) \mathrm{e}^{\sin(x)} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\sin(x)}$. Therefore equation (4) becomes

$$\mathrm{e}^{\sin(x)} = \mathrm{e}^{\sin(x)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y - \sin(x) + 1) \mathrm{e}^{\sin(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y - \sin(x) + 1) \mathrm{e}^{\sin(x)}$$

The solution becomes

$$y = e^{-\sin(x)} \left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-\sin(x)} \left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1\right) \tag{1}$$

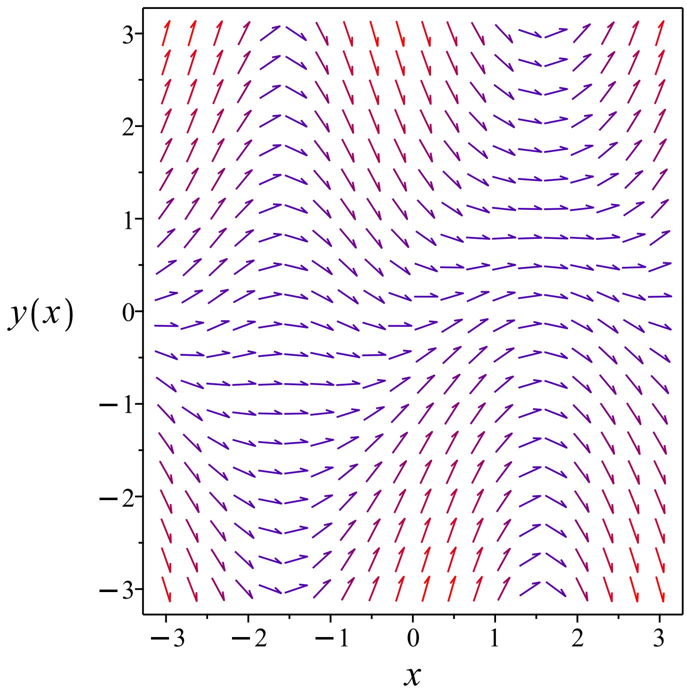


Figure 43: Slope field plot

Verification of solutions

$$y = e^{-\sin(x)} \left(\sin(x) e^{\sin(x)} - e^{\sin(x)} + c_1\right)$$

Verified OK.

1.17.4 Maple step by step solution

Let's solve

$y' + y\cos(x) = \frac{\sin(2x)}{2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y\cos(x) + \frac{\sin(2x)}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\cos(x) = \frac{\sin(2x)}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' + y\cos(x)) = \frac{\mu(x)\sin(2x)}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)(y' + y\cos(x)) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\cos(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\sin(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \frac{\mu(x)\sin(2x)}{2}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)\sin(2x)}{2}dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\sin(2x)}{2}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\sin(x)}$

  $y = \frac{\int \frac{\sin(2x)\mathrm{e}^{\sin(x)}}{2}dx + c_1}{\mathrm{e}^{\sin(x)}}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x)\mathrm{e}^{\sin(x)} - \mathrm{e}^{\sin(x)} + c_1}{\mathrm{e}^{\sin(x)}}$

- Simplify

$$y = \sin(x) - 1 + c_1 e^{-\sin(x)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)*cos(x)=1/2*sin(2*x),y(x), singsol=all)
```

$$y(x) = \sin(x) - 1 + e^{-\sin(x)} c_1$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 18

```
DSolve[y'[x]+y[x]*Cos[x]==1/2*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x) + c_1 e^{-\sin(x)} - 1$$

1.18 problem 6.5

Internal problem ID [4372]
Internal file name [`OUTPUT/3865_Sunday_June_05_2022_11_32_17_AM_76211693/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 6.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x^2+1)\, y'+y=\arctan(x)}$$

1.18.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{1}{x^2+1}$$
$$q(x)=\frac{\arctan(x)}{x^2+1}$$

Hence the ode is

$$y'+\frac{y}{x^2+1}=\frac{\arctan(x)}{x^2+1}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x^2+1}dx} \\
&= e^{\arctan(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\arctan(x)}{x^2+1}\right) \\
\frac{d}{dx}\left(e^{\arctan(x)} y\right) &= \left(e^{\arctan(x)}\right)\left(\frac{\arctan(x)}{x^2+1}\right) \\
d\left(e^{\arctan(x)} y\right) &= \left(\frac{\arctan(x)\, e^{\arctan(x)}}{x^2+1}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{\arctan(x)} y &= \int \frac{\arctan(x)\, e^{\arctan(x)}}{x^2+1}\, dx \\
e^{\arctan(x)} y &= \arctan(x)\, e^{\arctan(x)} - e^{\arctan(x)} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{\arctan(x)}$ results in

$$
y = e^{-\arctan(x)}\left(\arctan(x)\, e^{\arctan(x)} - e^{\arctan(x)}\right) + c_1 e^{-\arctan(x)}
$$

which simplifies to

$$
y = \arctan(x) - 1 + c_1 e^{-\arctan(x)}
$$

Summary
The solution(s) found are the following

$$
y = \arctan(x) - 1 + c_1 e^{-\arctan(x)} \tag{1}
$$

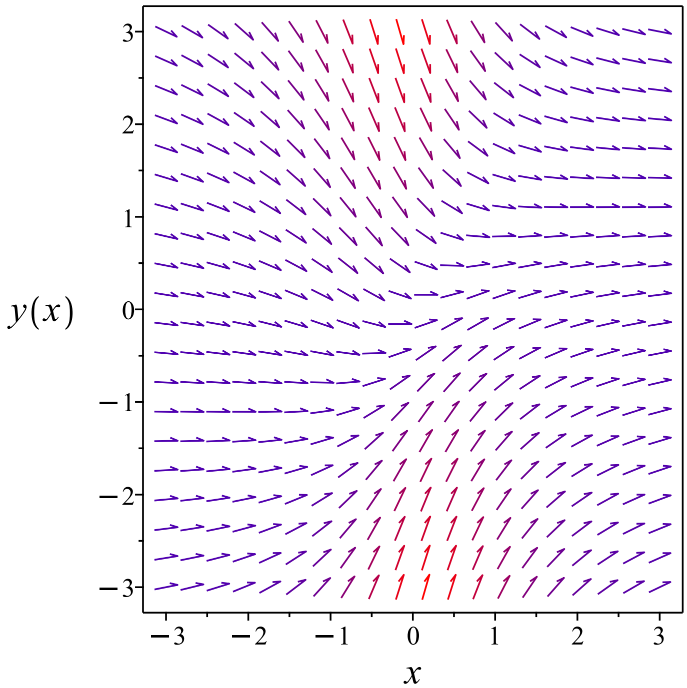


Figure 44: Slope field plot

Verification of solutions

$$y = \arctan(x) - 1 + c_1 e^{-\arctan(x)}$$

Verified OK.

1.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y + \arctan(x)}{x^2 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 33: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-\arctan(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\arctan(x)}} dy \end{aligned}$$

Which results in

$$S = e^{\arctan(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-y + \arctan(x)}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^{\arctan(x)} y}{x^2 + 1} \\ S_y &= e^{\arctan(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\arctan(x) e^{\arctan(x)}}{x^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\arctan(R) e^{\arctan(R)}}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R)\,e^{\arctan(R)} - e^{\arctan(R)} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\arctan(x)} y = \arctan(x)\,e^{\arctan(x)} - e^{\arctan(x)} + c_1$$

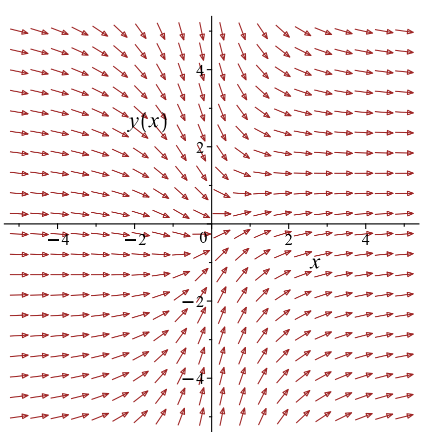
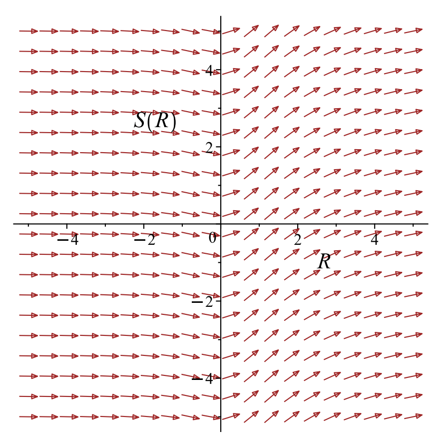
Which simplifies to

$$(y - \arctan(x) + 1)\,e^{\arctan(x)} - c_1 = 0$$

Which gives

$$y = \left(\arctan(x)\,e^{\arctan(x)} - e^{\arctan(x)} + c_1\right)e^{-\arctan(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = \frac{-y+\arctan(x)}{x^2+1}$

</td><td>$R = x$
$S = e^{\arctan(x)} y$</td><td>$\frac{dS}{dR} = \frac{\arctan(R)e^{\arctan(R)}}{R^2+1}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \left(\arctan(x)\,e^{\arctan(x)} - e^{\arctan(x)} + c_1\right)e^{-\arctan(x)} \tag{1}$$

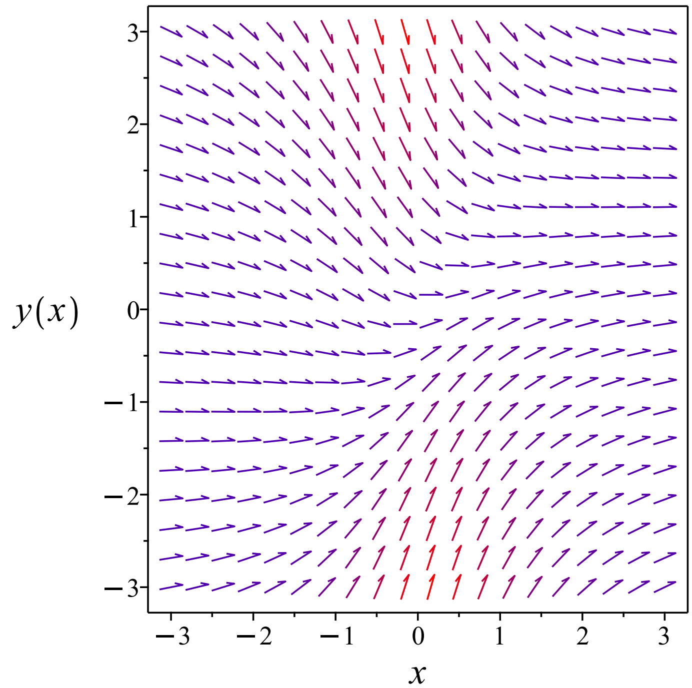


Figure 45: Slope field plot

Verification of solutions

$$y = \left(\arctan(x)\, e^{\arctan(x)} - e^{\arctan(x)} + c_1\right) e^{-\arctan(x)}$$

Verified OK.

1.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + 1)\,\mathrm{d}y &= (-y + \arctan(x))\,\mathrm{d}x\\ (y - \arctan(x))\,\mathrm{d}x + (x^2 + 1)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y - \arctan(x)$$
$$N(x, y) = x^2 + 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \arctan(x))\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + 1)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x^2+1}((1) - (2x)) \\ &= \frac{1-2x}{x^2+1} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1-2x}{x^2+1} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(x^2+1)+\arctan(x)} \\ &= \frac{\mathrm{e}^{\arctan(x)}}{x^2+1} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{\mathrm{e}^{\arctan(x)}}{x^2+1}(y - \arctan(x)) \\ &= \frac{(y - \arctan(x))\,\mathrm{e}^{\arctan(x)}}{x^2+1} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{\mathrm{e}^{\arctan(x)}}{x^2+1}(x^2+1) \\ &= \mathrm{e}^{\arctan(x)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{(y - \arctan(x))\,\mathrm{e}^{\arctan(x)}}{x^2+1}\right) + \left(\mathrm{e}^{\arctan(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{(y - \arctan(x))\, \mathrm{e}^{\arctan(x)}}{x^2+1} \,\mathrm{d}x$$

$$\phi = (y - \arctan(x) + 1)\, \mathrm{e}^{\arctan(x)} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\arctan(x)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\arctan(x)}$. Therefore equation (4) becomes

$$\mathrm{e}^{\arctan(x)} = \mathrm{e}^{\arctan(x)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (y - \arctan(x) + 1)\, \mathrm{e}^{\arctan(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y - \arctan(x) + 1)\, \mathrm{e}^{\arctan(x)}$$

Summary
The solution(s) found are the following

$$(y - \arctan(x) + 1)\, e^{\arctan(x)} = c_1 \tag{1}$$

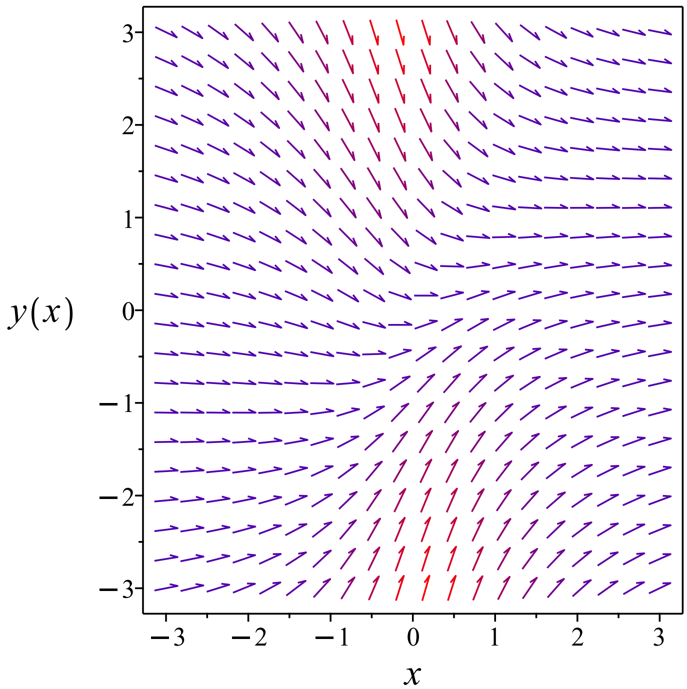


Figure 46: Slope field plot

Verification of solutions

$$(y - \arctan(x) + 1)\, e^{\arctan(x)} = c_1$$

Verified OK.

1.18.4 Maple step by step solution

Let's solve

$(x^2 + 1)\, y' + y = \arctan(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x^2+1} + \frac{\arctan(x)}{x^2+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x^2+1} = \frac{\arctan(x)}{x^2+1}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x^2+1}\right) = \frac{\mu(x)\arctan(x)}{x^2+1}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{x^2+1}\right) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x^2+1}$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\arctan(x)}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)\arctan(x)}{x^2+1} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)\arctan(x)}{x^2+1} dx + c_1$
- Solve for y

  $y = \frac{\int \frac{\mu(x)\arctan(x)}{x^2+1} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{\arctan(x)}$

  $y = \frac{\int \frac{\arctan(x)\mathrm{e}^{\arctan(x)}}{x^2+1} dx + c_1}{\mathrm{e}^{\arctan(x)}}$
- Evaluate the integrals on the rhs

  $y = \frac{\arctan(x)\mathrm{e}^{\arctan(x)} - \mathrm{e}^{\arctan(x)} + c_1}{\mathrm{e}^{\arctan(x)}}$
- Simplify

  $y = \arctan(x) - 1 + c_1 \mathrm{e}^{-\arctan(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x)+y(x)=arctan(x),y(x), singsol=all)
```

$$y(x) = \arctan(x) - 1 + e^{-\arctan(x)} c_1$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 18

```
DSolve[(1+x^2)*y'[x]+y[x]==ArcTan[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arctan(x) + c_1 e^{-\arctan(x)} - 1$$

1.19 problem 10.1

Internal problem ID [4373]
Internal file name [`OUTPUT/3866_Sunday_June_05_2022_11_32_25_AM_87361032/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 10.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(-x^2+1)\, z' - xz - axz^2 = 0}$$

1.19.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} z' &= F(x,z) \\ &= f(x)g(z) \\ &= -\frac{xz(az+1)}{x^2-1} \end{aligned}$$

Where $f(x) = -\frac{x}{x^2-1}$ and $g(z) = z(az+1)$. Integrating both sides gives

$$\frac{1}{z\,(az+1)}\, dz = -\frac{x}{x^2-1}\, dx$$

$$
\int \frac{1}{z\left(az+1\right)}dz = \int -\frac{x}{x^2-1}dx
$$

$$
-\ln\left(az+1\right)+\ln\left(z\right) = -\frac{\ln\left(x-1\right)}{2}-\frac{\ln\left(x+1\right)}{2}+c_1
$$

Raising both side to exponential gives

$$
\mathrm{e}^{-\ln(az+1)+\ln(z)} = \mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}+c_1}
$$

Which simplifies to

$$
\frac{z}{az+1} = c_2\mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}}
$$

Which simplifies to

$$
z = -\frac{c_2}{\sqrt{x+1}\,\sqrt{x-1}\,\left(-1+\frac{c_2 a}{\sqrt{x+1}\,\sqrt{x-1}}\right)}
$$

Summary

The solution(s) found are the following

$$
z = -\frac{c_2}{\sqrt{x+1}\,\sqrt{x-1}\,\left(-1+\frac{c_2 a}{\sqrt{x+1}\,\sqrt{x-1}}\right)} \tag{1}
$$

Verification of solutions

$$
z = -\frac{c_2}{\sqrt{x+1}\,\sqrt{x-1}\,\left(-1+\frac{c_2 a}{\sqrt{x+1}\,\sqrt{x-1}}\right)}
$$

Verified OK.

1.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
z' &= -\frac{xz(az+1)}{x^2-1} \\
z' &= \omega(x,z)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_z-\xi_x)-\omega^2\xi_z-\omega_x\xi-\omega_z\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 36: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, z) &= -\frac{x^2 - 1}{x} \\
\eta(x, z) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, z) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dz}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial z}\right) S(x,z) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = z$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x^2-1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,z) S_z}{R_x + \omega(x,z) R_z} \tag{2}$$

Where in the above R_x, R_z, S_x, S_z are all partial derivatives and $\omega(x,z)$ is the right hand side of the original ode given by

$$\omega(x,z) = -\frac{xz(az+1)}{x^2-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_z &= 1 \\ S_x &= -\frac{x}{x^2-1} \\ S_z &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{z(az+1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, z in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(Ra+1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(Ra+1) + \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, z coordinates. This results in

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} = -\ln(az+1) + \ln(z) + c_1$$

Which simplifies to

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} = -\ln(az+1) + \ln(z) + c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} = -\ln(az+1) + \ln(z) + c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} = -\ln(az+1) + \ln(z) + c_1$$

Verified OK.

1.19.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} z' &= F(x,z) \\ &= -\frac{xz(az+1)}{x^2-1} \end{aligned}$$

This is a Bernoulli ODE.

$$z' = -\frac{x}{x^2-1}z - \frac{ax}{x^2-1}z^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$z' = f_0(x)z + f_1(x)z^n \tag{2}$$

The first step is to divide the above equation by z^n which gives

$$\frac{z'}{z^n} = f_0(x)z^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $r = z^{1-n}$ in equation (3) which generates a new ODE in $r(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $z(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= -\frac{x}{x^2-1} \\
f_1(x) &= -\frac{ax}{x^2-1} \\
n &= 2
\end{aligned}$$

Dividing both sides of ODE (1) by $z^n = z^2$ gives

$$z'\frac{1}{z^2} = -\frac{x}{(x^2-1)\,z} - \frac{ax}{x^2-1} \tag{4}$$

Let

$$\begin{aligned}
r &= z^{1-n} \\
&= \frac{1}{z}
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$r' = -\frac{1}{z^2}z' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
-r'(x) &= -\frac{xr(x)}{x^2-1} - \frac{ax}{x^2-1} \\
r' &= \frac{xr}{x^2-1} + \frac{ax}{x^2-1}
\end{aligned} \tag{7}$$

The above now is a linear ODE in $r(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r'(x) + p(x)r(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{x}{x^2-1} \\
q(x) &= \frac{ax}{x^2-1}
\end{aligned}$$

Hence the ode is

$$r'(x) - \frac{xr(x)}{x^2-1} = \frac{ax}{x^2-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{x}{x^2-1}dx}\\ &= e^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x+1}\,\sqrt{x-1}}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu r) &= (\mu)\left(\frac{ax}{x^2-1}\right)\\ \frac{d}{dx}\left(\frac{r}{\sqrt{x+1}\,\sqrt{x-1}}\right) &= \left(\frac{1}{\sqrt{x+1}\,\sqrt{x-1}}\right)\left(\frac{ax}{x^2-1}\right)\\ d\left(\frac{r}{\sqrt{x+1}\,\sqrt{x-1}}\right) &= \left(\frac{ax}{(x^2-1)\sqrt{x+1}\,\sqrt{x-1}}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{r}{\sqrt{x+1}\,\sqrt{x-1}} &= \int \frac{ax}{(x^2-1)\sqrt{x+1}\,\sqrt{x-1}}\, dx\\ \frac{r}{\sqrt{x+1}\,\sqrt{x-1}} &= -\frac{\sqrt{x-1}\,\sqrt{x+1}\,a}{x^2-1} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x+1}\,\sqrt{x-1}}$ results in

$$r(x) = -\frac{(x+1)(x-1)a}{x^2-1} + c_1\sqrt{x+1}\,\sqrt{x-1}$$

which simplifies to

$$r(x) = -a + c_1\sqrt{x+1}\,\sqrt{x-1}$$

Replacing r in the above by $\frac{1}{z}$ using equation (5) gives the final solution.

$$\frac{1}{z} = -a + c_1\sqrt{x+1}\,\sqrt{x-1}$$

Or

$$z = \frac{1}{-a + c_1\sqrt{x+1}\,\sqrt{x-1}}$$

Summary

The solution(s) found are the following

$$z = \frac{1}{-a + c_1\sqrt{x+1}\,\sqrt{x-1}} \tag{1}$$

Verification of solutions

$$z = \frac{1}{-a + c_1\sqrt{x+1}\,\sqrt{x-1}}$$

Verified OK.

1.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,z)\,\mathrm{d}x + N(x,z)\,\mathrm{d}z = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(-\frac{1}{z\,(az+1)}\right)\mathrm{d}z &= \left(\frac{x}{x^2-1}\right)\mathrm{d}x \\
\left(-\frac{x}{x^2-1}\right)\mathrm{d}x + \left(-\frac{1}{z\,(az+1)}\right)\mathrm{d}z &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,z) &= -\frac{x}{x^2-1} \\
N(x,z) &= -\frac{1}{z\,(az+1)}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial z} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial z} &= \frac{\partial}{\partial z}\left(-\frac{x}{x^2-1}\right) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{z\,(az+1)}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial z} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,z)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial z} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{x}{x^2-1}\,\mathrm{d}x$$
$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + f(z) \tag{3}$$

Where $f(z)$ is used for the constant of integration since ϕ is a function of both x and z. Taking derivative of equation (3) w.r.t z gives

$$\frac{\partial \phi}{\partial z} = 0 + f'(z) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial z} = -\frac{1}{z(az+1)}$. Therefore equation (4) becomes

$$-\frac{1}{z(az+1)} = 0 + f'(z) \tag{5}$$

Solving equation (5) for $f'(z)$ gives

$$f'(z) = -\frac{1}{z(az+1)}$$

Integrating the above w.r.t z gives

$$\int f'(z)\,\mathrm{d}z = \int \left(-\frac{1}{z(az+1)}\right)\mathrm{d}z$$
$$f(z) = \ln(az+1) - \ln(z) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(z)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(az+1) - \ln(z) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(az+1) - \ln(z)$$

Summary
The solution(s) found are the following

$$
-\frac{\ln (x-1)}{2}-\frac{\ln (x+1)}{2}+\ln (az+1)-\ln (z)=c_{1} \tag{1}
$$

Verification of solutions

$$
-\frac{\ln (x-1)}{2}-\frac{\ln (x+1)}{2}+\ln (az+1)-\ln (z)=c_{1}
$$

Verified OK.

1.19.5 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
z' &= F(x,z) \\
&= -\frac{xz(az+1)}{x^{2}-1}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
z'=-\frac{x\,z^{2}a}{x^{2}-1}-\frac{xz}{x^{2}-1}
$$

With Riccati ODE standard form

$$
z'=f_{0}(x)+f_{1}(x)z+f_{2}(x)z^{2}
$$

Shows that $f_0(x)=0$, $f_1(x)=-\frac{x}{x^2-1}$ and $f_2(x)=-\frac{ax}{x^2-1}$. Let

$$
\begin{aligned}
z &= \frac{-u'}{f_{2}u} \\
&= \frac{-u'}{-\frac{axu}{x^{2}-1}} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_{2}u''(x)-\left(f_{2}'+f_{1}f_{2}\right)u'(x)+f_{2}^{2}f_{0}u(x)=0 \tag{2}
$$

But

$$
\begin{aligned}
f_{2}' &= -\frac{a}{x^{2}-1}+\frac{2a\,x^{2}}{\left(x^{2}-1\right)^{2}} \\
f_{1}f_{2} &= \frac{a\,x^{2}}{\left(x^{2}-1\right)^{2}} \\
f_{2}^{2}f_{0} &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$-\frac{axu''(x)}{x^2-1} - \left(-\frac{a}{x^2-1} + \frac{3a\,x^2}{\left(x^2-1\right)^2}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{\sqrt{x^2-1}}$$

The above shows that

$$u'(x) = -\frac{c_2 x}{\left(x^2-1\right)^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$z = -\frac{c_2}{\sqrt{x^2-1}\,a\left(c_1 + \frac{c_2}{\sqrt{x^2-1}}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$z = -\frac{1}{a\left(c_3\sqrt{x^2-1}+1\right)}$$

Summary

The solution(s) found are the following

$$z = -\frac{1}{a\left(c_3\sqrt{x^2-1}+1\right)} \tag{1}$$

Verification of solutions

$$z = -\frac{1}{a\left(c_3\sqrt{x^2-1}+1\right)}$$

Verified OK.

1.19.6 Maple step by step solution

Let's solve

$\left(-x^{2}+1\right) z'-x z-a x z^{2}=0$

- Highest derivative means the order of the ODE is 1

z'

- Separate variables

$\frac{z'}{z(az+1)}=-\frac{x}{(x-1)(x+1)}$

- Integrate both sides with respect to x

$\int \frac{z'}{z(az+1)}dx=\int -\frac{x}{(x-1)(x+1)}dx+c_1$

- Evaluate integral

$-\ln \left(a z+1\right)+\ln \left(z\right)=-\frac{\ln((x-1)(x+1))}{2}+c_1$

- Solve for z

$\left\{z=-\frac{e^{2c_1}a-\sqrt{e^{2c_1}x^2-e^{2c_1}}}{e^{2c_1}a^2-x^2+1}, z=-\frac{e^{2c_1}a+\sqrt{e^{2c_1}x^2-e^{2c_1}}}{e^{2c_1}a^2-x^2+1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve((1-x^2)*diff(z(x),x)-x*z(x)=a*x*z(x)^2,z(x), singsol=all)
```

$$z(x)=\frac{1}{\sqrt{x-1}\,\sqrt{1+x}\,c_1-a}$$

✓ Solution by Mathematica

Time used: 3.943 (sec). Leaf size: 47

```
DSolve[(1-x^2)*z'[x]-x*z[x]==a*x*z[x]^2,z[x],x,IncludeSingularSolutions -> True]
```

$$z(x) \to -\frac{e^{c_1}}{-\sqrt{1-x^2}+ae^{c_1}}$$
$$z(x) \to 0$$
$$z(x) \to -\frac{1}{a}$$

1.20 problem 10.2

Internal problem ID [4374]
Internal file name [`OUTPUT/3867_Sunday_June_05_2022_11_32_35_AM_44700999/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 10.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{3z^2z' - az^3 = x + 1}$$

1.20.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$z' = \frac{a\,z^3 + x + 1}{3z^2}$$
$$z' = \omega(x, z)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_z - \xi_x) - \omega^2\xi_z - \omega_x\xi - \omega_z\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 39: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, z) &= 0 \\
\eta(x, z) &= \frac{e^{ax}}{z^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, z) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dz}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial z}\right) S(x, z) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{ax}}{z^2}} dy \end{aligned}$$

Which results in

$$S = \frac{z^3 e^{-ax}}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, z) S_z}{R_x + \omega(x, z) R_z} \tag{2}$$

Where in the above R_x, R_z, S_x, S_z are all partial derivatives and $\omega(x, z)$ is the right hand side of the original ode given by

$$\omega(x, z) = \frac{a\, z^3 + x + 1}{3z^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_z &= 0 \\ S_x &= -\frac{z^3 a\, e^{-ax}}{3} \\ S_z &= z^2 e^{-ax} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-ax}(x + 1)}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, z in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{-aR}(R + 1)}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{(aR + a + 1)\,\mathrm{e}^{-aR}}{3a^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, z coordinates. This results in

$$\frac{z^3\mathrm{e}^{-ax}}{3} = -\frac{(ax + a + 1)\,\mathrm{e}^{-ax}}{3a^2} + c_1$$

Which simplifies to

$$\frac{z^3\mathrm{e}^{-ax}}{3} = -\frac{(ax + a + 1)\,\mathrm{e}^{-ax}}{3a^2} + c_1$$

Summary

The solution(s) found are the following

$$\frac{z^3\mathrm{e}^{-ax}}{3} = -\frac{(ax + a + 1)\,\mathrm{e}^{-ax}}{3a^2} + c_1 \tag{1}$$

Verification of solutions

$$\frac{z^3\mathrm{e}^{-ax}}{3} = -\frac{(ax + a + 1)\,\mathrm{e}^{-ax}}{3a^2} + c_1$$

Verified OK.

1.20.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} z' &= F(x, z) \\ &= \frac{a\,z^3 + x + 1}{3z^2} \end{aligned}$$

This is a Bernoulli ODE.

$$z' = \frac{a}{3}z + \frac{x}{3} + \frac{1}{3}\frac{1}{z^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$z' = f_0(x)z + f_1(x)z^n \tag{2}$$

The first step is to divide the above equation by z^n which gives

$$\frac{z'}{z^n} = f_0(x)z^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $r = z^{1-n}$ in equation (3) which generates a new ODE in $r(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $z(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{a}{3} \\ f_1(x) &= \frac{x}{3} + \frac{1}{3} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $z^n = \frac{1}{z^2}$ gives

$$z' z^2 = \frac{a\, z^3}{3} + \frac{x}{3} + \frac{1}{3} \tag{4}$$

Let

$$\begin{aligned} r &= z^{1-n} \\ &= z^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$r' = 3z^2 z' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{r'(x)}{3} &= \frac{ar(x)}{3} + \frac{x}{3} + \frac{1}{3} \\ r' &= ar + x + 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $r(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r'(x) + p(x)r(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -a \\ q(x) &= x + 1 \end{aligned}$$

Hence the ode is

$$r'(x) - ar(x) = x + 1$$

The integrating factor μ is

$$\mu = e^{\int -a dx}$$
$$= e^{-ax}$$

The ode becomes

$$\frac{d}{dx}(\mu r) = (\mu)(x + 1)$$
$$\frac{d}{dx}\left(e^{-ax} r\right) = \left(e^{-ax}\right)(x + 1)$$
$$d\left(e^{-ax} r\right) = \left(e^{-ax}(x + 1)\right) dx$$

Integrating gives

$$e^{-ax} r = \int e^{-ax}(x + 1) dx$$
$$e^{-ax} r = -\frac{(ax + a + 1) e^{-ax}}{a^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-ax}$ results in

$$r(x) = -\frac{e^{ax}(ax + a + 1) e^{-ax}}{a^2} + c_1 e^{ax}$$

which simplifies to

$$r(x) = \frac{c_1 e^{ax} a^2 - 1 + (-1 - x) a}{a^2}$$

Replacing r in the above by z^3 using equation (5) gives the final solution.

$$z^3 = \frac{c_1 e^{ax} a^2 - 1 + (-1 - x) a}{a^2}$$

Solving for z gives

$$z(x) = \frac{\left(\left(c_1 e^{ax} a^2 - 1 + (-1 - x) a\right) a\right)^{\frac{1}{3}}}{a}$$
$$z(x) = \frac{\left(\left(c_1 e^{ax} a^2 - 1 + (-1 - x) a\right) a\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2a}$$
$$z(x) = -\frac{\left(\left(c_1 e^{ax} a^2 - 1 + (-1 - x) a\right) a\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2a}$$

Summary
The solution(s) found are the following

$$z = \frac{\left(\left(c_1 \mathrm{e}^{ax} a^2 - 1 + \left(-1 - x\right) a\right) a\right)^{\frac{1}{3}}}{a} \tag{1}$$

$$z = \frac{\left(\left(c_1 \mathrm{e}^{ax} a^2 - 1 + \left(-1 - x\right) a\right) a\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2a} \tag{2}$$

$$z = -\frac{\left(\left(c_1 \mathrm{e}^{ax} a^2 - 1 + \left(-1 - x\right) a\right) a\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2a} \tag{3}$$

Verification of solutions

$$z = \frac{\left(\left(c_1 \mathrm{e}^{ax} a^2 - 1 + \left(-1 - x\right) a\right) a\right)^{\frac{1}{3}}}{a}$$

Verified OK.

$$z = \frac{\left(\left(c_1 \mathrm{e}^{ax} a^2 - 1 + \left(-1 - x\right) a\right) a\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2a}$$

Verified OK.

$$z = -\frac{\left(\left(c_1 \mathrm{e}^{ax} a^2 - 1 + \left(-1 - x\right) a\right) a\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2a}$$

Verified OK.

1.20.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, z)\,\mathrm{d}x + N(x, z)\,\mathrm{d}z = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(3z^2\right)\mathrm{d}z &= \left(a\,z^3 + x + 1\right)\mathrm{d}x \\ \left(-a\,z^3 - x - 1\right)\mathrm{d}x + \left(3z^2\right)\mathrm{d}z &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, z) &= -a\,z^3 - x - 1 \\ N(x, z) &= 3z^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial z} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial z} &= \frac{\partial}{\partial z}\left(-a\,z^3 - x - 1\right) \\ &= -3a\,z^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3z^2\right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial z} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial z} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{3z^2}\left(\left(-3a\,z^2\right) - (0)\right) \\
&= -a
\end{aligned}$$

Since A does not depend on z, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x} \\
&= e^{\int -a\,\mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-ax} \\
&= \mathrm{e}^{-ax}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-ax}\left(-a\,z^3 - x - 1\right) \\
&= -\mathrm{e}^{-ax}\left(a\,z^3 + x + 1\right)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-ax}\left(3z^2\right) \\
&= 3z^2\mathrm{e}^{-ax}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}z}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-ax}\left(a\,z^3 + x + 1\right)\right) + \left(3z^2\mathrm{e}^{-ax}\right)\frac{\mathrm{d}z}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, z)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial z} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\mathrm{e}^{-ax} \left(a\, z^3 + x + 1\right) \mathrm{d}x$$

$$\phi = \frac{\left(z^3 a^2 + ax + a + 1\right) \mathrm{e}^{-ax}}{a^2} + f(z) \tag{3}$$

Where $f(z)$ is used for the constant of integration since ϕ is a function of both x and z. Taking derivative of equation (3) w.r.t z gives

$$\frac{\partial \phi}{\partial z} = 3z^2 \mathrm{e}^{-ax} + f'(z) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial z} = 3z^2 \mathrm{e}^{-ax}$. Therefore equation (4) becomes

$$3z^2 \mathrm{e}^{-ax} = 3z^2 \mathrm{e}^{-ax} + f'(z) \tag{5}$$

Solving equation (5) for $f'(z)$ gives

$$f'(z) = 0$$

Therefore

$$f(z) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(z)$ into equation (3) gives ϕ

$$\phi = \frac{\left(z^3 a^2 + ax + a + 1\right) \mathrm{e}^{-ax}}{a^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\left(z^3 a^2 + ax + a + 1\right) \mathrm{e}^{-ax}}{a^2}$$

Summary

The solution(s) found are the following

$$\frac{\left(z^3 a^2 + ax + a + 1\right) \mathrm{e}^{-ax}}{a^2} = c_1 \tag{1}$$

Verification of solutions

$$\frac{(z^3a^2 + ax + a + 1)\,\mathrm{e}^{-ax}}{a^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 106

```
dsolve(3*z(x)^2*diff(z(x),x)-a*z(x)^3=x+1,z(x), singsol=all)
```

$$z(x) = \frac{((\mathrm{e}^{ax}c_1a^2 - 1 + (-1-x)\,a)\,a)^{\frac{1}{3}}}{a}$$
$$z(x) = -\frac{((\mathrm{e}^{ax}c_1a^2 - 1 + (-1-x)\,a)\,a)^{\frac{1}{3}}\,(1 + i\sqrt{3})}{2a}$$
$$z(x) = \frac{((\mathrm{e}^{ax}c_1a^2 - 1 + (-1-x)\,a)\,a)^{\frac{1}{3}}\,(i\sqrt{3} - 1)}{2a}$$

✓ Solution by Mathematica

Time used: 14.566 (sec). Leaf size: 111

```
DSolve[3*z[x]^2*z'[x]-a*z[x]^3==x+1,z[x],x,IncludeSingularSolutions -> True]
```

$$z(x) \to \frac{\sqrt[3]{a^2c_1e^{ax} - a(x+1) - 1}}{a^{2/3}}$$
$$z(x) \to -\frac{\sqrt[3]{-1}\,\sqrt[3]{a^2c_1e^{ax} - a(x+1) - 1}}{a^{2/3}}$$
$$z(x) \to \frac{(-1)^{2/3}\,\sqrt[3]{a^2c_1e^{ax} - a(x+1) - 1}}{a^{2/3}}$$

1.21 problem 10.3

Internal problem ID [4375]
Internal file name [`OUTPUT/3868_Sunday_June_05_2022_11_32_51_AM_63725271/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 10.3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{z' + 2xz - 2a\,x^3 z^3 = 0}$$

1.21.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} z' &= 2a\,x^3 z^3 - 2xz \\ z' &= \omega(x, z) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_z - \xi_x) - \omega^2 \xi_z - \omega_x \xi - \omega_z \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 41: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, z) &= 0 \\
\eta(x, z) &= z^3 \mathrm{e}^{2x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, z) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dz}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial z}\right) S(x, z) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{z^3 e^{2x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{-2x^2}}{2z^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,z) S_z}{R_x + \omega(x,z) R_z} \tag{2}$$

Where in the above R_x, R_z, S_x, S_z are all partial derivatives and $\omega(x,z)$ is the right hand side of the original ode given by

$$\omega(x,z) = 2a\, x^3 z^3 - 2xz$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_z &= 0 \\ S_x &= \frac{2x\, e^{-2x^2}}{z^2} \\ S_z &= \frac{e^{-2x^2}}{z^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x^3 e^{-2x^2} a \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, z in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R^3 e^{-2R^2} a$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{(2R^2+1)\,\mathrm{e}^{-2R^2} a}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, z coordinates. This results in

$$-\frac{\mathrm{e}^{-2x^2}}{2z^2} = -\frac{(2x^2+1)\,\mathrm{e}^{-2x^2} a}{4} + c_1$$

Which simplifies to

$$-\frac{\mathrm{e}^{-2x^2}}{2z^2} = -\frac{(2x^2+1)\,\mathrm{e}^{-2x^2} a}{4} + c_1$$

Summary

The solution(s) found are the following

$$-\frac{\mathrm{e}^{-2x^2}}{2z^2} = -\frac{(2x^2+1)\,\mathrm{e}^{-2x^2} a}{4} + c_1 \tag{1}$$

Verification of solutions

$$-\frac{\mathrm{e}^{-2x^2}}{2z^2} = -\frac{(2x^2+1)\,\mathrm{e}^{-2x^2} a}{4} + c_1$$

Verified OK.

1.21.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} z' &= F(x,z) \\ &= 2a\,x^3z^3 - 2xz \end{aligned}$$

This is a Bernoulli ODE.

$$z' = -2xz + 2a\,x^3z^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$z' = f_0(x)z + f_1(x)z^n \tag{2}$$

The first step is to divide the above equation by z^n which gives

$$\frac{z'}{z^n} = f_0(x)z^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $r = z^{1-n}$ in equation (3) which generates a new ODE in $r(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $z(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -2x \\ f_1(x) &= 2a\,x^3 \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $z^n = z^3$ gives

$$z'\frac{1}{z^3} = -\frac{2x}{z^2} + 2a\,x^3 \tag{4}$$

Let

$$\begin{aligned} r &= z^{1-n} \\ &= \frac{1}{z^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$r' = -\frac{2}{z^3}z' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{r'(x)}{2} &= -2r(x)\,x + 2a\,x^3 \\ r' &= -4a\,x^3 + 4xr \end{aligned} \tag{7}$$

The above now is a linear ODE in $r(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r'(x) + p(x)r(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -4x \\ q(x) &= -4a\,x^3 \end{aligned}$$

Hence the ode is

$$r'(x) - 4r(x)\,x = -4a\,x^3$$

The integrating factor μ is

$$\mu = e^{\int -4x dx}$$
$$= e^{-2x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu r) = (\mu)\left(-4a\,x^3\right)$$
$$\frac{d}{dx}\left(e^{-2x^2} r\right) = \left(e^{-2x^2}\right)\left(-4a\,x^3\right)$$
$$d\left(e^{-2x^2} r\right) = \left(-4x^3 e^{-2x^2} a\right)\,dx$$

Integrating gives

$$e^{-2x^2} r = \int -4x^3 e^{-2x^2} a\,dx$$
$$e^{-2x^2} r = \frac{(2x^2+1)\,e^{-2x^2} a}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-2x^2}$ results in

$$r(x) = \frac{e^{2x^2}(2x^2+1)\,e^{-2x^2} a}{2} + c_1 e^{2x^2}$$

which simplifies to

$$r(x) = a\,x^2 + \frac{a}{2} + c_1 e^{2x^2}$$

Replacing r in the above by $\frac{1}{z^2}$ using equation (5) gives the final solution.

$$\frac{1}{z^2} = a\,x^2 + \frac{a}{2} + c_1 e^{2x^2}$$

Solving for z gives

$$z(x) = \frac{2}{\sqrt{4a\,x^2 + 4c_1 e^{2x^2} + 2a}}$$
$$z(x) = -\frac{2}{\sqrt{4a\,x^2 + 4c_1 e^{2x^2} + 2a}}$$

Summary

The solution(s) found are the following

$$z = \frac{2}{\sqrt{4a\,x^2 + 4c_1 e^{2x^2} + 2a}} \tag{1}$$
$$z = -\frac{2}{\sqrt{4a\,x^2 + 4c_1 e^{2x^2} + 2a}} \tag{2}$$

Verification of solutions

$$z = \frac{2}{\sqrt{4a\,x^2 + 4c_1 e^{2x^2} + 2a}}$$

Verified OK.

$$z = -\frac{2}{\sqrt{4a\,x^2 + 4c_1 e^{2x^2} + 2a}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
dsolve(diff(z(x),x)+2*x*z(x)=2*a*x^3*z(x)^3,z(x), singsol=all)
```

$$z(x) = -\frac{2}{\sqrt{4a\,x^2 + 4\,e^{2x^2} c_1 + 2a}}$$
$$z(x) = \frac{2}{\sqrt{4a\,x^2 + 4\,e^{2x^2} c_1 + 2a}}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 29

```
DSolve[z'[x]+2*x*z[x]==2*a*x^3*z[x],z[x],x,IncludeSingularSolutions -> True]
```

$$z(x) \to c_1 e^{\frac{ax^4}{2} - x^2}$$
$$z(x) \to 0$$

1.22 problem 10.4

Internal problem ID [4376]
Internal file name [`OUTPUT/3869_Sunday_June_05_2022_11_33_02_AM_57925054/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 10.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{z' + z\cos(x) - z^n \sin(2x) = 0}$$

1.22.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} z' &= -z\cos(x) + z^n \sin(2x) \\ z' &= \omega(x, z) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_z - \xi_x) - \omega^2 \xi_z - \omega_x \xi - \omega_z \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 43: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, z) &= 0 \\
\eta(x, z) &= z^n \mathrm{e}^{(n-1)\sin(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, z) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dz}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial z}\right) S(x, z) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{z^n e^{(n-1)\sin(x)}} dy \end{aligned}$$

Which results in

$$S = -\frac{z\, z^{-n} e^{-(n-1)\sin(x)}}{n-1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,z) S_z}{R_x + \omega(x,z) R_z} \tag{2}$$

Where in the above R_x, R_z, S_x, S_z are all partial derivatives and $\omega(x,z)$ is the right hand side of the original ode given by

$$\omega(x,z) = -z\cos(x) + z^n \sin(2x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_z &= 0 \\ S_x &= z^{-n+1} \cos(x)\, e^{-(n-1)\sin(x)} \\ S_z &= z^{-n} e^{-(n-1)\sin(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\, e^{-(n-1)\sin(x)} \cos(x) \sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, z in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\, e^{-(n-1)\sin(R)} \cos(R) \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{c_1 (n-1)^2 - 2\,\mathrm{e}^{-(n-1)\sin(R)} (1 + (n-1)\sin(R))}{(n-1)^2} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, z coordinates. This results in

$$-\frac{z^{-n+1} \mathrm{e}^{-(n-1)\sin(x)}}{n-1} = \frac{c_1 (n-1)^2 - 2\,\mathrm{e}^{-(n-1)\sin(x)} (1 + (n-1)\sin(x))}{(n-1)^2}$$

Which simplifies to

$$\frac{((-n+1)\, z^{-n+1} + 2 + (2n-2)\sin(x))\, \mathrm{e}^{-(n-1)\sin(x)} - c_1 (n-1)^2}{(n-1)^2} = 0$$

Which gives

$$z = \mathrm{e}^{-\frac{\sin(x)n + \ln\left(\frac{2\sin(x)\mathrm{e}^{-(n-1)\sin(x)} n - c_1 n^2 - 2\sin(x)\mathrm{e}^{-(n-1)\sin(x)} + 2c_1 n + 2\,\mathrm{e}^{-(n-1)\sin(x)} - c_1}{n-1}\right) - \sin(x)}{n-1}}$$

Summary

The solution(s) found are the following

$$z = \mathrm{e}^{-\frac{\sin(x)n + \ln\left(\frac{2\sin(x)\mathrm{e}^{-(n-1)\sin(x)} n - c_1 n^2 - 2\sin(x)\mathrm{e}^{-(n-1)\sin(x)} + 2c_1 n + 2\,\mathrm{e}^{-(n-1)\sin(x)} - c_1}{n-1}\right) - \sin(x)}{n-1}} \tag{1}$$

Verification of solutions

$$z = \mathrm{e}^{-\frac{\sin(x)n + \ln\left(\frac{2\sin(x)\mathrm{e}^{-(n-1)\sin(x)} n - c_1 n^2 - 2\sin(x)\mathrm{e}^{-(n-1)\sin(x)} + 2c_1 n + 2\,\mathrm{e}^{-(n-1)\sin(x)} - c_1}{n-1}\right) - \sin(x)}{n-1}}$$

Verified OK.

1.22.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} z' &= F(x, z) \\ &= -z\cos(x) + z^n \sin(2x) \end{aligned}$$

This is a Bernoulli ODE.

$$z' = -\cos(x)\, z + 2\sin(x)\cos(x)\, z^n \tag{1}$$

The standard Bernoulli ODE has the form

$$z' = f_0(x)z + f_1(x)z^n \tag{2}$$

The first step is to divide the above equation by z^n which gives

$$\frac{z'}{z^n} = f_0(x)z^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $r = z^{1-n}$ in equation (3) which generates a new ODE in $r(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $z(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\cos(x) \\ f_1(x) &= 2\sin(x)\cos(x) \\ n &= n \end{aligned}$$

Dividing both sides of ODE (1) by $z^n = z^n$ gives

$$z'z^{-n} = -\cos(x)\, z^{-n+1} + 2\sin(x)\cos(x) \tag{4}$$

Let

$$\begin{aligned} r &= z^{1-n} \\ &= z^{-n+1} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$r' = (-n+1)\, z^{-n} z' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{r'(x)}{-n+1} &= -\cos(x)\, r(x) + 2\sin(x)\cos(x) \\ r' &= -(-n+1)\cos(x)\, r + 2(-n+1)\sin(x)\cos(x) \end{aligned} \tag{7}$$

The above now is a linear ODE in $r(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r'(x) + p(x)r(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -(n-1)\cos(x) \\
q(x) &= -(n-1)\sin(2x)
\end{aligned}
$$

Hence the ode is

$$
r'(x) - (n-1)\cos(x)\, r(x) = -(n-1)\sin(2x)
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -(n-1)\cos(x)dx} \\
&= \mathrm{e}^{(-n+1)\sin(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \mathrm{e}^{-(n-1)\sin(x)}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu r) &= (\mu)\left(-(n-1)\sin(2x)\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-(n-1)\sin(x)} r\right) &= \left(\mathrm{e}^{-(n-1)\sin(x)}\right)\left(-(n-1)\sin(2x)\right) \\
\mathrm{d}\left(\mathrm{e}^{-(n-1)\sin(x)} r\right) &= \left(-(n-1)\sin(2x)\,\mathrm{e}^{-(n-1)\sin(x)}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-(n-1)\sin(x)} r &= \int -(n-1)\sin(2x)\,\mathrm{e}^{-(n-1)\sin(x)}\,\mathrm{d}x \\
\mathrm{e}^{-(n-1)\sin(x)} r &= \frac{2\,\mathrm{e}^{(-n+1)\sin(x)}(-n+1)\sin(x) - 2\,\mathrm{e}^{(-n+1)\sin(x)}}{-n+1} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-(n-1)\sin(x)}$ results in

$$
r(x) = \frac{2\,\mathrm{e}^{(n-1)\sin(x)}\left(\mathrm{e}^{(-n+1)\sin(x)}(-n+1)\sin(x) - \mathrm{e}^{(-n+1)\sin(x)}\right)}{-n+1} + c_1 \mathrm{e}^{(n-1)\sin(x)}
$$

which simplifies to

$$
r(x) = \frac{(n-1)\, c_1 \mathrm{e}^{(n-1)\sin(x)} + 2 + (2n-2)\sin(x)}{n-1}
$$

Replacing r in the above by z^{-n+1} using equation (5) gives the final solution.

$$
z^{-n+1} = \frac{(n-1)\, c_1 \mathrm{e}^{(n-1)\sin(x)} + 2 + (2n-2)\sin(x)}{n-1}
$$

Summary

The solution(s) found are the following

$$z^{-n+1} = \frac{(n-1)\, c_1 \mathrm{e}^{(n-1)\sin(x)} + 2 + (2n-2)\sin(x)}{n-1} \tag{1}$$

Verification of solutions

$$z^{-n+1} = \frac{(n-1)\, c_1 \mathrm{e}^{(n-1)\sin(x)} + 2 + (2n-2)\sin(x)}{n-1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
dsolve(diff(z(x),x)+z(x)*cos(x)=z(x)^n*sin(2*x),z(x), singsol=all)
```

$$z(x) = \left(\frac{\mathrm{e}^{\sin(x)(n-1)} c_1 n - \mathrm{e}^{\sin(x)(n-1)} c_1 + 2\sin(x)\, n - 2\sin(x) + 2}{n-1}\right)^{-\frac{1}{n-1}}$$

✓ Solution by Mathematica

Time used: 6.964 (sec). Leaf size: 36

```
DSolve[z'[x]+z[x]*Cos[x]==z[x]^n*Sin[2*x],z[x],x,IncludeSingularSolutions -> True]
```

$$z(x) \to \left(c_1 e^{(n-1)\sin(x)} + \frac{2}{n-1} + 2\sin(x)\right)^{\frac{1}{1-n}}$$

1.23 problem 10.5

Internal problem ID [4377]
Internal file name [`OUTPUT/3870_Sunday_June_05_2022_11_33_11_AM_64842096/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 2
Problem number: 10.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{xy' + y - \ln(x)\, y^2 = 0}$$

1.23.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(\ln(x)\, y - 1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 45: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^2 x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 x} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(\ln(x)\, y - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y\, x^2} \\ S_y &= \frac{1}{y^2 x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\ln(x)}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\ln(R)}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{R} - \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{yx} = -\frac{\ln(x)}{x} - \frac{1}{x} + c_1$$

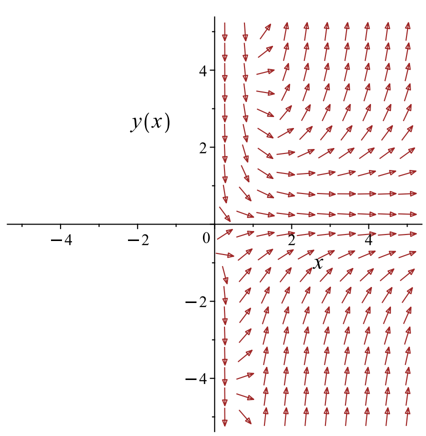
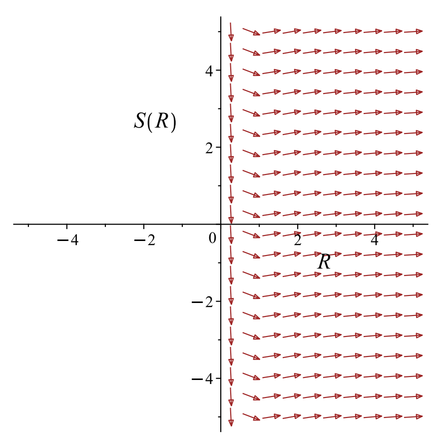
Which simplifies to

$$\frac{-yc_1x + \ln(x)\, y + y - 1}{xy} = 0$$

Which gives

$$y = \frac{1}{-c_1x + \ln(x) + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(\ln(x)y-1)}{x}$ 	$R = x$ $S = -\frac{1}{yx}$	$\frac{dS}{dR} = \frac{\ln(R)}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{1}{-c_1 x + \ln(x) + 1} \tag{1}$$

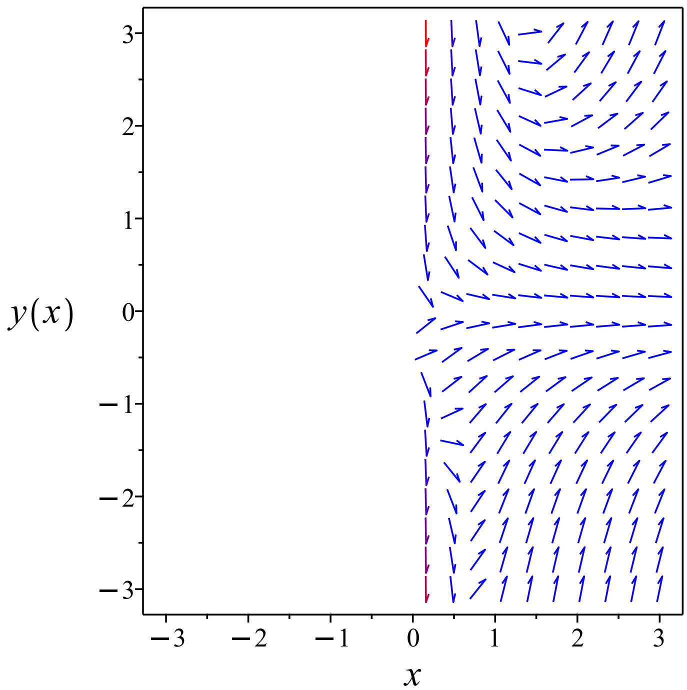


Figure 47: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

Verified OK.

1.23.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(\ln(x) y - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x} y + \frac{\ln(x)}{x} y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{\ln(x)}{x} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{yx} + \frac{\ln(x)}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} + \frac{\ln(x)}{x} \\ w' &= \frac{w}{x} - \frac{\ln(x)}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= -\frac{\ln(x)}{x}\end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -\frac{\ln(x)}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx}\\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{\ln(x)}{x}\right)\\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x}\right)\\ d\left(\frac{w}{x}\right) &= \left(-\frac{\ln(x)}{x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int -\frac{\ln(x)}{x^2} dx\\ \frac{w}{x} &= \frac{\ln(x)}{x} + \frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x\left(\frac{\ln(x)}{x} + \frac{1}{x}\right) + c_1 x$$

which simplifies to

$$w(x) = c_1 x + \ln(x) + 1$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = c_1 x + \ln(x) + 1$$

Or

$$y = \frac{1}{c_1 x + \ln(x) + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{c_1 x + \ln(x) + 1} \tag{1}$$

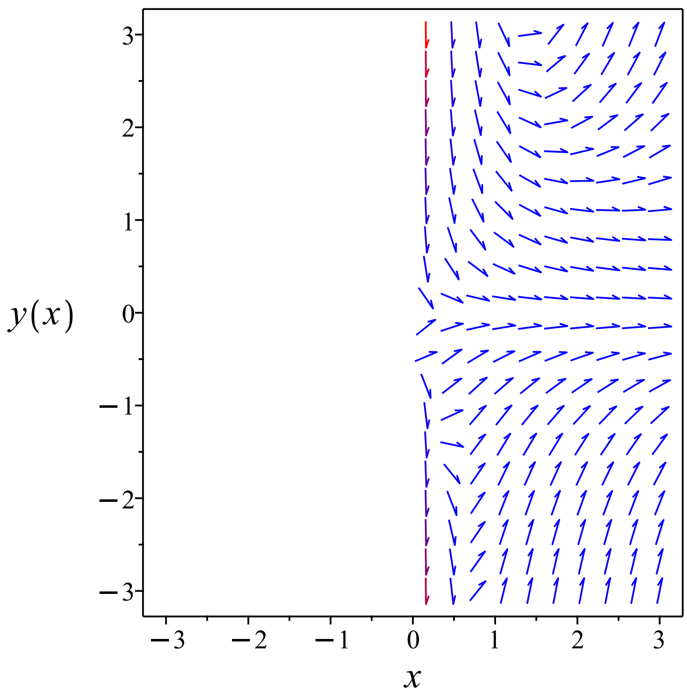


Figure 48: Slope field plot

Verification of solutions

$$y = \frac{1}{c_1 x + \ln(x) + 1}$$

Verified OK.

1.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(-y+\ln(x)\,y^2\right)\mathrm{d}x\\ \left(-\ln(x)\,y^2+y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\ln(x)\,y^2+y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\ln (x)\, y^2 + y\right) \\ &= -2 \ln (x)\, y + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-2 \ln (x)\, y + 1) - (1)) \\ &= -\frac{2 \ln (x)\, y}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{y\,(\ln (x)\, y - 1)}((1) - (-2 \ln (x)\, y + 1)) \\ &= -\frac{2 \ln (x)}{\ln (x)\, y - 1}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (-2\ln(x)\,y + 1)}{x\left(-\ln(x)\,y^2 + y\right) - y\,(x)} \\
&= -\frac{2}{yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2y^2}\left(-\ln(x)\,y^2 + y\right) \\
&= \frac{-\ln(x)\,y + 1}{y\,x^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2y^2}(x) \\
&= \frac{1}{y^2x}
\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-\ln(x)\, y + 1}{y\, x^2}\right) + \left(\frac{1}{y^2 x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-\ln(x)\, y + 1}{y\, x^2}\,\mathrm{d}x$$
$$\phi = \frac{\ln(x)\, y + y - 1}{xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1 + \ln(x)}{xy} - \frac{\ln(x)\, y + y - 1}{x\, y^2} + f'(y) \tag{4}$$
$$= \frac{1}{y^2 x} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2 x}$. Therefore equation (4) becomes

$$\frac{1}{y^2 x} = \frac{1}{y^2 x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x)\, y + y - 1}{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x)\, y + y - 1}{xy}$$

The solution becomes

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-c_1 x + \ln(x) + 1} \tag{1}$$

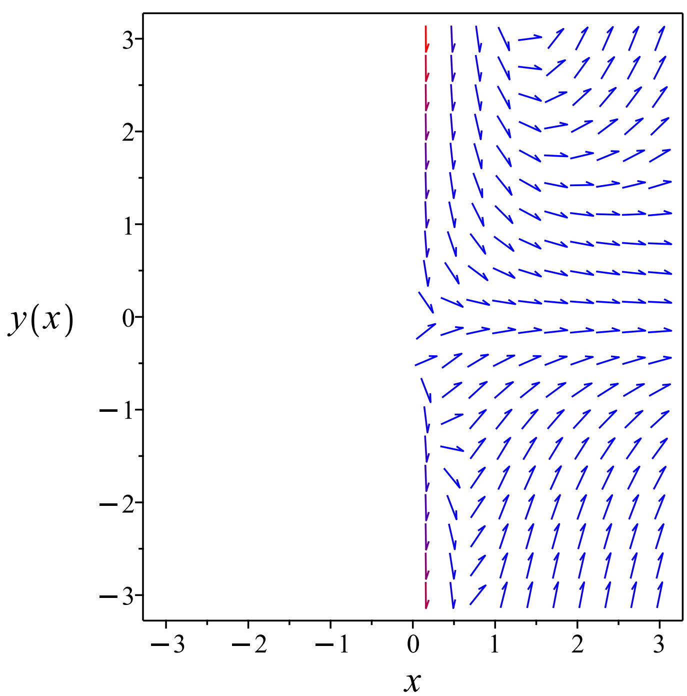


Figure 49: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

Verified OK.

1.23.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(\ln(x) y - 1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{\ln(x) y^2}{x} - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{\ln(x)}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{\ln(x)u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{\ln(x)}{x^2} + \frac{1}{x^2} \\ f_1 f_2 &= -\frac{\ln(x)}{x^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\ln(x) u''(x)}{x} - \left(-\frac{2\ln(x)}{x^2} + \frac{1}{x^2}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{-c_2 \ln(x) + c_1 x - c_2}{x}$$

The above shows that

$$u'(x) = \frac{c_2 \ln(x)}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{-c_2 \ln(x) + c_1 x - c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{-c_3 x + \ln(x) + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{-c_3 x + \ln(x) + 1} \tag{1}$$

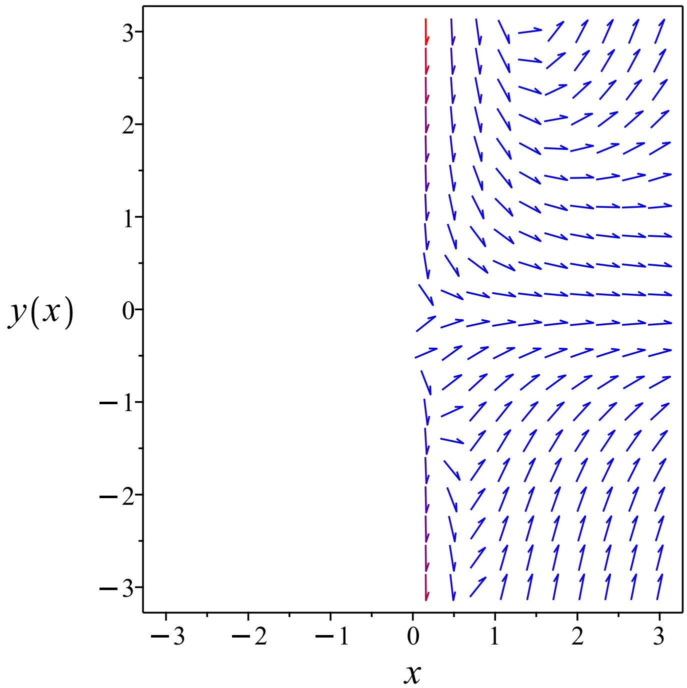


Figure 50: Slope field plot

Verification of solutions

$$y = \frac{1}{-c_3 x + \ln(x) + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x*diff(y(x),x)+y(x)=y(x)^2*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{1}{1 + c_1 x + \ln(x)}$$

✓ Solution by Mathematica

Time used: 0.157 (sec). Leaf size: 20

```
DSolve[x*y'[x]+y[x]==y[x]^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{\log(x) + c_1 x + 1}$$
$$y(x) \to 0$$

2 Chapter 3

2.1 problem 1

Internal problem ID [4378]
Internal file name [`OUTPUT/3871_Sunday_June_05_2022_11_33_19_AM_2745998/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 3
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{3xy^2 + (y^3 + 3yx^2)\, y' = -x^3}$$

2.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(3y\,x^2 + y^3)\,\mathrm{d}y &= (-x^3 - 3y^2x)\,\mathrm{d}x \\
(x^3 + 3y^2x)\,\mathrm{d}x + (3y\,x^2 + y^3)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= x^3 + 3y^2x \\
N(x,y) &= 3y\,x^2 + y^3
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^3 + 3y^2x) \\
&= 6xy
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3y\,x^2 + y^3) \\
&= 6xy
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int x^3 + 3y^2 x \mathrm{d}x \\
\phi &= \frac{(x^2+3y^2)^2}{4} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 3(x^2+3y^2)\, y + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y\, x^2 + y^3$. Therefore equation (4) becomes

$$
3y\, x^2 + y^3 = 3(x^2+3y^2)\, y + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -8y^3
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int (-8y^3) \mathrm{d}y \\
f(y) &= -2y^4 + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{(x^2+3y^2)^2}{4} - 2y^4 + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{(x^2+3y^2)^2}{4} - 2y^4
$$

Summary

The solution(s) found are the following

$$\frac{(x^2+3y^2)^2}{4}-2y^4=c_1 \tag{1}$$

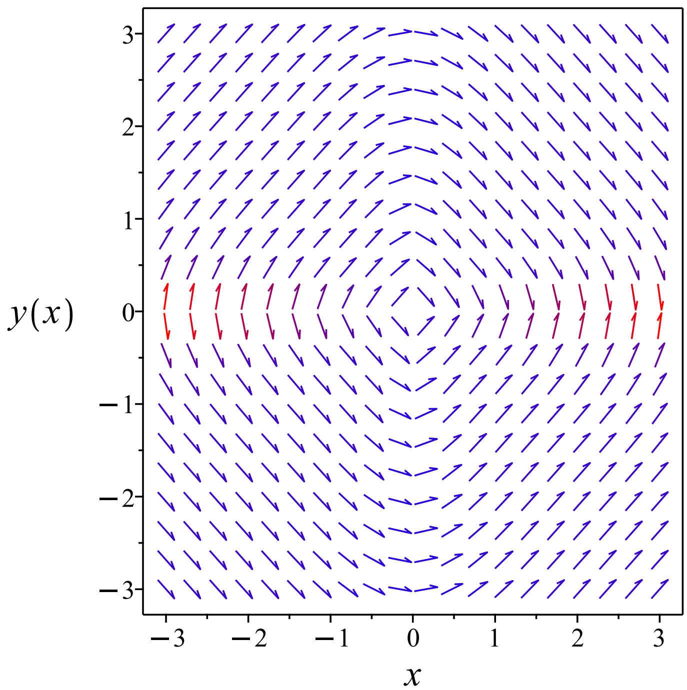


Figure 51: Slope field plot

Verification of solutions

$$\frac{(x^2+3y^2)^2}{4}-2y^4=c_1$$

Verified OK.

2.1.2 Maple step by step solution

Let's solve

$3xy^2+(y^3+3yx^2)\,y'=-x^3$

- Highest derivative means the order of the ODE is 1

y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y) = 0$

- Compute derivative of lhs

  $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

- Evaluate derivatives

  $6xy = 6xy$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (x^3 + 3y^2 x)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{(x^2+3y^2)^2}{4} + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $3y\,x^2 + y^3 = 3(x^2 + 3y^2)\,y + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -3(x^2 + 3y^2)\,y + 3y\,x^2 + y^3$

- Solve for $f_1(y)$

  $f_1(y) = -2y^4$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{(x^2+3y^2)^2}{4} - 2y^4$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{(x^2+3y^2)^2}{4} - 2y^4 = c_1$

- Solve for y

  $\left\{y = \sqrt{-3x^2 - 2\sqrt{2x^4 + c_1}}, y = \sqrt{-3x^2 + 2\sqrt{2x^4 + c_1}}, y = -\sqrt{-3x^2 - 2\sqrt{2x^4 + c_1}}, y = -\sqrt{-\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 119

```
dsolve((x^3+3*x*y(x)^2)+(y(x)^3+3*x^2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-3c_1x^2 - \sqrt{8c_1^2x^4 + 1}}}{\sqrt{c_1}}$$

$$y(x) = \frac{\sqrt{-3c_1x^2 + \sqrt{8c_1^2x^4 + 1}}}{\sqrt{c_1}}$$

$$y(x) = -\frac{\sqrt{-3c_1x^2 - \sqrt{8c_1^2x^4 + 1}}}{\sqrt{c_1}}$$

$$y(x) = -\frac{\sqrt{-3c_1x^2 + \sqrt{8c_1^2x^4 + 1}}}{\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 8.383 (sec). Leaf size: 245

```
DSolve[(x^3+3*x*y[x]^2)+(y[x]^3+3*x^2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\sqrt{-3x^2 - \sqrt{8x^4 + e^{4c_1}}}$$
$$y(x) \to \sqrt{-3x^2 - \sqrt{8x^4 + e^{4c_1}}}$$
$$y(x) \to -\sqrt{-3x^2 + \sqrt{8x^4 + e^{4c_1}}}$$
$$y(x) \to \sqrt{-3x^2 + \sqrt{8x^4 + e^{4c_1}}}$$
$$y(x) \to -\sqrt{-2\sqrt{2}\sqrt{x^4} - 3x^2}$$
$$y(x) \to \sqrt{-2\sqrt{2}\sqrt{x^4} - 3x^2}$$
$$y(x) \to -\sqrt{2\sqrt{2}\sqrt{x^4} - 3x^2}$$
$$y(x) \to \sqrt{2\sqrt{2}\sqrt{x^4} - 3x^2}$$

2.2 problem 2

Internal problem ID [4379]
Internal file name [`OUTPUT/3872_Sunday_June_05_2022_11_33_25_AM_40343060/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 3
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Bernoulli]
```

$$\boxed{\frac{y^2}{x^2} - \frac{2yy'}{x} = -1}$$

2.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{2y}{x}\right)\mathrm{d}y = \left(-1-\frac{y^2}{x^2}\right)\mathrm{d}x$$
$$\left(1+\frac{y^2}{x^2}\right)\mathrm{d}x + \left(-\frac{2y}{x}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= 1+\frac{y^2}{x^2} \\ N(x,y) &= -\frac{2y}{x} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(1+\frac{y^2}{x^2}\right) \\ &= \frac{2y}{x^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{2y}{x}\right) \\ &= \frac{2y}{x^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 1 + \frac{y^2}{x^2} \mathrm{d}x$$

$$\phi = x - \frac{y^2}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$x - \frac{y^2}{x} = c_1 \tag{1}$$

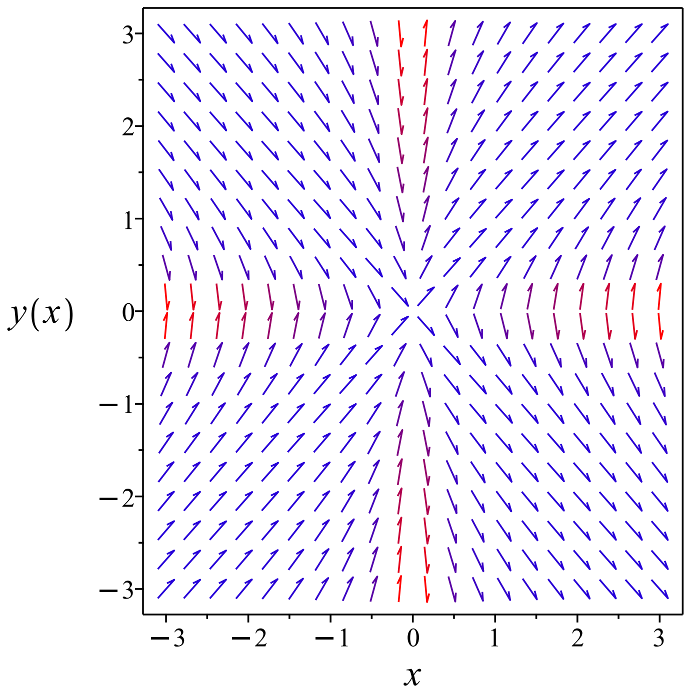


Figure 52: Slope field plot

Verification of solutions

$$x - \frac{y^2}{x} = c_1$$

Verified OK.

2.2.2 Maple step by step solution

Let's solve

$\frac{y^2}{x^2} - \frac{2yy'}{x} = -1$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $\frac{2y}{x^2} = \frac{2y}{x^2}$
  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(1 + \frac{y^2}{x^2}\right) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x - \frac{y^2}{x} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $-\frac{2y}{x} = -\frac{2y}{x} + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x, y) = x - \frac{y^2}{x}$

- Substitute $F(x, y)$ into the solution of the ODE

  $x - \frac{y^2}{x} = c_1$

- Solve for y

  $\{y = \sqrt{-c_1 x + x^2}, y = -\sqrt{-c_1 x + x^2}\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve((1+y(x)^2/x^2)-2*y(x)/x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{(x + c_1)\, x}$$
$$y(x) = -\sqrt{(x + c_1)\, x}$$

✓ Solution by Mathematica

Time used: 0.186 (sec). Leaf size: 38

```
DSolve[(1+y[x]^2/x^2)-2*y[x]/x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x}\sqrt{x + c_1}$$
$$y(x) \to \sqrt{x}\sqrt{x + c_1}$$

2.3 problem 3

Internal problem ID [4380]
Internal file name [`OUTPUT/3873_Sunday_June_05_2022_11_33_30_AM_61368743/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 3
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{\frac{3x}{y^3}+\left(\frac{1}{y^2}-\frac{3x^2}{y^4}\right)y'=0}$$

2.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right)\mathrm{d}y &= \left(-\frac{3x}{y^3}\right)\mathrm{d}x\\ \left(\frac{3x}{y^3}\right)\mathrm{d}x + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \frac{3x}{y^3}\\ N(x,y) &= \frac{1}{y^2} - \frac{3x^2}{y^4}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3x}{y^3}\right)\\ &= -\frac{9x}{y^4}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right)\\ &= -\frac{6x}{y^4}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{\frac{1}{y^2} - \frac{3x^2}{y^4}}\left(\left(-\frac{9x}{y^4}\right) - \left(-\frac{6x}{y^4}\right)\right) \\ &= \frac{3x}{3x^2 - y^2} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{y^3}{3x}\left(\left(-\frac{6x}{y^4}\right) - \left(-\frac{9x}{y^4}\right)\right) \\ &= \frac{1}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int \frac{1}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(y)} \\ &= y \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= y\left(\frac{3x}{y^3}\right) \\ &= \frac{3x}{y^2} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y\left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right) \\
&= \frac{-3x^2 + y^2}{y^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{3x}{y^2}\right) + \left(\frac{-3x^2 + y^2}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \frac{3x}{y^2} \,\mathrm{d}x \\
\phi &= \frac{3x^2}{2y^2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{3x^2}{y^3} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-3x^2+y^2}{y^3}$. Therefore equation (4) becomes

$$
\frac{-3x^2 + y^2}{y^3} = -\frac{3x^2}{y^3} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3x^2}{2y^2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3x^2}{2y^2} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(-3x^2\mathrm{e}^{-2c_1}\right)}{2}+c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(-3x^2\mathrm{e}^{-2c_1}\right)}{2}+c_1} \tag{1}$$

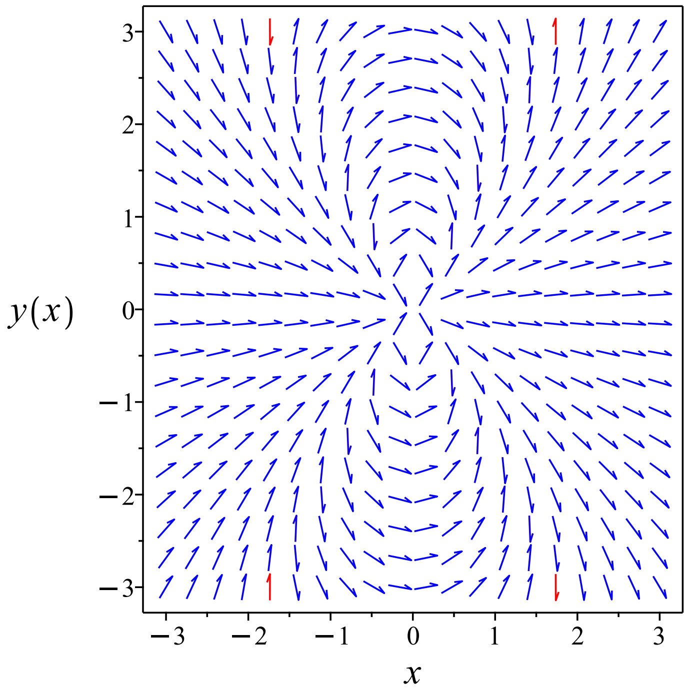


Figure 53: Slope field plot

Verification of solutions

$$y = e^{\frac{\mathrm{LambertW}\left(-3x^2 e^{-2c_1}\right)}{2}+c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve((3*x/y(x)^3)+(1/y(x)^2-3*x^2/y(x)^4)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{3}\,\sqrt{-\frac{1}{\text{LambertW}\left(-3c_1x^2\right)}}\,x$$

✓ Solution by Mathematica

Time used: 6.543 (sec). Leaf size: 66

```
DSolve[(3*x/y[x]^3)+(1/y[x]^2-3*x^2/y[x]^4)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{i\sqrt{3}x}{\sqrt{W\left(-3e^{-2c_1}x^2\right)}}$$
$$y(x) \to \frac{i\sqrt{3}x}{\sqrt{W\left(-3e^{-2c_1}x^2\right)}}$$
$$y(x) \to 0$$

2.4 problem 4

Internal problem ID [4381]
Internal file name [OUTPUT/3874_Sunday_June_05_2022_11_33_39_AM_69767660/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 3
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _exact, _rational]
```

$$\boxed{y'y+\frac{xy'}{x^2+y^2}-\frac{y}{x^2+y^2}=-x}$$

2.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(y + \frac{x}{x^2+y^2}\right)\mathrm{d}y = \left(-x + \frac{y}{x^2+y^2}\right)\mathrm{d}x$$

$$\left(x - \frac{y}{x^2+y^2}\right)\mathrm{d}x + \left(y + \frac{x}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x - \frac{y}{x^2+y^2}$$
$$N(x, y) = y + \frac{x}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x - \frac{y}{x^2+y^2}\right)\\ &= \frac{-x^2+y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y + \frac{x}{x^2+y^2}\right)\\ &= \frac{-x^2+y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x - \frac{y}{x^2 + y^2} \, \mathrm{d}x$$
$$\phi = \frac{x^2}{2} - \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2 \left(\frac{x^2}{y^2} + 1\right)} + f'(y) \tag{4}$$
$$= \frac{x}{x^2 + y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y + \frac{x}{x^2+y^2}$. Therefore equation (4) becomes

$$y + \frac{x}{x^2 + y^2} = \frac{x}{x^2 + y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y) \, \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{2} - \arctan\left(\frac{x}{y}\right) + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{2} - \arctan\left(\frac{x}{y}\right) + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$\frac{x^2}{2} - \arctan\left(\frac{x}{y}\right) + \frac{y^2}{2} = c_1 \tag{1}$$

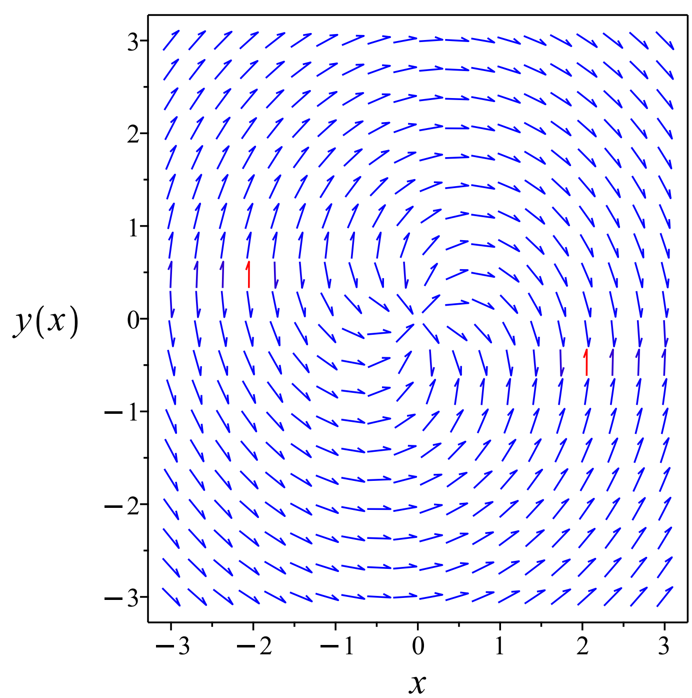


Figure 54: Slope field plot

Verification of solutions

$$\frac{x^2}{2} - \arctan\left(\frac{x}{y}\right) + \frac{y^2}{2} = c_1$$

Verified OK.

2.4.2 Maple step by step solution

Let's solve

$y'y + \frac{xy'}{x^2+y^2} - \frac{y}{x^2+y^2} = -x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right)y' = 0$
  - Evaluate derivatives

    $-\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$
  - Simplify

    $\frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$
  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(x - \frac{y}{x^2+y^2}\right)dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{x^2}{2} - \arctan\left(\frac{x}{y}\right) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $y + \frac{x}{x^2+y^2} = \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = y + \frac{x}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)}$$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{x^2}{2} - \arctan\left(\frac{x}{y}\right) + \frac{y^2}{2}$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{x^2}{2} - \arctan\left(\frac{x}{y}\right) + \frac{y^2}{2} = c_1$

- Solve for y

  $y = -\frac{x}{\tan\left(\mathit{RootOf}\left(-x^2\tan\left(\_Z\right)^2+2c_1\tan\left(\_Z\right)^2-2\tan\left(\_Z\right)^2\_Z-x^2\right)\right)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -x/y(x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      trying Bernoulli
      <- Bernoulli successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 26

```
dsolve(x+y(x)*diff(y(x),x)+x/(x^2+y(x)^2)*diff(y(x),x)- y(x)/(x^2+y(x)^2)=0,y(x), singsol=al
```

$$y(x) = \cot\left(\mathrm{RootOf}\left(2c_1 \sin\left(\_Z\right)^2 - 2\_Z \sin\left(\_Z\right)^2 + x^2\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 31

```
DSolve[x+y[x]*y'[x]+x/(x^2+y[x]^2)*y'[x]- y[x]/(x^2+y[x]^2)==0,y[x],x,IncludeSingularSolutio
```

$$\mathrm{Solve}\left[-\arctan\left(\frac{x}{y(x)}\right) + \frac{x^2}{2} + \frac{y(x)^2}{2} = c_1, y(x)\right]$$

2.5 problem 5

Internal problem ID [4382]
Internal file name [`OUTPUT/3875_Sunday_June_05_2022_11_33_46_AM_38092062/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 3
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _dAlembert]
```

$$\boxed{e^{\frac{x}{y}}+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)y'=-1}$$

2.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \left(\mathrm{e}^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\right)\mathrm{d}y &= \left(-1 - \mathrm{e}^{\frac{x}{y}}\right)\mathrm{d}x \\ \left(\mathrm{e}^{\frac{x}{y}} + 1\right)\mathrm{d}x + \left(\mathrm{e}^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\right)\mathrm{d}y &= 0 \qquad (2A) \end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \mathrm{e}^{\frac{x}{y}} + 1 \\ N(x, y) &= \mathrm{e}^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{\frac{x}{y}} + 1\right) \\ &= -\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\right) \\ &= -\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \mathrm{e}^{\frac{x}{y}} + 1 \, \mathrm{d}x$$

$$\phi = y \, \mathrm{e}^{\frac{x}{y}} + x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= \mathrm{e}^{\frac{x}{y}} - \frac{x \, \mathrm{e}^{\frac{x}{y}}}{y} + f'(y) \qquad (4) \\ &= -\frac{\mathrm{e}^{\frac{x}{y}} (x - y)}{y} + f'(y) \end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)$. Therefore equation (4) becomes

$$\mathrm{e}^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) = -\frac{\mathrm{e}^{\frac{x}{y}} (x - y)}{y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y \, \mathrm{e}^{\frac{x}{y}} + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{\frac{x}{y}} + x$$

The solution becomes

$$y = -\frac{x}{\mathrm{LambertW}\left(-\frac{x}{-x+c_1}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\mathrm{LambertW}\left(-\frac{x}{-x+c_1}\right)} \tag{1}$$

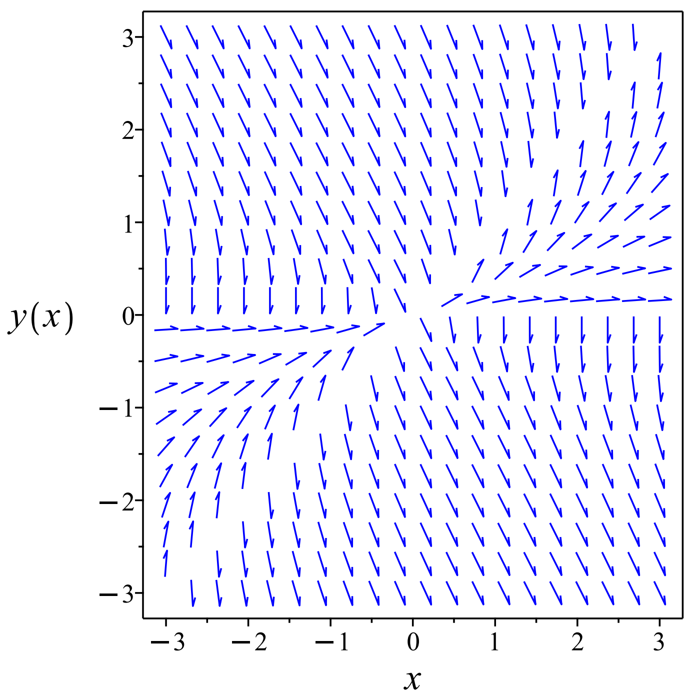


Figure 55: Slope field plot

Verification of solutions

$$y = -\frac{x}{\mathrm{LambertW}\left(-\frac{x}{-x+c_1}\right)}$$

Verified OK.

2.5.2 Maple step by step solution

Let's solve

$\mathrm{e}^{\frac{x}{y}}+\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right) y^{\prime}=-1$

- Highest derivative means the order of the ODE is 1

  $y^{\prime}$

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F^{\prime}(x, y)=0$
  - Compute derivative of lhs

    $F^{\prime}(x, y)+\left(\frac{\partial}{\partial y} F(x, y)\right) y^{\prime}=0$
  - Evaluate derivatives

    $-\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2}=\frac{\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{y}-\frac{\mathrm{e}^{\frac{x}{y}}}{y}$
  - Simplify

    $-\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2}=-\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2}$
  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y)=c_1, M(x, y)=F^{\prime}(x, y), N(x, y)=\frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y)=\int\left(\mathrm{e}^{\frac{x}{y}}+1\right) d x+f_1(y)$

- Evaluate integral

  $F(x, y)=y\,\mathrm{e}^{\frac{x}{y}}+x+f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y)=\frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)=\mathrm{e}^{\frac{x}{y}}-\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y}+\frac{d}{d y} f_1(y)$

- Isolate for $\frac{d}{d y} f_1(y)$

$$\frac{d}{dy}f_1(y) = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right) - e^{\frac{x}{y}} + \frac{x\,e^{\frac{x}{y}}}{y}$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = y\,e^{\frac{x}{y}} + x$$

- Substitute $F(x,y)$ into the solution of the ODE

$$y\,e^{\frac{x}{y}} + x = c_1$$

- Solve for y

$$y = -\frac{x}{\mathit{LambertW}\left(-\frac{x}{-x+c_1}\right)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve((1+exp(x/y(x)))+exp(x/y(x))*(1-x/y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{\text{LambertW}\left(\frac{xc_1}{c_1x-1}\right)}$$

✓ Solution by Mathematica

Time used: 1.182 (sec). Leaf size: 34

```
DSolve[(1+Exp[x/y[x]])+Exp[x/y[x]]*(1-x/y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to -\frac{x}{W\left(\frac{x}{x-e^{c_1}}\right)}$$
$$y(x) \to -\frac{x}{W(1)}$$

2.6 problem 6

Internal problem ID [4383]
Internal file name [`OUTPUT/3876_Sunday_June_05_2022_11_33_52_AM_65941304/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 3
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _exact, _rational, _Bernoulli]
```

$$\boxed{e^{x}\left(x^{2}+y^{2}+2x\right)+2y\,e^{x}y'=0}$$

2.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y\,\mathrm{e}^x)\,\mathrm{d}y &= \left(-\mathrm{e}^x\left(x^2 + y^2 + 2x\right)\right)\mathrm{d}x\\ \left(\mathrm{e}^x\left(x^2 + y^2 + 2x\right)\right)\mathrm{d}x + (2y\,\mathrm{e}^x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \mathrm{e}^x\left(x^2 + y^2 + 2x\right)\\ N(x, y) &= 2y\,\mathrm{e}^x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^x\left(x^2 + y^2 + 2x\right)\right)\\ &= 2y\,\mathrm{e}^x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y\,\mathrm{e}^x)\\ &= 2y\,\mathrm{e}^x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \mathrm{e}^{x} \left(x^{2}+y^{2}+2x\right) \mathrm{d}x$$
$$\phi = \left(x^{2}+y^{2}\right) \mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2y\,\mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y\,\mathrm{e}^{x}$. Therefore equation (4) becomes

$$2y\,\mathrm{e}^{x} = 2y\,\mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(x^{2}+y^{2}\right) \mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(x^{2}+y^{2}\right) \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$\left(x^{2}+y^{2}\right) \mathrm{e}^{x} = c_1 \tag{1}$$

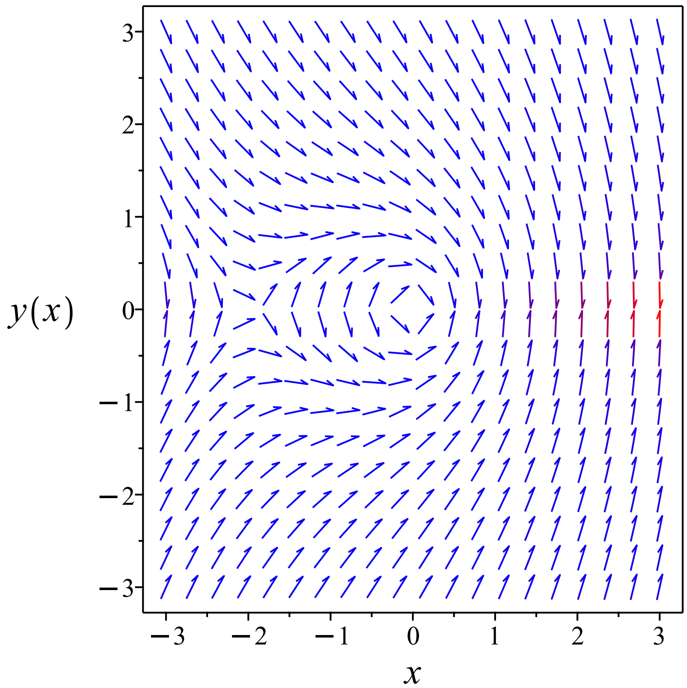


Figure 56: Slope field plot

Verification of solutions

$$\left(x^2+y^2\right) e^x = c_1$$

Verified OK.

2.6.2 Maple step by step solution

Let's solve

$e^x(x^2+y^2+2x)+2y\,e^x y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

  ◦ Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $2y\,\mathrm{e}^{x} = 2y\,\mathrm{e}^{x}$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int \mathrm{e}^{x}(x^2 + y^2 + 2x)\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = (x^2 + y^2)\,\mathrm{e}^{x} + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $2y\,\mathrm{e}^{x} = 2y\,\mathrm{e}^{x} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = (x^2 + y^2)\,\mathrm{e}^{x}$

- Substitute $F(x,y)$ into the solution of the ODE

  $(x^2 + y^2)\,\mathrm{e}^{x} = c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{-\mathrm{e}^{x}(\mathrm{e}^{x}x^2 - c_1)}}{\mathrm{e}^{x}}, y = -\frac{\sqrt{-\mathrm{e}^{x}(\mathrm{e}^{x}x^2 - c_1)}}{\mathrm{e}^{x}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(exp(x)*(x^2+y(x)^2+2*x)+2*y(x)*exp(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{e^{-x}c_1 - x^2}$$
$$y(x) = -\sqrt{e^{-x}c_1 - x^2}$$

✓ Solution by Mathematica

Time used: 5.67 (sec). Leaf size: 47

```
DSolve[Exp[x]*(x^2+y[x]^2+2*x)+2*y[x]*Exp[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \rightarrow -\sqrt{-x^2 + c_1 e^{-x}}$$
$$y(x) \rightarrow \sqrt{-x^2 + c_1 e^{-x}}$$

2.7 problem 7

Internal problem ID [4384]
Internal file name [`OUTPUT/3877_Sunday_June_05_2022_11_33_58_AM_24987842/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 3
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{n\cos(nx+my)-m\sin(mx+ny)+(m\cos(nx+my)-n\sin(mx+ny))\,y'=0}$$

2.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(m\cos(my + nx) - n\sin(mx + ny))\,\mathrm{d}y = (-n\cos(my + nx$$

$$(-m\sin(mx + ny) + n\cos(my + nx))\,\mathrm{d}x + (m\cos(my + nx) - n\sin(mx + ny))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -m\sin(mx + ny) + n\cos(my + nx) \\
N(x, y) &= m\cos(my + nx) - n\sin(mx + ny)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-m\sin(mx + ny) + n\cos(my + nx)) \\
&= mn(-\sin(my + nx) - \cos(mx + ny))
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(m\cos(my + nx) - n\sin(mx + ny)) \\
&= mn(-\sin(my + nx) - \cos(mx + ny))
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -m \sin(mx+ny) + n\cos(my+nx) \,\mathrm{d}x$$

$$\phi = \cos(mx+ny) + \sin(my+nx) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = m\cos(my+nx) - n\sin(mx+ny) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = m\cos(my+nx) - n\sin(mx+ny)$. Therefore equation (4) becomes

$$m\cos(my+nx) - n\sin(mx+ny) = m\cos(my+nx) - n\sin(mx+ny) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(mx+ny) + \sin(my+nx) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(mx+ny) + \sin(my+nx)$$

Summary

The solution(s) found are the following

$$\cos(mx+ny) + \sin(nx+my) = c_1 \tag{1}$$

Verification of solutions

$$\cos(mx+ny) + \sin(nx+my) = c_1$$

Verified OK.

2.7.2 Maple step by step solution

Let's solve

$n\cos(nx+my)-m\sin(mx+ny)+(m\cos(nx+my)-n\sin(mx+ny))\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$
  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - Evaluate derivatives

    $-mn\sin(my+nx)-nm\cos(mx+ny)=-mn\sin(my+nx)-nm\cos(mx+ny)$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(-m\sin(mx+ny)+n\cos(my+nx))\,dx+f_1(y)$
- Evaluate integral

  $F(x,y)=\cos(mx+ny)+\sin(my+nx)+f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $m\cos(my+nx)-n\sin(mx+ny)=-n\sin(mx+ny)+m\cos(my+nx)+\frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=0$
- Solve for $f_1(y)$

  $f_1(y)=0$
- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \cos(mx+ny) + \sin(my+nx)$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\cos(mx+ny) + \sin(my+nx) = c_1$$

- Solve for y

$$y = \frac{-mx+RootOf\left(-m^2x+n^2x-\arcsin\left(-\cos\left(\_Z\right)+c_1\right)n+\_Zm\right)}{n}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve((n*cos(n*x+m*y(x))-m*sin(m*x+n*y(x)))+(m*cos(n*x+m*y(x))-n*sin(m*x+n*y(x)))*diff(y(x)
```

$$y(x) = \frac{-nx + \text{RootOf}\left(2m^2x - 2n^2x - 2\arcsin\left(\sin\left(\_Z\right) + c_1\right)m - m\pi + 2\_Zn\right)}{m}$$

✓ Solution by Mathematica

Time used: 0.741 (sec). Leaf size: 50

```
DSolve[(n*Cos[n*x+m*y[x]]-m*Sin[m*x+n*y[x]])+(m*Cos[n*x+m*y[x]]-n*Sin[m*x+n*y[x]])*y'[x]==0,
```

$$\text{Solve}[\sin(mx)\sin(ny(x)) - \cos(mx)\cos(ny(x)) - \sin(nx)\cos(my(x)) - \cos(nx)\sin(my(x)) = c_1, y(x)]$$

2.8 problem 8.1

Internal problem ID [4385]
Internal file name [OUTPUT/3878_Sunday_June_05_2022_11_34_13_AM_49205020/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 3
Problem number: 8.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _exact]
```

$$\boxed{\frac{x}{\sqrt{1+x^2+y^2}}+\frac{yy'}{\sqrt{1+x^2+y^2}}+\frac{y}{x^2+y^2}-\frac{xy'}{x^2+y^2}=0}$$

2.8.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y}{\sqrt{x^2+y^2+1}} - \frac{x}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{x}{\sqrt{x^2+y^2+1}} - \frac{y}{x^2+y^2}\right)\mathrm{d}x$$

$$\left(\frac{x}{\sqrt{x^2+y^2+1}} + \frac{y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{y}{\sqrt{x^2+y^2+1}} - \frac{x}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= \frac{x}{\sqrt{x^2+y^2+1}} + \frac{y}{x^2+y^2} \\
N(x, y) &= \frac{y}{\sqrt{x^2+y^2+1}} - \frac{x}{x^2+y^2}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+1}} + \frac{y}{x^2+y^2}\right) \\
&= -\frac{xy}{(x^2+y^2+1)^{\frac{3}{2}}} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+1}} - \frac{x}{x^2+y^2}\right) \\
&= -\frac{xy}{(x^2+y^2+1)^{\frac{3}{2}}} - \frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2}
\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int \frac{x}{\sqrt{x^2+y^2+1}}+\frac{y}{x^2+y^2}\,\mathrm{d}x$$
$$\phi=\sqrt{x^2+y^2+1}+\arctan\left(\frac{x}{y}\right)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=\frac{y}{\sqrt{x^2+y^2+1}}-\frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)}+f'(y) \tag{4}$$
$$=\frac{y}{\sqrt{x^2+y^2+1}}-\frac{x}{x^2+y^2}+f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{y}{\sqrt{x^2+y^2+1}}-\frac{x}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{y}{\sqrt{x^2+y^2+1}}-\frac{x}{x^2+y^2}=\frac{y}{\sqrt{x^2+y^2+1}}-\frac{x}{x^2+y^2}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\sqrt{x^2+y^2+1}+\arctan\left(\frac{x}{y}\right)+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sqrt{x^2 + y^2 + 1} + \arctan\left(\frac{x}{y}\right)$$

Summary
The solution(s) found are the following

$$\sqrt{1 + x^2 + y^2} + \arctan\left(\frac{x}{y}\right) = c_1 \tag{1}$$

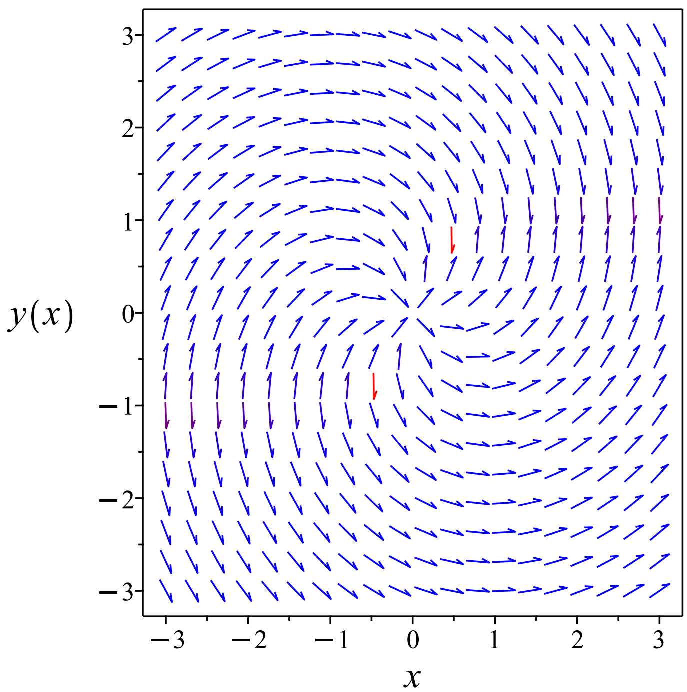


Figure 57: Slope field plot

Verification of solutions

$$\sqrt{1 + x^2 + y^2} + \arctan\left(\frac{x}{y}\right) = c_1$$

Verified OK.

2.8.2 Maple step by step solution

Let's solve

$\frac{x}{\sqrt{1+x^2+y^2}}+\frac{yy'}{\sqrt{1+x^2+y^2}}+\frac{y}{x^2+y^2}-\frac{xy'}{x^2+y^2}=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

    $-\frac{xy}{(x^2+y^2+1)^{\frac{3}{2}}}+\frac{1}{x^2+y^2}-\frac{2y^2}{(x^2+y^2)^2}=-\frac{xy}{(x^2+y^2+1)^{\frac{3}{2}}}-\frac{1}{x^2+y^2}+\frac{2x^2}{(x^2+y^2)^2}$

  ◦ Simplify

    $-\frac{xy}{(x^2+y^2+1)^{\frac{3}{2}}}+\frac{1}{x^2+y^2}-\frac{2y^2}{(x^2+y^2)^2}=-\frac{xy}{(x^2+y^2+1)^{\frac{3}{2}}}-\frac{1}{x^2+y^2}+\frac{2x^2}{(x^2+y^2)^2}$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(\frac{x}{\sqrt{x^2+y^2+1}}+\frac{y}{x^2+y^2}\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\sqrt{x^2+y^2+1}+\arctan\left(\frac{x}{y}\right)+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\frac{y}{\sqrt{x^2+y^2+1}}-\frac{x}{x^2+y^2}=\frac{y}{\sqrt{x^2+y^2+1}}-\frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = -\frac{x}{x^2+y^2} + \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)}$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \sqrt{x^2+y^2+1} + \arctan\left(\frac{x}{y}\right)$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\sqrt{x^2+y^2+1} + \arctan\left(\frac{x}{y}\right) = c_1$$

- Solve for y

$$y = \frac{x}{\tan\left(RootOf\left(-\_Z - \sqrt{\frac{x^2\tan(\_Z)^2+x^2+\tan(\_Z)^2}{\tan(\_Z)^2}} + c_1\right)\right)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 25

```
dsolve( x/sqrt(1+x^2+y(x)^2) + y(x)/sqrt(1+x^2+y(x)^2)*diff(y(x),x)+  y(x)/(x^2+y(x)^2) - x/
```

$$\arctan\left(\frac{x}{y(x)}\right) + \sqrt{1+x^2+y(x)^2} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.264 (sec). Leaf size: 27

```
DSolve[ x/Sqrt[1+x^2+y[x]^2] + y[x]/Sqrt[1+x^2+y[x]^2]*y'[x]+y[x]/(x^2+y[x]^2) - x/(x^2+y[x]
```

$$\text{Solve}\left[\arctan\left(\frac{x}{y(x)}\right)+\sqrt{x^2+y(x)^2+1}=c_1,y(x)\right]$$

2.9 problem 10

Internal problem ID [4386]
Internal file name [`OUTPUT/3879_Sunday_June_05_2022_11_34_20_AM_19298257/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 3
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{\frac{x^n y'}{b y^2 - c\,x^{2a}} - \frac{a y x^{a-1}}{b y^2 - c\,x^{2a}} = -x^{a-1}}$$

2.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \left(-b\,y^2 + c\,x^{2a} + ya\right) x^{a-1} x^{-n} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{x^a x^{-n} b\,y^2}{x} + \frac{x^{3a} x^{-n} c}{x} + \frac{x^a x^{-n} y a}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = x^{-n} x^{2a} x^{a-1} c$, $f_1(x) = a\,x^{a-1} x^{-n}$ and $f_2(x) = -b\,x^{a-1} x^{-n}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-b\,x^{a-1} x^{-n} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{b\, x^{a-1}(a-1)\, x^{-n}}{x} + \frac{b\, x^{a-1} x^{-n} n}{x} \\ f_1 f_2 &= -a\, x^{2a-2} x^{-2n} b \\ f_2^2 f_0 &= b^2 x^{3a-3} x^{-3n} x^{2a} c \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-b\, x^{a-1} x^{-n} u''(x) - \left(-\frac{b\, x^{a-1}(a-1)\, x^{-n}}{x} + \frac{b\, x^{a-1} x^{-n} n}{x} - a\, x^{2a-2} x^{-2n} b\right) u'(x) + b^2 x^{3a-3} x^{-3n} x^{2a} c u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)\,(1 - a + n - x^{a-n} a)}{x} - bc\_Y(x)\, x^{4a-2n-2}\right\}, \{\_Y(x)\}\right)$$

The above shows that

$$u'(x) = \frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)\,(1 - a + n - x^{a-n} a)}{x} - bc\_Y(x)\, x^{4a-2n-2}\right\}, \{\_Y(x)\}\right)$$

Using the above in (1) gives the solution

$$y = \frac{\left(\frac{\partial}{\partial x} \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)(1-a+n-x^{a-n}a)}{x} - bc\_Y(x)\, x^{4a-2n-2}\right\}, \{\_Y(x)\}\right)\right) x^{-a+1} x^n}{b\, \mathrm{DESol}\left(\left\{\_Y''(x) + \frac{\_Y'(x)(1-a+n-x^{a-n}a)}{x} - bc\_Y(x)\, x^{4a-2n-2}\right\}, \{\_Y(x)\}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^{-a+n+1}\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x)+\frac{\_Y'(x)\left(1-a+n-x^{a-n}a\right)}{x}-bc\_Y(x)\,x^{4a-2n-2}\right\},\{\_Y(x)\}\right)\right)}{b\operatorname{DESol}\left(\left\{\frac{-bc\_Y(x)x^{4a-2n}-a\,x^{a-n+1}\_Y'(x)-x\left(-\_Y''(x)x+\_Y'(x)(a-1-n)\right)}{x^2}\right\},\{\_Y(x)\}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{x^{-a+n+1}\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x)+\frac{\_Y'(x)\left(1-a+n-x^{a-n}a\right)}{x}-bc\_Y(x)\,x^{4a-2n-2}\right\},\{\_Y(x)\}\right)\right)}{b\operatorname{DESol}\left(\left\{\frac{-bc\_Y(x)x^{4a-2n}-a\,x^{a-n+1}\_Y'(x)-x\left(-\_Y''(x)x+\_Y'(x)(a-1-n)\right)}{x^2}\right\},\{\_Y(x)\}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{x^{-a+n+1}\left(\frac{\partial}{\partial x}\operatorname{DESol}\left(\left\{\_Y''(x)+\frac{\_Y'(x)\left(1-a+n-x^{a-n}a\right)}{x}-bc\_Y(x)\,x^{4a-2n-2}\right\},\{\_Y(x)\}\right)\right)}{b\operatorname{DESol}\left(\left\{\frac{-bc\_Y(x)x^{4a-2n}-a\,x^{a-n+1}\_Y'(x)-x\left(-\_Y''(x)x+\_Y'(x)(a-1-n)\right)}{x^2}\right\},\{\_Y(x)\}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (x^(a-1-n)*a*x+a-n-1)*(diff(y(
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
```

✗ Solution by Maple

```
dsolve( x^n/(b*y(x)^2-c*x^(2*a))*diff(y(x),x) - a*y(x)*x^(a-1)/(b*y(x)^2-c*x^(2*a)) + x^(a-1
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x^n/(b*y[x]^2-c*x^(2*a))*y'[x] - a*y[x]*x^(a-1)/(b*y[x]^2-c*x^(2*a)) + x^(a-1)==0,y[x
```

Not solved

3 Chapter 4

3.1 problem 2

Internal problem ID [4387]
Internal file name [`OUTPUT/3880_Sunday_June_05_2022_11_34_56_AM_27826988/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 4
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2xy + \left(y^2 - 2x^2\right) y' = 0}$$

3.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2x^2 + y^2)\,\mathrm{d}y &= (-2xy)\,\mathrm{d}x \\ (2xy)\,\mathrm{d}x + (-2x^2 + y^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 2xy \\ N(x, y) &= -2x^2 + y^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy) \\ &= 2x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2x^2 + y^2) \\ &= -4x \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{-2x^2 + y^2}((2x) - (-4x)) \\ &= -\frac{6x}{2x^2 - y^2} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{2yx}((-4x) - (2x))\\ &= -\frac{3}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int -\frac{3}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(y)}\\ &= \frac{1}{y^3}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y^3}(2xy)\\ &= \frac{2x}{y^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{y^3}(-2x^2 + y^2)\\ &= \frac{-2x^2 + y^2}{y^3}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{2x}{y^2}\right) + \left(\frac{-2x^2 + y^2}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{2x}{y^2} \, \mathrm{d}x$$

$$\phi = \frac{x^2}{y^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2x^2}{y^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-2x^2+y^2}{y^3}$. Therefore equation (4) becomes

$$\frac{-2x^2 + y^2}{y^3} = -\frac{2x^2}{y^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$

$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{y^2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{y^2} + \ln(y)$$

The solution becomes

$$y = e^{\frac{\text{LambertW}\left(-2x^2 e^{-2c_1}\right)}{2} + c_1}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{\text{LambertW}\left(-2x^2 e^{-2c_1}\right)}{2} + c_1} \tag{1}$$

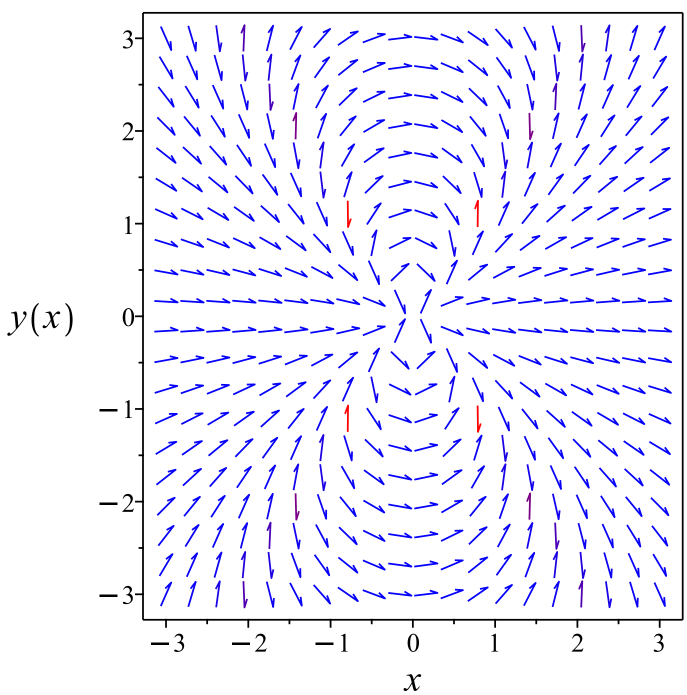


Figure 58: Slope field plot

Verification of solutions

$$y = e^{\frac{\mathrm{LambertW}\left(-2x^2 e^{-2c_1}\right)}{2} + c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(2*x*y(x)+(y(x)^2-2*x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{2}\,\sqrt{-\frac{1}{\mathrm{LambertW}\left(-2c_1x^2\right)}}\,x$$

✓ Solution by Mathematica

Time used: 7.214 (sec). Leaf size: 66

```
DSolve[2*x*y[x]+(y[x]^2-2*x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{i\sqrt{2}x}{\sqrt{W\left(-2e^{-2c_1}x^2\right)}}$$
$$y(x) \to \frac{i\sqrt{2}x}{\sqrt{W\left(-2e^{-2c_1}x^2\right)}}$$
$$y(x) \to 0$$

3.2 problem 4

Internal problem ID [4388]
Internal file name [`OUTPUT/3881_Sunday_June_05_2022_11_35_02_AM_61121252/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 4
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{\frac{2}{y} - \frac{2y'}{x} = -\frac{1}{x} - \frac{y'}{y}}$$

3.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x+2y)\,\mathrm{d}y &= (2x+y)\,\mathrm{d}x\\ (-2x-y)\,\mathrm{d}x + (-x+2y)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2x-y\\ N(x,y) &= -x+2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x-y)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x+2y)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2x - y \, \mathrm{d}x$$
$$\phi = -x(x+y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x + 2y$. Therefore equation (4) becomes

$$-x + 2y = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (2y) \, \mathrm{d}y$$
$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x(x+y) + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x(x+y) + y^2$$

Summary
The solution(s) found are the following

$$-x(x+y)+y^2=c_1 \tag{1}$$

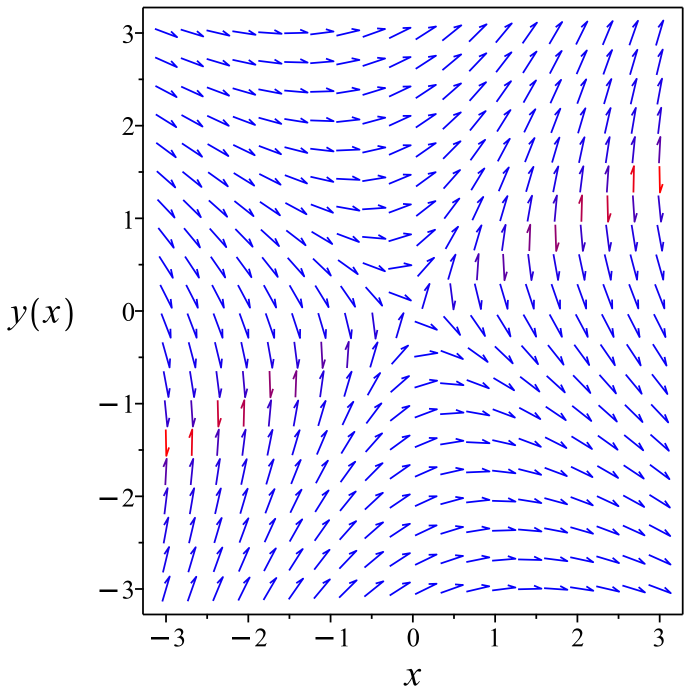


Figure 59: Slope field plot

Verification of solutions

$$-x(x+y)+y^2=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
dsolve(1/x+1/y(x)*diff(y(x),x)+2*(1/y(x)-1/x*diff(y(x),x))=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - \sqrt{5x^2 c_1^2 + 4}}{2c_1}$$
$$y(x) = \frac{c_1 x + \sqrt{5x^2 c_1^2 + 4}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.46 (sec). Leaf size: 102

```
DSolve[1/x+1/y[x]*y'[x]+2*(1/y[x]-1/x*y'[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x - \sqrt{5x^2 - 4e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(x + \sqrt{5x^2 - 4e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(x - \sqrt{5}\sqrt{x^2}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{5}\sqrt{x^2} + x\right)$$

3.3 problem 5.1

Internal problem ID [4389]
Internal file name [`OUTPUT/3882_Sunday_June_05_2022_11_35_08_AM_62087181/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 4
Problem number: 5.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy' - y - \sqrt{x^2 + y^2} = 0}$$

3.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y + \sqrt{x^2 + y^2}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{\left(y+\sqrt{x^2+y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{x^2+y^2}\right)^2 a_3}{x^2} \\
&- \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{y+\sqrt{x^2+y^2}}{x^2}\right)(x a_2 + y a_3 + a_1) \\
&- \frac{\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)(x b_2 + y b_3 + b_1)}{x} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{\left(x^2+y^2\right)^{\frac{3}{2}} a_3 + x^3 a_2 - x^3 b_3 + 2x^2 y a_3 + x^2 y b_2 + y^3 a_3 + \sqrt{x^2+y^2}\, x b_1 - \sqrt{x^2+y^2}\, y a_1 + x y b_1 - y^2 a_1}{\sqrt{x^2+y^2}\, x^2}
$$
$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-\left(x^2+y^2\right)^{\frac{3}{2}} a_3 - x^3 a_2 + x^3 b_3 - 2x^2 y a_3 - x^2 y b_2 - y^3 a_3 \\
&- \sqrt{x^2+y^2}\, x b_1 + \sqrt{x^2+y^2}\, y a_1 - x y b_1 + y^2 a_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-\left(x^2+y^2\right)^{\frac{3}{2}} a_3 + \left(x^2+y^2\right) x b_3 - \left(x^2+y^2\right) y a_3 - x^3 a_2 - x^2 y a_3 - x^2 y b_2 \\
&- x\, y^2 b_3 + \left(x^2+y^2\right) a_1 - \sqrt{x^2+y^2}\, x b_1 + \sqrt{x^2+y^2}\, y a_1 - x^2 a_1 - x y b_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-x^3 a_2 + x^3 b_3 - x^2 \sqrt{x^2+y^2}\, a_3 - 2x^2 y a_3 - x^2 y b_2 - \sqrt{x^2+y^2}\, y^2 a_3 \\
&- y^3 a_3 - \sqrt{x^2+y^2}\, x b_1 - x y b_1 + \sqrt{x^2+y^2}\, y a_1 + y^2 a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^2+y^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-v_1^3 a_2 - 2v_1^2 v_2 a_3 - v_1^2 v_3 a_3 - v_2^3 a_3 - v_3 v_2^2 a_3 - v_1^2 v_2 b_2 \\ &+ v_1^3 b_3 + v_2^2 a_1 + v_3 v_2 a_1 - v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &(b_3 - a_2) v_1^3 + (-2a_3 - b_2) v_1^2 v_2 - v_1^2 v_3 a_3 - v_1 v_2 b_1 \\ &- v_3 v_1 b_1 - v_2^3 a_3 - v_3 v_2^2 a_3 + v_2^2 a_1 + v_3 v_2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2a_3 - b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y + \sqrt{x^2+y^2}}{x}\right)(x) \\
&= -\sqrt{x^2+y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{x^2+y^2}} dy
\end{aligned}
$$

Which results in

$$
S = -\ln\left(y + \sqrt{x^2+y^2}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + \sqrt{x^2+y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x}{\sqrt{x^2+y^2}\,\left(y+\sqrt{x^2+y^2}\right)} \\
S_y &= -\frac{1}{\sqrt{x^2+y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2\left(\sqrt{x^2+y^2}\,y + x^2+y^2\right)}{x\sqrt{x^2+y^2}\,\left(y+\sqrt{x^2+y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y+\sqrt{x^2+y^2}\right) = -2\ln(x) + c_1$$

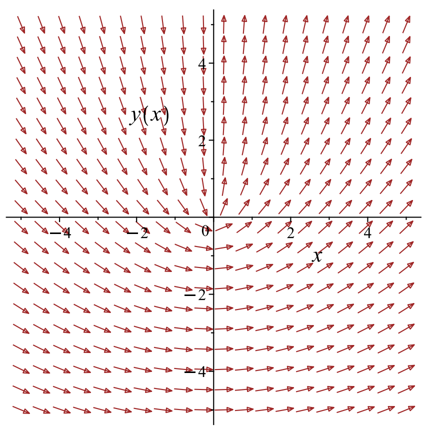
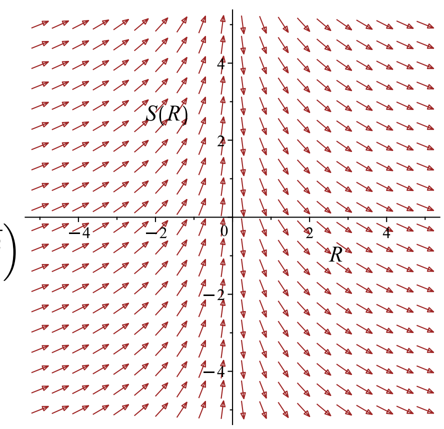
Which simplifies to

$$-\ln\left(y+\sqrt{x^2+y^2}\right) = -2\ln(x) + c_1$$

Which gives

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ 	$R = x$ $S = -\ln\left(y + \sqrt{x^2+y^2}\right)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2} \tag{1}$$

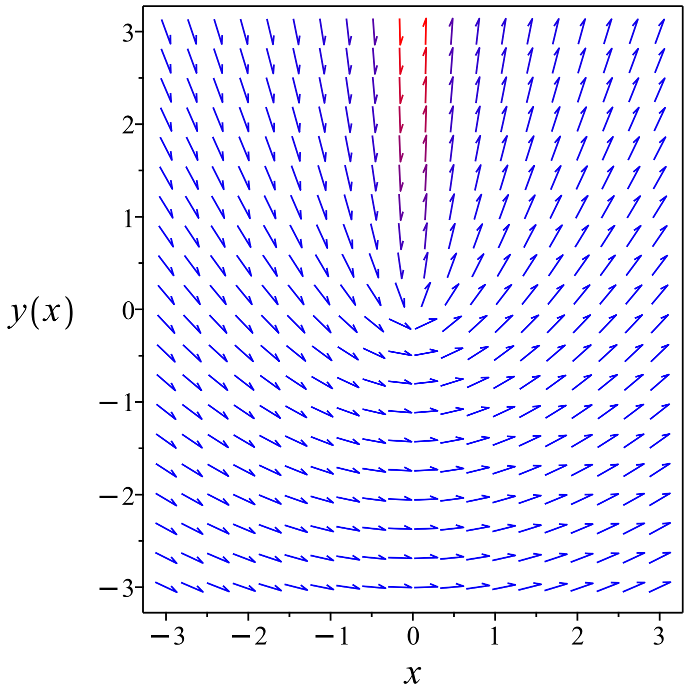


Figure 60: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_1}(e^{2c_1} - x^2)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x*diff(y(x),x)-y(x)=sqrt(x^2+y(x)^2),y(x), singsol=all)
```

$$\frac{-c_1 x^2 + \sqrt{x^2 + y(x)^2} + y(x)}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.336 (sec). Leaf size: 27

```
DSolve[x*y'[x]-y[x]==Sqrt[x^2+y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-c_1} \left(-1 + e^{2c_1} x^2\right)$$

3.4 problem 5.2

Internal problem ID [4390]
Internal file name [OUTPUT/3883_Sunday_June_05_2022_11_35_18_AM_24872058/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 4
Problem number: 5.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{8y + (5y + 7x)\, y' = -10x}$$

3.4.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$8u(x)\, x + (5u(x)\, x + 7x)\, (u'(x)\, x + u(x)) = -10x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{5(u^2 + 3u + 2)}{x\,(5u + 7)} \end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = \frac{u^2+3u+2}{5u+7}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+3u+2}{5u+7}}\, du = -\frac{5}{x}\, dx$$

$$\int \frac{1}{\frac{u^2+3u+2}{5u+7}} du = \int -\frac{5}{x} dx$$

$$2\ln(u+1) + 3\ln(u+2) = -5\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u+1)+3\ln(u+2)} = e^{-5\ln(x)+c_2}$$

Which simplifies to

$$(u+1)^2 (u+2)^3 = \frac{c_3}{x^5}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= \mathrm{RootOf}\left(\_Z^5 + 8x\_Z^4 + 25x^2\_Z^3 + 38x^3\_Z^2 + 28x^4\_Z + 8x^5 - c_3\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{RootOf}\left(\_Z^5 + 8x\_Z^4 + 25x^2\_Z^3 + 38x^3\_Z^2 + 28x^4\_Z + 8x^5 - c_3\right) \tag{1}$$

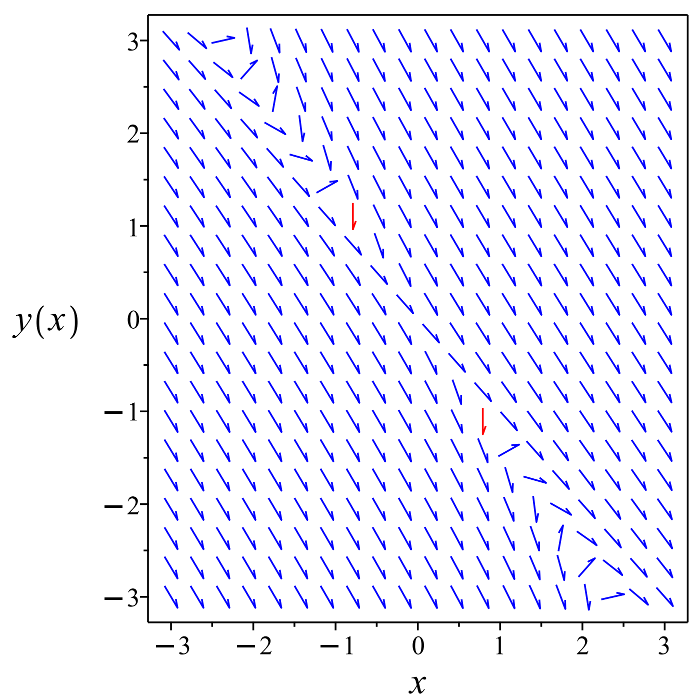


Figure 61: Slope field plot

Verification of solutions

$$y = \text{RootOf}\left(\_Z^5 + 8x\_Z^4 + 25x^2\_Z^3 + 38x^3\_Z^2 + 28x^4\_Z + 8x^5 - c_3\right)$$

Verified OK.

3.4.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2(4y+5x)}{5y+7x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{2(4y+5x)(b_3-a_2)}{5y+7x} - \frac{4(4y+5x)^2 a_3}{(5y+7x)^2} \\ &- \left(-\frac{10}{5y+7x} + \frac{56y+70x}{(5y+7x)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{8}{5y+7x} + \frac{40y+50x}{(5y+7x)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{70x^2a_2 - 100x^2a_3 + 55x^2b_2 - 70x^2b_3 + 100xya_2 - 160xya_3 + 70xyb_2 - 100xyb_3 + 40y^2a_2 - 70y^2a_3 + 25y}{(5y+7x)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&70x^2a_2 - 100x^2a_3 + 55x^2b_2 - 70x^2b_3 + 100xya_2 - 160xya_3 + 70xyb_2 \\
&- 100xyb_3 + 40y^2a_2 - 70y^2a_3 + 25y^2b_2 - 40y^2b_3 + 6xb_1 - 6ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&70a_2v_1^2 + 100a_2v_1v_2 + 40a_2v_2^2 - 100a_3v_1^2 - 160a_3v_1v_2 - 70a_3v_2^2 + 55b_2v_1^2 \\
&+ 70b_2v_1v_2 + 25b_2v_2^2 - 70b_3v_1^2 - 100b_3v_1v_2 - 40b_3v_2^2 - 6a_1v_2 + 6b_1v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(70a_2 - 100a_3 + 55b_2 - 70b_3)\, v_1^2 + (100a_2 - 160a_3 + 70b_2 - 100b_3)\, v_1v_2 \\
&+ 6b_1v_1 + (40a_2 - 70a_3 + 25b_2 - 40b_3)\, v_2^2 - 6a_1v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-6a_1 &= 0 \\
6b_1 &= 0 \\
40a_2 - 70a_3 + 25b_2 - 40b_3 &= 0 \\
70a_2 - 100a_3 + 55b_2 - 70b_3 &= 0 \\
100a_2 - 160a_3 + 70b_2 - 100b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 3a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= -2a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{2(4y+5x)}{5y+7x}\right)(x) \\
&= \frac{10x^2 + 15xy + 5y^2}{5y+7x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{10x^2+15xy+5y^2}{5y+7x}} dy
\end{aligned}
$$

Which results in

$$S = \frac{2\ln(x+y)}{5} + \frac{3\ln(2x+y)}{5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2(4y+5x)}{5y+7x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x + \frac{8y}{5}}{(x+y)(2x+y)} \\
S_y &= \frac{5y+7x}{5(x+y)(2x+y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

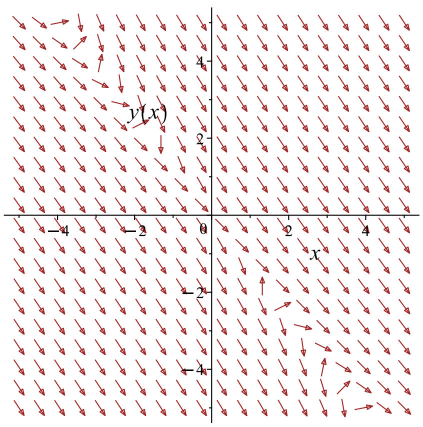
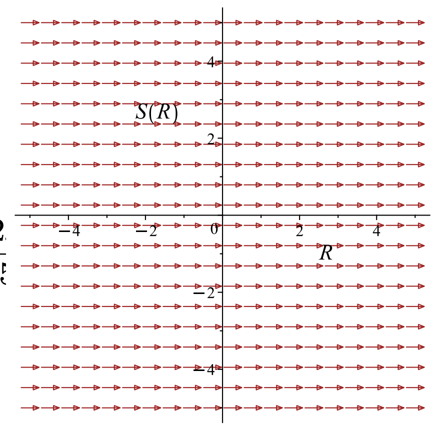
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(x+y)}{5} + \frac{3\ln(2x+y)}{5} = c_1$$

Which simplifies to

$$\frac{2\ln(x+y)}{5} + \frac{3\ln(2x+y)}{5} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2(4y+5x)}{5y+7x}$ 	$R = x$ $S = \frac{2\ln(x+y)}{5} + \frac{3\ln(2}{5}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{2\ln(x+y)}{5} + \frac{3\ln(2x+y)}{5} = c_1 \tag{1}$$

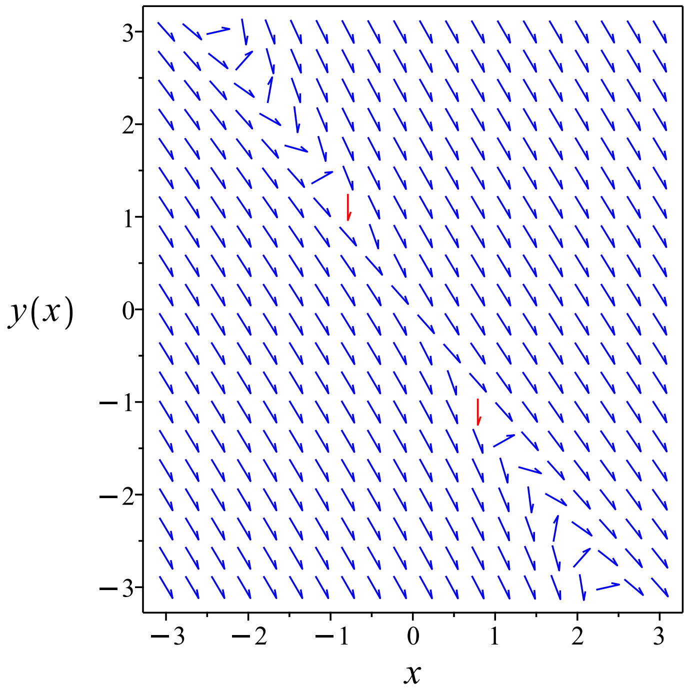


Figure 62: Slope field plot

Verification of solutions

$$\frac{2\ln\left(x+y\right)}{5}+\frac{3\ln\left(2x+y\right)}{5}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 38

```
dsolve((8*y(x)+10*x)+(5*y(x)+7*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x\left(\text{RootOf}\left(\_Z^{25}c_1x^5 - 2\_Z^{20}c_1x^5 + \_Z^{15}c_1x^5 - 1\right)^5 - 2\right)$$

✓ Solution by Mathematica

Time used: 2.162 (sec). Leaf size: 276

```
DSolve[(8*y[x]+10*x)+(5*y[x]+7*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \text{Root}\left[\#1^5 + 8\#1^4x + 25\#1^3x^2 + 38\#1^2x^3 + 28\#1x^4 + 8x^5 - e^{c_1}\&, 1\right]$$
$$y(x) \rightarrow \text{Root}\left[\#1^5 + 8\#1^4x + 25\#1^3x^2 + 38\#1^2x^3 + 28\#1x^4 + 8x^5 - e^{c_1}\&, 2\right]$$
$$y(x) \rightarrow \text{Root}\left[\#1^5 + 8\#1^4x + 25\#1^3x^2 + 38\#1^2x^3 + 28\#1x^4 + 8x^5 - e^{c_1}\&, 3\right]$$
$$y(x) \rightarrow \text{Root}\left[\#1^5 + 8\#1^4x + 25\#1^3x^2 + 38\#1^2x^3 + 28\#1x^4 + 8x^5 - e^{c_1}\&, 4\right]$$
$$y(x) \rightarrow \text{Root}\left[\#1^5 + 8\#1^4x + 25\#1^3x^2 + 38\#1^2x^3 + 28\#1x^4 + 8x^5 - e^{c_1}\&, 5\right]$$

3.5 problem 5.3

Internal problem ID [4391]
Internal file name [OUTPUT/3884_Sunday_June_05_2022_11_35_28_AM_14982222/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 4
Problem number: 5.3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2xy - y^2 + (y^2 + 2xy - x^2)\, y' = -x^2}$$

3.5.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$2x^2u(x) - u(x)^2\, x^2 + (u(x)^2\, x^2 + 2x^2u(x) - x^2)\, (u'(x)\, x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(u+1)\,(u^2+1)}{x\,(u^2+2u-1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{(u+1)(u^2+1)}{u^2+2u-1}$. Integrating both sides gives

$$\frac{1}{\frac{(u+1)(u^2+1)}{u^2+2u-1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{(u+1)(u^2+1)}{u^2+2u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$-\ln(u+1)+\ln\left(u^2+1\right) = -\ln(x)+c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(u+1)+\ln\left(u^2+1\right)} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u^2+1}{u+1} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)^2+1}{u(x)+1} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\frac{y^2}{x^2}+1}{\frac{y}{x}+1} = \frac{c_3}{x}$$

$$\frac{x^2+y^2}{x(x+y)} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{x^2+y^2}{x+y} = c_3$$

Summary

The solution(s) found are the following

$$\frac{x^2+y^2}{x+y} = c_3 \tag{1}$$

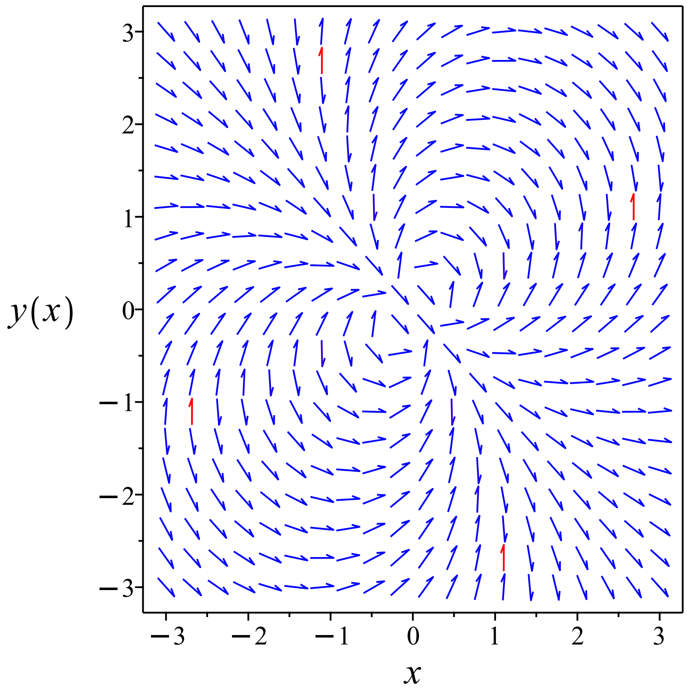


Figure 63: Slope field plot

Verification of solutions

$$\frac{x^2 + y^2}{x + y} = c_3$$

Verified OK.

3.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-x^2 - 2xy + y^2}{-x^2 + 2xy + y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-x^2 - 2xy + y^2)(b_3 - a_2)}{-x^2 + 2xy + y^2} - \frac{(-x^2 - 2xy + y^2)^2 a_3}{(-x^2 + 2xy + y^2)^2} - \left(\frac{-2x - 2y}{-x^2 + 2xy + y^2} - \frac{(-x^2 - 2xy + y^2)(-2x + 2y)}{(-x^2 + 2xy + y^2)^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{-2x + 2y}{-x^2 + 2xy + y^2} - \frac{(-x^2 - 2xy + y^2)(2y + 2x)}{(-x^2 + 2xy + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4a_2 + x^4a_3 + 3x^4b_2 - x^4b_3 - 4x^3ya_2 + 4x^3ya_3 + 4x^3yb_2 + 4x^3yb_3 - 6x^2y^2a_2 - 2x^2y^2a_3 + 2x^2y^2b_2 + 6x}{(x^2}$$

$$= 0$$

Setting the numerator to zero gives

$$-x^4a_2 - x^4a_3 - 3x^4b_2 + x^4b_3 + 4x^3ya_2 - 4x^3ya_3 - 4x^3yb_2 - 4x^3yb_3 + 6x^2y^2a_2 + 2x^2y^2a_3 - 2x^2y^2b_2 - 6x^2y^2b_3 + 4x\,y^3a_2 + 4x\,y^3a_3 + 4x\,y^3b_2 - 4x\,y^3b_3 - y^4a_2 + 3y^4a_3 + y^4b_2 + y^4b_3 - 4x^3b_1 + 4x^2ya_1 - 4x\,y^2b_1 + 4y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^4 + 4a_2v_1^3v_2 + 6a_2v_1^2v_2^2 + 4a_2v_1v_2^3 - a_2v_2^4 - a_3v_1^4 - 4a_3v_1^3v_2 + 2a_3v_1^2v_2^2 + 4a_3v_1v_2^3 + 3a_3v_2^4 - 3b_2v_1^4 - 4b_2v_1^3v_2 - 2b_2v_1^2v_2^2 + 4b_2v_1v_2^3 + b_2v_2^4 + b_3v_1^4 - 4b_3v_1^3v_2 - 6b_3v_1^2v_2^2 - 4b_3v_1v_2^3 + b_3v_2^4 + 4a_1v_1^2v_2 + 4a_1v_2^3 - 4b_1v_1^3 - 4b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_2 - a_3 - 3b_2 + b_3)\, v_1^4 + (4a_2 - 4a_3 - 4b_2 - 4b_3)\, v_1^3 v_2 - 4b_1 v_1^3 \\ + (6a_2 + 2a_3 - 2b_2 - 6b_3)\, v_1^2 v_2^2 + 4a_1 v_1^2 v_2 + (4a_2 + 4a_3 + 4b_2 - 4b_3)\, v_1 v_2^3 \\ - 4b_1 v_1 v_2^2 + (-a_2 + 3a_3 + b_2 + b_3)\, v_2^4 + 4a_1 v_2^3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}4a_1 &= 0 \\ -4b_1 &= 0 \\ -a_2 - a_3 - 3b_2 + b_3 &= 0 \\ -a_2 + 3a_3 + b_2 + b_3 &= 0 \\ 4a_2 - 4a_3 - 4b_2 - 4b_3 &= 0 \\ 4a_2 + 4a_3 + 4b_2 - 4b_3 &= 0 \\ 6a_2 + 2a_3 - 2b_2 - 6b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x \\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{-x^2-2xy+y^2}{-x^2+2xy+y^2}\right)(x) \\
&= \frac{-x^3-y\,x^2-y^2x-y^3}{x^2-2xy-y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{-x^3-y\,x^2-y^2x-y^3}{x^2-2xy-y^2}}dy
\end{aligned}
$$

Which results in

$$
S = -\ln(x+y) + \ln\left(x^2+y^2\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{-x^2-2xy+y^2}{-x^2+2xy+y^2}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= -\frac{1}{x+y}+\frac{2x}{x^2+y^2}\\ S_y &= -\frac{1}{x+y}+\frac{2y}{x^2+y^2}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

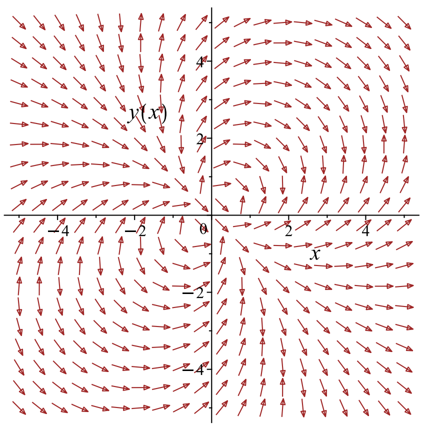
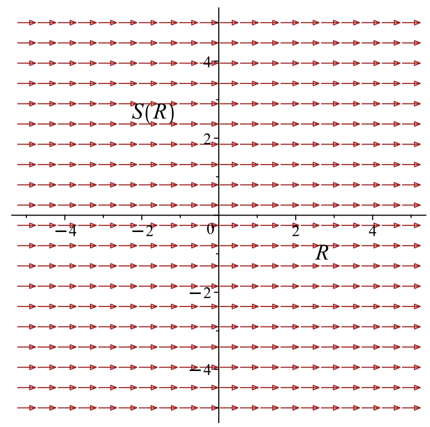
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(x+y)+\ln\left(x^2+y^2\right) = c_1$$

Which simplifies to

$$-\ln(x+y)+\ln\left(x^2+y^2\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x^2-2xy+y^2}{-x^2+2xy+y^2}$ 	$R = x$ $S = -\ln(x+y) + \ln(x^2$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\ln(x+y) + \ln(x^2+y^2) = c_1 \tag{1}$$

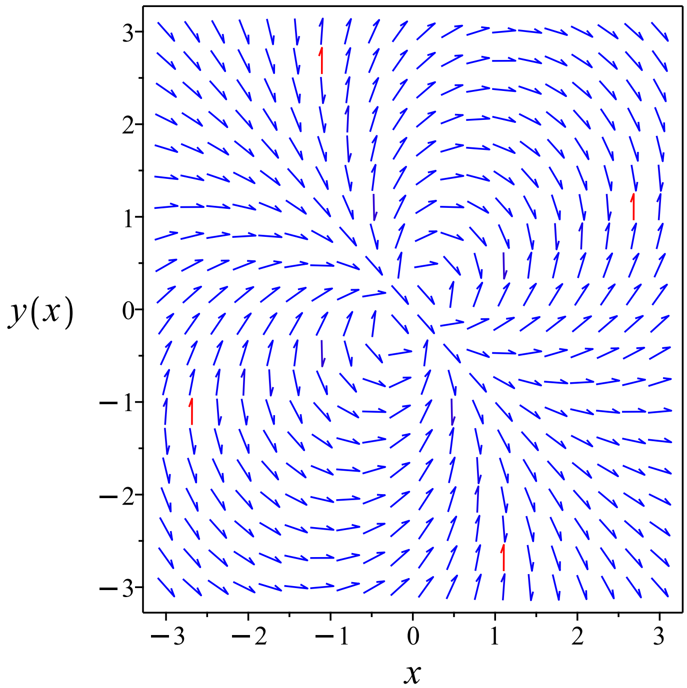


Figure 64: Slope field plot

Verification of solutions

$$-\ln(x+y)+\ln\left(x^2+y^2\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 55

```
dsolve((x^2+2*x*y(x)-y(x)^2)+(y(x)^2+2*x*y(x)-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1 - \sqrt{-4x^2c_1^2 + 4c_1x + 1}}{2c_1}$$
$$y(x) = \frac{1 + \sqrt{-4x^2c_1^2 + 4c_1x + 1}}{2c_1}$$

✓ Solution by Mathematica

Time used: 1.304 (sec). Leaf size: 75

```
DSolve[(x^2+2*x*y[x]-y[x]^2)+(y[x]^2+2*x*y[x]-x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{2}\left(e^{c_1} - \sqrt{-4x^2 + 4e^{c_1}x + e^{2c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{-4x^2 + 4e^{c_1}x + e^{2c_1}} + e^{c_1}\right)$$

3.6 problem 5.4

Internal problem ID [4392]
Internal file name [`OUTPUT/3885_Sunday_June_05_2022_11_35_41_AM_43678614/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 4
Problem number: 5.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2 + (xy + x^2)\, y' = 0}$$

3.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\, x^2 + (x^2 u(x) + x^2)\, (u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{2u^2 + u}{x\,(u + 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2+u}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+u}{u+1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{2u^2+u}{u+1}}\,du = \int -\frac{1}{x}\,dx$$

$$\ln(u) - \frac{\ln(2u+1)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\ln(u) - \frac{\ln(2u+1)}{2}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u}{\sqrt{2u+1}} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)}{\sqrt{2u(x)+1}} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x\sqrt{\frac{2y}{x}+1}} = \frac{c_3}{x}$$

$$\frac{y}{\sqrt{\frac{2y+x}{x}}\,x} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{y}{\sqrt{\frac{2y+x}{x}}} = c_3$$

Summary

The solution(s) found are the following

$$\frac{y}{\sqrt{\frac{2y+x}{x}}} = c_3 \tag{1}$$

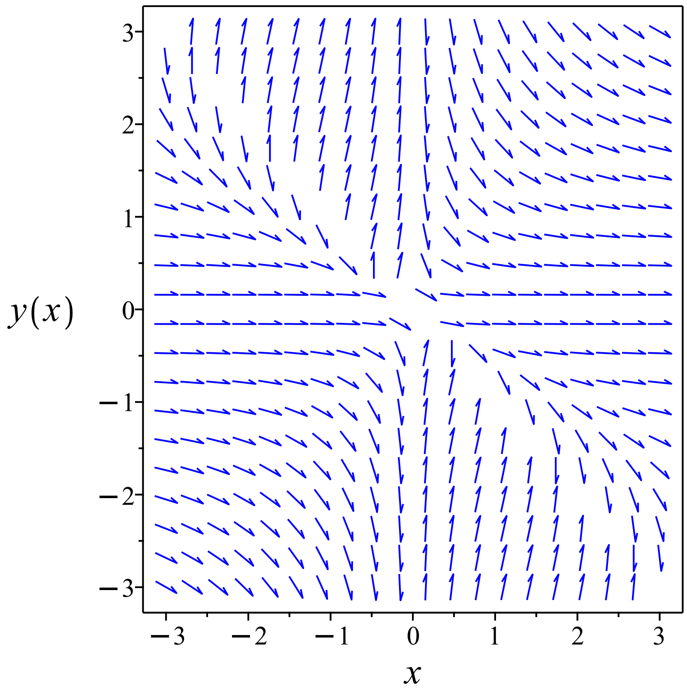


Figure 65: Slope field plot

Verification of solutions

$$\frac{y}{\sqrt{\frac{2y+x}{x}}} = c_3$$

Verified OK.

3.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y^2}{x(x+y)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y^2(b_3 - a_2)}{x(x+y)} - \frac{y^4 a_3}{x^2(x+y)^2} - \left(\frac{y^2}{x^2(x+y)} + \frac{y^2}{x(x+y)^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{2y}{x(x+y)} + \frac{y^2}{x(x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^4 b_2 + 4x^3 y b_2 - x^2 y^2 a_2 + 2x^2 y^2 b_2 + x^2 y^2 b_3 - 2x\,y^3 a_3 - 2y^4 a_3 + 2x^2 y b_1 - 2x\,y^2 a_1 + x\,y^2 b_1 - y^3 a_1}{x^2(x+y)^2} = 0$$

Setting the numerator to zero gives

$$x^4 b_2 + 4x^3 y b_2 - x^2 y^2 a_2 + 2x^2 y^2 b_2 + x^2 y^2 b_3 - 2x\,y^3 a_3 - 2y^4 a_3 + 2x^2 y b_1 - 2x\,y^2 a_1 + x\,y^2 b_1 - y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_1^2 v_2^2 - 2a_3 v_1 v_2^3 - 2a_3 v_2^4 + b_2 v_1^4 + 4b_2 v_1^3 v_2 + 2b_2 v_1^2 v_2^2 + b_3 v_1^2 v_2^2 - 2a_1 v_1 v_2^2 - a_1 v_2^3 + 2b_1 v_1^2 v_2 + b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&b_2v_1^4 + 4b_2v_1^3v_2 + (-a_2 + 2b_2 + b_3)\, v_1^2v_2^2 + 2b_1v_1^2v_2 \\ &- 2a_3v_1v_2^3 + (-2a_1 + b_1)\, v_1v_2^2 - 2a_3v_2^4 - a_1v_2^3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}b_2 &= 0\\ -a_1 &= 0\\ -2a_3 &= 0\\ 2b_1 &= 0\\ 4b_2 &= 0\\ -2a_1 + b_1 &= 0\\ -a_2 + 2b_2 + b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi\\ &= y - \left(-\frac{y^2}{x\,(x+y)}\right)(x)\\ &= \frac{xy + 2y^2}{x+y}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{xy+2y^2}{x+y}} dy \end{aligned}$$

Which results in

$$S = \ln(y) - \frac{\ln(2y + x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2}{x(x + y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{4y + 2x} \\ S_y &= \frac{x + y}{y(2y + x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

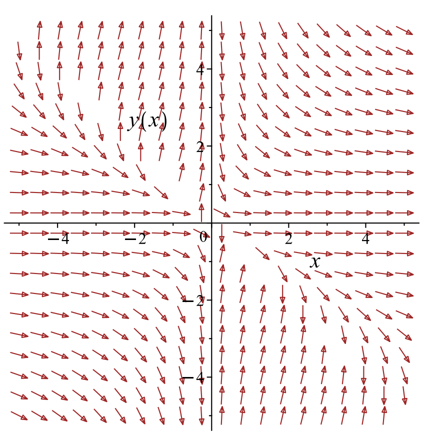
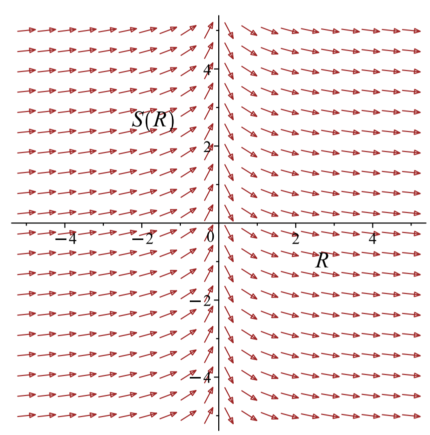
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - \frac{\ln(2y + x)}{2} = -\frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$\ln(y) - \frac{\ln(2y + x)}{2} = -\frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y^2}{x(x+y)}$
</td><td>$R = x$
$S = \ln(y) - \frac{\ln(2y + x)}{2}$</td><td>$\frac{dS}{dR} = -\frac{1}{2R}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$\ln(y) - \frac{\ln(2y+x)}{2} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

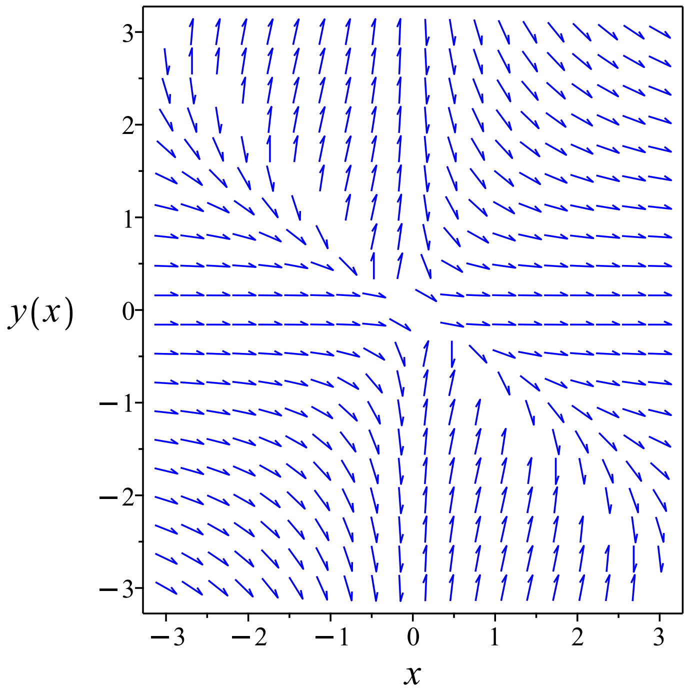


Figure 66: Slope field plot

Verification of solutions

$$\ln(y) - \frac{\ln(2y+x)}{2} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
dsolve(y(x)^2+(x*y(x)+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1 + \sqrt{c_1 x^2 + 1}}{c_1 x}$$
$$y(x) = \frac{1 - \sqrt{c_1 x^2 + 1}}{c_1 x}$$

✓ Solution by Mathematica

Time used: 2.31 (sec). Leaf size: 80

```
DSolve[y[x]^2+(x*y[x]+x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2c_1} - \sqrt{e^{2c_1}\left(x^2 + e^{2c_1}\right)}}{x}$$
$$y(x) \to \frac{\sqrt{e^{2c_1}\left(x^2 + e^{2c_1}\right)} + e^{2c_1}}{x}$$
$$y(x) \to 0$$

3.7 problem 5.4

Internal problem ID [4393]
Internal file name [`OUTPUT/3886_Sunday_June_05_2022_11_35_49_AM_26170709/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 4
Problem number: 5.4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{\left(x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right)y+\left(x\cos\left(\frac{y}{x}\right)-y\sin\left(\frac{y}{x}\right)\right)xy'=0}$$

3.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\left(x\cos\left(\frac{y}{x}\right) - y\sin\left(\frac{y}{x}\right)\right)x\right)\mathrm{d}y = \left(-y\left(x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right)\right)\mathrm{d}x$$

$$\left(y\left(x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right)\right)\mathrm{d}x + \left(\left(x\cos\left(\frac{y}{x}\right) - y\sin\left(\frac{y}{x}\right)\right)x\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= y\left(x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right) \\ N(x,y) &= \left(x\cos\left(\frac{y}{x}\right) - y\sin\left(\frac{y}{x}\right)\right)x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\left(x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right)\right) \\ &= \frac{(x^2+y^2)\cos\left(\frac{y}{x}\right) + \sin\left(\frac{y}{x}\right)xy}{x} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\left(x\cos\left(\frac{y}{x}\right) - y\sin\left(\frac{y}{x}\right)\right)x\right) \\ &= \frac{\cos\left(\frac{y}{x}\right)(2x^2+y^2)}{x} \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{\left(x\cos\left(\frac{y}{x}\right) - y\sin\left(\frac{y}{x}\right)\right)x}\left(\left(x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right) + \frac{y^2\cos\left(\frac{y}{x}\right)}{x}\right) - \left(\left(\cos\left(\frac{y}{x}\right) + \frac{y\sin\left(\frac{y}{x}\right)}{x} + \frac{y^2 \mathrm{co}\ldots}{x \ldots}\right.\right.\right. \\
&= -\frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(x)} \\
&= \frac{1}{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x}\left(y\left(x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right)\right) \\
&= \frac{y\left(x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right)}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x}\left(\left(x\cos\left(\frac{y}{x}\right) - y\sin\left(\frac{y}{x}\right)\right)x\right) \\
&= x\cos\left(\frac{y}{x}\right) - y\sin\left(\frac{y}{x}\right)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y\left(x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right)}{x}\right) + \left(x\cos\left(\frac{y}{x}\right) - y\sin\left(\frac{y}{x}\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{y\left(x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right)}{x} \, \mathrm{d}x$$

$$\phi = y \cos\left(\frac{y}{x}\right) x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x \cos\left(\frac{y}{x}\right) - y \sin\left(\frac{y}{x}\right) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x \cos\left(\frac{y}{x}\right) - y \sin\left(\frac{y}{x}\right)$. Therefore equation (4) becomes

$$x \cos\left(\frac{y}{x}\right) - y \sin\left(\frac{y}{x}\right) = x \cos\left(\frac{y}{x}\right) - y \sin\left(\frac{y}{x}\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y \cos\left(\frac{y}{x}\right) x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y \cos\left(\frac{y}{x}\right) x$$

Summary

The solution(s) found are the following

$$\cos\left(\frac{y}{x}\right) y x = c_1 \tag{1}$$

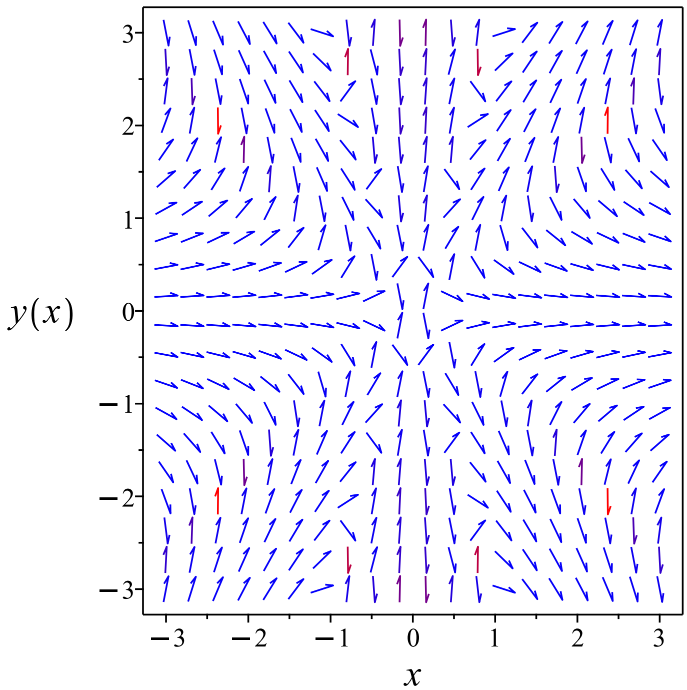


Figure 67: Slope field plot

Verification of solutions

$$\cos\left(\frac{y}{x}\right) y x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 18

```
dsolve((x*cos(y(x)/x)+y(x)*sin(y(x)/x))*y(x)+(x*cos(y(x)/x)-y(x)*sin(y(x)/x))*x*diff(y(x),x)
```

$$y(x) = x \operatorname{RootOf}\left(\_Z \cos\left(\_Z\right) x^{2} - c_{1}\right)$$

✓ Solution by Mathematica

Time used: 0.347 (sec). Leaf size: 31

```
DSolve[(x*Cos[y[x]/x]+y[x]*Sin[y[x]/x])*y[x]+(x*Cos[y[x]/x]-y[x]*Sin[y[x]/x])*x*y'[x]==0,y[x
```

$$\text{Solve}\left[-\log\left(\frac{y(x)}{x}\right) - \log\left(\cos\left(\frac{y(x)}{x}\right)\right) = 2\log(x) + c_1, y(x)\right]$$

3.8 problem 7.1

Internal problem ID [4394]
Internal file name [`OUTPUT/3887_Sunday_June_05_2022_11_35_57_AM_27411867/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 4
Problem number: 7.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{(y^2x^2 + xy)\, y + (y^2x^2 - 1)\, xy' = 0}$$

3.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y^2}{xy - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2-\frac{y^2(b_3-a_2)}{xy-1}-\frac{y^4a_3}{(xy-1)^2}-\frac{y^3(xa_2+ya_3+a_1)}{(xy-1)^2} \\ & -\left(-\frac{2y}{xy-1}+\frac{y^2x}{(xy-1)^2}\right)(xb_2+yb_3+b_1)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2y^2b_2-2y^4a_3+x\,y^2b_1-y^3a_1-4xyb_2-y^2a_2-y^2b_3-2yb_1+b_2}{(xy-1)^2}=0$$

Setting the numerator to zero gives

$$2x^2y^2b_2-2y^4a_3+x\,y^2b_1-y^3a_1-4xyb_2-y^2a_2-y^2b_3-2yb_1+b_2=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$-2a_3v_2^4+2b_2v_1^2v_2^2-a_1v_2^3+b_1v_1v_2^2-a_2v_2^2-4b_2v_1v_2-b_3v_2^2-2b_1v_2+b_2=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^2v_2^2+b_1v_1v_2^2-4b_2v_1v_2-2a_3v_2^4-a_1v_2^3+(-a_2-b_3)\,v_2^2-2b_1v_2+b_2=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_1 &= 0 \\
b_2 &= 0 \\
-a_1 &= 0 \\
-2a_3 &= 0 \\
-2b_1 &= 0 \\
-4b_2 &= 0 \\
2b_2 &= 0 \\
-a_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{y^2}{xy - 1}\right)(-x) \\
&= -\frac{y}{xy - 1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{y}{xy-1}} dy \end{aligned}$$

Which results in

$$S = -xy + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2}{xy - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y \\ S_y &= -x + \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-xy + \ln(y) = c_1$$

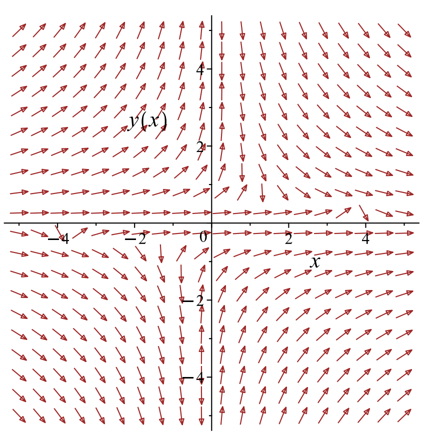
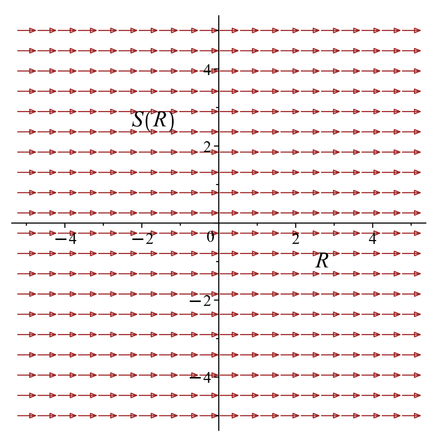
Which simplifies to

$$-xy + \ln(y) = c_1$$

Which gives

$$y = e^{-\operatorname{LambertW}(-x\,e^{c_1})+c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2}{xy-1}$ 	$R = x$ $S = -xy + \ln(y)$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}(-x e^{c_1})+c_1} \tag{1}$$

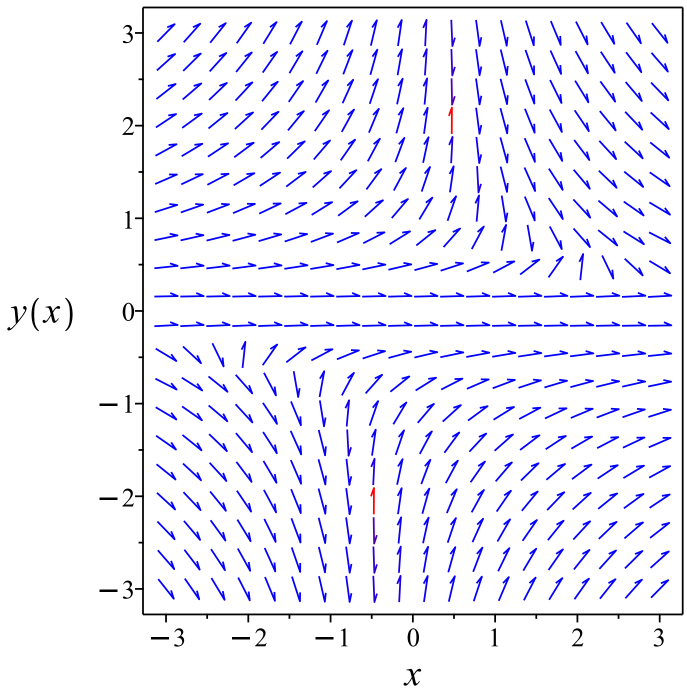


Figure 68: Slope field plot

Verification of solutions

$$y = e^{-\operatorname{LambertW}(-x e^{c_1})+c_1}$$

Verified OK.

3.8.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x+N(x, y) \,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\left(y^2 x^2-1\right) x\right) \mathrm{d}y &= \left(-\left(y^2 x^2+x y\right) y\right) \mathrm{d}x \\
\left(\left(y^2 x^2+x y\right) y\right) \mathrm{d}x+\left(\left(y^2 x^2-1\right) x\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= \left(y^2 x^2+x y\right) y \\
N(x, y) &= \left(y^2 x^2-1\right) x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\left(y^2 x^2+x y\right) y\right) \\
&= 3 y^2 x^2+2 x y
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\left(y^2x^2-1\right)x\right)\\
&= 3y^2x^2-1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\
&= \frac{1}{y^2x^3-x}\left(\left(\left(2y\,x^2+x\right)y+y^2x^2+xy\right)-\left(3y^2x^2-1\right)\right)\\
&= \frac{2xy+1}{y^2x^3-x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\\
&= \frac{1}{x\,y^2\left(xy+1\right)}\left(\left(3y^2x^2-1\right)-\left(\left(2y\,x^2+x\right)y+y^2x^2+xy\right)\right)\\
&= \frac{-2xy-1}{x\,y^2\left(xy+1\right)}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{xM-yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{xM-yN}\\
&= \frac{\left(3y^2x^2-1\right)-\left(\left(2y\,x^2+x\right)y+y^2x^2+xy\right)}{x\left(\left(y^2x^2+xy\right)y\right)-y\left(\left(y^2x^2-1\right)x\right)}\\
&= \frac{-2xy-1}{xy\left(xy+1\right)}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = \frac{-2t-1}{t\left(t+1\right)}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}\mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(\frac{-2t-1}{t(t+1)}\right) \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(t(t+1))} \\ &= \frac{1}{t\left(t+1\right)}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{xy\left(xy+1\right)}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{xy\left(xy+1\right)}\left(\left(y^2x^2+xy\right)y\right) \\ &= y\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{xy\left(xy+1\right)}\left(\left(y^2x^2-1\right)x\right) \\ &= \frac{xy-1}{y}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (y)+\left(\frac{xy-1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y \, \mathrm{d}x$$
$$\phi = xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{xy-1}{y}$. Therefore equation (4) becomes

$$\frac{xy - 1}{y} = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = xy - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)-c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)-c_1} \tag{1}$$

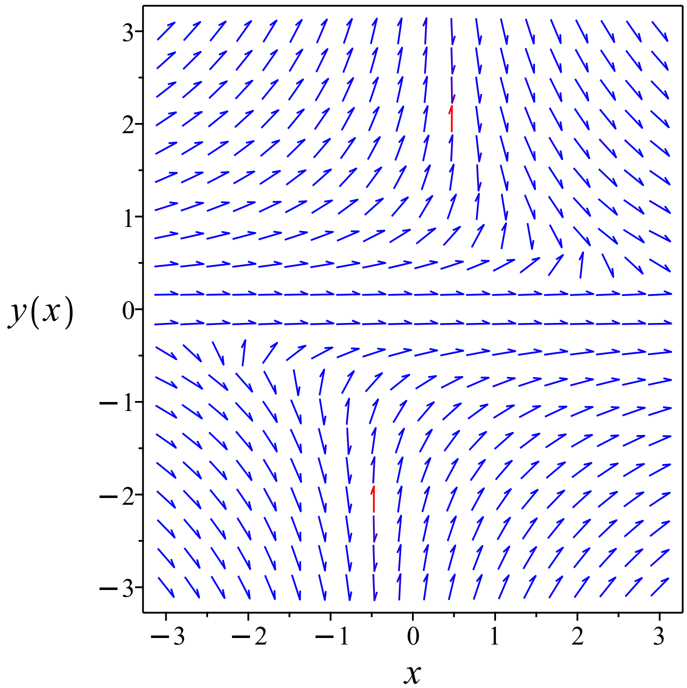


Figure 69: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)-c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve((x^2*y(x)^2+x*y(x))*y(x)+(x^2*y(x)^2-1)*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{x}$$
$$y(x) = -\frac{\mathrm{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)}{x}$$

✓ Solution by Mathematica

Time used: 2.043 (sec). Leaf size: 43

```
DSolve[(x^2*y[x]^2+x*y[x])*y[x]+(x^2*y[x]^2-1)*x*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{1}{x}$$
$$y(x) \to -\frac{W(-e^{-c_1}x)}{x}$$
$$y(x) \to 0$$
$$y(x) \to -\frac{1}{x}$$

3.9 problem 7.1

Internal problem ID [4395]
Internal file name [`OUTPUT/3888_Sunday_June_05_2022_11_36_05_AM_89436757/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 4
Problem number: 7.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{\left(x^3y^3+y^2x^2+xy+1\right)y+\left(x^3y^3-y^2x^2-xy+1\right)xy'=0}$$

3.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{(y^2x^2+1)\,y}{x\,(y^2x^2-2xy+1)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(y^2x^2+1)\, y(b_3-a_2)}{x\,(y^2x^2-2xy+1)} - \frac{(y^2x^2+1)^2\, y^2 a_3}{x^2\,(y^2x^2-2xy+1)^2} - \left(-\frac{2y^3}{y^2x^2-2xy+1} + \frac{(y^2x^2+1)\, y}{x^2\,(y^2x^2-2xy+1)} + \frac{(y^2x^2+1)\, y(2y^2x-2y)}{x\,(y^2x^2-2xy+1)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{2y^2x}{y^2x^2-2xy+1} - \frac{y^2x^2+1}{x\,(y^2x^2-2xy+1)} + \frac{(y^2x^2+1)\, y(2y\,x^2-2x)}{x\,(y^2x^2-2xy+1)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^6y^4b_2 - 2x^4y^6a_3 + x^5y^4b_1 - x^4y^5a_1 - 8x^5y^3b_2 - 2x^4y^4a_2 - 2x^4y^4b_3 - 4x^4y^3b_1 + 8x^4y^2b_2 - 4x^2y^4a_3 + 2}{(y^2x^2-2xy+}$$
$$= 0$$

Setting the numerator to zero gives

$$2x^6y^4b_2 - 2x^4y^6a_3 + x^5y^4b_1 - x^4y^5a_1 - 8x^5y^3b_2 - 2x^4y^4a_2 - 2x^4y^4b_3 - 4x^4y^3b_1 + 8x^4y^2b_2 - 4x^2y^4a_3 + 2x^3y^2b_1 - 2x^2y^3a_1 - 4x^3yb_2 + 2x^2y^2a_2 + 2x^2y^2b_3 + 4x\,y^3a_3 + 4x\,y^2a_1 + 2b_2x^2 - 2y^2a_3 + xb_1 - ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_3v_1^4v_2^6 + 2b_2v_1^6v_2^4 - a_1v_1^4v_2^5 + b_1v_1^5v_2^4 - 2a_2v_1^4v_2^4 - 8b_2v_1^5v_2^3 - 2b_3v_1^4v_2^4 - 4b_1v_1^4v_2^3 - 4a_3v_1^2v_2^4 + 8b_2v_1^4v_2^2 - 2a_1v_1^2v_2^3 + 2b_1v_1^3v_2^2 + 2a_2v_1^2v_2^2 + 4a_3v_1v_2^3 - 4b_2v_1^3v_2 + 2b_3v_1^2v_2^2 + 4a_1v_1v_2^2 - 2a_3v_2^2 + 2b_2v_1^2 - a_1v_2 + b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&2b_2v_1^6v_2^4 + b_1v_1^5v_2^4 - 8b_2v_1^5v_2^3 - 2a_3v_1^4v_2^6 - a_1v_1^4v_2^5 + (-2a_2 - 2b_3)\, v_1^4v_2^4 \\
&- 4b_1v_1^4v_2^3 + 8b_2v_1^4v_2^2 + 2b_1v_1^3v_2^2 - 4b_2v_1^3v_2 - 4a_3v_1^2v_2^4 - 2a_1v_1^2v_2^3 \\
&+ (2a_2 + 2b_3)\, v_1^2v_2^2 + 2b_2v_1^2 + 4a_3v_1v_2^3 + 4a_1v_1v_2^2 + b_1v_1 - 2a_3v_2^2 - a_1v_2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_1 &= 0 \\
-2a_1 &= 0 \\
-a_1 &= 0 \\
4a_1 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
4a_3 &= 0 \\
-4b_1 &= 0 \\
2b_1 &= 0 \\
-8b_2 &= 0 \\
-4b_2 &= 0 \\
2b_2 &= 0 \\
8b_2 &= 0 \\
-2a_2 - 2b_3 &= 0 \\
2a_2 + 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{(y^2x^2+1)\,y}{x\,(y^2x^2-2xy+1)}\right)(-x) \\ &= -\frac{2y^2x}{y^2x^2-2xy+1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{2y^2x}{y^2x^2-2xy+1}} dy\end{aligned}$$

Which results in

$$S = -\frac{xy}{2} + \ln(y) + \frac{1}{2yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{(y^2x^2+1)\, y}{x\,(y^2x^2-2xy+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{-y^2x^2-1}{2y\,x^2}\\
S_y &= -\frac{(xy-1)^2}{2y^2x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

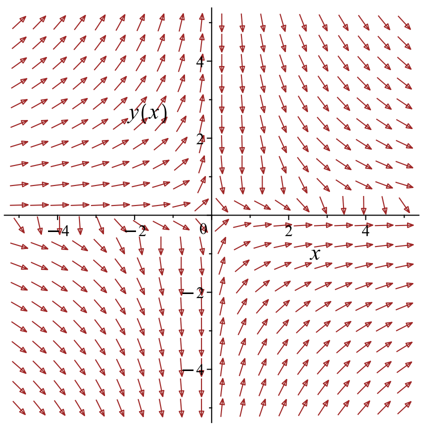
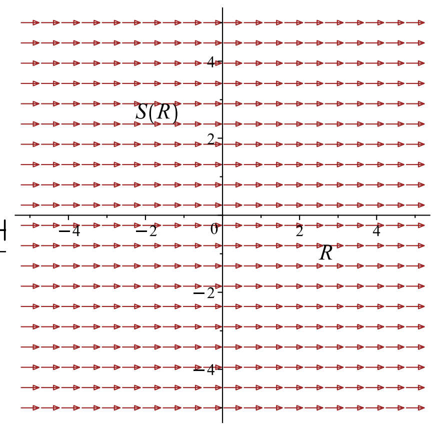
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{-y^2x^2+2\ln(y)\, xy+1}{2xy} = c_1$$

Which simplifies to

$$\frac{-y^2x^2+2\ln(y)\, xy+1}{2xy} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(y^2x^2+1)y}{x(y^2x^2-2xy+1)}$ 	$R = x$ $S = \frac{-y^2x^2 + 2\ln(y)\,xy +}{2xy}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{-y^2x^2 + 2\ln(y)\,xy + 1}{2xy} = c_1 \tag{1}$$

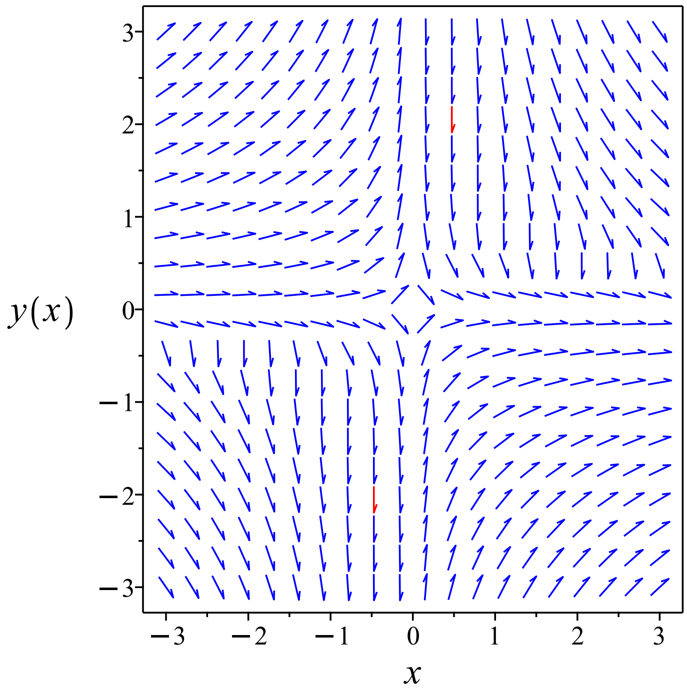


Figure 70: Slope field plot

Verification of solutions

$$\frac{-y^2x^2 + 2\ln(y)\,xy + 1}{2xy} = c_1$$

Verified OK.

3.9.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\left(y^3x^3 - y^2x^2 - xy + 1\right)x\right)\mathrm{d}y = \left(-y\left(y^3x^3 + y^2x^2 + xy + 1\right)\right)\mathrm{d}x$$
$$\left(y\left(y^3x^3 + y^2x^2 + xy + 1\right)\right)\mathrm{d}x + \left(\left(y^3x^3 - y^2x^2 - xy + 1\right)x\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = y\left(y^3x^3 + y^2x^2 + xy + 1\right)$$
$$N(x,y) = \left(y^3x^3 - y^2x^2 - xy + 1\right)x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\left(y^3x^3 + y^2x^2 + xy + 1\right)\right)\\ &= 4y^3x^3 + 3y^2x^2 + 2xy + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\left(y^3x^3 - y^2x^2 - xy + 1\right)x\right)\\ &= 4y^3x^3 - 3y^2x^2 - 2xy + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{(xy+1)(xy-1)^2 x}\left(\left(y^3x^3 + y^2x^2 + xy + 1 + y\left(3y^2x^3 + 2y\,x^2 + x\right)\right) - \left(\left(3x^2y^3 - 2y^2x - y\right)x + y^3x\right.\right. \\
&= \frac{6y^2x + 4y}{(xy+1)(xy-1)^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y(xy+1)(y^2x^2+1)}\left(\left(\left(3x^2y^3 - 2y^2x - y\right)x + y^3x^3 - y^2x^2 - xy + 1\right) - \left(y^3x^3 + y^2x^2 + xy + 1 + y\right.\right. \\
&= \frac{-6y\,x^2 - 4x}{(xy+1)(y^2x^2+1)}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{\left(\left(3x^2y^3 - 2y^2x - y\right)x + y^3x^3 - y^2x^2 - xy + 1\right) - \left(y^3x^3 + y^2x^2 + xy + 1 + y\left(3y^2x^3 + 2y\,x^2 + x\right)\right)}{x\left(y\left(y^3x^3 + y^2x^2 + xy + 1\right)\right) - y\left(\left(y^3x^3 - y^2x^2 - xy + 1\right)x\right)} \\
&= \frac{-3xy - 2}{xy(xy+1)}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = \frac{-3t - 2}{t(t+1)}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(\frac{-3t-2}{t(t+1)}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(t+1)-2\ln(t)} \\
&= \frac{1}{(t+1)\,t^2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{(xy+1)\,x^2y^2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(xy+1)\,x^2y^2}\left(y\left(y^3x^3+y^2x^2+xy+1\right)\right) \\
&= \frac{y^2x^2+1}{y\,x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(xy+1)\,x^2y^2}\left(\left(y^3x^3-y^2x^2-xy+1\right)x\right) \\
&= \frac{(xy-1)^2}{y^2x}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{dy}{dx} &= 0 \\
\left(\frac{y^2x^2+1}{y\,x^2}\right)+\left(\frac{(xy-1)^2}{y^2x}\right)\frac{dy}{dx} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{y^2x^2+1}{y\,x^2} \mathrm{d}x \\
\phi &= \frac{y^2x^2-1}{xy} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= 2x - \frac{y^2x^2-1}{x\,y^2} + f'(y) \qquad (4) \\
&= \frac{y^2x^2+1}{y^2x} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(xy-1)^2}{y^2x}$. Therefore equation (4) becomes

$$
\frac{(xy-1)^2}{y^2x} = \frac{y^2x^2+1}{y^2x} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{2}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(-\frac{2}{y}\right) \mathrm{d}y \\
f(y) &= -2\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{y^2x^2-1}{xy} - 2\ln(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^2x^2 - 1}{xy} - 2\ln(y)$$

Summary

The solution(s) found are the following

$$\frac{y^2x^2 - 1}{xy} - 2\ln(y) = c_1 \tag{1}$$

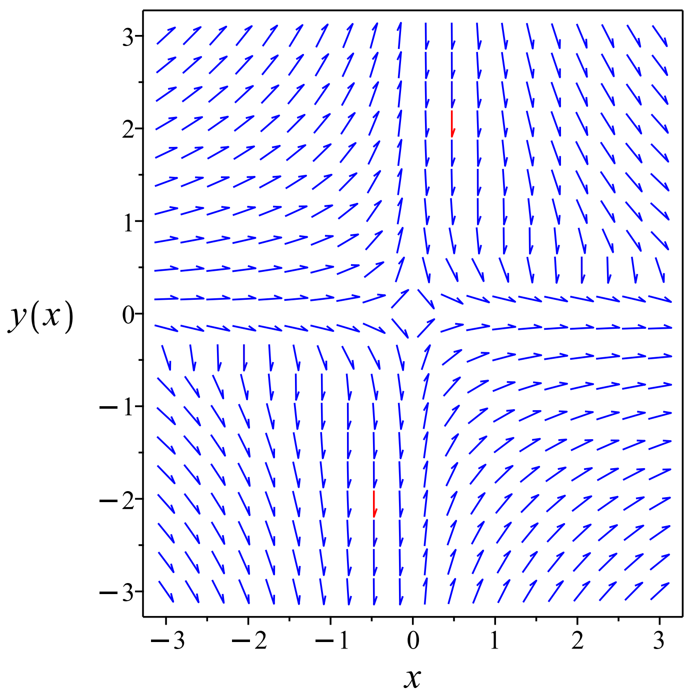


Figure 71: Slope field plot

Verification of solutions

$$\frac{y^2x^2 - 1}{xy} - 2\ln(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 42

```
dsolve((x^3*y(x)^3+x^2*y(x)^2+x*y(x)+1)*y(x)+(x^3*y(x)^3-x^2*y(x)^2-x*y(x)+1)*x*diff(y(x),x)
```

$$y(x) = -\frac{1}{x}$$
$$y(x) = \frac{e^{\mathrm{RootOf}\left(-e^{2\_Z}-2\ln(x)e^{\_Z}+2c_1e^{\_Z}+2\_Ze^{\_Z}+1\right)}}{x}$$

✓ Solution by Mathematica

Time used: 0.219 (sec). Leaf size: 35

```
DSolve[(x^3*y[x]^3+x^2*y[x]^2+x*y[x]+1)*y[x]+(x^3*y[x]^3-x^2*y[x]^2-x*y[x]+1)*x*y'[x]==0,y[x
```

$$y(x) \to -\frac{1}{x}$$
$$\text{Solve}\left[xy(x) - \frac{1}{xy(x)} - 2\log(y(x)) = c_1, y(x)\right]$$

4 Chapter 5

4.1 problem 1.1

Internal problem ID [4396]
Internal file name [`OUTPUT/3889_Sunday_June_05_2022_11_36_14_AM_82116322/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 5
Problem number: 1.1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{y^2 + 2y'y = -x^2 - 2x}$$

4.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y)\,\mathrm{d}y &= \left(-x^2 - y^2 - 2x\right)\mathrm{d}x\\ \left(x^2 + y^2 + 2x\right)\mathrm{d}x + (2y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 + y^2 + 2x\\ N(x, y) &= 2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2 + 2x\right)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2y}((2y) - (0))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int 1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x} \\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{x}\left(x^{2}+y^{2}+2x\right) \\ &= \mathrm{e}^{x}\left(x^{2}+y^{2}+2x\right)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{x}(2y) \\ &= 2y\,\mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\mathrm{e}^{x}\left(x^{2}+y^{2}+2x\right)\right)+\left(2y\,\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\mathrm{e}^{x}\left(x^{2}+y^{2}+2x\right)\mathrm{d}x\end{aligned}$$

$$\phi = \left(x^{2}+y^{2}\right)\mathrm{e}^{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2y\,\mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y\,\mathrm{e}^{x}$. Therefore equation (4) becomes

$$2y\,\mathrm{e}^{x} = 2y\,\mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(x^2 + y^2\right)\mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(x^2 + y^2\right)\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$\left(x^2 + y^2\right)\mathrm{e}^{x} = c_1 \tag{1}$$

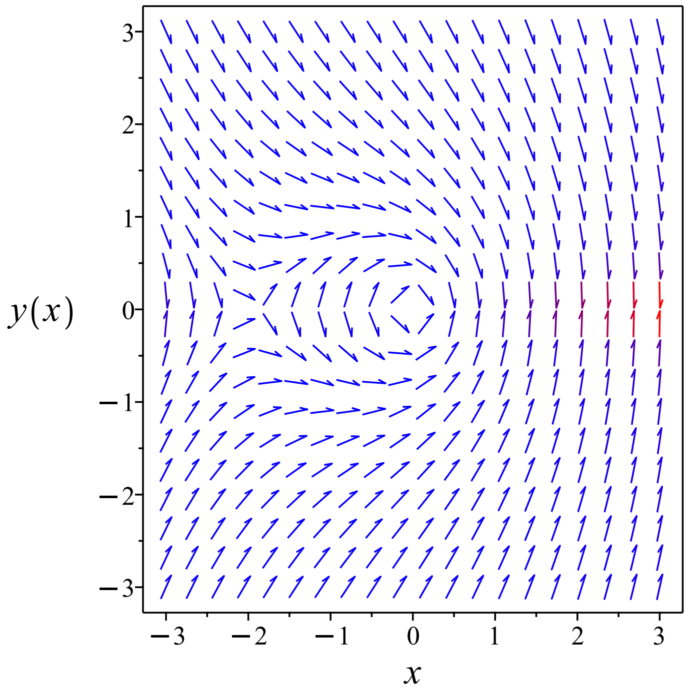


Figure 72: Slope field plot

Verification of solutions

$$\left(x^2+y^2\right) e^x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve((x^2+y(x)^2+2*x)+2*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{e^{-x}c_1 - x^2}$$
$$y(x) = -\sqrt{e^{-x}c_1 - x^2}$$

✓ Solution by Mathematica

Time used: 5.675 (sec). Leaf size: 47

```
DSolve[(x^2+y[x]^2+2*x)+2*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2 + c_1 e^{-x}}$$
$$y(x) \to \sqrt{-x^2 + c_1 e^{-x}}$$

4.2 problem 1.2

Internal problem ID [4397]
Internal file name [OUTPUT/3890_Sunday_June_05_2022_11_36_19_AM_16649650/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 5
Problem number: 1.2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - 2xyy' = -x^2}$$

4.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2xy)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x\\ \left(x^2 + y^2\right)\mathrm{d}x + (-2xy)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 + y^2\\ N(x, y) &= -2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2\right)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2xy)\\ &= -2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{2yx}((2y) - (-2y))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(x^2+y^2) \\
&= \frac{x^2+y^2}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-2xy) \\
&= -\frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2+y^2}{x^2}\right) + \left(-\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{x^2 + y^2}{x^2} \mathrm{d}x$$
$$\phi = x - \frac{y^2}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$x - \frac{y^2}{x} = c_1 \tag{1}$$

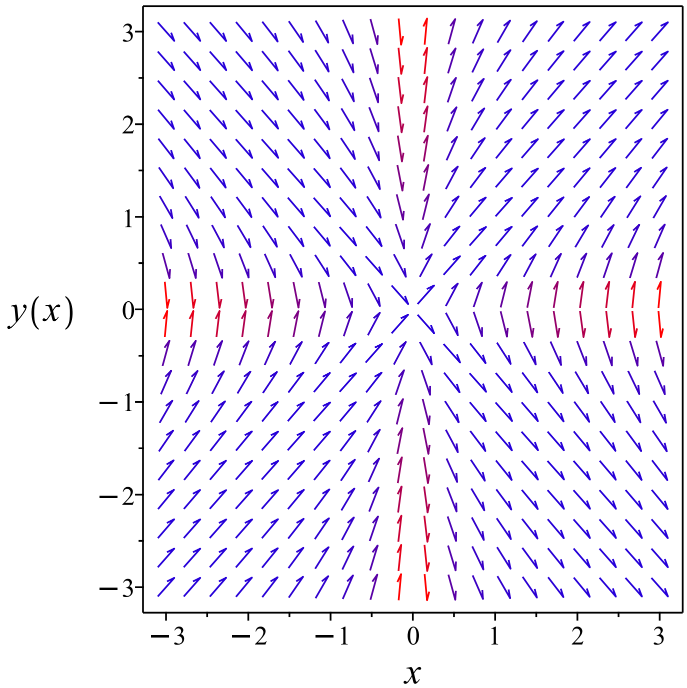


Figure 73: Slope field plot

Verification of solutions

$$x - \frac{y^2}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve((x^2+y(x)^2)-2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{(x + c_1)\,x}$$
$$y(x) = -\sqrt{(x + c_1)\,x}$$

✓ Solution by Mathematica

Time used: 0.209 (sec). Leaf size: 38

```
DSolve[(x^2+y[x]^2)-2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x}\sqrt{x + c_1}$$
$$y(x) \to \sqrt{x}\sqrt{x + c_1}$$

4.3 problem 2

Internal problem ID [4398]
Internal file name [OUTPUT/3891_Sunday_June_05_2022_11_36_25_AM_59487004/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 5
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2xy + \left(y^2 - 3x^2\right) y' = 0}$$

4.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(-3x^2 + y^2)\,\mathrm{d}y &= (-2xy)\,\mathrm{d}x \\
(2xy)\,\mathrm{d}x + (-3x^2 + y^2)\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 2xy \\
N(x, y) &= -3x^2 + y^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy) \\
&= 2x
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-3x^2 + y^2) \\
&= -6x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-3x^2 + y^2}((2x) - (-6x)) \\
&= -\frac{8x}{3x^2 - y^2}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{2yx}((-6x) - (2x)) \\ &= -\frac{4}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{4}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-4\ln(y)} \\ &= \frac{1}{y^4} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^4}(2xy) \\ &= \frac{2x}{y^3} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^4}(-3x^2 + y^2) \\ &= \frac{-3x^2 + y^2}{y^4} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{2x}{y^3}\right) + \left(\frac{-3x^2 + y^2}{y^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{2x}{y^3} \,\mathrm{d}x$$
$$\phi = \frac{x^2}{y^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{3x^2}{y^4} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-3x^2+y^2}{y^4}$. Therefore equation (4) becomes

$$\frac{-3x^2+y^2}{y^4} = -\frac{3x^2}{y^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{y^2}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{y^3} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{y^3} - \frac{1}{y}$$

Summary

The solution(s) found are the following

$$\frac{x^2}{y^3} - \frac{1}{y} = c_1 \tag{1}$$

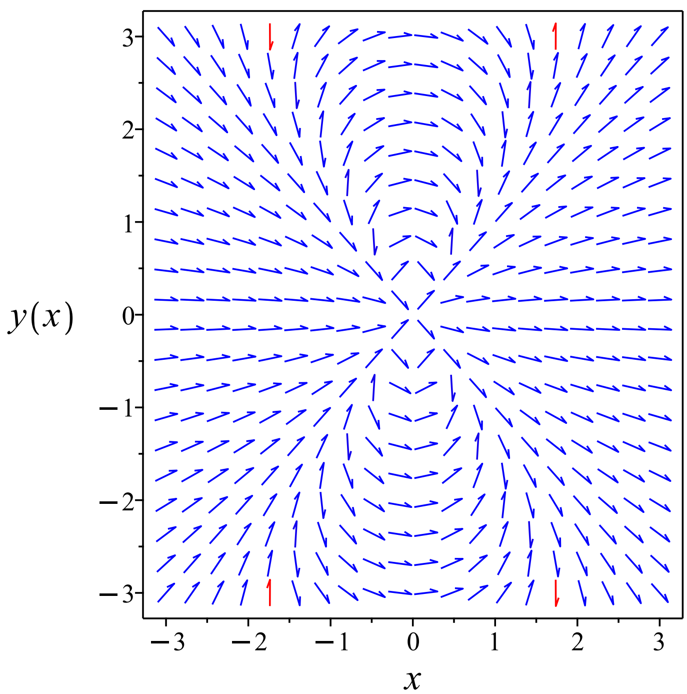


Figure 74: Slope field plot

Verification of solutions

$$\frac{x^2}{y^3} - \frac{1}{y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 317

```
dsolve((2*x*y(x))+(y(x)^2-3*x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{1 + \frac{\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1 - 108x^2c_1^2+8\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1 - 108x^2c_1^2+8\right)^{\frac{1}{3}}}}{3c_1}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1 - 108x^2c_1^2+8\right)^{\frac{2}{3}} - 4i\sqrt{3} - 4\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1 - 108x^2c_1^2 \cdots\right.}{12\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1 - 108x^2c_1^2+8\right)^{\frac{1}{3}} c_1}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1 - 108x^2c_1^2+8\right)^{\frac{2}{3}} - 4i\sqrt{3} + 4\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1 - 108x^2c_1^2 + 8 \cdots\right.}{12\left(12\sqrt{3}\,x\sqrt{27x^2c_1^2-4}\,c_1 - 108x^2c_1^2+8\right)^{\frac{1}{3}} c_1}$$

✓ Solution by Mathematica

Time used: 60.189 (sec). Leaf size: 458

```
DSolve[(2*x*y[x])+(y[x]^2-3*x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(\frac{\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}e^{2c_1}}{\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}} - e^{c_1}\right)$$

$$y(x) \to \frac{i(\sqrt{3}+i)\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}}{6\sqrt[3]{2}} - \frac{i(\sqrt{3}-i)e^{2c_1}}{3\ 2^{2/3}\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}} - \frac{e^{c_1}}{3}$$

$$y(x) \to -\frac{i(\sqrt{3}-i)\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}}{6\sqrt[3]{2}} + \frac{i(\sqrt{3}+i)e^{2c_1}}{3\ 2^{2/3}\sqrt[3]{27e^{c_1}x^2+3\sqrt{81e^{2c_1}x^4-12e^{4c_1}x^2}-2e^{3c_1}}} - \frac{e^{c_1}}{3}$$

4.4 problem 3

Internal problem ID [4399]
Internal file name [`OUTPUT/3892_Sunday_June_05_2022_11_36_33_AM_76381200/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 5
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (-x + 2y)\, y' = 0}$$

4.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x+2y)\,\mathrm{d}y &= (-y)\,\mathrm{d}x\\ (y)\,\mathrm{d}x + (-x+2y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\\ N(x,y) &= -x+2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x+2y)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{-x+2y}((1)-(-1))\\ &= -\frac{2}{x-2y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y}((-1) - (1)) \\
&= -\frac{2}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{2}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(y)} \\
&= \frac{1}{y^2}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2}(y) \\
&= \frac{1}{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(-x + 2y) \\
&= \frac{-x + 2y}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{1}{y}\right) + \left(\frac{-x + 2y}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{1}{y} \, \mathrm{d}x$$
$$\phi = \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+2y}{y^2}$. Therefore equation (4) becomes

$$\frac{-x+2y}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{2}{y}\right) \mathrm{d}y$$
$$f(y) = 2\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x}{y} + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x}{y} + 2\ln(y)$$

The solution becomes

$$y = e^{\mathrm{LambertW}\left(-\frac{x\,e^{-\frac{c_1}{2}}}{2}\right)+\frac{c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{\mathrm{LambertW}\left(-\frac{x\,e^{-\frac{c_1}{2}}}{2}\right)+\frac{c_1}{2}} \tag{1}$$

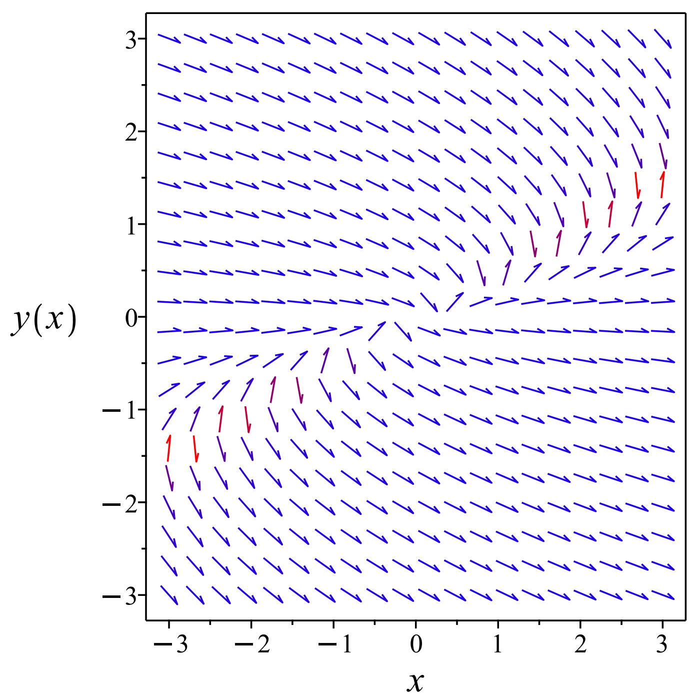


Figure 75: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}\left(-\frac{x\,e^{-\frac{c_1}{2}}}{2}\right)+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(y(x)+(2*y(x)-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{2\,\mathrm{LambertW}\left(-\frac{x\,\mathrm{e}^{-\frac{c_1}{2}}}{2}\right)}$$

✓ Solution by Mathematica

Time used: 4.711 (sec). Leaf size: 31

```
DSolve[y[x]+(2*y[x]-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{2W\left(-\frac{1}{2}e^{-\frac{c_1}{2}}x\right)}$$

$$y(x) \to 0$$

5 Chapter 6

5.1 problem 1

Internal problem ID [4400]
Internal file name [OUTPUT/3893_Sunday_June_05_2022_11_36_39_AM_95241541/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 6
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{xy' - ya + y^2 = x^{-2a}}$$

5.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-ya + y^2 - x^{-2a}}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{ya}{x} - \frac{y^2}{x} + \frac{x^{-2a}}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{x^{-2a}}{x}$, $f_1(x) = \frac{a}{x}$ and $f_2(x) = -\frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{x^2} \\ f_1 f_2 &= -\frac{a}{x^2} \\ f_2^2 f_0 &= \frac{x^{-2a}}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x} - \left(\frac{1}{x^2} - \frac{a}{x^2}\right) u'(x) + \frac{x^{-2a} u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x^a \left(c_1 \sinh\left(\frac{x^{-a}}{a}\right) + c_2 \cosh\left(\frac{x^{-a}}{a}\right)\right)$$

The above shows that

$$u'(x) = \frac{(a c_2 x^a - c_1) \cosh\left(\frac{x^{-a}}{a}\right) + \sinh\left(\frac{x^{-a}}{a}\right)(a c_1 x^a - c_2)}{x}$$

Using the above in (1) gives the solution

$$y = \frac{\left((a c_2 x^a - c_1) \cosh\left(\frac{x^{-a}}{a}\right) + \sinh\left(\frac{x^{-a}}{a}\right)(a c_1 x^a - c_2)\right) x^{-a}}{c_1 \sinh\left(\frac{x^{-a}}{a}\right) + c_2 \cosh\left(\frac{x^{-a}}{a}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left((x^a a - c_3) \cosh\left(\frac{x^{-a}}{a}\right) + \sinh\left(\frac{x^{-a}}{a}\right)(a c_3 x^a - 1)\right) x^{-a}}{c_3 \sinh\left(\frac{x^{-a}}{a}\right) + \cosh\left(\frac{x^{-a}}{a}\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\left(x^{a} a - c_{3}\right) \cosh\left(\frac{x^{-a}}{a}\right) + \sinh\left(\frac{x^{-a}}{a}\right) \left(a c_{3} x^{a} - 1\right)\right) x^{-a}}{c_{3} \sinh\left(\frac{x^{-a}}{a}\right) + \cosh\left(\frac{x^{-a}}{a}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\left(x^{a} a - c_{3}\right) \cosh\left(\frac{x^{-a}}{a}\right) + \sinh\left(\frac{x^{-a}}{a}\right) \left(a c_{3} x^{a} - 1\right)\right) x^{-a}}{c_{3} \sinh\left(\frac{x^{-a}}{a}\right) + \cosh\left(\frac{x^{-a}}{a}\right)}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a-1)*(diff(y(x), x))/x+x^(-2*
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Group is reducible or imprimitive
         <- Kovacics algorithm successful
      <- Equivalence, under non-integer power transformations successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 74

```
dsolve(x*diff(y(x),x)-a*y(x)+y(x)^2=x^(-2*a),y(x), singsol=all)
```

$$y(x) = \frac{(-x^{-a}c_1 + a)\sinh\left(\frac{x^{-a}}{a}\right) + (c_1 a - x^{-a})\cosh\left(\frac{x^{-a}}{a}\right)}{\cosh\left(\frac{x^{-a}}{a}\right)c_1 + \sinh\left(\frac{x^{-a}}{a}\right)}$$

✓ Solution by Mathematica

Time used: 0.393 (sec). Leaf size: 112

```
DSolve[x*y'[x]-a*y[x]+y[x]^2==x^(-2*a),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^{-a}\left((ax^a + ic_1)\cosh\left(\frac{x^{-a}}{a}\right) - i(ac_1x^a - i)\sinh\left(\frac{x^{-a}}{a}\right)\right)}{\cosh\left(\frac{x^{-a}}{a}\right) - ic_1\sinh\left(\frac{x^{-a}}{a}\right)}$$

$$y(x) \to a - x^{-a}\coth\left(\frac{x^{-a}}{a}\right)$$

5.2 problem 2

Internal problem ID [4401]
Internal file name [OUTPUT/3894_Sunday_June_05_2022_11_36_47_AM_40904880/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 6
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_rational, _Riccati]
```

$$\boxed{xy' - ya + y^2 = x^{-\frac{2a}{3}}}$$

5.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= -\frac{-ya + y^2 - x^{-\frac{2a}{3}}}{x}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{ya}{x} - \frac{y^2}{x} + \frac{x^{-\frac{2a}{3}}}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{x^{-\frac{2a}{3}}}{x}$, $f_1(x) = \frac{a}{x}$ and $f_2(x) = -\frac{1}{x}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{-\frac{u}{x}}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{x^2} \\ f_1 f_2 &= -\frac{a}{x^2} \\ f_2^2 f_0 &= \frac{x^{-\frac{2a}{3}}}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x} - \left(\frac{1}{x^2} - \frac{a}{x^2}\right) u'(x) + \frac{x^{-\frac{2a}{3}} u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(c_2\left(3\sqrt{x^{-\frac{2a}{3}}} + a\right) \mathrm{e}^{-\frac{3x^{-\frac{a}{3}}}{a}} + \mathrm{e}^{\frac{3x^{-\frac{a}{3}}}{a}} c_1\left(-3\sqrt{x^{-\frac{2a}{3}}} + a\right)\right) x^a$$

The above shows that

$$u'(x) = \frac{\left(c_2\left(\left(2a + 3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}} + a\left(a + x^{-\frac{a}{3}}\right)\right) \mathrm{e}^{-\frac{3x^{-\frac{a}{3}}}{a}} + \mathrm{e}^{\frac{3x^{-\frac{a}{3}}}{a}} c_1\left(\left(-2a + 3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}} + a\left(a - x^{-\frac{a}{3}}\right)\right)\right) x^a}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_2\left(\left(2a + 3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}} + a\left(a + x^{-\frac{a}{3}}\right)\right) \mathrm{e}^{-\frac{3x^{-\frac{a}{3}}}{a}} + \mathrm{e}^{\frac{3x^{-\frac{a}{3}}}{a}} c_1\left(\left(-2a + 3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}} + a\left(a - x^{-\frac{a}{3}}\right)\right)}{c_2\left(3\sqrt{x^{-\frac{2a}{3}}} + a\right) \mathrm{e}^{-\frac{3x^{-\frac{a}{3}}}{a}} + \mathrm{e}^{\frac{3x^{-\frac{a}{3}}}{a}} c_1\left(-3\sqrt{x^{-\frac{2a}{3}}} + a\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3\left(\left(-2a + 3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}} + a\left(a - x^{-\frac{a}{3}}\right)\right) \mathrm{e}^{\frac{6x^{-\frac{a}{3}}}{a}} + \left(2a + 3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}} + a\left(a + x^{-\frac{a}{3}}\right)}{c_3\left(-3\sqrt{x^{-\frac{2a}{3}}} + a\right) \mathrm{e}^{\frac{6x^{-\frac{a}{3}}}{a}} + 3\sqrt{x^{-\frac{2a}{3}}} + a}$$

Summary

The solution(s) found are the following

$$
y = \frac{c_3\left(\left(-2a+3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}}+a\left(a-x^{-\frac{a}{3}}\right)\right){\mathrm e}^{\frac{6x^{-\frac{a}{3}}}{a}}+\left(2a+3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}}+a\left(a+x^{-\frac{a}{3}}\right)}{c_3\left(-3\sqrt{x^{-\frac{2a}{3}}}+a\right){\mathrm e}^{\frac{6x^{-\frac{a}{3}}}{a}}+3\sqrt{x^{-\frac{2a}{3}}}+a} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_3\left(\left(-2a+3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}}+a\left(a-x^{-\frac{a}{3}}\right)\right){\mathrm e}^{\frac{6x^{-\frac{a}{3}}}{a}}+\left(2a+3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}}+a\left(a+x^{-\frac{a}{3}}\right)}{c_3\left(-3\sqrt{x^{-\frac{2a}{3}}}+a\right){\mathrm e}^{\frac{6x^{-\frac{a}{3}}}{a}}+3\sqrt{x^{-\frac{2a}{3}}}+a}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (a-1)*(diff(y(x), x))/x+x^(-1-
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying an equivalence, under non-integer power transformations,
         to LODEs admitting Liouvillian solutions.
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Group is reducible or imprimitive
         <- Kovacics algorithm successful
      <- Equivalence, under non-integer power transformations successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 119

```
dsolve(x*diff(y(x),x)-a*y(x)+y(x)^2=x^(-2*a/3),y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(-2a+3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}}+a\left(a-x^{-\frac{a}{3}}\right)\right)\mathrm{e}^{\frac{6x^{-\frac{a}{3}}}{a}}+\left(\left(2a+3x^{-\frac{a}{3}}\right)\sqrt{x^{-\frac{2a}{3}}}+a\left(a+x^{-\frac{a}{3}}\right)\right)c_1}{\left(-3\sqrt{x^{-\frac{2a}{3}}}+a\right)\mathrm{e}^{\frac{6x^{-\frac{a}{3}}}{a}}+c_1\left(3\sqrt{x^{-\frac{2a}{3}}}+a\right)}$$

✓ Solution by Mathematica

Time used: 0.427 (sec). Leaf size: 270

```
DSolve[x*y'[x]-a*y[x]+y[x]^2==x^(-2*a/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^{-a/3}\left(\left(a^2x^{2a/3}-3iac_1x^{a/3}+3\right)\cosh\left(\frac{3x^{-a/3}}{a}\right)+i\left(a^2c_1x^{2a/3}+3iax^{a/3}+3c_1\right)\sinh\left(\frac{3x^{-a/3}}{a}\right)\right)}{\left(ax^{a/3}-3ic_1\right)\cosh\left(\frac{3x^{-a/3}}{a}\right)+i\left(ac_1x^{a/3}+3i\right)\sinh\left(\frac{3x^{-a/3}}{a}\right)}$$

$$y(x) \to \frac{\left(a^2x^{2a/3}+3\right)\sinh\left(\frac{3x^{-a/3}}{a}\right)-3ax^{a/3}\cosh\left(\frac{3x^{-a/3}}{a}\right)}{ax^{2a/3}\sinh\left(\frac{3x^{-a/3}}{a}\right)-3x^{a/3}\cosh\left(\frac{3x^{-a/3}}{a}\right)}$$

5.3 problem 3

Internal problem ID [4402]
Internal file name [`OUTPUT/3895_Sunday_June_05_2022_11_36_56_AM_11134300/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 6
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_rational, [_Riccati, _special]]`

$$\boxed{u' + u^2 = \frac{c}{x^{\frac{4}{3}}}}$$

5.3.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= -\frac{u^2 x^{\frac{4}{3}} - c}{x^{\frac{4}{3}}} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$u' = -u^2 + \frac{c}{x^{\frac{4}{3}}}$$

With Riccati ODE standard form

$$u' = f_0(x) + f_1(x)u + f_2(x)u^2$$

Shows that $f_0(x) = \frac{c}{x^{\frac{4}{3}}}$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned} u &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{c}{x^{\frac{4}{3}}} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + \frac{c u(x)}{x^{\frac{4}{3}}} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 \mathrm{e}^{3x^{\frac{1}{3}}\sqrt{c}} \left(3x^{\frac{1}{3}}\sqrt{c} - 1\right) + 3c_1 \mathrm{e}^{-3x^{\frac{1}{3}}\sqrt{c}} \left(x^{\frac{1}{3}}\sqrt{c} + \frac{1}{3}\right)$$

The above shows that

$$u'(x) = \frac{3c\left(c_2 \mathrm{e}^{3x^{\frac{1}{3}}\sqrt{c}} - c_1 \mathrm{e}^{-3x^{\frac{1}{3}}\sqrt{c}}\right)}{x^{\frac{1}{3}}}$$

Using the above in (1) gives the solution

$$u = \frac{3c\left(c_2 \mathrm{e}^{3x^{\frac{1}{3}}\sqrt{c}} - c_1 \mathrm{e}^{-3x^{\frac{1}{3}}\sqrt{c}}\right)}{x^{\frac{1}{3}} \left(c_2 \mathrm{e}^{3x^{\frac{1}{3}}\sqrt{c}} \left(3x^{\frac{1}{3}}\sqrt{c} - 1\right) + 3c_1 \mathrm{e}^{-3x^{\frac{1}{3}}\sqrt{c}} \left(x^{\frac{1}{3}}\sqrt{c} + \frac{1}{3}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$u = -\frac{3c\left(-\mathrm{e}^{6x^{\frac{1}{3}}\sqrt{c}} + c_3\right)}{x^{\frac{1}{3}} \left(3x^{\frac{1}{3}}\sqrt{c}\, \mathrm{e}^{6x^{\frac{1}{3}}\sqrt{c}} + 3\sqrt{c}\, x^{\frac{1}{3}} c_3 - \mathrm{e}^{6x^{\frac{1}{3}}\sqrt{c}} + c_3\right)}$$

Summary
The solution(s) found are the following

$$u = -\frac{3c\left(-e^{6x^{\frac{1}{3}}\sqrt{c}}+c_3\right)}{x^{\frac{1}{3}}\left(3x^{\frac{1}{3}}\sqrt{c}\,e^{6x^{\frac{1}{3}}\sqrt{c}}+3\sqrt{c}\,x^{\frac{1}{3}}c_3-e^{6x^{\frac{1}{3}}\sqrt{c}}+c_3\right)} \tag{1}$$

Verification of solutions

$$u = -\frac{3c\left(-e^{6x^{\frac{1}{3}}\sqrt{c}}+c_3\right)}{x^{\frac{1}{3}}\left(3x^{\frac{1}{3}}\sqrt{c}\,e^{6x^{\frac{1}{3}}\sqrt{c}}+3\sqrt{c}\,x^{\frac{1}{3}}c_3-e^{6x^{\frac{1}{3}}\sqrt{c}}+c_3\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(u(x),x)+u(x)^2=c*x^(-4/3),u(x), singsol=all)
```

$$u(x) = -\frac{3c}{x^{\frac{1}{3}}\left(3x^{\frac{1}{3}}\tan\left(3\sqrt{-c}\left(x^{\frac{1}{3}}-c_1\right)\right)\sqrt{-c}+1\right)}$$

✓ Solution by Mathematica

Time used: 0.286 (sec). Leaf size: 183

```
DSolve[u'[x]+u[x]^2==c*x^(-4/3),u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{3c\left(3i\sinh\left(3\sqrt{c}\sqrt[3]{x}\right) + 8c_1\cosh\left(3\sqrt{c}\sqrt[3]{x}\right)\right)}{\sqrt[3]{x}\left(\left(9i\sqrt{c}\sqrt[3]{x} - 8c_1\right)\cosh\left(3\sqrt{c}\sqrt[3]{x}\right) + 3\left(8\sqrt{c}c_1\sqrt[3]{x} - i\right)\sinh\left(3\sqrt{c}\sqrt[3]{x}\right)\right)}$$

$$u(x) \to -\frac{3c\cosh\left(3\sqrt{c}\sqrt[3]{x}\right)}{\sqrt[3]{x}\left(\cosh\left(3\sqrt{c}\sqrt[3]{x}\right) - 3\sqrt{c}\sqrt[3]{x}\sinh\left(3\sqrt{c}\sqrt[3]{x}\right)\right)}$$

5.4 problem 4

Internal problem ID [4403]
Internal file name [OUTPUT/3896_Sunday_June_05_2022_11_37_04_AM_85092418/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 6
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_rational, [_Riccati, _special]]
```

$$\boxed{u' + bu^2 = \frac{c}{x^4}}$$

5.4.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= -\frac{b\,u^2x^4 - c}{x^4} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$u' = -b\,u^2 + \frac{c}{x^4}$$

With Riccati ODE standard form

$$u' = f_0(x) + f_1(x)u + f_2(x)u^2$$

Shows that $f_0(x) = \frac{c}{x^4}$, $f_1(x) = 0$ and $f_2(x) = -b$. Let

$$\begin{aligned} u &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-bu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{b^2 c}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-b u''(x) + \frac{b^2 c u(x)}{x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x \left(c_1 \sinh\left(\frac{\sqrt{bc}}{x}\right) + c_2 \cosh\left(\frac{\sqrt{bc}}{x}\right) \right)$$

The above shows that

$$u'(x) = \frac{-c_2\sqrt{bc}\,\sinh\left(\frac{\sqrt{bc}}{x}\right) - c_1\sqrt{bc}\,\cosh\left(\frac{\sqrt{bc}}{x}\right) + c_1 x \sinh\left(\frac{\sqrt{bc}}{x}\right) + c_2 x \cosh\left(\frac{\sqrt{bc}}{x}\right)}{x}$$

Using the above in (1) gives the solution

$$u = \frac{-c_2\sqrt{bc}\,\sinh\left(\frac{\sqrt{bc}}{x}\right) - c_1\sqrt{bc}\,\cosh\left(\frac{\sqrt{bc}}{x}\right) + c_1 x \sinh\left(\frac{\sqrt{bc}}{x}\right) + c_2 x \cosh\left(\frac{\sqrt{bc}}{x}\right)}{x^2 b \left(c_1 \sinh\left(\frac{\sqrt{bc}}{x}\right) + c_2 \cosh\left(\frac{\sqrt{bc}}{x}\right)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$u = \frac{\left(-c_3\sqrt{bc} + x\right)\cosh\left(\frac{\sqrt{bc}}{x}\right) + \sinh\left(\frac{\sqrt{bc}}{x}\right)\left(c_3 x - \sqrt{bc}\right)}{x^2 b \left(c_3 \sinh\left(\frac{\sqrt{bc}}{x}\right) + \cosh\left(\frac{\sqrt{bc}}{x}\right)\right)}$$

Summary
The solution(s) found are the following

$$u = \frac{\left(-c_3\sqrt{bc}+x\right)\cosh\left(\frac{\sqrt{bc}}{x}\right)+\sinh\left(\frac{\sqrt{bc}}{x}\right)\left(c_3x-\sqrt{bc}\right)}{x^2b\left(c_3\sinh\left(\frac{\sqrt{bc}}{x}\right)+\cosh\left(\frac{\sqrt{bc}}{x}\right)\right)} \tag{1}$$

Verification of solutions

$$u = \frac{\left(-c_3\sqrt{bc}+x\right)\cosh\left(\frac{\sqrt{bc}}{x}\right)+\sinh\left(\frac{\sqrt{bc}}{x}\right)\left(c_3x-\sqrt{bc}\right)}{x^2b\left(c_3\sinh\left(\frac{\sqrt{bc}}{x}\right)+\cosh\left(\frac{\sqrt{bc}}{x}\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(u(x),x)+b*u(x)^2=c*x^(-4),u(x), singsol=all)
```

$$u(x) = \frac{-\sqrt{-bc}\,\tan\left(\frac{\sqrt{-bc}\,(c_1x-1)}{x}\right)+x}{b\,x^2}$$

✓ Solution by Mathematica

Time used: 0.308 (sec). Leaf size: 98

```
DSolve[u'[x]+b*u[x]^2==x^(-4),u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{-2bc_1e^{\frac{2\sqrt{b}}{x}} + \sqrt{b}\left(1 + 2c_1xe^{\frac{2\sqrt{b}}{x}}\right) + x}{x^2\left(b + 2b^{3/2}c_1e^{\frac{2\sqrt{b}}{x}}\right)}$$
$$u(x) \to \frac{x - \sqrt{b}}{bx^2}$$

5.5 problem 5

Internal problem ID [4404]
Internal file name [OUTPUT/3897_Sunday_June_05_2022_11_37_14_AM_82274080/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 6
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_rational, [_Riccati, _special]]
```

$$\boxed{u' - u^2 = \frac{2}{x^{\frac{8}{3}}}}$$

5.5.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= \frac{u^2 x^{\frac{8}{3}} + 2}{x^{\frac{8}{3}}} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$u' = u^2 + \frac{2}{x^{\frac{8}{3}}}$$

With Riccati ODE standard form

$$u' = f_0(x) + f_1(x)u + f_2(x)u^2$$

Shows that $f_0(x) = \frac{2}{x^{\frac{8}{3}}}$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} u &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{2}{x^{\frac{8}{3}}} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{2u(x)}{x^{\frac{8}{3}}} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{6\left(c_1\left(-\frac{x^{\frac{1}{3}}\sqrt{2}}{6} + i\right)\sqrt{x^{\frac{1}{3}}\sqrt{2} - 6i}\, \mathrm{e}^{\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}} + \frac{c_2 \mathrm{e}^{-\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}\sqrt{x^{\frac{2}{3}}+18}\,\sqrt{x^{\frac{1}{3}}\sqrt{2}+6i}}{6}\right)x^{\frac{2}{3}}}{\sqrt{x^{\frac{1}{3}}\sqrt{2} - 6i}}$$

The above shows that

$$u'(x) = \frac{36\left(i x^{\frac{2}{3}}\sqrt{2} + 18i\sqrt{2} + \frac{5x}{3} + 12x^{\frac{1}{3}} + \frac{x^{\frac{5}{3}}}{18}\right) c_2 \mathrm{e}^{-\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}} + 12\left(i x^{\frac{2}{3}}\sqrt{2} - 3i\sqrt{2} - \frac{x}{6} + 4x^{\frac{1}{3}}\right)\sqrt{x^{\frac{1}{3}}\sqrt{2} - 6i}\, \mathrm{e}^{\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}\, c_1}{\sqrt{x^{\frac{2}{3}}+18}\,\left(x^{\frac{1}{3}}\sqrt{2} - 6i\right)^{\frac{3}{2}}\sqrt{x^{\frac{1}{3}}\sqrt{2}+6i}\, x^{\frac{2}{3}}}$$

Using the above in (1) gives the solution

$$u = -\frac{2\left(3\left(i x^{\frac{2}{3}}\sqrt{2} + 18i\sqrt{2} + \frac{5x}{3} + 12x^{\frac{1}{3}} + \frac{x^{\frac{5}{3}}}{18}\right) c_2 \mathrm{e}^{-\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}} + \left(i x^{\frac{2}{3}}\sqrt{2} - 3i\sqrt{2} - \frac{x}{6} + 4x^{\frac{1}{3}}\right)\sqrt{x^{\frac{1}{3}}\sqrt{2} - 6i}\, \mathrm{e}^{\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}}{\sqrt{x^{\frac{2}{3}}+18}\,\left(x^{\frac{1}{3}}\sqrt{2} - 6i\right)\sqrt{x^{\frac{1}{3}}\sqrt{2}+6i}\, x^{\frac{4}{3}}\left(c_1\left(-\frac{x^{\frac{1}{3}}\sqrt{2}}{6} + i\right)\sqrt{x^{\frac{1}{3}}\sqrt{2} - 6i}\, \mathrm{e}^{\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}} + \frac{c_2 \mathrm{e}^{-\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}\sqrt{}}{}\right.}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$u = \frac{\left(3i x^{\frac{2}{3}}\sqrt{2}+54i\sqrt{2}+5x+36x^{\frac{1}{3}}+\frac{x^{\frac{5}{3}}}{6}\right) e^{-\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}+\left(i x^{\frac{2}{3}}\sqrt{2}-3i\sqrt{2}-\frac{x}{6}+4x^{\frac{1}{3}}\right)\sqrt{x^{\frac{1}{3}}\sqrt{2}-6i}\, e^{\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}} c_3 \sqrt{x^{\frac{2}{3}} \cdots}}{3\sqrt{x^{\frac{2}{3}}+18}\, x^{\frac{4}{3}}\sqrt{x^{\frac{1}{3}}\sqrt{2}+6i}\left(-\frac{x^{\frac{1}{3}}\sqrt{2}}{6}+i\right)\left(c_3\left(-\frac{x^{\frac{1}{3}}\sqrt{2}}{6}+i\right)\sqrt{x^{\frac{1}{3}}\sqrt{2}-6i}\, e^{\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}+\frac{e^{-\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}\sqrt{x^{\frac{2}{3}}+18 \cdots}}{6} \cdots\right.}$$

Summary

The solution(s) found are the following

$$u = \frac{\left(3i x^{\frac{2}{3}}\sqrt{2}+54i\sqrt{2}+5x+36x^{\frac{1}{3}}+\frac{x^{\frac{5}{3}}}{6}\right) e^{-\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}+\left(i x^{\frac{2}{3}}\sqrt{2}-3i\sqrt{2}-\frac{x}{6}+4x^{\frac{1}{3}}\right)\sqrt{x^{\frac{1}{3}}\sqrt{2}-6i}\, e^{\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}} c_3 \sqrt{x^{\frac{2}{3}} \cdots}}{3\sqrt{x^{\frac{2}{3}}+18}\, x^{\frac{4}{3}}\sqrt{x^{\frac{1}{3}}\sqrt{2}+6i}\left(-\frac{x^{\frac{1}{3}}\sqrt{2}}{6}+i\right)\left(c_3\left(-\frac{x^{\frac{1}{3}}\sqrt{2}}{6}+i\right)\sqrt{x^{\frac{1}{3}}\sqrt{2}-6i}\, e^{\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}+\frac{e^{-\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}\sqrt{x^{\frac{2}{3}}+18 \cdots}}{6} \cdots\right.} \quad (1)$$

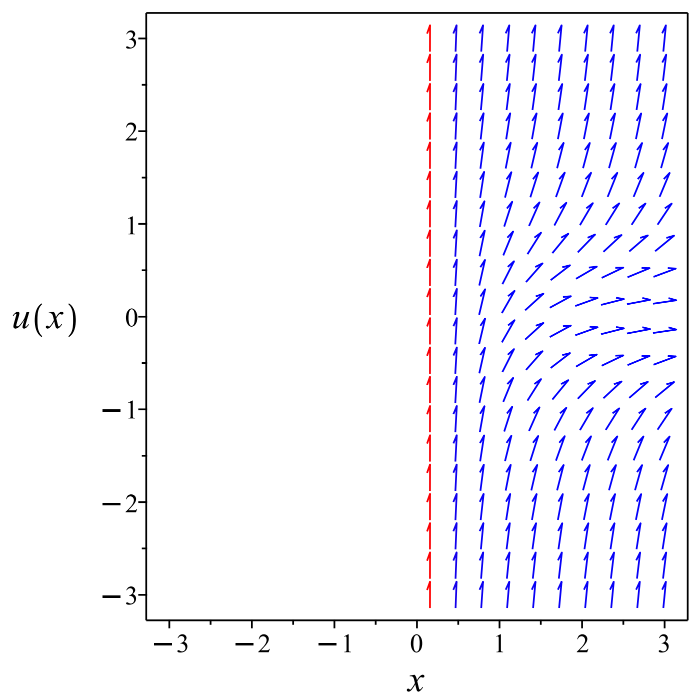


Figure 76: Slope field plot

Verification of solutions

$$u = \frac{\left(3i x^{\frac{2}{3}}\sqrt{2}+54i\sqrt{2}+5x+36x^{\frac{1}{3}}+\frac{x^{\frac{5}{3}}}{6}\right)\mathrm{e}^{-\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}+\left(i x^{\frac{2}{3}}\sqrt{2}-3i\sqrt{2}-\frac{x}{6}+4x^{\frac{1}{3}}\right)\sqrt{x^{\frac{1}{3}}\sqrt{2}-6i}\,\mathrm{e}^{\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}c_3\sqrt{x^{\frac{2}{3}}}}{3\sqrt{x^{\frac{2}{3}}+18}\,x^{\frac{4}{3}}\sqrt{x^{\frac{1}{3}}\sqrt{2}+6i}\left(-\frac{x^{\frac{1}{3}}\sqrt{2}}{6}+i\right)\left(c_3\left(-\frac{x^{\frac{1}{3}}\sqrt{2}}{6}+i\right)\sqrt{x^{\frac{1}{3}}\sqrt{2}-6i}\,\mathrm{e}^{\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}+\frac{\mathrm{e}^{-\frac{3i\sqrt{2}}{x^{\frac{1}{3}}}}\sqrt{x^{\frac{2}{3}}+18}}{6}\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 78

```
dsolve(diff(u(x),x)-u(x)^2=2*x^(-8/3),u(x), singsol=all)
```

$$u(x) = -\frac{3\left(\tan\left(3\sqrt{2}\left(\left(\frac{1}{x}\right)^{\frac{1}{3}}-c_1\right)\right)\sqrt{2}\,x\left(\frac{1}{x}\right)^{\frac{2}{3}}+\frac{x\left(\frac{1}{x}\right)^{\frac{1}{3}}}{3}-2\right)}{\left(\frac{1}{x}\right)^{\frac{1}{3}}x^2\left(3\left(\frac{1}{x}\right)^{\frac{1}{3}}\sqrt{2}\tan\left(3\sqrt{2}\left(\left(\frac{1}{x}\right)^{\frac{1}{3}}-c_1\right)\right)+1\right)}$$

✓ Solution by Mathematica

Time used: 0.266 (sec). Leaf size: 215

```
DSolve[u'[x]-u[x]^2==x^(-8/3),u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to -\frac{\left(-9\sqrt[3]{\frac{1}{x}}+c_1\left(8-24\left(\frac{1}{x}\right)^{2/3}\right)\right)\cos\left(3\sqrt[3]{\frac{1}{x}}\right)+3\left(-3\left(\frac{1}{x}\right)^{2/3}+8c_1\sqrt[3]{\frac{1}{x}}+1\right)\sin\left(3\sqrt[3]{\frac{1}{x}}\right)}{x\left(\left(-9\sqrt[3]{\frac{1}{x}}+8c_1\right)\cos\left(3\sqrt[3]{\frac{1}{x}}\right)+3\left(1+8c_1\sqrt[3]{\frac{1}{x}}\right)\sin\left(3\sqrt[3]{\frac{1}{x}}\right)\right)}$$

$$u(x) \to \frac{\left(3\left(\frac{1}{x}\right)^{2/3}-1\right)\cos\left(3\sqrt[3]{\frac{1}{x}}\right)-3\sqrt[3]{\frac{1}{x}}\sin\left(3\sqrt[3]{\frac{1}{x}}\right)}{x\left(3\sqrt[3]{\frac{1}{x}}\sin\left(3\sqrt[3]{\frac{1}{x}}\right)+\cos\left(3\sqrt[3]{\frac{1}{x}}\right)\right)}$$

5.6 problem 12

Internal problem ID [4405]
Internal file name [`OUTPUT/3898_Sunday_June_05_2022_11_37_22_AM_10384012/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 6
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\frac{\sqrt{f\,x^4+c\,x^3+c\,x^2+bx+a}\,y'}{\sqrt{a+by+cy^2+cy^3+fy^4}}=-1}$$

5.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= -\frac{\sqrt{f\,y^4+c\,y^3+c\,y^2+by+a}}{\sqrt{f\,x^4+c\,x^3+c\,x^2+bx+a}}
\end{aligned}$$

Where $f(x)=-\frac{1}{\sqrt{f\,x^4+c\,x^3+c\,x^2+bx+a}}$ and $g(y)=\sqrt{f\,y^4+c\,y^3+c\,y^2+by+a}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\sqrt{f\,y^4+c\,y^3+c\,y^2+by+a}}\,dy &= -\frac{1}{\sqrt{f\,x^4+c\,x^3+c\,x^2+bx+a}}\,dx \\
\int\frac{1}{\sqrt{f\,y^4+c\,y^3+c\,y^2+by+a}}\,dy &= \int-\frac{1}{\sqrt{f\,x^4+c\,x^3+c\,x^2+bx+a}}\,dx
\end{aligned}$$

$$\int^{y} \frac{1}{\sqrt{\_a^{4} f+\_a^{3} c+\_a^{2} c+\_ab+a}}d\_a = \int -\frac{1}{\sqrt{f\,x^{4}+c\,x^{3}+c\,x^{2}+bx+a}}dx + c_1$$

Which results in

$$\int^{y} \frac{1}{\sqrt{\_a^{4} f+\_a^{3} c+\_a^{2} c+\_ab+a}}d\_a = \int -\frac{1}{\sqrt{f\,x^{4}+c\,x^{3}+c\,x^{2}+bx+a}}dx + c_1$$

The solution is

$$\int^{y} \frac{1}{\sqrt{\_a^{4} f+\_a^{3} c+\_a^{2} c+\_ab+a}}d\_a - \left(\int -\frac{1}{\sqrt{f\,x^{4}+c\,x^{3}+c\,x^{2}+bx+a}}dx\right) - c_1 = 0$$

Summary

The solution(s) found are the following

$$\int^{y} \frac{1}{\sqrt{\_a^{4} f+\_a^{3} c+\_a^{2} c+\_ab+a}}d\_a - \left(\int -\frac{1}{\sqrt{f\,x^{4}+c\,x^{3}+c\,x^{2}+bx+a}}dx\right) - c_1 = 0 \tag{1}$$

Verification of solutions

$$\int^{y} \frac{1}{\sqrt{\_a^{4} f+\_a^{3} c+\_a^{2} c+\_ab+a}}d\_a - \left(\int -\frac{1}{\sqrt{f\,x^{4}+c\,x^{3}+c\,x^{2}+bx+a}}dx\right) - c_1 = 0$$

Verified OK.

5.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\sqrt{f\,y^{4}+c\,y^{3}+c\,y^{2}+by+a}}{\sqrt{f\,x^{4}+c\,x^{3}+c\,x^{2}+bx+a}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 54: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\sqrt{f\, x^4 + c\, x^3 + c\, x^2 + bx + a} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta=0$ then in this special case

$$R=y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\sqrt{f\,x^4+c\,x^3+c\,x^2+bx+a}} dx
\end{aligned}$$

Which results in

$$S=\text{Expression too large to display}$$

5.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{\sqrt{f\,y^4 + c\,y^3 + c\,y^2 + by + a}}\right)\mathrm{d}y = \left(\frac{1}{\sqrt{f\,x^4 + c\,x^3 + c\,x^2 +}}\right.$$

$$\left(-\frac{1}{\sqrt{f\,x^4 + c\,x^3 + c\,x^2 + bx + a}}\right)\mathrm{d}x + \left(-\frac{1}{\sqrt{f\,y^4 + c\,y^3 + c\,y^2 + by + a}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -\frac{1}{\sqrt{f\,x^4 + c\,x^3 + c\,x^2 + bx + a}} \\
N(x,y) &= -\frac{1}{\sqrt{f\,y^4 + c\,y^3 + c\,y^2 + by + a}}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{\sqrt{f\,x^4 + c\,x^3 + c\,x^2 + bx + a}}\right) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{\sqrt{f\,y^4 + c\,y^3 + c\,y^2 + by + a}}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{1}{\sqrt{f\,x^4 + c\,x^3 + c\,x^2 + bx + a}} \mathrm{d}x$$
$$\phi = \int^x -\frac{1}{\sqrt{\_a^4 f + \_a^3 c + \_a^2 c + \_ab + a}} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{\sqrt{f\,y^4 + c\,y^3 + c\,y^2 + by + a}}$. Therefore equation (4) becomes

$$-\frac{1}{\sqrt{f\,y^4 + c\,y^3 + c\,y^2 + by + a}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{\sqrt{f\,y^4 + c\,y^3 + c\,y^2 + by + a}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(-\frac{1}{\sqrt{f\,y^4 + c\,y^3 + c\,y^2 + by + a}}\right) \mathrm{d}y$$
$$f(y) = \text{Expression too large to display} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \text{Expression too large to display} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \text{Expression too large to display}$$

Summary

The solution(s) found are the following

$$\text{Expression too large to display} \tag{1}$$

Verification of solutions

$$\text{Expression too large to display}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve((sqrt(a+b*x+c*x^2+c*x^3+f*x^4))/(sqrt(a+b*y(x)+c*y(x)^2+c*y(x)^3+f*y(x)^4))*diff(y(x)
```

$$\int \frac{1}{\sqrt{f\,x^{4}+x^{3}c+x^{2}c+xb+a}}dx+\int^{y(x)}\frac{1}{\sqrt{\_a^{4}f+\_a^{3}c+\_a^{2}c+\_ab+a}}d\_a+c_{1} = 0$$

✓ Solution by Mathematica

Time used: 21.472 (sec). Leaf size: 2239

```
DSolve[Sqrt[a+b*x+c*x^2+c*x^3+f*x^4]/Sqrt[a+b*y[x]+c*y[x]^2+c*y[x]^3+f*y[x]^4]*y'[x]==-1,y[x
```

Too large to display

6 Chapter 7

6.1 problem 1

Internal problem ID [4406]
Internal file name [OUTPUT/3899_Sunday_June_05_2022_11_37_32_AM_31421343/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 1.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 - 5y' = -6}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 3 \tag{1}$$
$$y' = 2 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int 3 \, \mathrm{d}x \\ &= 3x + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = 3x + c_1 \tag{1}$$

Verification of solutions

$$y = 3x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int 2 \, \mathrm{d}x \\ &= 2x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = 2x + c_2 \tag{1}$$

Verification of solutions

$$y = 2x + c_2$$

Verified OK.

6.1.1 Maple step by step solution

Let's solve

$y'^2 - 5y' = -6$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(y'^2 - 5y'\right) dx = \int (-6) \, dx + c_1$

- Cannot compute integral

  $\int \left(y'^2 - 5y'\right) dx = -6x + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((diff(y(x),x))^2-5*diff(y(x),x)+6=0,y(x), singsol=all)
```

$$y(x) = 3x + c_1$$
$$y(x) = 2x + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 21

```
DSolve[(y'[x])^2-5*y'[x]+6==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x + c_1$$
$$y(x) \to 3x + c_1$$

6.2 problem 2

Internal problem ID [4407]
Internal file name [`OUTPUT/3900_Sunday_June_05_2022_11_37_40_AM_32836822/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 2.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 = \frac{a^2}{x^2}}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{a}{x} \tag{1}$$

$$y' = -\frac{a}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{a}{x} \,\mathrm{d}x \\ &= a\ln(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = a\ln(x) + c_1 \tag{1}$$

Verification of solutions

$$y = a\ln(x) + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{a}{x}\,\mathrm{d}x \\ &= -a\ln(x) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -a\ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = -a\ln(x) + c_2$$

Verified OK.

6.2.1 Maple step by step solution

Let's solve

$y'^2 = \frac{a^2}{x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'^2 dx = \int \frac{a^2}{x^2} dx + c_1$

- Cannot compute integral

  $\int y'^2 dx = -\frac{a^2}{x} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve((diff(y(x),x))^2-a^2/x^2=0,y(x), singsol=all)
```

$$y(x) = a \ln(x) + c_1$$
$$y(x) = -a \ln(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 24

```
DSolve[(y'[x])^2-a^2/x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -a \log(x) + c_1$$
$$y(x) \to a \log(x) + c_1$$

6.3 problem 3

Internal problem ID [4408]
Internal file name [OUTPUT/3901_Sunday_June_05_2022_11_37_49_AM_18271090/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 3.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 = \frac{1-x}{x}}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{-x(x-1)}}{x} \tag{1}$$

$$y' = -\frac{\sqrt{-x(x-1)}}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sqrt{-x(x-1)}}{x}\, dx \\ &= \sqrt{-x^2+x} + \frac{\arcsin(2x-1)}{2} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \sqrt{-x^2+x} + \frac{\arcsin(2x-1)}{2} + c_1 \tag{1}$$

Verification of solutions

$$y = \sqrt{-x^2 + x} + \frac{\arcsin(2x - 1)}{2} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{\sqrt{-x(x-1)}}{x}\, \mathrm{d}x \\ &= -\sqrt{-x^2 + x} - \frac{\arcsin(2x - 1)}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\sqrt{-x^2 + x} - \frac{\arcsin(2x - 1)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = -\sqrt{-x^2 + x} - \frac{\arcsin(2x - 1)}{2} + c_2$$

Verified OK.

6.3.1 Maple step by step solution

Let's solve

$y'^2 = \frac{1-x}{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'^2 dx = \int \frac{1-x}{x} dx + c_1$

- Cannot compute integral

  $\int y'^2 dx = -x + \ln(x) + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve((diff(y(x),x))^2=(1-x)/x,y(x), singsol=all)
```

$$y(x) = \sqrt{-x(x-1)} + \frac{\arcsin(2x-1)}{2} + c_1$$
$$y(x) = -\sqrt{-x(x-1)} - \frac{\arcsin(2x-1)}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 81

```
DSolve[(y'[x])^2==(1-x)/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2\arctan\left(\frac{\sqrt{1-x}}{\sqrt{x}+1}\right) + \sqrt{-((x-1)x)} + c_1$$
$$y(x) \to 2\arctan\left(\frac{\sqrt{1-x}}{\sqrt{x}+1}\right) - \sqrt{-((x-1)x)} + c_1$$

6.4 problem 4

Internal problem ID [4409]
Internal file name [`OUTPUT/3902_Sunday_June_05_2022_11_37_57_AM_52028092/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 4.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y'^2 = -\frac{2xy'}{y} + 1}$$

6.4.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 = -\frac{2xp}{y} + 1$$

Solving for y from the above results in

$$y = -\frac{2xp}{p^2 - 1} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -\frac{2p}{p^2 - 1} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p + \frac{2p}{p^2 - 1} = x\left(-\frac{2}{p^2 - 1} + \frac{4p^2}{\left(p^2 - 1\right)^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p + \frac{2p}{p^2 - 1} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) + \frac{2p(x)}{p(x)^2 - 1}}{x\left(-\frac{2}{p(x)^2 - 1} + \frac{4p(x)^2}{\left(p(x)^2 - 1\right)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(-\frac{2}{p^2 - 1} + \frac{4p^2}{\left(p^2 - 1\right)^2}\right)}{p + \frac{2p}{p^2 - 1}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{2}{p^3 - p}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{2x(p)}{p^3 - p} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{p^3-p}dp} \\ &= e^{-\ln(p+1)-\ln(p-1)+2\ln(p)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{p^2}{p^2 - 1}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p^2 x}{p^2 - 1}\right) = 0$$

Integrating gives

$$\frac{p^2 x}{p^2 - 1} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{p^2}{p^2-1}$ results in

$$x(p) = \frac{c_3(p^2 - 1)}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{-x + \sqrt{x^2 + y^2}}{y}$$
$$p = -\frac{x + \sqrt{x^2 + y^2}}{y}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{2c_3 x}{-x + \sqrt{x^2 + y^2}}$$
$$x = \frac{2c_3 x}{x + \sqrt{x^2 + y^2}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 0 \tag{1}$$
$$y = -ix \tag{2}$$
$$y = ix \tag{3}$$
$$x = -\frac{2c_3 x}{-x + \sqrt{x^2 + y^2}} \tag{4}$$
$$x = \frac{2c_3 x}{x + \sqrt{x^2 + y^2}} \tag{5}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$x = -\frac{2c_3x}{-x + \sqrt{x^2 + y^2}}$$

Verified OK.

$$x = \frac{2c_3x}{x + \sqrt{x^2 + y^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying homogeneous B
   trying homogeneous types:
   trying homogeneous C
   1st order, trying the canonical coordinates of the invariance group
      -> Calling odsolve with the ODE`, diff(y(x), x) = y(x)/x, y(x)`         *** Sublevel 3
         Methods for first order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         trying 1st order linear
         <- 1st order linear successful
   <- 1st order, canonical coordinates successful
   <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 45

```
dsolve((diff(y(x),x))^2+2*x/y(x)*diff(y(x),x)-1=0,y(x), singsol=all)
```

$$y(x) = -ix$$
$$y(x) = ix$$
$$y(x) = -\frac{2\sqrt{c_1 x + 1}}{c_1}$$
$$y(x) = \frac{2\sqrt{c_1 x + 1}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.466 (sec). Leaf size: 126

```
DSolve[(y'[x])^2+2*x/y[x]*y'[x]-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{\frac{c_1}{2}}\sqrt{-2x+e^{c_1}}$$
$$y(x) \to e^{\frac{c_1}{2}}\sqrt{-2x+e^{c_1}}$$
$$y(x) \to -e^{\frac{c_1}{2}}\sqrt{2x+e^{c_1}}$$
$$y(x) \to e^{\frac{c_1}{2}}\sqrt{2x+e^{c_1}}$$
$$y(x) \to 0$$
$$y(x) \to -ix$$
$$y(x) \to ix$$

6.5 problem 5

Internal problem ID [4410]
Internal file name [`OUTPUT/3903_Sunday_June_05_2022_11_38_11_AM_57655295/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 5.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y - ay' - by'^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-a + \sqrt{a^2 + 4by}}{2b} \tag{1}$$

$$y' = -\frac{a + \sqrt{a^2 + 4by}}{2b} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2b}{-a + \sqrt{a^2 + 4by}} dy = \int dx$$

$$\frac{\ln(y) a}{2} + \frac{a \ln\left(-a + \sqrt{a^2 + 4by}\right)}{2} - \frac{a \ln\left(a + \sqrt{a^2 + 4by}\right)}{2} + \sqrt{a^2 + 4by} = x + c_1$$

Summary
The solution(s) found are the following

$$\frac{\ln(y) a}{2} + \frac{a \ln\left(-a + \sqrt{a^2 + 4by}\right)}{2} - \frac{a \ln\left(a + \sqrt{a^2 + 4by}\right)}{2} + \sqrt{a^2 + 4by} = x + c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(y)\,a}{2}+\frac{a\ln\left(-a+\sqrt{a^2+4by}\right)}{2}-\frac{a\ln\left(a+\sqrt{a^2+4by}\right)}{2}+\sqrt{a^2+4by}=x+c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2b}{a+\sqrt{a^2+4by}}dy=\int dx$$

$$\frac{\ln(y)\,a}{2}-\frac{a\ln\left(-a+\sqrt{a^2+4by}\right)}{2}+\frac{a\ln\left(a+\sqrt{a^2+4by}\right)}{2}-\sqrt{a^2+4by}=x+c_2$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)\,a}{2}-\frac{a\ln\left(-a+\sqrt{a^2+4by}\right)}{2}+\frac{a\ln\left(a+\sqrt{a^2+4by}\right)}{2}-\sqrt{a^2+4by}=x+c_2 \quad (1)$$

Verification of solutions

$$\frac{\ln(y)\,a}{2}-\frac{a\ln\left(-a+\sqrt{a^2+4by}\right)}{2}+\frac{a\ln\left(a+\sqrt{a^2+4by}\right)}{2}-\sqrt{a^2+4by}=x+c_2$$

Verified OK.

6.5.1 Maple step by step solution

Let's solve

$y-ay'-by'^2=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-a+\sqrt{a^2+4by}}=\frac{1}{2b}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-a+\sqrt{a^2+4by}}dx=\int \frac{1}{2b}dx+c_1$

- Evaluate integral

  $\frac{a\ln(y)}{4b}+\frac{2\sqrt{a^2+4by}+a\ln\left(-a+\sqrt{a^2+4by}\right)-a\ln\left(a+\sqrt{a^2+4by}\right)}{4b}=\frac{x}{2b}+c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 207

```
dsolve(y(x)=a*diff(y(x),x)+b*(diff(y(x),x))^2,y(x), singsol=all)
```

$$y(x) = e^{\frac{-a \operatorname{LambertW}\left(\frac{2\,e^{\frac{-c_1-a+x}{a}}}{a\sqrt{\frac{1}{b}}}\right)-a+x-c_1}{a}}\left(a\sqrt{\frac{1}{b}}+e^{\frac{-a \operatorname{LambertW}\left(\frac{2\,e^{\frac{-c_1-a+x}{a}}}{a\sqrt{\frac{1}{b}}}\right)-a+x-c_1}{a}}\right)$$

$$y(x) = \frac{a^2\left(\operatorname{LambertW}\left(-\frac{2\sqrt{b}\,e^{\frac{-c_1-a+x}{a}}}{a}\right)+2\right)\operatorname{LambertW}\left(-\frac{2\sqrt{b}\,e^{\frac{-c_1-a+x}{a}}}{a}\right)}{4b}$$

$$y(x) = \frac{a^2\left(\operatorname{LambertW}\left(\frac{2\sqrt{b}\,e^{\frac{-c_1-a+x}{a}}}{a}\right)+2\right)\operatorname{LambertW}\left(\frac{2\sqrt{b}\,e^{\frac{-c_1-a+x}{a}}}{a}\right)}{4b}$$

✓ Solution by Mathematica

Time used: 0.803 (sec). Leaf size: 123

```
DSolve[y[x]==a*y'[x]+b*(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{\sqrt{4\#1b+a^2}+a\log\left(b\left(a-\sqrt{4\#1b+a^2}\right)\right)}{2b}\&\right]\left[\frac{x}{2b}+c_1\right]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\sqrt{4\#1b+a^2}-a\log\left(\sqrt{4\#1b+a^2}+a\right)}{2b}\&\right]\left[-\frac{x}{2b}+c_1\right]$$

$$y(x) \to 0$$

6.6 problem 6

Internal problem ID [4411]
Internal file name [`OUTPUT/3904_Sunday_June_05_2022_11_38_21_AM_13572593/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 6.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{-ay' - by'^2 = -x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-a + \sqrt{a^2 + 4bx}}{2b} \tag{1}$$

$$y' = -\frac{a + \sqrt{a^2 + 4bx}}{2b} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{-a + \sqrt{a^2 + 4bx}}{2b}\, \mathrm{d}x \\ &= \frac{\frac{(a^2+4bx)^{\frac{3}{2}}}{6b} - ax}{2b} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\frac{(a^2+4bx)^{\frac{3}{2}}}{6b} - ax}{2b} + c_1 \tag{1}$$

Verification of solutions

$$y = \frac{\frac{(a^2+4bx)^{\frac{3}{2}}}{6b} - ax}{2b} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{a + \sqrt{a^2 + 4bx}}{2b} \, \mathrm{d}x \\ &= -\frac{ax + \frac{(a^2+4bx)^{\frac{3}{2}}}{6b}}{2b} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{ax + \frac{(a^2+4bx)^{\frac{3}{2}}}{6b}}{2b} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{ax + \frac{(a^2+4bx)^{\frac{3}{2}}}{6b}}{2b} + c_2$$

Verified OK.

6.6.1 Maple step by step solution

Let's solve

$-ay' - by'^2 = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(-ay' - by'^2\right) dx = \int -x dx + c_1$

- Cannot compute integral

  $\int \left(-ay' - by'^2\right) dx = -\frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 80

```
dsolve(x=a*diff(y(x),x)+b*(diff(y(x),x))^2,y(x), singsol=all)
```

$$y(x) = \frac{(a^2+4xb)^{\frac{3}{2}} + 12c_1b^2 - 6axb}{12b^2}$$
$$y(x) = \frac{12c_1b^2 - a^2\sqrt{a^2+4xb} - 6axb - 4bx\sqrt{a^2+4xb}}{12b^2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 74

```
DSolve[x==a*y'[x]+b*(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(a^2+4bx)^{3/2} - 6abx + 12b^2c_1}{12b^2}$$
$$y(x) \to -\frac{\frac{(a^2+4bx)^{3/2}}{6b} + ax}{2b} + c_1$$

6.7 problem 7

Internal problem ID [4412]
Internal file name [`OUTPUT/3905_Sunday_June_05_2022_11_38_30_AM_10840435/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 7.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y - \sqrt{1 + y'^2} - ay' = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{ya + \sqrt{y^2 + a^2 - 1}}{a^2 - 1} \tag{1}$$

$$y' = \frac{ya - \sqrt{y^2 + a^2 - 1}}{a^2 - 1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{a^2 - 1}{ya + \sqrt{a^2 + y^2 - 1}} dy = \int dx$$

$$\int^y \frac{a^2 - 1}{\_aa + \sqrt{\_a^2 + a^2 - 1}} d\_a = x + c_1$$

Summary
The solution(s) found are the following

$$\int^y \frac{a^2 - 1}{\_aa + \sqrt{\_a^2 + a^2 - 1}} d\_a = x + c_1 \tag{1}$$

Verification of solutions

$$\int^{y} \frac{\_a^2-1}{\_a a+\sqrt{\_a^2+a^2-1}}d\_a = x+c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int \frac{a^2-1}{ya-\sqrt{a^2+y^2-1}}dy = \int dx$$
$$\int^{y} \frac{a^2-1}{\_aa-\sqrt{\_a^2+a^2-1}}d\_a = x+c_2$$

Summary

The solution(s) found are the following

$$\int^{y} \frac{a^2-1}{\_aa-\sqrt{\_a^2+a^2-1}}d\_a = x+c_2 \tag{1}$$

Verification of solutions

$$\int^{y} \frac{a^2-1}{\_aa-\sqrt{\_a^2+a^2-1}}d\_a = x+c_2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 112

```
dsolve(y(x)=a*diff(y(x),x)+sqrt(1+(diff(y(x),x))^2),y(x), singsol=all)
```

$$-\left(\int^{y(x)}\frac{1}{a\_a+\sqrt{\_a^2+a^2-1}}d\_a\right)a^2+\int^{y(x)}\frac{1}{a\_a+\sqrt{\_a^2+a^2-1}}d\_a-c_1+x=0$$

$$\left(\int^{y(x)}\frac{1}{-a\_a+\sqrt{\_a^2+a^2-1}}d\_a\right)a^2-\left(\int^{y(x)}\frac{1}{-a\_a+\sqrt{\_a^2+a^2-1}}d\_a\right)-c_1+x=0$$

✓ Solution by Mathematica

Time used: 0.597 (sec). Leaf size: 210

```
DSolve[y[x]==a*y'[x]+Sqrt[1+(y'[x])^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\text{InverseFunction}\left[\frac{a\left(\log\left(\sqrt{\#1^2+a^2-1}-\#1-a+1\right)+\log\left(\sqrt{\#1^2+a^2-1}-\#1+a-1\right)\right)-(a}{a^2-1}+c_1\right]$$

$$y(x)\to\text{InverseFunction}\left[\frac{a\left(\log\left(\sqrt{\#1^2+a^2-1}-\#1-a-1\right)+\log\left(\sqrt{\#1^2+a^2-1}-\#1+a+1\right)\right)-(a}{a^2-1}+c_1\right]$$

$$y(x)\to 1$$

6.8 problem 8

Internal problem ID [4413]
Internal file name [`OUTPUT/3906_Sunday_June_05_2022_11_38_52_AM_79502661/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 8.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{-\sqrt{1+y'^2}-ay'=-x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y'=\frac{ax+\sqrt{a^2+x^2-1}}{a^2-1} \tag{1}$$

$$y'=-\frac{-ax+\sqrt{a^2+x^2-1}}{a^2-1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{ax+\sqrt{a^2+x^2-1}}{a^2-1}\, \mathrm{d}x \\ &= \frac{\frac{x\sqrt{a^2+x^2-1}}{2}+\frac{(4a^2-4)\ln\left(x+\sqrt{a^2+x^2-1}\right)}{8}+\frac{a\,x^2}{2}}{a^2-1}+c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y=\frac{\frac{x\sqrt{a^2+x^2-1}}{2}+\frac{(4a^2-4)\ln\left(x+\sqrt{a^2+x^2-1}\right)}{8}+\frac{a\,x^2}{2}}{a^2-1}+c_1 \tag{1}$$

Verification of solutions

$$y = \frac{\frac{x\sqrt{a^2+x^2-1}}{2} + \frac{(4a^2-4)\ln\left(x+\sqrt{a^2+x^2-1}\right)}{8} + \frac{a\,x^2}{2}}{a^2-1} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{-ax + \sqrt{a^2+x^2-1}}{a^2-1}\,\mathrm{d}x \\ &= -\frac{\frac{x\sqrt{a^2+x^2-1}}{2} + \frac{(4a^2-4)\ln\left(x+\sqrt{a^2+x^2-1}\right)}{8} - \frac{a\,x^2}{2}}{a^2-1} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\frac{x\sqrt{a^2+x^2-1}}{2} + \frac{(4a^2-4)\ln\left(x+\sqrt{a^2+x^2-1}\right)}{8} - \frac{a\,x^2}{2}}{a^2-1} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{\frac{x\sqrt{a^2+x^2-1}}{2} + \frac{(4a^2-4)\ln\left(x+\sqrt{a^2+x^2-1}\right)}{8} - \frac{a\,x^2}{2}}{a^2-1} + c_2$$

Verified OK.

6.8.1 Maple step by step solution

Let's solve

$-\sqrt{1+y'^2} - ay' = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(-\sqrt{1+y'^2} - ay'\right) dx = \int -x dx + c_1$

- Cannot compute integral

  $\int \left(-\sqrt{1+y'^2} - ay'\right) dx = -\frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 113

```
dsolve(x=a*diff(y(x),x)+sqrt(1+(diff(y(x),x))^2),y(x), singsol=all)
```

$$y(x) = \frac{a\,x^2 + x\sqrt{a^2+x^2-1} + \left(\ln\left(x+\sqrt{a^2+x^2-1}\right)+2c_1\right)(1+a)\,(a-1)}{2a^2-2}$$
$$y(x) = \frac{a\,x^2 - x\sqrt{a^2+x^2-1} - (1+a)\,(a-1)\left(\ln\left(x+\sqrt{a^2+x^2-1}\right)-2c_1\right)}{2a^2-2}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 113

```
DSolve[x==a*y'[x]+Sqrt[1+(y'[x])^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\frac{x\left(ax-\sqrt{a^2+x^2-1}\right)}{a^2-1} + \log\left(\sqrt{a^2+x^2-1}-x\right)\right) + c_1$$
$$y(x) \to \frac{1}{2}\left(\frac{x\left(\sqrt{a^2+x^2-1}+ax\right)}{a^2-1} - \log\left(\sqrt{a^2+x^2-1}-x\right)\right) + c_1$$

6.9 problem 9

Internal problem ID [4414]
Internal file name [`OUTPUT/3907_Sunday_June_05_2022_11_39_21_AM_33254320/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 9.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \frac{\sqrt{1 + y'^2}}{x} = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1}{\sqrt{x^2 - 1}} \tag{1}$$

$$y' = -\frac{1}{\sqrt{x^2 - 1}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{\sqrt{x^2 - 1}}\, \mathrm{d}x \\ &= \ln\left(x + \sqrt{x^2 - 1}\right) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln\left(x + \sqrt{x^2 - 1}\right) + c_1 \tag{1}$$

Verification of solutions

$$y = \ln\left(x + \sqrt{x^2 - 1}\right) + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{\sqrt{x^2 - 1}}\,\mathrm{d}x \\ &= -\ln\left(x + \sqrt{x^2 - 1}\right) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\ln\left(x + \sqrt{x^2 - 1}\right) + c_2 \tag{1}$$

Verification of solutions

$$y = -\ln\left(x + \sqrt{x^2 - 1}\right) + c_2$$

Verified OK.

6.9.1 Maple step by step solution

Let's solve

$y' - \frac{\sqrt{1+y'^2}}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(y' - \frac{\sqrt{1+y'^2}}{x}\right) dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int \left(y' - \frac{\sqrt{1+y'^2}}{x}\right) dx = c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(diff(y(x),x)-1/x*sqrt(1+(diff(y(x),x))^2)=0,y(x), singsol=all)
```

$$y(x) = \ln \left(x+\sqrt{x^2-1}\right)+c_1$$
$$y(x) = -\ln \left(x+\sqrt{x^2-1}\right)+c_1$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 89

```
DSolve[y'[x]-1/x*Sqrt[1+(y'[x])^2]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\log \left(1-\frac{x}{\sqrt{x^2-1}}\right)-\log \left(\frac{x}{\sqrt{x^2-1}}+1\right)+2c_1\right)$$
$$y(x) \to \frac{1}{2}\left(-\log \left(1-\frac{x}{\sqrt{x^2-1}}\right)+\log \left(\frac{x}{\sqrt{x^2-1}}+1\right)+2c_1\right)$$

6.10 problem 10

Internal problem ID [4415]
Internal file name [`OUTPUT/3908_Sunday_June_05_2022_11_39_52_AM_42926693/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 10.
ODE order: 1.
ODE degree: 6.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x^2\left(1+{y'}^2\right)^3 = a^2}$$

Solving the given ode for y' results in 6 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{x\left(\left(a^2x\right)^{\frac{1}{3}} - x\right)}}{x} \tag{1}$$

$$y' = -\frac{\sqrt{x\left(\left(a^2x\right)^{\frac{1}{3}} - x\right)}}{x} \tag{2}$$

$$y' = \frac{\sqrt{2}\,\sqrt{x\left(i\sqrt{3}\,\left(a^2x\right)^{\frac{1}{3}} - \left(a^2x\right)^{\frac{1}{3}} - 2x\right)}}{2x} \tag{3}$$

$$y' = -\frac{\sqrt{2}\,\sqrt{x\left(i\sqrt{3}\,\left(a^2x\right)^{\frac{1}{3}} - \left(a^2x\right)^{\frac{1}{3}} - 2x\right)}}{2x} \tag{4}$$

$$y' = \frac{\sqrt{-2x\left(i\sqrt{3}\,\left(a^2x\right)^{\frac{1}{3}} + \left(a^2x\right)^{\frac{1}{3}} + 2x\right)}}{2x} \tag{5}$$

$$y' = -\frac{\sqrt{-2x\left(i\sqrt{3}\,\left(a^2x\right)^{\frac{1}{3}} + \left(a^2x\right)^{\frac{1}{3}} + 2x\right)}}{2x} \tag{6}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sqrt{x\left(\left(a^2x\right)^{\frac{1}{3}} - x\right)}}{x}\,\mathrm{d}x \\ &= \frac{\sqrt{-\frac{\left(a^2x\right)^{\frac{4}{3}}\left(\left(a^2x\right)^{\frac{2}{3}} - a^2\right)}{a^4}}\left(\left(a^2x\right)^{\frac{2}{3}} - a^2\right)}{\left(a^2x\right)^{\frac{2}{3}}} + c_1 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-\frac{\left(a^2x\right)^{\frac{4}{3}}\left(\left(a^2x\right)^{\frac{2}{3}} - a^2\right)}{a^4}}\left(\left(a^2x\right)^{\frac{2}{3}} - a^2\right)}{\left(a^2x\right)^{\frac{2}{3}}} + c_1 \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{-\frac{(a^2x)^{\frac{4}{3}}\left((a^2x)^{\frac{2}{3}}-a^2\right)}{a^4}}\left((a^2x)^{\frac{2}{3}}-a^2\right)}{(a^2x)^{\frac{2}{3}}} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{\sqrt{x\left((a^2x)^{\frac{1}{3}}-x\right)}}{x}\,\mathrm{d}x \\ &= -\frac{\sqrt{-\frac{(a^2x)^{\frac{4}{3}}\left((a^2x)^{\frac{2}{3}}-a^2\right)}{a^4}}\left((a^2x)^{\frac{2}{3}}-a^2\right)}{(a^2x)^{\frac{2}{3}}} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-\frac{(a^2x)^{\frac{4}{3}}\left((a^2x)^{\frac{2}{3}}-a^2\right)}{a^4}}\left((a^2x)^{\frac{2}{3}}-a^2\right)}{(a^2x)^{\frac{2}{3}}} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{\sqrt{-\frac{(a^2x)^{\frac{4}{3}}\left((a^2x)^{\frac{2}{3}}-a^2\right)}{a^4}}\left((a^2x)^{\frac{2}{3}}-a^2\right)}{(a^2x)^{\frac{2}{3}}} + c_2$$

Verified OK.

Solving equation (3)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sqrt{2}\,\sqrt{x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}}-(a^2x)^{\frac{1}{3}}-2x\right)}}{2x}\,\mathrm{d}x \\ &= -\frac{\sqrt{2}\,\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{a^4}}\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{4\,(a^2x)^{\frac{2}{3}}} + c_3 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{\sqrt{2}\,\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{a^4}}\,\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{4\,(a^2x)^{\frac{2}{3}}}+c_3 \tag{1}$$

Verification of solutions

$$y = -\frac{\sqrt{2}\,\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{a^4}}\,\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{4\,(a^2x)^{\frac{2}{3}}}+c_3$$

Verified OK.

Solving equation (4)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{\sqrt{2}\,\sqrt{x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}}-(a^2x)^{\frac{1}{3}}-2x\right)}}{2x}\,\mathrm{d}x \\ &= \frac{\sqrt{2}\,\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{a^4}}\,\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{4\,(a^2x)^{\frac{2}{3}}}+c_4 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{2}\,\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{a^4}}\,\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{4\,(a^2x)^{\frac{2}{3}}}+c_4 \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\,\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{a^4}}\,\left(i\sqrt{3}\,a^2-2(a^2x)^{\frac{2}{3}}-a^2\right)}{4\,(a^2x)^{\frac{2}{3}}}+c_4$$

Verified OK.

Solving equation (5)

Integrating both sides gives

$$y = \int \frac{\sqrt{-2x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}} + (a^2x)^{\frac{1}{3}} + 2x\right)}}{2x}\,\mathrm{d}x$$

$$= \frac{\sqrt{-2x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}} + (a^2x)^{\frac{1}{3}} + 2x\right)}\,\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2+2(a^2x)^{\frac{2}{3}}+a^2\right)}{a^4}}\,\left(i\sqrt{3}\,a^2 + 2(a^2x)^{\frac{2}{3}} + a^2\right)}{4\sqrt{x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}} + (a^2x)^{\frac{1}{3}} + 2x\right)}\,(a^2x)^{\frac{2}{3}}} + c_5$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-2x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}} + (a^2x)^{\frac{1}{3}} + 2x\right)}\,\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2+2(a^2x)^{\frac{2}{3}}+a^2\right)}{a^4}}\,\left(i\sqrt{3}\,a^2 + 2(a^2x)^{\frac{2}{3}} + a^2\right)}{4\sqrt{x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}} + (a^2x)^{\frac{1}{3}} + 2x\right)}\,(a^2x)^{\frac{2}{3}}} + c_5 \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{-2x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}} + (a^2x)^{\frac{1}{3}} + 2x\right)}\,\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2+2(a^2x)^{\frac{2}{3}}+a^2\right)}{a^4}}\,\left(i\sqrt{3}\,a^2 + 2(a^2x)^{\frac{2}{3}} + a^2\right)}{4\sqrt{x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}} + (a^2x)^{\frac{1}{3}} + 2x\right)}\,(a^2x)^{\frac{2}{3}}} + c_5$$

Verified OK.

Solving equation (6)

Integrating both sides gives

$$y = \int -\frac{\sqrt{-2x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}} + (a^2x)^{\frac{1}{3}} + 2x\right)}}{2x}\,\mathrm{d}x$$

$$= -\frac{\sqrt{-2x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}} + (a^2x)^{\frac{1}{3}} + 2x\right)}\,\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2+2(a^2x)^{\frac{2}{3}}+a^2\right)}{a^4}}\,\left(i\sqrt{3}\,a^2 + 2(a^2x)^{\frac{2}{3}} + a^2\right)}{4\sqrt{x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}} + (a^2x)^{\frac{1}{3}} + 2x\right)}\,(a^2x)^{\frac{2}{3}}} + c_6$$

Summary
The solution(s) found are the following

$$y = -\frac{\sqrt{-2x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}}+(a^2x)^{\frac{1}{3}}+2x\right)}\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2+2(a^2x)^{\frac{2}{3}}+a^2\right)}{a^4}}\left(i\sqrt{3}\,a^2+2(a^2x)^{\frac{2}{3}}+a^2\right)}{4\sqrt{x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}}+(a^2x)^{\frac{1}{3}}+2x\right)}\,(a^2x)^{\frac{2}{3}}} + c_6 \tag{1}$$

Verification of solutions

$$y = -\frac{\sqrt{-2x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}}+(a^2x)^{\frac{1}{3}}+2x\right)}\sqrt{\frac{(a^2x)^{\frac{4}{3}}\left(i\sqrt{3}\,a^2+2(a^2x)^{\frac{2}{3}}+a^2\right)}{a^4}}\left(i\sqrt{3}\,a^2+2(a^2x)^{\frac{2}{3}}+a^2\right)}{4\sqrt{x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}}+(a^2x)^{\frac{1}{3}}+2x\right)}\,(a^2x)^{\frac{2}{3}}} + c_6$$

Verified OK.

6.10.1 Maple step by step solution

Let's solve

$x^2\left(1+y'^2\right)^3 = a^2$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int x^2\left(1+y'^2\right)^3 dx = \int a^2 dx + c_1$

- Cannot compute integral

  $\int x^2\left(1+y'^2\right)^3 dx = a^2x + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 605

```
dsolve(x^2*(1+(diff(y(x),x))^2)^3-a^2=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{\frac{x(a^2x)^{\frac{1}{3}}\left(a^2-(a^2x)^{\frac{2}{3}}\right)}{a^2}}\, a^2 + c_1(a^2x)^{\frac{2}{3}} + \sqrt{\frac{x(a^2x)^{\frac{1}{3}}\left(a^2-(a^2x)^{\frac{2}{3}}\right)}{a^2}}\, (a^2x)^{\frac{2}{3}}}{(a^2x)^{\frac{2}{3}}}$$

$$y(x) = \frac{\left(a^2-(a^2x)^{\frac{2}{3}}\right)\sqrt{\frac{x(a^2x)^{\frac{1}{3}}\left(a^2-(a^2x)^{\frac{2}{3}}\right)}{a^2}} + c_1(a^2x)^{\frac{2}{3}}}{(a^2x)^{\frac{2}{3}}}$$

$$y(x) = -\frac{\sqrt{2}\sqrt{-x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}}+(a^2x)^{\frac{1}{3}}+2x\right)}\sqrt{\frac{\left(2i(a^2x)^{\frac{2}{3}}+ia^2-\sqrt{3}\,a^2\right)x(a^2x)^{\frac{1}{3}}}{a^2}}\left(2(a^2x)^{\frac{2}{3}}+a^2+i\sqrt{3}\,a^2\right)}{4\sqrt{\left(i\,(a^2x)^{\frac{1}{3}}+2ix-\sqrt{3}\,(a^2x)^{\frac{1}{3}}\right)x}\,(a^2x)^{\frac{2}{3}}} + c_1$$

$$y(x) = \frac{\sqrt{2}\sqrt{-x\left(i\sqrt{3}\,(a^2x)^{\frac{1}{3}}+(a^2x)^{\frac{1}{3}}+2x\right)}\sqrt{\frac{\left(2i(a^2x)^{\frac{2}{3}}+ia^2-\sqrt{3}\,a^2\right)x(a^2x)^{\frac{1}{3}}}{a^2}}\left(2(a^2x)^{\frac{2}{3}}+a^2+i\sqrt{3}\,a^2\right)}{4\sqrt{\left(i\,(a^2x)^{\frac{1}{3}}+2ix-\sqrt{3}\,(a^2x)^{\frac{1}{3}}\right)x}\,(a^2x)^{\frac{2}{3}}} + c_1$$

$$y(x) = \frac{\left(-2(a^2x)^{\frac{2}{3}}\sqrt{2}+\left(i\sqrt{6}-\sqrt{2}\right)a^2\right)\sqrt{\frac{\left(\left(i\sqrt{3}-1\right)a^2-2(a^2x)^{\frac{2}{3}}\right)x(a^2x)^{\frac{1}{3}}}{a^2}} + 4c_1(a^2x)^{\frac{2}{3}}}{4\,(a^2x)^{\frac{2}{3}}}$$

$$y(x) = -\frac{\left(-2(a^2x)^{\frac{2}{3}}\sqrt{2}+\left(i\sqrt{6}-\sqrt{2}\right)a^2\right)\sqrt{\frac{\left(\left(i\sqrt{3}-1\right)a^2-2(a^2x)^{\frac{2}{3}}\right)x(a^2x)^{\frac{1}{3}}}{a^2}} - 4c_1(a^2x)^{\frac{2}{3}}}{4\,(a^2x)^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 18.927 (sec). Leaf size: 375

```
DSolve[x^2*(1+(y'[x])^2)^3-a^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{x}\sqrt{\frac{a^{2/3}}{x^{2/3}}-1}\left(x^{2/3}-a^{2/3}\right)+c_1$$
$$y(x) \to \sqrt[3]{x}\sqrt{\frac{a^{2/3}}{x^{2/3}}-1}\left(a^{2/3}-x^{2/3}\right)+c_1$$
$$y(x) \to c_1-\frac{1}{2}\sqrt[3]{x}\sqrt{-1+\frac{i\left(\sqrt{3}+i\right)a^{2/3}}{2x^{2/3}}}\left(2x^{2/3}+\left(1-i\sqrt{3}\right)a^{2/3}\right)$$
$$y(x) \to \frac{1}{2}\sqrt[3]{x}\sqrt{-1+\frac{i\left(\sqrt{3}+i\right)a^{2/3}}{2x^{2/3}}}\left(2x^{2/3}+\left(1-i\sqrt{3}\right)a^{2/3}\right)+c_1$$
$$y(x) \to c_1-\frac{1}{2}\sqrt[3]{x}\sqrt{-1-\frac{i\left(\sqrt{3}-i\right)a^{2/3}}{2x^{2/3}}}\left(2x^{2/3}+\left(1+i\sqrt{3}\right)a^{2/3}\right)$$
$$y(x) \to \frac{1}{2}\sqrt[3]{x}\sqrt{-1-\frac{i\left(\sqrt{3}-i\right)a^{2/3}}{2x^{2/3}}}\left(2x^{2/3}+\left(1+i\sqrt{3}\right)a^{2/3}\right)+c_1$$

6.11 problem 11

Internal problem ID [4416]
Internal file name [`OUTPUT/3909_Sunday_June_05_2022_11_40_07_AM_14979362/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 11.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 = -1 + \frac{(a+x)^2}{2ax+x^2}}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{a}{\sqrt{2ax+x^2}} \tag{1}$$

$$y' = -\frac{a}{\sqrt{2ax+x^2}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{a}{\sqrt{2ax+x^2}}\, \mathrm{d}x \\ &= a \ln\left(a+x+\sqrt{2ax+x^2}\right) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = a \ln\left(a+x+\sqrt{2ax+x^2}\right) + c_1 \tag{1}$$

Verification of solutions

$$y = a \ln\left(a + x + \sqrt{2ax + x^2}\right) + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{a}{\sqrt{2ax + x^2}} \,\mathrm{d}x \\ &= -a \ln\left(a + x + \sqrt{2ax + x^2}\right) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -a \ln\left(a + x + \sqrt{2ax + x^2}\right) + c_2 \tag{1}$$

Verification of solutions

$$y = -a \ln\left(a + x + \sqrt{2ax + x^2}\right) + c_2$$

Verified OK.

6.11.1 Maple step by step solution

Let's solve

$y'^2 = -1 + \frac{(a+x)^2}{2ax+x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'^2 dx = \int \left(-1 + \frac{(a+x)^2}{2ax+x^2}\right) dx + c_1$

- Cannot compute integral

  $\int y'^2 dx = -\frac{a \ln(2a+x)}{2} + \frac{a \ln(x)}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
dsolve(1+(diff(y(x),x))^2=(x+a)^2/(x^2+2*a*x),y(x), singsol=all)
```

$$y(x) = a\ln\left(x + a + \sqrt{x\,(2a + x)}\right) + c_1$$
$$y(x) = -a\ln\left(x + a + \sqrt{x\,(2a + x)}\right) + c_1$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 107

```
DSolve[1+(y'[x])^2==(x+a)^2/(x^2+2*a*x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2a\sqrt{x}\sqrt{2a+x}\log\left(\sqrt{2a+x}-\sqrt{x}\right)}{\sqrt{x(2a+x)}} + c_1$$
$$y(x) \to \frac{2a\sqrt{x}\sqrt{2a+x}\log\left(\sqrt{2a+x}-\sqrt{x}\right)}{\sqrt{x(2a+x)}} + c_1$$

6.12 problem 12

Internal problem ID [4417]
Internal file name [`OUTPUT/3910_Sunday_June_05_2022_11_40_15_AM_98915764/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 12.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y - xy' - y' + {y'}^2 = 0}$$

6.12.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$p^2 - xp - p + y = 0$$

Solving for y from the above results in

$$y = -p^2 + xp + p \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= -p^2 + xp + p \\ &= -p^2 + xp + p \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -p^2 + p$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = -c_1^2 + c_1x + c_1$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -p^2 + p$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - 2p + 1 \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{1}{2} + \frac{x}{2}$$

Substituting the above back in (1) results in

$$y_1 = \frac{(x+1)^2}{4}$$

Summary
The solution(s) found are the following

$$y = -c_1^2 + c_1 x + c_1 \tag{1}$$

$$y = \frac{(x+1)^2}{4} \tag{2}$$

Verification of solutions

$$y = -c_1^2 + c_1 x + c_1$$

Verified OK.

$$y = \frac{(x+1)^2}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(y(x)=x*diff(y(x),x)+diff(y(x),x)-(diff(y(x),x))^2,y(x), singsol=all)
```

$$y(x) = \frac{(1+x)^2}{4}$$
$$y(x) = c_1(-c_1 + x + 1)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 28

```
DSolve[y[x]==x*y'[x]+y'[x]-(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x + 1 - c_1)$$
$$y(x) \to \frac{1}{4}(x+1)^2$$

6.13 problem 13

Internal problem ID [4418]
Internal file name [`OUTPUT/3911_Sunday_June_05_2022_11_40_28_AM_68233945/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 13.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _rational, _Clairaut]`

$$\boxed{y - xy' - \sqrt{b^2 - a^2 y'^2} = 0}$$

6.13.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - xp - \sqrt{-a^2p^2 + b^2} = 0$$

Solving for y from the above results in

$$y = xp + \sqrt{-a^2p^2 + b^2} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \sqrt{-a^2p^2 + b^2} \\ &= xp + \sqrt{-a^2p^2 + b^2} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \sqrt{-a^2p^2 + b^2}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1x + \sqrt{-a^2c_1^2 + b^2}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \sqrt{-a^2p^2 + b^2}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{a^2p}{\sqrt{-a^2p^2 + b^2}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{xb}{\sqrt{a^2+x^2}\,a}$$
$$p_2 = -\frac{xb}{\sqrt{a^2+x^2}\,a}$$

Substituting the above back in (1) results in

$$y_1 = \frac{\sqrt{\frac{a^2b^2}{a^2+x^2}}\,\sqrt{a^2+x^2}\,a + b\,x^2}{\sqrt{a^2+x^2}\,a}$$
$$y_2 = \frac{\sqrt{\frac{a^2b^2}{a^2+x^2}}\,\sqrt{a^2+x^2}\,a - b\,x^2}{\sqrt{a^2+x^2}\,a}$$

Summary

The solution(s) found are the following

$$y = c_1x + \sqrt{-a^2c_1^2 + b^2} \tag{1}$$
$$y = \frac{\sqrt{\frac{a^2b^2}{a^2+x^2}}\,\sqrt{a^2+x^2}\,a + b\,x^2}{\sqrt{a^2+x^2}\,a} \tag{2}$$
$$y = \frac{\sqrt{\frac{a^2b^2}{a^2+x^2}}\,\sqrt{a^2+x^2}\,a - b\,x^2}{\sqrt{a^2+x^2}\,a} \tag{3}$$

Verification of solutions

$$y = c_1x + \sqrt{-a^2c_1^2 + b^2}$$

Verified OK.

$$y = \frac{\sqrt{\frac{a^2b^2}{a^2+x^2}}\,\sqrt{a^2+x^2}\,a + b\,x^2}{\sqrt{a^2+x^2}\,a}$$

Verified OK.

$$y = \frac{\sqrt{\frac{a^2b^2}{a^2+x^2}}\,\sqrt{a^2+x^2}\,a - b\,x^2}{\sqrt{a^2+x^2}\,a}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 22

```
dsolve(y(x)=x*diff(y(x),x)+sqrt(b^2-a^2*(diff(y(x),x))^2),y(x), singsol=all)
```

$$y(x) = c_1 x + \sqrt{-a^2 c_1^2 + b^2}$$

✓ Solution by Mathematica

Time used: 0.349 (sec). Leaf size: 38

```
DSolve[y[x]==x*y'[x]+Sqrt[b^2-a^2*(y'[x])^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{b^2 - a^2 c_1{}^2} + c_1 x$$
$$y(x) \to \sqrt{b^2}$$

6.14 problem 14

Internal problem ID [4419]
Internal file name [`OUTPUT/3912_Sunday_June_05_2022_11_40_53_AM_721168/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "dAlembert", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y - xy' - x\sqrt{1 + y'^2} = 0}$$

6.14.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x - x(u'(x)\,x + u(x)) - x\sqrt{1 + (u'(x)\,x + u(x))^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{2ux} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u}}\,du = -\frac{1}{2x}\,dx$$

$$\int \frac{1}{\frac{u^2+1}{u}}\,du = \int -\frac{1}{2x}\,dx$$

$$\frac{\ln (u^2+1)}{2} = -\frac{\ln (x)}{2} + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = \mathrm{e}^{-\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = \frac{c_3}{\sqrt{x}}$$

Which simplifies to

$$\sqrt{u(x)^2+1} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

The solution is

$$\sqrt{u(x)^2+1} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+1} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}} \tag{1}$$

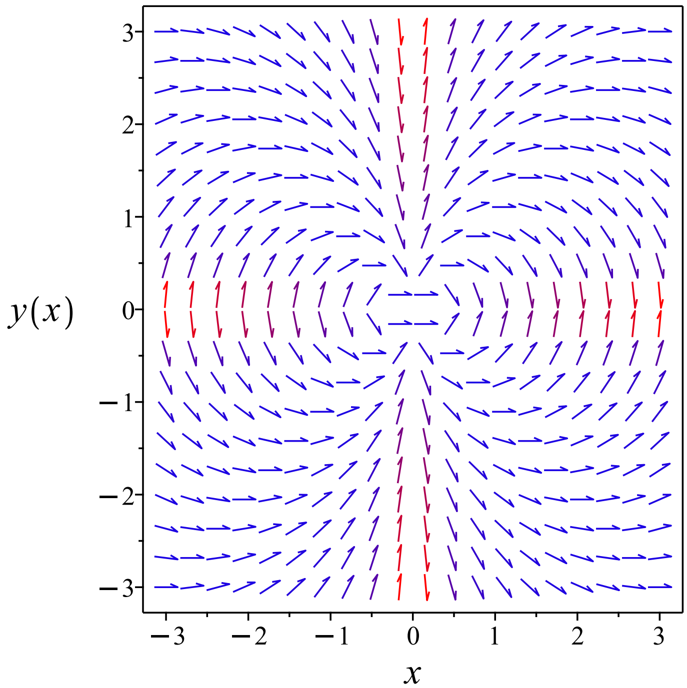


Figure 77: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2+y^2}{x^2}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

Verified OK.

6.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2 - y^2}{2xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 65: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^2 - y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 \tag{4}$$

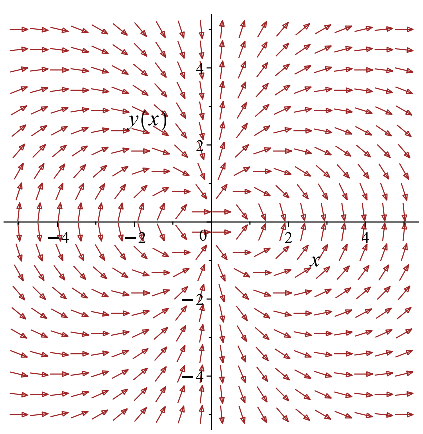
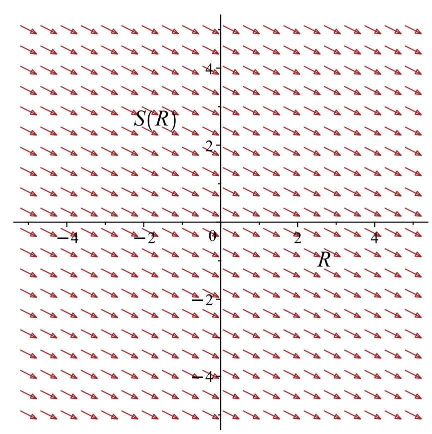
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = -\frac{x}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = -\frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2-y^2}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = -\frac{1}{2}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x} = -\frac{x}{2} + c_1 \tag{1}$$

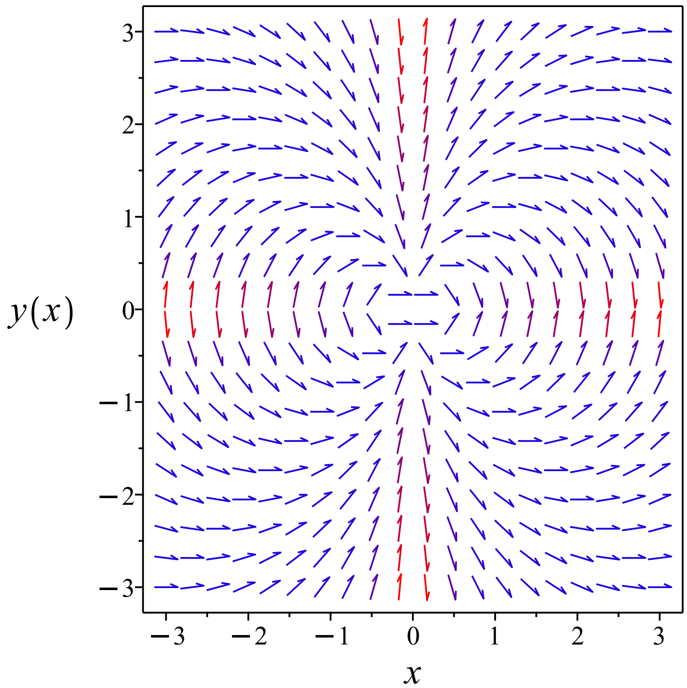


Figure 78: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = -\frac{x}{2} + c_1$$

Verified OK.

6.14.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^2 - y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y - \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{2x} \\
f_1(x) &= -\frac{x}{2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{y^2}{2x} - \frac{x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{w(x)}{2x} - \frac{x}{2} \\
w' &= \frac{w}{x} - x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-x) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(-x) \\ d\left(\frac{w}{x}\right) &= -1\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int -1\, dx \\ \frac{w}{x} &= -x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x - x^2$$

which simplifies to

$$w(x) = x(-x + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x(-x + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{x(-x + c_1)} \\ y(x) &= -\sqrt{x(-x + c_1)}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x(-x+c_1)} \tag{1}$$
$$y = -\sqrt{x(-x+c_1)} \tag{2}$$

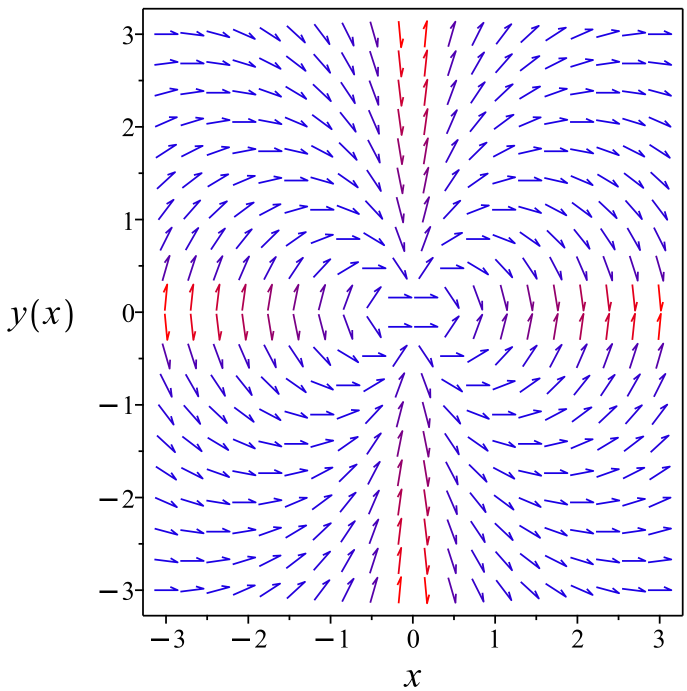


Figure 79: Slope field plot

Verification of solutions

$$y = \sqrt{x(-x+c_1)}$$

Verified OK.

$$y = -\sqrt{x(-x+c_1)}$$

Verified OK.

6.14.4 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - xp - x\sqrt{p^2+1} = 0$$

Solving for y from the above results in

$$y = \left(\sqrt{p^2+1}+p\right)x \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \sqrt{p^2+1}+p \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$-\sqrt{p^2+1} = x\left(\frac{p}{\sqrt{p^2+1}}+1\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-\sqrt{p^2+1} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= i \\ p &= -i \end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned} y &= -ix \\ y &= ix \end{aligned}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{\sqrt{p(x)^2+1}}{x\left(\frac{p(x)}{\sqrt{p(x)^2+1}}+1\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{x(p)\left(\frac{p}{\sqrt{p^2+1}}+1\right)}{\sqrt{p^2+1}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{-\sqrt{p^2+1}-p}{p^2+1}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{\left(-\sqrt{p^2+1}-p\right)x(p)}{p^2+1} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-\sqrt{p^2+1}-p}{p^2+1}dp}$$

The ode becomes

$$\frac{d}{dp}\mu x = 0$$
$$\frac{d}{dp}\left(e^{\int -\frac{-\sqrt{p^2+1}-p}{p^2+1}dp}x\right) = 0$$

Integrating gives

$$e^{\int -\frac{-\sqrt{p^2+1}-p}{p^2+1}dp}x = c_2$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{-\sqrt{p^2+1}-p}{p^2+1}dp}$ results in

$$x(p) = c_2 e^{-\left(\int \frac{\sqrt{p^2+1}+p}{p^2+1}dp\right)}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Summary

The solution(s) found are the following

$$y = -ix \tag{1}$$
$$y = ix \tag{2}$$

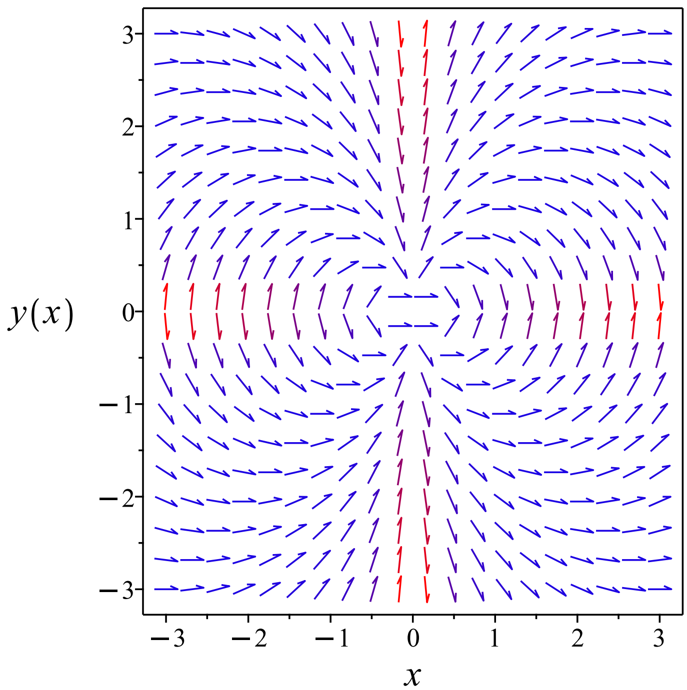


Figure 80: Slope field plot

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 97

```
dsolve(y(x)=x*diff(y(x),x)+x*sqrt(1+(diff(y(x),x))^2),y(x), singsol=all)
```

$$y(x) = \frac{\left(\sqrt{-\frac{c_1^2}{x(-2c_1+x)}}\,\sqrt{-x\,(-2c_1+x)} - x + c_1\right) x}{\sqrt{-x\,(-2c_1+x)}}$$

$$y(x) = \frac{\left(\sqrt{-\frac{c_1^2}{x(-2c_1+x)}}\,\sqrt{-x\,(-2c_1+x)} + x - c_1\right) x}{\sqrt{-x\,(-2c_1+x)}}$$

✓ Solution by Mathematica

Time used: 0.269 (sec). Leaf size: 37

```
DSolve[y[x]==x*y'[x]+x*Sqrt[1+(y'[x])^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x(x-c_1)}$$
$$y(x) \to \sqrt{-x(x-c_1)}$$

6.15 problem 15

Internal problem ID [4420]
Internal file name [OUTPUT/3913_Sunday_June_05_2022_11_41_28_AM_42177893/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 15.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y - xy' - ax\sqrt{1 + y'^2} = 0}$$

6.15.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - xp - ax\sqrt{p^2 + 1} = 0$$

Solving for y from the above results in

$$y = \left(\sqrt{p^2 + 1}\, a + p\right) x \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned}f &= \sqrt{p^2+1}\, a + p\\ g &= 0\end{aligned}$$

Hence (2) becomes

$$-\sqrt{p^2+1}\, a = x\left(\frac{ap}{\sqrt{p^2+1}}+1\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-\sqrt{p^2+1}\, a = 0$$

Solving for p from the above gives

$$\begin{aligned}p &= i\\ p &= -i\end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned}y &= -ix\\ y &= ix\end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{\sqrt{p(x)^2+1}\, a}{x\left(\frac{ap(x)}{\sqrt{p(x)^2+1}}+1\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{x(p)\left(\frac{ap}{\sqrt{p^2+1}}+1\right)}{\sqrt{p^2+1}\, a} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{-ap - \sqrt{p^2+1}}{(p^2+1)\,a} \\
q(p) &= 0
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) - \frac{\left(-ap - \sqrt{p^2+1}\right)x(p)}{(p^2+1)\,a} = 0
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int -\frac{-ap-\sqrt{p^2+1}}{(p^2+1)a}dp}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathrm{e}^{\int -\frac{-ap-\sqrt{p^2+1}}{(p^2+1)a}dp} x\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\mathrm{e}^{\int -\frac{-ap-\sqrt{p^2+1}}{(p^2+1)a}dp} x = c_2
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-ap-\sqrt{p^2+1}}{(p^2+1)a}dp}$ results in

$$
x(p) = c_2 \mathrm{e}^{-\frac{\int \frac{ap+\sqrt{p^2+1}}{p^2+1}dp}{a}}
$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= -ix \qquad (1)\\
y &= ix \qquad (2)
\end{aligned}
$$

Verification of solutions

$$
y = -ix
$$

Verified OK.

$$
y = ix
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 340

```
dsolve(y(x)=x*diff(y(x),x)+a*x*sqrt(1+(diff(y(x),x))^2),y(x), singsol=all)
```

$$\frac{x\sqrt{\frac{-x^2a^2+y(x)^2a^2+2\sqrt{-x^2a^2+x^2+y(x)^2}\,ay(x)+x^2+y(x)^2}{(a^2-1)^2x^2}}-\mathrm{e}^{\frac{\operatorname{arcsinh}\left(\frac{\sqrt{-x^2a^2+x^2+y(x)^2}\,a+y(x)}{(a^2-1)x}\right)}{a}}c_1}{\sqrt{\frac{-x^2a^2+y(x)^2a^2+2\sqrt{-x^2a^2+x^2+y(x)^2}\,ay(x)+x^2+y(x)^2}{(a^2-1)^2x^2}}}=0$$

$$\frac{x\sqrt{\frac{-x^2a^2+y(x)^2a^2-2\sqrt{-x^2a^2+x^2+y(x)^2}\,ay(x)+x^2+y(x)^2}{(a^2-1)^2x^2}}-\mathrm{e}^{\frac{\operatorname{arcsinh}\left(\frac{-\sqrt{-x^2a^2+x^2+y(x)^2}\,a+y(x)}{(a^2-1)x}\right)}{a}}c_1}{\sqrt{\frac{-x^2a^2+y(x)^2a^2-2\sqrt{-x^2a^2+x^2+y(x)^2}\,ay(x)+x^2+y(x)^2}{(a^2-1)^2x^2}}}=0$$

✓ Solution by Mathematica

Time used: 0.993 (sec). Leaf size: 223

```
DSolve[y[x]==x*y'[x]+a*x*Sqrt[1+(y'[x])^2],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2i\arctan\left(\frac{y(x)}{x\sqrt{a^2-\frac{y(x)^2}{x^2}-1}}\right)-2ia\arctan\left(\frac{ay(x)}{x\sqrt{a^2-\frac{y(x)^2}{x^2}-1}}\right)+a\log\left(\frac{y(x)^2}{x^2}+1\right)}{2a^2-2}=\frac{a\log(x-a^2x)}{1-a^2}+c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{-2i\arctan\left(\frac{y(x)}{x\sqrt{a^2-\frac{y(x)^2}{x^2}-1}}\right)+2ia\arctan\left(\frac{ay(x)}{x\sqrt{a^2-\frac{y(x)^2}{x^2}-1}}\right)+a\log\left(\frac{y(x)^2}{x^2}+1\right)}{2a^2-2}=\frac{a\log(x-a^2x)}{1-a^2}+c_1,y(x)\right]$$

6.16 problem 16

Internal problem ID [4421]
Internal file name [`OUTPUT/3914_Sunday_June_05_2022_11_41_54_AM_95615808/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 16.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_dAlembert]`

$$\boxed{-y'y - ay'^2 = -x}$$

6.16.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$-a\,p^2 - py = -x$$

Solving for y from the above results in

$$y = \frac{x}{p} - ap \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{1}{p}$$
$$g = -ap$$

Hence (2) becomes

$$p - \frac{1}{p} = \left(-\frac{x}{p^2} - a\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{1}{p} = 0$$

Solving for p from the above gives

$$p = 1$$
$$p = -1$$

Substituting these in (1A) gives

$$y = a - x$$
$$y = -a + x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{1}{p(x)}}{-\frac{x}{p(x)^2} - a} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{-\frac{x(p)}{p^2} - a}{p - \frac{1}{p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$
\begin{aligned}
p(p) &= \frac{1}{p^{3}-p} \\
q(p) &= -\frac{ap}{p^{2}-1}
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) + \frac{x(p)}{p^{3}-p} = -\frac{ap}{p^{2}-1}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{p^{3}-p}dp} \\
&= \mathrm{e}^{\frac{\ln(p+1)}{2}+\frac{\ln(p-1)}{2}-\ln(p)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\sqrt{p+1}\,\sqrt{p-1}}{p}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{ap}{p^{2}-1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p}\right) &= \left(\frac{\sqrt{p+1}\,\sqrt{p-1}}{p}\right)\left(-\frac{ap}{p^{2}-1}\right) \\
\mathrm{d}\left(\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p}\right) &= \left(-\frac{a\sqrt{p+1}\,\sqrt{p-1}}{p^{2}-1}\right)\mathrm{d}p
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p} &= \int -\frac{a\sqrt{p+1}\,\sqrt{p-1}}{p^{2}-1}\,\mathrm{d}p \\
\frac{\sqrt{p+1}\,\sqrt{p-1}\,x}{p} &= -\frac{a\sqrt{p+1}\,\sqrt{p-1}\,\ln\left(p+\sqrt{p^{2}-1}\right)}{\sqrt{p^{2}-1}} + c_{1}
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p+1}\,\sqrt{p-1}}{p}$ results in

$$
x(p) = -\frac{pa\ln\left(p+\sqrt{p^{2}-1}\right)}{\sqrt{p^{2}-1}} + \frac{c_{1}p}{\sqrt{p+1}\,\sqrt{p-1}}
$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$
\begin{aligned}
p &= \frac{-y+\sqrt{y^2+4ax}}{2a} \\
p &= -\frac{y+\sqrt{y^2+4ax}}{2a}
\end{aligned}
$$

Substituting the above in the solution for x found above gives

$$
x = \left(-y+\sqrt{y^2+4ax}\right)\left(-\frac{\sqrt{2}\left(-\ln(2)+\ln\left(\frac{\sqrt{2}\sqrt{\frac{-y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}\,a+\sqrt{y^2+4ax}-y}{a}\right)\right)}{2\sqrt{\frac{-y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}} + \frac{c_1}{\sqrt{\frac{-y+\sqrt{y^2+4ax}+2a}{a}}\sqrt{\frac{-y+\sqrt{y^2+4ax}-2a}{a}}\,a}\right)
$$

$$
x = \left(y+\sqrt{y^2+4ax}\right)\left(\frac{\sqrt{2}\left(-\ln(2)+\ln\left(\frac{\sqrt{2}\sqrt{\frac{y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}\,a-\sqrt{y^2+4ax}-y}{a}\right)\right)}{2\sqrt{\frac{y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}} - \frac{c_1}{\sqrt{\frac{-y-\sqrt{y^2+4ax}+2a}{a}}\sqrt{\frac{-y-\sqrt{y^2+4ax}-2a}{a}}\,a}\right)
$$

Summary

The solution(s) found are the following

$$y = a - x \tag{1}$$

$$y = -a + x \tag{2}$$

$$x = \left(-y + \sqrt{y^2+4ax}\right)\left(-\frac{\sqrt{2}\left(-\ln\left(2\right)+\ln\left(\frac{\sqrt{2}\,\sqrt{\frac{-y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}\,a+\sqrt{y^2+4ax}-y}{a}\right)\right)}{2\sqrt{\frac{-y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}} + \frac{c_1}{\sqrt{\frac{-y+\sqrt{y^2+4ax}+2a}{a}}\,\sqrt{\frac{-y+\sqrt{y^2+4ax}-2a}{a}}\,a}\right) \tag{3}$$

$$x = \left(y + \sqrt{y^2+4ax}\right)\left(\frac{\sqrt{2}\left(-\ln\left(2\right)+\ln\left(\frac{\sqrt{2}\,\sqrt{\frac{y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}\,a-\sqrt{y^2+4ax}-y}{a}\right)\right)}{2\sqrt{\frac{y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}} - \frac{c_1}{\sqrt{\frac{-y-\sqrt{y^2+4ax}+2a}{a}}\,\sqrt{\frac{-y-\sqrt{y^2+4ax}-2a}{a}}\,a}\right) \tag{4}$$

Verification of solutions

$$y = a - x$$

Verified OK.

$$y = -a + x$$

Verified OK.

$$x = \left(-y + \sqrt{y^2+4ax}\right)\left(-\frac{\sqrt{2}\left(-\ln(2)+\ln\left(\frac{\sqrt{2}\sqrt{\frac{-y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}\,a+\sqrt{y^2+4ax}-y}{a}\right)\right)}{2\sqrt{\frac{-y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}} + \frac{c_1}{\sqrt{\frac{-y+\sqrt{y^2+4ax}+2a}{a}}\sqrt{\frac{-y+\sqrt{y^2+4ax}-2a}{a}}\,a}\right)$$

Warning, solution could not be verified

$$x = \left(y + \sqrt{y^2+4ax}\right)\left(\frac{\sqrt{2}\left(-\ln(2)+\ln\left(\frac{\sqrt{2}\sqrt{\frac{y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}\,a-\sqrt{y^2+4ax}-y}{a}\right)\right)}{2\sqrt{\frac{y\sqrt{y^2+4ax}-2a^2+2ax+y^2}{a^2}}} - \frac{c_1}{\sqrt{\frac{-y-\sqrt{y^2+4ax}+2a}{a}}\sqrt{\frac{-y-\sqrt{y^2+4ax}-2a}{a}}\,a}\right)$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 396

```
dsolve(x-y(x)*diff(y(x),x)=a*(diff(y(x),x))^2,y(x), singsol=all)
```

$$\frac{c_1\left(y(x)-\sqrt{4ax+y(x)^2}\right)}{\sqrt{\frac{-y(x)+\sqrt{4ax+y(x)^2}-2a}{a}}\,\sqrt{\frac{-y(x)+\sqrt{4ax+y(x)^2}+2a}{a}}}+x-\frac{\left(y(x)-\sqrt{4ax+y(x)^2}\right)\left(-3\ln(2)+2\ln\left(\frac{2\sqrt{\frac{y(x)^2-y(x)\sqrt{4ax+y(x)^2}-2a^2+2ax}{a^2}}\,a-\left(y(x)-\sqrt{4ax+y(x)^2}\right)\sqrt{2}}{a}\right)\right)\sqrt{2}}{4\sqrt{\frac{y(x)^2-y(x)\sqrt{4ax+y(x)^2}-2a^2+2ax}{a^2}}}=0$$

$$\frac{c_1\left(y(x)+\sqrt{4ax+y(x)^2}\right)}{2\sqrt{\frac{-y(x)-\sqrt{4ax+y(x)^2}-2a}{a}}\,\sqrt{\frac{-y(x)-\sqrt{4ax+y(x)^2}+2a}{a}}}+x-\frac{\left(-\frac{3\ln(2)}{2}+\ln\left(\frac{2\sqrt{\frac{y(x)\sqrt{4ax+y(x)^2}-2a^2+2ax+y(x)^2}{a^2}}\,a-\left(y(x)+\sqrt{4ax+y(x)^2}\right)\sqrt{2}}{a}\right)\right)\left(y(x)+\sqrt{4ax+y(x)^2}\right)\sqrt{2}}{2\sqrt{\frac{y(x)\sqrt{4ax+y(x)^2}-2a^2+2ax+y(x)^2}{a^2}}}=0$$

✓ Solution by Mathematica

Time used: 0.55 (sec). Leaf size: 79

```
DSolve[x-y[x]*y'[x]==a*(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\left\{x=-\frac{2aK[1]\arctan\left(\frac{\sqrt{1-K[1]^2}}{K[1]+1}\right)}{\sqrt{1-K[1]^2}}+\frac{c_1K[1]}{\sqrt{1-K[1]^2}},y(x)=\frac{x}{K[1]}-aK[1]\right\},\{y(x),K[1]\}\right]$$

6.17 problem 17

Internal problem ID [4422]
Internal file name [`OUTPUT/3915_Sunday_June_05_2022_11_44_35_AM_53176799/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 17.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y'y - a\sqrt{1 + y'^2} = -x}$$

6.17.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$py - a\sqrt{p^2 + 1} = -x$$

Solving for y from the above results in

$$y = -\frac{x}{p} + \frac{a\sqrt{p^2 + 1}}{p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\frac{1}{p}$$
$$g = \frac{a\sqrt{p^2+1}}{p}$$

Hence (2) becomes

$$p + \frac{1}{p} = \left(\frac{x}{p^2} + \frac{a}{\sqrt{p^2+1}} - \frac{a\sqrt{p^2+1}}{p^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{1}{p} = 0$$

Solving for p from the above gives

$$p = i$$
$$p = -i$$

Substituting these in (1A) gives

$$y = -ix$$
$$y = ix$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{1}{p(x)}}{\frac{x}{p(x)^2} + \frac{a}{\sqrt{p(x)^2+1}} - \frac{a\sqrt{p(x)^2+1}}{p(x)^2}} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{\frac{x(p)}{p^2} + \frac{a}{\sqrt{p^2+1}} - \frac{a\sqrt{p^2+1}}{p^2}}{p + \frac{1}{p}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{1}{p\left(p^{2}+1\right)} \\
q(p) &= -\frac{a}{p\left(p^{2}+1\right)^{\frac{3}{2}}}
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) - \frac{x(p)}{p\left(p^{2}+1\right)} = -\frac{a}{p\left(p^{2}+1\right)^{\frac{3}{2}}}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{p\left(p^{2}+1\right)}dp} \\
&= \mathrm{e}^{\frac{\ln\left(p^{2}+1\right)}{2}-\ln(p)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\sqrt{p^{2}+1}}{p}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{a}{p\left(p^{2}+1\right)^{\frac{3}{2}}}\right) \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\sqrt{p^{2}+1}\,x}{p}\right) &= \left(\frac{\sqrt{p^{2}+1}}{p}\right)\left(-\frac{a}{p\left(p^{2}+1\right)^{\frac{3}{2}}}\right) \\
\mathrm{d}\left(\frac{\sqrt{p^{2}+1}\,x}{p}\right) &= \left(-\frac{a}{p^{2}\left(p^{2}+1\right)}\right)\mathrm{d}p
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\sqrt{p^{2}+1}\,x}{p} &= \int -\frac{a}{p^{2}\left(p^{2}+1\right)}\,\mathrm{d}p \\
\frac{\sqrt{p^{2}+1}\,x}{p} &= a\arctan\left(p\right) + \frac{a}{p} + c_{1}
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{p^{2}+1}}{p}$ results in

$$
x(p) = \frac{p\left(a\arctan\left(p\right)+\frac{a}{p}\right)}{\sqrt{p^{2}+1}} + \frac{c_{1}p}{\sqrt{p^{2}+1}}
$$

which simplifies to

$$x(p) = \frac{a \arctan(p) p + c_1 p + a}{\sqrt{p^2+1}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{xy + \sqrt{y^2a^2 - a^4 + a^2x^2}}{a^2 - y^2}$$
$$p = -\frac{-xy + \sqrt{y^2a^2 - a^4 + a^2x^2}}{a^2 - y^2}$$

Substituting the above in the solution for x found above gives

$$x = \frac{a\left(xy + \sqrt{-a^2(-y^2+a^2-x^2)}\right) \arctan\left(\frac{xy+\sqrt{-a^2(-y^2+a^2-x^2)}}{a^2-y^2}\right) + a^3 - y^2a + yc_1x + \sqrt{-a^2(-y^2+a^2-}}{\sqrt{\frac{2\sqrt{-a^2(-y^2+a^2-x^2)}\,xy+y^4+(-a^2+x^2)y^2+a^2x^2}{(a^2-y^2)^2}}\,(a^2-y^2)}$$

$$x = \frac{-a\left(xy - \sqrt{-a^2(-y^2+a^2-x^2)}\right) \arctan\left(\frac{-xy+\sqrt{-a^2(-y^2+a^2-x^2)}}{a^2-y^2}\right) + a^3 - y^2a + yc_1x - \sqrt{-a^2(-y^2+a}}{\sqrt{\frac{-2\sqrt{-a^2(-y^2+a^2-x^2)}\,xy+y^4+(-a^2+x^2)y^2+a^2x^2}{(a^2-y^2)^2}}\,(a^2-y^2)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -ix \tag{1}$$
$$y = ix \tag{2}$$
$$x = \frac{a\left(xy + \sqrt{-a^2(-y^2+a^2-x^2)}\right) \arctan\left(\frac{xy+\sqrt{-a^2(-y^2+a^2-x^2)}}{a^2-y^2}\right) + a^3 - y^2a + yc_1x + \sqrt{-a^2(-y^2+a^2-}}{\sqrt{\frac{2\sqrt{-a^2(-y^2+a^2-x^2)}\,xy+y^4+(-a^2+x^2)y^2+a^2x^2}{(a^2-y^2)^2}}\,(a^2-y^2)} \tag{3}$$
$$x = \frac{-a\left(xy - \sqrt{-a^2(-y^2+a^2-x^2)}\right) \arctan\left(\frac{-xy+\sqrt{-a^2(-y^2+a^2-x^2)}}{a^2-y^2}\right) + a^3 - y^2a + yc_1x - \sqrt{-a^2(-y^2+a}}{\sqrt{\frac{-2\sqrt{-a^2(-y^2+a^2-x^2)}\,xy+y^4+(-a^2+x^2)y^2+a^2x^2}{(a^2-y^2)^2}}\,(a^2-y^2)} \tag{4}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$x = \frac{a\left(xy + \sqrt{-a^2(-y^2+a^2-x^2)}\right)\arctan\left(\frac{xy+\sqrt{-a^2(-y^2+a^2-x^2)}}{a^2-y^2}\right) + a^3 - y^2 a + y c_1 x + \sqrt{-a^2(-y^2+a^2-}{\sqrt{\frac{2\sqrt{-a^2(-y^2+a^2-x^2)}\,xy+y^4+(-a^2+x^2)y^2+a^2x^2}{(a^2-y^2)^2}}\,(a^2-y^2)}$$

Verified OK.

$$x = \frac{-a\left(xy - \sqrt{-a^2(-y^2+a^2-x^2)}\right)\arctan\left(\frac{-xy+\sqrt{-a^2(-y^2+a^2-x^2)}}{a^2-y^2}\right) + a^3 - y^2 a + y c_1 x - \sqrt{-a^2(-y^2+a^2}{\sqrt{\frac{-2\sqrt{-a^2(-y^2+a^2-x^2)}\,xy+y^4+(-a^2+x^2)y^2+a^2x^2}{(a^2-y^2)^2}}\,(a^2-y^2)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 237

```
dsolve(x+y(x)*diff(y(x),x)=a*sqrt(1+(diff(y(x),x))^2),y(x), singsol=all)
```

$$y(x) = \csc\left(\operatorname{RootOf}\left((\sin(\_Z)\_Za+\sin(\_Z)\,c_1-\cos(\_Z)\,a-x)\,(\sin(\_Z)\_Za+\sin(\_Z)\,c_1+\cos(\_Z)\,a-x)\right)\right)\,c$$
$$-\cot\left(\operatorname{RootOf}\left((\sin(\_Z)\_Za+\sin(\_Z)\,c_1-\cos(\_Z)\,a-x)\,(\sin(\_Z)\_Za+\sin(\_Z)\,c_1+\cos(\_Z)\,a-x)\right.\right.$$

$$y(x) = \csc\left(\operatorname{RootOf}\left((\sin(\_Z)\_Za+\sin(\_Z)\,c_1+\cos(\_Z)\,a+x)\,(\sin(\_Z)\_Za+\sin(\_Z)\,c_1-\cos(\_Z)\,a+x)\right)\right)\,c$$
$$-\cot\left(\operatorname{RootOf}\left((\sin(\_Z)\_Za+\sin(\_Z)\,c_1+\cos(\_Z)\,a+x)\,(\sin(\_Z)\_Za+\sin(\_Z)\,c_1-\cos(\_Z)\,a+x)\right.\right.$$

✓ Solution by Mathematica

Time used: 3.538 (sec). Leaf size: 388

```
DSolve[x+y[x]*y'[x]==a*Sqrt[1+(y'[x])^2],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\frac{2a\sqrt{a^2y(x)^2-a^4}\arctan\left(\frac{ax\sqrt{y(x)^2-a^2}}{y(x)\left(\sqrt{a^2(y(x)^2-a^2)}-\sqrt{a^2(-a^2+x^2+y(x)^2)}\right)+a^2x}\right)}{\sqrt{y(x)^2-a^2}}-\sqrt{a^2(-a^2+x^2+y(x)^2)}}{a^2}\right.$$
$$\left.-\frac{a\sqrt{y(x)^2-a^2}\arctan\left(\frac{\sqrt{y(x)^2-a^2}}{a}\right)}{\sqrt{a^2(y(x)^2-a^2)}}=c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{a\sqrt{y(x)^2-a^2}\arctan\left(\frac{\sqrt{y(x)^2-a^2}}{a}\right)}{\sqrt{a^2(y(x)^2-a^2)}}\right.$$
$$\left.+\frac{\sqrt{a^2(-a^2+x^2+y(x)^2)}-\frac{2a\sqrt{a^2y(x)^2-a^4}\arctan\left(\frac{ax\sqrt{y(x)^2-a^2}}{y(x)\left(\sqrt{a^2(-a^2+x^2+y(x)^2)}-\sqrt{a^2(y(x)^2-a^2)}\right)+a^2x}\right)}{\sqrt{y(x)^2-a^2}}}{a^2}=c_1,y(x)\right]$$

6.18 problem 18

Internal problem ID [4423]
Internal file name [OUTPUT/3916_Sunday_June_05_2022_11_48_17_AM_21845302/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 18.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

Unable to solve or complete the solution.

$$\boxed{y'y - y^2 + y^2 y'^2 = x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-\frac{1}{2} + \frac{\sqrt{1+4y^2+4x}}{2}}{y} \tag{1}$$

$$y' = \frac{-\frac{1}{2} - \frac{\sqrt{1+4y^2+4x}}{2}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (2)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 3
   `, `-> Computing symmetries using: way = 4`[1, -1/2/y], [1/2+x, 1/2*(2*y^2+x)/y], [1/2*x^
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 77

```
dsolve(y(x)*diff(y(x),x)=x+(y(x)^2-y(x)^2*(diff(y(x),x))^2),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-1-4x}}{2}$$
$$y(x) = \frac{\sqrt{-1-4x}}{2}$$
$$y(x) = -\frac{\sqrt{4x^2+(-8c_1-4)\,x+4c_1^2-1}}{2}$$
$$y(x) = \frac{\sqrt{4x^2+(-8c_1-4)\,x+4c_1^2-1}}{2}$$

✓ Solution by Mathematica

Time used: 0.236 (sec). Leaf size: 69

```
DSolve[y[x]*y'[x]==x+(y[x]^2-y[x]^2*(y'[x])^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}\sqrt{4x^2 - 4(1+4c_1)x - 1 + 16{c_1}^2}$$
$$y(x) \to \frac{1}{2}\sqrt{4x^2 - 4(1+4c_1)x - 1 + 16{c_1}^2}$$

6.19 problem 19

Internal problem ID [4424]
Internal file name [`OUTPUT/3917_Sunday_June_05_2022_11_48_39_AM_42490421/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 19.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y-\frac{1}{\sqrt{1+y'^2}}-\frac{y'}{\sqrt{1+y'^2}}=x}$$

6.19.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$y-\frac{1}{\sqrt{p^2+1}}-\frac{p}{\sqrt{p^2+1}}=x$$

Solving for y from the above results in

$$y=x+\frac{p+1}{\sqrt{p^2+1}} \tag{1A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f+(xf'+g')\frac{dp}{dx} \\ p-f &= (xf'+g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 1$$
$$g = \frac{p+1}{\sqrt{p^2+1}}$$

Hence (2) becomes

$$p - 1 = \left(\frac{1}{\sqrt{p^2+1}} - \frac{(p+1)\,p}{(p^2+1)^{\frac{3}{2}}}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - 1 = 0$$

Solving for p from the above gives

$$p = 1$$

Substituting these in (1A) gives

$$y = x + \sqrt{2}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - 1}{\frac{1}{\sqrt{p(x)^2+1}} - \frac{(p(x)+1)p(x)}{\left(p(x)^2+1\right)^{\frac{3}{2}}}} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{\frac{1}{\sqrt{p^2+1}} - \frac{(p+1)p}{(p^2+1)^{\frac{3}{2}}}}{p-1} \tag{4}$$

This ODE is now solved for $x(p)$. Integrating both sides gives

$$\begin{aligned} x(p) &= \int -\frac{1}{(p^2+1)^{\frac{3}{2}}}\,\mathrm{d}p \\ &= -\frac{p}{\sqrt{p^2+1}} + c_2 \end{aligned}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$
\begin{aligned}
p &= \frac{y\left(-x+y+\sqrt{-y^2+2xy-x^2+2}\right)}{x^2-2xy+y^2-1} - \frac{x\left(-x+y+\sqrt{-y^2+2xy-x^2+2}\right)}{x^2-2xy+y^2-1} - 1 \\
p &= \frac{x\left(-y+x+\sqrt{-y^2+2xy-x^2+2}\right)}{x^2-2xy+y^2-1} - \frac{y\left(-y+x+\sqrt{-y^2+2xy-x^2+2}\right)}{x^2-2xy+y^2-1} - 1
\end{aligned}
$$

Substituting the above in the solution for x found above gives

$$
\begin{aligned}
x &= \frac{\sqrt{2}\left(-1+(x-y)\sqrt{-y^2+2xy-x^2+2}\right)}{2(x-y+1)(x-y-1)\sqrt{\frac{-x\sqrt{-y^2+2xy-x^2+2}+y\sqrt{-y^2+2xy-x^2+2}+1}{(x^2-2xy+y^2-1)^2}}} + c_2 \\
x &= -\frac{\sqrt{2}\left(1+(x-y)\sqrt{-y^2+2xy-x^2+2}\right)}{2(x-y+1)(x-y-1)\sqrt{\frac{x\sqrt{-y^2+2xy-x^2+2}-y\sqrt{-y^2+2xy-x^2+2}+1}{(x^2-2xy+y^2-1)^2}}} + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x + \sqrt{2} \tag{1}
$$

$$
x = \frac{\sqrt{2}\left(-1+(x-y)\sqrt{-y^2+2xy-x^2+2}\right)}{2(x-y+1)(x-y-1)\sqrt{\frac{-x\sqrt{-y^2+2xy-x^2+2}+y\sqrt{-y^2+2xy-x^2+2}+1}{(x^2-2xy+y^2-1)^2}}} + c_2 \tag{2}
$$

$$
x = -\frac{\sqrt{2}\left(1+(x-y)\sqrt{-y^2+2xy-x^2+2}\right)}{2(x-y+1)(x-y-1)\sqrt{\frac{x\sqrt{-y^2+2xy-x^2+2}-y\sqrt{-y^2+2xy-x^2+2}+1}{(x^2-2xy+y^2-1)^2}}} + c_2 \tag{3}
$$

Verification of solutions

$$y = x + \sqrt{2}$$

Verified OK.

$$x = \frac{\sqrt{2}\,\left(-1+(x-y)\sqrt{-y^2+2xy-x^2+2}\right)}{2\,(x-y+1)\,(x-y-1)\sqrt{\frac{-x\sqrt{-y^2+2xy-x^2+2}+y\sqrt{-y^2+2xy-x^2+2}+1}{(x^2-2xy+y^2-1)^2}}} + c_2$$

Warning, solution could not be verified

$$x = -\frac{\sqrt{2}\,\left(1+(x-y)\sqrt{-y^2+2xy-x^2+2}\right)}{2\,(x-y+1)\,(x-y-1)\sqrt{\frac{x\sqrt{-y^2+2xy-x^2+2}-y\sqrt{-y^2+2xy-x^2+2}+1}{(x^2-2xy+y^2-1)^2}}} + c_2$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 49

```
dsolve(y(x)-1/sqrt(1+(diff(y(x),x))^2)=(x+diff(y(x),x)/sqrt(1+(diff(y(x),x))^2)),y(x), sings
```

$$y(x) = \frac{c_1\sqrt{-\frac{1}{(-c_1+x+1)(x-c_1-1)}}+1}{\sqrt{-\frac{1}{(-c_1+x+1)(x-c_1-1)}}}$$

✓ Solution by Mathematica

Time used: 42.598 (sec). Leaf size: 15753

```
DSolve[y[x]-1/Sqrt[1+(y'[x])^2]==(x+y'[x]/Sqrt[1+(y'[x])^2]),y[x],x,IncludeSingularSolutions
```

Too large to display

6.20 problem 20

Internal problem ID [4425]
Internal file name [`OUTPUT/3918_Sunday_June_05_2022_11_48_56_AM_11174133/index.tex`]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 20.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{y - 2xy' - xy'^2 = 0}$$

6.20.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$-x\,p^2 - 2xp + y = 0$$

Solving for y from the above results in

$$y = \left(p^2 + 2p\right)x \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p^2 + 2p \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
-p^2 - p = x(2p + 2)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p^2 - p = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= -1 \\
p &= 0
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -x \\
y &= 0
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{-p(x)^2 - p(x)}{x\,(2p\,(x) + 2)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{2x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$
p'(x) + \frac{p(x)}{2x} = 0
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{2x}dx} \\
&= \sqrt{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu p &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\, p\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\sqrt{x}\, p = c_1
$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$
p(x) = \frac{c_1}{\sqrt{x}}
$$

Substituing the above solution for p in (2A) gives

$$
y = \left(\frac{c_1^2}{x} + \frac{2c_1}{\sqrt{x}}\right) x
$$

Summary

The solution(s) found are the following

$$
y = -x \tag{1}
$$

$$
y = 0 \tag{2}
$$

$$
y = \left(\frac{c_1^2}{x} + \frac{2c_1}{\sqrt{x}}\right) x \tag{3}
$$

Verification of solutions

$$
y = -x
$$

Verified OK.

$$
y = 0
$$

Verified OK.

$$
y = \left(\frac{c_1^2}{x} + \frac{2c_1}{\sqrt{x}}\right) x
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve(y(x)-2*x*diff(y(x),x)=(x*(diff(y(x),x))^2),y(x), singsol=all)
```

$$y(x) = -x$$
$$y(x) = c_1 + 2\sqrt{c_1 x}$$
$$y(x) = c_1 - 2\sqrt{c_1 x}$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 63

```
DSolve[y[x]-2*x*y'[x]==(x*(y'[x])^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{c_1} - 2e^{\frac{c_1}{2}}\sqrt{x}$$
$$y(x) \to 2e^{-\frac{c_1}{2}}\sqrt{x} + e^{-c_1}$$
$$y(x) \to 0$$
$$y(x) \to -x$$

6.21 problem 21

Internal problem ID [4426]
Internal file name [OUTPUT/3919_Sunday_June_05_2022_11_49_07_AM_64437950/index.tex]

Book: Differential Equations, By George Boole F.R.S. 1865
Section: Chapter 7
Problem number: 21.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "riccati", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{y - xy'}{y^2 + y'} - \frac{y - xy'}{1 + x^2 y'} = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y}{x} \tag{1}$$

$$y' = \frac{y^2 - 1}{x^2 - 1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y}\,dy &= \frac{1}{x}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\ \ln(y) &= \ln(x) + c_1 \\ y &= \mathrm{e}^{\ln(x)+c_1} \\ &= c_1 x\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

Verification of solutions

$$y = c_1 x$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2-1}{x^2-1}\end{aligned}$$

Where $f(x) = \frac{1}{x^2-1}$ and $g(y) = y^2 - 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2-1}\,dy &= \frac{1}{x^2-1}\,dx \\ \int \frac{1}{y^2-1}\,dy &= \int \frac{1}{x^2-1}\,dx \\ -\operatorname{arctanh}(y) &= -\operatorname{arctanh}(x) + c_2\end{aligned}$$

Which results in

$$y = -\tanh(-\operatorname{arctanh}(x) + c_2)$$

Summary

The solution(s) found are the following

$$y = -\tanh(-\operatorname{arctanh}(x) + c_2) \tag{1}$$

Verification of solutions

$$y = -\tanh\left(-\operatorname{arctanh}\left(x\right)+c_2\right)$$

Verified OK.

6.21.1 Maple step by step solution

Let's solve

$\frac{y-xy'}{y^2+y'}-\frac{y-xy'}{1+x^2y'}=0$

- Highest derivative means the order of the ODE is 1

  y'
- Separate variables

  $\frac{y'}{y}=\frac{1}{x}$
- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx=\int \frac{1}{x}dx+c_1$
- Evaluate integral

  $\ln\left(y\right)=\ln\left(x\right)+c_1$
- Solve for y

  $y = x\,\mathrm{e}^{c_1}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
<- symmetries for implicit equations successful
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 19

```
dsolve((y(x)-x*diff(y(x),x))/(y(x)^2+diff(y(x),x))=(y(x)-x*diff(y(x),x))/(1+x^2*diff(y(x),x)
```

$$y(x) = c_1 x$$
$$y(x) = -\tanh\left(-\operatorname{arctanh}\left(x\right) + c_1\right)$$

✓ Solution by Mathematica

Time used: 60.122 (sec). Leaf size: 45

```
DSolve[(y[x]-x*y'[x])/(y[x]^2+y'[x])==(y[x]-x*y'[x])/(1+x^2*y'[x]),y[x],x,IncludeSingularSol
```

$$y(x) \to -\frac{x + e^{2c_1}(x-1) + 1}{-x + e^{2c_1}(x-1) - 1}$$
$$y(x) \to c_1 x$$